J. Noncommut. Geom. 16 (2022), 677–716
DOI 10.4171/JNCG/470

A detailed look on actions on Hochschild complexes especially the degree 1 coproduct and actions on loop spaces

Ralph M. Kaufmann

Abstract. We explain our previous results about Hochschild actions (2007/2008) pertaining, in particular, to the coproduct, which appeared in a different form in a work by Goresky and Hingston (2009), and provide a fresh look at the results. We recall the general action, specialize to the aforementioned coproduct and prove that the assumption of commutativity, made for convenience in a previous article (2008), is not needed. We give detailed background material on loop spaces, Hochschild complexes and dualizations, and discuss details and extensions of these techniques which work for all operations given in two previous articles (2007/2008).

With respect to loop spaces, we show that the coproduct is well defined modulo constant loops and going one step further that in the case of a graded Gorenstein Frobenius algebra, the coproduct is well defined on the reduced normalized Hochschild complex. We discuss several other aspects such as "time reversal" duality and several homotopies of operations induced by it. This provides a cohomology operation which is a homotopy of the anti-symmetrization of the coproduct. The obstruction again vanishes on the reduced normalized Hochschild complex if the Frobenius algebra is graded Gorenstein. Further structures such as "animation", the BV structure, a coloring for operations on chains and cochains, and a Gerstenhaber double bracket are briefly treated.

Introduction

In [20,22] (first published on arXiv in June 2006), we gave an action of a dg-PROP of cellular chains of a CW complex, based on arc systems, on the Hochschild cochain complex of a Frobenius algebra, algebraically realizing and expanding the Chas–Sullivan string topology [7] operations. Among many other operations, this includes a product of degree 0, a coproduct of degree 1 and a pre-Lie operation of degree 1 whose cellular representatives together with the computation of composition of product and coproduct appear in [20, Figures 4 and 5]. The action of the open version of the degree 1 coproduct was explicitly given the generalization to the open/closed context in [27, §5.4.2].

2020 Mathematics Subject Classification. Primary 16E40; Secondary 55P50, 16S38, 18D50.
Keywords. Hochschild chains and cochains, Goresky–Hingston coproduct, string topology, loop spaces, Frobenius algebras, coproduct.

Such a coproduct was constructed by Goresky–Hingston [13] in a geometric setting and has found its way into a symplectic setting [8], see also [40, 43] for further developments after the announcement of our results presented here. There are precedents for such operations going back to the basic string topology [7] with further clarifications and developments in [47–49]. For the coproduct to descend to homology of the loop space, one has to work relative to constant loops. This idea can be traced back to Sullivan [46]. The vanishing of the obstruction for descent to cohomology has geometric meaning and has been used to distinguish homotopy equivalent non diffeomorphic manifolds [4].

We will recall our coproduct, explain the background and give the geometric interpretations and show that the geometry of the product of [13] agrees with the our previously defined algebraic version using the cosimplicial setup of [9, 16]. Specifically, the relevant cellular chain complex whose cellular chains act as a dg-PROP on the Hochschild chain complex $CH^*(A, A)$ of a Frobenius algebra A was defined in [20, Definition 5.31]. The action of cells was proven in [22, Theorem B]. Specializing to the case of $A = H^*(M)$, with M a compact simply connected manifold, we make the action explicit. We show that using translation, provided by [9, 16], the coproduct geometry agrees with that of [13] on the E_1-page. This matches with the original foliation geometry for the whole gamut of operations which goes back to [30, §4], see also [18, §5.11] and [31, §1 esp. Figure 1].

In the course of this discussion, we give many details for the calculations and interpretations as well as generalizations that are universal and useful for other operations contained in the PROP and the algebras over it. We prove that the assumption of commutativity for the Frobenius algebra, made out of convenience in [22], is not needed, by showing that all the equations that need to be satisfied for the action to be well defined and independent of choices hold for a general associative Frobenius algebra. This also yields a succinct formula for correlation functions in terms of 2d OTFT correlation functions. Lastly, we consider restricting the PROP to operations which are already defined for associative algebras.

There are several stages to defining the actions. The first is to simply define individual operations, the next is to give compatible operations, that is an operad, PROP or modular operad structure, and last stage provides dg-actions. In [20, 22], we provided modular operad actions for cell complexes from moduli spaces and dg-PROP actions for Sullivan-type surfaces yielding the dg-operations under discussion. Iterated k-fold operations, as considered, also already appear in our framework and are readily treated using our formalism. As we prove below, if one restricts to the reduced Hochschild complex, one automatically discards constant loops and hence the results of [14] follow algebraically from our formalism. We also discuss different methods for lifting the operations from the Frobenius algebra level to a chain level, e.g. from $H^*(M)$ to $C^*(M)$ and $C_*(M)$.

The individual operations inherently have a naïve duality by virtue of being defined as correlation functions given by switching input/output designations. For instance, the degree 0 product is dual to a degree 0 coproduct, which is different from the natural degree 1 product. However, there is PROPic "time reversal symmetry", basically rooted the asymmetric treatment of "in" and "out" boundaries. The prime example of being related by this symmetry are the degree 0 product and the degree 1 coproduct. The symmetric partners are obtained from the same underlying arc system, but differ by switching the "in" and "out" boundaries. Moreover, there is an operadic structure, which is in/out symmetric, used for moduli space operations [20, 22] in which this symmetry is natural. For the moduli space actions, this is part of a modular operad.

The string topology operations are recorded by applying degeneracy maps to "outs". This asymmetry is needed to obtain the correct dg-operations, see [22].

Just as Gerstenhaber's bracket comes from the homotopy of $\cup$ and $\cup^{\mathrm{op}}$, so too can the coproducts, as well many other operations, be identified as operations from homotopies—a point stressed by B. Tsygan. Our formalism also naturally identifies such homotopies for instance the coproduct, its anti-symmetrization, the Gerstenhaber double bracket and with extra decorations the BV structure in the setting of "animation" [44].

Finally, to incorporate the extra dualities from the naïve duality operadically, or better PROPicly, we introduce a two colored PROP to keep track of extra dualization which specifies co (cohomological) and ho (homological) inputs and outputs. In the actions this gives operations on Hochschild chains and cochains. This subsumes the operations of [45] into the correlation function formalism of [22]. The details of these computations are consigned to [32], where, in particular, we show that the mixed m_3 operations of [45] stem from a natural homotopy which is a double Gerstenhaber bibracket of degree 2 in the sense of [42, 50].

Organization

The paper is organized in a formula forward way, first giving the algebraic formulas and then going deeper into their origin which at the lowest level is rooted in the cell geometry.

After fixing notation and giving essential remarks in Section 1, we give the formula for the coproduct and its boundary in Section 2, which is based on the cell [20, Figure 4] with action according to [22] in Theorem 2.1. With the explicit form of the action, one can see in which ways this (co)chain operation descends to an operation in (co)homology. This is made explicit in Proposition 2.3 and Theorem 2.4. The technical discussion on how to define the correlation functions and dualize them is contained in Section 2.2. The application to the coproduct is in Section 2.3 and a discussion of generalizations of the particular actions follows in Section 2.4.

In Section 3, we review the Hochschild chain and cochain models for loop space according to [9, 16]. This allows to complete the geometric identification of the action, and hence the coproduct, in the case of loop spaces. We identify the constant loops with $\overline{CH}^0(C^*(M), C_*(M))$ in Proposition 3.5 and which allows us to deduce that the coproduct is well defined on $\bar{H}_*(LM)$ in Corollary 3.6. We also discuss several ways regarding the operations we defined in other natural contexts in Section 3.3. In Section 3.3.3 and Section 3.3.4 we give the geometry of the coproduct and its boundary terms.

The geometry of the CW complexes and dg-action of the cellular chains is discussed in Section 4, where we also succinctly define the action in terms of local OTFT correlators.

The concrete calculations are performed in Section 5. This contains the various dualizations and relaxations for the conditions of existence of the basic operations (Section 5.1) and the proof that the commutativity assumption is superfluous; see Corollary 5.2, which also contains an explicit formula for the local OTFT correlators.

Several dualities are defined in Section 6.1, in particular the naïve and "time reversal symmetry", which bridges the different treatment of inputs and outputs in the actions and string topology. Further topics, such as dualization, A_∞ versions and "animation" are briefly discussed in Section 6.2. Finally, Section 6.2.2 contains a preview of the upgrading of the naïve duality into a colored action on Hochschild chain, cochains and the Hochschild–Tate complex and the Gerstenhaber double bracket. The full details are relegated to [32].

1. Preliminary remarks and notations

1.1. Removing assumptions

In [20,22], we used the notation k for the coefficients thinking about fields. This made life easier, due to the Künneth formula. However, we can take $\mathbb{Z}$ coefficients throughout. In order to not confuse with the references, we set $k = \mathbb{Z}$. This also conforms to the notation of [38].

In [22] commutativity of A was assumed, see [22, Assumption 4.1.2]. This is not necessary as had been announced and detailed in several talks and discussions over the years. Here we write out the proof. Indeed all the needed equations, see [22, Remark 4.2], hold for any Frobenius algebra. This follows from a direct verification by calculation, which is done in Section 5.2 and the resulting expression is (5.10). With hindsight, it also follows from the well-definedness of 2d Open Topological Field Theory (OTFT) and the equivalence of OTFTs with Frobenius algebras, see Remark 4.1.

1.2. Notation for the various complexes

For an A-A bimodule M, we let $CH_*(A, M)$, $HH_*(A, M)$ be the Hochschild chain complex and homology and set $CH_*(A) := CH_*(A, A)$, $HH_*(A) = HH_*(A, A)$. Thus, $CH_n(A) = A^{\otimes n+1}$. For $A = 1 \oplus \bar{A}$, where k is generated by the unit, the normalized complex is $\overline{CH}_n(A, M) = M \otimes \bar{A}^{\otimes n}$.

Dually, $CH^n(A, M) = \mathrm{Hom}(A^{\otimes n}, M) = M \otimes \check{A}^{\otimes n}$ denotes Hochschild cochains and HH^* Hochschild cohomology. An element $f \in CH^n(A, A)$ is a linear function $f : A^{\otimes n} \to A$. We use the short hand $CH^*(A) = CH^*(A, \check{A})$ and $HH^*(A) = HH^*(A, \check{A})$. The normalized cochains $\overline{CH}^*(A, M)$ are those functions $f(a_1 \otimes \cdots \otimes a_n)$ which vanish if one of the $a_i = 1$.

If $M = A$ then $CH^n(A, A) = A \otimes \check{A}^{\otimes n}$, and if $M = \check{A}$ then $CH^n(A) := CH^n(A, \check{A}) \simeq \check{A}^{\otimes n+1} \simeq \mathrm{Hom}(A^{\otimes n+1}, k)$. In particular, as complexes $CH^*(A) = \mathrm{Hom}(CH_*(A), k)$, see [38, Section 1.5.5]. See [22, Lemma 3.5], and [38, Section 2.5.9] for the relation to the cyclic complex and cyclic cohomology.

The reduced Hochschild complex is defined as $\widetilde{CH}_n(A) = \overline{CH}_n(A) = A \otimes \bar{A}^{\otimes n}, n > 0$ and $\widetilde{CH}_0 = \bar{A}$. Its homology is denoted by $\widetilde{HH}_n(A)$. The reduced complex $\widetilde{CH}^*(A)$ is the dual to $\widetilde{CH}_*(A)$ and is also the normalized complex modulo the constants in $\overline{CH}^0(A)$.

If A is Frobenius, then

$$A^{\otimes n+1} \simeq CH_n(A) \simeq CH_n(A, \check{A}) \simeq \check{A}^{\otimes n} \simeq CH^n(A, \check{A}) \simeq CH^n(A),$$

see Section 2.2 for more details. The duality $A \simeq \check{A}$ extends to the duality between $CH_*(A)$ and $CH^*(A)$ as complexes.

We will call graded algebra A of finite type if all the graded pieces are finite dimensional. In this case, we consider $\check{A}$ as the graded dual.

1.3. Levels of action

There are three levels to the actions of [20, 22]:

1.3.1. Dg-PROP action on $CH^*(A, A)$ for a Frobenius algebra A. This was established in [22, Theorem B]. The definition of the action uses that linearly $CH^*(A)$ is isomorphic to the reduced tensor algebra $\bar{T}A$ on A.

The operations are defined via correlators, which are morphisms $Y : (\bar{T}A)^{\otimes n} \to k$. These dualize to the dg-PROP action as detailed in Section 2.2. This entails specifying inputs and outputs which yields a morphism in $\mathrm{Hom}(CH^{\otimes n_1}, CH^{\otimes n_2})$, for a specification of n_1 inputs and n_2 outputs with $n_1 + n_2 = n$. These are compatible with the differentials and give a dg-PROP action if the input/output designation is the one specified by the cell model. The asymmetric treatment of inputs and outputs gives rise to two types of duality, a naïve one which works on the level of operations—allowing to assign inputs and outputs in the operations arbitrarily—and a time reversal duality, see Section 6.1.

Since the two complexes $CH^*(A)$ and $CH_*(A)$ are duals, we can furthermore identify the complexes $\mathrm{Hom}(CH^{\otimes n_1}, CH^{\otimes n_2}) \simeq (CH^*)^{\otimes n_1} \otimes (CH_*)^{\otimes n_2}$. This allows one to dualize CH^* outputs, as specified by the cell, as CH_* inputs and likewise dualize factors of CH^*, which are inputs according to the cell marking, to CH_* outputs, augmenting the naïve duality. Structurally this is handled by a two colored PROP, which we introduce in Section 6.2.2—more details and examples will be given in [32].

1.3.2. PROP actions on $CH^*(D, \check{D})$, for D a quasi Frobenius algebra and a lift of the Casimir aka. diagonal. A quasi Frobenius algebra, see [22, Definition 2.7] is a unital associative dg-algebra (D, d) with a trace $\int$, i.e. a cyclically invariant counit, such that $\int da = 0$ and $A = (H(D, d), \int_H)$ is Frobenius. In [22, Theorem A and B], we lifted the cochain operations to the cocycles of such a dg-algebra using a lift of the Casimir from H to A. The prototypical example is $D = C^*(M)$, for a compact simply connected manifold M, with the lift being a choice of a lift of the diagonal. The cocycle condition was introduced to avoid the ambiguity introduced by the choice of lift. This also means that the induced operations on cohomology are well defined and independent of the lift. However, fixing a lift it is clear that the operadic correlation functions [22, §2, §2.3] actually lift to all of D, that is to $C^*(D) \simeq CH^*(D, \check{D})$.

One has to be careful with the dualizations if D is not finite dimensional or of finite type. In the case of $D = C^*(M)$ the (degree shifting) quasi isomorphism

$$CH^*(C^*(M), C^*(M)) \simeq CH^{*+d}(C^*(M), C_*(M))$$

was established in [9]. The double complex gives rise to a spectral sequence, whose E^1-term is isomorphic to $CH^*(H^*(M), H^*(M))$ and the action via correlation functions gives a PROP action on $CH^*(C^*(M), C^*(M)$ which induces the dg-action on E^1. The discussions pertaining to the coproduct are in Section 2.4 and Section 3.3.

1.3.3. Subsets of operations which do not need dualization. Finally, upon inspection of operations or sub-PROPs operads, dualizations may not be necessary. As remarked, e.g. in [22, §4] this is the case for the suboperad action yielding Deligne's conjecture [21]. More details are given in Section 2.2.1 and Section 2.4, specific, relevant examples are in Section 5.1, while general background is discussed briefly in Section 4.3.2 and Section 5.1.

1.4. Frobenius algebras

A Frobenius algebra A is an associative, unital (possibly $\mathbb{Z}/2\mathbb{Z}$ graded) algebra over a commutative ring k, with a non-degenerate even symmetric perfect pairing η, commonly written as $\langle\, ,\rangle$ which is invariant, that is $\langle a, bc\rangle = \langle ab, c\rangle$.

Remark 1.1. It is possible to work with an odd pairing as well. This introduces extra Koszul signs. This happens for non-geometric situations, for instance algebraically if one shifts a complex, see [34] for the induced odd structures. For string topology applications, we will be mainly concerned with the geometric case and will omit this extra layer of sign complexity.

$A \otimes A$ is again a Frobenius algebra with the usual multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$, and $\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle$ as the perfect even symmetric invariant pairing. Here and often in the following, for simplicity, we omitted appropriate Koszul sign stemming from the use of the commutator τ_{23}, or simply add a $\pm$ sign. There are several schemes for sign conventions for operations discussed at length in [21, 22], see Section 4.2.2 and Section 4.3.1 for the sign convention for operations.

