J. Noncommut. Geom. 16 (2022), 787–837
DOI 10.4171/JNCG/453

Iterated Hopf Ore extensions in positive characteristic

Kenneth A. Brown and James J. Zhang

Abstract. Iterated Hopf Ore extensions (IHOEs) over an algebraically closed base field $\Bbbk$ of positive characteristic p are studied. We show that every IHOE over $\Bbbk$ satisfies a polynomial identity (PI), with PI-degree a power of p, and that it is a filtered deformation of a commutative polynomial ring. We classify all 2-step IHOEs over $\Bbbk$, thus generalising the classification of 2-dimensional connected unipotent algebraic groups over $\Bbbk$. Further properties of 2-step IHOEs are described: for example their simple modules are classified, and every 2-step IHOE is shown to possess a large Hopf center and hence an analog of the restricted enveloping algebra of a Lie $\Bbbk$-algebra. As one of a number of questions listed, we propose that such a restricted Hopf algebra may exist for every IHOE over $\Bbbk$.

1. Introduction

1.1. First main result

Let p be a prime and let $\Bbbk$ be an algebraically closed field of characteristic p. A famous result of Jacobson [20] states that the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over $\Bbbk$ is a finitely generated module over its center, denoted by $Z(\mathfrak{g})$. In fact, it is easy to see from Jacobson's proof that the *PI-degree* d of $U(\mathfrak{g})$ is a power of p (the PI-degree being by definition the square root of $\dim_{Q(Z(\mathfrak{g}))} Q(U(\mathfrak{g}))$, where $Q(U(\mathfrak{g}))$ is the central simple quotient division ring of $U(\mathfrak{g})$).

Our first aim is to extend Jacobson's result to certain connected Hopf $\Bbbk$-algebras, as follows. Hopf Ore extensions $A[x;\sigma,\delta]$ were defined and studied by Panov [40] in 2003. His definition was refined and extended in [7], and then in [17], to *iterated Hopf Ore extensions* (IHOEs) over a field F. (The field F may have characteristic zero.) An n-step IHOE over F is a Hopf F-algebra H with a finite chain of Hopf subalgebras

$$F = H_0 \subset \cdots \subset H_n = H,$$

such that, for $i = 1,\dots,n$, $H_i = H_{i-1}[x_i;\sigma_i,\delta_i]$, for an algebra automorphism σ_i and a σ_i-derivation δ_i. See Definition 2.1 for details, and recall that every IHOE is *connected*, meaning that its coradical is F, by [7, Proposition 2.5]. Our first main result is the following theorem.

2020 Mathematics Subject Classification. Primary 16T05; Secondary 16S36, 16R50.
Keywords. Hopf Ore extension, Ore extension, Hopf algebra, PBW deformation, Nakayama automorphism, positive characteristic, polynomial identity.

Theorem 1.1. *Let p and $\Bbbk$ be as above, and let H be an n-step $\Bbbk$-IHOE.*

(1) (Corollary 3.4) *H is a finitely generated module over its center $Z(H)$, which is a finitely generated normal $\Bbbk$-algebra.*

(2) (Theorem 4.3) *The PI-degree of H is a power of p.*

(3) (Theorem 4.8) *The order of the antipode of H divides $4p^{n-1}$, and the order of the Nakayama automorphism of H divides $2p^n$.*

The proofs of (1) and (2) of the theorem in Sections 3 and 4 make use of results of [10, 11, 27] on Ore extensions satisfying a polynomial identity (PI), and ultimately hinge on fundamental results of Kharchenko [23]; some aspects of the argument may have relevance for the study of general Ore extensions, that is, beyond the realm of Hopf algebras.

A number of important homological consequences follow by routine arguments from Theorem 1.1, with details included in Corollary 3.4—namely, H has global and Gel'fand–Kirillov dimensions equal to n, and is a homologically homogeneous maximal order, and hence is skew Calabi–Yau.

1.2. Filtrations on an IHOE

By contrast with Theorem 1.1 (1), an IHOE over a field F of characteristic 0 does not satisfy a PI unless it is commutative [7, Theorem 5.4]. Our second main result, however, generalises an aspect of Lie theory where there is no such dichotomy between characteristic 0 and positive characteristic: namely, it is of course fundamental to the study of enveloping algebras that if $\mathfrak{g}$ is a Lie algebra of finite dimension n over a field F, then $U(\mathfrak{g})$ has an ascending filtration whose associated graded algebra is the commutative polynomial algebra in n variables over F. Zhuang proved in [61, Theorem 6.10] that the same conclusion holds for a connected Hopf F-algebra of finite Gel'fand–Kirillov dimension n, when F is algebraically closed of characteristic 0. Zhuang's result does not extend to positive characteristic—for example, the group algebra over $\Bbbk$ of the cyclic group of order p is a connected Hopf algebra. Nevertheless, we prove here the following theorem.

Theorem 1.2 (Theorem 5.3). *Let p and $\Bbbk$ be as in Theorem 1.1, and let H be an n-step $\Bbbk$-IHOE. Then positive integer degrees can be assigned to the defining variables of H so that the corresponding filtration has associated graded algebra which is the commutative polynomial $\Bbbk$-algebra on n variables.*

One can succinctly state this result as follows: every H as in the theorem is a filtered deformation of a polynomial algebra. Etingof asks in [13, Question 1.1] whether *every* filtered deformation of a commutative domain in positive characteristic has to satisfy a PI. Taken together, Theorems 1.1 and 1.2 give some support for a positive answer to this question.

1.3. A classification of 2-step IHOEs

In Section 6, we classify IHOEs in the lowest dimensions. As noted in Lemma 6.1, there is only one 1-step IHOE, whatever the characteristic of the field, namely $\Bbbk[x]$ with x primitive. But in dimension two the contrast between characteristic 0 and positive characteristic is stark. Zhuang showed in [61, Proposition 7.6] that in characteristic 0, there are only two connected Hopf algebras of Gel'fand–Kirillov dimension two, and both are cocommutative IHOEs: namely $\Bbbk[x, y]$ with x and y primitive, and the enveloping algebra of the 2-dimensional non-abelian Lie algebra. In positive characteristic, however, it is a different story, as we now explain.

Let $\mathbf{d_s} = \{d_s\}_{s\geq 0}$, $\mathbf{b_s} = \{b_s\}_{s\geq 0}$, and $\mathbf{c_{s,t}} = \{c_{s,t}\}_{0\leq s<t}$ be sequences of scalars in $\Bbbk$ with only finitely many nonzero elements. Let $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ denote the Ore extension $\Bbbk[X_1][X_2; \mathrm{Id}, \delta]$ where

$$\delta(X_1) = \sum_{s\geq 0} d_s X_1^{p^s}. \tag{1.2.1}$$

Define maps Δ, ε, and S on $\{X_1, X_2\}$ by setting

$$\begin{aligned}
\Delta(X_1) &= X_1 \otimes 1 + 1 \otimes X_1,\\
\Delta(X_2) &= X_2 \otimes 1 + 1 \otimes X_2 + w,\\
\varepsilon(X_1) &= \varepsilon(X_2) = 0,\\
S(X_1) &= -X_1,\\
S(X_2) &= -X_2 - m(\mathrm{Id} \otimes S)(w),
\end{aligned}$$

where m denotes the multiplication operator, and

$$\begin{aligned}
w = \sum_{s\geq 0} b_s \Big(\sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (X_1^{p^s})^i \otimes (X_1^{p^s})^{p-i} \Big)\\
+ \sum_{0\leq s<t} c_{s,t} (X_1^{p^s} \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1^{p^s}).
\end{aligned} \tag{1.2.2}$$

Theorem 1.3 (Propositions 6.6 and 6.11). *Let p and $\Bbbk$ be as in Theorem 1.1. Let H be a 2-step IHOE over $\Bbbk$, so $H = \Bbbk[X_1][X_2; \sigma, \delta]$ for $\sigma \in \mathrm{Aut}_{\Bbbk\text{-alg}}(\Bbbk[X_1])$ and a σ-derivation δ.*

(1) (Proposition 6.6 (1)) *For all choices of the scalars $\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}}$, $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ is a Hopf algebra.*

(2) (Proposition 6.6 (2)) *H is isomorphic to $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ for some choice of scalars $\{d_s\}_{s\geq 0}$, $\{b_s\}_{s\geq 0}$, and $\{c_{s,t}\}_{0\leq s<t}$.*

(3) (Proposition 6.11) *Two such $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ and $H(\mathbf{d'_s}, \mathbf{b'_s}, \mathbf{c'_{s,t}})$ are isomorphic as Hopf algebras if and only if there are nonzero scalars α, β in $\Bbbk$ such that, for all the sequences of scalars,*

$$d'_s = d_s \alpha^{p^s - 1} \beta^{-1}, \quad b'_s = b_s \alpha^{p^{s+1}} \beta^{-1}, \quad c'_{s,t} = c_{s,t} \alpha^{p^s + p^t} \beta^{-1}. \tag{1.3.1}$$

If (1.3.1) *holds, there is a Hopf algebra isomorphism*

$$\phi : H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}}) \to H(\mathbf{d'_s}, \mathbf{b'_s}, \mathbf{c'_{s,t}})$$

such that

$$\phi(X_1) = \alpha X_1 \quad \textit{and} \quad \phi(X_2) = \beta X_2.$$

Restricting the above result to the cases where H is commutative, that is where $\mathbf{d_s} = \mathbf{0}$, yields the classification of connected unipotent algebraic groups over $\Bbbk$ of dimension 2. This is discussed in Section 7.4.

1.4. Other properties of 2-step IHOEs

We study further properties of the 2-step IHOEs over $\Bbbk$ in Section 9. Thus we describe their antipodes, determine the Calabi–Yau members of the family in Proposition 9.1, and examine the finite dimensional representation theory of all the 2-step $\Bbbk$-IHOEs in Proposition 9.2, specifying the Azumaya locus, and determining all the simple modules and the extensions between them.

The final subsection, Section 9.3, is inspired by another seminal paper of Jacobson [19], where he showed that every restricted Lie algebra $\mathfrak{g}$ of finite dimension n has a *restricted enveloping algebra* $u(\mathfrak{g})$, a Hopf algebra of dimension p^n which is a Hopf factor of $U(\mathfrak{g})$. We show that a similar phenomenon occurs for the 2-step IHOEs. The following is an abbreviated version of Theorem 9.8 and Propositions 9.9 and 9.10 together.

Theorem 1.4. *Let* $H = H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ *be a* 2*-step IHOE over* $\Bbbk$ *and assume that* H *is not commutative.*

(1) H *contains a unique maximal central Hopf subalgebra* $C(H)$, *and* H *is a free* $C(H)$*-module of rank* p^2 *when* $b_0 = 0$, *and rank* p^3 *otherwise.*

(2) *Let* $C(H)_+$ *denote the augmentation ideal of* $C(H)$. *The resulting Hopf algebras* $H/C(H)_+H$, *of dimensions* p^2 *or* p^3, *are classified.*

Finite dimensional connected Hopf $\Bbbk$-algebras have as dimension a power of p [31, Proposition 1.1 (1)]. Those of dimension at most p^3 have been classified, in a series of papers [37, 38, 54, 55] by Xingting Wang and colleagues. We explain in Section 9.3 how the Hopf algebras of Theorem 1.4 fit into their classification.

1.5. Comments

There have been a number of recent papers on infinite dimensional connected Hopf algebras of finite Gel'fand–Kirillov dimension, starting with [61] and including the aforementioned [7, 17], but also, for example, [4, 9, 52, 60]. Much of this work has focused on the characteristic 0 setting (although *not*, notably, [17], on which we rely heavily). Thus the present paper can be viewed as taking some further steps to open the topic in positive characteristic. As befits a gathering of first steps, the paper contains a number of open questions, scattered throughout. We can also point out here an open question of a more general, less precise type. Namely, it is possible that many or even all of the results proved here for positive characteristic IHOEs are valid, when rephrased appropriately, in the wider context of connected Hopf domains over $\Bbbk$ of finite Gel'fand–Kirillov dimension.

1.6. Organization

The paper is organized as follows. Section 2 contains some definitions and preliminaries, leading to Proposition 2.2, where it is shown that the automorphisms used to construct a positive characteristic IHOE have finite order, a key ingredient in the proof of Theorem 1.1. In Section 3, we use the Frobenius map and Noether's theorem on the finite generation of invariants to obtain the characterisation (Theorem 3.3) of when an iterated Ore extension in positive characteristic satisfies a PI. From this, it is easy to invoke Proposition 2.2 to deduce Theorem 1.1 (1) (Corollary 3.4). Theorem 1.2 is proved in Section 4, using results of Chuang and Lee [11], depending ultimately on the work of Kharchenko in [23] to obtain the needed information on the PI-degree of an iterated Ore extension which is known to satisfy a PI. In Section 5, we show that every IHOE has a filtration such that the associated graded ring is isomorphic to the commutative polynomial algebra. The classification of 2-step IHOEs is given in Section 6, including an analysis of their automorphism groups. A list of comments and questions are collected in Section 7. A description of the Hopf center of all 2-step IHOEs is obtained in Section 9, enabling the construction and classification of their restricted Hopf algebra factors. The description of the Hopf center requires a special case of the noncommutative binomial theorem of Jacobson, which we recall in Section 8.

2. Preliminaries

2.1. Definitions and their consequences

Throughout the paper, we shall use the standard notation Δ, μ, ε, and S for the comultiplication, multiplication, counit, and antipode of a Hopf algebra H, with $\Delta(h) = h_1 \otimes h_2$ for $h \in H$. Unexplained Hopf algebra terminology can be found in [36], for example. All Hopf algebras appearing in this paper will be (factors of) noetherian domains, so S is necessarily bijective by [46, Theorem A (ii)]. For an algebra A, $Z(A)$ will denote the center of A, and if a group G acts on A as automorphisms, A^G will denote the fixed subring of this action; when $G = \langle\sigma\rangle$ generated by a single element σ, $A^{\langle\sigma\rangle}$ will be abbreviated to A^σ.

The following definition of the algebras in the paper's title is—on the face of it—significantly weaker than the original definition given in [40, Definition 1.0] and the modified definition in [7, Definition 2.1]. The improvement here is due to recent work of Huang [17], as we explain after the definitions. Recall that, given an F-algebra automorphism σ of an F-algebra R, a σ-derivation δ of R is an F-linear endomorphism of R such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$.

Definition 2.1. Let F be a field of any characteristic.

(1) Let R be a Hopf F-algebra. A *Hopf Ore extension* (abbreviated to *HOE*) of R is an algebra H such that

(1a) H is a Hopf F-algebra with Hopf subalgebra R;

(1b) there exist an algebra automorphism σ of R and a σ-derivation δ of R such that $H = R[x; \sigma, \delta]$.

(2) An *(n-step) iterated Hopf Ore extension of F* (abbreviated to *(n-step) IHOE (of F)*) is a Hopf algebra

$$H = F[X_1][X_2; \sigma_2, \delta_2] \cdots [X_n; \sigma_n, \delta_n], \tag{2.1.1}$$

where

(2a) H is a Hopf algebra;

(2b) $H_{(i)} := F\langle X_1, \ldots, X_i \rangle$ is a Hopf subalgebra of H for $i = 1, \ldots, n$;

(2c) σ_i is an algebra automorphism of $H_{(i-1)}$, and δ_i is a σ_i-derivation of $H_{(i-1)}$, for $i = 2, \ldots, n$.

Throughout the paper, whenever H is an HOE (respectively, an n-step IHOE), we assume that H satisfies the conditions of Definition 2.1 (1) (respectively, (2)), with the same notation.

The definition of an HOE in [7, Definition 2.1] is required, in addition to Definition 2.1 (1a), (1b), that

(1c) there are $a, b \in R$ and $v, w \in R \otimes R$ such that

$$\Delta(x) = a \otimes x + x \otimes b + v(x \otimes x) + w. \tag{2.1.2}$$

However, by Huang's theorem [17, Theorem 1.3], if $R \subseteq T$ are noetherian F-algebras satisfying hypotheses (1a) and (1b) of Definition 2.1 (1), and $R \otimes R$ is a domain, then, up to a change of variable,

(1d) there are $a \in R$ and $w \in R \otimes R$ such that

$$\Delta(x) = a \otimes x + x \otimes 1 + w. \tag{2.1.3}$$

This means that under the conditions in [17, Theorem 1.3], namely that $R \otimes R$ is a noetherian domain, Definition 2.1 (1) is equivalent to [7, Definition 2.1]. In turn, this yields the following consequences for IHOEs. Recall that a Hopf F-algebra is *connected* if its coradical is F; see e.g. [36, Definition 5.1.5].

Proposition 2.2. *Let H be an IHOE of F.*

(1) *H is noetherian and $H \otimes H$ is a domain.*

(2) *After a change of variables (but not of the subalgebras $H_{(i)}$),*

$$\Delta(X_i) = 1 \otimes X_i + X_i \otimes 1 + w_i \tag{2.2.1}$$

for some $w_i \in H_{(i-1)} \otimes H_{(i-1)}$, for $i = 1, \ldots, n$.

(3) *H is a connected Hopf algebra.*

(4) *For $i = 1, \ldots, n$, there is a character χ_i of $H_{(i-1)}$ such that*

$$\sigma_i = \tau^{\ell}_{\chi_i} = \tau^{r}_{\chi_i}$$

is a (left and right) winding automorphism of $H_{(i-1)}$. In particular, for each j with $j < i$, there exists $a_{ji} \in H_{(j-1)}$ such that

$$\sigma_i(X_j) = X_j + a_{ji}.$$

Proof. (1) This follows by induction on n using [33, Theorem 1.2.9].

(2) Fix i, $1 \le i \le n$ (where we take $H_{(0)} = F$, $\sigma_1 = \mathrm{id}_F$, and $\delta_1 = 0$). By (1), $H_{(i-1)}$ is noetherian and $H_{(i-1)} \otimes H_{(i-1)}$ is a domain. Hence, by [17, Theorem 1.3],

$$\Delta(X_i) = a_i \otimes X_i + X_i \otimes 1 + w_i$$

for a group-like element of a_i of $H_{(i-1)}$ and $w_i \in H_{(i-1)} \otimes H_{(i-1)}$. But H has no non-trivial invertible elements. Therefore, noting that $\mu \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}_{H_{(i-1)}}$, $a_i = 1$, yielding (2.2.1).

(3) This is proved in [7, Proposition 2.5].

(4) That the map σ_i is a (left and right) winding automorphism (with the same character) of $H_{(i-1)}$ is proved in [7, Theorem 1.2] and its improvement [17, Theorem 1.3]. From (2.1.3) applied to X_j we see that

$$a_{ji} = \chi_i(X_j) + \sum w_{j\,(1)} \chi_i(w_{j\,(2)}) \in H_{(j-1)}. \tag{2.2.2}$$

■

Since it is needed for the proof of Theorem 1.1, we recall a result in classical invariant theory due to E. Noether [39], and also see [35, Theorem 1.1] and [47, Theorem 2.3.1].

Theorem 2.3. *Let A be an affine commutative algebra over F and let G be a finite subgroup of $\mathrm{Aut}_{F\text{-alg}}(A)$. Then the fixed subring A^G is affine over F and A is a finitely generated module over A^G.*

2.2. Positive characteristic aspects

Given Proposition 2.2, the following result is key to our main results.

Proposition 2.4. *Suppose that $\Bbbk$ has positive characteristic p. Let d be a positive integer.*

(1) *Let R be a Hopf $\Bbbk$-algebra such that the order of every left or right winding automorphism of R divides d. Suppose that $H := R[x, \sigma, \delta]$ is an HOE of R such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1 + w \tag{2.4.1}$$

for some $w \in R \otimes R$. Then the order of every left or right winding automorphism of H divides dp.

(2) *Let H be an n-step IHOE of $\Bbbk$. Then every left or right winding automorphism of H has order dividing p^n.*

Proof. We prove the results for left winding automorphisms.

(1) Let $\pi : H \to \Bbbk$ be an algebra map and let Ξ^l_π be the corresponding left winding automorphism of H, so $\Xi^l_\pi(h) = \pi(h_1)h_2$ for $h \in H$. Since R is a Hopf subalgebra of H and $\pi|_R$ is a character of R, then Ξ^l_π restricted to R, still denoted by Ξ^l_π, is a left winding automorphism of R. By assumption, $(\Xi^l_\pi)^d$ is the identity of R. It remains to show that $(\Xi^l_\pi)^{dp}$ is the identity when applied to x. Using (2.4.1),

$$\Xi^l_\pi(x) = x + \pi(x) + \mu \circ (\pi \otimes 1)(w) = x + s,$$

where $s = \pi(x) + \mu \circ (\pi \otimes 1)(w) \in R$. Then

$$(\Xi^l_\pi)^d(x) = x + t$$

for some $t \in R$. Note that $(\Xi^l_\pi)^d(t) = t$ as the order of Ξ^l_π restricted to R divides d. Then

$$(\Xi^l_\pi)^{dp}(x) = \big((\Xi^l_\pi)^d\big)^p(x) = x + pt = x.$$

Therefore, $(\Xi^l_\pi)^{dp}$ is the identity map on H as required.

