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An equivariant pullback structure of trimmable graph C^*-algebras

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Abstract. To unravel the structure of fundamental examples studied in noncommutative topology, we prove that the graph C^*-algebra $C^*(E)$ of a trimmable graph E is $U(1)$-equivariantly isomorphic to a pullback C^*-algebra of a subgraph C^*-algebra $C^*(E'')$ and the C^*-algebra of functions on a circle tensored with another subgraph C^*-algebra $C^*(E')$. This allows us to approach the structure and K-theory of the fixed-point subalgebra $C^*(E)^{U(1)}$ through the (typically simpler) C^*-algebras $C^*(E')$, $C^*(E'')$ and $C^*(E'')^{U(1)}$. As examples of trimmable graphs, we consider one-loop extensions of the standard graphs encoding respectively the Cuntz algebra $\mathcal{O}_2$ and the Toeplitz algebra $\mathcal{T}$. Then we analyze equivariant pullback structures of trimmable graphs yielding the C^*-algebras of the Vaksman–Soibelman quantum sphere S_q^{2n+1} and the quantum lens space $L_q^3(l;1,l)$, respectively.

1. Introduction and preliminaries

Graph C^*-algebras are remarkable examples of "operator algebras that one can see" [35]. In particular, they proved to be extremely useful in determining the K-theory of noncommutative deformations of interesting topological spaces. They come equipped with a natural circle action (called the gauge action), and their gauge-invariant subalgebras describe some fundamental examples of noncommutative topology [9, 27, 28].

The goal of this paper is to analyze trimmable graphs (see Definition 2.1) naturally occurring in noncommutative topology, such as graphs giving the C^*-algebras of the Vaksman–Soibelman quantum sphere S_q^{2n+1} and the quantum lens space $L_q^3(l;1,l)$, respectively. The gauge-invariant subalgebra of the former C^*-algebra defines the Vaksman–Soibelman quantum complex projective space $\mathbb{C}\mathrm{P}_q^n$, and the gauge-invariant subalgebra of the latter C^*-algebra defines the quantum teardrop $\mathbb{W}\mathrm{P}_q^1(1,l)$. Although much is already known (generators included) about both the K-theory of $C(\mathbb{C}\mathrm{P}_q^n)$ and $C(\mathbb{W}\mathrm{P}_q^1(1,l))$, revealing pullback structures of these C^*-algebras allows us to view projections whose classes generate the even K-groups as Milnor idempotents. Thus we obtain the noncommutative vector bundles defined by these projections as given by a clutching of noncommutative vector bundles.

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We enter a new territory by approaching graph C^*-algebras as pullback algebras. More precisely, we present $U(1)$-C^*-algebras from a large class of trimmable graph C^*-algebras as $U(1)$-equivariant pullbacks, so as to determine a pullback structure of their fixed-point subalgebras. The following is the main theorem of the paper.

Theorem. *Let E be a $\bar{v}$-trimmable graph. Denote by E' the subgraph of E obtained by removing the unique outgoing edge (loop) of $\bar{v}$, and by E'' the subgraph of E' obtained by removing the vertex $\bar{v}$ and all edges ending in $\bar{v}$. Then, there exist $U(1)$-equivariant *-homomorphisms making the diagram*

$$\begin{array}{ccccc} & & C^*(E) & & \\ & \swarrow & & \searrow & \\ C^*(E'') & & & & C^*(E')\otimes C(S^1) \\ & \searrow & & \swarrow & \\ & & C^*(E'')\otimes C(S^1) & & \end{array}$$

a pullback diagram of $U(1)$-C^*-algebras. Here $C^*(E)$ and $C^*(E'')$ are considered as $U(1)$-C^*-algebras with respect to the gauge action, whereas the tensor product algebras are viewed as $U(1)$-C^*-algebras with respect to the standard $U(1)$-action on $C(S^1)$.

To focus attention and to follow motivating examples, we prove the above theorem in the setting of finite one-rank graphs and with only one loop being trimmed. We hope that this opens up a new venue of research with the setting enlarged to infinite higher-rank graphs with many loops to trim.

The induced pullback structure of the fixed-point subalgebra of a trimmable graph C^*-algebra yields a Mayer–Vietoris six-term exact sequence in K-theory allowing us to express the even K-group of this C^*-algebra in terms of the K-theory of simpler C^*-algebras.

Corollary. *There is the following Mayer–Vietoris six-term exact sequence in K-theory:*

$$\begin{array}{ccccc} K_0\big(C^*(E)^{U(1)}\big) & \longrightarrow & K_0\big(C^*(E'')^{U(1)}\big)\oplus K_0\big(C^*(E')\big) & \longrightarrow & K_0\big(C^*(E'')\big) \\ {\scriptstyle\partial_{10}}\big\uparrow & & & & \big\downarrow \\ K_1\big(C^*(E'')\big) & \longleftarrow & K_1\big(C^*(E')\big) & \longleftarrow & 0. \end{array}$$

The K-theory of graph C^*-algebras is very well developed by now [10, 14, 35–37]. However, our approach to the matter does bring its own benefits as illustrated in the following:

- The K-theory of fixed-point subalgebras of higher-rank graph C^*-algebras [24] can be quite difficult to study. It was shown in [17,21] how equivariant pullback structures of graph C^*-algebras provided by the above theorem can be used to reduce the K-theory computations for higher-rank graphs to far more doable K-theory calculations for usual graphs.

- Although the gauge-invariant subalgebra of a graph C^*-algebra is an approximately finite-dimensional (AF) algebra, and thus its K-groups are, in principle, computable [39], the above corollary makes this computation more effective by shifting it to simpler graphs.
- The Milnor connecting homomorphism ∂_{10} in the above corollary unravels generators of the even K-group of the gauge-invariant subalgebra of a graph C^*-algebra, so it might be useful even when the gauge-invariant algebra is again a graph C^*-algebra.

Furthermore, when the gauge-invariant subalgebra of a graph C^*-algebra is only Morita equivalent to a graph C^*-algebra [39, Theorem 1], it is tricky to describe it explicitly, so having it as a pullback C^*-algebra (see Corollary 2.5) remedies the problem whenever the gauge-invariant subalgebra of the trimmed graph C^*-algebra is known.

A one-loop extension (see Definition 2.2) of any finite graph E'' automatically becomes a trimmable graph E, and E' is simply a one-sink extension of E'' (see [37]). Thus we can "grow" trimmable graphs from any finite graph. In one case we take as E'' the standard graph encoding the Cuntz algebra $\mathcal{O}_2$, and in another case we take as E'' the standard graph encoding the Toeplitz algebra $\mathcal{T}$. Then, in both cases, using the above corollary, not only can we determine the even K-group of the gauge-invariant subalgebra $C^*(E)^{U(1)}$ (the odd K-group vanishes), but also see where its generators come from.

1.1. Introduction

Pushout diagrams in topology provide a systematic way of "gluing" of topological spaces, in particular in the category of compact Hausdorff spaces. The latter is dualized by the Gelfand transform to the category of commutative unital C^*-algebras with pushout diagrams of spaces translated into pullback diagrams of algebras.

Our motivating example came from the following $U(1)$-equivariant pushout diagram concerning spheres and the $2n$-ball:

$$\begin{array}{ccccc} & & S^{2n+1} & & \\ & \nearrow & & \nwarrow & \\ S^{2n-1} & & & & B^{2n}\times S^{1} \\ & \nwarrow & & \nearrow & \\ & & S^{2n-1}\times S^{1}. & & \end{array} \tag{1.1}$$

The equivariance of the above diagram allows it to descend to the quotient spaces:

$$\begin{array}{ccccc} & & \mathbb{C}\mathrm{P}^{n} & & \\ & \nearrow & & \nwarrow & \\ \mathbb{C}\mathrm{P}^{n-1} & & & & B^{2n} \\ & \nwarrow & & \nearrow & \\ & & S^{2n-1}. & & \end{array} \tag{1.2}$$

Thus we obtain a diagram that manifests the CW-complex structure of complex projective spaces $\mathbb{C}\mathrm{P}^n$.

For $n = 1$, a q-deformed version of the diagram (1.1) was considered in [25] and proved to be a $U(1)$-equivariant pullback diagram of C^*-algebras. This pullback structure provided therein an alternative way for computing an index pairing for noncommutative line bundles over the standard Podleś quantum sphere [22, 34]. For $n = 2$, an analogous result was called for in [21] in the context of the multipullback noncommutative deformation of complex projective spaces. This led to obtaining the general result for an arbitrary n in this paper: a $U(1)$-equivariant pullback structure of the C^*-algebra of the Vaksman–Soibelman quantum sphere S_q^{2n+1} is a prototype of our main theorem herein.

The theorem is based on a general concept of a trimmable graph [23]: a finite graph E is $\bar{v}$-trimmable iff it consists of a subgraph E'' emitting at least one edge to an external vertex $\bar{v}$ whose only outgoing edge $\bar{e}$ is a loop and such that all edges other than $\bar{e}$ that end in $\bar{v}$ begin in a vertex emitting an edge that ends not in $\bar{v}$. A trimmable graph E can be trimmed to its subgraph E''. There is an intermediate subgraph E' of E that is a one-sink extension of E''. Much as one defines a one-sink extension, we can define a one-loop extension, so that a one-loop extension of any finite graph is automatically a trimmable graph.