Using these pairings, a product μ has an adjoint Δ_A defined by $\langle \Delta_A(a), b \otimes c\rangle = \langle a, bc\rangle$. The pairing η defines a counit for this comultiplication ε via $\varepsilon(a) = \langle 1, a\rangle = \langle a, 1\rangle$. Alternate notations in use are $\langle a\rangle := \varepsilon(a) =: \int a$. In this notation: $\langle a, b\rangle = \langle ab\rangle$. If A is not commutative, Δ_A is not cocommutative in general. The relationship between μ_A and Δ_A in this convention is

$$\Delta_A(ab) = \Delta_A(a)(b \otimes 1) = (1 \otimes a)\Delta_A(b), \tag{1.1}$$

as $\langle \Delta_A(ab), c \otimes d\rangle = \langle ab, cd\rangle = \langle a, bcd\rangle = \langle \Delta_A(a), bc \otimes d\rangle = \langle \Delta_A(a), (b \otimes 1)(c \otimes d)\rangle = \langle \Delta_A(a)(b \otimes 1), c \otimes d\rangle$. Since $\Delta_A(a) = \Delta_A(a1) = \Delta_A(1a)$,

$$\Delta_A(a) = \Delta_A(1)(a \otimes 1) = (1 \otimes a)\Delta_A(1). \tag{1.2}$$

The element $e = \mu\Delta(1)$, called the Euler element, will play an important role. The quantum dimension of A is $\mathrm{tr}(\mathrm{id}_A) = \varepsilon(e)$.

We set $\Delta_A(1) =: C = \sum C^{(1)} \otimes C^{(2)} \in A \otimes A$ and call it the Casimir element. We will use Sweedler notation throughout. In particular,

$$a = \sum \langle a, C^{(1)}\rangle C^{(2)} = \sum \langle C^{(1)}, a\rangle C^{(2)}. \tag{1.3}$$

Explicitly, if A is free as a k-module and e_i is a basis for A, $g_{ij} = \langle e_i, e_j\rangle$ and g^{ij} is the inverse matrix then $C = \sum_{i,j} g^{ij} e_i \otimes e_j = \sum_i e_i \otimes e^i$ with $e^i = \sum_j g^{ij} e_j$ and $e = \mu_A \circ \Delta_A(1) = \mu_A(C) = \sum_{i,j} g^{ij} e_i e_j$.

A is isomorphic to its dual $\check{A} = \mathrm{Hom}(A, k)$ via $a \mapsto \langle a, \cdot\rangle$. Via this duality $\eta \in \check{A}^{\otimes 2}$ is dual to C. The Casimir element C allows to express the dual perfect pairing $\check{\eta}$ on $\check{A}$ via $\check{\eta}(\phi, \psi) = (\phi \otimes \psi)(C)$. As usual, Δ_A defines a multiplication $\mu_{\check{A}}$ on $\check{A}$ via $(\phi\psi)(a) = (\phi \otimes \psi)(\Delta_A(a))$ and μ_A a comultiplication $\Delta_{\check{A}}(\phi)(a \otimes b) = \phi(ab)$.

1.4.1. Geometric/Gorenstein A. In case that $A = H^*(X)$ for a compact oriented connected d-dimensional Poincaré duality space X, or more generally if A is graded Gorenstein with socle d, we set $e_0 = 1$ and e_{top} the unique degree d element with $\varepsilon(e_{\mathrm{top}}) = 1$. In this case $e = \mathrm{sdim}(A)e_{\mathrm{top}}$, where sdim is the super or $\mathbb{Z}/2\mathbb{Z}$ dimension.

In particular, if $A = H^*(M)$, where M is a compact oriented manifold, and $\varepsilon = \int_{[M]} \cdot = \varepsilon_{\mathrm{aug}} \circ \cdot \cap [M]$, where $\varepsilon_{\mathrm{aug}}$ is the augmentation map, then e is the Euler class, 1 is Poincaré dual to the fundamental class of $[M]$ and e_{top} is Poincaré dual to a point, $\varepsilon(e) = \chi(M)$ is the Euler-characteristic and $e = \varepsilon(e)e_{\mathrm{top}}$.

2. The algebraic formula for the coproduct and its boundary

The coproduct is defined by the action of a particular cell, which was already given in [20, Figure 4]. It is depicted in Figure 1. We will state the algebraic result and then give the derivation of the explicit formula for the operation from the general setting of [22]. We will use the short hand $CH = CH^*(A, A)$.

2.1. The coproduct on CH^*

Theorem 2.1. *Given a Frobenius algebra A consider $CH := CH^*(A, A)$. The cell for the coproduct given in Figure 1 acts, according to [22, §3.2.1], as a coproduct morphism*

$$\Delta_{CH} \in \mathrm{Hom}(CH, CH^{\otimes 2}). \tag{2.1}$$

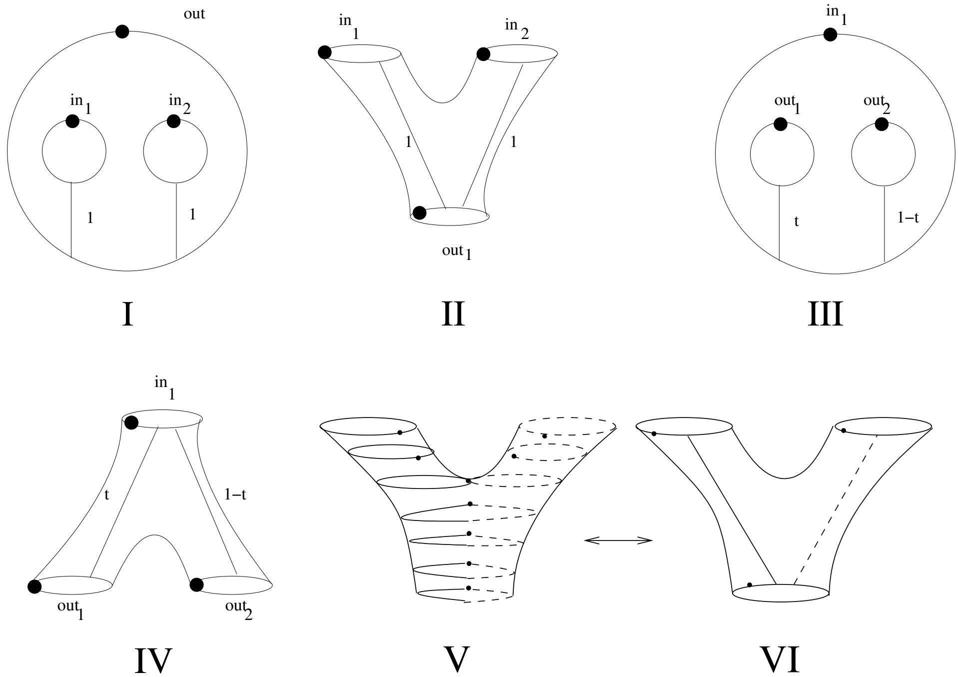

Figure 1. The cell for the product I and a non-planar depiction II, the coproduct III and a non-planar depiction of it IV. The weights in the product case are both normalized to one, since each arc is incident to one input boundary. Asymmetrically, for the coproduct both arcs are incident to a single input boundary, so that only their combined weight is normalized to 1, yielding a cell of dimension 1. These operations are time reversal dual to each other. In the string picture V of [18] the two strings merge for the product moving down and moving up for the coproduct one string forms a figure 8 and breaks apart into two strings with the relative lengths t and $1-t$. The transformation to arcs is in VI where each arc represents a piece of string of the indicated length.

The formulas for its non-zero components $\Delta_{CH}(f) \in \bigoplus_{p+q=n-1} CH^p \otimes CH^q$ *are explicitly given by*

$$\begin{aligned}\Delta_{CH}(f)&[(a_1 \otimes \cdots \otimes a_p) \otimes (a_{p+1} \otimes \cdots \otimes a_{n-1})] \\ &= (-1)^p \sum_{C_1, C_2} C_1^{(1)} f(a_1 \otimes \cdots \otimes a_p \otimes C_2^{(1)} C_1^{(2)} \otimes a_{p+1} \otimes \cdots \otimes a_{n-1}) \otimes C_2^{(2)}.\end{aligned} \tag{2.2}$$

According to [22, Theorem B] *the boundary of this chain operation is given by the operation of the boundary of the cell, given in Figure* 2. *It has two components and the operations corresponding to these are*

$$\partial_0 \Delta_{CH} : CH^n \to CH^n \otimes CH^0 \quad \textit{and} \quad \partial_1 \Delta_{CH} : CH^n \to CH^0 \otimes CH^n$$

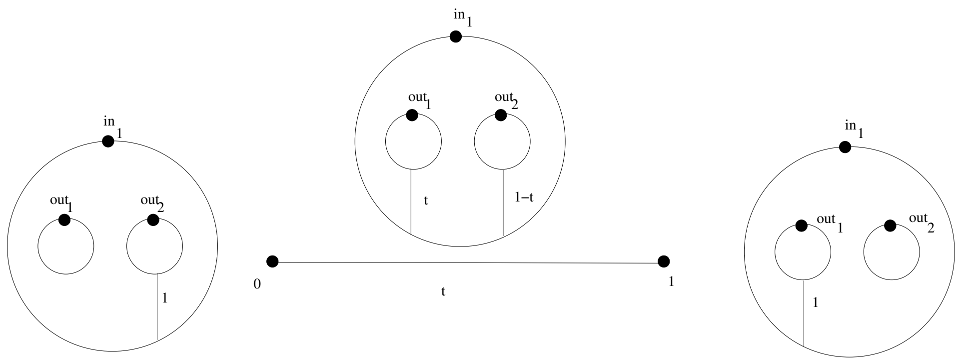

Figure 2. The 1-dimensional cell and its two boundary points.

which are given by the following explicit formulas, using $CH^0(A, A) = \mathrm{Hom}(k, A)$*, choosing* $a_i \in A$ *and* $\lambda \in k$*:*

$$\begin{aligned} \partial_0 \Delta_{CH}(f)(\lambda \otimes (d_1 \otimes \cdots \otimes d_n)) &= \lambda(1 \otimes f(a_1, \ldots, a_n))\Delta(1)^2, \\ \partial_1 \Delta_{CH}(f)((a_1 \otimes \cdots \otimes a_n) \otimes \lambda) &= \lambda \Delta(1)^2 (f(a_1, \ldots, a_n) \otimes 1). \end{aligned} \tag{2.3}$$

Furthermore, the coproduct is well defined as a cohomology operation if $\Delta(1)^2 = 0$*. It is also a well-defined cohomology operation modulo the "constant term"* CH^0 *or relative to the constant term.*

Proof. The proof is in Section 2.3.3, which also contains equivalent forms of the boundary operation involving $\Delta_A(f)$, see (2.14). ∎

2.1.1. Geometric/Gorenstein case.

Lemma 2.2. *Let* A *be graded Gorenstein—in particular this is the case if* $A = H^*(X)$ *for a connected Poincaré duality space* X*. Then,*

$$\Delta(1)^2 = \varepsilon(e)e_{\mathrm{top}} \otimes e_{\mathrm{top}} = (1 \otimes e)\Delta(1) = \Delta_A(e).$$

Furthermore, the following are equivalent:

(i) $\Delta(1)^2 = 0$,

(ii) $\varepsilon(e) = 0$ *and*

(iii) $e = 0$.

Proof. If A has socle in degree d, the total degree of $\Delta(1)^2$ is in degree $2d$ and this space is spanned by $e_{\mathrm{top}} \otimes e_{\mathrm{top}}$. It suffices to compute $(\varepsilon \otimes \varepsilon)(\Delta_A(1)^2) = \langle \Delta(1)^2, 1 \otimes 1 \rangle = \langle \Delta_A(1), \Delta_A(1) \rangle = \langle \mu(\Delta(1)), 1 \rangle = \varepsilon(e)$. The other equation follows in similar fashion, using (1.2). Since $e = \varepsilon(e)e_{\mathrm{top}}$, the equivalences follow. ∎

Proposition 2.3. *If A is graded Gorenstein, the action corresponding to the boundary components, which is the boundary Δ_{CH}, factors through maps to the degree 0 part A_0 of A, that is $CH^*(A, \bar{A}) \subset \ker(\partial_{0/1}(\Delta_{CH}))$ and*

$$\begin{aligned} \operatorname{Im}(\partial_0 \Delta_{CH}) &\subset CH^n(A, A_0) \otimes CH^0(A, A_0) \subset CH^n(A, A) \otimes CH^0(A, A), \\ \operatorname{Im}(\partial_1 \Delta_{CH}) &\subset CH^0(A, A_0) \otimes CH^n(A, A_0) \subset CH^0(A, A) \otimes CH^n(A, A). \end{aligned} \tag{2.4}$$

Proof. This follows from Corollary 2.12. ■

Summing up:

Theorem 2.4. *If A is graded Gorenstein, the coproduct induces an operation on cohomology relative to or modulo the constants $CH^0(A, A_0) \simeq \operatorname{Hom}(k, k)$ and thus is well defined on the reduced complex $\widetilde{CH}^*(A)$. In particular, this is the case for $A = H^*(X)$ for a connected Poincaré duality space X.*

Furthermore, if the Euler characteristic $\varepsilon(e) = 0$ vanishes, the coproduct is a cohomology operation on $HH^*(A, A)$ directly. ■

Remark 2.5. If A is graded Gorenstein then split A as $A = A_0 \oplus \bar{A}$, with $\bar{A} = \bigoplus_{k>0} A_k$. $\bar{A}$ is an A-A bimodule, and the constant maps, that is maps to $A_0 = k$, can be identified with $CH^*(A, A)/CH^*(A, \bar{A})$.

Remark 2.6. There is another way in which the constants in CH^0 appear in quotients. If $A = 1 \oplus \bar{A}$ is augmented, then $CH_*(\bar{A})$ which is linearly given by $\bar{T}\bar{A}$ computes the naïve Hochschild (co)homology of $\bar{A}$ [38, §1.4.3], and in the Gorenstein case the coproduct is well defined on this complex and consequentially on its dual as well.

2.1.2. The coproduct as a homotopy. The two boundary terms are homotopic and so are the operations. In fact, the coproduct *is* the homotopy between the left and right multiplication by elements of CH^0. The boundary terms are also homotopic to the algebraic version of the pointwise coproduct of [48], see Section 6.1.1.

Similar to the brace operation, which is obtained from anti-symmetrizing the pre-Lie product, we can regard the symmetrized coproduct $\Delta_{\mathrm{sym}} := \Delta_{CH} + \Delta_{CH}^{\mathrm{op}}$. Note that if one adopts signs as for the usual bracket, that is shifted degrees with the operation in the middle, see [21, §4.4], the operation is actually the anti-symmetrization of Δ.

Proposition 2.7. *Δ_{sym} is also a well defined homology operation. It is null-homotopic modulo $CH^0(A)$ or the constants in $CH^0(A)$ in the case of A being graded Gorenstein. Thus, the coproduct is cocommutative modulo $CH^0(A)$ or in $\widetilde{CH}^*(A)$ if A is graded Gorenstein.*

Proof. See Example 6.4. ■

Note, one does not really need to assume that the algebra is connected. It could be the direct sum of connected (graded Gorenstein) components.

2.2. Correlation functions and operations on Hochschild (co)chains

2.2.1. Hom spaces and correlation functions. The power $\eta^{\otimes n} \otimes \check{\eta}^{\otimes m}$ is a perfect pairing for $A^{\otimes n} \otimes \check{A}^{\otimes m}$. For simplicity, we denote all these by $\langle\ ,\ \rangle$. Which precise form is used is determined by the type of elements the form is applied to; e.g. $\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle$.

Using the various dualities:

$$\mathrm{Hom}(A^{\otimes n}, A^{\otimes m}) \simeq A^{\otimes m} \otimes \check{A}^{\otimes n} = \check{A}^{\otimes n+m} \simeq \mathrm{Hom}(A^{\otimes n+m}, k). \tag{2.5}$$

Maps $Y \in \mathrm{Hom}(A^{\otimes n}, k)$ are called correlation functions. Explicitly, a correlation function $Y : A^{\otimes n} \to k$ defines an element in $\mathrm{Hom}(A^{\otimes p}, A^{\otimes q})$ for any (p,q)-shuffle σ via

$$\mathrm{sign}_{Z/2\mathbb{Z}}(\sigma) \sum_{C_1,\dots,C_q} Y\big(\sigma(a_1 \otimes \cdots \otimes a_p \otimes C_1^{(1)} \otimes \cdots \otimes C_q^{(1)})\big) C_1^{(2)} \otimes \cdots \otimes C_q^{(2)}, \tag{2.6}$$

where the sum is the multiple Sweedler sum for q copies of the Casimir element and $\mathrm{sign}_{Z/2\mathbb{Z}}(\sigma)$ is the Koszul sign for the shuffle. These dualities extend to the tensor algebra $\mathrm{Hom}(TA^{\otimes n}, TA^{\otimes m}) \simeq \mathrm{Hom}(TA^{\otimes n+m}, k)$.