(2) This follows from part (1), Proposition 2.2 (2), and induction on n. ■

Note that there is an anti-monomorphism of groups from the character group $X(H)$ of H (that is, the algebra homomorphisms from H to $\Bbbk$) to the group of left winding automorphisms of H, [8, Section 2.5]. Hence, viewing H in dual language, as a quantum group, Proposition 2.4 (2) can be interpreted as giving a bound on the exponent of the maximal classical subgroup of H, which is a unipotent algebraic group.

3. Proof of Theorem 1.1

3.1. HOEs and IHOEs in positive characteristic

If C is a commutative $\Bbbk$-algebra, where $\Bbbk$ has positive characteristic p, we denote the Frobenius map $c \mapsto c^p$ on C by F. In this section, we always assume the following.

Convention 3.1. When $\Bbbk$ is a field of positive characteristic and C is a commutative $\Bbbk$-algebra, we use $F(C)$ to denote the $\Bbbk$-subalgebra of C generated by the image of C. Thus, in general, $F(C)$ is larger than the image of C under F.

Given an automorphism σ and a σ-derivation δ of an algebra A, a subspace E of A is called *(σ, δ)-trivial* if $\sigma(e) = e$ and $\delta(e) = 0$ for all $e \in E$.

Lemma 3.2. *Let B be an Ore extension $A[x; \sigma, \delta]$ of an affine $\Bbbk$-algebra A.*

(1) *Suppose that no nonzero element of $Z(A)$ is a zero divisor on A, and that $Z(A)^\sigma \subsetneq Z(A)$. Then*

$$zx = xz$$

for all $z \in Z(A)^\sigma$. As a consequence, B contains the commutative subalgebra $Z(A)^\sigma[x]$.

(2) *Suppose that* $Z(A) = Z(A)^\sigma$ *and* $\operatorname{char} \Bbbk = p > 0$. *Then*

$$z^p x = x z^p$$

for all $z \in Z(A)$. *As a consequence,* B *contains the commutative subalgebra* $F(Z(A))[x]$.

(3) *Suppose that*

(a) A *is a prime ring that is a finitely generated module over* $Z(A)$*;*

(b) $Z(A)$ *is an affine* $\Bbbk$*-algebra;*

(c) $\sigma|_{Z(A)}$ *has finite order;*

(d) $\operatorname{char} \Bbbk = p > 0$.

Then B *is a prime noetherian algebra satisfying a PI, and* A *is a finite module over a* (σ, δ)*-trivial subalgebra of its center.*

(4) *Suppose, in addition to hypotheses* (3)(a)–(3)(d)*, that*

(e) A *is a maximal order.*

Then B *is a maximal order and is a finitely generated module over* $Z(B)$*, which is an affine normal* $\Bbbk$*-algebra.*

Proof. (1) Let $z \in Z(A)^\sigma$. Then $xz = zx + \delta(z)$, so we only need to show that $\delta(z) = 0$ for all $z \in Z(A)^\sigma$. Pick any element $y \in Z(A) \setminus Z(A)^\sigma$. Then $zy = yz$. Applying δ to this equation we have

$$\delta(z)y + \sigma(z)\delta(y) = \delta(y)z + \sigma(y)\delta(z).$$

Since $z \in Z(A)^\sigma$, we have $\sigma(z)\delta(y) = z\delta(y) = \delta(y)z$, which implies that

$$\delta(z)y = \sigma(y)\delta(z).$$

Hence, since $\sigma(y) \in Z(A)$,

$$\delta(z)\big(y - \sigma(y)\big) = 0.$$

But $y - \sigma(y) \in Z(A) \setminus \{0\}$ and is thus, by hypothesis, not a zero divisor in A. So $\delta(z) = 0$ as required, and hence $Z(A)^\sigma[x]$ is commutative.

(2) Suppose that $Z(A) = Z(A)^\sigma$. Then, for every $z \in Z(A)$,

$$xz = \sigma(z)x + \delta(z) = zx + \delta(z).$$

By induction on n and since $z \in Z(A)$, we have, for all $n \geq 1$,

$$xz^n = z^n x + nz^{n-1}\delta(z).$$

The assertion follows because $pz^{p-1}\delta(z) = 0$. Once again, the consequence is clear.

(3) It is clear from [33, Theorem 1.2.9] and hypotheses (a) and (b) that B is noetherian, and it is prime by [33, Theorem 1.2.9 (iii)]. By Theorem 2.3, and hypotheses (a) and (c),

A is a finitely generated $Z(A)^\sigma$-module. When $Z(A) \neq Z(A)^\sigma$, B is a finitely generated left module over its commutative affine subalgebra $Z(A)^\sigma[x]$ by part (1). When $Z(A) = Z(A)^\sigma$, B is a finitely generated left module over its commutative affine subalgebra $F(Z(A))[x]$ by part (2). In both cases, B thus satisfies a PI by [33, Corollary 13.4.9 (i)]. The second claim in the final sentence of (3) is clear from (1), (2), and Theorem 2.3.

(4) Suppose that A satisfies hypothesis (e). Then B is also a maximal order by [32, Proposition V.2.5]. Therefore, since it is a prime noetherian PI ring by part (3), B equals its own trace ring and is thus a finite module over its affine normal center by [33, Propositions 13.9.8(i) and 13.9.11(ii)]. ■

Theorem 1.1 is a corollary of the following more general result applying to iterated Ore extensions in positive characteristic.

Theorem 3.3. *Suppose $\Bbbk$ has positive characteristic p, let n be a non-negative integer, and let R be an n-step iterated Ore extension,*

$$R = \Bbbk[X_1][X_2;\sigma_2,\delta_2]\cdots[X_n;\sigma_n,\delta_n]. \tag{3.3.1}$$

For $i = 1,\ldots,n-1$, denote $\Bbbk\langle X_1,\ldots,X_i\rangle$ by $R_{(i)}$.

(1) *R satisfies a PI if and only if $\sigma_i|_{Z(R_{(i-1)})}$ has finite order for all $i = 2,\ldots,n$.*

(2) *Suppose that R satisfies a PI. Then it is a finitely generated module over its center, which is a normal $\Bbbk$-affine domain. Moreover, R is a homologically homogeneous and GK-Cohen–Macaulay domain, with*

$$\operatorname{gldim} R = \operatorname{GKdim} R = n.$$

Proof. (1) Suppose that R satisfies a PI. Then so also does $R_{(i)} = R_{(i-1)}[X_i;\sigma_i,\delta_i]$ for $i = 2,\ldots,n$. For each such i, since $R_{(i-1)}$ is a domain satisfying a PI, [27, Theorem 2.7] then requires that $|\sigma_i|_{Z(R_{(i-1)})}| < \infty$, as claimed.

Conversely, assume that each $\sigma_i|_{Z(R_{(i-1)})}$ has finite order. By induction on n we may assume that $A := R_{(n-1)}$ satisfies a PI, and that part (2) of the theorem has been proved for A. In particular, A is a finitely generated $Z(A)$-module. Since A is an iterated Ore extension of a field, it is a domain. By our induction hypothesis and since $\sigma_n|_{Z(A)}$ has finite order, hypotheses (3) (a)–(d) of Lemma 3.2 are satisfied. By [32, Proposition V.2.5], A is a maximal order, so (e) of Lemma 3.2 (4) holds. Therefore, by Lemma 3.2 (4), R is a finitely generated $Z(R)$-module and $Z(R)$ is an affine normal domain.

(2) The statements in the second sentence have already been proved in (1). The properties listed in the last sentence are also proved by induction on n. By the induction hypothesis, $\operatorname{gldim} A = n-1$, so $\operatorname{gldim} R$ is $n-1$ or n by [33, Theorem 7.5.3 (i)]. Thanks to [33, Corollary 7.1.14], there is a simple (left) A-module V with $\operatorname{projdim} V = n-1$. Choose a maximal left ideal I of A with $V \cong A/I$. Let W be a simple R-module which is a factor of the nonzero cyclic R-module $R/RI \cong R \otimes_A V$, so $W \cong R/L$ for a left ideal L of R containing RI. Since R is a noetherian affine $\Bbbk$-algebra satisfying a PI, by

part (1), $\dim_{\Bbbk} W < \infty$ by Kaplansky's theorem [33, Theorem 13.10.3 (i)]. In particular, W is a finitely generated A-module, and $L \cap A = I$ by maximality of I. Thus V is an A-submodule of W. Hence, choosing an A-module X such that $\mathrm{Ext}_A^{n-1}(V, X) \neq 0$, the long exact sequence of Ext yields

$$\mathrm{Ext}_A^{n-1}(W, X) \to \mathrm{Ext}_A^{n-1}(V, X) \to 0,$$

so that $\mathrm{projdim}_A W = n - 1$. Therefore, $\mathrm{gldim}\, R = n$ by [33, Corollary 7.9.18].

Since A is homologically homogeneous, so is R, by [59, Theorem 2.3], noting that the hypotheses of Yi's theorem are satisfied in view of Lemma 3.2 (1), (2), which guarantee that A is a finite module over a central subalgebra which is (σ_n, δ_n)-trivial. Finally R, being a homologically homogeneous algebra which is a finite module over an affine central domain, is GK-Cohen–Macaulay by [6, Corollary 5.4 and Theorem 4.8]. ■

Theorem 1.1 is now an immediate consequence of the above theorem combined with Proposition 2.4 (2) as follows:

Corollary 3.4. *Suppose that $\Bbbk$ has positive characteristic p and let n be a non-negative integer. Then every n-step IHOE H of $\Bbbk$ is a finitely generated module over its center $Z(H)$ and $Z(H)$ is a normal $\Bbbk$-affine domain. Moreover, H is a homologically homogeneous and GK-Cohen–Macaulay algebra with*

$$\mathrm{gldim}\, H = \mathrm{GKdim}\, H = n.$$

3.2. First questions around Theorem 1.1

We briefly consider two possible improvements of Theorem 1.1.

Remarks 3.5. (1) As a large family of examples to which Corollary 3.4 applies, let $\mathfrak{g}$ be a finite dimensional completely solvable Lie algebra over a field $\Bbbk$ of positive characteristic p. (That is, $\mathfrak{g}$ has a full flag of ideals.) Thus the enveloping algebra $U(\mathfrak{g})$ is an IHOE, so Corollary 3.4 tells us that $U(\mathfrak{g})$ is a finite module over its center. This is a special case of a fundamental result due to Jacobson [20]:

> *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$. Then $U(\mathfrak{g})$ contains a central Hopf subalgebra over which $U(\mathfrak{g})$ is a free module of finite rank equal to a power of p.*

Note that Jacobson's theorem does not require $\mathfrak{g}$ to be restricted. In language not then available to Jacobson, his result implies that the PI-degree of $U(\mathfrak{g})$ is a power of p, prefiguring Theorem 4.3 below. Moreover, the Lie algebra case thus suggests a possible strengthened version of Corollary 3.4:

Question 3.6. Is every IHOE over a field of positive characteristic a finite module over a central Hopf subalgebra?

(2) Since the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ is not an IHOE when $\mathfrak{g}$ contains a non-abelian (classical) simple subalgebra[1] other than $\mathfrak{sl}(2, \Bbbk)$, one might wonder, in the light of Jacobson's theorem discussed in (1), whether *every* connected affine Hopf algebra of finite Gel'fand–Kirillov dimension is finite over its center, or even finite over a central Hopf subalgebra, when $\Bbbk$ has positive characteristic. As a first test case for this, consider an affine connected graded Hopf algebra H, meaning that

$$H = \bigoplus_{i \geq 0} H_i \tag{3.6.1}$$

is connected graded as both an algebra *and* as a coalgebra. Two recent papers, [4, 60], study such Hopf algebras. (In [4] the possibly weaker hypothesis that H is only connected graded *as an algebra* is also treated, but we do not discuss that here.) Theorem A of [60] states that if $\Bbbk$ has characteristic 0 and H is an affine connected graded Hopf $\Bbbk$-algebra, then H is an n-step IHOE, where n is the Gel'fand–Kirillov dimension of H. The hypothesis on $\Bbbk$ is necessary here—consider $H = \Bbbk[x]/\langle x^p \rangle$ where $\Bbbk$ has characteristic p. Nevertheless, Corollary 3.4 and [60, Theorem A] prompt the following question.

Question 3.7. Suppose that H is an affine connected graded Hopf $\Bbbk$-algebra of finite Gel'fand–Kirillov dimension, where $\Bbbk$ has positive characteristic. Is $Z(H)$ affine, and is H a finite $Z(H)$-module?

4. The PI-degree and other invariants

4.1. PI-degree

Theorem 4.3, the main result of this subsection, gives information on the PI-degree of an n-step IHOE in characteristic p. Recall that the *PI-degree* of a prime PI ring R is denoted by PIdeg R and defined to be d, where d^2 is the dimension over the field $Z(Q(R))$ of the central simple quotient ring $Q(R)$; see [33, Sections 13.3.6 and 13.6.7] and [5, p. 115]. When R is in addition an affine $\Bbbk$-algebra with the field $\Bbbk$ algebraically closed, PIdeg R equals the maximum dimension over $\Bbbk$ of the simple R-modules [33, Theorem 13.10.3], [5, Theorem 1.13.5 (2)]. To prove Theorem 4.3, we need the following definitions and theorem, adapted from [11]. Note that the definitions in [11] are expressed in terms of the left Martindale quotient ring of a ring R, but when R is a prime PI ring, as here, this coincides with the total quotient ring $Q(R) = R[Z(R) \setminus \{0\}]^{-1}$, as follows easily from the definition of the Martindale quotient ring and Posner's theorem [33, Section 10.3.5 and Theorem 13.6.5].

Definition 4.1 ([11]). Let R be a prime algebra satisfying a PI, $\sigma \in \mathrm{Aut}_{\Bbbk\text{-alg}}(R)$, and δ a σ-derivation of R. Let Q be the total quotient ring of R.

[1] See [7, Example 3.1 (iv)], where the characteristic 0 hypothesis is not necessary.

(1) σ is called *X-inner* if its extension to Q is inner, that is, there exists a unit b of Q such that

$$\sigma(x) = b^{-1}xb$$

for all $x \in Q$. Otherwise, σ is called *X-outer*.

(2) σ is called *quasi-inner* if there exists an integer $n \geq 1$ such that σ^n is X-inner. The least such integer n is called the *outer degree* of σ and is denoted by $\operatorname{Outdeg}\sigma$.

(3) δ is called *X-inner* if its extension to Q is inner, that is, there exists $b \in Q$ such that

$$\delta(x) = bx - \sigma(x)b$$

for all $x \in Q$. Otherwise, δ is called *X-outer*.

(4) δ is called *quasi-algebraic* if there exist a positive integer n, an automorphism g of Q, and $b_1, \ldots, b_{n-1}, b \in Q$ such that for all $x \in R$,

$$\delta^n(x) + b_1\delta^{n-1}(x) + \cdots + b_{n-1}\delta(x) = bx - g(x)b.$$

The least such integer n is called the *quasi-algebraic degree* or the *outer degree* of the σ-derivation δ and is denoted by $\operatorname{Outdeg}\delta$. In particular, $\operatorname{Outdeg}\delta = 1$ if and only if δ is X-inner.

The main result of Chuang–Lee [11] is the following.

Theorem 4.2 ([11, Theorem 2.5]). *Let R be a prime PI algebra, $\sigma \in \operatorname{Aut}_{\Bbbk\text{-alg}}(R)$, and δ a σ-derivation of R. Then $R[x;\sigma,\delta]$ is a PI ring if and only if δ is quasi-algebraic and σ is quasi-inner. In this case,*

(a) *if δ is X-outer, then* $\operatorname{PIdeg} R[x;\sigma,\delta] = \operatorname{PIdeg} R \times \operatorname{Outdeg}\delta$*;*

(b) *if δ is X-inner, then* $\operatorname{PIdeg} R[x;\sigma,\delta] = \operatorname{PIdeg} R \times \operatorname{Outdeg}\sigma$.

We can now deduce our second main theorem from Corollary 3.4 and Theorem 4.2:

Theorem 4.3. *Suppose that $\Bbbk$ is a field of characteristic $p > 0$, and let H be an IHOE over $\Bbbk$. Then* $\operatorname{PIdeg} H$ *is a power of p.*

Proof. We argue by induction on n, where H is an n-step IHOE. When $n = 1$, $H = \Bbbk[x]$ and the result is clear. For the induction step, write B for $H_{(n-1)}$, so, by the induction hypothesis,

$$\operatorname{PIdeg} B = p^t, \tag{4.3.1}$$

for some $t \geq 0$. After relabeling, $H = B[x;\sigma,\delta]$ for $\sigma \in \operatorname{Aut}_{\Bbbk\text{-alg}}(B)$ and a σ-derivation δ of B. By Proposition 2.2 (1) and Corollary 3.4, B and H are both prime PI-algebras. Thus, by Theorem 4.2, σ is quasi-inner and δ is quasi-algebraic, and there are two cases to consider.

Case (a). δ is X-outer. Then, since B is a prime PI ring, a result of Kharchenko, see [10, Lemmas 5 and 6], guarantees the existence of a unit u of the quotient ring $Q(B)$ and an X-outer derivation d of $Q(B)$ such that, for all $b \in B$,

$$\sigma(b) = ubu^{-1} \quad \text{and} \quad \delta = ud.$$

Note that σ and δ extend, respectively, to an automorphism and a σ-derivation of $Q(B)$. Then, by [11, Lemma 1.3 (2)],

$$Q(B)[x;\sigma,\delta] \cong Q(B)[x;d]. \tag{4.3.2}$$

Now

$$\text{PIdeg}\, B = \text{PIdeg}\, Q(B) \tag{4.3.3}$$

and

$$\text{PIdeg}\, H = \text{PIdeg}\, Q(B)[x;\sigma,\delta]. \tag{4.3.4}$$

To prove the induction step in Case (a) it is therefore enough, by (4.3.1), (4.3.2), (4.3.3), and (4.3.4), to prove that

$$\text{PIdeg}\, Q(B)[x;d] = \text{PIdeg}\, B \times p^{\ell} \tag{4.3.5}$$

for some $\ell \geq 0$. But (4.3.5) follows immediately from Theorem 4.2 (a) and a theorem of Kharchenko, stated and proved as [10, Theorem, p. 60].

Case (b). δ is X-inner. Now Theorem 4.2 (b) applies. But clearly, from the definition,

$$\text{Outdeg}\, \sigma \mid |\sigma|.$$

Moreover, by [17, Theorem 1.3 (ii)] and Proposition 2.4 (2),

$$|\sigma| \mid p^n.$$

Therefore, (4.3.1) is proved in this case also, and so the theorem follows. ∎

Remarks 4.4. (1) A second proof of Theorem 4.3 will be given Section 5, where we obtain it as a corollary of Theorem 5.3 together with a result of Etingof [13] on filtered deformations of commutative domains in positive characteristic.

(2) Faced with Theorem 4.3, it is natural to ask the following question:

Question 4.5. What is the power of p occurring in Theorem 4.3?