The paper is organized as follows. To make it self-contained, we start by recalling some results on graph C^*-algebras in the preliminaries. Then we proceed to Section 2 where we prove the main result and study general K-theoretical benefits of decomposing a trimmable graph C^*-algebra into simpler building blocks. The last two sections are devoted to four examples of two different types. The first two examples are in the spirit of abstract graph algebras (one-loop extensions of the Cuntz algebra $\mathcal{O}_2$ and the Toeplitz algebra $\mathcal{T}$), whereas the remaining two are our major applications. These are examples in the spirit of noncommutative topology (q-deformations of spheres, balls, complex projective spaces, lens spaces, and teardrops).

1.2. Preliminaries

In this section, we recall some general results from the theory of graph C^*-algebras. Our main references are [1, 4, 35]. We adopt the conventions of [4], i.e., the roles of source and range maps are exchanged with respect to [35].

Let $E := (E_0, E_1, s, r)$ be a directed graph, where E_0 is the set of vertices, E_1 is the set of edges, and $s : E_1 \to E_0$ and $r : E_1 \to E_0$ are the source and range map, respectively. A directed graph is called *row-finite* iff $s^{-1}(v) := \{e \in E_1 \mid s(e) = v\}$ is a finite set for every $v \in E_0$. It is called *finite* if both sets E_0 and E_1 are finite. A *sink* is a vertex v with no outgoing edges, that is with $s^{-1}(v) = \varnothing$. By a *path* e of length $|e| = k \geq 1$ we mean a directed path, i.e., a sequence of edges $e := e_1 \cdots e_k$, with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, k-1$. It is standard to view vertices as paths of length zero, and one commonly refers to any positive-length path that begins and ends at the same vertex as a *cycle* or a *loop*. However, throughout this paper a loop is always an edge that begins and ends at the same vertex. We denote the set of all paths in E by $\text{Path}(E)$. We extend the maps r and s to $\text{Path}(E)$ by setting $s_P(e) := s(e_1)$ and $r_P(e) := r(e_k)$ for all e of length $k \geq 1$, and $s_P(v) := v =: r_P(v)$ for all paths v of length zero.

Definition 1.1. The *graph C^*-algebra* $C^*(E)$ of a row-finite graph E is the universal C^*-algebra generated by mutually orthogonal projections $\{P_v \mid v \in E_0\} =: P$ and partial isometries $\{S_e \mid e \in E_1\} =: S$ satisfying the *Cuntz–Krieger relations*:

$$S_e^* S_e = P_{r(e)} \quad \text{for all } e \in E_1, \text{ and} \tag{CK1}$$

$$\sum_{e \in s^{-1}(v)} S_e S_e^* = P_v \quad \text{for all } v \in E_0 \text{ that are not sinks.} \tag{CK2}$$

The datum $\{S, P\}$ is called a *Cuntz–Krieger E-family*.

Any graph C^*-algebra $C^*(E)$ can be endowed with a natural circle action

$$\alpha : U(1) \to \operatorname{Aut}\big(C^*(E)\big), \tag{1.3}$$

called the *gauge action*. Using the universality of $C^*(E)$, it is defined by being set on the generators as follows:

$$\alpha_\lambda(P_v) = P_v, \quad \alpha_\lambda(S_e) = \lambda S_e, \quad \text{where } \lambda \in U(1),\ v \in E_0,\ e \in E_1. \tag{1.4}$$

The fixed-point subalgebra under the gauge action is an AF-subalgebra of the form (e.g., see [35, Corollary 3.3])

$$C^*(E)^{U(1)} = \overline{\operatorname{span}}\big\{S_x S_y^* \mid x, y \in \operatorname{Path}(E),\ r_P(x) = r_P(y),\ |x| = |y|\big\}. \tag{1.5}$$

Here for a path $x := x_1 x_2 \cdots x_n$ we set $S_x := S_{x_1} S_{x_2} \cdots S_{x_n}$, and for a path v of length 0 we put $S_v := P_v$.

The gauge action is a central ingredient in the gauge-invariant uniqueness theorem proved by an Huef and Raeburn [1, Theorem 2.1] in the context of Cuntz–Krieger algebras [16], and then generalized to graph C^*-algebras of row-finite graphs by Bates, Pask, Raeburn, and Szymański [4, Theorem 2.1]. This theorem, together with the universality of graph C^*-algebras with respect to the Cuntz–Krieger relations, is an essential tool in proving that a given C^*-algebra is isomorphic to a graph C^*-algebra. We give here a slight reformulation of the result, more suitable for the purposes of this work.

Theorem 1.2 (Gauge-invariant uniqueness theorem [1,4], [35, Theorem 2.2]). *Let E be a row-finite graph with the Cuntz–Krieger family $\{S, P\}$, let A be a C^*-algebra with a continuous action of $U(1)$, and let $\rho : C^*(E) \to A$ be a $U(1)$-equivariant *-homomorphism. If $\rho(P_v) \neq 0$ for all $v \in E_0$, then ρ is injective.*

To understand gauge-invariant ideals of graph C^*-algebras, we need to introduce two kinds of subsets of the set of vertices. Recall that given two vertices $v, w \in E_0$, whenever w is *reachable* from v, that is whenever there is a path from v to w, we write $v \geq w$. A subset H of E_0 is called *hereditary* iff $v \geq w$ and $v \in H$ imply $w \in H$. A subset H is *saturated* iff every vertex which feeds into H and only into H is again in H. We denote by $\overline{H}$ the *saturation* of a subset H, that is the smallest saturated subset containing H.

It follows from [4, Lemma 4.3] that, for any hereditary subset H, the (algebraic) ideal generated by $\{P_v \mid v \in H\}$ is of the form

$$I_E(H) = \operatorname{span}\{S_x S_y^* \mid x, y \in \mathrm{Path}(E), r_P(x) = r_P(y) \in \bar{H}\}. \tag{1.6}$$

Equation (1.6) will play an essential role in the proof of Theorem 2.4.

By [4, Theorem 4.1 (a)], given a row-finite graph E, the gauge-invariant closed ideals in the graph algebra $C^*(E)$ are in one-to-one correspondence with saturated hereditary subsets of E_0. By [4, Theorem 4.1 (b)], quotients by closed ideals generated by saturated hereditary subsets can be realised also as graph C^*-algebras by constructing a suitable subgraph. Given a saturated hereditary subset H of E_0, the subgraph E/H is the graph obtained by removing from E all the vertices in H and all the edges whose range is in H, i.e., $(E/H)_0 := E_0 \setminus H$ and $(E/H)_1 := \{e \in E_1 \mid r(e) \notin H\}$. As a consequence, we have a $U(1)$-equivariant isomorphism

$$C^*(E)/\overline{I_E(H)} \cong C^*(E/H), \tag{1.7}$$

where $\overline{I_E(H)}$ is the norm closure of $I_E(H)$.

2. Trimmable graph C^*-algebras

The following notion of a trimmable graph was introduced in [23, Definition 3.1] in the context of Leavitt path algebras.

Definition 2.1 ([23]). Let E be a finite graph with a distinguished vertex $\bar{v}$ emitting a loop $\bar{e}$. A graph E is called $\bar{v}$-*trimmable* iff the pair $(E, \bar{v})$ satisfies the following conditions:

$$s^{-1}(\bar{v}) = \{\bar{e}\}, \quad r^{-1}(\bar{v}) \setminus \{\bar{e}\} \neq \varnothing, \tag{T1}$$

$$\forall v \in s\big(r^{-1}(\bar{v}) \setminus \{\bar{e}\}\big): \quad s^{-1}(v) \setminus r^{-1}(\bar{v}) \neq \varnothing. \tag{T2}$$

We call $C^*(E)$ a $\bar{v}$-*trimmable graph* C^*-*algebra* iff E is $\bar{v}$-trimmable.

Note that conditions (T1) and (T2) imply that $\{\bar{v}\}$ is a saturated hereditary subset of E_0. Furthermore, if (T2) were not satisfied, i.e., for some $v \in s(r^{-1}(\bar{v}) \setminus \{\bar{e}\})$ the set difference $s^{-1}(v) \setminus r^{-1}(\bar{v})$ was empty, then the quotient map $C^*(E) \to C^*(E/\{\bar{v}\})$ would not be well defined as it would map all elements S_y, where $y \in r^{-1}(\bar{v})$, to zero, thus violating the Cuntz–Krieger relations for $C^*(E/\{\bar{v}\})$.

There is an ample supply of trimmable graphs because, given any finite graph E'', we can create a trimmable graph E by taking a *one-loop extension* of E''. We define one-loop extensions in the spirit of *one-sink extensions* defined in [37, Definition 1.1].