Remark 2.8. By (2.6), Y gives rise to different morphisms $\hat{Y}_{p,q} \in \mathrm{Hom}(A^{\otimes p}, A^{\otimes q})$ for each $p + q = n$, which will be called *forms of* Y. If A is Frobenius then all these are equivalent. If it is not, some of these forms might exist apart from the others, see Section 5.1 for explicit examples and in particular Section 5.1.2 for the calculations relevant for the coproduct.

2.2.2. Dualization to functions. An element in $CH^n(A, M)$ is a sum of expressions $m \otimes \check{a}_n \otimes \cdots \otimes \check{a}_1$ with $m \in M$ and the $\check{a}_i \in \check{A}$. For a Frobenius algebra, $\mathrm{Hom}(A^{\otimes n}, A) \simeq A \otimes \check{A}^{\otimes n} \simeq A^{\otimes n+1}$. A function $f = a_0 \otimes \check{a}_n \otimes \cdots \otimes \check{a}_1$ with $\check{a}_i = \langle a_i, .\rangle$ dualizes to $\hat{f} = a_0 \otimes \cdots \otimes a_n \in A^{\otimes n+1} \in TA$. The first tensor factor plays a special role and will be called the module variable. Vice versa, given $\hat{f}$, we recover f as $f(b_1, \dots, b_n) = a_0 \prod_{i=1}^{n} \langle a_i, b_i\rangle$. The particular ordering is chosen to avoid an extra Koszul sign, see e.g. [21].

2.2.3. Operations on $CH^*(A, A)$. The dg-PROP action of [22] on CH has homogeneous components defined via correlation functions whose definition proceeds as follows: Via the procedure given in Section 4 a cell c defines correlation functions (4.3), which are morphisms

$$Y(c)_{p_i,q_j} \in \mathrm{Hom}(A^{\otimes p_1+1} \otimes \cdots \otimes A^{\otimes p_n+1} \otimes A^{\otimes q_1+1} \otimes \cdots \otimes A^{\otimes q_m+1}, k), \tag{2.7}$$

where n and m are part of the given data of c. Dualizing the $A^{\otimes q_i+1}$ according to (2.6) one obtains a PROP action on $\overline{T}A$:

$$\hat{Y}(c)_{p_i,q_j} \in \mathrm{Hom}(A^{\otimes p_1+1} \otimes \cdots \otimes A^{\otimes p_n+1}, A^{\otimes q_1+1} \otimes \cdots \otimes A^{\otimes q_m+1}) \tag{2.8}$$

Finally, identifying the $A^{\otimes k+1} \simeq CH^k(A,A)$ as in Section 2.2.2, one obtains a dg-PROP action

$$\mathrm{op}_{CH}(c)_{p_i,q_j} \in \mathrm{Hom}\left(\bigotimes_{i=1}^{n} CH^{p_i}, \bigotimes_{j=1}^{m} CH^{q_j}\right). \tag{2.9}$$

Remark 2.9. By Section 2.2.1 there are additional possible dualizations for the *individual operations*, see Section 6.1.1. For composable (PROPic) versions one needs to dualize these operations using Hochschild homology, see Section 6.2.2.

2.3. Correlation functions and action on CH^* from c_Δ

The PROP cell for the coproduct 1-dimensional cell c_Δ is parameterized by an interval. The cell and its boundary 0-cells are given in Figure 2. Notice that $\partial_1 C = \tau_{12}\partial_0 C$ where τ_{12} switches the "out" labels 1 and 2. Switching these two labels produces the cell for Δ^{op}.

2.3.1. The coproduct correlation function. Using the procedure reviewed in §4 one duplicates arcs, assigns a local correlation function for each complementary region, and then takes the product of the local correlation functions to obtain the correlation function of the cell. For the cell c_Δ one obtains one summand for each pair (k,n) where the left arc is duplicated n and the right arc is duplicated $k-n$ times. The complementary regions are a central octagon and $n+k$ quadrilaterals. This homogeneous component corresponds to a map $CH^{\otimes k} \to CH^{\otimes n} \otimes CH^{\otimes k-n-1}$. The $(8,4)$ term is depicted in Figure 3.

Proposition 2.10. *The total correlation function for the cell c_Δ defining the degree 1 coproduct Δ_{CH} is a product over the local correlation functions $Y(c_\Delta) = Y_A(P_8) \otimes Y_A(P_4) \otimes \cdots \otimes Y_A(P_4) \circ \sigma$, where the $Y(P_{2n})$ are given by (4.2) and σ is a permutation. The formula of the operation on homogeneous components is given by*

$$\begin{aligned} &Y(c_\Delta)\big((a_0 \otimes \cdots \otimes a_n) \otimes (b_0 \otimes \cdots \otimes b_p) \otimes (c_0 \otimes \cdots \otimes c_{n-p-1})\big) \\ &\quad = \pm\langle a_0 b_0 a_{p+1} c_0\rangle \prod_{i=1}^{p} \langle a_i c_i\rangle \cdot \prod_{j=1}^{q} \langle a_{p+1+j} b_j\rangle. \end{aligned} \tag{2.10}$$

Proof. Decorating according to Section 4.3.1, the input pieces of the boundary are decorated by $a_0, \ldots, a_n$ starting at the marked point going clockwise, that is in the opposite orientation of the boundary as it is an input. The two outputs are decorated by $b_0, \ldots, b_p$ and $c_0, \ldots, c_q$ respectively, also going clockwise, which is the induced orientation, used for outputs. Cutting on the arcs, one sees a central octagon whose sides are decorated by (a_0, b_0, a_{p+1}, c_0) in this cyclic order. The alternating sides of the quadrilaterals P_4 are decorated by a_i, b_i on the left and by a_{p+1+j}, c_j on the right. The sign comes from the shuffle, shuffling the tensors into the given place according to Section 4.2.2. ∎

2.3.2. The boundary correlation functions. For the boundary of c_Δ the correlation function is more complicated as the complementary regions are not simply polygons. The action according to [22] is given by introducing in a system of extra cut-arcs decomposing

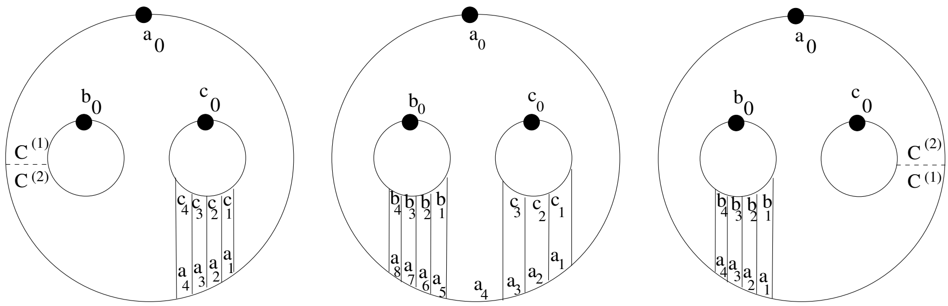

Figure 3. Middle: The $(8,3)$ summand of the component of the degree 1 operation on CH^* corresponding to $CH^8 \to CH^3 \otimes CH^4$. Cutting at the arcs yields one octagon P_8 and 7 quadrilaterals (P_4s). Left: the component of the ∂_0 boundary operation $CH^4 \to CH^4 \otimes CH^0$. Right: The component of the ∂_1 boundary operation $CH^4 \to CH^0 \otimes CH^4$. The extra cut for the annulus is the dotted line and decorated by $C = \Delta(1) = C^{(1)} \otimes C^{(2)}$.

each of the non-polygonal regions into polygons and decorating the two sides of the extra cut-arcs by Casimir elements. The procedure is reviewed in Section 4, see Figure 3 for the relevant example. In Section 5.2, we prove different cut systems yield the same correlation function and show that the assumption of commutativity of A made in [22] is unnecessary. The formula for the local correlation function is in (5.10). The appearance of the Casimir element is what makes the boundary factor through the constants. After cutting these extra arcs, one is again left with a decorated polygon as above. In the case of the boundary of c_Δ one cut-arc suffices, see Figure 3.

Decorating by elements of A, reading off the cyclic word, and integrating, one obtains the following operations:

Proposition 2.11. *Let $\partial_0 c_\Delta$ be the boundary at $t = 0$ and $\partial_1 c_D$ the boundary at $t = 1$, then they define the following correlation functions:*

$$
\begin{aligned}
Y(\partial_0 c_\Delta)(a_0 \otimes \cdots \otimes a_n \otimes b_0 \otimes c_0 \otimes \cdots \otimes c_n) &= \pm \sum \langle a_0 C^{(1)} b_0 C^{(2)} c_0 \rangle \prod_{i=1}^{n} \langle a_i c_i \rangle, \\
Y(\partial_1 c_\Delta)(a_0 \otimes \cdots \otimes a_n \otimes b_0 \otimes \cdots \otimes b_n \otimes c_0) &= \pm \sum \langle a_0 b_0 C^{(1)} c_0 C^{(2)} \rangle \prod_{i=1}^{n} \langle a_i b_i \rangle.
\end{aligned}
\tag{2.11}
$$

Proof. Treating boundary at $t = 1$, there is only one arc to replicate. The input is decorated by $a_0, \ldots, a_n$, the first output by $b_0, \ldots, b_m$, and the second output by c_0. After cutting, besides the P_4 quadrilaterals labelled by a_i, b_i, there is a central surface which is an annulus. One of the two boundary components labelled by a_0, b_0 and the other by c_0.

Inserting one cut-arc and decorating it by the Casimir element yields the given correlation function according to (5.10), see Figure 3. The boundary at $t = 0$ is analogous, albeit that the sole arc now runs to the other input. ■

2.3.3. Proof of Theorem 2.1. Let $a_0 \otimes \cdots \otimes a_n$ be the input tensor and $b_0 \otimes \cdots \otimes b_p$ and $c_0 \otimes \cdots \otimes c_{n-p-1}$ be the two output tensors. Using the calculations for the dualization of $\int_2$ and $\int_4$ given in Section 5.1.2, (1) and (3), and summing up the contributions

$$
\begin{aligned}
&\hat{Y}(c_\Delta)(a_0 \otimes \cdots \otimes a_n)\\
&\quad = \sum_{p=0}^{n} \sum \pm (a_0^{(1)} \otimes a_1 \otimes \cdots \otimes a_p) \otimes (a_0^{(2)} a_{p+1} \otimes a_{p+2} \otimes \cdots \otimes a_n)\\
&\quad = \sum_{p=0}^{n} \sum \pm (C^{(1)} a_0 \otimes a_1 \otimes \cdots \otimes a_p) \otimes (C^{(2)} a_{p+1} \otimes a_{p+2} \otimes \cdots \otimes a_n) \quad (2.12)
\end{aligned}
$$

where $\pm$ is the sign coming from shuffling in $a_0^{(2)}$.

Identifying the tensors with elements of CH^*, this translates to $\Delta_{CH} := \mathrm{op}(c_\Delta)$ according to Section 2.2.2. Since

$$
(C_1^{(2)} \check{a}_{p+1})(C_2^{(1)}) = \langle C_1^{(2)} a_{p+1}, C_2^{(1)} \rangle = \langle a_{p+1}, C_2^{(1)} C_1^{(2)} \rangle = \check{a}_{p+1}(C_2^{(1)} C_1^{(2)})
$$

for $f \in CH^n$ the components $\Delta_{CH}(f) \in \bigoplus_{p+q=n-1} CH^p \otimes CH^q$ is given by (2.2).

For the boundary the calculation for $\int_5$ in Section 5.1.2 (4) results in

$$
\hat{Y}(\partial_1 c_\Delta)(a_0 \otimes \cdots \otimes a_n) = \sum (C^{(1)} a_0^{(1)} \otimes a_1 \otimes \cdots \otimes a_n) \otimes C^{(2)} a_0^{(2)}, \qquad (2.13)
$$

where we used Sweedler's notation. This in turn yields the operation

$$
\begin{aligned}
\partial_1 \Delta_{CH}(f)((d_1 \otimes \cdots \otimes d_n) \otimes \lambda) &= (-1)^n \sum \lambda [(C^{(1)} \otimes C^{(2)}) \Delta_A(f(d_1, \ldots, d_n))]\\
&= \lambda \Delta(1) \Delta_A(f(d_1, \ldots, d_n))\\
&= \lambda \Delta(1)^2 (f(d_1, \ldots, d_n) \otimes 1) \qquad (2.14)
\end{aligned}
$$

and similarly for $\partial_0 \Delta_{CH}$, where the last equality comes from (1.2), viz.

$$
\sum (C^{(1)} \otimes C^{(2)}) \Delta_A(a) = \Delta_A(1) \Delta_A(a) = \Delta(1)^2 (a \otimes 1).
$$

■

Corollary 2.12. *If A is graded Gorenstein, then the boundary correlation functions vanish unless $a_0, b_0, c_0 \in A_0$. Dually, $\partial_{0/1} \Delta_{CH}(f) = 0$ unless $f : A^{\otimes n+1} \to A_0 \simeq k$ is a constant map and the image of $\partial_{0/1} \Delta_{CH}(f)$ has image as specified in* (2.4).

Proof. Let A have socle in dimension d, we see that each term $a_0 C^{(1)} b_0 C^{(2)} c_0$ has degree at least d and hence all the terms of the correlation functions are 0 unless a_0, b_0 and c_0 are of degree 0 and hence all multiples of the unit $1 \in A_0$.

In particular, the condition that $a_0 \in A_0$ implies that on CH the operation is zero on any map f not having A_0 as image, and $b_0, c_0 \in A_0$ implies that the output functions are also maps to A_0. ■

Corollary 2.13. *If A is graded Gorenstein, the coproduct is a well-defined cohomology operation, in the complex $\overline{CH}^*(A, \bar{A})$.* ■

2.4. Generalizing the actions

Using the point of view of Section 5.1.1, the operations generalize from a Frobenius algebra in several ways. As the $\int_2$ terms represent identity morphisms, see Section 5.1.2 (1), the term $\int_4$ is the only interesting one in $\hat{Y}(c_\Delta)$. In the form presented in (2.12), the module variable a_0 needs to be of the type Ω_A with the rest of the tensor variables lying in A, see Section 5.1.2 (3). This means that the equation is well defined as a morphism $CH_*(A, {}_A\Omega_A) \to CH_*(A, {}_A\Omega_A) \otimes CH_*(A, {}_A\Omega_A)$, for instance on $CH_*(C^*(M), C_*(M))$.

If $\check{A}$ is a coalgebra, for instance if A is finite dimensional of finite type, then the operation exists as a morphism $CH_*(A, \check{A}) \to CH_*(A, \check{A})$, where A only needs to be associative. If this is not the case and one cannot identify $(A \otimes A)^\vee$ with $\check{A} \otimes \check{A}$ then one can still specify a special element C, see Section 5.1.2 (1), and use the formalism of operadic correlation functions [22, §2]. Such an element is given if the coalgebra ${}_A\Omega_A$ is pointed in Quillen's sense.

Using the alternative form, (5.4), one obtains a map

$$CH^*(A, A) \to CH^*(A, {}_A\Omega_A) \otimes CH^*(A, {}_A\Omega_A)),$$

thus in particular,

$$CH^*(C^*(X), C^*(X)) \to CH^*(C^*(X), C_*(X)) \otimes CH^*(C^*(X), C_*(X)).$$

If $X = M$ is a manifold using the isomorphisms (3.11) and (3.7) one obtains a map $H_{*+d}(LM) \to H_*(LM) \otimes H_*(LM)$.

For the boundary operations, the discussion is analogous using Section 5.1.2 (4), but in any formulation, due to the cut, there is the need for the special element C.

3. Actions on (co)chains of loop spaces and their geometric interpretation

3.1. Manifolds, Poincaré duality and intersection

Let M be a compact oriented connected manifold, then $A = H^*(M, k)$ is a Frobenius algebra over K with $\mu_A = \cup$, $\varepsilon = \int_M$ is the cap product with the fundamental class of $[M]$ followed by the augmentation map. The duality between A and $\check{A} = H_*(M, k)$ is known as Poincaré duality.