This is a delicate matter, as can be seen from the case of enveloping algebras. Assume that $\Bbbk$ is algebraically closed of characteristic $p > 0$, and let $\mathfrak{g}$ be a finite dimensional Lie $\Bbbk$-algebra. For $f \in \mathfrak{g}^*$ define

$$\text{stab}_{\mathfrak{g}}(f) := \{x \in \mathfrak{g} \mid f([x,-]) = 0\},$$

and set

$$\text{ind}\, \mathfrak{g} := \min\{\dim_{\Bbbk} \text{stab}_{\mathfrak{g}}(f) \mid f \in \mathfrak{g}^*\}.$$

The first Kac–Weisfeiler conjecture [57] proposed that when $\mathfrak{g}$ is restricted,

$$\operatorname{PIdeg} U(\mathfrak{g}) = p^{\frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})}. \tag{4.5.1}$$

This was already known, due to Rudakov [43], when $\mathfrak{g}$ is the Lie algebra of a reductive group (in this case $\operatorname{ind}\mathfrak{g}$ is the rank of $\mathfrak{g}$); and Strade [48] proved the conjecture for $\mathfrak{g}$ solvable in 1978, extending the completely solvable case done in [57]. Premet and Skryabin [41] confirmed the conjecture for all restricted $\mathfrak{g}$ admitting a toral stabilizer, and also showed that $p^{\frac{1}{2}(\dim \mathfrak{g} - \operatorname{ind} \mathfrak{g})}$ is a lower bound for the PI-degree for all restricted $\mathfrak{g}$. The general case, however, remains open, although it has recently been confirmed in [30] for all restricted subalgebras $\mathfrak{g}$ of $\mathfrak{gl}_n(\Bbbk)$, for $p \gg 0$. Note, however, that the conjecture is not true if the hypothesis that $\mathfrak{g}$ is restricted is omitted—Topley [50] exhibits pairs of Lie $\Bbbk$-algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, both solvable, with $\mathfrak{g}_1 \ncong \mathfrak{g}_2$ and indeed $\operatorname{ind}\mathfrak{g}_1 \neq \operatorname{ind}\mathfrak{g}_2$, but with $U(\mathfrak{g}_1) \cong U(\mathfrak{g}_2)$.

It might be easier to approach Question 4.5 by first seeking an upper bound:

Question 4.6. Is the PI-degree of an n-step IHOE over a field $\Bbbk$ of characteristic p bounded above by $p^{\frac{n}{2}}$?

We show in Section 9.2 that this is true when $n \leq 2$.

(3) As recalled at the start of Section 4.1, when $\Bbbk$ is algebraically closed the PI-degree is the maximum dimension of a simple H-module, so Question 4.5 leads one to the broader issue of the possible dimensions of all the simple H-modules, when H is an n-step IHOE over an algebraically closed field of characteristic p. One might hope that every simple module has dimension a power of p, and we confirm that this is the case for all 2-step IHOEs in Proposition 9.2. Further positive evidence is provided by the enveloping algebras of completely solvable Lie algebras. Such a Lie algebra $\mathfrak{g}$ of dimension n has a chain of $n+1$ ideals from $\{0\}$ to $\mathfrak{g}$, so that $U(\mathfrak{g})$ is an n-step IHOE. Kac and Weisfeler showed in [57] that the dimension of every simple $U(\mathfrak{g})$-module is a power of p. But the naive hope fails for $\mathfrak{sl}(2,\Bbbk)$ when $\Bbbk$ has characteristic $p > 2$, since $U(\mathfrak{sl}(2,\Bbbk))$ is a 3-step IHOE [7, Example 3.1(iv)], and Rudakov and Shafarevitch showed [44] that the dimensions of the simple $U(\mathfrak{sl}(2,\Bbbk))$-modules are $1,\dots,p$. These examples and results suggest the following possibility:

Question 4.7. Suppose that $\Bbbk$ is an algebraically closed field of characteristic $p > 0$. Let H be an n-step IHOE over $\Bbbk$, which has a chain of Hopf subalgebras as in (2.1.1) with each σ_i being the identity. Is the dimension of every simple H-module a power of p?

The above question has a positive answer for all 2-step IHOEs—see Proposition 9.2.

4.2. The antipode and Nakayama automorphism

The antipode S of a Hopf algebra that is a finite module over its center has finite order up to inner automorphisms by [8, Corollary 0.6 (c)]. In the case of IHOEs in positive characteristic, we can easily obtain information about the order, as we show below. The

Nakayama automorphism ν of an AS-Gorenstein Hopf algebra provides the twist in its rigid dualizing complex; see [8, Sections 0.2 and 0.3] for details. It is defined up to an inner automorphism. The *Nakayama order*, that is the lowest power ℓ such that ν^ℓ is inner, is finite in the current setting thanks to [8, Corollary 0.6 (c)]. Again, more can be said about its precise value.

Theorem 4.8. *Let $n \geq 1$ and let H be an n-step IHOE over a field $\Bbbk$ of characteristic $p > 0$.*

(1) *The order of the antipode S divides $4p^{n-1}$.*

(2) *The Nakayama order of H divides $2p^n$.*

Proof. Note that every invertible element in H is in $\Bbbk$, so the Nakayama order of H coincides with the (well-defined) order of the Nakayama automorphism ν of H.

(1) The proof of part (1) is similar to the proof of Proposition 2.4 (2). We use induction on n to show that the order of S^4 divides p^{n-1}. The assertion holds trivially for $n = 1$. Now assume that the assertion holds for $(n-1)$-step IHOEs. Let H be an n-step IHOE and write it as $B[x; \sigma, \delta]$, where B is an $(n-1)$-step IHOE. Suppose that the order of S^4 restricted to B is p^t for some $t \leq n-2$.

Let $\eta : H \to \Bbbk$ be the right character of the left homological integral of H [29]. Let Ξ^l_η (respectively, Ξ^r_η) be the corresponding left (respectively, right) winding automorphism of H. By [8, Theorem 0.6] and the fact that H has no non-trivial inner automorphism, $S^4 = \Xi^l_\eta \circ (\Xi^r_\eta)^{-1}$. Since B is a Hopf subalgebra of H and $\eta|_B$ is a character of B, then Ξ^l_η (respectively, Ξ^r_η) restricted to B, still denoted by Ξ^l_η (respectively, Ξ^r_η), is a left (respectively, right) winding automorphism of B. By the induction hypothesis, $(S^4)^{p^t}$ is the identity of B. It remains to show that $(S^4)^{p^{t+1}}$ is the identity when applied to x. By Proposition 2.2 (2), we may assume that (2.4.1) holds, so that

$$\Xi^l_\eta(x) = x + \eta(x) + \mu \circ (\eta \otimes 1)(w) = x + s,$$

where $s = \eta(x) + \mu \circ (\eta \otimes 1)(w) \in B$. Similarly,

$$(\Xi^r_\eta)^{-1}(x) = x + s'$$

for some $s' \in B$. These imply that

$$(S^4)^{p^t}(x) = \left(\Xi^l_\eta (\Xi^r_\eta)^{-1}\right)^{p^t}(x) = x + s''$$

for some $s'' \in B$. Note that $(S^4)^{p^t}(s'') = s''$ as the order of S^4 restricted to B is p^t. Then

$$(S^4)^{p^{t+1}}(x) = \left((S^4)^{p^t}\right)^p(x) = x + ps'' = x.$$

Therefore, $(S^4)^{p^{t+1}}$ is the identity map on H as required.

(2) By [8, Theorem 0.3], the Nakayama automorphism of H is of the form $S^2 \circ \Xi^r_\eta = \Xi^r_\eta \circ S^2$. The order of S^2 divides $2p^{n-1}$ by part (1) and the order of Ξ^r_η divides p^n by Proposition 2.4 (2). Then the assertion follows. ∎

Remark 4.9. Recall that η is the right character of the left integral of H. With a slightly more careful analysis along the above lines, combined also with Proposition 9.1, one can show that the order of S divides $\max\{1, 4p^{n-t}\}$, where

$$t := \max\{2, n_0\} \quad \text{and} \quad n_0 := \#\big\{i \mid \Xi^l_\eta(w_i) = \Xi^r_\eta(w_i)\big\},$$

where w_i is as in (2.2.1).

5. Associated graded algebra of an IHOE

The main result in this section is Theorem 5.3. In this section, we do not assume that $\operatorname{char} \Bbbk > 0$ though we keep using $\Bbbk$ as the base field.

5.1. Construction of a filtration

Recall that, by [61, Theorem 6.9], if H is an affine connected Hopf algebra of finite Gel'fand–Kirillov dimension n over an algebraically closed field $\Bbbk$ of characteristic 0, then the associated graded algebra of H with respect to the coradical filtration is a commutative polynomial algebra in n variables. This result does not extend to positive characteristic—in the first place there are non-trivial finite dimensional connected Hopf algebras in characteristic p, for example $\Bbbk[x]/\langle x^p\rangle$; and, second, the coradical filtration is typically much coarser in this setting. Thus, for instance, there is often an infinite dimensional space of primitive elements, as in $\Bbbk[x]$ for example. At least for IHOEs there is nevertheless an analogue of Zhuang's result, provided one uses a more suitable filtration. We begin by describing such a filtration and proving its required properties.

Let H be an n-step IHOE with notation as in Definition 2.1. The ordered monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $m_i \in \mathbb{Z}_{\geq 0}$ constitute a $\Bbbk$-basis $\mathcal{B}(H)$ of H, which we call its *PBW basis*. (This holds true for all iterated Ore extensions, not just for Hopf algebras.) For $\alpha \in H$, call the presentation of α in terms of the PBW basis the *PBW expression of* α, and let

$$\operatorname{PBWsupp}(\alpha) := \big\{\mathbf{m} \in \mathcal{B}(H) \mid \mathbf{m} \text{ occurs in the PBW expression for } \alpha\big\}.$$

We proceed now to define a degree function $\deg : H \to \mathbb{Z}_{\geq 0}$. Recall from Proposition 2.2 (4) the elements $a_{ji} \in H_{(j-1)}$ for $i, j = 1, \ldots, n$ with $i > j$, such that

$$\sigma_i(x_j) = x_j + a_{ji},$$

where a_{ji} is given as in (2.2.2). Moreover, we can define elements $c_{ji} \in H_{(i-1)}$ such that, for all $i, j = 1, \ldots, n$ with $i > j$,

$$\delta_i(x_j) = c_{ji}.$$

Thus the defining relations of H are

$$x_i x_j = x_j x_i + a_{ji} x_i + c_{ji}, \quad (1 \leq j < i \leq n). \tag{5.0.1}$$

Recall that w_i is defined in (2.2.1); fix an expression

$$w_i = \sum w_{i(1)} \otimes w_{i(2)},$$

where $w_{i(1)}, w_{i(2)}$ are in $H_{(i-1)} \cap \ker \varepsilon$. For each $\ell > i$, let χ_ℓ be the character of $H_{(\ell-1)}$ such that $\sigma_\ell = \tau^r_{\chi_\ell}$ as given in Proposition 2.2 (4). Then by (2.2.2),

$$a_{i\ell} = \chi_\ell(x_i) + \sum w_{i(1)} \chi_\ell(w_{i(2)}) \in H_{(i-1)}.$$

Define the degree $\deg(\lambda) := 0$ for $\lambda \in \Bbbk$, and inductively define the degree $\deg(x_i) := d_i \in \mathbb{Z}_{>0}$ and $\deg(\alpha)$ for $\alpha \in H_{(i)}$, as follows:

- $d_1 = 1$;
- suppose that $i > 1$ and that $d_1, \ldots, d_{i-1}$ have been defined. For each PBW basis monomial $\mathbf{m} = x_1^{m_1} x_2^{m_2} \cdots x_{i-1}^{m_{i-1}}$, set

$$\deg(\mathbf{m}) := \sum_{j=1}^{i-1} m_j d_j; \tag{5.0.2}$$

- for $\alpha \in H_{(i-1)}$ define

$$\deg(\alpha) := \max \left\{ \deg(\mathbf{m}) \mid \mathbf{m} \in \mathrm{PBWsupp}(\alpha) \right\}; \tag{5.0.3}$$

- since deg is defined on $H_{(i-1)}$, let

$$D(w_i) = \max_{i(1)} \left\{ \deg(w_{i(1)}) \right\}$$

  which is only dependent on i. Then we have

$$D(w_i) \geq \max_{\ell > i} \left\{ \deg(a_{i\ell}) \right\}.$$

  Now set $d_i \in \mathbb{Z}_{>0}$ such that

$$d_i > D(w_i) \quad \left(\geq \max_{\ell > i} \left\{ \deg(a_{i\ell}) \right\} \right), \tag{5.0.4}$$

  and such that

$$d_i \geq \max_{j < i} \left\{ \deg(c_{ji}) \right\}. \tag{5.0.5}$$

Finally, after the above steps are completed up to $i = n$, extend (5.0.2) (resp. (5.0.3)) to *all* PBW basis monomials (resp. every element) of H. Clearly, this procedure yields a well-defined degree for all $\alpha \in H$. We define a filtration $\mathscr{C} := \{\mathscr{C}_i\}_{i \geq 0}$ by setting, for all $i \geq 0$,

$$\mathscr{C}_i := \left\{ \alpha \in H \mid \deg(\alpha) \leq i \right\}.$$

It is clear that $\mathscr{C}_0 = \Bbbk$.

Lemma 5.1. *Keep the above notation.*

(1) *For* $1 \leq j < i \leq n$*,*

$$x_i x_j = x_j x_i + (\textit{lower degree terms}).$$

(2) *For* $1 \leq j < i \leq n$*,*

$$\deg(x_i x_j) = \deg(x_j x_i).$$

(3) *For all monomials* $\mathbf{m} = x_{i_1}^{t_1} x_{i_2}^{t_2} \cdots x_{i_m}^{t_m}$*, where* $i_\ell \in \{1, \ldots, n\}$ *and* $t_\ell \in \mathbb{Z}_{\geq 0}$ *for all* $\ell = 1, \ldots, m$*,*

$$\deg(\mathbf{m}) = \sum_{\ell=1}^{m} t_\ell d_{i_\ell}. \tag{5.1.1}$$

Proof. (1) Let $j < i$. In the relation (5.0.1) $a_{ji} \in H_{(j-1)}$ and $c_{ji} \in H_{(i-1)}$. Thus

$$\deg(a_{ji} x_i + c_{ji}) = \max\left\{\deg(a_{ji} x_i), \deg(c_{ji})\right\} < d_j + d_i = \deg(x_j x_i),$$

where the first equality follows from (5.0.3) and the fact that there is empty intersection between the PBW expressions for $a_{ji} x_i$ and c_{ji}, so that no cancellation can occur, and the inequality follows from (5.0.4) and (5.0.5), since $a_{ji} x_i$ is already in PBW form.

(2) This follows from part (1) and the definition of the degree of elements of H.

(3) Define the *weight* of a not necessarily PBW-ordered monomial $\mathbf{m} = x_{i_1}^{t_1} \cdots x_{i_m}^{t_m}$ to be

$$\mathrm{wt}(\mathbf{m}) := \sum_{\ell=1}^{m} t_\ell d_{i_\ell}.$$

Suppose that assertion (3) is false, and choose a counterexample $\mathbf{m}$ of minimal weight, say $\mathrm{wt}(\mathbf{m}) = d > 0$. Then $\mathbf{m}$ cannot be an ordered monomial by the definition (5.0.2). So amongst counterexamples to assertion (3) of weight d, choose $\mathbf{m}$ to be one with the minimal number ω of *badly ordered pairs*—that is, pairs of generators occurring in $\mathbf{m}$ as $\mathbf{m} = \cdots x_r \cdots x_s \cdots$ with $r > s$. Clearly, there must be an adjacent bad pair in $\mathbf{m}$. That is, there exist s, r with $1 \leq s < r \leq n$, with

$$\begin{aligned}
\mathbf{m} &= x_{i_1} \cdots x_r x_s \cdots x_{i_m} \\
&= x_{i_1} \cdots (x_s x_r + a_{sr} x_r + c_{sr}) \cdots x_{i_m} \\
&= (x_{i_1} \cdots x_s x_r \cdots x_{i_m}) + (x_{i_1} \cdots a_{sr} x_r \cdots x_{i_m}) + (x_{i_1} \cdots c_{sr} \cdots x_{i_m}).
\end{aligned}$$

Here, the first monomial on the right side has fewer than ω badly ordered pairs, so assertion (3) is true for it by choice of $\mathbf{m}$. The second and third brackets on the right consist of monomials of weight strictly less than d, by (5.0.4) and (5.0.5). So, again by choice of $\mathbf{m}$, assertion (3) holds for all the monomials in the second and third brackets on the right; in particular, their degree is strictly less than the degree of $x_{i_1} \cdots x_s x_r \cdots x_{i_m}$. Thus

$$\deg(\mathbf{m}) = \deg(x_{i_1} \cdots x_s x_r \cdots x_{i_m}) = \sum_{\ell=1}^{m} t_\ell d_{i_\ell},$$

and $\mathbf{m}$ is not a counterexample. This proves (3). ∎

Lemma 5.2. *Continue with the above notation.*

(1) *The filtration $\mathscr{C}$ defined before Lemma 5.1 is an algebra filtration of H.*

(2) *The associated graded algebra $\operatorname{gr}_{\mathscr{C}} H$ is a factor ring of the commutative polynomial ring $\Bbbk[\bar{x}_1, \dots, \bar{x}_n]$ where $\bar{x}_i$ is the principal symbol of x_i.*

Proof. (1) We need to prove that, for all $r, s \in \mathbb{Z}_{\geq 0}$,

$$\mathscr{C}_r \mathscr{C}_s \subseteq \mathscr{C}_{r+s}.$$

For this it is enough to show that if $\mathbf{r}$, $\mathbf{s}$ are ordered monomials in $\mathscr{C}_r$ and $\mathscr{C}_s$, respectively, then $\deg(\mathbf{rs}) \leq r + s$. This is immediate from Lemma 5.1 (3).

(2) It is clear from the definitions that $0 \neq \bar{x}_i \in \operatorname{gr}_{\mathscr{C}} H$ for all $i = 1, \dots, n$. From this and (5.1.1), one sees immediately that $\operatorname{gr}_{\mathscr{C}} H$ is generated by $\{\bar{x}_i\}_{i=1}^n$.

Commutativity of $\operatorname{gr}_{\mathscr{C}} H$ follows from Lemma 5.1 (1). Therefore, there is an algebra epimorphism ϕ from the commutative polynomial ring $\Bbbk[\bar{x}_1, \dots, \bar{x}_n]$ to $\operatorname{gr}_{\mathscr{C}} H$. ■

5.2. The associated graded algebra

We can now prove the main result of this section.

Theorem 5.3. *Let $\Bbbk$ be an algebraically closed field, and let H be an n-step IHOE over $\Bbbk$. Then positive integer degrees can be assigned to the defining variables of H so that the corresponding filtration, constructed from this assignment by using the PBW basis as above, has associated graded algebra which is the commutative polynomial $\Bbbk$-algebra on n variables.*

Proof. Let H be given by Definition 2.1 and define its filtration $\mathscr{C}$ by the recipe given in Section 5.1. By Lemma 5.2, $\operatorname{gr}_{\mathscr{C}} H$ is a factor of the polynomial $\Bbbk$-algebra on n generators. On the other hand, $\operatorname{GKdim}(H) = n$ by Corollary 3.4. Since $\mathscr{C}$ is discrete and finite, it follows from [25, Proposition 6.6] that $\operatorname{gr}_{\mathscr{C}} H$ grows at the same rate; that is, $\operatorname{GKdim}(\operatorname{gr}_{\mathscr{C}} H) = n$. Since proper factors of the polynomial algebra on n generators have GKdimension strictly less than n, the result follows. ■

Remarks 5.4. (1) An alternative proof that $\operatorname{gr}_{\mathscr{C}} H$ is a polynomial algebra uses the PBW basis directly. Namely, by the definition of $\mathscr{C}$, H and $\operatorname{gr}_{\mathscr{C}} H$ have the same PBW basis, which agrees with the PBW basis of $\Bbbk[\bar{x}_1, \dots, \bar{x}_n]$. Therefore, the map ϕ from the proof of Lemma 5.2 (2) is an isomorphism.

(2) A theorem of Etingof, [13, Corollary 3.2 (ii)], states that if $\Bbbk$ is a field of characteristic $p > 0$ and A is any $\Bbbk$-algebra with filtration $\mathscr{C} = \{\mathscr{C}_i : i \geq 0\}$ such that A satisfies a PI and $\operatorname{gr}_{\mathscr{C}} A$ is a commutative domain, then the PI-degree of A is a power of p. In view of Corollary 3.4 and Theorem 5.3, these hypotheses both are satisfied by any IHOE over the field $\Bbbk$, yielding a second proof of Theorem 4.3.

(3) Etingof asks in [13, Question 1.1] whether every filtered deformation of an affine commutative domain in positive characteristic has to satisfy a PI. Theorem 5.3 coupled with Corollary 3.4 provide some evidence in favour of a positive answer.

(4) Theorem 5.3 can be restated as follows: every n-step IHOE over the field $\Bbbk$ is a PBW-deformation of the polynomial ring in n variables over $\Bbbk$ in the sense of [3, Section 3]. It follows from Zhuang's theorem [61, Theorem 6.9] that a connected Hopf algebra over a field $\Bbbk$ of characteristic zero is a PBW-deformation of the polynomial ring in n variables if $n = \operatorname{GKdim} H < \infty$. Given these facts, together with the classical PBW theorem for enveloping algebras of finite dimensional Lie algebras, and the fact that the underlying variety of every connected unipotent group of dimension n is affine n-space over $\Bbbk$, [45, Chapter VII, Section 6, Corollary and Remark 1, p. 170], it is natural to ask the following question:

Question 5.5. Is every connected Hopf $\Bbbk$-algebra domain of finite Gel'fand–Kirillov dimension a PBW-deformation of a polynomial algebra over $\Bbbk$?