Definition 2.2. Let E'' be a finite graph. A finite graph $E := (E_0, E_1, s, r)$ is called a *one-loop extension* of E'' iff the following conditions are satisfied:

(1) E'' is a subgraph of E,

(2) $E_0 \setminus E''_0 = \{\bar{v}\}$ (there is only one vertex outside of E''),

(3) $s^{-1}(\bar{v}) = \{\bar{e}\}$ and $r(\bar{e}) = \bar{v}$ (the only edge outgoing from $\bar{v}$ is a loop),

(4) $r^{-1}(\bar{v}) \setminus \{\bar{e}\} \neq \varnothing$ (there is at least edge connecting E'' with $\bar{v}$),

(5) if v is a sink in E'', then it remains a sink in E (equivalent to the condition (T2)).

Note that, for any trimmable graph E, there is an intermediate graph E' that is a subgraph of E and a one-sink extension of E''.

2.1. A K_1-generator for trimmable graphs without sinks

Given a $\bar{v}$-trimmable graph E, the Cuntz–Krieger relations imply that the partial isometry associated to the loop $\bar{e}$ based at $\bar{v}$ is a normal operator. This fact can be used to construct a distinguished class in $K_1(C^*(E))$.

Proposition 2.3. *Let E be a $\bar{v}$-trimmable graph without sinks. Then the element*

$$U := S_{\bar{e}} + (1 - S_{\bar{e}} S_{\bar{e}}^*)$$

of $C^*(E)$ is unitary and generates a copy of $\mathbb{Z}$ as a direct summand in $K_1(C^*(E))$.

Proof. Let $e_i, i = 1, \dots, n-1$, be the edges of E different from $\bar{e}$ (recall that E is finite). By the trimmability conditions (T1) and (T2) in Definition 2.1, the edge matrix B_E for the graph E (e.g., see [35, p. 18]) is an $n \times n$ matrix of the form

$$B_E = \left(\begin{array}{c|ccccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline B(1,0) & & & & & \\ \vdots & & & B_{E'} & & \\ B(n-1,0) & & & & & \end{array}\right). \tag{2.1}$$

Here the edge $\bar{e}$ is listed first, and $B_{E'}$ is the edge matrix of the subgraph E'. The values of the entries $B(i,0)$ do not matter in our proof. It follows from (2.1) that the first column of the matrix $1 - B_E^t$ contains only zeros. Therefore, the vector $(1, 0, \dots, 0)$ generates a direct summand in $\ker(1 - B_E^t)$ isomorphic to $\mathbb{Z}$. Furthermore, since the graph E has no sinks, the C^*-algebra $C^*(E)$ is isomorphic to the Cuntz–Krieger algebra of the edge matrix B_E of the graph (again, see [35, p. 18]), so $\ker(1 - B_E^t) \cong K_1(C^*(E))$ by [14, Proposition 3.1].

Next, let us consider the partial isometry $S_{\bar{e}}$ associated to the loop $\bar{e}$ based at the distinguished vertex $\bar{v}$. Since $\bar{v}$ does not emit any other edge besides the loop $\bar{e}$, we deduce from the Cuntz–Krieger relations that $S_{\bar{e}}$ is a normal element. Then one readily checks that the element U is unitary. Finally, the argument outlined in [38, Section 2] allows us to conclude that U generates a copy of $\mathbb{Z}$ as a direct summand in $K_1(C^*(E))$ corresponding to the vector $(1, 0, \dots, 0)$. ■

2.2. An equivariant pullback structure

In this section, we prove that every $\bar{v}$-trimmable graph C^*-algebra $C^*(E)$ is $U(1)$-equivariantly isomorphic to the pullback C^*-algebra of $C^*(E') \otimes C(S^1)$ and $C^*(E'')$ over $C^*(E'') \otimes C(S^1)$, where E' is the subgraph of E obtained by removing the loop $\bar{e}$ and $E'' := E/\{\bar{v}\}$. For this statement, we need to introduce $U(1)$-equivariant *-homomorphisms which give the aforementioned pullback structure.

We begin with the map that dualizes the gauge action α, namely the gauge coaction

$$\delta : C^*(E'') \to C^*(E'') \otimes C(S^1), \tag{2.2}$$

given on generators by

$$\delta(S_e) = S_e \otimes u, \quad \delta(P_v) = P_v \otimes 1, \quad \text{for all } e \in E''_1 \text{ and } v \in E''_0. \tag{2.3}$$

Here we denote by u the standard unitary generator of $C(S^1)$ and by $\{S, P\}$ the Cuntz–Krieger E''-family. The gauge coaction is $U(1)$-equivariant with respect to the gauge action on $C^*(E'')$ and the action on $C^*(E'') \otimes C(S^1)$ given by the standard action on the rightmost tensorand.

The singleton set $\{\bar{v}\}$ is a saturated hereditary subset of both E_0 and E'_0 (note that $E/\{\bar{v}\} = E'/\{\bar{v}\}$) and, by the one-to-one correspondence of gauge-invariant ideals and saturated hereditary subsets, the quotient maps

$$\pi_1 : C^*(E) \to C^*(E'') \quad \text{and} \quad \pi_2 : C^*(E') \to C^*(E'') \tag{2.4}$$

are $U(1)$-equivariant with respect to the gauge actions on $C^*(E)$, $C^*(E')$, and $C^*(E'')$. In the forthcoming theorem, we will consider the *-homomorphism

$$\pi_2 \otimes \mathrm{id} : C^*(E') \otimes C(S^1) \to C^*(E'') \otimes C(S^1) \tag{2.5}$$

viewed as a $U(1)$-equivariant map with respect to the standard $U(1)$-action on $C(S^1)$.

Finally, using the condition (T1), one readily verifies that the assignment

$$f(P_v) = P_v \otimes 1,\ v \in E_0, \quad f(S_e) = \begin{cases} P_{\bar{v}} \otimes u & \text{if } e = \bar{e}, \\ S_e \otimes u & \text{if } e \in E'_1, \end{cases} \tag{2.6}$$

defines a map

$$f : C^*(E) \to C^*(E') \otimes C(S^1), \tag{2.7}$$

as it preserves the relations (CK1)–(CK2) of the graph C^*-algebra $C^*(E)$. It is equally straightforward to check that f is equivariant with respect to the gauge action on $C^*(E)$ and the action on $C^*(E') \otimes C(S^1)$ given by the standard action on the rightmost tensorand.

Theorem 2.4. *Let E be a $\bar{v}$-trimmable graph, E' the subgraph of E obtained by removing the unique outgoing loop $\bar{e}$, and E'' the subgraph of E' obtained by removing the vertex $\bar{v}$ and all edges ending in $\bar{v}$. Then the following diagram of the above-defined $U(1)$-equivariant *-homomorphisms*

$$\begin{array}{ccccc} & & C^*(E) & & \\ & \swarrow^{\pi_1} & & \searrow^{f} & \\ C^*(E'') & & & & C^*(E') \otimes C(S^1) \\ & \searrow_{\delta} & & \swarrow_{\pi_2 \otimes \mathrm{id}} & \\ & & C^*(E'') \otimes C(S^1) & & \end{array} \tag{2.8}$$

is a pullback diagram of $U(1)$-C^*-algebras.

Proof. The above diagram is clearly commutative, so there is a *-homomorphism F mapping $C^*(E)$ into the pullback C^*-algebra. The injectivity of F follows from the injectivity of f, which is a consequence of the gauge-invariant uniqueness theorem (Theorem 1.2). Furthermore, since π_1 and $\pi_2 \otimes \mathrm{id}$ are surjective, using a C^*-algebraic incarnation [33, Proposition 3.1] of a well-known characterization of when a commutative diagram is a pullback diagram (e.g., see [23, Lemma 4.1]) to prove the surjectivity of F, it only remains to check whether

$$\ker(\pi_2 \otimes \mathrm{id}) \subseteq f\big(\ker(\pi_1)\big). \tag{2.9}$$

Again, we observe that $\{\bar{v}\}$ is a saturated hereditary subset of both E_0 and E'_0. Therefore, it generates gauge-invariant ideals $\overline{I_E(\bar{v})}$ and $\overline{I_{E'}(\bar{v})}$ in $C^*(E)$ and $C^*(E')$, respectively. It follows from (1.7) that

$$\ker(\pi_1) = \overline{I_E(\bar{v})} \quad \text{and} \quad \ker(\pi_2) = \overline{I_{E'}(\bar{v})}, \tag{2.10}$$

so, by (1.6), every element in $I_{E'}(\bar{v}) \otimes \mathbb{C}[u, u^{-1}]$ is a linear combination of elements of the form

$$\Big(\sum_{i=1}^{n} k_i S_{x_i} S_{y_i}^*\Big) \otimes u^m, \quad k_i \in \mathbb{C},\ x_i, y_i \in \mathrm{Path}(E'),\ r_P(x_i) = r_P(y_i) = \bar{v},\ m \in \mathbb{Z}.$$

Given an element as above, we can show that it belongs to $f(I_E(\bar{v}))$ using the following equality:

$$\sum_{i=1}^{n} f\big(k_i S_{x_i} S_{\bar{e}}^{m-(|x_i|-|y_i|)} S_{y_i}^*\big) = \Big(\sum_{i=1}^{n} k_i S_{x_i} S_{y_i}^*\Big) \otimes u^m. \tag{2.11}$$

Hence $I_{E'}(\bar{v}) \otimes \mathbb{C}[u, u^{-1}] \subseteq f(I_E(\bar{v}))$. Finally, to prove (2.9), we compute

$$\ker(\pi_2 \otimes \mathrm{id}) = \ker(\pi_2) \otimes C(S^1) = \overline{I_{E'}(\bar{v}) \otimes \mathbb{C}[u, u^{-1}]} \subseteq \overline{f\big(I_E(\bar{v})\big)} = f\big(\ker(\pi_1)\big).$$

Here the last equality follows from the fact that the image of a C^*-algebra under any *-homomorphism is closed. ∎

As a corollary, by the equivariance of all maps in diagram (2.8) and the compactness of $U(1)$, we obtain a pullback diagram at the level of fixed-point subalgebras illustrated in the following corollary.

