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Spectral triples with multitwisted real structure

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Abstract. We generalize the notion of spectral triple with reality structure to spectral triples with multitwisted real structure, the class of which is closed under the tensor product composition. In particular, we introduce a multitwisted first-order condition (characterizing the Dirac operators as an analogue of first-order differential operator). This provides a unified description of the known examples, which include rescaled triples with the conformal factor from the commutant of the algebra and (on the algebraic level) triples on quantum disc and on quantum cone, that satisfy twisted first-order condition of Brzeziński et al. (2016, 2019), as well as asymmetric tori, non-scalar conformal rescaling and noncommutative circle bundles. In order to deal with them, we allow twists that do not implement automorphisms of the algebra of spectral triple.

1. Introduction

Spectral triples were introduced in [3] as a setup to generalize differential geometry to noncommutative algebras, that carries topological information and allows explicit analytic computations of index parings [8]. The concept of *real spectral triples* [4] was motivated by successful applications to the Standard Model of particle physics and also by the quest for the equivalence in the commutative case with the geometry of spin manifolds, culminating in the reconstruction theorem [5]. The role of the real structure in noncommutative examples became evident in the relation between the classes of equivariant real spectral triples and the spin structures on noncommutative tori [17].

While the theory of real spectral triples gained more and more examples [7, 10, 15], some interesting noncommutative geometries did not fit into the original set of axioms for real spectral triples (unbounded K-cycles). Remarkable ones were the twisted (or modular) spectral triples on the curved noncommutative torus [9], intensively studied afterwards (see [14] for a review of results of curvature computations). A scheme to incorporate the noncommutative analogue of conformally rescaled geometries in the framework of usual spectral triples was proposed in [1]. Therein the Dirac operator is rescaled by a positive element from the commutant of the algebra, thus maintaining the bounded commutator with elements of the algebra, but leading to a twisted reality structure, together with a generalized first-order condition. This construction was further studied in [2], where the relation between spectral triples with twisted real structure and real twisted spectral triples [16] was uncovered.

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Yet even these generalizations do not embrace the recent examples of partially rescaled conformal torus [13] and spectral triples over a circle bundle with the Dirac operator compatible with a given connection [12]. Moreover, neither the class of spectral triples with a twisted first-order condition nor the twisted spectral triples is closed under the tensor product composition of spectral triples. We propose here a construction which complies with tensor product of spectral triples, allows for fluctuations, and covers almost all known interesting and geometrically motivated examples. To avoid confusion with *twisted spectral triples* (and *real twisted spectral triples*), we use the name *spectral triples with (multi)twisted real structure* or, for brevity, *(multi)twisted-real spectral triples*.

2. Multitwisted real structure for spectral triples

Consider a spectral triple (A, H, D), where A is an $*$-algebra identified with a subalgebra of bounded operators $B(H)$ on a Hilbert space H, and D is a densely defined self-adjoint operator on H such that D has a compact resolvent and for each $a \in A$ the commutator $[D, a]$ is bounded. Let J be an antilinear isometry on H, such that $J^2 = \pm 1$ and

$$[a, JbJ^{-1}] = 0, \tag{2.1}$$

in which case (with a slight abuse of terminology) we call (A, H, D, J) a real spectral triple. If in addition there is a grading γ of H, $\gamma^2 = 1$, such that $D\gamma = -\gamma D$, $[\gamma, a] = 0$ for all a in A and $\gamma J = \pm J\gamma$, we call (A, H, D, J, γ) a real even spectral triple.

Definition 2.1. We say that a real spectral triple (A, H, D, J) is a *spectral triple with multitwisted real structure* if there are N densely defined operators D_ℓ, $\ell = 1, \dots, N$, the domains of which contain the domain of D, such that $\sum_{\ell=1}^{N} D_\ell = D$ and for every ℓ there exists an operator $\nu_\ell \in B(H)$, with bounded inverse, such that for every $a, b \in A$, the *multitwisted zero-order condition* holds:

$$\big[a, J\bar{\nu}_\ell(b)J^{-1}\big] = 0 = \big[a, J\bar{\nu}_\ell^{-1}(b)J^{-1}\big], \tag{2.2}$$

where $\bar{\nu}_\ell := \mathrm{Ad}_{\nu_\ell} \in \mathrm{Aut}(B(H))$. Additionally, if the spectral triple is even, we assume that