(5) An ∞-step IHOE is defined to be

$$\lim_{n\to\infty} H_n$$

if there is an infinite sequence of n-step IHOEs

$$\Bbbk = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots$$

such that each H_n satisfies the conditions in Definition 2.1 (2). If $\{d_i\}_{i\geq 1}$ is a strictly increasing sequence of integers defined by using the process given before Lemma 5.1, then we can define a locally finite filtration $\mathcal{C}$ of H such that the associated graded algebra $\operatorname{gr}_{\mathcal{C}} H$ is isomorphic to $\Bbbk[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \dots]$—the polynomial ring of infinitely many variables.

(6) Let χ be a character of H. Then the right winding automorphism τ^r_χ of H preserves the filtration $\mathcal{C}$ constructed in Section 5.1. It follows that the winding automorphism τ^r_χ induces an automorphism of $\operatorname{gr}_{\mathcal{C}} H$, which one can show to be the identity map.

6. Classification of 1- and 2-step IHOEs in positive characteristic

For the rest of the paper, we would like to take the first steps in a project to classify the Hopf algebra domains of Gel'fand–Kirillov dimension at most two in positive characteristic. In characteristic 0 considerable progress has already been made towards this classification, as we shall briefly recall at the start of Section 6.2. First, we deal with the well-known case of Gel'fand–Kirillov dimension one.

6.1. Hopf domains of GKdimension one

Recall that the only connected algebraic groups of dimension 1 over an algebraically closed field $\Bbbk$ are the additive and multiplicative groups of $\Bbbk$ [18, Theorem 20.5]. It is easy to remove the hypothesis of commutativity from this result, as follows. Let $\Bbbk^\times$ denote $\Bbbk \setminus \{0\}$.

Lemma 6.1. *Let $\Bbbk$ be any algebraically closed field. The only affine Hopf $\Bbbk$-algebra domains of Gel'fand–Kirillov dimension one are the coordinate rings $\Bbbk[X]$ and $\Bbbk[X^{\pm 1}]$ of $(\Bbbk, +)$ and $(\Bbbk^{\times}, \times)$, respectively. In particular, the only 1-step IHOE over $\Bbbk$ is $\Bbbk[X]$ with X primitive.*

Proof. By [29, Corollary 7.8 (a)⇒(b)], H is commutative. Therefore, H is the coordinate ring of an affine connected algebraic group over $\Bbbk$, and the result follows from [18, Theorem 20.5]. ■

By [14, Proposition 2.1], in characteristic 0 the hypothesis that the Hopf algebra is affine can be removed from Lemma 6.1, at the cost of adding the group algebras of non-cyclic subgroups of $\mathbb{Q}$ to the list. But the proof of [14, Proposition 2.1] does not work in positive characteristic. We refer to a recent survey paper [9] for some facts about the prime Hopf algebras of GKdimension one and two.

6.2. Hopf domains of GKdimension two

When $\Bbbk$ has characteristic 0, the classification of affine Hopf $\Bbbk$-algebra domains H of Gel'fand–Kirillov dimension 2 was achieved in [14, 15] with the imposition of the extra hypothesis that $\mathrm{Ext}^1_H(\Bbbk, \Bbbk) \neq 0$ (or equivalently, that the quantum group H contains a non-trivial classical subgroup). Then, in [51], a family of dimension two $\Bbbk$-affine Hopf PI domains is constructed which fail to satisfy the extra hypothesis, but it is not yet known whether, with the addition of this family, the list is complete. Staying with $\Bbbk$ of characteristic 0, if one restricts attention to *connected* Hopf $\Bbbk$-algebras of finite Gel'fand–Kirillov dimension, then they are all affine by [61, Theorem 6.9], and the classification is complete up dimension at most four [52, 61]. In all these classification results in characteristic 0, the outcome takes a similar form—namely, there is a finite number of families in each list, with each family being given by a finite set of discrete or continuously varying parameters.

Turning now to positive characteristic, and restricting attention to 2-step IHOEs, we find that even in this very confined setting the situation is completely different from that pertaining in characteristic 0. This is shown by the main result of this subsection, Proposition 6.6, which lists all the two-step IHOEs in characteristic p. Together with Proposition 6.11 in Section 6.3, which describes the isomorphisms and automorphisms between these algebras, this classifies 2-step IHOEs in positive characteristic. It transpires that their description entails an infinite dimensional space of parameters. The contrast with characteristic 0 could not be starker—when $\Bbbk$ has characteristic 0, a result of Zhuang [61, Proposition 7.4 (III)] shows that there are only two connected Hopf $\Bbbk$-algebras of Gel'fand–Kirillov dimension 2, namely the enveloping algebras of the two 2-dimensional Lie algebras, both of which are obviously IHOEs. Conversely, by [7, Theorem 1.3], every IHOE is connected, so that [61, Proposition 7.4 (III)] provides a list of the (two) 2-step IHOEs in characteristic 0.

To understand 2-step IHOEs $\Bbbk[X_1][X_2; \sigma, \delta]$ in positive characteristic, we employ the primitive cohomology studied in [53], from which we first recall some definitions. Let (C, Δ) be a coalgebra with a fixed grouplike element 1_C. Let $T(C)$ be the tensor algebra over the vector space C in cohomological degree 1 with differential ∂ determined by

$$\partial(x) = -1_C \otimes x + \Delta(x) - x \otimes 1_C \in C \otimes C \tag{6.1.1}$$

for all $x \in C$. Then ∂ can uniquely be extended to a derivation of $T(C)$ such that $(T(C), \partial)$ is a differential graded algebra. Let $B^i(C)$ be the image of $\partial^{n-1} : C^{\otimes(n-1)} \to C^{\otimes n}$ and let $Z^n(C)$ be the kernel of $\partial^n : C^{\otimes n} \to C^{\otimes(n+1)}$. By [53, Definition 1.2 (1)], the nth primitive cohomology of C (associated to $g = h = 1_C$) is defined to be

$$\mathfrak{P}^n_{1_C,1_C}(C) = H^n\big(T(C), \partial\big) = \ker \partial^n / \operatorname{im} \partial^{n-1} = Z^n(C)/B^n(C).$$

If C is $\mathbb{N}$-graded locally finite, then $T(C)$ is $\mathbb{Z}^2$-graded and locally finite. As a consequence, each $\mathfrak{P}^n_{1_C,1_C}(C)$ is $\mathbb{N}$-graded and locally finite. We will use the following lemma in the computation of primitive cohomology.

Lemma 6.2. *Let C be a connected $\mathbb{N}$-graded coalgebra and let A be the graded dual algebra of C, namely, $A_i = \operatorname{Hom}_{\Bbbk}(C_i, \Bbbk)$ for all $i \geq 0$. Then A is a connected graded algebra with trivial graded module $\Bbbk$ and*

$$\dim \mathfrak{P}^n_{1_C,1_C}(C)_i = \dim \operatorname{Ext}^n_A(\Bbbk, \Bbbk)_i$$

for all n and i.

Proof. After we identify $\dim \operatorname{Ext}^n_A(\Bbbk, \Bbbk)_i$ with $\dim \operatorname{Tor}^A_n(\Bbbk, \Bbbk)_{-i}$ for all i, the assertion is a consequence of [53, Lemma 3.6 (2)]. ■

For the rest of this section, we assume that $\Bbbk$ has positive characteristic p. We are now ready to compute the primitive cohomology of the coalgebra $\Bbbk[X_1]$ from Lemma 6.1. The divided power Hopf algebra (of one variable) was introduced in [49]; also see [12, Example 5 in Section 4.3]. By definition, the divided power Hopf algebra $\mathcal{T}$ is a $\Bbbk$-vector space with basis $\{t_n\}_{n \geq 0}$ and with its bialgebra structure determined by

$$\Delta(t_n) = \sum_{i=0}^{n} t_i \otimes t_{n-i}, \quad \varepsilon(t_n) = \delta_{0,n}, \quad \text{and} \quad t_n t_m = \binom{n+m}{n} t_{n+m} \tag{6.2.1}$$

for all $n, m \geq 0$. By comparing the structure coefficients of the multiplications and comultiplications of $\mathcal{T}$ and $\Bbbk[X_1]$ (as in Lemma 6.1) respectively, one can easily see that $\mathcal{T}$ is the graded $\Bbbk$-linear dual of $\Bbbk[X_1]$, giving part (1) of Proposition 6.4.

Convention 6.3. Since we are using various different algebra and/or coalgebra structures on the same or similar spaces, it is convenient to fix some notation.

(1) Let A denote the algebra $\mathcal{T}$ obtained by forgetting the coalgebra structure; namely, $A = \bigoplus_{d \geq 0} \Bbbk t_d$ is the divided power algebra of one variable with multiplication determined by $t_d t_e = \binom{d+e}{d} t_{d+e}$ for all $d, e \geq 0$.

(2) Let C be the graded coalgebra $\Bbbk[X_1]$ given in Lemma 6.1 by forgetting its algebra structure.

Proposition 6.4. *Retain the above notation.*

(1) *A is isomorphic to the graded dual algebra of the coalgebra C.*

(2) *As a graded algebra A is generated by $\{t_{p^s} \mid s \geq 0\}$ subject to the relations*

$$(t_{p^s})^p = 0 \quad \text{and} \quad t_{p^s} t_{p^t} = t_{p^t} t_{p^s}$$

for all $s < t$. As a consequence,

$$\dim \operatorname{Ext}^1_A(\Bbbk, \Bbbk)_i = \begin{cases} 1 & i = p^s \quad \text{for all } s \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\dim \operatorname{Ext}^2_A(\Bbbk, \Bbbk)_i = \begin{cases} 1 & i = p^{s+1} \text{ for all } s \geq 0, \\ 1 & i = p^s + p^t \text{ for all } 0 \leq s < t, \\ 0 & \text{otherwise}. \end{cases}$$

(3) *Consider C as a graded coalgebra with 1_C being the identity of $\Bbbk[X_1]$. Then*

$$\dim \mathfrak{P}^1_{1_C,1_C}(C)_i = \begin{cases} 1 & i = p^s \text{ for all } s \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\dim \mathfrak{P}^2_{1_C,1_C}(C)_i = \begin{cases} 1 & i = p^{s+1} \text{ for all } s \geq 0, \\ 1 & i = p^s + p^t \text{ for all } 0 \leq s < t, \\ 0 & \text{otherwise}. \end{cases}$$

(4) *The following elements in $C \otimes C$ generate a $\Bbbk$-linear basis of $\mathfrak{P}^2_{1_C,1_C}(C)$:*

$$Z_s := \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (X_1^{p^s})^i \otimes (X_1^{p^s})^{p-i} \tag{6.4.1}$$

for each $s \geq 0$, and

$$Y_{s,t} := X_1^{p^s} \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1^{p^s} \tag{6.4.2}$$

for all $s < t$.

Proof. (2) The assertion follows from the fact that $\operatorname{Ext}^1_A(\Bbbk, \Bbbk)$ can be identified with a minimal set of generators and $\operatorname{Ext}^2_A(\Bbbk, \Bbbk)$ can be identified with a minimal set of relations.

(3) This follows from Lemma 6.2 and parts (1), (2).

(4) By part (3), it suffices to show that elements Z_s and $Y_{s,t}$ are in $Z^2(C)$ but not in $B^2(C)$, which can be verified by some straightforward computations. ■

In the next lemma, Z_s and $Y_{s,t}$ are as given in (6.4.1)–(6.4.2).

Lemma 6.5. *Let H be a 2-step IHOE generated by X_1 and X_2 as in Definition 2.1 (2); that is, $H = \Bbbk[X_1][X_2; \sigma, \delta]$. Retain the notation introduced in Proposition 6.4 (4). Then the following hold.*

(1) $\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1$ *and* $\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w$, *where*

$$w = \sum_{s \geq 0} b_s Z_s + \sum_{s<t} c_{s,t} Y_{s,t} \tag{6.5.1}$$

for some scalars $b_s, c_{s,t} \in \Bbbk$.

(2) $\sigma(X_1) = X_1 + \chi$, *where* $\chi \in \Bbbk$.

(3) δ *is a σ-derivation of* $\Bbbk[X_1]$ *such that*

$$\Delta\big(\delta(X_1)\big) = \delta(X_1) \otimes 1 + 1 \otimes \delta(X_1) - \chi w,$$

where w is given in (6.5.1).

(4) $\chi w = 0$.

(5) $\delta(X_1) = \sum_{s \geq 0} d_s X_1^{p^s}$ *for some* $d_s \in \Bbbk$.

Proof. Recall that C is the coalgebra $\Bbbk[X_1]$ and 1_C is the identity 1 in $\Bbbk[X_1]$.

(1) Let w be $\partial(X_2) := \Delta(X_2) - (X_2 \otimes 1 + 1 \otimes X_2)$. By [17, Theorem 1.3 (iii)] or [7, Theorem 2.4 (i)(f)], $w \in \Bbbk[X_1] \otimes \Bbbk[X_1]$ and

$$w \otimes 1 + (\Delta \otimes \mathrm{Id})(w) = 1 \otimes w + (\mathrm{Id} \otimes \Delta)(w).$$

This means that w is a 2-cocycle in $Z^2(C)$. Up to a change of variable X_2, one can assume that w is an element in $\mathfrak{P}^2_{1_C,1_C}(C)$. By Proposition 6.4 (3), (4), we can assume that w is of the form (6.5.1).

(2) This follows from [7, Theorem 2.4 (i)(d)].

(3) This follows from the fact that $\Bbbk[X_1]$ is commutative and by taking $r = X_1$ in [7, Theorem 2.4 (i)(d)].

(4) Let ∂ be the differential of the differential graded algebra $T(C)$ as in (6.1.1), so ∂ is determined by

$$\partial(f) = \Delta(f) - f \otimes 1 - 1 \otimes f$$

for all $f \in C = \Bbbk[X_1]$. By part (3), $\partial(\delta(X_1)) = -\chi w$. This means that $\chi w = 0$ in $\mathfrak{P}^2_{1_C,1_C}(C)$. By (6.5.1) and the fact that $\{Z_s\}_{s\geq 0} \cup \{Y_{s,t}\}_{(s<t)}$ form a basis of $\mathfrak{P}^2_{1_C,1_C}(C)$, it follows that $\chi w = 0$.

(5) By parts (3) and (4), $\delta(X_1)$ is primitive. It is well known that every primitive element in $\Bbbk[X_1]$ is of the form $\sum_{s\geq 0} d_s X_1^{p^s}$ for some $d_s \in \Bbbk$. This is also a consequence of Proposition 6.4 (3). ■

Recall now the construction of 2-step IHOEs from the introduction. Namely, let $\mathbf{d_s} = \{d_s\}_{s\geq 0}$, $\mathbf{b_s} = \{b_s\}_{s\geq 0}$, and $\mathbf{c_{s,t}} = \{c_{s,t}\}_{0\leq s<t}$ be sequences of scalars in $\Bbbk$ with only finitely many nonzero elements. Let $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ denote the Ore extension $\Bbbk[X_1][X_2; \mathrm{Id}, \delta]$,

where

$$\delta(X_1) = \sum_{s\geq 0} d_s X_1^{p^s}.$$

Moreover, the comultiplication $\Delta : H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}}) \to H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}}) \otimes H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ is determined by

$$\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1,$$
$$\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w,$$

where

$$w = \sum_{s\geq 0} b_s \left(\sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (X_1^{p^s})^i \otimes (X_1^{p^s})^{p-i} \right) + \sum_{0\leq s<t} c_{s,t} (X_1^{p^s} \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1^{p^s}).$$

Similarly, define maps ε and S from $\{X_1, X_2\}$ to $\Bbbk$ and $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$, respectively, by

$$\varepsilon(X_1) = \varepsilon(X_2) = 0,$$
$$S(X_1) = -X_1,$$
$$S(X_2) = -X_2 - m(\mathrm{Id} \otimes S)(w).$$

Now we are ready to prove Theorem 1.3 (1), (2).

Proposition 6.6. *Retain the above definitions and notation.*

(1) *The definitions above of Δ, ε, and S extend uniquely to $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ so that it is a Hopf algebra.*

(2) *Let $H := \Bbbk[X_1][X_2; \sigma, \delta]$ be a 2-step IHOE generated by X_1 and X_2 as in Definition* 2.1 (2). *Then H is isomorphic to $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$, for a suitable choice of scalars.*

Proof. (1) This follows from [7, Theorem 2.4 (ii)] and an easy computation (similar to the proof of Lemma 6.5).

(2) Let χ be the scalar given in Lemma 6.5 (2). Up to a change of variable, we can assume that χ is either 0 or 1.

Case 1. $\chi = 0$. The assertion follows from Lemma 6.5 (1), (5).

Case 2. $\chi = 1$. In this case $\sigma(X_1) = X_1 + 1$ by Lemma 6.5 (2). By Lemma 6.5 (4), $w = 0$. By Lemma 6.5 (1), both X_1 and X_2 are primitive. Since $\sigma(X_1) = X_1 + 1$, we have a relation

$$X_2 X_1 = X_1 X_2 + X_2 + \sum_{s\geq 0} d_s X_1^{p^s}.$$

Replacing X_2 by the new primitive generator $\widehat{X_2} := X_2 + \sum_{s\geq 0} d_s X_1^{p^s}$, the above relation becomes

$$\widehat{X_2}X_1 = X_1\widehat{X_2} + \widehat{X_2}.$$

Exchanging X_1 and $\widehat{X_2}$ and changing the sign of X_1, one now sees that H is isomorphic as a Hopf algebra to $H(\mathbf{d_s}, \mathbf{0}, \mathbf{0})$, where $\mathbf{d_s} = \{d_0 = 1,\ d_s = 0 : s \geq 0\}$. This completes the proof. ■

We shall see in the next subsection that Proposition 6.6 implies that there is an immense zoo of isomorphism classes of 2-step IHOEs in positive characteristic. But, in contrast to this plethora, the classification up to birational equivalence is very simple. In fact, it exactly parallels the story in characteristic 0, where—in the light of Zhuang's result [61, Proposition 7.4 (III)], there are 2 birational equivalence classes, with the quotient division rings of the field of rational functions $\Bbbk(X_1, X_2)$ and the quotient division ring $Q(A_1(\Bbbk))$ of the first Weyl algebra in the commutative and noncommutative cases, respectively.

Corollary 6.7. *Let $\Bbbk$ be algebraically closed of positive characteristic and let H be a 2-step IHOE over $\Bbbk$. If H is not commutative, that is if $\mathbf{d_s} \neq \mathbf{0}$, then the quotient division ring of H is isomorphic to $Q(A_1(\Bbbk))$, the first Weyl skew field over $\Bbbk$.*

Proof. By Proposition 6.6 (2), $H \equiv H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ for some choice of the parameters, with $\mathbf{d_s} \neq \mathbf{0}$ since H is by assumption not commutative. Thus $H = \Bbbk\langle X_1, X_2\rangle$ with

$$[X_2, X_1] = \delta(X_1) = \sum_{s\geq 0} d_s X_1^{p^s}.$$

The quotient division ring Q of H is, therefore, generated by $X_2(\delta(X_1))^{-1}$ and X_1, and these generators satisfy the defining relation of the Weyl skew field, as required. ■

6.3. Classification of 2-step IHOEs: Isomorphisms and automorphisms

This subsection has two interconnected purposes: we complete the classification begun in Proposition 6.6 by determining when any two of the algebras listed there are isomorphic; and in so doing we describe all the automorphisms of these Hopf algebras. The proof of the main result requires three preliminary lemmas.

Lemma 6.8. *Let $H = H(\mathbf{0}, \mathbf{b_s}, \mathbf{c_{s,t}})$ be a commutative 2-step IHOE as described in Proposition 6.6, so*

$$\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1,$$
$$\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w,$$

where w is given in (1.2.2). Let $P(H)$ denote its subspace of primitive elements.