Corollary 2.5. *The diagram of *-homomorphisms*

$$\begin{array}{ccccc} & & C^*(E)^{U(1)} & & \\ & \swarrow^{\widetilde{\pi_1}} & & \searrow^{\widetilde{f}} & \\ C^*(E'')^{U(1)} & & & & C^*(E') \\ & \searrow_{\iota} & & \swarrow_{\pi_2} & \\ & & C^*(E'') & & \end{array} \tag{2.12}$$

is a pullback diagram of C^*-algebras. Here $\widetilde{\pi_1}$ and $\widetilde{f}$ denote *-homomorphisms that are restrictions of π_1 and f, respectively, and ι is the subalgebra inclusion.

2.3. Mayer–Vietoris exact sequences in K-theory

Any one-surjective pullback diagram of unital C^*-algebras induces a six-term exact sequence in K-theory that goes under the name of the Mayer–Vietoris exact sequence (see for example [5, Section 1.3], [6, Section 1.2.3], and [7, Theorem 21.5.1]). In this subsection, we describe the Mayer–Vietoris exact sequence for trimmable graph C^*-algebras and their gauge-invariant subalgebras.

Let E be a $\bar{v}$-trimmable graph. The Mayer–Vietoris six-term exact sequence associated to the diagram (2.8) reads

$$\begin{array}{ccccc}
K_0(C^*(E)) & \xrightarrow{(\pi_{1*},f_*)} & K_0(C^*(E''))\oplus K_0(C^*(E')\otimes C(S^1)) & \xrightarrow{(\delta_*,-(\pi_2\otimes\mathrm{id})_*)} & K_0(C^*(E'')\otimes C(S^1)) \\
\partial_{10}\Big\uparrow & & & & \Big\downarrow\partial_{01} \\
K_1(C^*(E'')\otimes C(S^1)) & \xleftarrow[(\delta_*,-(\pi_2\otimes\mathrm{id})_*)]{} & K_1(C^*(E''))\oplus K_1(C^*(E')\otimes C(S^1)) & \xleftarrow[(\pi_{1*},f_*)]{} & K_1(C^*(E)).
\end{array} \tag{2.13}$$

Here ∂_{10} is a Milnor connecting homomorphism and ∂_{01} is a Bott connecting homomorphism. Furthermore, the pullback diagram (2.12) of fixed-point subalgebras leads to another six-term exact sequence in K-theory:

$$\begin{array}{ccccc}
K_0\big(C^*(E)^{U(1)}\big) & \xrightarrow{(\widetilde{\pi}_{1*},\widetilde{f}_*)} & K_0\big(C^*(E'')^{U(1)}\big)\oplus K_0\big(C^*(E')\big) & \xrightarrow{(\iota_*,-\pi_{2*})} & K_0\big(C^*(E'')\big) \\
\partial_{10}\Big\uparrow & & & & \Big\downarrow\partial_{01} \\
K_1\big(C^*(E'')\big) & \xleftarrow[(\iota_*,-\pi_{2*})]{} & K_1\big(C^*(E'')^{U(1)}\big)\oplus K_1\big(C^*(E')\big) & \xleftarrow[(\widetilde{\pi}_{1*},\widetilde{f}_*)]{} & K_1\big(C^*(E)^{U(1)}\big).
\end{array} \tag{2.14}$$

Next, since gauge-invariant subalgebras of graph C^*-algebras are always AF-algebras, their odd-K-groups vanish, so we obtain a simpler six-term exact sequence:

$$\begin{array}{ccccc}
K_0\big(C^*(E)^{U(1)}\big) & \xrightarrow{(\widetilde{\pi}_{1*},\widetilde{f}_*)} & K_0\big(C^*(E'')^{U(1)}\big)\oplus K_0\big(C^*(E')\big) & \xrightarrow{(\iota_*,-\pi_{2*})} & K_0\big(C^*(E'')\big) \\
\partial_{10}\Big\uparrow & & & & \Big\downarrow \\
K_1\big(C^*(E'')\big) & \xleftarrow[(\iota_*,-\pi_{2*})]{} & K_1\big(C^*(E')\big) & \longleftarrow & 0.
\end{array} \tag{2.15}$$

3. Examples

3.1. A one-loop extension of the Cuntz algebra $\mathcal{O}_2$

Recall that the Cuntz algebra $\mathcal{O}_2$ can be viewed as the graph C^*-algebra of the graph Λ'' consisting of two loops starting at the unique vertex w. Let us now consider the one-loop extension Λ of this graph obtained by adding one outgoing edge from w to $\bar{v}$ and one loop at $\bar{v}$ (see Figure 1).

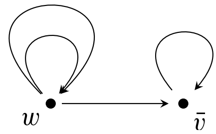

Figure 1. The graph Λ.

Denote by Λ' the one-sink extension of Λ'' obtained by adding one outgoing edge from w to $\bar{v}$. By Theorem 2.4, we have the following $U(1)$-equivariant pullback structure:

$$\begin{array}{ccc} & C^*(\Lambda) & \\ \swarrow & & \searrow \\ \mathcal{O}_2 & & C^*(\Lambda') \otimes C(S^1). \\ \searrow & & \swarrow \\ & \mathcal{O}_2 \otimes C(S^1) & \end{array} \tag{3.1}$$

The fixed-point subalgebra $\mathcal{O}_2^{U(1)}$ is isomorphic to the CAR algebra $M_{2^\infty}(\mathbb{C})$ (see [12, §1.5]). Hence, by Corollary 2.5, we have another pullback diagram:

$$\begin{array}{ccc} & C^*(\Lambda)^{U(1)} & \\ \swarrow & & \searrow \\ M_{2^\infty}(\mathbb{C}) & & C^*(\Lambda'). \\ \searrow & & \swarrow \\ & \mathcal{O}_2 & \end{array} \tag{3.2}$$

The exact sequence (2.15) for this example reads

$$\begin{array}{ccccc} K_0\big(C^*(\Lambda)^{U(1)}\big) & \longrightarrow & K_0\big(M_{2^\infty}(\mathbb{C})\big) \oplus K_0\big(C^*(\Lambda')\big) & \longrightarrow & K_0(\mathcal{O}_2) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_2) & \longleftarrow & K_1\big(C^*(\Lambda')\big) & \longleftarrow & 0. \end{array} \tag{3.3}$$

Now, to compute $K_0(C^*(\Lambda)^{U(1)})$, we observe that

(1) $K_0(\mathcal{O}_2) = 0 = K_1(\mathcal{O}_2)$ (see [13], [15, Theorems 3.7 and 3.8]),

(2) $K_0(M_{2^\infty}(\mathbb{C})) \cong \mathbb{Z}(\frac{1}{2})$, where $\mathbb{Z}(\frac{1}{2})$ is the group of dyadic rationals (e.g., see [19, Example IV.3.4]), and

(3) $K_0(C^*(\Lambda')) = \mathbb{Z}[P_{\bar{v}}]$ and $K_1(C^*(\Lambda')) = 0$, where $P_{\bar{v}}$ is the vertex projection of $\bar{v}$.

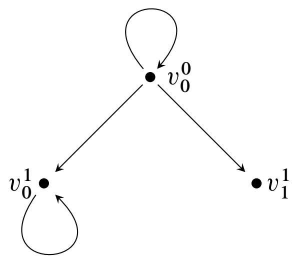

Figure 2. The graph Q_2.

By the general theory of graph C^*-algebras, here the last statement follows from the facts that Λ' is a one-sink extension of Λ'' and $K_i(C^*(\Lambda'')) = 0$, $i = 0, 1$. Hence, from the diagram (3.3), we conclude the following proposition.

Proposition 3.1. *Let $C^*(\Lambda)$ be the graph C^*-algebra of the graph given by Figure 1. The gauge-invariant subalgebra $C^*(\Lambda)^{U(1)}$ has the following even K-group:*

$$K_0\big(C^*(\Lambda)^{U(1)}\big) \cong K_0\big(M_{2^\infty}(\mathbb{C})\big) \oplus K_0\big(C^*(\Lambda')\big) \cong \mathbb{Z}\left(\frac{1}{2}\right) \oplus \mathbb{Z}[P_{\bar{v}}].$$

Since Λ is finite and without sinks, the above result must agree with [32, Proposition 4.1.2], where the K-theory of gauge-invariant subalgebras of arbitrary Cuntz–Krieger algebras was computed.