The integral $\int ab$ has the following dual geometric interpretation. Let $\check{a}$ and $\check{b}$ be Poincaré dual cycles intersecting transversally then $\int ab$ is zero unless $\check{a}$ and $\check{b}$ have complementary dimensions and then $\int ab = \#$ of intersection points $a \pitchfork b$.

3.2. Loop space models using Hochschild (co)chains

3.2.1. Geometric motivation. If we regard a singular chain c on the free loop space $LM = C(S^1, M)$, we get a chain $b_*(c)$ in M by the push-forward with respect to the base-point map $b : LM \to M$ which sends ϕ to $\phi(0)$. Evaluating at different points of S^1 gives similar maps.

The algebraic structure of Hochschild cochains is given by sampling S^1 by sequences of $n+1$ points that are cyclically ordered and coherent—the first point always being 0. The $n+1$ points yield $n+1$ singular chains. This gives a sequence parameterized by n. The i-th point may collide with the $i+1$-st point lowering the point count in which case one should obtain the family with less points. This is the coherence. Both points 1 and n can collide with 0 which gives the extra degeneracy.

In terms of elements of $A^{\otimes n+1}$ the element $a_0 \otimes \cdots \otimes a_n$ represents the dual homology classes swept out by the $n+1$ points, that is a_i is dual to the homology cycle swept out by the i-th point and a_0 is dual to the base points of the loop.

3.2.2. Cosimplicial viewpoint according to Jones/Cohen–Jones [9, 16]. The sampling is formalized as follows: using a simplicial structure $S^1_\bullet$ on S^1 one obtains a cosimplicial structure on $\mathrm{Hom}(S^1_\bullet, X)$ whose totalization gives back the loop space. In fact, $\mathrm{Hom}(S^1_\bullet, X)$ is cocyclic since $S^1_\bullet$ is cyclic. This cyclic structure is the reason for the existence of the BV operator. More precisely, one has maps

$$\begin{aligned} f_k : \Delta^k \times LX &\to X^{k+1}, \\ (0 \le t_1 \le \cdots \le t_k \le 1) &\mapsto (\gamma(0), \gamma(t_1), \ldots, \gamma(t_k)) \end{aligned} \tag{3.1}$$

which one can think of as discretizing the loop. These maps dualize to

$$\begin{aligned} \bar{f}_k : LX &\to \mathrm{Hom}(\Delta^k, X^{k+1}), \\ \gamma &\mapsto (t_0, \ldots, t_k) \mapsto (\gamma(t_0), \ldots, \gamma(t_k)) \end{aligned} \tag{3.2}$$

which are compatible with coface and codegeneracy maps; see Section 3.2.3 for details.

Theorem 3.1 ([9, 16]). *Let X be a space and $f : LX \to \prod_{k \ge 0} \mathrm{Map}(\Delta^k, X^k)$ be the product of the maps $\bar{f}_k$ then f is a homeomorphism onto its image. The image is the subspace $\mathrm{Tot}(\mathrm{Map}(S^1_\bullet, X))$, whose elements are those sequences that commute with the coface and codegeneracy maps.*

A singular l-chain $c_l : \Delta^l \to LX$ can be regarded as a family of loops $\gamma_{\mathbf{t}}$ depending on $\mathbf{t} \in \Delta^l$. Its discretization gives a family of maps $\Delta^l \times \Delta^k \to X^{k+1}$ which is an $l+k$ chain on X^{k+1}. The chain is given by the usual shuffle product formula which expresses the bi-simplicial $\Delta^l \times \Delta^k$ as a union of simplices.

Therefore, pulling back along the f_k and using the Alexander Whitney map $\mathrm{AW} : C_*(X^{k+1}) \to C_*(X)^{\otimes k+1}$, one obtains maps

$$f_k^* : C^*(X)^{\otimes k+1} \to C^{*-k}(LX). \tag{3.3}$$

Theorem 3.2 ([9, 16]). *The homomorphisms* f_k^* *define a chain map*

$$f^* : CH_*(C^*(X)) \to C^*(LX), \tag{3.4}$$

which is a chain homotopy equivalence when X *is simply connected. Hence it induces an isomorphism*

$$f^* : HH_*(C^*(X)) \xrightarrow{\simeq} H^*(LX), \tag{3.5}$$

dualizing these maps and using that $HH_*(C^*(X)) = HH_*(C^*(X); C^*(X))$ *yields*

$$f_* : C_*(LX) \to CH^*(C^*(X); C_*(X)), \tag{3.6}$$

which is a chain homotopy equivalence when X *is simply connected. Hence it induces an isomorphism*

$$f_* : H_*(LX) \xrightarrow{\simeq} HH^*(C^*(X), C_*(X)). \tag{3.7}$$

Remark 3.3. The direct dualization yields the dual of the complex

$$\mathrm{Hom}(CH_*(C^*(X)), k).$$

If A is finite dimensional or of finite type, then as remarked previously, up to signs $CH^*(A) = \mathrm{Hom}(CH_*(A), k) \simeq CH^*(A, \check{A})$ [38, Section 1.1.5], with the isomorphism given by $F \leftrightarrow f$ as defined by

$$F(a_0, \dots, a_n) = f(a_1, \dots, a_n)(a_0). \tag{3.8}$$

Taking (3.8) as a definition of F given f always defines a map $CH^*(A, \check{A}) \to CH^*(A)$. In total, the map f_* can be seen as a the map that takes an l-dimensional family of loops $\gamma_{\mathbf{t}}$ to the evaluation maps

$$F_k = \mathrm{ev}_{\mathrm{AW}(\gamma_{\mathbf{t}}(0), \dots, \gamma_{\mathbf{t}}(t_k))} \in \mathrm{Hom}(CH^*(C_*(X)), k), \tag{3.9}$$

where on the right hand side the degree is $k+1$ and the total degree is $k+l$. This gives the explicit description with homological coefficients.

Using the same kind of rationale Cohen–Jones also prove a second description with cohomological coefficients.

Theorem 3.4 ([9, Corollary 11, Theorem 1]). *For any closed (simply connected)* d*-dimensional manifold* M*:* $H_{*+d}(LM) \simeq H_*(LM^{-TM})$*. And, there are naturally defined chain maps* $f_{k,*}$ *which fit together to define a chain homotopy equivalence*

$$f_* : C_*(LM^{-TM}) \to CH^*(C^*(M), C^*(M)) \tag{3.10}$$

inducing an isomorphism

$$f_* : H_*(LM^{-TM}) \simeq HH^*(C^*(M), C^*(M)) \simeq H_{*+d}(LM). \tag{3.11}$$

3.2.3. Discretizing and dualizing. We give the explicit (co)face and (co)degeneracy maps of the simplicial/cosimplicial structures at the various level. This allows us to identify the constant loops in the Hochschild cochain complex. They may also be used to find the Hochschild cochain representations of families of loops used in the arguments of string topology [7, 13] using the totalization.

For a discretized loop γ these are

$$
\begin{aligned}
\delta_i : \Delta^k &\to \Delta^{k+1} & \mathrm{Hom}(\Delta^k, X^{k+1}) &\to \mathrm{Hom}(\Delta^{k+1}, X^{k+2}) \\
(\dots, t_i, \dots) &\mapsto (\dots, t_i, t_i, \dots), & \gamma(\dots, t_i, \dots) &\mapsto \gamma(\dots, t_i, t_i, \dots), \\
\sigma_i : \Delta^{k+1} &\to \Delta^k & \mathrm{Hom}(\Delta^{k+1}, X^{k+2}) &\to \mathrm{Hom}(\Delta^k, X^{k+1}) \\
(\dots, t_i, \dots) &\mapsto (\dots, \hat{t}_i, \dots), & \gamma(\dots, t_i, \dots) &\mapsto \gamma(\dots, \hat{t}_i, \dots).
\end{aligned}
$$

Thus, the map δ_i induces the map $\Delta_{i,*}$ which after applying the AW map $C_*(X^k) \to C_*(X^{k+1})$ is just the coproduct.

For families/homology classes using the diagonal maps Δ_i which repeat the i-th entry and projections π_i which omit the i-th entry, we have that $\Delta_i : X^k \to X^{k+1}$ induces the map

$$(\dots, \gamma_i, \dots) \mapsto (\dots, \gamma(t_i), \gamma(t_i), \dots).$$

This translates to a map $\Delta_{i,*} : C_*(X^k) \to C_*(X^{k+1})$ given by

$$\gamma_{\mathbf{t}}(\dots, t_i, \dots) \mapsto \gamma_{\mathbf{t}}(\dots, t_i, t_i, \dots),$$

and a map $\Delta_i := \mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id} : C_*(X)^{\otimes k} \to C_*(X)^{\otimes k+1}$ which is given by

$$\gamma_0 \otimes \dots \otimes \gamma_k \to \gamma_0 \otimes \dots \otimes \gamma_i^{(1)} \otimes \gamma_i^{(2)} \otimes \dots \otimes \gamma_k.$$

Similarly, $\pi_i : X^{k+1} \to X^k$ induces the map

$$(\dots, \gamma(t_i), \dots) \mapsto (\dots, \widehat{\gamma(t_i)}, \dots).$$

This translates to a map $C_*(X^{k+1}) \to C_*(X^k)$ which is given by

$$\gamma_{\mathbf{t}}(\dots, t_i, \dots) \mapsto \gamma_{\mathbf{t}}(\dots, \hat{t}_i, \dots)$$

and maps $\varepsilon_i := \mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes \varepsilon \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id} : C_*(X)^{\otimes k+1} \to C_*(X)^{\otimes k}$ given by

$$\gamma_0 \otimes \dots \otimes \gamma_k \to \gamma_0 \otimes \dots \otimes \varepsilon(\gamma_i) \otimes \dots \otimes \gamma_k.$$

Finally, dualizing, in the manifold setting, we see that these morphisms go to

$$
\begin{aligned}
\mu_i &:= \mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes \mu \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id} : C^*(M)^{\otimes k+1} \to C^*(M)^{\otimes k}, \\
\eta_i &:= \mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes \eta \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id} : C^*(M)^{\otimes k} \to C^*(M)^{\otimes k+1},
\end{aligned}
$$

where μ is the multiplication given by the $\cup$ product and $\eta : \mathbb{Z} \to C^*(M)$ is the unit.

3.2.4. Constant loops. The discretized series for a constant loop $\gamma(t) \equiv x \in M$ is given using the maps δ_i

$$(\gamma(0), \dots, \gamma(t_k)) = \delta_{k-1} \cdots \delta_0 \gamma(0).$$

Thus a constant family of loops has the series

$$\mathrm{AW}(\gamma_0 \otimes \cdots \otimes \gamma_k) = \Delta_{k-1} \circ \cdots \circ \Delta(\gamma_0)$$

which can be reconstructed from

$$\gamma_0 = \varepsilon_1 \circ \cdots \circ \varepsilon_1(\gamma_0 \otimes \cdots \otimes \gamma_k).$$

Dually the cochain/cohomology sequence is given by

$$(\check{\gamma}_0 \otimes \cdots \otimes \check{\gamma}_k) = \mathrm{AW}^*(\eta_k \circ \cdots \circ \eta_1(\check{\gamma}_0)).$$

From these formulas one obtains that evaluation at a constant loop in degrees bigger than 0 is in the degenerate subcomplex and these do not appear in the normalized complex.

Proposition 3.5. *In higher degrees, the image of constant loops is in the degenerate subcomplex. In the normalized complex their image is $\overline{CH}^0(C^*(M), C_*(M)) = C_*(M)$. Moreover, choosing a base point for M defines a constant loop as a base point for LM and the reduced homology of $\bar{H}_*(LM)$ is quasi isomorphic to the dual of the reduced chain complexes $\widetilde{HH}_*(C^*(M))$ and $\widetilde{HH}^*(C^*(M))$ computed by the reduced (co)chain complexes.*

Proof. The only thing left to prove is the surjectivity. For this, one identifies a singular chain $f : \Delta^k \to M$ as a family of constant loops. ∎

3.3. Geometric interpretation for loop spaces

The preceding theorems and corollaries translate the algebraic results to a geometric interpretation in terms of loops. By this we mean that the given algebraic operations reflect a geometric situation, in which usually transversality is assumed. This is analogous to the discussion of transversal intersection in Section 3.1 and quantum cohomology [39], where the true operations are the Gromov–Witten invariants and the geometry they reflect is the enumerative geometry, which is itself elusive.

For the loop space geometry this agreement with the geometry that applying discretization given via the totalization to a geometric input family, e.g. constant loops or figure 8 loops, is commensurate with the algebraic operations.

3.3.1. Figure 8 loops. We define the subspace of figure 8 loops $F_8 \subset \mathrm{Tot}(\mathrm{Map}(S^1_\bullet, X))$, those maps that factor through $\mathrm{Tot}(\mathrm{Map}(S^1_\bullet \vee S^1_\bullet, X))$ for a given simplicial model of the map $S^1_\bullet \to S^1_\bullet \vee S^1_\bullet$. These can be represented by sequences $(\gamma(0), \gamma(t_1), \dots, \gamma(t_k))$ for which $\gamma(t_i) = \gamma(0)$ for some i. That is, they are in the image of the small diagonal map $\Delta_{0,i} : X^{k+1} \to X^{k+2}$ duplicating the first and $i+1$-st factors. Decomposing $\Delta^k = \Delta^{k_1} * \Delta^{k-k_1-1}$ induces maps

$$\mathrm{Hom}(\Delta^k, X^{k+1}) \to \mathrm{Hom}(\Delta^{k_1}, X^{k_1+1}) \times \mathrm{Hom}(\Delta^{k-k_1-1}, X^{k-k_1-1}).$$

When restricted to F_8 these maps yield coherent families and yield a map $F_8 \to \mathrm{Tot}(S^1_\bullet, X) \times \mathrm{Tot}(S^1_\bullet, X)$. Let $L_8M \subset LM$ be the space of these loops and $\Delta_8 : L_8M \to LM \times_M LM \subset LM \times LM$ the map constructed above via the totalization. That is, we obtain models for the maps

$$LM \stackrel{i_8}{\leftarrow} F_8 \stackrel{\Delta_8}{\to} LM \times_M LM \subset LM \times LM,$$

where $i_8 : F_8 \to \mathrm{Tot}(S^1_\bullet, X) \approx LM$.

3.3.2. Levels of action. By Proposition 3.5, one can identify the constant loops with $\overline{CH}^0$ in the normalized chain complex and hence Corollary 2.1 tells us that the operations are well defined modulo constant loops as in [13]. We even have more, namely that the coproduct already descends to operations relative to a base point constant loop. The three levels of Section 1.3 as they relate to loop spaces are:

(1) Since

$$E^1(CH^*(C^*(M)), C_*(M)) = CH^*(H^*(M), H_*(M)) \simeq CH^*(H^*(M)),$$

an action on this page is exactly the case discussed above for the action on $CH^*(A, A)$ for the Frobenius algebra $A = H^*(M)$.

(2) Note that in the formulas using $\varepsilon = \int$ lifts to the chain level as it can be replaced by capping with the fundamental class $\cap[M]$. The multiplication $\cup$-product also lifts the chain level. This means that the correlation functions all lift to

$$CH^*(C^*(M)) = CH^*(C^*(M), C_*(M)) \simeq CH^*(C^*(M), C^*(M)),$$

which means they are well defined and induce the operations on the E^1 page. To obtain PROP action one has to "dualize" the outputs. This can be done by *choosing a cochain representative* of the diagonal $C = \sum C^{(1)} \otimes C^{(2)}$ and simply using (2.6) as a definition.

The formalism of using a propagator C to define actions is discussed in detail in [22, §2] under the name of operadic correlation functions. The relevant result is [22, Theorem 4.15].

(3) Lastly, if one does not look at the whole PROP of operations on one space, one can pick individual operations and see if picking cleverly from the descriptions $CH^*(C^*(M), C_*(M))$, $CH^*(C^*(M), C^*(M))$ or $CH_*(C^*(M))$ for $C_*(LM)$ yields a formula that does not utilize dualization.

The classical example is the product. In this case, one can take the coefficient module to be an algebra. This was the motivation for [9]. In fact, it is clear from our formulas, that the whole little discs suboperad will act when picking cohomological coefficients [21]. For the BV action, the natural space is $\overline{CH}^*(A)$, for a Frobenius algebra A together with its cyclic structure [24]. For the coproduct the natural morphism is

$$CH_*(C^*(M), C_*(M)) \to CH^*(CH^*(M), CH_*(M))$$

as now the coefficients have a coproduct structure, see Section 2.4.