(1) *If $w = 0$, that is if $\mathbf{b_s} = \mathbf{c_{s,t}} = \mathbf{0}$, then $P(H)$ has $\Bbbk$-basis $\{X_1^{p^i}, X_2^{p^j} : i, j \geq 0\}$, so that $\Bbbk\langle P(H)\rangle = H$.*

(2) *If $w \neq 0$, then $P(H)$ has $\Bbbk$-basis $\{X_1^{p^i} : i \geq 0\}$, and $\Bbbk\langle P(H)\rangle = \Bbbk[X_1]$.*

Proof. Let $f := \sum_{i=0}^{n} f_i X_2^i \in P(H)$, where $f_i \in \Bbbk[X_1]$. If $n = 0$, then $f = f_0 \in \Bbbk[X_1]$. It then follows from a direct computation that f is of the form $\sum_{s\geq 0} \alpha_s X_1^{p^s}$ for some finite sequence of scalars α_s. (This fact is also well known.)

Next assume that the X_2-degree of f is n, with $n \geq 1$, and consider the equation

$$f \otimes 1 + 1 \otimes f = \Delta(f) = \sum_{i=0}^{n} \Delta(f_i)\Delta(X_2)^i. \tag{6.8.1}$$

For $i = p^s$, then using commutativity of H and the fact that char $\Bbbk = p$,

$$\Delta(X_2)^{p^s} = X_2^{p^s} \otimes 1 + 1 \otimes X_2^{p^s} + w^{p^s}, \tag{6.8.2}$$

where $w^{p^s} \in \Bbbk[X_1] \otimes \Bbbk[X_1]$.

On the other hand, if $i \leq n$ and $i \neq p^s$, then the expansion of $\Delta(X_2^i)$ has at least one nonzero term of the form $cX_2^j \otimes X_2^{i-j}$, where $0 < j < i$ and $0 \neq c \in \Bbbk$. Suppose that $f_i \neq 0$ for such an integer i. Since $H \otimes H$ is a domain, this implies that the expansion of $\Delta(f)$ has a nonzero term of the form $\Delta(f_i)cX_2^j \otimes X_2^{i-j}$, where $0 < j < i$ and $0 \neq c \in \Bbbk$. The total degree in X_2 of this term is i, so it cannot cancel with any other term in $f \otimes 1 + 1 \otimes f$ $(= \Delta(f))$. This contradicts the hypothesis that f is primitive. Therefore, $f = f_0 + \sum_{s\geq 0} g_s X_2^{p^s}$ for some $f_0, g_s \in \Bbbk[X_1]$. Since f is primitive, (6.8.1) and (6.8.2) imply that

$$f \otimes 1 + 1 \otimes f = \Delta(f_0) + \sum_{s\geq 0} \Delta(g_s)[X_2^{p^s} \otimes 1 + 1 \otimes X_2^{p^s} + w^{p^s}],$$

and hence $\Delta(g_s) = 1 \otimes g_s = g_s \otimes 1$ for all s. The counital Hopf algebra axiom now forces $\beta_s := g_s \in \Bbbk$ for all s.

(1) If $w = 0$, then $\sum_{s\geq 0} \beta_s X_2^{p^s}$ is clearly primitive. Hence f_0 is primitive, $f_0 = \sum_{s\geq 0} \alpha_s X_1^{p^s}$ for some $\alpha_s \in \Bbbk$, and the claims follow.

(2) If $w \neq 0$, then it is of the form (1.2.2). Since char $\Bbbk = p$, for every $s \geq 0$, w^{p^s} is also of the form of (1.2.2), but of higher X_1-degree. If $\beta_s \neq 0$ for some s, by counting the X_1-degree one sees that $\sum_{s\geq 0} \beta_s w^{p^s}$ is another nonzero element of the form (1.2.2). In particular, the class of $\sum_{s\geq 0} \beta_s w^{p^s}$ in $\mathfrak{P}^2_{1,1}(\Bbbk[X_1])$ is nonzero, by Proposition 6.4 (4).

Recall that $f = f_0 + \sum_{i\geq 1} f_i X_2^i = f_0 + \sum_{s\geq 0} \beta_s X_2^{p^s}$. By (6.8.2),

$$\begin{aligned} 0 &= \Delta(f) - 1 \otimes f - f \otimes 1 \\ &= \left[\Delta(f_0) - 1 \otimes f_0 - f_0 \otimes 1\right] + \sum_{s\geq 0} \beta_s w^{p^s} \\ &= \partial(f_0) + \sum_{s\geq 0} \beta_s w^{p^s}. \end{aligned}$$

This implies that the class $\sum_{s\geq 0} \beta_s w^{p^s}$ in $\mathfrak{P}^2_{1,1}(\Bbbk[X_1])$ is equal to the class of $-\partial(f_0)$, which is zero by definition. This yields a contradiction. Therefore, all $\beta_s = 0$ and the result follows. ∎

Lemma 6.9. *Let $H = H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ be a noncommutative 2-step IHOE as in Proposition 6.6. That is, $\delta(X_1) \neq 0$, equivalently $\mathbf{d_s} \neq \mathbf{0}$.*

(1) *The commutator ideal $[H, H]$ is a nonzero Hopf ideal of H.*

(2) *Let K denote the Hopf algebra $\Bbbk[t]$, so t is primitive. Suppose that $\phi : H \to K$ is a Hopf algebra epimorphism. Then $\ker\phi$ is the principal ideal generated by X_1.*

(3) *For any Hopf algebra epimorphism, $\phi : H \to K$, $H^{\operatorname{co} K}$ is the Hopf subalgebra $\Bbbk[X_1]$ of H.*

Proof. (1) It is well known that the commutator ideal is a Hopf ideal [14, Lemma 3.7]. Since H is noncommutative, $[H, H] \neq 0$.

(2) Let $f(X_1)$ denote the polynomial $\delta(X_1) \in \Bbbk[X_1]$. Then $f(X_1)$ is a nonzero element in $[H, H]$. Since $\operatorname{im}\phi$ is commutative, $[H, H]$ is a subspace of $\ker\phi$, so $\phi(f(X_1)) = 0$. We claim that $\phi(X_1) = 0$. Suppose not, and let $y = \phi(X_1)$. Since X_1 is primitive, so is y. Then $0 \neq y = \sum_{s\geq 0} \alpha_s t^{p^s}$ for a finite sequence of scalars α_s, so that

$$\phi\big(f(X_1)\big) = f\big(\phi(X_1)\big) = f(y) = f\Big(\sum_{s\geq 0} \alpha_s t^{p^s}\Big)$$

which is nonzero. This yields a contradiction. Thus $y = 0$ as required.

(3) As

$$(\mathrm{Id} \otimes \phi) \circ \Delta(X_1) = X_1 \otimes \phi(1),$$

$X_1 \in H^{\operatorname{co} K}$ and so $\Bbbk[X_1] \subseteq H^{\operatorname{co} K}$. To prove the reverse inclusion, let $g(X_1, X_2)$ be any element in $H^{\operatorname{co} K}$. Since $\phi(X_1) = 0$ by part (2) and ϕ is an epimorphism, $\phi(X_2)$ is an algebra generator of K. Now

$$\begin{aligned} g(X_1 \otimes 1, X_2 \otimes 1) &= g(X_1, X_2) \otimes 1 \\ &= (\mathrm{Id} \otimes \phi)\Delta g(X_1, X_2) \\ &= g\big((\mathrm{Id} \otimes \phi)\Delta(X_1), (\mathrm{Id} \otimes \phi)\Delta(X_2)\big) \\ &= g\Big(X_1 \otimes 1, X_2 \otimes 1 + 1 \otimes \phi(X_2) - \sum_{t>0} c_{0,t} X_1^{p^t} \otimes 1\Big). \end{aligned}$$

This implies that X_2-degree of g is zero, so $g \in \Bbbk[X_1]$ as required. ∎

Lemma 6.10. *Let $H = \Bbbk\langle X_1, X_2\rangle$ and $H' = \Bbbk\langle X_1', X_2'\rangle$ be two 2-step IHOEs as listed in Proposition 6.6. Let $\phi : H \to H'$ be a Hopf algebra isomorphism such that $\phi(\Bbbk[X_1]) = \Bbbk[X_1']$. Then $\phi(X_2) = cX_2' + v(X_1')$ for some $0 \neq c \in \Bbbk$ and $v(X_1') \in \Bbbk[X_1']$.*

Proof. We will not need any Hopf algebra structure for this proof. So up to a change of variable, we can assume that $\phi(X_1) = X_1'$.

Let a be the X_2'-degree of $\phi(X_2)$ and let b be the X_2-degree of $\phi^{-1}(X_2')$. Since $\phi(X_1) = X_1'$ and both H and H' are Ore extensions, it is easy to see that $ab = 1$. This forces $a = b = 1$. Write $\phi(X_2) = c(X_1')X_2' + v(X_1')$ and $\phi^{-1}(X_2') = d(X_1)X_2 + u(X_1)$. Then $c(X_1')d(X_1') = 1$, so $0 \neq c := c(X_1') \in \Bbbk$ is a nonzero scalar in $\Bbbk$. ∎

We can now describe the Hopf algebra isomorphisms and automorphisms between 2-step IHOEs. In what follows, we shall use the adjective *trivial* to specify the IHOE $H(\mathbf{0},\mathbf{0},\mathbf{0})$—in other words, the *trivial* 2-step IHOE is the coordinate ring of $(\Bbbk,+)\times(\Bbbk,+)$. To describe $\mathrm{Aut}(H(\mathbf{0},\mathbf{0},\mathbf{0}))$, let F denote the automorphism of $(\Bbbk,+)$ which maps $\lambda\in\Bbbk$ to λ^p. Thus F and $\Bbbk\setminus\{0\}$ are in $\mathrm{Aut}((\Bbbk,+))$, where elements of $\Bbbk\setminus\{0\}$ act by left multiplication, and together they generate in $\mathrm{End}((\Bbbk,+))$ the skew polynomial subalgebra $A:=\Bbbk[F;\sigma]$, where $\sigma(\lambda)=\lambda^{-p}$ for $\lambda\in\Bbbk$.

In the next proposition, let H and $\Bbbk\langle X_1,X_2\rangle$ denote $H(\mathbf{d_s},\mathbf{b_s},\mathbf{c_{s,t}})$ and let H' and $\Bbbk\langle X_1',X_2'\rangle$ denote $H(\mathbf{d'_s},\mathbf{b'_s},\mathbf{c'_{s,t}})$ as listed in Proposition 6.6.

Proposition 6.11. *Retain the above notation.*

(1) *Let $\phi: H\to H'$ be a Hopf algebra isomorphism. Then there are nonzero scalars α,β in $\Bbbk$ such that, for all $s\geq 0$, $s<t$,*

$$d_s'=d_s\alpha^{p^s-1}\beta^{-1},\quad b_s'=b_s\alpha^{p^{s+1}}\beta^{-1},\quad c_{s,t}'=c_{s,t}\alpha^{p^s+p^t}\beta^{-1}. \tag{6.11.1}$$

Conversely, if α and β are nonzero scalars such that (6.11.1) holds, then there is a Hopf algebra isomorphism $\phi: H\to H'$ such that

$$\phi(X_1)=\alpha X_1' \quad \textit{and} \quad \phi(X_2)=\beta X_2'.$$

(2) *Suppose that H is not trivial. Then every Hopf algebra isomorphism from H to H' has the form*

$$\phi(X_1)=\alpha X_1' \quad \textit{and} \quad \phi(X_2)=\beta X_2'+\sum_{s\geq 0} e_s X_1'^{p^s},$$

for an arbitrary finite sequence of scalars $\{e_s : s\geq 0\}$, and nonzero scalars α and β satisfying (6.11.1).

(3) *If H is not trivial, then every Hopf algebra automorphism of H is of the form*

$$\phi(X_1)=\alpha X_1,$$
$$\phi(X_2)=\beta X_2+\sum_{s\geq 0} e_s X_1^{p^s},$$

where $\{e_s : s\geq 0\}$ is an arbitrary finite sequence of scalars, and α,β are nonzero scalars satisfying

$$d_s=d_s\alpha^{p^s-1}\beta^{-1},\quad b_s=b_s\alpha^{p^{s+1}}\beta^{-1},\quad c_{s,t}=c_{s,t}\alpha^{p^s+p^t}\beta^{-1}; \tag{6.11.2}$$

equivalently,

$$\beta=\begin{cases}\alpha^{p^s-1} & \textit{if } d_s\neq 0,\\ \alpha^{p^{s+1}} & \textit{if } b_s\neq 0,\\ \alpha^{p^s+p^t} & \textit{if } c_{s,t}\neq 0.\end{cases}$$

(4) *Suppose that H is trivial, so $H \cong \mathcal{O}((\Bbbk,+)^2)$. Then, in the notation introduced before the proposition,*

$$\mathrm{Aut}(H) \cong \mathrm{GL}_2(A).$$

Proof. (1) If both H and H' are trivial, then the assertions are obvious. Now we assume that H is not trivial.

If H is noncommutative, then H' is noncommutative, and so H' is not trivial. If H is commutative, then $w \neq 0$ and, by Lemma 6.8, H' is not trivial. Thus, in both cases, H' is not trivial.

By Lemma 6.8 (2) and Lemma 6.9 (3), the Hopf algebra isomorphism $\phi : H \to H'$ maps $\Bbbk[X_1]$ to $\Bbbk[X_1']$. In particular, $\Bbbk[X_1'] = \Bbbk[\phi(X_1)]$, so $\phi(X_1) = \alpha X_1' + a_0$ for a nonzero scalar α and a scalar a_0. Since X_1 is primitive, $a_0 = 0$ and $\phi(X_1) = \alpha X_1'$. By Lemma 6.10, $\phi(X_2) = \beta X_2' + v(X_1')$, where β is a nonzero scalar and $v(X_1') \in \Bbbk[X_1']$. Applying ϕ to the defining relation of H, namely

$$X_2X_1 - X_1X_2 = \delta(X_1) = \sum_{s\geq 0} d_s X_1^{p^s},$$

we obtain the following equation in H':

$$\alpha\beta(X_2'X_1' - X_1'X_2') = \sum_{s\geq 0} d_s \alpha^{p^s} X_1'^{p^s},$$

which must agree with

$$X_2'X_1' - X_1'X_2' = \delta'(X_1') = \sum_{s\geq 0} d_s' X_1'^{p^s}.$$

Therefore, $d_s' = d_s \alpha^{p^s-1}\beta^{-1}$ for all s.

Recall that ∂ denotes the map defined in (6.1.1). Applying $\phi \otimes \phi$ to the following equation in $H \otimes H$,

$$\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + w,$$

we obtain an equation in $H' \otimes H'$, namely

$$\beta\Delta(X_2') + \partial\big(v(X_1')\big) = \beta(X_2' \otimes 1 + 1 \otimes X_2') + (\phi \otimes \phi)(w).$$

With the obvious notation $w' := \partial(X_2')$, the above equation must agree with

$$\Delta(X_2') = X_2' \otimes 1 + 1 \otimes X_2' + w'.$$

Using the explicit form of $(\phi \otimes \phi)(w)$, one can show that $\partial(v(X_1')) = 0$ and $w' = \beta^{-1}(\phi \otimes \phi)(w)$. The former implies that $v(X_1') \in \Bbbk[X_1'] \subseteq H'$ is primitive and the latter implies that

$$b_s' = b_s \alpha^{p^{s+1}} \beta^{-1}, \quad c_{s,t}' = c_{s,t} \alpha^{p^s+p^t} \beta^{-1}$$

for all $s < t$. Therefore, (6.11.1) holds.

The converse is clear.

(2) By the proof of part (1), $\phi(X_1) = \alpha X_1'$ and $\phi(X_2) = \beta X_2' + v(X_1')$, with $v(X_1')$ primitive. Since every primitive element in $\Bbbk[X_1]$ is of the form $\sum_{s\geq 0} e_s X_1^{p^s}$, the assertion follows.

(3) This is a special case of part (2).

(4) Suppose that $H = H(\mathbf{0}, \mathbf{0}, \mathbf{0})$ is trivial. That $\mathrm{Aut}(H) \cong \mathrm{GL}_2(A)$ is an immediate consequence of the equivalence of categories between the category of unipotent algebraic groups over $\Bbbk$ which are subgroups of $(\Bbbk, +)^r$ for some r, and the category of finitely generated A-modules, under which $(k, +)$ corresponds to the free A-module of rank 1, [34, Theorem 14.46 and Example 14.40]. ∎

7. Comments and questions

We gather here a number of observations related to the classification results in Section 6.3.

7.1. Algebraic groups

To recast Sections 6.2 and 6.3 in the language of affine algebraic $\Bbbk$-groups, first consider the overuse in this context of the word "connected": a Hopf algebra H is by definition *connected* if its coradical H_0 is $\Bbbk$, equivalently if it has a unique simple comodule [36, Definition 5.1.5]; an affine algebraic $\Bbbk$-group G is *unipotent* if and only if its coordinate ring $H = \mathcal{O}(G)$ is a connected Hopf algebra [34, Theorem 14.5]; and an affine algebraic $\Bbbk$-group G is *connected* if and only if it equals G°, its connected component of 1_G. Equivalently, G is connected if and only if $\mathcal{O}(G)$ is a domain[2]. In characteristic 0, every unipotent group U is connected—that is, $\mathcal{O}(U)$ is always a domain; but the cyclic group of order p shows that this is false in characteristic p.

The key part (2)⇒(1) of the following fundamental result is due to Lazard [26]; a short proof valid in all characteristics can be found in [24, Section 4]. For the full statement of the theorem, see for example [22, Section 8].

Theorem 7.1. *Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic, let G be an affine algebraic $\Bbbk$-group, and let n be a positive integer. Then the following are equivalent.*

(1) *G is a connected unipotent group over $\Bbbk$ of dimension n.*

(2) *$\mathcal{O}(G) \cong \Bbbk[X_1, \ldots, X_n]$ as $\Bbbk$-algebras.*

(3) *$\mathcal{O}(G)$ is an affine connected Hopf domain of Gel'fand–Kirillov dimension n.*

(4) *$\mathcal{O}(G)$ is an n-step IHOE.*

(5) *G has a subnormal series of length n with factors isomorphic to $(\Bbbk, +)$.*

(6) *G has a central series of length n with factors isomorphic to $(\Bbbk, +)$.*

[2] In the algebraic groups literature, this confusion is sometimes avoided by using *coconnected* for the first of these usages.

7.2. Classification of connected unipotent groups of dimension two

Revert to our usual hypothesis that the base field is $\Bbbk$, algebraically closed of characteristic $p > 0$. The coordinate ring of every 2-dimensional connected unipotent group over $\Bbbk$ is a 2-step IHOE, by (1)$\Leftrightarrow$(4) of Theorem 7.1, so the classification of Section 6.3 incorporates a classification of these groups. Clearly, in the notation of Section 6.2, $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ is commutative if and only if $\mathbf{d_s} = \mathbf{0}$, so the 2-dimensional connected unipotent $\Bbbk$-groups are classified by

$$H(\mathbf{0}, \mathbf{b_s}, \mathbf{c_{s,t}})/\!\sim,$$

where the equivalence is provided by the isomorphisms of Proposition 6.11 (1), (2). A version of the same classification, over an arbitrary field, in group-theoretic language, is given ("rather formal" in the words of the authors) at [22, Section 3.7].

It is easy to write down the group G such that $H(\mathbf{0}, \mathbf{b_s}, \mathbf{c_{s,t}}) \cong \mathcal{O}(G)$. Namely, $G = \mathbb{A}^2(\Bbbk)$, and for (a, e) and (f, g) in G

$$(a, e) * (f, g) = \left(a + f, e + g + \sum_{s \geq 0} b_s \left(\sum_{i=1}^{p-1} \lambda_i a^{ip^s} e^{(p-i)p^s} \right) + \sum_{0 \leq s < t} c_{s,t} (a^{p^s} e^{p^t} - a^{p^t} e^{p^s}) \right). \tag{7.1.1}$$

From this, it is easy to see that

- G is abelian $\Leftrightarrow$ $\mathbf{c_{s,t}} = \mathbf{0}$ or $p = 2$,
- G has exponent p $\Leftrightarrow$ $\mathbf{b_s} = \mathbf{0}$.

Combining these two statements yields the conclusion that the only abelian connected 2-dimensional unipotent group of exponent p is $(\Bbbk, +)^2$, a special case of a result valid for all dimensions (see [45, Proposition VII.11] and [22, Lemma 1.7.1]).

7.3. Ore extensions of dimension two

The Ore extensions of the polynomial ring $\Bbbk[X_1]$ are classified in [2], so one can view Proposition 6.6 as determining which of these Ore extensions admit Hopf algebra structures. Notice that Propositions 6.6 and 6.11 together show that, for every Ore extension $R = \Bbbk[X_1][X_2; \sigma, \delta]$, the number of distinct Hopf structures which R can carry is either 0 or ∞. This is in stark contrast to the situation in characteristic 0, as noted in the opening sentence of Section 6.2.