3.2. A one-loop extension of the Toeplitz algebra $\mathcal{T}$

Let us now consider an example that goes beyond [32, Proposition 4.1.2] and connects with Section 4.2.2. To this end, we take the standard graph Q_2'' encoding the Toeplitz algebra $\mathcal{T}$, that is the graph consisting of two vertices v_0^0 and v_1^1, one loop e_0^0 based at v_0^0, and one edge e_1^{01} from v_0^0 to v_1^1. Note that v_1^1 is a sink, so $\mathcal{T}$ is not a Cuntz–Krieger algebra. Next, we consider the one-loop extension Q_2 of this graph obtained by adding a vertex v_0^1, an edge from v_0^0 to v_0^1, and a loop at v_0^1 (see Figure 2).

Denote by Q_2' the one-sink extension of Q_2'' obtained by removing from Q_2 the loop at v_0^1. The C^*-algebra of the graph Q_2' is isomorphic to the C^*-algebra of the equatorial Podleś quantum sphere $S^2_{q\infty}$ (see [27, 34]). By Theorem 2.4, we have the following $U(1)$-equivariant pullback structure:

$$\begin{array}{ccccc} & & C^*(Q_2) & & \\ & \swarrow & & \searrow & \\ \mathcal{T} & & & & C(S^2_{q\infty}) \otimes C(S^1). \\ & \searrow & & \swarrow & \\ & & \mathcal{T} \otimes C(S^1) & & \end{array} \tag{3.4}$$

To compute the fixed-point subalgebra $\mathcal{T}^{U(1)}$, we take advantage of (1.5) and combine it with the fact that $\mathcal{T}$ is the unital universal C^*-algebra generated by a single isometry s [11]. Indeed, identifying the isometry s with the sum of partial isometries $S_{e_0^0} + S_{e_1^{01}}$, one easily computes that $\mathcal{T}^{U(1)} = \overline{\operatorname{span}}\{s^k(s^*)^k \mid k \in \mathbb{N}\}$, where $s^0(s^*)^0 = 1$. Hence, $\mathcal{T}^{U(1)}$ is a unital commutative AF-algebra generated by 1 and countably infinitely many orthogonal projections $\{p_k\}_{k\in\mathbb{N}}$, where $p_k := s^k(1-ss^*)(s^*)^k$, so it is isomorphic to the C^*-algebra of continuous complex-valued functions on the one-point compactification of natural numbers $\mathbb{N}^*$. Thus, by Corollary 2.5, we have another pullback diagram:

$$\begin{array}{ccccc} & & C^*(Q_2)^{U(1)} & & \\ & \swarrow & & \searrow & \\ C(\mathbb{N}^*) & & & & C(S^2_{q\infty}). \\ & \searrow & & \swarrow & \\ & & \mathcal{T} & & \end{array} \tag{3.5}$$

The exact sequence (2.15) for this example reads

$$\begin{array}{ccccc} K_0\big(C^*(Q_2)^{U(1)}\big) & \longrightarrow & K_0\big(C(\mathbb{N}^*)\big) \oplus K_0\big(C(S^2_{q\infty})\big) & \longrightarrow & K_0(\mathcal{T}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}) & \longleftarrow & K_1\big(C(S^2_{q\infty})\big) & \longleftarrow & 0. \end{array} \tag{3.6}$$

The K-groups of all the algebras involved here, except for $C^*(Q_2)^{U(1)}$, are as follows:

(1) $K_0(\mathcal{T}) = \mathbb{Z}[1]$ and $K_1(\mathcal{T}) = 0$,

(2) $K_0(C(\mathbb{N}^*)) = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}[p_k] \oplus \mathbb{Z}[1]$, and

(3) $K_0(C(S^2_{q\infty})) = \mathbb{Z}[\beta] \oplus \mathbb{Z}[1]$ (where β is the Bott generator) and $K_1(C(S^2_{q\infty})) = 0$ (see [31, Section 4]).

Hence, there is a short exact sequence

$$0 \to K_0\big(C^*(Q_2)^{U(1)}\big) \to \bigoplus_{k\in\mathbb{N}} \mathbb{Z}[p_k] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[\beta] \oplus \mathbb{Z}[1] \to \mathbb{Z}[1] \to 0, \tag{3.7}$$

so $K_0(C^*(Q_2)^{U(1)})$ is a countably generated subgroup of a free abelian group. Furthermore, one can easily check that both $[p_k]$, $k \in \mathbb{N}$, and $[\beta]$ are mapped to 0 in (3.7). Consequently, we conclude the following proposition.

Proposition 3.2. *Let $C^*(Q_2)$ be the graph C^*-algebra of the graph given by Figure 2. The gauge-invariant subalgebra $C^*(Q_2)^{U(1)}$ has the following even K-group:*

$$K_0\big(C^*(Q_2)^{U(1)}\big) \cong \bigoplus_{k\in\mathbb{N}} \mathbb{Z}\big[(p_k, 0)\big] \oplus \mathbb{Z}\big[(0, \beta)\big] \oplus \mathbb{Z}\big[(1,1)\big] \subseteq K_0\big(C(\mathbb{N}^*) \oplus C(S^2_{q\infty})\big).$$

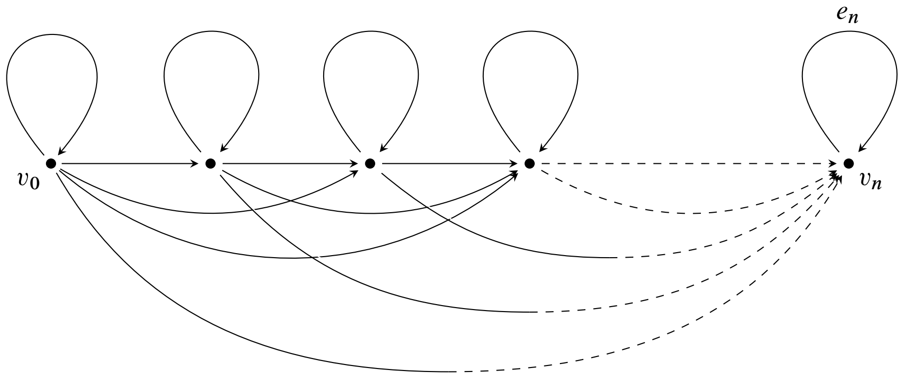

Figure 3. The graph L_{2n+1} of $C(S_q^{2n+1})$.

4. Applications

4.1. The Vaksman–Soibelman quantum spheres and projective spaces

In 1990, Vaksman and Soibelman [40] defined a class of odd-dimensional quantum spheres S_q^{2n+1} for a non-negative integer n. Their C^*-algebras can be viewed as q-deformations of the C^*-algebras of continuous functions on odd-dimensional spheres S^{2n+1}, where $q \in [0, 1]$ is a deformation parameter. About a decade later, Hong and Szymański [27] showed that, in any dimension and for any $q \in [0, 1)$, these spheres can be realised as graph C^*-algebras. In the same paper, they define even-dimensional noncommutative balls $C(B_q^{2n})$ using noncommutative double suspension, and in [30] they give their graph-algebraic presentations.

4.1.1. Spheres. The C^*-algebra $C(S_q^{2n+1})$ of the $(2n + 1)$-dimensional quantum sphere is isomorphic, for any $q \in [0, 1)$, to the graph C^*-algebra of the graph L_{2n+1} [27, Theorem 4.4] (see Figure 3) with

- $n + 1$ vertices $\{v_0, v_1, \dots, v_n\}$,
- one edge $e_{i,j}$ from v_i to v_j for all $0 \leq i < j \leq n$,
- one loop e_i over each vertex v_i for all $0 \leq i \leq n$.

To simplify notation for the Cuntz–Krieger L_{2n+1}-family, we set

$$S_{i,j} := S_{e_{i,j}}, \quad S_k := S_{e_k}, \quad P_k := P_{v_k}, \text{ where } 0 \leq i < j \leq n,\ 0 \leq k \leq n. \tag{4.1}$$

Furthermore, the C^*-algebra $C(B_q^{2n})$ (see Figure 4) of the Hong–Szymański $2n$-dimensional quantum ball [29, 30] can be viewed as the graph C^*-algebra of the graph Γ_{2n} obtained from L_{2n+1} by removing the loop e_n.

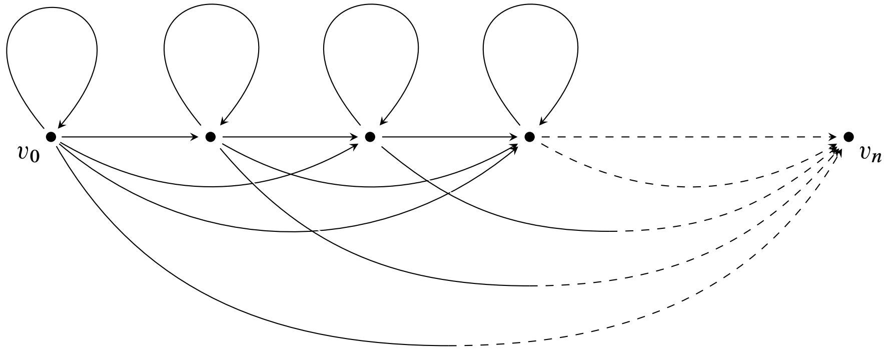

Figure 4. The graph Γ_{2n} of $C(B_q^{2n})$.