$$\gamma\nu_\ell^2 = \nu_\ell^2\gamma, \quad \forall \ell. \tag{2.3}$$

We say that *multitwisted first-order condition* holds if

$$[D_\ell, a]J\bar{\nu}_\ell(b)J^{-1} = J\bar{\nu}_\ell^{-1}(b)J^{-1}[D_\ell, a], \tag{2.4}$$

and that *multitwisted ε'-condition* holds if

$$D_\ell J\nu_\ell = \varepsilon'\nu_\ell JD_\ell, \quad \text{where } \varepsilon' = \pm 1, \tag{2.5}$$

and we call the multitwisted real spectral triple *regular* if

$$\nu_\ell J\nu_\ell = J, \tag{2.6}$$

for each ℓ.

Remark 2.2. We assumed that the domain of the full Dirac operator D is contained in the domains of all operators D_ℓ so the decomposition makes sense at least on this domain.

However, in principle it is not required that individually each D_ℓ is self-adjoint with compact resolvent and each $[D_\ell, a]$ is bounded as in order to obtain a spectral triple only the sum D of all D_ℓ is required to have these properties. It will be interesting to find examples of this situation.

Remark 2.3. The notion of a spectral triple with a twisted real structure in [1,2] fits this definition as a special case when $N = 1$ and $\bar{\nu}_1$ is an automorphism of A, since then the multitwisted zero-order condition (2.2) is equivalent with (2.1), and the relation (2.4), though it appears slightly different, is equivalent with the previous twisted first-order condition by taking $b = \bar{\nu}(c)$. Definition 2.1 is, however, slightly more general as we do not assume that $\bar{\nu}_\ell$ are automorphisms of the algebra A, and in order that the multitwisted first-order condition (2.4) be satisfied for all one-forms, that is if $\omega_\ell = \sum_i a_i[D_\ell, b_i]$, then $\omega_\ell J\bar{\nu}_\ell(b)J^{-1} = J\bar{\nu}_\ell^{-1}(b)J^{-1}\omega_\ell$, we require that besides (2.1) also $[a, J\bar{\nu}_\ell(b)J^{-1}] = 0$ holds. Furthermore, the consistency with the A-bimodule structure of one-forms requires also that $[a, J\bar{\nu}_\ell^{-1}(b)J^{-1}] = 0$; however, thanks to the consistency under the adjoint operation in A, if $\nu_\ell^* = \nu_\ell$, it suffices to impose one of these two conditions.

2.1. Properties of spectral triples with multitwisted real structure

The important feature of the spectral triples which are multitwisted real is that they are closed under the product.

Proposition 2.4. *Let $(A', H', D', J', \gamma')$ and (A'', H'', D'', J'') be spectral triples with multitwisted real structure (the first one even and satisfying $J'\gamma' = \gamma' J'$), with $D' = \sum_{j=1}^{N'} D'_j$ and $D'' = \sum_{k=1}^{N''} D''_k$, for the twists $\nu'_j \in B(H')$ and $\nu''_k \in B(H'')$, respectively. Then*

$$(A' \otimes A'', H' \otimes H'', D' \otimes \mathrm{id} + \gamma' \otimes D'', J' \otimes J'') \tag{2.7}$$

is a multitwisted real spectral triple with the Dirac operator decomposing as a sum of

$$D_\ell = \begin{cases} D'_\ell \otimes \mathrm{id}, & 1 \le \ell \le N', \\ \gamma' \otimes D''_{\ell - N'}, & N'+1 \le \ell \le N' + N'', \end{cases} \tag{2.8}$$

for the twists

$$\nu_\ell = \begin{cases} \nu'_\ell \otimes \mathrm{id}, & 1 \le \ell \le N', \\ \mathrm{id} \otimes \nu''_{\ell - N'}, & N'+1 \le \ell \le N' + N''. \end{cases} \tag{2.9}$$

Furthermore, if both triples satisfy the multitwisted zero- or first-order conditions, and are regular, then this holds for their tensor product.