A second noteworthy feature is this: every such R admitting a Hopf algebra structure can be presented with $\sigma = \mathrm{Id}_{\Bbbk[X_1]}$. (The enveloping algebra of the non-abelian 2-dimensional Lie algebra can be presented in 2 distinct ways as an Ore extension, only one of which can be written as an Ore extension with $\sigma = \mathrm{Id}_{\Bbbk[X_1]}$.) But this does not extend to higher dimensions: $U(\mathfrak{sl}(2, \Bbbk))$ is a 3-step IHOE [7, Section 3.1, Examples (iii)], but cannot be so presented without using automorphisms.

Another case to consider is HOE $\Bbbk[X_1^{\pm 1}][X_2, \sigma, \delta]$, where X_1 is a grouplike element. Let $C = \Bbbk[X_1^{\pm 1}]$. By [17, Theorem 1.3 (i)], there is a grouplike element $\alpha = X_1^a \in C$ such that

$$\Delta(X_2) = \alpha \otimes X_2 + X_2 \otimes 1 + w,$$

where w is an element in $C \otimes C$. By [53, Lemma 2.3 (1)], w is an $(\alpha, 1)$-2-cocycle. Since C is cosemisimple, it has primitive cohomology dimension 0 and

$$\mathfrak{P}^2_{\alpha,1}(C) = 0.$$

This implies that w is an $(\alpha, 1)$-2-coboundary. Up to a change of variable X_2, we can assume that $w = 0$ by [53, Lemma 2.3 (3)]. By [17, Theorem 1.3 (ii)], there is a nonzero $c \in \Bbbk$ such that $\sigma(X_1) = cX_1$. Let $\delta(X_1) = \sum_n b_n X_1^n$. By [17, Theorem 1.3 (iii)], we have

$$\sum_n b_n X_1^n \otimes X_1^n - \Big(\sum_n b_n X_1^n\Big) \otimes X_1 - X_1^{a+1} \otimes \Big(\sum_n b_n X_1^n\Big) = 0$$

which implies that

$$\delta(X_1) = b_1(X_1 - X_1^{a+1}).$$

In summary, we have the following family of Hopf algebras:

$$K(a, b, c) := \Bbbk[X_1^{\pm 1}][X_2; \sigma, \delta],$$

where $\sigma(X_1) = cX_1$ for some nonzero $c \in \Bbbk$ and $\delta(X_1) = b(X_1 - X_1^{a+1})$ for some $b \in \Bbbk$, $a \in \mathbb{Z}$, and the coalgebra structure of $K(a, b, c)$ is determined by

$$\Delta(X_1) = X_1 \otimes X_1, \quad \varepsilon(X_1) = 0,$$

and

$$\Delta(X_2) = X_1^a \otimes X_2 + 1 \otimes X_2, \quad \varepsilon(X_2) = 0.$$

The following lemma is clear.

Lemma 7.2. *Retain the above notation. Then $K(a, b, c)$ is commutative if and only if $b = 0$ and $c = 1$. In this case, there is a unique Hopf ideal I such that $K(a, 0, 1)/I \cong \Bbbk[X_2]$.*

7.4. Classification of connected algebraic groups of dimension two

The following is a well-known classification of connected algebraic groups of dimension two. Assume that $\Bbbk$ is algebraically closed and that G is a connected algebraic group of dimension two. Since no quotient group of G can be semisimple, G is solvable. Let G_u be the unipotent radical of G. Then G_u is normal and unipotent, with G/G_u connected and with no unipotent elements. By [18, Section 19.1], G_u is connected, and there is a short exact sequence

$$1 \to G_u \to G \to T \to 1, \tag{7.2.1}$$

where T is a torus. There are three cases to consider.

Case 1: $\dim G_u = 0$. Then $G = T$ is a torus and $\mathcal{O}(G) = \Bbbk[X_1^{\pm 1}, X_2^{\pm 1}]$.

Case 2: $\dim G_u = 2$. Then G is unipotent, so G is classified in (7.1.1).

Case 3: $\dim G_u = 1$. By (7.2.1), G is a semidirect product $(\Bbbk, +) \rtimes (\Bbbk \setminus \{0\}, \times)$. Dual to (7.2.1), there is a short exact sequence of commutative Hopf algebras

$$\Bbbk \to \Bbbk[X_1^{\pm 1}] \to \mathcal{O}(G) \to \Bbbk[X_2] \to \Bbbk.$$

Then $\mathcal{O}(G)$ is one of the Hopf algebras $K(a, 0, 1)$ given in the last section.

7.5. Connected affine Hopf algebra domains of GKdimension 2

We do not know whether Propositions 6.6 and 6.11 give a classification of all connected affine Hopf $\Bbbk$-algebra domains of Gel'fand–Kirillov dimension 2. More precisely, one can ask the following question:

Question 7.3. If $\Bbbk$ has positive characteristic, is every affine connected Hopf $\Bbbk$-algebra domain H of Gel'fand–Kirillov dimension 2 an IHOE?

The answer is "yes" when H is commutative, and not only in dimension 2—this follows from the structure of connected unipotent groups, Theorem 7.1. If one drops the restriction to dimension 2 in Question 7.3, then the answer is "no"—for example the enveloping algebra of $\mathfrak{sl}(3, \Bbbk)$ endowed with its standard cocommutative coproduct is not an IHOE, since $\mathfrak{sl}(3, \Bbbk)$ does not contain a full flag of Lie subalgebras.

7.6. Automorphism groups of 2-step IHOEs

From Proposition 6.11 (3), one can easily read off the structure of the group $\mathrm{Aut}(H)$ of Hopf algebra automorphisms of $H := H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$, for each non-trivial H. Namely, define subgroups T and N of $\mathrm{Aut}(H)$ by

$$T = \{\phi \in \mathrm{Aut}(H) \mid \phi(X_1) = \alpha X_1,\ \phi(X_2) = \beta X_2\},$$

where $\alpha, \beta \in (\Bbbk, \times)$ satisfy (6.11.2); and

$$N = \Big\{\phi \in \mathrm{Aut}(H) \mid \phi(X_1) = X_1,\ \phi(X_2) = X_2 + \sum_{s \geq 0} e_s X_1^{p^s}\Big\},$$

for an arbitrary sequence $(e_s) \in \Bbbk^{\mathbb{N}}$ with only finitely many nonzero entries. Thus N is a normal subgroup of $\mathrm{Aut}(H)$ with $N \cong (\Bbbk, +)^{\mathbb{N}}$. The constraints (6.11.2) mean that $T \cong (\Bbbk, \times)$ if H has only one nonzero parameter, while T is a finite (possibly trivial) subgroup of $(\Bbbk, \times)$ if there are two or more defining parameters for H. It is clear from Proposition 6.11 that $\mathrm{Aut}(H)$ is the semidirect product of N by T.

8. Noncommutative binomial theorem in characteristic p

In this short section, we derive a corollary of an important theorem due to Jacobson, which is needed in Section 9. Stated in our notation, Jacobson's theorem is as follows:

Theorem 8.1 ([21, pp. 186–187]). *Let F be a field of characteristic $p > 0$ and let R be an F-algebra. For elements a, b of R,*

$$(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a,b),$$

where, for $i = 1, \dots, p-1$ and $\lambda \in F \setminus \{0\}$, $s_i(a,b)$ is the coefficient of $\frac{1}{i}\lambda^{i-1}$ in $\mathrm{ad}^{p-1}_{(\lambda a+b)}(a)$.

Here is the required corollary.

Corollary 8.2. *In the situation of the theorem, suppose that the elements $\mathrm{ad}^i_b(a)$ of R commute with a for $i = 1, \dots, p-1$. Then*

$$(a+b)^p = a^p + b^p + \mathrm{ad}^{p-1}_b(a).$$

Proof. The extra hypothesis of the corollary ensures that, for $j > 0$, λ^j does not occur in the expansion of $\mathrm{ad}^{p-1}_{(\lambda a+b)}(a)$. So Theorem 8.1 states that only $s_1(a,b)$ is nonzero; namely,

$$s_1(a,b) = \mathrm{ad}^{p-1}_b(a),$$

as claimed. ∎

9. Properties of 2-step IHOEs

9.1. The antipode, the center, and the Calabi–Yau property

In this subsection, we study properties of the Hopf algebras listed in Proposition 6.6. The definition and relevant properties of the PI-degree are recalled in Section 4.1; regarding the Nakayama automorphism and the skew Calabi–Yau property, see Section 4.2 and [8].

Proposition 9.1. *Let H be a 2-step IHOE $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ as in Proposition 6.6.*

(1) *S^2 is the identity of H.*

(2) *H is commutative if and only if $\mathbf{d_s} = \mathbf{0}$.*

(3) *H is cocommutative if and only if $\mathbf{c_{s,t}} = \mathbf{0}$ or $p = 2$.*

(4) *Suppose that H is not commutative. Then the center of H is $\Bbbk[X_1^p, X_2^p - d_0^{p-1}X_2]$. Hence the PI-degree of H is p.*

(5) *The Nakayama automorphism μ of H is determined by*

$$\mu(X_1) = X_1 \quad \text{and} \quad \mu(X_2) = X_2 + d_0.$$

Hence, H is Calabi–Yau if and only if $d_0 = 0$.

(6) *The left homological integral $\int^l_H$ of H is the 1-dimensional H-bimodule which is trivial as left module and with $\int^l_H \cong H/(X_1, X_2 - d_0)$ as right module.*

Proof. (1) Using the comultiplication formula of $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ or the definition given before Theorem 1.3, one finds that

$$S(X_1) = -X_1 \quad \text{and} \quad S(X_2) = -X_2 + f(X_1)$$

for some polynomial f. It is clear that $S^2(X_1) = X_1$. If $p = 2$, then

$$S^2(X_2) = X_2 + 2f(X_1) = X_2$$

which implies that S^2 is the identity. (In fact, when $p=2$, $S(X_2) = X_2 + \sum_{s\geq 0} b_s X_1^{2^{s+1}}$.) Suppose now that $p \geq 3$. Again by the comultiplication formula,

$$S(X_2) = -X_2 - m\big((S \otimes \mathrm{Id})(w)\big),$$

where w is defined in (6.5.1) and m denotes multiplication in H. Since p is odd,

$$m\big((S \otimes \mathrm{Id})(Z_s)\big) = 0 \quad \text{and} \quad m\big((S \otimes \mathrm{Id})(Y_{s,t})\big) = 0.$$

Therefore, $S(X_2) = -X_2$ and S^2 is the identity.

(2) This is clear from (1.2.1).

(3) This is clear from (1.2.2).

(4) Since $\sigma = \mathrm{Id}$, δ is a derivation of $\Bbbk[X_1]$. Thus $\delta(X_1^p) = pX_1^{p-1}\delta(X_1) = 0$. This implies that $X_2X_1^p = X_1^p X_2$, so that X_1^p is central.

One easily checks by induction that

$$\delta^n(X_1) = d_0^{n-1}\delta(X_1) \tag{9.1.1}$$

for all $n \geq 2$. Write the derivation $\delta = [X_2, -]$ of $\Bbbk[X_1]$ as $\lambda_{X_2} - \rho_{X_2}$ where the symbols λ_{X_2} (resp. ρ_{X_2}) denote left (resp. right) multiplication by X_2 in H. Note that these linear maps commute in $\mathrm{End}_{\Bbbk}(H)$, so one has

$$(\lambda_{X_2} - \rho_{X_2})^p = \lambda_{X_2}^p - \rho_{X_2}^p.$$

In other words,

$$\delta^p = [X_2^p, -], \tag{9.1.2}$$

so that

$$X_2^p X_1 = X_1 X_2^p + \delta^p(X_1) = X_1 X_2^p + d_0^{p-1}\delta(X_1).$$

Combining this with the relation

$$X_2 X_1 = X_1 X_2 + \delta(X_1),$$

it follows that $X_2^p - d_0^{p-1}X_2$ is central. Thus H is a free module of rank p^2 over the central polynomial subalgebra $Z_0 := \Bbbk[X_1^p, X_2^p - d_0^{p-1}X_2]$. Hence, using $Q(-)$ to denote quotient division algebras,

$$\dim_{Q(Z_0)} Q(H) = p^2.$$

But $\dim_{Q(Z(H))} Q(H)$ is an even power of p, by Theorem 4.3 and the discussion at the start of Section 4.1. Therefore, since H is not commutative, the PI-degree of H is p and $Q(Z(H)) = Q(Z_0)$. However, $Z(H)$ is a finite module over Z_0 by noetherianity of H as a Z_0-module, and $Z(H)$ is contained in $Q(Z_0)$. Since Z_0 is normal, it follows that $Z(H) = Z_0$. That the PI-degree of H is p is now clear.

(5) By [28, Theorem 4.2], $\mu(X_1) = X_1$ and

$$\mu(X_2) = X_2 + \frac{d}{dX_1}\big(\delta(X_1)\big) = X_2 + d_0.$$

The consequence is clear (and it also follows from [28, Corollary 4.3]).

(6) By [8, Theorem 0.3], $\mu = S^2 \circ \Xi^l_\eta$, where η is the right character of the left homological integral $\int^l$ of H and Ξ^l_η is the corresponding left winding automorphism. By (1), $S^2 = \mathrm{Id}$, so that $\mu = \Xi^l_\eta$. Thus (5) implies (6). ■

9.2. Representation theory of 2-step IHOEs

In this subsection, we describe the simple representations of the 2-step IHOEs $H := H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$. Recall that if $\Bbbk$ is any algebraically closed field and A is an affine $\Bbbk$-algebra which is a finite module over its center Z, then Z is also affine by the Artin–Tate lemma [33, Lemma 13.9.10], and a version of Schur's lemma applies [33, Theorem 13.10.3]: namely, if V is a simple A-module, then $\mathrm{End}_A(V) = \Bbbk$, and so V is annihilated by a maximal ideal $\mathfrak{m}$ of Z, so that V is a (necessarily finite dimensional) simple module over the finite dimensional algebra $H/\mathfrak{m}H$. If A is prime (as in the current setting, when $A = H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$ is a domain), with $\mathrm{PIdeg}(A) = d$, then for $\mathfrak{m}$ in a non-empty (and hence dense) open subset $\mathcal{A}(A)$ of $\mathrm{maxspec}(A)$,

$$A/\mathfrak{m}A \cong M_d(\Bbbk).$$

The set $\mathcal{A}(A)$ is called the *Azumaya locus* of A. For convenience, we shall denote the *non-Azumaya locus* by $\mathcal{N}\mathcal{A}(A)$; that is, $\mathcal{N}\mathcal{A}(A) := \mathrm{maxspec}(Z) \setminus \mathcal{A}(A)$, a proper closed subset of $\mathrm{maxspec}(Z)$. If the simple A-module V has $\mathrm{Ann}_Z(V) \in \mathcal{N}\mathcal{A}(A)$, then $\dim_{\Bbbk}(V) < d$. For more details on this circle of ideas, see for example [5, Part III].

Suppose now that $H := H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}}) = \Bbbk[X_1][X_2; \delta]$, so that

$$Z := Z(H) = \Bbbk[X_1^p, X_2^p - d_0^{p-1} X_2]$$

by Theorem 9.1 (4). In view of the discussion in the previous paragraph, our task is to describe the algebras

$$H_{\alpha,\beta} := H/\mathfrak{m}_{\alpha,\beta} H,$$

where $\mathfrak{m}_{\alpha,\beta}$ denotes the maximal ideal $\langle X_1^p - \alpha, X_2^p - d_0^{p-1} X_2 - \beta\rangle$ of Z and (α, β) ranges through $\mathbb{A}^2(\Bbbk)$. Notice that, by the PBW theorem for H,

$$\dim_{\Bbbk}(H_{\alpha,\beta}) = p^2$$

for all $(\alpha, \beta) \in \mathbb{A}^2(\Bbbk)$. Therefore, the maximal dimension of a simple H-module V is p, and this value is attained by V if and only if V is the (unique) simple $H/\mathfrak{m}_{\alpha,\beta}H$-module for some $(\alpha, \beta) \in \mathcal{A}(H)$.

Recall from (1.2.1) that $\delta(X_1) = d_0 X_1 + \sum_{s\geq 1} d_s X_1^{p^s}$. It is convenient to define a polynomial $d(x) = \sum_{s\geq 1} d_s x^{p^{s-1}}$, so that

$$\delta(X_1)^p = d_0^p X_1^p + \Big(\sum_{s\geq 1} d_s X_1^{p^s}\Big)^p = d_0^p X_1^p + d(X_1^p)^p.$$

Proposition 9.2. *Suppose that $\Bbbk$ is algebraically closed of characteristic $p > 0$. Let H be a 2-step IHOE as given in Proposition 6.6, and fix the notation as above. Suppose that H is not commutative, that is $\mathbf{d_s} \neq \mathbf{0}$.*

(1) *The defining ideal of $\mathcal{NA}(H)$ in Z is $\sqrt{\langle d_0^p X_1^p + d(X_1^p)^p \rangle}$. That is,*

$$H_{\alpha,\beta} \cong M_p(\Bbbk) \Leftrightarrow d_0^p \alpha + d(\alpha)^p \neq 0.$$

(2) *Suppose that $d_0 = 0$. Let $\mathfrak{m}_{\alpha,\beta} \in \mathcal{NA}(H)$; that is, $d_0^p \alpha + d(\alpha)^p = 0$, or equivalently,*

$$d(\alpha) = 0.$$

Then

$$H_{\alpha,\beta} \cong \Bbbk[X, Y]/\langle X^p, Y^p \rangle.$$

Thus $H_{\alpha,\beta}$ has a unique simple module, of dimension 1.

(3) *Suppose that $d_0 \neq 0$. Let $\mathfrak{m}_{\alpha,\beta} \in \mathcal{NA}(H)$, so $d_0^p \alpha + d(\alpha)^p = 0$. Then there are elements $u, w \in H_{\alpha,\beta}$ such that $H_{\alpha,\beta} = \Bbbk\langle u, w \rangle$, with relations*

$$u^p = 0, \quad wu - uw = u, \quad w^p - w + \alpha\beta d(\alpha)^{-p} = 0. \tag{9.2.1}$$

In particular, $H_{\alpha,\beta}$ has a single block of p simple modules, each of dimension 1.

(4) *The non-Azumaya locus $\mathcal{NA}(H)$ is the disjoint union of r copies of $\mathbb{A}^1(\Bbbk)$, the affine line, where r is the number of distinct roots of the equation*

$$\sum_{s\geq 0} d_s^p x^{p^s} = 0.$$

Note that in the setting of part (3) the third equation in (9.2.1) is equivalent to

$$w^p - w = \beta d_0^{-p}.$$

Proof. (1), (2), (3) Let $(\alpha, \beta) \in \mathbb{A}^2(\Bbbk)$. The defining relation of $H := H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}})$, namely $[X_2, X_1] = \delta(X_1)$, yields in $H_{\alpha,\beta}$ the relation

$$X_2 X_1 - X_1 X_2 = \delta(X_1) = d_0 X_1 + \sum_{s\geq 1} d_s \alpha^{p^{s-1}} = d_0 X_1 + d(\alpha). \tag{9.2.2}$$

Suppose first that $d_0^p\alpha + d(\alpha)^p \neq 0$. If $d_0 = 0$, then (9.2.2) yields the defining relation for the Weyl algebra. Thus $H_{\alpha,\beta}$ is a factor of the first Weyl algebra over $\mathbb{k}$, of dimension p^2. But the Weyl $\mathbb{k}$-algebra is Azumaya of PI-degree p by [42, Theorem 2], so each of its p^2-dimensional factors, in particular $H_{\alpha,\beta}$, is isomorphic to $M_p(\mathbb{k})$.

Next consider the case where $d_0^p\alpha + d(\alpha)^p \neq 0$ and $d_0 \neq 0$. Let u and w denote the images in $H_{\alpha,\beta}$ of $d_0X_1 + d(\alpha)$ and $d_0^{-1}X_2$, respectively. These satisfy the following relations in $H_{\alpha,\beta}$:

$$wu - uw = u, \quad u^p = d_0^p\alpha + d(\alpha)^p, \quad w^p - w = d_0^{-p}\beta.$$

The second of these relations shows that u is a unit, so post-multiplying the first relation by u^{-1} again yields the defining relation of the first Weyl algebra. As before, we find that $H_{\alpha,\beta} \cong M_p(\mathbb{k})$. We have therefore shown that $\mathcal{N}\mathcal{A}(H)$ is contained in the subvariety of maxspec(Z) defined by $d_0^pX_1^p + d(X_1^p)^p$.