Since the graph L_{2n+1} giving the Vaksman–Soibelman $(2n+1)$-sphere is v_n-trimmable, we immediately conclude from Theorem 2.4 that the diagram

$$\begin{array}{ccccc} & & C(S_q^{2n+1}) & & \\ & \swarrow^{\pi_1^n} & & \searrow^{f^n} & \\ C(S_q^{2n-1}) & & & & C(B_q^{2n}) \otimes C(S^1) \\ & \searrow_{\delta^n} & & \swarrow_{\pi_2^n \otimes \mathrm{id}} & \\ & & C(S_q^{2n-1}) \otimes C(S^1) & & \end{array} \tag{4.2}$$

is a pullback diagram of $U(1)$-C^*-algebras. All the maps in the above diagram are special cases of the maps in the diagram (2.8). In this example, the six-term exact sequence (2.13) reads

$$\begin{array}{ccccc} K_0(C(S_q^{2n+1})) & \xrightarrow{({\pi_1^n}_*, {f^n}_*)} & K_0(C(S_q^{2n-1})) \oplus K_0(C(B_q^{2n}) \otimes C(S^1)) & \xrightarrow{(\delta_*^n, -(\pi_2^n \otimes \mathrm{id})_*)} & K_0(C(S_q^{2n-1}) \otimes C(S^1)) \\ \uparrow{\scriptstyle \partial_{10}} & & & & \downarrow{\scriptstyle \partial_{01}} \\ K_1(C(S_q^{2n-1}) \otimes C(S^1)) & \xleftarrow[(\delta_*^n, -(\pi_2^n \otimes \mathrm{id})_*)]{} & K_1(C(S_q^{2n-1})) \oplus K_1(C(B_q^{2n}) \otimes C(S^1)) & \xleftarrow[({\pi_1^n}_*, {f^n}_*)]{} & K_1(C(S_q^{2n+1})). \end{array}$$

Using Proposition 2.3, one obtains explicit formulas for the K_1-generators of the quantum odd spheres, which can be compared with results in [26, Section 4.3]. Then one can use the above exact sequence to unravel inductive relations between these generators.

4.1.2. Projective spaces. The C^*-algebra $C(\mathbb{C}\mathrm{P}_q^n)$ of the Vaksman–Soibelman quantum complex projective space $\mathbb{C}\mathrm{P}_q^n$ (see [40]) is defined as the fixed-point subalgebra of

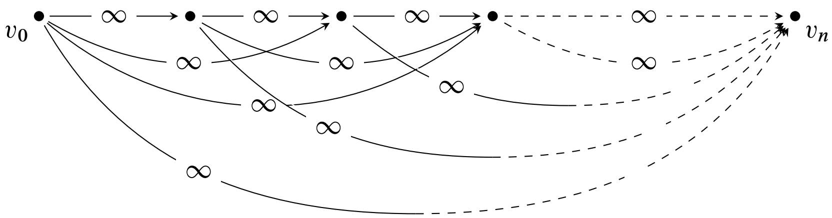

Figure 5. The graph M_n of $C(\mathbb{C}\mathrm{P}^n_q)$.

$C(S^{2n+1}_q)$ under the gauge action of $U(1)$. It can be viewed as the graph C^*-algebra of the graph M_n consisting of the same vertices $v_0, \dots, v_n$ as in the graph L_{2n+1} (vertex projections are gauge invariant), with no loops, and countably infinitely many edges from v_i to v_j for any $i < j$ (see Figure 5).

From the equivariance of the diagram (4.2), we conclude the following proposition.

Proposition 4.1. *The C^*-algebra of the Vaksman–Soibelman quantum complex projective space $C(\mathbb{C}\mathrm{P}^n_q)$ has the following pullback structure:*

$$\begin{array}{ccccc} & & C(\mathbb{C}\mathrm{P}^n_q) & & \\ & \overset{\widetilde{\pi^n_1}}{\swarrow} & & \overset{\widetilde{f^n}}{\searrow} & \\ C(\mathbb{C}\mathrm{P}^{n-1}_q) & & & & C(B^{2n}_q), \\ & \underset{\iota^n}{\searrow} & & \underset{\pi^n_2}{\swarrow} & \\ & & C(S^{2n-1}_q) & & \end{array} \tag{4.3}$$

where all the maps above are special cases of the maps in the diagram (2.12). Furthermore, we obtain

$$K_0\big(C(\mathbb{C}\mathrm{P}^n_q)\big) \cong K_0\big(C(\mathbb{C}\mathrm{P}^{n-1}_q)\big) \oplus \partial_{10}\big(K_1\big(C(S^{2n-1}_q)\big)\big), \tag{4.4}$$

where ∂_{10} is Milnor's connecting homomorphism.

Proof. The first part of the statement follows from Corollary 2.5. Recall from [40] and [29, 30] that, for all $n \geq 1$, we have that $K_0(C(S^{2n-1}_q)) = \mathbb{Z}[1]$, $K_1(C(S^{2n-1}_q)) \cong \mathbb{Z}$, and that $K_0(C(B^{2n}_q)) = \mathbb{Z}[1]$ and $K_1(C(B^{2n}_q)) = 0$. Note also that there is a short exact sequence (e.g., see [27])

$$0 \longrightarrow \overline{I(v_n)} \cong \mathcal{K} \longrightarrow C(\mathbb{C}\mathrm{P}^n_q) \xrightarrow{\widetilde{\pi^n_1}} C(\mathbb{C}\mathrm{P}^{n-1}_q) \longrightarrow 0, \tag{4.5}$$

where $\mathcal{K}$ is the C^*-algebra of compact operators. Now, the associated six-term exact sequence and the vanishing of $K_1(\mathcal{K})$ imply that

$$K_0\big(C(\mathbb{C}\mathrm{P}^n_q)\big) \cong \mathbb{Z}^{n+1} \quad \text{and} \quad K_1\big(C(\mathbb{C}\mathrm{P}^n_q)\big) = 0, \tag{4.6}$$

and that the map $\widetilde{\pi^n_1}_* : K_0(C(\mathbb{C}\mathrm{P}^n_q)) \to K_0(C(\mathbb{C}\mathrm{P}^{n-1}_q))$ is surjective.

Let us consider the Mayer–Vietoris six-term exact sequence induced by the pullback diagram (4.3):

$$\begin{array}{ccccc} K_0\big(C(\mathbb{C}\mathrm{P}^n_q)\big) & \xrightarrow{(\widetilde{\pi^n_1}_*, \widetilde{f^n}_*)} & K_0\big(C(\mathbb{C}\mathrm{P}^{n-1}_q)\big) \oplus K_0\big(C(B^{2n}_q)\big) & \longrightarrow & K_0\big(C(S^{2n-1}_q)\big) \\ {\scriptstyle \partial_{10}}\uparrow & & & & \downarrow \\ K_1\big(C(S^{2n-1}_q)\big) & \longleftarrow & K_1\big(C(\mathbb{C}\mathrm{P}^{n-1}_q)\big) \oplus K_1\big(C(B^{2n}_q)\big) & \longleftarrow & K_1\big(C(\mathbb{C}\mathrm{P}^n_q)\big). \end{array} \tag{4.7}$$

We are going to prove (4.4) by extracting from (4.7) the following split short exact sequence:

$$0 \longrightarrow K_1\big(C(S^{2n-1}_q)\big) \xrightarrow{\partial_{10}} K_0\big(C(\mathbb{C}\mathrm{P}^n_q)\big) \xrightarrow{\widetilde{\pi^n_1}_*} K_0\big(C(\mathbb{C}\mathrm{P}^{n-1}_q)\big) \longrightarrow 0. \tag{4.8}$$

We already know that $\widetilde{\pi^n_1}_*$ is surjective, so to prove the exactness of (4.8), it suffices to show that the kernel of $(\widetilde{\pi^n_1}_*, \widetilde{f^n}_*)$ is the same as the kernel of $\widetilde{\pi^n_1}_*$. To this end, since

$$\ker(\widetilde{\pi^n_1}_*, \widetilde{f^n}_*) = \ker \widetilde{\pi^n_1}_* \cap \ker \widetilde{f^n}_*, \tag{4.9}$$

we want to show the inclusion $\ker \widetilde{\pi^n_1}_* \subseteq \ker \widetilde{f^n}_*$. It follows from the pullback diagram (4.3) and the functoriality of K-theory that

$$\ker \widetilde{\pi^n_1}_* \subseteq \ker(\iota^n_* \circ \widetilde{\pi^n_1}_*) = \ker\big((\pi^n_2)_* \circ \widetilde{f^n_*}\big) = \ker(\widetilde{f^n}_*). \tag{4.10}$$

Here the last equality holds because $(\pi^n_2)_*$ is an isomorphism. Finally, the exact sequence (4.8) splits by the freeness of the $\mathbb{Z}$-module $K_0(C(\mathbb{C}\mathrm{P}^{n-1}_q))$. ∎