Proof. It is well known that (2.7) is a real spectral triple. The first equality in condition (2.2) for $1 \le \ell \le N'$ follows from (2.2) for the first spectral triple and (2.1) for the second one

$$(a' \otimes a'')\big(J'\bar{\nu}_\ell(b')J'^{-1} \otimes J''b''J''^{-1}\big) = \big(J'\bar{\nu}_\ell(b')J'^{-1} \otimes J''b''J''^{-1}\big)(a' \otimes a'')$$

and analogously for $N'+1 \le \ell \le N' + N''$, and similarly for the second equality.

Condition (2.4) for $1 \le \ell \le N'$ reads

$$\big([D_\ell, a'] \otimes a''\big)\big(J'\bar{\nu}_\ell(b')J'^{-1} \otimes J''b''J''^{-1}\big)$$
$$= \big(J'\bar{\nu}_\ell(b')J'^{-1} \otimes J''b''J''^{-1}\big)\big([D_\ell, a'] \otimes a''\big)$$

and is satisfied by (2.4) for the first spectral triple and (2.1) for the second one, and analogously for $N'+1 \le \ell \le N'+N''$ using the properties of γ'. Condition (2.5) for $1 \le \ell \le N'$ (respectively, $N'+1 \le \ell \le N'+N''$) follows from (2.2) for the first (respectively, second) spectral triple. Finally, (2.6) is immediate. ∎

Remark 2.5. Note that the resulting spectral triple is not even (as a product of an even and an odd triple). All other cases of the product of even and odd spectral triples can be also considered and we postpone the full discussion till future work.

In [1] we have demonstrated that, with an appropriate definition of the fluctuated Dirac operator, a perturbation of D by a one form and its appropriate image in the commutant of the algebra A yields the Dirac operator with the same properties. This functorial property holds also in the multitwisted case.

Proposition 2.6. *Assume that (A, H, D, J), where $D = \sum_{\ell=1}^N D_\ell$, is a spectral triple with multitwisted real structure satisfying the twisted zero- and first-order conditions (2.2), (2.4). Let $\omega = \sum_i a_i[D, b_i]$ be a self-adjoint one-form. Then (A, H, D_ω, J), where $D_\omega = D + \omega$, is again a multitwisted real spectral triple satisfying the twisted zero- (2.2) and first-order conditions (2.4), with $D_\omega = \sum_{\ell=1}^N (D_\omega)_\ell$, where $(D_\omega)_\ell = D_\ell + \omega_\ell$ and $\omega_\ell = \sum_i a_i[D_\ell, b_i]$, and with the same twists. Moreover, if (A, H, D, J) is regular, then so is (A, H, D_ω, J).*

Proof. Observe that neither (2.2) nor the regularity condition (2.6) changes, so we need to verify only the multitwisted first-order condition. Further, for any ω_ℓ we have that

$$\omega_\ell J\bar{\nu}_\ell(b)J^{-1} = \sum_i a_i[D_\ell, b_i]J\bar{\nu}_\ell(b)J^{-1} = \sum_i a_i J\bar{\nu}_\ell^{-1}(b)J^{-1}[D_\ell, b_i]$$
$$= \sum_i J\bar{\nu}_\ell^{-1}(b)J^{-1}a_i[D_\ell, b_i] = J\bar{\nu}_\ell^{-1}(b)J^{-1}\omega_\ell.$$

Since $[\omega_\ell, a] = \sum_i a_i[D_\ell, b_i a] - \sum_i (a_i b_i)[D_\ell, a] - a\sum_i a_i[D_\ell, b_i]$, for any $a \in A$, then it is of the same form as ω_ℓ, and, in consequence, we have that

$$\begin{aligned}\big[(D_\omega)_\ell, a\big]J\bar{\nu}_\ell(b)J^{-1} &= [D_\ell + \omega_\ell, a]J\bar{\nu}_\ell(b)J^{-1} \\ &= [D_\ell, a]J\bar{\nu}_\ell(b)J^{-1} + [\omega_\ell, a]J\bar{\nu}_\ell(b)J^{-1} \\ &= J\bar{\nu}_\ell^{-1}(b)J^{-1}[D_\ell, a] + J\bar{\nu}_\ell^{-1}(b)[\omega_\ell, a] \\ &= J\bar{\nu}_\ell^{-1}(b)\big[(D_\omega)_\ell, a\big].\end{aligned}$$

∎

Remark 2.7. Note that the above construction does not preserve the multitwisted ε'-condition (2.5). To cure this problem, we modify the manner of fluctuations of D.