Now assume that $d_0^p\alpha + d(\alpha)^p = 0$. Suppose first that $d_0 = 0$. Then $Z = \mathbb{k}[X_1^p, X_2^p]$, and so the images in $H_{\alpha,\beta}$ of $X_1 - \alpha^{\frac{1}{p}}$ and $X_2 - \beta^{\frac{1}{p}}$ are mutually commuting nilpotent elements, which together generate $H_{\alpha,\beta}$ and have index of nilpotency p. Therefore, $H_{\alpha,\beta}$ is the local algebra $\mathbb{k}[X,Y]/\langle X^p, Y^p\rangle$ in this case.

Finally, suppose that $d_0^p\alpha + d(\alpha)^p = 0$, with $d_0 \neq 0$. Let u and w denote the images in $H_{\alpha,\beta}$ of $d_0X_1 + d(\alpha)$ and $d_0^{-1}X_2$, respectively. Then we obtain the three relations in (9.2.1). The first two relations of (9.2.1) now show that u is a nilpotent normal element, with index of nilpotency at most p. Now $H_{\alpha,\beta}/uH_{\alpha,\beta} = \mathbb{k}\langle w\rangle$, and the third relation of (9.2.1) has the form $f(w) = 0$, where $f(x)$ is a polynomial with $f'(x) = -1$. Hence $f(w) = 0$ has no repeated roots (or $f(w) = 0$ has root $w, w+1, \dots, w+p-1$ when one root w is chosen), so that

$$H_{\alpha,\beta}/uH_{\alpha,\beta} \cong \mathbb{k}^{\oplus p}.$$

Since $\dim_{\mathbb{k}} H_{\alpha,\beta} = p^2$, this shows that $u^p = 0 \neq u^{p-1}$. Moreover, the second relation of (9.2.1) shows that the p distinct 1-dimensional simple $H_{\alpha,\beta}$-modules form a single block, with the Ext-quiver being a circle. This completes the proof of parts (1), (2), and (3).

(4) From the above proof, we see that a maximal ideal $\mathfrak{m}_{\alpha,\beta}$ of Z is in $\mathcal{N}\mathcal{A}(H)$ if and only if α is a root of the equation $d_0^px + d(x)^p = 0$, that is of the equation

$$d_0^px + \sum_{s\geq 1} d_s^p x^{p^s} = 0.$$

This proves (4). ∎

9.3. The Hopf center and restricted Hopf algebras

Recall the following definition, due to Andruskiewitsch [1, Definition 2.2.3].

Definition 9.3. Let H be a Hopf algebra. The *Hopf center* of H, denoted by $C(H)$, is the unique largest central Hopf subalgebra of H.

It is easy to see that $C(H)$ exists for any Hopf algebra H. As already discussed in Remark 3.5 (1), when $\Bbbk$ has positive characteristic p and $\mathfrak{g}$ is an n-dimensional Lie algebra over $\Bbbk$, $C(U(\mathfrak{g}))$ exists and is a polynomial algebra in n variables, with $U(\mathfrak{g})$ a free $C(U(\mathfrak{g}))$-module of rank a power of p. In general, even when H is a finite module over its center, $C(H)$ can be very small—consider, for instance, the group algebra $H = FG$ over any field F of the dihedral group

$$G = \langle a, b : b^2 = 1, bab = a^{-1} \rangle,$$

where $C(H) = F$. Nevertheless, current evidence suggests that when H is a connected Hopf $\Bbbk$-algebra, $C(H)$ may always be large, and Question 3.7 proposes that this may be the case for IHOEs over $\Bbbk$. We shall show in this subsection that this is indeed the case for all 2-step $\Bbbk$-IHOEs. First, however, we show that whenever a $\Bbbk$-IHOE *is* a finite module over its Hopf center, some of the desirable features of the Lie algebra case immediately follow. In this subsection, we denote the augmentation ideal of a Hopf algebra T by T_+.

Proposition 9.4. *Let $\Bbbk$ be algebraically closed of characteristic $p > 0$, let n be a positive integer, and let H be a noncommutative n-step IHOE over $\Bbbk$. Suppose that H is a finite $C(H)$-module. Then $C(H)$ is a polynomial algebra in n variables, and H is a free $C(H)$-module of rank p^ℓ for some $\ell \geq 2$.*

Proof. By [7, Proposition 2.5], H is connected as a Hopf algebra, so $C(H)$ is also connected. Moreover, $C(H)$ is affine of Gel'fand–Kirillov dimension n, by the Artin–Tate lemma and Corollary 3.4. By Theorem 7.1 ((3)⇒(2)), $C(H)$ is a commutative polynomial ring $\Bbbk[X_1, \dots, X_n]$. By [58, Theorem 0.3], H is a finitely generated projective module over $C(H)$. Hence H is a free module over $C(H)$ of finite rank. Let r denote the rank of H over $C(H)$. By definition, $C(H)$ is central in H, then $\bar{H} := H/(C(H))_+H$ is a Hopf algebra of dimension r. By [36, Corollary 5.3.5], $\bar{H}$ is connected. Since $\bar{H}$ is finite dimensional and connected, its dimension is of the form p^ℓ for some $\ell > 0$ [54, Proposition 2.2 (7)].

Since H is prime, the rank H as a $Z(H)$-module is a square of some integer. This implies that r is not a prime number. Therefore, $\ell \geq 2$. ∎

Observe that, in the setting of the proposition, the factor algebra

$$\bar{H} := H/C(H)_+H$$

is a connected Hopf algebra of dimension p^ℓ. It seems reasonable to call $\bar{H}$ the *restricted Hopf algebra of H*.

In the rest of this subsection, we confirm that the hypotheses of the proposition are satisfied by all noncommutative 2-step IHOEs, with ℓ equal to 2 or 3 in all cases, and we determine their restricted Hopf algebras. The following notation will remain in force throughout the rest of Section 9.3.

Notation 9.5. The field $\Bbbk$ is algebraically closed of characteristic $p > 0$, and H denotes a 2-step IHOE $H(\mathbf{d_s}, \mathbf{b_s}, \mathbf{c_{s,t}}) = \Bbbk\langle X_1, X_2\rangle$ as defined in Proposition 6.6; see also Section 1.3. The element $X_2^p - d_0^{p-1}X_2$ of H, which is central by Proposition 9.1 (4), will be denoted by z.

We will always denote the element of $H \otimes H$ defined in (1.2.2) by w, so that, writing λ_i for $\frac{(p-1)!}{i!(p-i)!}$ $(1 \le i \le p-1)$,

$$w = \sum_{s\geq 0} b_s\Bigg(\sum_{i=1}^{p-1} \lambda_i (X_1^{p^s})^i \otimes (X_1^{p^s})^{p-i}\Bigg) + \sum_{0\leq s<t} c_{s,t}(X_1^{p^s} \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1^{p^s}).$$

We denote the element $X_2 \otimes 1 + 1 \otimes X_2$ of $H \otimes H$ by b.

Let s be a positive integer divisible by p, and let $\equiv_s$ denote equality of elements of $H \otimes H$ modulo the subspace $X_1^s H \otimes H + H \otimes X_1^p H$. Since this subspace is preserved by ad_b, it follows that, for elements f and g of $H \otimes H$,

$$f \equiv_s g \Rightarrow ad_b(f) \equiv_s ad_b(g). \tag{9.5.1}$$

Using the fact that

$$ad_{X_2}(X_1^i) = iX_1^{i-1}\delta(X_1)$$

for all $i \geq 1$, one can check that

$$\begin{aligned} &ad_b(w) \\ &\quad = \big[\delta(X_1)\otimes 1\big]\Big(b_0\{(X_1\otimes 1+1\otimes X_1)^{p-1} - X_1^{p-1}\otimes 1\} + \sum_{t>0} c_{0,t} 1\otimes X_1^{p^t}\Big) \\ &\qquad + \big[1\otimes\delta(X_1)\big]\Big(b_0\{(X_1\otimes 1+1\otimes X_1)^{p-1} - 1\otimes X_1^{p-1}\} - \sum_{t>0} c_{0,t} X_1^{p^t}\otimes 1\Big). \end{aligned} \tag{9.5.2}$$

Lemma 9.6. *Retain Notation 9.5.*

(1) $\Delta(z) = z \otimes 1 + 1 \otimes z + w^p - d_0^{p-1}w + ad_b^{p-1}(w)$. *Moreover, the element* $w^p - d_0^{p-1}w + ad_b^{p-1}(w)$ *of* $H \otimes H$ *is in* $\Bbbk[X_1] \otimes \Bbbk[X_1]$.

(2) *The subalgebra* $\Bbbk[X_1^p, z]$ *of* H *is a Hopf subalgebra if and only if*

$$-d_0^{p-1}w + ad_b^{p-1}(w) \in \Bbbk[X_1^p] \otimes \Bbbk[X_1^p].$$

(3) *If* $b_0 = 0$, *then* $\Bbbk[X_1^p, z]$ *is a Hopf subalgebra of* H.

Proof. (1) First of all, for every $i \geq 0$, $ad_b^i(w) \in \Bbbk[X_1] \otimes \Bbbk[X_1]$. Hence $\{ad_b^i(w)\}_{i\geq 0}$ are in a commutative subalgebra of $H \otimes H$. Thus all the hypotheses of Corollary 8.2 are satisfied. Therefore, by that corollary and the definitions of H and z,

$$\begin{aligned} \Delta(z) &= \Delta(X_2^p) - d_0^{p-1}\Delta(X_2) \\ &= \big(\Delta(X_2)\big)^p - d_0^{p-1}\Delta(X_2) \\ &= (w+b)^p - d_0^{p-1}(w+b) \end{aligned}$$

$$= w^p + b^p + ad_b^{p-1}(w) - d_0^{p-1}(w + b)$$
$$= w^p + (X_2 \otimes 1 + 1 \otimes X_2)^p + ad_b^{p-1}(w) - d_0^{p-1}(X_2 \otimes 1 + 1 \otimes X_2 + w)$$
$$= w^p + (X_2 \otimes 1)^p + (1 \otimes X_2)^p + ad_b^{p-1}(w) - d_0^{p-1}(X_2 \otimes 1 + 1 \otimes X_2 + w)$$
$$= z \otimes 1 + 1 \otimes z + w^p - d_0^{p-1} w + ad_b^{p-1}(w).$$

That $w^p - d_0^{p-1} w + ad_b^{p-1}(w)$ is in $\Bbbk[X_1] \otimes \Bbbk[X_1]$ is clear from the definition of w.

(2) Denote $\Bbbk[X_1^p, z]$ by Z. Since a connected bialgebra is a Hopf algebra by [16, Corollary 3.5.4 (a)], it is enough to check when $\Delta(Z) \subseteq Z \otimes Z$. As X_1^p is primitive, this amounts to checking whether $\Delta(z) \in Z \otimes Z$. The result now follows from part (1) and the fact that $w^p \in \Bbbk[X_1^p] \otimes \Bbbk[X_1^p]$.

(3) If $b_0 = 0$, then, by (9.5.2),

$$ad_b(w) = [\delta(X_1) \otimes 1]\Big(\sum_{t>0} c_{0,t} 1 \otimes X_1^{p^t}\Big) + [1 \otimes \delta(X_1)]\Big(-\sum_{t>0} c_{0,t} X_1^{p^t} \otimes 1\Big).$$

By (9.1.1) and the fact that $ad_{X_2}(X_1^{pi}) = 0$ for all $i \geq 0$, we have that

$$ad_b^n(w) = [d_0^{n-1}\delta(X_1) \otimes 1]\Big(\sum_{t>0} c_{0,t} 1 \otimes X_1^{p^t}\Big) + [1 \otimes d_0^{n-1}\delta(X_1)]\Big(-\sum_{t>0} c_{0,t} X_1^{p^t} \otimes 1\Big)$$

for all $n \geq 1$. Then

$$ad_b^{p-1}(w) = d_0^{p-2}\Big\{(\delta(X_1) \otimes 1)\Big(\sum_{t>0} c_{0,t} 1 \otimes X_1^{p^t}\Big) - (1 \otimes \delta(X_1))\Big(\sum_{t>0} c_{0,t} X_1^{p^t} \otimes 1\Big)\Big\}$$
$$= d_0^{p-1}\Big\{\sum_{t>0} c_{0,t}(X_1 \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1)\Big\} + \mathcal{T}_1,$$

where $\mathcal{T}_1 \in \Bbbk[X_1^p] \otimes \Bbbk[X_1^p]$. On the other hand, by definition,

$$d_0^{p-1} w = d_0^{p-1}\Big\{\sum_{t>0} c_{0,t}(X_1 \otimes X_1^{p^t} - X_1^{p^t} \otimes X_1)\Big\} + \mathcal{T}_2,$$

where $\mathcal{T}_2 \in \Bbbk[X_1^p] \otimes \Bbbk[X_1^p]$. Hence

$$-d_0^{p-1} w + ad_b^{p-1}(w) \in \Bbbk[X_1^p] \otimes \Bbbk[X_1^p].$$

The assertion follows from part (2). ∎

Lemma 9.7. *Retain Notation 9.5.*

(1) *For all* $i > 0$,

$$ad_b^i(w) \equiv_p 0. \tag{9.7.1}$$

(2) *Suppose that* $b_0 \neq 0$ *and* $d_0 \neq 0$. *Then*

$$-d_0^{p-1} w + ad_b^{p-1}(w) \notin \Bbbk[X_1^p] \otimes \Bbbk[X_1^p].$$

Hence $\Bbbk[X_1^p, z]$ *is not a Hopf subalgebra of* H.

(3) *Suppose that* $d_0 = 0$, *that* $b_0 \neq 0$, *and that* $\delta(X_1) \neq 0$. *Then* $\Bbbk[X_1^p, z]$ *is not a Hopf subalgebra of* H.

Proof. (1) Using (9.5.2),

$$\begin{aligned} ad_b(w) &\equiv_p \big(\delta(X_1) \otimes 1\big) b_0 \{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1\} \\ &\quad + \big(1 \otimes \delta(X_1)\big) b_0 \{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1}\} \\ &\equiv_p d_0 b_0 (X_1 \otimes 1)\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1\} \\ &\quad + d_0 b_0 (1 \otimes X_1)\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1}\} \\ &\equiv_p d_0 b_0 (X_1 \otimes 1 + 1 \otimes X_1)^p - d_0 b_0 (X_1^p \otimes 1) - d_0 b_0 (1 \otimes X_1^p) \\ &\equiv_p 0. \end{aligned}$$

This proves (9.7.1) for $i = 1$, and the general case follows from (9.5.1).

(2) Assume that d_0 and b_0 are nonzero. Then, using (9.7.1),

$$\begin{aligned} -d_0^{p-1} w + ad_b^{p-1}(w) &\equiv_p -d_0^{p-1} w \\ &\equiv_p -d_0^{p-1} b_0 \Big(\sum_{i=1}^{p-1} \lambda_i X_1^i \otimes X_1^{p-i} \Big). \end{aligned}$$

This implies that

$$-d_0^{p-1} w + ad_b^{p-1}(w) \notin \Bbbk[X_1^p] \otimes \Bbbk[X_1^p].$$

The second claim now follows from Lemma 9.6 (2).

(3) Using (9.5.2) and the hypothesis $d_0 = 0$, we obtain

$$\begin{aligned} ad_b(w) = {} & \big[\delta(X_1) \otimes 1\big]\big(b_0\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - X_1^{p-1} \otimes 1\}\big) \\ & + \big[1 \otimes \delta(X_1)\big]\big(b_0\{(X_1 \otimes 1 + 1 \otimes X_1)^{p-1} - 1 \otimes X_1^{p-1}\}\big) + \mathcal{T}, \end{aligned}$$

where $\mathcal{T} \in \Bbbk[X_1^p] \otimes \Bbbk[X_1^p]$. Since $d_0 = 0$, $\delta(X_1) \in \Bbbk[X_1^p]$ and hence is central. Therefore, by induction, we can show that

$$\begin{aligned} ad_b^{p-1}(w) = {} & b_0(p-1)!\big[\delta(X_1) \otimes 1\big]\big\{\big(\delta(X_1) \otimes 1 + 1 \otimes \delta(X_1)\big)^{p-2}(X_1 \otimes 1 + 1 \otimes X_1) \\ & - \big(\delta(X_1) \otimes 1\big)^{p-2}(X_1 \otimes 1)\big\} \\ & + b_0(p-1)!\big[1 \otimes \delta(X_1)\big]\big\{\big(\delta(X_1) \otimes 1 + 1 \otimes \delta(X_1)\big)^{p-2}(X_1 \otimes 1 + 1 \otimes X_1) \\ & - \big(1 \otimes \delta(X_1)\big)^{p-2}(1 \otimes X_1)\big\} + \delta_{2,\operatorname{char}\Bbbk} \mathcal{T} \\ = {} & b_0(p-1)!\big[\big(\delta(X_1) \otimes 1 + 1 \otimes \delta(X_1)\big)^{p-1}(X_1 \otimes 1 + 1 \otimes X_1) \\ & - \big(\delta(X_1) \otimes 1\big)^{p-1}(X_1 \otimes 1) - \big(1 \otimes \delta(X_1)\big)^{p-1}(1 \otimes X_1)\big] + \delta_{2,\operatorname{char}\Bbbk} \mathcal{T}. \end{aligned}$$

Observe that this element is not a member of $\Bbbk[X_1^p]\otimes\Bbbk[X_1^p]$. Since $d_0=0$, $d_0^{p-1}w=0$. Therefore,

$$-d_0^{p-1}w+ad_b^{p-1}(w)\notin\Bbbk[X_1^p]\otimes\Bbbk[X_1^p].$$

The assertion thus follows from Lemma 9.6 (2). ■

Lemmas 9.6 and 9.7 can now be applied to determine $C(H)$ for all 2-step IHOEs:

Theorem 9.8. *Retain Notation 9.5. Assume that H is not commutative.*

(1) *The subalgebra $\Bbbk[X_1^p,z^p]$ is a central Hopf subalgebra of H.*

(2) *The Hopf center $C(H)$ of H is either $\Bbbk[X_1^p,z^p]$ or $\Bbbk[X_1^p,z]$.*

(3) *$C(H)=\Bbbk[X_1^p,z]$ if and only if $b_0=0$.*

Proof. (1) Since H is connected, it is enough to show that $\Bbbk[X_1^p,z^p]$ is a bialgebra. First, X_1^p is primitive. To see that

$$\Delta(z^p)\in\Bbbk[X_1^p,z^p]\otimes\Bbbk[X_1^p,z^p],$$

note that z is central and apply Lemma 9.6 (1) to determine $\Delta(z)^p$.

(2) In view of part (1) and Proposition 9.1 (4),

$$\Bbbk[X_1^p,z^p]\subseteq C(H)\subseteq\Bbbk[X_1^p,z].$$

Moreover, H is a free module over each of these subalgebras, with the ranks over the outer two algebras being p^3 and p^2. But the rank of H over $C(H)$ is also a power of p by Proposition 9.4. Thus equality must hold at some point in the chain of inclusions.

(3) The assertion follows from Lemmas 9.6 (3) and 9.7 (2), (3). ■

The following two propositions examine the restricted Hopf algebra of H in each of the two cases distinguished by Theorem 9.8.

Proposition 9.9. *Retain Notation 9.5. Assume that H is not commutative. Suppose that $b_0=0$. In parts (2) and (3), let $\bar{H}$ denote the quotient Hopf algebra H/Z_+H.*

(1) *The center $Z:=\Bbbk[X_1^p,z]$ is a Hopf subalgebra of H.*

(2) *Suppose that $d_0=0$. Then $\bar{H}$ is*

$$H_{0,0}\cong\Bbbk[x,y]/\langle x^p,y^p\rangle,$$

a commutative and cocommutative Hopf algebra of dimension p^2, with x and y primitive.

(3) *Suppose that $d_0\neq 0$. Then $\bar{H}$ is*

$$H_{0,0}\cong\Bbbk\langle x,y:[y,x]-x,\ x^p,\ y^p-y\rangle,$$

a cocommutative Hopf algebra of dimension p^2 with x and y primitive, which is the restricted enveloping algebra of the 2-dimensional non-abelian restricted Lie algebra.

Proof. The center Z is as stated, by Proposition 9.1 (4).

(1) This is Theorem 9.8 (3).