4.1.3. Milnor's clutching construction for generators of $K_0(C(\mathbb{C}\mathrm{P}^n_q))$. Throughout this section, we keep the notation of (4.1). First, recall that there are $(n+1)$-many projections $P_0, P_1, \ldots, P_n \in C^*(M_n) \cong C(\mathbb{C}\mathrm{P}^n_q)$. Therefore, since $K_0(C(\mathbb{C}\mathrm{P}^n_q)) \cong \mathbb{Z}^{n+1}$ and the K_0-group of a graph C^*-algebra is generated by its vertex projections (see [10, Proposition 3.8 (1)]), we infer that

$$K_0\big(C(\mathbb{C}\mathrm{P}^n_q)\big) = \mathbb{Z}[P_0] \oplus \mathbb{Z}[P_1] \oplus \cdots \oplus \mathbb{Z}[P_n]. \tag{4.11}$$

We will now compute the explicit value of Milnor's connecting homomorphism ∂_{10} on a generator of $K_1(C(S^{2n-1}_q))$. Let us first recall that, by Proposition 2.3, the generator of $K_1(C(S^{2n-1}_q)) \cong \mathbb{Z}$ is given by the K_1-class of the unitary

$$U = S_{n-1} + (1 - P_{n-1}). \tag{4.12}$$

Next, following [20, Section 2.1], we use the pullback structure of $C(\mathbb{C}\mathrm{P}^n_q)$ to find $C, D \in C(B^{2n}_q)$ such that $\pi^n_2(C) = U$ and $\pi^n_2(D) = U^*$, i.e.

$$C = S_{n-1} + S_{n-1,n} + (1 - P_{n-1} - P_n), \tag{4.13}$$

$$D = C^* = S^*_{n-1} + S^*_{n-1,n} + (1 - P_{n-1} - P_n), \tag{4.14}$$

and compute the following 2 by 2 matrix with entries in $C(\mathbb{C}\mathrm{P}^n_q)$:

$$p_U = \begin{bmatrix} \big(1, C(2-DC)D\big) & \big(0, C(2-DC)(1-DC)\big) \\ \big(0, (1-DC)D\big) & \big(0, (1-DC)^2\big) \end{bmatrix} = \begin{bmatrix} (1, 1-P_n) & (0,0) \\ (0,0) & (0,0) \end{bmatrix}. \tag{4.15}$$

By [20, Theorem 2.2], the element $\partial_{10}([U]) = [p_U] - [1]$ is a generator of $K_0(C(\mathbb{C}\mathrm{P}^n_q))$. Observe that in the above formula for p_U we can remove all the entries except the top left one without changing its class in $K_0(C(\mathbb{C}\mathrm{P}^{n-1}_q))$, namely $[1] - [p_U] = [1] - [p]$, where $p := (1, 1 - P_n) = (1, 1) - (0, P_n)$ is a projection in $C(\mathbb{C}\mathrm{P}^n_q)$. Furthermore, since p and $1 - p$ are orthogonal, we have $[p] + [1 - p] = [1]$, so

$$-\partial_{10}\big([U]\big) = [1] - [p] = [1 - p] = \big[(0, P_n)\big] = [P_n]. \tag{4.16}$$

Here the rightmost projection P_n is viewed as the vertex projection corresponding to the vertex v_n in the graph of $C(\mathbb{C}\mathrm{P}^n_q)$, and the rightmost equality follows from the fact that the isomorphism from $C(\mathbb{C}\mathrm{P}^n_q)$ to the pullback C^*-algebra of the diagram (4.3) maps P_n to $(0, P_n)$.

Finally, let us observe that Milnor's idempotent in the above calculation is

$$p_U \cong 1 - P_n = P_0 + P_1 + \cdots + P_{n-1}. \tag{4.17}$$

Hence the projective module it defines can be understood as the section module of a noncommutative vector bundle obtained by the Milnor clutching construction.

4.2. Quantum lens spaces and quantum teardrops

Quantum lens spaces, both weighted and unweighted, have been the subject of increasing interest in the last years. Their realisation as graph C^*-algebras has been first proven in [28], and then further generalised in [9] under less stringent assumptions. In the rest of the paper, we focus on the three-dimensional quantum lens spaces $L^3_q(l; 1, l)$. In [18], generators for the K-theory and K-homology of multi-dimensional quantum weighted projective spaces were constructed, leading to an extension of the K-theoretic computations for quantum weighted lens spaces in [2] in the Cuntz–Pimsner picture [3].

4.2.1. Lens spaces. Our starting point is the C^*-algebra $C(L^3_q(l; 1, l))$ of the quantum lens space $L^3_q(l; 1, l)$. As explained in [9, Example 2.1], $C(L^3_q(l; 1, l))$, for any $q \in [0, 1)$, can be viewed as the graph C^*-algebra of the graph L^3_l (see Figure 6) with

- $l + 1$ vertices $\{v^0_0, v^1_0, \ldots, v^1_{l-1}\}$,
- one edge e^{01}_i from v^0_0 to v^1_i for all $0 \le i \le l - 1$,
- a loop e^0_0 over the vertex v^0_0 and one loop e^1_i over each vertex v^1_i for all $0 \le i \le l - 1$.

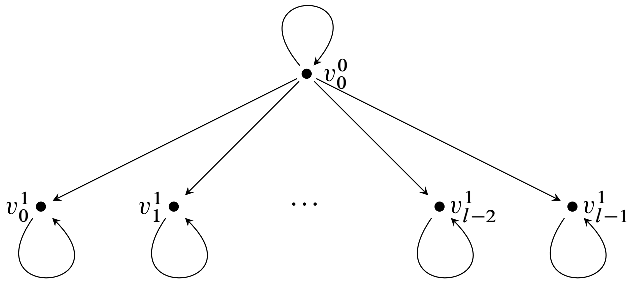

Figure 6. The graph L_l^3.

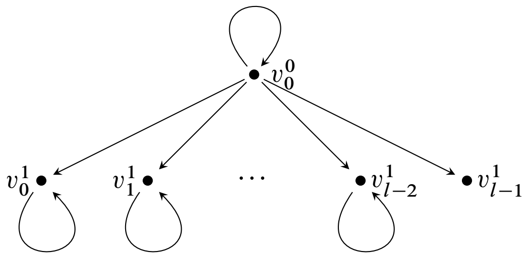

Figure 7. The graph Q_l.

Observe that $C^*(L_1^3) \cong C(S_q^3)$ (e.g., see [27]). To simplify notation for the Cuntz–Krieger L_l^3-family, we set

$$S_j^{01} := S_{e_j^{01}},\ S_j^i := S_{e_j^i} \text{ and } P_j^i := P_{v_j^i}, \text{ where } 0 \le i \le 1 \text{ and } 0 \le j \le l-1. \tag{4.18}$$

Clearly, the graph L_l^3 is v_j^1-trimmable for $0 \le j \le l-1$. Let us choose the vertex v_{l-1}^1, and construct a new graph by removing the loop e_{l-1}^1. We denote the thus obtained graph by Q_l (see Figure 7). Note that the graph Q_l and its C^*-algebra do not depend on our choice of a vertex.

By Theorem 2.4, we obtain the following $U(1)$-equivariant pullback structure of the C^*-algebra $C(L_q^3(l;1,l)) \cong C^*(L_l^3)$:

$$\begin{array}{ccc} & C^*(L_l^3) & \\ \swarrow & & \searrow \\ C^*(L_{l-1}^3) & & C^*(Q_l) \otimes C(S^1). \\ \searrow & & \swarrow \\ & C^*(L_{l-1}^3) \otimes C(S^1) & \end{array} \tag{4.19}$$

Here all the maps are special cases of the maps used in Theorem 2.4.

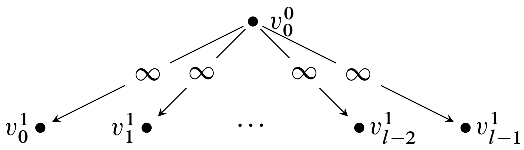

Figure 8. The graph W_l of $C(\mathbb{W}\mathrm{P}^1_q(1,l))$.

4.2.2. Teardrops. Recall that the C^*-algebra $C(\mathbb{W}\mathrm{P}^1_q(1,l))$ of the weighted projective space $\mathbb{W}\mathrm{P}^1_q(1,l)$ (see [8]) is defined as a $U(1)$-fixed-point subalgebra of $C(L^3_q(l;1,l))$. It can be viewed as the graph C^*-algebra of the graph W_l consisting of the same vertices $v^0_0,\ldots,v^1_{l-1}$ as in the graph L^3_l (vertex projections are gauge invariant), with no loops, and countably infinitely many edges from v^0_0 to all the other vertices [9] (see Figure 8).