3.3.3. Coproduct on loop space. We will now discuss the degree 1 coproduct from all three different points of view.

(1) In terms of dual classes, we see that the first term says that a_0, b_0, a_{p+1} and c_0 "coincide" in the sense that if we use the interpretation of the $\cup$ product as intersection of the dual homology chains, see Section 3.1. The degree count says that the loci need to intersect in points (counted with multiplicity).

This means that all the base points and the $p+1$-st point coincide, which is indeed the situation of [13,47–49]. Summing over all p re-parameterizes the loop. The map sends the first loop which is a figure 8, to the two loops as in Figure 4.

(2) Lifting to chains, we see from (2.12), that the coincidence conditions for spawning off of a the loop via the coproduct are being forced by the intersection with the diagonal—again forcing the situation of [13] that encodes [47–49].

(3) As discussed in Section 2.4 the operations lift to operations

$$CH^*(C^*(M), C^*(M)) \to CH^*(C^*(M), C_*(M)) \otimes CH^*(C^*(M), C_*(M))$$

which also give a chain model for the loop space homology. Interpreting this as homology classes given by discretizations of loops, and uses $C \in C_*(M) \otimes C_*(M)$ as a chain representative of the diagonal the terms $C^{(1)}a_0$ and $C^{(2)}a_{p+1}$ becomes the intersections $C \pitchfork (a_0 \otimes a_{p+1})$. Restricting to the space where there is such intersections is the starting point of [13].

3.3.4. Boundary operations. We again have the three points of view as above:

(1) On the level of classes and dual intersections, we see that (2.13) says that the loop itself is left alone and spawns off a second constant loop at its base point. We furthermore see that due to degree reasons a_0, b_0, c_0 all must be of degree 0. This is due to the fact that the coproduct and the intersection with the diagonal produce a term $\Delta(1)$ which is already in top degree. This means that dually a_0 and hence b_0 and c_0 which coincide up to scalars

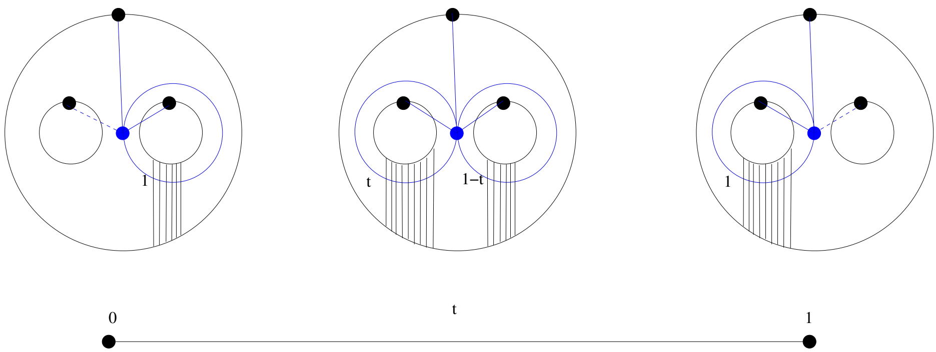

Figure 4. The blue graph is the dual graph to the weighted arc system. The tails and dotted tail keep track of the almost ribbon structure and the base points [25, Appendix A1].

have to sweep out the entire M. The dual interpretation is consistent with the intersection interpretation. Indeed, just like the cup product with 1 is trivial, so is the intersection with all of M. The loop that spawned off is a constant loop.

(2) The lift to chains is possible along the same lines as in the coproduct case and the geometric statements are those made above.

(3) The interpretation of C as a chain representative of the diagonal intersected with the relevant homology classes applies as in the coproduct case.

3.3.5. Identifying coproducts. An algebraic realization of the loop space is given by a Frobenius dgA model for M. This exists for instance if M is formal. Such a model has also more generally been provided in [36].

A transversal realization of the string topology operations is a geometric construction which on transversal families of loops induces the type string topology operations of [7]. The coproduct of [13] is of this type. Transversal realization is also the input for Umkehr maps [10] and guarantees a Cohen–Jones [9] type of setup as postulated in [30, §4.6]. Umkehr on (co)homology uses Poincaré duality [10] and as in Section 3.1 turns intersection into cup products. The map should be $\Delta^!_* i_{8!} : H_*(LM) \to H_*(LM) \otimes H_*(LM)$where i_8 is a Umkehr map, that means a map going the "wrong way". This and other geometric schemes can be traced through the discretization and starting at E^1 the transversal intersection of loops can hence be characterized via the previous calculations.

Corollary 3.6. *For an algebraic realization of the coproduct or a transversal realization, the coproduct descends to a cohomology operation on the reduced complex $\widetilde{HH}^*(C^*(M))$, inducing a coproduct on $\bar{H}_*(LM)$. Such a realization induces a morphism on the E^1 page of the spectral sequence which is given by the Δ_{CH}.*

Proof. Following through the discretization as detailed above, we see that the formula for the coproduct is indeed the transversal intersection in the form of cup products. ∎

4. Geometry and actions of CW complexes and dualities

4.1. Cells

The correlation functions of [22] are given for cells in a CW complex $\mathcal{A}$ together with an interval marking via data indexing the cell. The complex $\mathcal{A}$ is a CW complex whose cells are indexed by classes represented by an oriented surface Σ, with enumerated boundary components $\partial\Sigma = \coprod_{i=1}^n S^1$, one marked point p_i in each boundary component, and an arc system. An arc system is a set α of nonintersecting embedded curves, aka. arcs, that run from boundary to boundary not hitting the marked points which do not intersect, are not parallel to each other and not parallel to the boundary. This configuration is considered up to isotopy and mapping class group action. The classes $c = [(\Sigma, p_i, \alpha)]$ index the cells of a CW complex $\mathcal{A}$. The dimension of a cell is $|\alpha| - 1$ and the interior of a cell is naturally identified with the open simplex $\dot{\Delta}^{|\alpha|-1} \subset \mathbb{R}^{|\alpha|}$. The attaching maps or equivalently the cell boundaries are given by removing arcs. This is induced by a simplicial differential for which all arcs are enumerated first according to the boundary components and then according to their order on the boundary, see [20, 30] for more details.

4.1.1. Discretization. An integer weight for a set of arcs α is given a map $\mathrm{wt} : \alpha \to \mathbb{N}_{>0}$. A discretized cell is given by $c_{\mathrm{disc}} = [(c, \mathrm{wt})]$. As an arc system, this is represented by replacing each arc a by $\mathrm{wt}(a)$ parallel arcs. The differential is again given by removing arcs, which now is a sum of lowering the degree wt of the arcs in α by 1 and removing the arc if the resulting weight is 0, see [22] for details.

4.1.2. Interval/angle marking and action. An interval is the part of the boundary between two arcs. In [22], the intervals are called angles as they are the angles of the arc graph [22, Section 1.1.2]. A marking is a morphism $\mathrm{mk} : \text{interval} \to \{0, 1\}$ with the condition that each interval that contains a marked point, also called the module-interval, has value 1. Intervals between parallel arcs are called splitting intervals. For simplicity we will restrict the value of mk to be 1 on splitting intervals. The other intervals are called inner intervals and the function mk is completely determined by its value on these. The data $(c, \mathrm{wt}, \mathrm{mk})$ defines a homogeneous correlation function (4.3). The correlation function for a marked cell (c, mk) is given by summing over all possible weights (4.4), with the marking being the one fixed by its value on inner intervals.

Intervals with value 1 will be referred to as marked or active and the intervals with value 0 as unmarked or inactive. This terminology avoids a possible confusion as marking by 0 means decorating by the unit $1 \in A$ for the correlation functions.

4.2. Correlation functions for a cell

4.2.1. Local OTFT correlators. Let S be a surface with enumerated boundary components $\partial F = \coprod_{j=1}^{b} S^1$ and d_j marked intervals on each boundary component. An OTFT based on a Frobenius algebra A assigns a correlation function

$$Y_A(S) : A^{\otimes d_1} \otimes \cdots \otimes A^{\otimes d_b} \to k \tag{4.1}$$

which is given by the formula (5.10). Note that the formula is invariant under cyclic rotations of the tensor factors at each boundary component, and is equivariant with respect to renumbering the boundary components, see Section 5.2. The simplest OTFT correlation functions, which suffice to define the product, coproduct, pre-Lie and braces, are the $Y_A(P_{2n})$ where P_{2n} is a $2n$-gon for which every other side is marked. The general correlation function specializes to

$$Y_A(P_{2n})(a_1 \otimes \cdots \otimes a_n) = \langle a_1 \cdots a_n \rangle. \tag{4.2}$$

Remark 4.1. Usually, OTFTs are defined as involutive functors Z from a cobordism category, see e.g. [31, 37]. The Frobenius algebra is $A = Z(I)$, $I = [0, 1]$. The correlation function $Y(S)$ is the value of Z on S as a cobordism with all intervals being inputs and an empty output. As $Z(\emptyset) = k$ this gives the map above. Vice versa, since the functor

Z is involutive, A and the correlation functions fixes all of Z up to equivalence. When specifying inputs and outputs, one has to be careful with the orders, this explains different versions of the Frobenius equation (1.1). Dualizing inputs and outputs yields the different forms discussed in Section 5.1.

4.2.2. Global correlators for a cell. Given (c, wt) represent each integer weighted arc $a \in \alpha$ with weight p is represented by p parallel arcs. These decompose the surface into sub-surfaces given by the complementary regions $\Sigma = \bigcup_{v \in V} S_v$ where the intersections are at the boundaries of the S_v along the arcs, see Figure 3 for an example. The set V is the set of vertices of a dual description in terms of almost ribbon graphs, [25, Appendix A1] and Figure 4. Let l_i be the number of intervals at boundary $\partial_i \Sigma, i = 1, \dots, n$ marked by 1 in $(c, \mathrm{mk}, \mathrm{wt})$, then

$$Y_A(c, \mathrm{mk}, \mathrm{wt}) := \bigotimes_{v \in V} Y_A(S_v) \circ \sigma : A^{\otimes l_1} \otimes \cdots \otimes A^{\otimes l_n} \to k \tag{4.3}$$

where σ is the shuffle that shuffles the factors of A into their relative position. We used indexing by sets to make the formula easier. There is a natural order on V given by enumerating each S_v by the first appearance of an interval that belongs to it. The intervals themselves, and hence the factors of A, are enumerated first by the boundary component and then in their natural orientation starting at the interval containing the marked point, see also Section 4.3.1. These homogeneous components (4.3) sum up to a correlation function

$$Y_A(c, \mathrm{mk}) = \sum_{\mathrm{wt}} Y_A(c, \mathrm{mk}, \mathrm{wt}). \tag{4.4}$$

4.3. PROP cells and their action

In order to obtain the relevant PROP, one partitions the boundaries of Σ into inputs $\mathrm{In}_1, \dots, \mathrm{In}_n$ and outputs $\mathrm{Out}_1, \dots, \mathrm{Out}_m$ enumerating them separately. Furthermore, one restricts the arcs to run from input to output only and requires that every input boundary has at least one incident arc. This is the Sullivan quasi PROP $\overline{\mathcal{A}rc}^{i\leftrightarrow o}$. Lastly, one retracts the cells to a normalized version by scaling the coordinates so that the sum of barycentric coordinates separately at each input boundary is 1. Let s_i be the number of arcs incident to In_i, then the retracted cell c_1 is a product of simplices $\Delta^{s_1-1} \times \cdots \times \Delta^{s_n-1}$. These cells make up the cell complex called $\overline{\mathcal{A}rc}_1^{i\leftrightarrow o}$, see [20] for details.

4.3.1. Standard marking and action. Each PROP cell has a standard marking dictated by the input/output designation, see [22]. All module intervals are active. All inner input intervals are active and all inner output intervals are inactive. This defines $Y_A(c_1)$. The operation has degree $\dim(c_1)$ which is the number of input intervals not containing the base point. The main result for this complex is that the cellular chains have a dg-action on $CH^*(A, A)$ see [22, Theorem B]. Here a cell c_1 with n inputs boundaries and m output boundaries acts via the operation $\mathrm{op}_{CH}(c_1) : CH^{\otimes n} \to CH^{\otimes m}$ with the graded components given in (2.9).

The standard order is as follows: The module variable is assigned to the module-interval. This is followed in the linear order of $\overline{T}A$ according to the following rules. The input intervals are enumerated in opposite order to the orientation and the output intervals are enumerated according to the orientation.

4.3.2. Standard decomposition. There are several ways to find standard decompositions of the operations into standard operations. The most useful for $CH^*(A, A)$ being the following:

Theorem 4.2 ([22, Proposition 4.13]). *All the operations of the Sullivan PROP or even those of moduli spaces are expressible in terms of shuffles, deconcatenation coproducts $\diamond$ (on TA) and integrals.*

This kind of decomposition can be rewritten easily in other contexts, to lift operations to the various versions of CH. If one has a different context, then one should simply keep track of the fact that in (2.7) the leading tensor is the one from the coefficients and which form of the operation defined by the integral in the Frobenius case is being used, see Section 5.1. Sample considerations are given in Sections 2.4, 3.3.3 and 3.3.4.

Remark 4.3. The deconcatenation coproduct $\diamond$ is a coproduct for the following monidal structure on A-Mod-A: $M \boxtimes N = M \otimes_k A \otimes_k N$. A similar coproduct appears in [3] as a cotensor product in a different, but maybe not unrelated, context. This is the coproduct for the inputs corresponding to the interval marking by 1. On the outputs, the product is the simple tensor product. Using the unit $u : k \to A$, there is an embedding $M \otimes N \to M \boxtimes N$, which is precisely the application of the degeneracy maps.

From a simplicial point of view, $\boxtimes$ corresponds to the join, see Section 6.2 and Section 3.3.1. The Joyal dual monoidal product "$+$"—see e.g. [11, Appendix B] and [11, §3.5] for explicit formulas—is what is used on the outputs.

4.3.3. Remarks on PROP and quasi PROPs. Although the relevant structures on the chain level are PROPs, that are strictly associative, on the topological level there are two complications. The first is, that there is a rescaling involved. This is possible without penalty for the operad part as a global scaling and expressed as a bicrossed product with a scaling operad [19]. For the multi-gluings in the PROP one has to perform local scalings and this results in associativity only up to homotopy, which is the content of the notion of quasi PROP [20, Definition 5.22]. The explicit homotopies are controlled by rather intricate flows on the geometric level [31] that even work in the more general modular operad setting. The second complications, which already appeared in the operad part [21, 24], is that in order to obtain a cell complex with cells of the right dimension, one needs to retract to a smaller complex given by normalization, which is also a local scaling. Hence, the second problem and the first one are of the same ilk. Already the normalized operad is only a quasi operad [19, 1.1.1 Definition]. The full statements for the topological level are contained in [20, Theorem D].

4.3.4. Geometry of the PROPs on the topological level. The cell itself can be viewed in different ways as giving "geometric actions", either as foliations [30,31] or as a cellular chains [20, 22]. Here it is helpful to regard the dual graph, marked in blue in Figure 4. Combinatorially this is a dual graph on the surface. Geometrically, the dual graph is the image of a map called $\mathcal{L}$oop defined in [30, Definition 4.3] which identifies the points of the various geometric in- and output circles using a foliation, see [30,31] for more details. In the particular example of Figure 4, this means that the outside circle gets identified with the figure 8 configuration in such a way that all the base points coincide and the length of the two parts is given by $1-t$ and t, yielding a 1-parameter family. The length in the picture is given via a partially measured foliation indicated by parallel lines. At the boundaries, one of the blue loops obtains length zero, the graph is however still embedded in the surface. The extra tails or spines give the base points and the dotted spine keeps track of the polycyclic structure [25, Appendix A] in which the extra tail pointing to the "empty boundary" is a cycle by itself. There is no extra genus, but an extra boundary component with one interval, which is a module-interval. Geometrically this means that there is a second constant loop that is identified with the input base point. The polycyclic structure and extra markings appear in the combinatorial compactification of moduli space, [20], that was axiomatized with graphs in [25] using polycyclic graphs aka. stable ribbon graphs [35]. It also related to non-Sigma modular operads [5, 17, 28, 41]. The extra decoration manifests itself in the action, which does not only involve polygon correlators. The interpretation of the partially measured foliations as moving pieces of string according to [18, §5.11] is in Figure 1, which also illustrates the time reversal.