Proposition 2.8. *Let (A, H, D, J) be a spectral triple with multitwisted real structure satisfying (2.2) and (2.4). Let ω be a one-form as in Proposition 2.6. If the sum*

$$\sum_{\ell=1}^{N} \nu_\ell J(\omega_\ell) J^{-1} \nu_\ell \tag{2.10}$$

is bounded, then taking

$$(D_\omega)'_\ell = D_\ell + \omega_\ell + \varepsilon' \nu_\ell J(\omega_\ell) J^{-1} \nu_\ell, \tag{2.11}$$

and

$$D'_\omega = \sum_{\ell=1}^{N} (D_\omega)'_\ell, \tag{2.12}$$

(A, H, D'_ω, J) is a spectral triple with a multitwisted real structure, with the same twists, satisfying the conditions (2.2), (2.4), and (2.5). Moreover, for each $a \in A$,

$$\big[(D_\omega)'_\ell, a\big] = \big[(D_\omega)_\ell, a\big]. \tag{2.13}$$

Proof. Let us take $\omega_\ell = \sum_i a_i [D_\ell, b_i]$. Then for any $a \in A$,

$$\begin{aligned}
\nu_\ell J(\omega_\ell) J^{-1} \nu_\ell a &= \nu_\ell J\Big(\sum_i a_i [D_\ell, b_i]\Big) J^{-1} \nu_\ell a \\
&= \nu_\ell J\Big(\sum_i a_i [D_\ell, b_i]\Big)\big(J^{-1} \bar{\nu}_\ell(a) J\big) J^{-1} \nu_\ell \\
&= \nu_\ell J\big(J^{-1} \bar{\nu}_\ell^{-1}(a) J\big)\Big(\sum_i a_i [D_\ell, b_i]\Big) J^{-1} \nu_\ell \\
&= a \nu_\ell J\Big(\sum_i a_i [D_\ell, b_i]\Big) J^{-1} \nu_\ell,
\end{aligned}$$

where, again, we have used (2.2) and (2.4). This shows that for any ω_ℓ and any a (2.13) holds and as a consequence the bimodule of one-forms remains unchanged if we pass from D_ω to D'_ω.

To see the multitwisted ε'-condition (2.5), we check that

$$\begin{aligned}
D'_\ell J \nu_\ell &= \big(D_\ell + \omega_\ell + \varepsilon' \nu_\ell J(\omega_\ell) J^{-1} \nu_\ell\big) J \nu_\ell \\
&= \varepsilon' \nu_\ell J D_\ell + \omega_\ell J \nu_\ell + \varepsilon' \nu_\ell J(\omega_\ell) J^{-1} \nu_\ell J \nu_\ell \\
&= \varepsilon' \nu_\ell J D_\ell + \omega_\ell J \nu_\ell + \varepsilon' \nu_\ell J \omega_\ell \\
&= \varepsilon' \nu_\ell J(D_\ell + \varepsilon' \nu_\ell J^{-1} \omega_\ell J \nu_\ell + \omega_\ell),
\end{aligned}$$

where we have used (2.6), $\varepsilon'^2 = 1$, and $J^2 = 1$. ∎

Remark 2.9. Observe that the additional assumption in (2.10) was necessary only to guarantee that the additional term is bounded. If this is not the case, all properties of the modified Dirac operator that were demonstrated in the proof above will still hold; however, since the fluctuation is not by a bounded operator, spectral properties of $(D_\omega)'$ (like the compactness of the resolvent) may be modified.

It is also worth noting that one can extend the possible fluctuations of the Dirac operator to the sums of partial fluctuations, that is, fluctuating each of D_ℓ by ω_ℓ, which may differ from each other, provided that the resulting full Dirac operator D'_ω is a bounded perturbation of D.

The notion of a spectral triple with a twisted real structure in [1, 2] was largely motivated by spectral triples conformally rescaled by a positive element in JAJ^{-1}. Below we propose a generalization of this construction.