As discussed in Section 2.2, since the $U(1)$-action defining $C(\mathbb{W}\mathrm{P}^1_q(1,l))$ is the gauge action on $C^*(L^3_l)$, from (4.19) we obtain another pullback diagram of fixed-point subalgebras:

$$\begin{array}{ccc} & C^*(W_l) & \\ \swarrow & & \searrow \\ C^*(W_{l-1}) & & C^*(Q_l). \\ \searrow & & \swarrow{\scriptstyle \chi_1} \\ & C^*(L^3_{l-1}) & \end{array} \tag{4.20}$$

Here χ_1 is the quotient map by the ideal $\overline{I(v^1_{l-1})}$.

Next, let us introduce two more quotient maps needed for Lemma 4.2 below. Define

$$\chi_2 : C^*(Q_l) \longrightarrow C^*(Q_l/H) \tag{4.21}$$

to be the quotient map given by the ideal generated by the saturated hereditary subset $H := \{v^1_0, v^1_1, \ldots, v^1_{l-2}\} \subset (Q_l)_0$, and $g : C^*(L^3_{l-1}) \to C^*(L^3_{l-1}/H)$ to be the quotient map given by the ideal generated by H considered as a subset in $(L^3_{l-1})_0$.

Lemma 4.2. *The following diagram of the above-defined $U(1)$-equivariant *-homomorphisms*

$$\begin{array}{ccc} & C^*(Q_l) & \\ {\scriptstyle \chi_1}\swarrow & & \searrow{\scriptstyle \chi_2} \\ C^*(L^3_{l-1}) & & \mathcal{T} \\ {\scriptstyle g}\searrow & & \swarrow{\scriptstyle \sigma} \\ & C(S^1) & \end{array} \tag{4.22}$$

is a pullback diagram. Here $\mathcal{T}$ is the Toeplitz algebra and $\sigma : \mathcal{T} \to C(S^1)$ is the symbol map.

Proof. Recall that the Toeplitz algebra $\mathcal{T}$ is isomorphic to the graph C^*-algebra corresponding to the subgraph Q_l/H and the isomorphism is given by $s \mapsto S_0^0 + S_{l-1}^{01}$ (e.g., see [30]), where we use the notation (4.18) and s is the isometry generating $\mathcal{T}$.

By [33, Proposition 3.1] and the surjectivity of g and σ, it suffices to show that $\ker \chi_1 \cap \ker \chi_2 = \{0\}$ and that $\chi_2(\ker \chi_1) \supseteq \ker \sigma$. To prove the first condition, recall that, since $\ker \chi_1$ and $\ker \chi_2$ are closed ideals in a C^*-algebra, we have that $\ker \chi_1 \cap \ker \chi_2 = \ker \chi_1 \ker \chi_2$. Next, $\{v_{l-1}^1\}$ and H are saturated hereditary subsets of $(Q_l)_0$, so

$$\ker \chi_1 = \overline{I_{Q_l}(v_{l-1}^1)} \quad \text{and} \quad \ker \chi_2 = \overline{I_{Q_l}(H)}. \tag{4.23}$$

Using (1.6), one can observe that $\ker \chi_1 \ker \chi_2$ is the closed linear span of elements of the form $S_\alpha S_\beta^* S_\gamma S_\delta^*$, where $\alpha, \beta \in \mathrm{Path}(Q_l)$ with $r_P(\alpha) = v_{l-1}^1 = r_P(\beta)$ and $\gamma, \delta \in \mathrm{Path}(Q_l)$ with $r_P(\gamma) = r_P(\delta) \in H$. The claim follows from the analysis of all possible paths satisfying the above conditions.

To prove the second condition, notice that $\ker \sigma = \overline{I_{Q_l/H}(v_{l-1}^1)}$. Any element of $I_{Q_l/H}(v_{l-1}^1)$ is an element of $I_{Q_l}(v_{l-1}^1)$, and $\chi_2(S_\alpha) = S_\alpha$ for all $\alpha \in \mathrm{Path}(Q_l/H)$. Hence $\chi_2(I_{Q_l}(v_{l-1}^1)) \supseteq I_{Q_l/H}(v_{l-1}^1)$. Furthermore, since χ_2 is a $*$-homomorphism, we can argue as in the proof of Theorem 2.4 to conclude that $\chi_2(\ker \chi_1) \supseteq \ker \sigma$. ■

Remark 4.3. Let $q \in [0, 1)$. Note that for $l = 2$, we get the graph Q_2 considered in Section 3.2. By Lemma 4.2, the C^*-algebra $C^*(Q_2)$ has the following $U(1)$-equivariant pullback structure:

$$\begin{array}{ccccc} & & C^*(Q_2) & & \\ & \swarrow & & \searrow & \\ C(S_q^3) & & & & \mathcal{T}. \\ & \searrow & & \swarrow & \\ & & C(S^1) & & \end{array} \tag{4.24}$$

The equivariance of the above diagram allows it to descend to the fixed-point subalgebras:

$$\begin{array}{ccccc} & & C^*(Q_2)^{U(1)} & & \\ & \swarrow & & \searrow & \\ C(\mathbb{C}\mathrm{P}_q^1) & & & & \mathcal{T}^{U(1)}. \\ & \searrow & & \swarrow & \\ & & \mathbb{C} & & \end{array} \tag{4.25}$$

The Mayer–Vietoris six-term exact sequence in K-theory associated to this diagram gives the K-groups of $C^*(Q_2)^{U(1)}$ as in Proposition 3.2.

Next, with the help of (4.20) and Lemma 4.2 along with [33, Proposition 2.7], the analogous reasoning as in the proof of Proposition 4.1 yields the following proposition.

Proposition 4.4. *The C^*-algebra $C(\mathbb{W}\mathrm{P}^1_q(1,l))$, $q \in [0,1)$, has the following pullback structure:*

$$\begin{array}{ccccc} & & C\big(\mathbb{W}\mathrm{P}^1_q(1,l)\big) & & \\ & \swarrow & & \searrow & \\ C\big(\mathbb{W}\mathrm{P}^1_q(1,l-1)\big) & & & & \mathcal{T}. \\ & \searrow & & \swarrow & \\ & & C(S^1) & & \end{array} \tag{4.26}$$

Furthermore, we obtain

$$K_0\Big(C\big(\mathbb{W}\mathrm{P}^1_q(1,l)\big)\Big) \cong K_0\Big(C\big(\mathbb{W}\mathrm{P}^1_q(1,l-1)\big)\Big) \oplus \partial_{10}\Big(K_1\big(C(S^1)\big)\Big) \cong \mathbb{Z}^{l+1}, \tag{4.27}$$

where ∂_{10} is Milnor's connecting homomorphism.

4.2.3. Milnor's clutching construction for generators of $K_0(C(\mathbb{W}\mathrm{P}^1_q(1,l)))$. Throughout this section, we keep the notation of (4.18). First, recall that there are $(l+1)$-many projections $P^0_0, P^1_0, \ldots, P^1_{l-1}$, in the graph W_l whose graph algebra is $C(\mathbb{W}\mathrm{P}^1(1,l))$. Therefore, since $K_0(C(\mathbb{W}\mathrm{P}^1(1,l))) \cong \mathbb{Z}^{l+1}$ and the K_0-group of a graph C^*-algebra is generated by its vertex projections (see [10, Proposition 3.8 (1)]), we infer that

$$K_0\Big(C\big(\mathbb{W}\mathrm{P}^1(1,l)\big)\Big) = \mathbb{Z}[P^0_0] \oplus \mathbb{Z}[P^1_0] \oplus \cdots \oplus \mathbb{Z}[P^1_{l-1}]. \tag{4.28}$$

Next, we compute the value of Milnor's connecting homomorphism on the generator $[u] \in K_1(C(S^1))$, where u is the standard generator of $C(S^1)$. This computation closely follows an analogous computation in Section 4.1.3. First, we find $c, d \in \mathcal{T}$ such that $\sigma(c) = u$ and $\sigma(d) = u^*$, i.e.

$$c = S^0_0 + S^1_{l-1}, \quad d = c^* = (S^0_0)^* + (S^1_{l-1})^*, \tag{4.29}$$

and, using the formula (4.15), we compute the following 2 by 2 matrix with entries in $C(\mathbb{W}\mathrm{P}^1_q(1,l))$:

$$p_u = \begin{bmatrix} (1, P^0_0) & (0,0) \\ (0,0) & (0,0) \end{bmatrix}. \tag{4.30}$$

The element $\partial_{10}([u]) = [p_u] - [1]$ is a generator of $K_0(C(\mathbb{W}\mathrm{P}^1_q(1,l)))$. We have that $[1] - [p_u] = [1] - [p]$, where $p := (1, P^0_0) = (1,1) - (0, P^1_{l-1})$ is a projection in $C(\mathbb{W}\mathrm{P}^1_q(1,l))$. Notice that

$$-\partial_{10}\big([u]\big) = [1] - [p] = \big[(0, P^1_{l-1})\big] = [P^1_{l-1}], \tag{4.31}$$

where the rightmost P^1_{l-1} is viewed as an element of $C(\mathbb{W}\mathrm{P}^1_q(1,l))$.

Finally, as we did in Section 4.1.3, let us observe that Milnor's idempotent in the above calculation is

$$p_u \cong 1 - P^1_{l-1} = P^0_0 + P^1_0 + \cdots + P^1_{l-2}. \tag{4.32}$$

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