(2) By Proposition 9.2 (2),

$$H/Z_+H = H_{0,0} \cong \Bbbk[x, y]/\langle x^p, y^p\rangle,$$

where x and y are the images of X_1 and X_2, respectively. Since X_1 is primitive so is x, and y is also primitive under the hypotheses.

(3) Assume that $d_0 \neq 0$. Then Proposition 9.2 (3) applies, yielding

$$H/Z_+H = H_{0,0} \cong \Bbbk\big\langle x, y : x^p,\ y^p - y,\ [y, x] = x\big\rangle,$$

where x and y are, respectively, the images of X_1 and $d_0^{-1}X_2$. As in (2), x and y are both primitive since $b_0 = 0$. ∎

Proposition 9.10. *Retain Notation 9.5. Assume that H is not commutative. Suppose that $b_0 \neq 0$. In parts (2) and (3), let $\bar{H}$ denote the quotient Hopf algebra H/C_+H.*

(1) *The center $\Bbbk[X_1^p, z]$ is not a Hopf subalgebra of H and the unique largest central Hopf subalgebra of H is $C := \Bbbk[X_1^p, z^p]$.*

(2) *Suppose that $d_0 = 0$. Then*

$$\bar{H} \cong \Bbbk[x, y]/(x^p, y^{p^2})$$

with x primitive and

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}.$$

This is a commutative and cocommutative Hopf algebra of dimension p^3.

(3) *Suppose that $d_0 \neq 0$. Then the quotient Hopf algebra $\bar{H}$ is $\Bbbk\langle x, y, z\rangle$, with relations*

$$[y, x] = x, \quad z = y^p - y, \quad [y, z] = [x, z] = 0, \quad x^p = z^p = 0.$$

This is noncommutative and cocommutative of dimension p^3, with x primitive and

$$\Delta(y) = y \otimes 1 + 1 \otimes y + b_0 d_0^{-1} \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i},$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z - b_0 d_0^{-1} \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}.$$

Proof. (1) This follows from Theorem 9.8 (2), (3).

(2) Assume that $d_0 = 0$. Let x be the image of X_1 and y the image of $b_0^{-1}X_2$. The result follows by direct computation.

(3) Assume that $d_0 \neq 0$. Let x be the image of X_1, y the image of $b_0^{-1}X_2$, and z the image of $(d_0^{-1}X_2)^p - (d_0^{-1}X_2)$. Most of the assertions follow by routine computations. For example, it is not hard to check that

$$\Delta(y) = y \otimes 1 + 1 \otimes y + eu,$$

where $u = \sum_{i=1}^{p-1} \lambda_i x^i \otimes x^{p-i}$ and $e := b_0 d_0^{-1}$. Here we only prove the formula for $\Delta(z)$. Repeating the computation in Lemma 9.6 (1), we have, for $ad_b := ad_{y\otimes 1+1\otimes y}$,

$$\begin{aligned}\Delta(z) &= z \otimes 1 + 1 \otimes z + (eu)^p - eu + ad_b^{p-1}(eu)\\ &= z \otimes 1 + 1 \otimes z - eu + ad_b^{p-1}(eu),\end{aligned}$$

where the last equation follows from the fact that $u^p = 0$ in H/C_+H. Similarly to (9.5.2), one checks that

$$\begin{aligned}ad_b(u) &= (x \otimes 1)\big[(x \otimes 1 + 1 \otimes x)^{p-1} - x^{p-1} \otimes 1\big]\\ &\quad + (1 \otimes x)\big[(x \otimes 1 + 1 \otimes x)^{p-1} - 1 \otimes x^{p-1}\big]\\ &= (x \otimes 1)(x \otimes 1 + 1 \otimes x)^{p-1} - x^p \otimes 1\\ &\quad + (1 \otimes x)(x \otimes 1 + 1 \otimes x)^{p-1} - 1 \otimes x^p\\ &= (x \otimes 1 + 1 \otimes x)^p - x^p \otimes 1 - 1 \otimes x^p\\ &= x^p \otimes 1 + 1 \otimes x^p - x^p \otimes 1 - 1 \otimes x^p\\ &= 0.\end{aligned}$$

Therefore, $\Delta(z) = z \otimes 1 + 1 \otimes z - eu$. This finishes the proof. ■

Remarks 9.11. (1) In the light of Propositions 9.9–9.10 and the known situation for enveloping algebras of finite dimensional Lie algebras over $\Bbbk$, we conjecture that the following question has a positive answer:

Question 9.12. Let $\Bbbk$ be of positive characteristic p. Does every n-step $\Bbbk$-IHOE H have a Hopf center $C(H)$ over which H is a finite module?

(2) For an algebraically closed field $\Bbbk$ of positive characteristic p, the connected Hopf $\Bbbk$-algebras of dimension p^n for $p > 2$ and $n \leq 3$ have been classified in a series of papers by Nguyen, L. Wang, and X. Wang [37, 54, 55], culminating in [38]. The Hopf algebra $\bar{H}$ in Proposition 9.9 (2) is the one in [54, Theorem 7.4 (1)], and the Hopf algebra $\bar{H}$ in Proposition 9.9 (3) is isomorphic to the one in [54, Theorem 7.4 (5)]. The Hopf algebra $\bar{H}$ in Proposition 9.10 (2) is isomorphic to the one listed in [38, Table 5 with type T1 in the third case, p. 858]. The Hopf algebra $\bar{H}$ in Proposition 9.10 (3) can be written as $\Bbbk\langle X, Y, Z\rangle$ (where $X = y + z$, $Y = x$, and $Z = -b_0^{-1}d_0 z$), with relations

$$[X, Y] = Y, \quad [X, Z] = 0, \quad [Y, Z] = 0$$

and

$$X^p = X, \quad Y^p = 0, \quad Z^p = 0,$$

with X, Y being primitive and

$$\Delta(Z) = Z \otimes 1 + 1 \otimes Z + \sum_{i=1}^{p-1} \lambda_i Y^i \otimes Y^{p-i}.$$

Hence this Hopf algebra is isomorphic to the one in [37, Theorem 1.3 (B1)].

The following questions are thus very natural:

Question 9.13. (a) Which finite dimensional connected Hopf $\Bbbk$-algebras can be realised as factors of IHOEs over $\Bbbk$?

(b) If Question 9.12 has a positive answer, then which connected Hopf $\Bbbk$-algebras can be realised as the restricted Hopf algebras $H/C(H)_+H$ of $\Bbbk$-IHOEs H?

Regarding (a), note that a finite dimensional Hopf algebra T which can be realised as a factor of an IHOE necessarily has some rather strict structural constraints: namely, it will have a chain of Hopf subalgebras T_i, for $i = 1, \dots, t$ such that $T_1 = \Bbbk$ and $T_{i+1} = \Bbbk\langle T_i, x_{i+1}\rangle$ for elements $x_2, \dots, x_n$, with a corresponding "PBW-structure". It is thus clear that not all finite dimensional connected Hopf $\Bbbk$-algebras can be so realised—for example, the restricted enveloping algebra of $\mathfrak{sl}(3, \Bbbk)$ presumably cannot be presented as a factor of an IHOE. Conversely, however, Xingting Wang has informed us [56] that he has checked case by case that all connected Hopf $\Bbbk$-algebras of dimension at most p^3 can be realised as Hopf factors of IHOEs over $\Bbbk$, at least when $p > 2$.

(3) All of the speculations in remarks (1) and (2) can be reformulated with IHOEs over $\Bbbk$ replaced by *all* connected Hopf $\Bbbk$-algebras of finite Gel'fand–Kirillov dimension.

Acknowledgements. The authors would like to thank Sara Billey, Jason Gaddis, Sean Griffin, Liyu Liu, and Xingting Wang for many useful conversations and valuable comments on the subject. The initial ideas for this paper were developed in March 2019 during the Oberwolfach miniworkshop "Cohomology of Hopf Algebras and Tensor Categories". We are extremely grateful to the Mathematisches Forschungsinstitut, and to the organisers of the workshop, for their hospitality.

Funding. K. A. Brown was supported by Leverhulme Emeritus Fellowship EM-2017-081-9 and J. J. Zhang by the US National Science Foundation (Nos. DMS-1700825 and DMS-2001015).

References

[1] N. Andruskiewitsch, Notes on extensions of Hopf algebras. *Canad. J. Math.* **48** (1996), no. 1, 3–42 Zbl 0857.16033 MR 1382474

[2] M. Awami, M. Van den Bergh, and F. Van Oystaeyen, Note on derivations of graded rings and classification of differential polynomial rings. Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987). *Bull. Soc. Math. Belg. Sér. A* **40** (1988), no. 2, 175–183 Zbl 0663.16001 MR 976632

[3] R. Berger and V. Ginzburg, Higher symplectic reflection algebras and non-homogeneous N-Koszul property. *J. Algebra* **304** (2006), no. 1, 577–601 Zbl 1151.16026 MR 2256407

[4] K. A. Brown, P. Gilmartin, and J. J. Zhang, Connected (graded) Hopf algebras. *Trans. Amer. Math. Soc.* **372** (2019), no. 5, 3283–3317 Zbl 07089862 MR 3988611

[5] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*. Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2002 Zbl 1027.17010 MR 1898492

[6] K. A. Brown and M. J. Macleod, The Cohen Macaulay property for noncommutative rings. *Algebr. Represent. Theory* **20** (2017), no. 6, 1433–1465 Zbl 1388.16009 MR 3735914

[7] K. A. Brown, S. O'Hagan, J. J. Zhang, and G. Zhuang, Connected Hopf algebras and iterated Ore extensions. *J. Pure Appl. Algebra* **219** (2015), no. 6, 2405–2433 Zbl 1312.16031 MR 3299738

[8] K. A. Brown and J. J. Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras. *J. Algebra* **320** (2008), no. 5, 1814–1850 Zbl 1159.16009 MR 2437632

[9] K. A. Brown and J. J. Zhang, Survey on Hopf algebras of GK-dimension 1 and 2. In *Hopf algebras, tensor categories and related topics*, pp. 43–62, Contemp. Math. 771, Amer. Math. Soc., Providence, RI, 2021 MR 4281365

[10] C.-L. Chuang and T.-K. Lee, Identities with a single skew derivation. *J. Algebra* **288** (2005), no. 1, 59–77 Zbl 1073.16021 MR 2138371

[11] C.-L. Chuang and T.-K. Lee, Ore extensions which are GPI-rings. *Manuscripta Math.* **124** (2007), no. 1, 45–58 Zbl 1138.16011 MR 2336054

[12] S. Dăscălescu, C. Năstăsescu, and c. Raianu, *Hopf algebras: An introduction*. Monogr. Textb. Pure Appl. Math. 235, Marcel Dekker, New York, 2001 Zbl 0962.16026 MR 1786197

[13] P. Etingof, A PI degree theorem for quantum deformations. *J. Algebra* **466** (2016), 308–313 Zbl 1378.16038 MR 3541688

[14] K. R. Goodearl and J. J. Zhang, Noetherian Hopf algebra domains of Gelfand–Kirillov dimension two. *J. Algebra* **324** (2010), no. 11, 3131–3168 Zbl 1228.16030 MR 2732991

[15] K. R. Goodearl and J. J. Zhang, Non-affine Hopf algebra domains of Gelfand–Kirillov dimension two. *Glasg. Math. J.* **59** (2017), no. 3, 563–593 Zbl 1396.16024 MR 3682000

[16] R. G. Heyneman and M. E. Sweedler, Affine Hopf algebras. I. *J. Algebra* **13** (1969), 192–241 Zbl 0203.31601 MR 245570

[17] H. Huang, Hopf Ore extensions. *Algebr. Represent. Theory* **23** (2020), no. 4, 1477–1486 Zbl 1455.16027 MR 4125587

[18] J. E. Humphreys, *Linear algebraic groups*. Grad. Texts in Math. 21, Springer, New York, 1975 Zbl 0325.20039 MR 0396773

[19] N. Jacobson, Restricted Lie algebras of characteristic p. *Trans. Amer. Math. Soc.* **50** (1941), 15–25 Zbl 0025.30301 MR 5118

[20] N. Jacobson, A note on Lie algebras of characteristic p. *Amer. J. Math.* **74** (1952), 357–359 Zbl 0046.03402 MR 47026

[21] N. Jacobson, *Lie algebras. Republication of the 1962 original*. Dover Publications, New York, 1979 MR 559927

[22] T. Kambayashi, M. Miyanishi, and M. Takeuchi, *Unipotent algebraic groups*. Lecture Notes in Math. 414, Springer, Berlin, 1974 Zbl 0294.14022 MR 0376696

[23] V. Kharchenko, Differential identities of prime rings. *Algebra i Logika* **17** (1978), 220–238; English translation in *Algebra Logic* **17** (1979), 155–168 Zbl 0423.16011 MR 541758

[24] P. Kowalski and A. Pillay, Pro-algebraic and differential algebraic group structures on affine spaces. *Amer. J. Math.* **122** (2000), no. 1, 213–221 Zbl 0983.14033 MR 1737261

[25] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand–Kirillov dimension*. Revised edn., Grad. Stud. Math. 22, Amer. Math. Soc., Providence, RI, 2000 Zbl 0957.16001 MR 1721834

[26] M. Lazard, Sur la nilpotence de certains groupes algébriques. *C. R. Acad. Sci. Paris* **241** (1955), 1687–1689 Zbl 0065.25401 MR 75210

[27] A. Leroy and J. Matczuk, Ore extensions satisfying a polynomial identity. *J. Algebra Appl.* **5** (2006), no. 3, 287–306 Zbl 1103.16017 MR 2235812

[28] L. Liu and W. Ma, Nakayama automorphisms of ore extensions over polynomial algebras. *Glasg. Math. J.* **62** (2020), no. 3, 518–530 Zbl 1455.16010 MR 4133335

[29] D.-M. Lu, Q.-S. Wu, and J. J. Zhang, Homological integral of Hopf algebras. *Trans. Amer. Math. Soc.* **359** (2007), no. 10, 4945–4975 Zbl 1145.16022 MR 2320655

[30] B. Martin, D. Stewart, and L. Topley, A proof of the first Kac–Weisfeiler conjecture in large characteristics. With an appendix by Akaki Tikaradze. *Represent. Theory* **23** (2019), 278–293 Zbl 07121934 MR 4007168

[31] A. Masuoka, Semisimplicity criteria for irreducible Hopf algebras in positive characteristic. *Proc. Amer. Math. Soc.* **137** (2009), no. 6, 1925–1932 Zbl 1169.16025 MR 2480272

[32] G. Maury and J. Raynaud, *Ordres maximaux au sens de K. Asano*. Lecture Notes in Math. 808, Springer, Berlin, 1980 Zbl 0432.16003 MR 588150

[33] J. C. McConnell and J. C. Robson, *Noncommutative noetherian rings*. Pure Appl. Math., John Wiley & Sons, Chichester, 1987 Zbl 0644.16008 MR 934572

[34] J. S. Milne, *Algebraic groups. The theory of group schemes of finite type over a field*. Cambridge Stud. Adv. Math. 170, Cambridge University Press, Cambridge, 2017 Zbl 1390.14004 MR 3729270

[35] S. Montgomery, Trace functions and affine fixed rings for groups acting on noncommutative rings. In *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 34th Year (Paris, 1981)*, pp. 356–374, Lecture Notes in Math. 924, Springer, Berlin, 1982 Zbl 0479.16026 MR 662267

[36] S. Montgomery, *Hopf algebras and their actions on rings*. CBMS Reg. Conf. Ser. Math. 82, Amer. Math. Soc., Providence, RI, 1993 Zbl 0793.16029 MR 1243637

[37] V. C. Nguyen, L. Wang, and X. Wang, Classification of connected Hopf algebras of dimension p^3 I. *J. Algebra* **424** (2015), 473–505 Zbl 1312.16032 MR 3293230

[38] V. C. Nguyen, L. Wang, and X. Wang, Primitive deformations of quantum p-groups. *Algebr. Represent. Theory* **22** (2019), no. 4, 837–865 Zbl 07107937 MR 3985142

[39] E. Noether, Der Eindlichkeitssatz der Invarianten endlicher Gruppen. *Math. Ann.* **77** (1916), 89–92

[40] A. N. Panov, Ore extensions of Hopf algebras. *Mat. Zametki* **74** (2003), no. 3, 425–434 Zbl 1071.16035 MR 2022506

[41] A. Premet and S. Skryabin, Representations of restricted Lie algebras and families of associative L-algebras. *J. Reine Angew. Math.* **507** (1999), 189–218 Zbl 0978.17011 MR 1670211

[42] P. Revoy, Algèbres de Weyl en caractéristique p. *C. R. Acad. Sci. Paris Sér. A-B* **276** (1973), A225–A228 Zbl 0265.16007 MR 335564

[43] A. N. Rudakov, The representations of classical semisimple Lie algebras in characteristic p. *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 735–743 MR 0266970

[44] A. N. Rudakov and I. R. Šafarevič, Irreducible representations of a simple three-dimensional Lie algebra over a field of finite characteristic. *Mat. Zametki* **2** (1967), 439–454 Zbl 0184.06002 MR 219583

[45] J.-P. Serre, *Algebraic groups and class fields. Translated from the French*. Grad. Texts in Math. 117, Springer, New York, 1988 Zbl 0703.14001 MR 918564

[46] S. Skryabin, New results on the bijectivity of antipode of a Hopf algebra. *J. Algebra* **306** (2006), no. 2, 622–633 Zbl 1125.16031 MR 2271358

[47] L. Smith, *Polynomial invariants of finite groups*. Res. Notes Math. 6, A K Peters, Wellesley, MA, 1995 Zbl 0864.13002 MR 1328644

[48] H. Strade, Darstellungen auflösbarer Lie-p-Algebren. *Math. Ann.* **232** (1978), no. 1, 15–32 Zbl 0351.17005 MR 481594

[49] M. E. Sweedler, Cocommutative Hopf algebras with antipode. *Bull. Amer. Math. Soc.* **73** (1967), 126–128 Zbl 0173.03101 MR 214646

[50] L. Topley, A non-restricted counterexample to the first Kac–Weisfeiler conjecture. *Proc. Amer. Math. Soc.* **145** (2017), no. 5, 1937–1942 Zbl 1418.17042 MR 3611310

[51] D.-G. Wang, J. J. Zhang, and G. Zhuang, Hopf algebras of GK-dimension two with vanishing Ext-group. *J. Algebra* **388** (2013), 219–247 Zbl 1355.16033 MR 3061686

[52] D.-G. Wang, J. J. Zhang, and G. Zhuang, Connected Hopf algebras of Gelfand–Kirillov dimension four. *Trans. Amer. Math. Soc.* **367** (2015), no. 8, 5597–5632 Zbl 1330.16022 MR 3347184

[53] D.-G. Wang, J. J. Zhang, and G. Zhuang, Primitive cohomology of Hopf algebras. *J. Algebra* **464** (2016), 36–96 Zbl 1402.16019 MR 3533424

[54] X. Wang, Connected Hopf algebras of dimension p^2. *J. Algebra* **391** (2013), 93–113 Zbl 1301.16036 MR 3081624

[55] X. Wang, Isomorphism classes of finite dimensional connected Hopf algebras in positive characteristic. *Adv. Math.* **281** (2015), 594–623 Zbl 1325.16030 MR 3366848

[56] X. Wang, Personal communication. 2020

[57] B. Y. Weisfeiler and V. G. Kac, The irreducible representations of Lie p-algebras. *Funkcional. Anal. i Priložen.* **5** (1971), no. 2, 28–36; English translation in *Functional Anal. Appl.* **5** (1971), 111–117 MR 0285575

[58] Q.-S. Wu and J. J. Zhang, Noetherian PI Hopf algebras are Gorenstein. *Trans. Amer. Math. Soc.* **355** (2003), no. 3, 1043–1066 Zbl 1019.16028 MR 1938745

[59] Z. Yi, Injective homogeneity and homological homogeneity of the Ore extensions. *Acta Math. Sinica (N.S.)* **13** (1997), no. 4, 433–442 Zbl 0922.16006 MR 1612694

[60] G.-S. Zhou, Y. Shen, and D.-M. Lu, The structure of connected (graded) Hopf algebras. *Adv. Math.* **372** (2020), 107292, 31 Zbl 1452.16033 MR 4125517

[61] G. Zhuang, Properties of pointed and connected Hopf algebras of finite Gelfand–Kirillov dimension. *J. Lond. Math. Soc. (2)* **87** (2013), no. 3, 877–898 Zbl 1365.16018 MR 3073681

Received 21 May 2020.

Kenneth A. Brown
School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QQ, Scotland, UK;
ken.brown@glasgow.ac.uk

James J. Zhang
Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA;
zhang@math.washington.edu