Proposition 2.10. *Suppose that (A, H, D, J) is a real spectral triple and $D = \sum_{\ell=1}^{N} D_\ell$ such that each D_ℓ satisfy the first-order condition for every $\ell = 1, \dots, N$. Let k_ℓ be positive elements from A with bounded inverses. Then $(A, H, \widetilde{D}, J)$, where*

$$\widetilde{D} := (D_1)_{k_1} + \cdots + (D_N)_{k_N}, \tag{2.14}$$

with $(D_\ell)_{k_\ell} = (Jk_\ell J^{-1}) D_\ell (Jk_\ell J^{-1})$ satisfying multitwisted zero- (2.2) and first-order (2.4) conditions with

$$\nu_\ell = k_\ell^{-1} J k_\ell J^{-1},$$

is regular (2.6) with the multitwisted ε'-condition (2.5). Furthermore, if D has a compact resolvent, it is a spectral triple with a multitwisted real structure.

Proof. Since the original real spectral triple satisfies the zero-order condition (2.1), so does the multitwisted-real spectral triple. Next, if all $k_\ell \in A$, then $\nu_\ell(a) = k_\ell^{-1} a k_\ell \in A$ for every $a \in A$ and (2.2) holds as well. We compute further using the first-order condition for the spectral triple:

$$\begin{aligned}
\big[(D_\ell)_{k_\ell}, a\big] J \bar{\nu}_\ell(b) J^{-1} &= (Jk_\ell J^{-1})[D_\ell, a](Jk_\ell J^{-1}) J \bar{\nu}_\ell(b) J^{-1} \\
&= (Jk_\ell J^{-1})[D_\ell, a](Jk_\ell J^{-1})(Jk_\ell^{-1} b k_\ell J^{-1}) \\
&= (Jk_\ell J^{-1})[D_\ell, a](JbJ^{-1})(Jk_\ell J^{-1}) \\
&= (Jk_\ell J^{-1})(JbJ^{-1})[D_\ell, a] J k_\ell J^{-1} \\
&= J \bar{\nu}_\ell^{-1}(b) J^{-1} J k_\ell J^{-1} [D_\ell, a] J k_\ell J^{-1} \\
&= J \bar{\nu}_\ell^{-1}(b) J^{-1} \big[(D_\ell)_{k_\ell}, a\big],
\end{aligned}$$

which proves (2.4). The regularity condition (2.6) follows directly:

$$\nu_\ell J \nu_\ell = (k_\ell^{-1} J k_\ell J^{-1}) J (k_\ell^{-1} J k_\ell J^{-1}) = k_\ell^{-1} J k_\ell k_\ell^{-1} J k_\ell J^{-1} = J,$$

using $J^2 = \varepsilon$ so that $J = \varepsilon J^{-1}$.

The multitwisted ε'-condition is again a simple consequence of the ε'-condition of the real spectral triple:

$$\begin{aligned}
(D_\ell)_{k_\ell} J\nu_\ell &= (D_\ell)_{k_\ell} J(k_\ell^{-1} Jk_\ell J^{-1}) \\
&= (Jk_\ell J^{-1}) D_\ell (Jk_\ell J^{-1}) J(k_\ell^{-1} Jk_\ell J^{-1}) \\
&= Jk_\ell J^{-1} D_\ell k_\ell J = \varepsilon' Jk_\ell D_\ell (Jk_\ell J^{-1}) \\
&= \varepsilon' Jk_\ell (Jk_\ell^{-1} J^{-1})(D_\ell)_{k_\ell} \\
&= \varepsilon' (k_\ell^{-1} Jk_\ell J^{-1}) J (D_\ell)_{k_\ell} = \varepsilon' \nu_\ell J (D_\ell)_{k_\ell}.
\end{aligned}$$ ■

Remark 2.11. It is worth noticing that the above construction mimics the conformal rescaling of the Dirac operator from [1], however, additionally one still has to assume that the resulting Dirac operator has a compact resolvent. It is an interesting problem to investigate whether and under what conditions on the operators D_ℓ this occurs for a spectral triple that allows such a splitting of D.

3. Examples

3.1. Multiconformally rescaled spectral triples

A specific example of the above construction was given by the asymmetric torus in [13]. It was motivated by the search of spectral triples over the noncommutative torus which can be interpreted as arising from a non-flat metric. In fact, in a certain precise sense it has a non-vanishing local scalar curvature, yet obeying a generalized Gauss–Bonnet theorem. We will supplement the construction in [13] by discussion of the real structure and twisted reality properties.