This is also exactly the action that is induced on loop spaces which is algebratized in Section 3. The totalization is the discretization of the map $\mathcal{L}$oop. By Section 3.3.3 and Section 3.3.4 is exactly realized via an intersection interpretation of Section 3.1 cosimplicially on the loop spaces, see Section 3.2.3.

5. Calculations

5.1. Correlators

We give the dualizations for the functions $\int_n := \varepsilon \circ \mu^{[n]} : A^{\otimes n} \to k$, which is given by $\int_n a_1 \otimes \cdots \otimes a_n = \langle a_1 \cdots a_n \rangle$, where $\mu_A^{[n]}$ is the iterated multiplication. This defines different forms of the operation, and we discuss which of these forms may exist, without the Frobenius assumption. These forms may break the cyclic symmetry. The tensor factors may simply be a k-module V or its dual $\check{V}$, an associative algebra A, or a coassociative coalgebra Ω, e.g. $\check{A}$ for A finite dimensional or of finite type. We will tacitly assume such a condition, when we use $\check{A}$ as a coalgebra variable. Subscripts indicate modules, e.g. ${}_AM_A$ stands for an A-A-bimodule M. The idea is that there is a hierarchy of operations on tensors: shuffles, contractions, multiplication, comultiplication, actions. This is the point of view underlying [12] and [22].

5.1.1. Relaxations of the Frobenius condition for $\int_n$. The basic form $\int_{n+1}$ exists for an associative algebra A with a morphism of k-modules $\varepsilon : A \to k$. This is cyclic if ε is a trace. It also exists for an algebra A as a morphism of k-modules

$$\mathrm{ev} \circ \mu_A^{[n]} : \check{A} \otimes A^{\otimes n} \to k, \quad \check{a}_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \check{a}_0(a_1 \cdots a_n)$$

as the multiplication followed by the dual pairing, aka. evaluation. The latter form is basis of the action of the little discs [21], since the restriction on the surfaces says that all the regions are polygonal and have one distinguished coalgebra tensor. Dualizing in the last slot generally yields the iterated multiplication map $\mu_A^{(n)} \in \mathrm{Hom}(A^{\otimes n}, A)$:

$$\mu_A^{(n)}(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_n,$$

which is defined for any associative algebra A. Dualizing all but the first entry yields the iterated co-multiplication $\Delta_\Omega^{(n)} \in \mathrm{Hom}(\Omega, \Omega^{\otimes n})$ which exists for any coalgebra Ω. Dualizing all entries yields the element $\Delta^{(n+1)}(1)$ which exists for a pointed coalgebra $(\Omega, 1)$.

5.1.2. Calculations for low n.

(1) $\int_2 = \eta$: Interpreted as a morphism $A \to A$ this is id_A. As a morphism $k \to A^{\otimes 2}$ this is $C = \Delta(1)$, that is the Casimir element dual to the form.
Restrictions: The form η is simply the dual pairing and exists as the evaluation map $\mathrm{ev} : \check{V} \otimes V \to k$. The form $\mathrm{id}_V : V \to V$ only needs a k-module V. As $k \to \check{V} \otimes \check{V}$ is a bilinear form. In the form $k \to V \otimes V$ it is simply a fixed element—sometimes called a propagator—which is needed to operadically compose correlation functions [22, §2].

(2) $\int_3$: By dualizing in the third slot, this represents $\mu_A \in \mathrm{Hom}(A^{\otimes 2}, A)$.

$$a \otimes b \mapsto \Big(\int_3 a \otimes b \otimes C^{(1)}\Big) \otimes C^{(2)} = \sum \langle ab, C^{(1)} \rangle C^{(2)} = ab = \mu(a,b). \quad (5.1)$$

By dualizing in the second and third slot this yields $\Delta_\Omega \in \mathrm{Hom}(\Omega, \Omega^{\otimes 2})$.

$$\begin{aligned} a \mapsto \sum_{C_1,C_2} \Big(\int_3 a \otimes C_1^{(1)} \otimes C_2^{(1)}\Big) C_1^{(2)} \otimes C_2^{(2)} &= \sum_{C_1,C_2} \langle a, C_1^{(1)} C_2^{(1)} \rangle C_1^{(2)} \otimes C_2^{(1)} \\ = \sum_{C_1,C_2} \langle \Delta(a), C_1^{(1)} \otimes C_2^{(1)} \rangle C_1^{(2)} \otimes C_2^{(2)} &= \Delta(a). \end{aligned} \quad (5.2)$$

(3) $\int_4$: We will give the dualization in the 2nd and 4th slot. The map $A^{\otimes 2} \to A^{\otimes 2}$ is

$$\begin{aligned} a \otimes b \mapsto & \sum_{C_1,C_2} \Big(\int_4 a \otimes C_1^{(1)} \otimes b \otimes C_2^{(1)}\Big) C_1^{(2)} \otimes C_2^{(2)} \\ &= \sum_{C_1,C_2} \langle a, C_1^{(1)} b C_2^{(1)} \rangle \, C_1^{(2)} \otimes C_2^{(2)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{C_1,C_2} \langle \Delta_A(a), C_1^{(1)} \otimes bC_2^{(1)} \rangle C^{(2)} \otimes C_2^{(1)} \\
&= \sum_{C_1,C_2} \langle \Delta_A(a)(1 \otimes b), C_1^{(1)} \otimes C_2^{(1)} \rangle C_1^{(2)} \otimes C_2^{(1)} \\
&= \sum_{C_1,C_2} \langle \Delta_A(a), (1 \otimes b)(C_1^{(1)} \otimes C_2^{(1)}) \rangle C_1^{(2)} \otimes C_2^{(1)} \\
&= \Delta(a)(1 \otimes b) = a^{(1)} \otimes a^{(2)} b. \qquad (5.3)
\end{aligned}
$$

This form exists as a morphism $\Omega_A \otimes A \to \Omega_A \otimes \Omega_A$, i.e. a is a decoration by a coalgebra element, where the coalgebra is a right A module, and b is an algebra element. Using (1.2), also $\Delta(a)(1 \otimes b) = \Delta(1)(a \otimes b) = C(a \otimes b)$, where now there are no restrictions of a, b at first, but there has to be some module structure. E.g., if $a, b \in A$ and $C \in \check{A} \otimes \check{A}$ then this is a morphism $A \otimes A \to \check{A} \otimes \check{A}$ etc. Switching the roles of a and b, one also has the form $A \otimes {}_A\Omega \to {}_A\Omega \otimes {}_A\Omega$:

$$
a \otimes b \mapsto ab^{(2)} \otimes b^{(1)} = (a \otimes 1)\Delta_\Omega^{\mathrm{op}}(b). \qquad (5.4)
$$

(4) $\int_5$: We compute the dualization in the 2nd and 4th slot. The map $A^{\otimes 3} \to A^{\otimes 2}$ is

$$
\begin{aligned}
a \otimes b \otimes c \mapsto &\sum_{C_1,C_2} \left(\int aC_1^{(1)} bC_2^{(1)} c \right) C_1^{(2)} \otimes C_2^{(2)} \\
&= \sum_{C_1,C_2} \langle C_1^{(1)} bC_2^{(1)} c, a \rangle \, C_1^{(2)} \otimes C_2^{(2)} \\
&= \sum_{C_1,C_2} \langle (C_1^{(1)} b) \otimes (C_2^{(1)} c), \Delta_A(a) \rangle \, C_1^{(2)} \otimes C_2^{(2)} \\
&= \sum_{C_1,C_2} \langle C_1^{(1)} \otimes C_2^{(1)}, (b \otimes c)\Delta_A(a) \rangle \, C_1^{(2)} \otimes C_2^{(2)} \\
&= (b \otimes c)\Delta(a) = \sum ba^{(1)} \otimes ca^{(2)}. \qquad (5.5)
\end{aligned}
$$

which exists as a morphism ${}_A\Omega \otimes A \otimes A \to {}_A\Omega \otimes {}_A\Omega$.
From (1.2) it follows that

$$
(b \otimes c)\Delta_A(a) = (b \otimes c)\Delta_A(1)(a \otimes 1) = (b \otimes ca)\Delta_A(1).
$$

5.2. OTFT from a Frobenius algebra

We will now show that [22, Assumption 4.1.2] of commutativity of A is not necessary and that the equations of [22, Remark 4.2] hold for any Frobenius algebra. Let A be a Frobenius algebra as in Section 1.4. Since $\int$ is cyclic, we have that

$$
\langle a_i \cdots a_n a_1 \cdots a_{i-1} \rangle = \langle a_j \cdots a_n a_1 \cdots a_{j-1} \rangle \quad \text{for all } 1 \le i, j \le n. \qquad (5.6)
$$

Using that $a = \sum \langle aC^{(1)} \rangle C^{(2)}$, we get the factorization

$$
\langle a_1 \cdots a_n \rangle = \sum \langle a_1 \cdots a_i C^{(1)} \rangle \langle C^{(2)} a_{i+1} \cdots a_n \rangle. \qquad (5.7)
$$

Proposition 5.1. *Let A be a Frobenius algebra. Using the notation of Section 1.4 and the one above it follows: For all $1 \le i, j \le n$ and $1 \le k, l \le m$*

$$\begin{aligned}\sum \langle a_1 \cdots a_i C_1^{(1)} b_k \cdots b_m b_1 \cdots b_{k-1} C_1^{(2)} a_{i+1} \cdots a_n \rangle \\ = \sum \langle a_1 \cdots a_j C_2^{(2)} b_l \dots b_m b_1 \cdots b_{l-1} C_2^{(1)} a_{j+1} \cdots a_n \rangle.\end{aligned} \tag{5.8}$$

Also,

$$\begin{aligned}\sum \langle a_1 \cdots a_i C_1^{(1)} a_{k+1} \cdots a_l C_2^{(1)} a_{j+1} \cdots a_k C_1^{(2)} a_{i+1} \cdots a_j C_2^{(2)} a_{l+1} \cdots a_n \rangle \\ = \sum \langle a_1 \cdots a_n C_1^{(1)} C_2^{(1)} C_1^{(2)} C_2^{(2)} \rangle.\end{aligned} \tag{5.9}$$

This fact is well known, albeit maybe not in this presentation, as it is equivalent the theorem that 2d Open Topological Field Theories are equivalent to Frobenius algebras, see Remark 4.1. The two equations correspond to cuts for the annulus and the torus with one boundary, see Figure 5.

Proof. For (5.8), assume without loss of generality $i < j$ and $k < l$,

$$\begin{aligned}&\sum \langle a_1 \cdots a_i C_1^{(1)} b_k \cdots b_m b_1 \cdots b_{k-1} C_1^{(2)} a_{i+1} \cdots a_n \rangle \\ &\quad = \sum_{C_1, C_2} \langle a_{j+1} \cdots a_n a_1 \cdots a_i C_1^{(1)} b_k \cdots b_{l-1} C_2^{(1)} \rangle \\ &\qquad\qquad \cdot \langle C_2^{(2)} b_l \cdots b_m b_1 \cdots b_{k-1} C_1^{(2)} a_{i+1} \cdots a_j \rangle \\ &\quad = \sum \langle a_1 \cdots a_j C_2^{(2)} b_l \dots b_m b_1 \cdots b_{l-1} C_2^{(1)} a_{j+1} \cdots a_n \rangle,\end{aligned}$$

where in the first step we used (5.6) and then (5.7) to first rotate until a_j is at the end and then split. In the second step, we rotated both expressions with (5.6) so that $C_1^{(1)}$ is on the

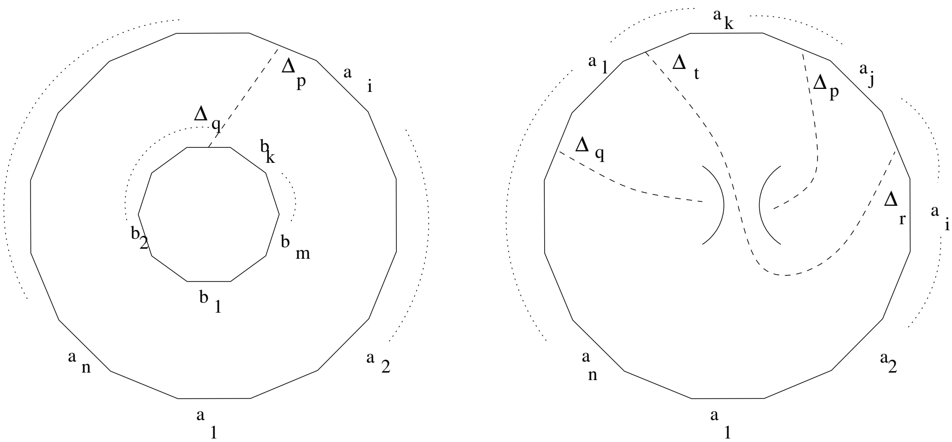

Figure 5. The cut on the annulus corresponding to (5.8) and the cuts on the torus with one boundary component (5.9). The equations say that the choice of endpoints of the cuts does not matter.

right and $C_1^{(2)}$ is on the left and then used (5.7) to merge them. For (5.9) we have

$$
\begin{aligned}
&\sum_{C_1,C_2} \langle a_1 \cdots a_i C_1^{(1)} a_{k+1} \cdots a_l C_2^{(1)} a_{j+1} \cdots a_k C_1^{(2)} a_{i+1} \cdots a_j C_2^{(2)} a_{l+1} \cdots a_n \rangle \\
&\quad = \sum_{C_1,C_2} \langle a_1 \cdots a_i a_{i+1} \cdots a_j \Delta_p a_{l+1} \cdots a_n C_2^{(1)} a_{k+1} \cdots a_l C_1^{(2)} a_{j+1} \cdots a_k C_2^{(2)} \rangle \\
&\quad = \sum_{C_1,C_2} \langle a_1 \cdots a_i a_{i+1} \cdots a_j a_{j+1} \cdots a_k C_1^{(1)} C_2^{(1)} a_{l+1} \cdots a_n C_1^{(2)} a_{k+1} \cdots a_l C_2^{(2)} \rangle \\
&\quad = \sum_{C_1,C_2} \langle a_1 \cdots a_i a_{i+1} \cdots a_j a_{j+1} \cdots a_k a_{k+1} \cdots a_l C_1^{(1)} C_2^{(1)} C_1^{(2)} a_{l+1} \cdots a_n C_2^{(2)} \rangle \\
&\quad = \sum_{C_1,C_2} \langle a_1 \cdots a_i a_{i+1} \cdots a_j a_{j+1} \cdots a_k a_{k+1} \cdots a_l a_{l+1} \cdots a_n C_1^{(1)} C_2^{(1)} C_1^{(2)} C_2^{(2)} \rangle,
\end{aligned}
$$

where we used (5.8) to move each block not yet in place by one in each step. ∎

Corollary 5.2. *The assumption of commutativity* [22, Assumption 4.1.2] *is unnecessary and all the operations of* [22] *are defined for the Hochschild cochains of any associative Frobenius algebra A. The local correlation function $Y(S)$ in* [22, (4.3)] *for a surface S of genus g with b decorated boundaries where the i-th boundary is decorated by elements $a_1^i, \ldots, a_{n_i}^i$ the non-commutative case is*

$$
Y(S)\left(\bigotimes_{j=1}^{b} \bigotimes_{i=1}^{n_j} a_i^j\right) = \left\langle (a_1^1 \cdots a_{n_1}^1) \prod_{l=2}^{b} \left(\sum C_l^{(1)} a_1^l \cdots a_{n_l}^l C_l^{(2)} \right) \right.
$$

$$
\left. \cdot \prod_{k=1}^{g} \left(\sum_{C_{k_1} C_{k_2}} C_{k1}^{(1)} C_{k2}^{(1)} C_{k1}^{(2)} C_{k2}^{(2)} \right) \right\rangle, \tag{5.10}
$$

which is simply the correlation function $Y_A(S)$ of the 2d-OTFT of the marked surface for the OTFT defined by A.

Proof. The operations *a priori* depend on a choice of triangulation by extra arcs/cuts. Since the two equations (5.8) and (5.9) hold, the result is independent of such a choice as they can be used to put the cuts into a standard position yielding (5.10). This follows from the transitivity of Whitehead moves on triangulations. By the gluing axioms of an OTFT this is the correlation function corresponding to the given surface. ∎

Note that (5.10) seems to depend on the enumeration of the boundary components but the result is independent of that ordering, again by applying (5.8). By the same equation, it also only depends on the cyclic order of the elements at each boundary. There is a standard order of all the elements given by the fact that the boundary components are labelled.