Let ∂_1 and ∂_2 denote the operators that extend the standard derivations of $C^\infty(T^2_\theta)$ to $H = L^2(T^2_\theta) \otimes \mathbb{C}^2$ as self-adjoint (unbounded) operators and let J be the usual antilinear isometry on H. Then, for any positive invertible $k_1, k_2 \in C^\infty(T^2_\theta)$, the Dirac operator

$$\widetilde{D} = Jk_1 J^{-1} \sigma^1 \partial_1 Jk_1 J^{-1} + Jk_2 J^{-1} \sigma^2 \partial_2 Jk_2 J^{-1}, \tag{3.1}$$

where σ^1 and σ^2 are the usual Pauli matrices, makes $(C^\infty(T^2_\theta), L^2(T^2_\theta) \otimes \mathbb{C}^2, \widetilde{D}, J)$ a multitwisted real spectral triple satisfying by Proposition 2.10 all conditions including (2.2), (2.4), and (2.6). In [13] we considered a particular case with $k_1 = 1$ (which is not a product spectral triple). A four-dimensional generalization (of product type) with two different scalings was studied in [6].

3.2. Conformal rescaling without an automorphism

Consider the following situation, which further generalizes the construction of conformally rescaled spectral triples[1] allowing conformal rescaling of the Dirac operator by an element, which is still from the commutant of A but not from JAJ^{-1}. We have the following proposition.

[1] Note that here conformally rescaled spectral triples are indeed *spectral triples* and not *twisted spectral triples*. We still call them conformally rescaled as they are such in the classical (commutative) situation.

Proposition 3.1. *Let (A, H, D, J) be a real spectral triple, which satisfies the usual first-order condition, regularity, and ε'-condition. Let $\mathrm{Cl}_D(A)$ be the algebra generated by A and $[D, A]$ and let $k \in \mathrm{Cl}_D(A)$ be an invertible element with bounded inverse. Then, with $\nu = k^{-1}JkJ^{-1}$ and $D_k = JkJ^{-1}DJkJ^{-1}$, (A, D_k, H, J, ν) with a twist ν is a spectral triple with a twisted (and thus a multitwisted, in the sense of Definition 2.1) real structure, which satisfies the twisted zero- and first-order conditions, twisted regularity, and twisted ε'-condition.*

Proof. First of all, observe that since $k \in \mathrm{Cl}_D(A)$ and the spectral triple satisfies the first-order condition, then $\bar{\nu}(b) = k^{-1}bk$ for any $b \in A$; however, $\bar{\nu}$ is not necessarily an automorphism of A. Still, a simple computation using the first-order condition and the fact that $k \in \mathrm{Cl}_D(A)$ shows that

$$\begin{aligned} aJ\bar{\nu}(b)J^{-1} &= aJk^{-1}bkJ^{-1} = J(J^{-1}aJ)(k^{-1}bk)J^{-1} \\ &= J(k^{-1}bk)J^{-1}a = J\bar{\nu}^{-1}(b)J^{-1}a, \end{aligned} \tag{3.2}$$

which is (2.2). Further, using again zero- and first-order conditions for the spectral triple we have that

$$[D_k, a]J\bar{\nu}(b)J^{-1} = J\bar{\nu}^{-1}(b)J^{-1}[D_k, a], \tag{3.3}$$

for any $a, b \in A$, which is precisely (2.4). Unlike in the case of the usual conformal rescaling, one cannot write this condition replacing b with $c = \bar{\nu}(b)$ as c is not guaranteed to be in A. The proof of regularity and the twisted ε'-condition are the same as in the standard situation of conformally rescaled spectral triple. ∎

The above construction is, of course, a case of single twisting; however, it can be easily extended to the situation of multitwisting and multiconformal scaling, which provides new examples of multitwisted real spectral triples. Interestingly, such objects do have a deep geometric motivation, arising from the Dirac operators over noncommutative circle bundles [11].