5.2.1. Pseudo-commutative Frobenius algebras. We call A pseudo-commutative if $\mu\Delta(ab) = eab$.

Lemma 5.3. *If one of the following conditions holds, A is pseudo-commutative.*

(1) *A is commutative, or*

(2) *A is graded Gorenstein, or*

(3) *$\Delta_A(1)(a \otimes 1) = \Delta_A(1)(1 \otimes a)$, or*

(4) *$(1 \otimes a)\Delta_A(1) = (a \otimes 1)\Delta_A(1)$.*

Proof. A is pseudo-commutative if and only if $\sum C^{(1)} a C^{(2)} b = \sum C^{(1)} C^{(2)} ab$, which is the case if A is commutative. If A is graded Gorenstein of degree d then $\sum C^{(1)} a C^{(2)} b$ is of degree at least d and unless $\deg(a) = \deg(b) = 0$ both sides are 0. As all elements in degree 0 lie in the center, the equation holds.

For (3) using (1.2), we have

$$\begin{aligned}\sum C^{(1)} a \otimes C^{(2)} b &= \Delta_A(1)(a \otimes 1)(1 \otimes b) = \Delta_A(1)(1 \otimes a)(1 \otimes b)\\ &= \Delta_A(1)(1 \otimes ab) = \sum C^{(1)} \otimes C^{(2)} ab,\end{aligned}$$

which after applying μ to both sides yields the defining equation. If A is graded Gorenstein of degree d then $\sum C^{(1)} a C^{(2)} b$ is of degree at least d and unless $\deg(a) = \deg(b) = 0$ or both sides are 0. As all elements in degree 0 lie in the center, the equation holds. The case (4) is similar. ■

Lemma 5.4. *In case that A is pseudo-commutative, equation* [22, (4.3)] *holds, that is*

$$Y(S)\Big(\bigotimes_{j=1}^{b} \bigotimes_{i=1}^{n_j} a_i^j\Big) = \Big\langle \prod_{j=1}^{b} \prod_{i=1}^{n_j} a_i^j e^{-\chi(S)+1} \Big\rangle. \tag{5.11}$$

Proof. If A is pseudo-commutative, we can move all the factors $C_i^{(1)} C_i^{(2)}$ next to each other to the right. $C^{(1)} C^{(2)} = \mu\Delta(1) = e$ and there are $b - 1 + 2g = -\chi(S) + 1$ such factors. ■

Remark 5.5. Note that if A is graded Gorenstein, for degree reasons, (5.11) is 0 unless $-\chi(S) + 1 \leq 1$ and if $\chi(S) = 0$, that is S is an annulus, then all the a_i^j must be of degree 0, so that in this case, the correlation function vanishes modulo the constants A_0.

5.2.2. Stabilization and the semi-simple case. We call A *E-unital* if $e = 1$. In this case, $\varepsilon(a) = \mathrm{tr}(L_a)$ where L_a is the left multiplication by a, as $\varepsilon(a) = \sum \langle a, C^{(1)} C^{(2)} \rangle = \sum \langle a C^{(1)}, C^{(2)} \rangle = \varepsilon(\mathrm{tr}(L_a))$.

Lemma 5.6. *A is commutative and E-unital if and only if A is* isometric, *i.e.* $\mu_A \Delta_A = \mathrm{id}_A$.

Proof. "$\Rightarrow$": (1.2) implies that if A is commutative and E-unital, it is isometric.

"$\Leftarrow$": By the assumption $1 = \mu\Delta(1) = e$ and using (1.2),

$$\mu(\Delta_A(a)) = \mu(\Delta_A(1)(a \otimes 1)) = \sum C^{(1)} a C^{(2)} = a.$$

Using this and Proposition 5.1,

$$\langle ab, c\rangle = \sum \langle C^{(1)} ab C^{(2)} c\rangle = \sum \langle C^{(1)} ba C^{(2)} c\rangle = \langle ba, c\rangle. \qquad \blacksquare$$

Remark 5.7. If A is isometric, then the operations pass through the stabilization [25] and contain an E_∞ structure [23].

Remark 5.8 (Semi-simplicity). If A is free of finite rank and semi-simple, there is a basis e_i with $e_i e_j = \delta_{ij} e_i$. This implies that $\sum_i e_i = 1$. Setting $\lambda_i = \varepsilon(e_i)$, $e^i = \frac{1}{\lambda_i} e_i$, and $e = \sum_i \frac{1}{\lambda_i} e_i$ is invertible with inverse $e^{-1} = \sum_i \lambda_i e_i$. If all $\lambda_i = 1$, which is sometimes called normalized semi-simple, then A is E-unital. In case A is semi-simple, there is a flow to a normalized A, see [23]. In [2] it is shown that if A is even commutative, then being semi-simple is equivalent to e being invertible.

6. Dualities and further topics

6.1. Dualities

6.1.1. Naïve duality. The operations were defined by dualizing the arguments of $\hat{Y}$, see Section 2.2.3. The choice of inputs and outputs is dictated by the cell. One can ask about the other forms of the operation $\hat{Y}_{CH}$ as $\overline{T}A$ is graded isomorphic to its dual. As operations these always exists, but their PROP structure is more complicated, see Section 6.2.2.

Switching all inputs to outputs for the PROP one obtains a naïve input/output dual operation that is an (m, n)-ary operation from every (n, m)-ary operation via

$$\mathrm{Hom}(CH^{\otimes n}, CH^{\otimes m}) \simeq \mathrm{Hom}(CH^{\otimes m+n}, k) \simeq \mathrm{Hom}(CH^{\otimes m}, CH^{\otimes n}).$$

Example 6.1. For instance, the degree 0 product is dual to a degree 0 coproduct, which is different from the natural degree 1 product. It corresponds to the pointwise coproduct of [47]. Similarly the degree 1 coproduct is dual to a degree 1 product which shares the same correlation function (2.10). This is the sum over the products $\sqcup$, see [22, §4.1.1, eq. (4.10)], where the degree $n \mapsto (p, q), n = p + q + 1$ part of the coproduct dualized to $\sqcup$ in degree $(p, q) \mapsto p + q + 1 = n$,

$$f \sqcup g(a_1 \otimes \cdots \otimes a_n) = f(a_1 \otimes \cdots \otimes a_p) a_{p+1} g(a_{p+2} \otimes \cdots \otimes a_n). \tag{6.1}$$

To obtain the usual cup product, one needs to apply a degeneracy, that is set $a_{p+1} = 1$.

6.1.2. Time reversal symmetry (TRS). Given a cell c represented by an arc family and a boundary input output marking, we define the TRS dual $\check{c}$ by reversing the "in" and "out" labels. This *changes the normalized cell* and the interval marking in the discretized PROP.

For a cell c with arcs only from inputs to outputs in which all boundaries are hit, that is a cell of $\mathcal{A}rc^{i\leftrightarrow o}$ in the notation of [20], the TRS dual $\check{c}$ is also a cell of $\mathcal{A}rc^{i\leftrightarrow o}$, and hence retracts to a normalized cell $\check{c}_1$ of $\overline{\mathcal{A}rc}_1^{i\leftrightarrow o}$. If c_1 is the normalized cell of c then the TRS dual of the operation $\mathrm{op}(c_1)$ is $\mathrm{TRS}(\mathrm{op}(c_1)) := \mathrm{op}(\check{c}_1)$. Unlike the naïve dual, the TRS dual operation usually has different degree. The degree of a normalized cell with n inputs and m outputs, and thus the degree of the corresponding operation, is $\#\,\text{arcs} - n$. The degree of the TRS dual which is an m to n operation is $\#\,\text{arcs} - m$. Thus the degree difference of the operations is $m - n$.

Remark 6.2. This can simplicially be understood as two different join decompositions. A cell $c = (\Sigma, \alpha)$ of $\overline{\mathcal{A}rc}^{i\leftrightarrow o}$ is a $\Delta^{|\alpha|-1}$. If Σ has n inputs and m outputs, then there are two partitions of α, $\alpha = \coprod_{i=1}^{n} \alpha_i$ and $\alpha = \coprod_{j=1}^{m} \alpha'_j$. This gives rise to two join decompositions $[\alpha_1 - 1] * \cdots * [\alpha_n - 1] = [\alpha - 1] = [\alpha'_1 - 1] * \cdots * [\alpha'_m - 1]$. The normalization drops the $*$ and replaces it with the polysimplicial product.

Example 6.3. The cell c_Δ for the comultiplication is the TRS dual of the cell c_μ for the multiplication, which has 2 arcs, see Figure 1. The $(2,1)$ multiplication of degree $2-2=0$ has as TRS dual the $(1,2)$ comultiplication of degree $2-1=1$.

Several interesting cells appear as homotopies for the Gerstenhaber and BV structure [30]. Their TRS duals give new interesting homotopies for the TRS dual operations.

Example 6.4. The TRS dual for the pre-Lie or Gerstenhaber product gives a homotopy relation for Δ and Δ^{op}. Further, $\Delta_{CH} + \Delta^{\mathrm{op}}_{CH} \sim C$ where C only has components $C : CH^n \to CH^{n-1} \otimes CH^0$ which, using notation as in Theorem 2.1, are given by

$$C(f^n)(a_1 \otimes \cdots \otimes a_{n-1} \otimes \lambda) = \sum_{i=1}^{n} \pm \lambda f(a_1 \otimes \cdots \otimes a_{p-1} \otimes C_1^{(1)} \otimes a_p \otimes \cdots \otimes a_{n-1}) \otimes C_2^{(2)} C_1^{(2)} C_2^{(1)}. \quad (6.2)$$

This follows from applying the general procedure laid out in Section 4 to Figure 6. The cell given in Figure 6 gives that homotopy of the sum of the two operations to the operation of the base side. Since the boundary of the operation C and $\Delta + \Delta^{\mathrm{op}}$ is 0 they are cohomology operations.

Note that in the graded Gorenstein case, by degree reasons, operation equals

$$\sum_{i=1}^{n} \pm \lambda f(a_1 \otimes \cdots \otimes a_{p-1} \otimes e_{\mathrm{top}} \otimes a_p \otimes \cdots \otimes a_{n-1}) \otimes e_{\mathrm{top}}. \quad (6.3)$$

This is a sort of Poincaré dual degeneracy map, which geometrically corresponds to spawning off a loop at some point of the loop. In the Gorenstein case, this operation is zero for the reduced cochains $\widetilde{CH}^*(A, A)$.

Example 6.5. Using the TRS dual of δ of [30, §2.2, Figure 7], see Figure 6, one obtains the following relation for the BV operator, see Figure 7—denoted by BV here to avoid confusion with the coproduct $\Delta_{CH}(BV(f)) = \check{\partial}(f) + \tau_{12}\check{\partial}(f)$, where

$$\begin{aligned}
&\hat{Y}(\check{\delta})(a_0 \otimes \cdots \otimes a_n)\\
&\quad= \sum_{p=1}^{n-2}\sum_{m=0}^{n-2} \pm\varepsilon(a_0)(a_p C^{(1)} \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes a_{p+m+2} \otimes \cdots \otimes a_n)\\
&\qquad \otimes (a_{p+m+1} C^{(2)} \otimes a_{p+1} \otimes \cdots \otimes a_{p+m}).
\end{aligned} \tag{6.4}$$

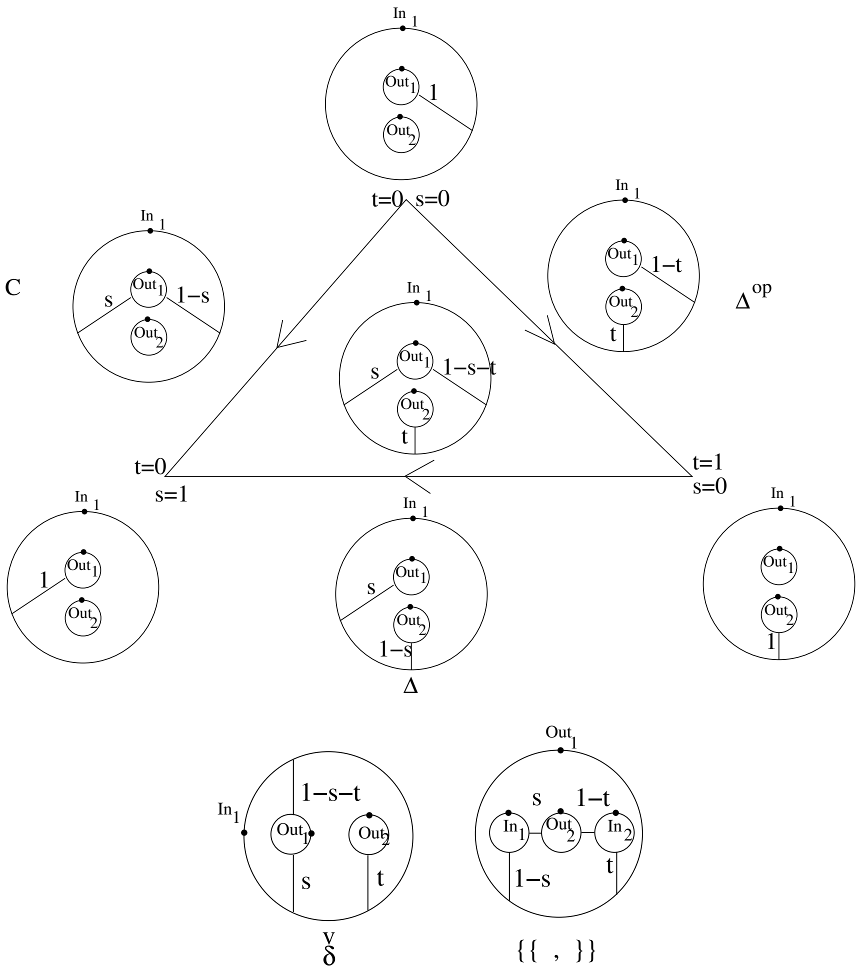

Figure 6. The TRS dual of the pre-Lie product as a homotopy, the TRS dual of the operator δ and the double bracket.

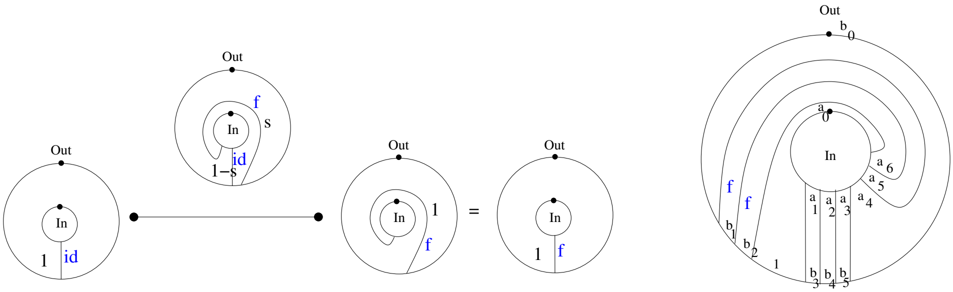

Figure 7. The animated BV as a homotopy and one discretized operation. The example is the morphism $a_0 \otimes \cdots \otimes a_6 \mapsto \varepsilon(a_0)\, a_4 \otimes f(a_5) \otimes f(a_6) \otimes a_1 \otimes a_2 \otimes a_3$. Omitting reference to f is the original BV.

Note that the TRS dual of the homotopies for the Gerstenhaber structure and the BV property [30, Figures 10, 11, 12] should also yield interesting operations.

6.1.3. Treating empty boundaries. More generally, there can be empty output boundaries allowing to "bubble off constant loops", while "in" boundaries all have to be hit. Upon reversal, this condition gets switched, to all "out" boundaries are hit. The correlation function is well defined as well in this case, by using the standard marking. This may lead to additional factors of A, which in the loop space operations stem from the inclusion of constant loops. These operations are also in the TRS dual PROP where the incidence conditions on the arcs on the input and outputs are switched. The TRS dual operations and correlations functions are summarized in Table 1.