An example is a three-dimensional noncommutative torus $\mathbb{T}^3_\theta$ seen as the noncommutative $U(1)$-bundle over the two-dimensional noncommutative torus $\mathbb{T}^2_\theta$. We consider the usual equivariant Dirac operator D over $\mathbb{T}^3_\theta$ and the bimodule of one-forms in the Clifford algebra $\mathrm{Cl}_D(\mathbb{T}^3_\theta)$. There exists a canonical action of $U(1)$ on $\mathbb{T}^3_\theta$, as described in [11], the invariant subalgebra of which is $\mathbb{T}^2_\theta$. A $U(1)$-connection over $C^\infty(\mathbb{T}^3_\theta)$ can be given by a one-form, which as an element of $\mathrm{Cl}_D(\mathbb{T}^3_\theta)$ is

$$\omega = \sigma^1\omega_1 + \sigma^2\omega_2 + \sigma^3, \tag{3.4}$$

where $\omega_1, \omega_2 \in \mathbb{T}^2_\theta$ are $U(1)$-invariant elements of the algebra $C^\infty(\mathbb{T}^3_\theta)$.

In [11] we have shown that for any self-adjoint connection ω (3.4) there exists a compatible (in the sense defined therein) Dirac operator over $A = C^\infty(\mathbb{T}^3_\theta)$, which has the form

$$\mathcal{D}_\omega = \sigma^1\partial_1 + \sigma^2\partial_2 + JwJ^{-1}\partial_3, \tag{3.5}$$

where J is the usual real structure on $C^\infty(\mathbb{T}^3_\theta)$, ∂_i, $i = 1, 2, 3$, are the usual derivations represented on the Hilbert space of the spectral triple, and the one-form w is

$$w = \sigma^3 - \sigma^1\omega_1 - \sigma^2\omega_2.$$

Although $\mathcal{D}_\omega$ does not satisfy a twisted first-order condition, we have the following proposition.

Proposition 3.2. *The spectral triple $(C^\infty(\mathbb{T}^3_\theta), H, \mathcal{D}_\omega, J)$ is a multitwisted real spectral triple, with splitting $\mathcal{D}_\omega = D_{(2)} + D_w$ and twists:*

$$\nu_1 = \mathrm{id}, \quad \nu_2 = w^{-\frac{1}{2}} J w^{\frac{1}{2}} J^{-1}.$$

Proof. The decomposition of D is natural, with

$$D_{(2)} = \sigma^1\partial_1 + \sigma^2\partial_2$$

being the usual Dirac operator over the noncommutative two-torus. Clearly, it satisfies the first-order condition and the ε'-condition for the trivial twist $\nu_1 = \mathrm{id}$. The second part of the splitting

$$D_w = J w J^{-1}\partial_3$$

can be rewritten as

$$D_w = J w^{\frac{1}{2}} J^{-1}\partial_3 J w^{\frac{1}{2}} J^{-1},$$

since w is invariant with respect to the $U(1)$-action and therefore commutes with ∂_3. Note that since w is Hermitian, it has a square root and we can use it to write D_w in a convenient form. It is easy to see that the decomposition and the twists satisfy

$$\begin{aligned} [D_{(2)}, a] J b J^{-1} &= J b J^{-1} [D_{(2)}, a], \\ [D_w, a] J \bar{\nu}(b) J^{-1} &= J \bar{\nu}^{-1}(b) J^{-1} [D_w, a], \end{aligned} \tag{3.6}$$

where $\bar{\nu}(x) = w^{-\frac{1}{2}} x w^{\frac{1}{2}}$ and hence the requirements of Definition 2.1. Note that condition (2.2) is also satisfied since w belongs to the completion of Clifford algebra and therefore $J\bar{\nu}(a)J^{-1}$ is in the commutant of A. ∎

Note that since by construction $w \notin C^\infty(\mathbb{T}^3_\theta)$, this multitwisted spectral triple is not of the same type as the example discussed in Section 3.1.

4. Conclusions and outlook

The new notion of spectral triples with a multitwisted real structure, which we propose here, has some major advantages. Firstly, it is consistent with the usual definition of *spectral triples* (unbounded Fredholm modules), thus allowing to use the power of Connes–Moscovici local index theorem. Secondly, it vastly extends the realm of examples, covering almost all known spectral triples, including those motivated by geometrical constructions, like conformal rescaling or noncommutative principle fiber bundles. Moreover, it is closed under the tensor product operation.

In particular, the multitwisted first-order condition may provide a better understanding of the notion of first-order differential operators in noncommutative geometry, which we hope will allow to finer apprehend the examples arising from the quantum groups and quantum homogeneous spaces as constituting noncommutative manifolds.

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