6.2. Further topics

Note that in this setting the intervals between parallel arcs all belong to quadrilaterals and the relevant form of correlation function $\int_2$ is id passing on the variable, see Section 5.1.2 (1). The interesting part of the action is therefore on the surfaces that are defined

	in → out	out → in
No empty boundary	$\mathrm{Hom}(CH^{\otimes n}, CH^{\otimes m})$	$\mathrm{Hom}(CH^{\otimes m}, CH^{\otimes n})$
Correlation function	$\mathrm{Hom}(CH^{\otimes n+m}, k)$	$\mathrm{Hom}(CH^{\otimes n+m}, k)$
r empty boundaries	$\mathrm{Hom}(CH^{\otimes n}, CH^{\otimes m} \otimes A^{\otimes r})$	$\mathrm{Hom}(CH^{\otimes m} \otimes A^{\otimes r}, CH^{\otimes n})$
	$\subset (CH^{\otimes n}, CH^{\otimes m+r})$	$\subset \mathrm{Hom}(CH^{\otimes m+r}, CH^{\otimes n})$
Correlation function	$\mathrm{Hom}(CH^{\otimes n+m} \otimes A^{\otimes r}, k)$	$\mathrm{Hom}(CH^{\otimes n+m} \otimes A^{\otimes r}, k)$
	$\subset \mathrm{Hom}(CH^{\otimes m+n+r}, k)$	$\subset \mathrm{Hom}(CH^{\otimes m+n+r}, k)$

Table 1. The TRS duals of operations and their correlation functions.

by the original, not replicated, arc system. The original choice is to use OTFTs and the pairing, we will briefly discuss other choices. A fuller discussion is relayed to [29].

6.2.1. Animation. In [44] a so called animation is introduced. This is naturally incorporated into the present framework. Given an A module M and a morphism $f : A \to A$, one has the new module structure $\rho_f(a,m) = f(a)m$. In this way, one can twist the coefficient bimodule ${}_AM_A$ by f. Furthermore, one can act on TA by powers of f thus inducing new twisted Hochschild complexes. In the presented formalism, this simply means allowing to replace the propagator for the quadrilaterals to be given by f. In particular, given a set of A-endomorphisms $\mathcal{F}$, we define an $\mathcal{F}$ marking for an arc system to be a map $f : \alpha \to \mathcal{F}$. We define the operation for an $\mathcal{F}$ marking for a cell c to be given by $f(a)$ on the quadrilaterals. In the general case, one should use self-adjoint maps f and set the marked quadrilateral function to be $\langle a, f(b)\rangle$ on quadrilaterals marked by f. The form which used to be id_A then becomes $a \mapsto f(a)$. Figure 7 shows the twisted B operator which is a homotopy from $\mathrm{id}_{A\otimes TA}$ to $\mathrm{id}_A \otimes f^{\otimes}$ as an example. The operation is

$$a_0 \otimes \cdots \otimes a_n \mapsto \pm \sum \varepsilon(a_0)\, a_p \otimes f(a_{p+1}) \otimes \cdots \otimes f(a_n) \otimes a_1 \otimes \cdots \otimes a_{p-1}. \quad (6.5)$$

6.2.2. Dualization to Hochschild chain operations. Further, following discussions with Z. Wang, we can also look at the dualization to $CH_*(A, A)$. This allows reinterpreting the naïve duality as an additional coloring for the PROP, and includes the products on Hochschild chains as found in [1, 45] into our package. For this, one labels naïvely dualized inputs and outputs by ho for homological and the ones keeping their original input/output designation as co for cohomological. This yields a bi-colored dg-PROP which acts on Hochschild chains, via the ho color, and Hochschild cochains via the co color. The correlation functions are the *old* correlation functions of [22]. The decoration is according the both in/out and ho/co marking. That is, co outputs and ho inputs are decorated in the induced orientation while co inputs and ho outputs are decorated in the opposite of the induced orientation.

In [32], we furthermore show that the action on the Tate–Hochschild complex [45] can be subsumed into the formalism of correlation functions [22] by a coloring keeping track of dualizations. This allows us to realize the homotopy transfer concretely. We naturally obtain the operations that are found in [1, 45]. For instance, dualizing the coproduct from co to ho colors for all three boundaries, one recovers the degree 1, (2, 1) product on CH_* given by the formula

$$\begin{aligned} &b_0 \otimes \cdots \otimes b_p \cup c_0 \otimes \cdots \otimes c_{n-p-1} \\ &\qquad = \pm \sum b_0 C^{(1)} \otimes c_1 \otimes \cdots \otimes c_{n-p-1} \otimes c_0 C^{(2)} \otimes b_1 \otimes \cdots \otimes b_p \end{aligned} \quad (6.6)$$

which can be read off (2.10).

Another upshot is a nice interpretation of the mixed m_3 products $CH^* \otimes CH_* \otimes CH^* \to CH^*$, $CH_* \otimes CH^* \otimes CH_* \to CH_*$ in terms of the dualization of a double bracket, which arises as a natural homotopy incorporating the coproduct and its opposite simultaneously and is a Gerstenhaber double bracket operation $CH^{\otimes 2} \to CH^{\otimes 2}$ in the sense of [42, 50]. The operation is given in Figure 6. The action is given by

$$\begin{aligned}
&(a_0 \otimes \cdots \otimes a_n) \otimes (b_0 \otimes \cdots \otimes b_m) \\
&\quad \mapsto \sum_{p,q} \pm \langle a_p, b_q \rangle (C^{(2)} b_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes b_{q+1} \otimes \cdots \otimes b_m) \\
&\qquad \otimes (C^{(1)} a_0 \otimes b_1 \otimes \cdots \otimes b_{q-1} \otimes a_{p+1} \otimes \cdots \otimes a_n). \qquad (6.7)
\end{aligned}$$

6.2.3. A_∞-version. In [33] we showed that one can relax the condition of A being associative to A_∞ for the brace operations, aka. A_∞-Deligne conjecture, and in [51] the same was done tor the BV operators, aka. cyclic A_∞ conjecture. As described in [26], this corresponds to introducing diagonals into the non-quadrilateral surfaces to specify an A_∞ version of the OTFT. This type of different theory for the S_v defined by α can be treated quite generally [6]. There should be a nontrivial relation to the A_∞ case to the double brackets above and those of [15].

Acknowledgments. We thank Gabriel Drummond-Cole who brought the GH coproduct to our attention during a visit to the IBS center for Geometry and Physics. We also thank him for the ensuing discussions. We furthermore thank Alexandru Oancea for his enlightening talk on the subject and Muriel Livernet for bringing me to that talk and encouraging me further to write this exposition. It is a pleasure to thank Dennis Sullivan for his comments and remarks. We thank Manuel Rivera and Zhengfeng Wang for discussions on the action on Hochschild chains and Boris Tsygan for his interest and questions about homotopies and "animation". Finally, we would like to thank the referee for the careful reading of the manuscript, the useful comments to improve the exposition and questions about further ramifications.

Funding. We thank the IHES, where this note was written, and the Simons Foundation for their support.

References

[1] H. Abbaspour, On the Hochschild homology of open Frobenius algebras. *J. Noncommut. Geom.* **10** (2016), no. 2, 709–743 Zbl 1380.16005 MR 3519050

[2] L. Abrams, The quantum Euler class and the quantum cohomology of the Grassmannians. *Israel J. Math.* **117** (2000), 335–352 Zbl 0954.53048 MR 1760598

[3] L. Abrams and C. Weibel, Cotensor products of modules. *Trans. Amer. Math. Soc.* **354** (2002), no. 6, 2173–2185 Zbl 1002.16033 MR 1885648

[4] S. Basu, *Transversal String Topology & Invariants of Manifolds*. ProQuest LLC, Ann Arbor, MI, 2011 MR 2942255

[5] C. Berger and R. M. Kaufmann, Comprehensive factorisation systems. *Tbilisi Math. J.* **10** (2017), no. 3, 255–277 Zbl 1397.18004 MR 3742580

[6] C. Berger and R. M. Kaufmann, Derived Decorated Feynman Categories: chain aspects. 2021, in preparation

[7] M. Chas and D. Sullivan, String topology. 1999, arXiv:math/9911159

[8] K. Cieliebak, N. Hingston, and A. Oancea, Poincaré duality for loop spaces. 2020, arXiv:2008.13161

[9] R. L. Cohen and J. D. S. Jones, A homotopy theoretic realization of string topology. *Math. Ann.* **324** (2002), no. 4, 773–798 Zbl 1025.55005 MR 1942249

[10] R. L. Cohen and J. R. Klein, Umkehr maps. *Homology Homotopy Appl.* **11** (2009), no. 1, 17–33 Zbl 1160.55005 MR 2475820

[11] I. Gálvez-Carrillo, R. M. Kaufmann, and A. Tonks, Three Hopf algebras from number theory, physics & topology, and their common background I: operadic & simplicial aspects. *Commun. Number Theory Phys.* **14** (2020), no. 1, 1–90 Zbl 1468.18021 MR 4060970

[12] M. Gerstenhaber, The cohomology structure of an associative ring. *Ann. of Math. (2)* **78** (1963), 267–288 Zbl 0131.27302 MR 161898

[13] M. Goresky and N. Hingston, Loop products and closed geodesics. *Duke Math. J.* **150** (2009), no. 1, 117–209 Zbl 1181.53036 MR 2560110

[14] N. Hingston and N. Wahl, Products and coproducts in string topology. 2017, arXiv:1709.06839

[15] N. Iyudu, M. Kontsevich, and Y. Vlassopoulos, Pre-Calabi-Yau algebras and double Poisson brackets. 2019, arXiv:1906.07134

[16] J. D. S. Jones, Cyclic homology and equivariant homology. *Invent. Math.* **87** (1987), no. 2, 403–423 Zbl 0644.55005 MR 870737

[17] R. Kaufmann and J. Lucas, Decorated Feynman categories. *J. Noncommut. Geom.* **11** (2017), no. 4, 1437–1464 Zbl 1410.18010 MR 3743229

[18] R. M. Kaufmann, Operads, moduli of surfaces and quantum algebras. In *Woods Hole mathematics*, pp. 133–224, Ser. Knots Everything 34, World Sci. Publ., Hackensack, NJ, 2004 MR 2123369

[19] R. M. Kaufmann, On several varieties of cacti and their relations. *Algebr. Geom. Topol.* **5** (2005), 237–300 Zbl 1083.55007 MR 2135554

[20] R. M. Kaufmann, Moduli space actions on the Hochschild co-chains of a Frobenius algebra. I. Cell operads. *J. Noncommut. Geom.* **1** (2007), no. 3, 333–384 Zbl 1145.55008 MR 2314100

[21] R. M. Kaufmann, On spineless cacti, Deligne's conjecture and Connes-Kreimer's Hopf algebra. *Topology* **46** (2007), no. 1, 39–88 Zbl 1140.55008 MR 2288726

[22] R. M. Kaufmann, Moduli space actions on the Hochschild co-chains of a Frobenius algebra. II. Correlators. *J. Noncommut. Geom.* **2** (2008), no. 3, 283–332 Zbl 1169.55005 MR 2411420

[23] R. M. Kaufmann, Noncommutative aspects of open/closed strings via foliations. *Rep. Math. Phys.* **61** (2008), no. 2, 281–293 Zbl 1164.81013 MR 2424095

[24] R. M. Kaufmann, A proof of a cyclic version of Deligne's conjecture via cacti. *Math. Res. Lett.* **15** (2008), no. 5, 901–921 Zbl 1161.55001 MR 2443991

[25] R. M. Kaufmann, Dimension vs. genus: a surface realization of the little k-cubes and an E_∞ operad. In *Algebraic topology—old and new*, pp. 241–274, Banach Center Publ. 85, Polish Acad. Sci. Inst. Math., Warsaw, 2009 Zbl 1171.55004 MR 2503531

[26] R. M. Kaufmann, Graphs, strings, and actions. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, pp. 127–178, Progr. Math. 270, Birkhäuser Boston, Boston, MA, 2009 Zbl 1206.18009 MR 2641189

[27] R. M. Kaufmann, Open/closed string topology and moduli space actions via open/closed Hochschild actions. *SIGMA Symmetry Integrability Geom. Methods Appl.* **6** (2010), Paper 036 Zbl 1197.55005 MR 2647315

[28] R. M. Kaufmann, Feynman categories and representation theory. In *Representations of algebras, geometry and physics*, pp. 11–84, Contemp. Math. 769, Amer. Math. Soc., [Providence], RI, [2021] ©2021 MR 4254096

[29] R. M. Kaufmann, Branes and derived operations of arcs. 2022, in preparation

[30] R. M. Kaufmann, M. Livernet, and R. C. Penner, Arc operads and arc algebras. *Geom. Topol.* **7** (2003), 511–568 Zbl 1034.18009 MR 2026541

[31] R. M. Kaufmann and R. C. Penner, Closed/open string diagrammatics. *Nuclear Phys. B* **748** (2006), no. 3, 335–379 Zbl 1214.32004 MR 2242677

[32] R. M. Kaufmann, M. Rivera, and Z. Wang, The algebraic structure of the Tate-Hochschild complex via a geometric moduli space action. 2021, in preparation

[33] R. M. Kaufmann and R. Schwell, Associahedra, cyclohedra and a topological solution to the A_∞ Deligne conjecture. *Adv. Math.* **223** (2010), no. 6, 2166–2199 Zbl 1194.55014 MR 2601011

[34] R. M. Kaufmann, B. C. Ward, and J. J. Zúñiga, The odd origin of Gerstenhaber brackets, Batalin-Vilkovisky operators, and master equations. *J. Math. Phys.* **56** (2015), no. 10, 103504 Zbl 1327.81268 MR 3412702

[35] M. Kontsevich, Feynman diagrams and low-dimensional topology. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, pp. 97–121, Progr. Math. 120, Birkhäuser, Basel, 1994 Zbl 0872.57001 MR 1341841

[36] P. Lambrechts and D. Stanley, Poincaré duality and commutative differential graded algebras. *Ann. Sci. Éc. Norm. Supér. (4)* **41** (2008), no. 4, 495–509 Zbl 1172.13009 MR 2489632

[37] A. D. Lauda and H. Pfeiffer, Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras. *Topology Appl.* **155** (2008), no. 7, 623–666 Zbl 1158.57038 MR 2395583

[38] J.-L. Loday, *Cyclic homology*. Second edn., Grundlehren Math. Wiss. 301, Springer, Berlin, 1998 MR 1600246

[39] Y. I. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*. Colloq. Publ., Am. Math. Soc. 47, American Mathematical Society, Providence, RI, 1999 Zbl 0952.14032 MR 1702284

[40] R. Manuel and Z. Wang, Invariance of the Goresky-Hingston algebra on reduced hochschild homology. 2019, arXiv:1912.13267

[41] M. Markl, Modular envelopes, OSFT and nonsymmetric (non-Σ) modular operads. *J. Noncommut. Geom.* **10** (2016), no. 2, 775–809 Zbl 1375.81205 MR 3519052

[42] G. Massuyeau and V. Turaev, Brackets in the Pontryagin algebras of manifolds. *Mém. Soc. Math. Fr. (N.S.)* (2017), no. 154, 138 Zbl 1416.17012 MR 3796434

[43] F. Naef and T. Willwacher, String topology and configuration spaces of two points. 2019, arXiv:1911.06202

[44] R. Nest and B. Tsygan, Cyclic homology. In preparation available at https://sites.math.northwestern.edu/~tsygan/

[45] M. Rivera and Z. Wang, Singular Hochschild cohomology and algebraic string operations. *J. Noncommut. Geom.* **13** (2019), no. 1, 297–361 Zbl 1448.16010 MR 3941481

[46] D. Sullivan, Private communication with D. Sullivan. 2001, about a conversation between D. Sullivan and K. Fukaya in 2001 at Sullivan's 60th birthday conference "Graphs and Patterns in Mathematics and Theoretical Physics". Stony Brook, 2001

[47] D. Sullivan, Open and closed string field theory interpreted in classical algebraic topology. In *Topology, geometry and quantum field theory*, pp. 344–357, London Math. Soc. Lecture Note Ser. 308, Cambridge Univ. Press, Cambridge, 2004 Zbl 1088.81082 MR 2079379

[48] D. Sullivan, Sigma models and string topology. In *Graphs and patterns in mathematics and theoretical physics*, pp. 1–11, Proc. Sympos. Pure Math. 73, Amer. Math. Soc., Providence, RI, 2005 Zbl 1080.53085 MR 2131009

[49] D. Sullivan, String topology background and present state. In *Current developments in mathematics, 2005*, pp. 41–88, Int. Press, Somerville, MA, 2007 Zbl 1171.55003 MR 2459297

[50] M. Van den Bergh, Double Poisson algebras. *Trans. Amer. Math. Soc.* **360** (2008), no. 11, 5711–5769 Zbl 1157.53046 MR 2425689

[51] B. C. Ward, Cyclic A_∞ structures and Deligne's conjecture. *Algebr. Geom. Topol.* **12** (2012), no. 3, 1487–1551 Zbl 1277.18013 MR 2966694

Received 1 August 2019.

Ralph M. Kaufmann
Department of Mathematics, Purdue University, 150 N. University St., West Lafayette, IN 47907, USA; rkaufman@purdue.edu