J. Noncommut. Geom. 16 (2022), 927–966
DOI 10.4171/JNCG/455

The strong homotopy structure of Poisson reduction

Chiara Esposito, Andreas Kraft, and Jonas Schnitzer

Abstract. In this paper, we propose a reduction scheme for multivector fields phrased in terms of L_∞-morphisms. Using well-known geometric properties of the reduced manifolds, we perform a Taylor expansion of multivector fields, which allows us to build up a suitable deformation retract of differential graded Lie algebras (DGLAs). We first obtained an explicit formula for the L_∞-projection and -inclusion of generic DGLA retracts. We then applied this formula to the deformation retract that we constructed in the case of multivector fields on reduced manifolds. This allows us to obtain the desired reduction L_∞-morphism. Finally, we perform a comparison with other reduction procedures.

1. Introduction

This paper aims to propose a reduction scheme for multivector fields that is phrased in terms of L_∞-morphisms and adapted to deformation quantization. Deformation quantization has been introduced in [1, 2] by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer and it relies on the idea that the quantization of a Poisson manifold M is described by a formal deformation of the commutative algebra of smooth complex-valued functions $\mathscr{C}^\infty(M)$, a so-called *star product*. The existence and classification of star products on Poisson manifolds has been provided by Kontsevich's formality theorem [18], whereas the invariant setting of Lie group actions has been treated by Dolgushev; see [10, 11]. In the last years, many developments have taken place; see, e.g., [5,6,20]. More explicitly, the formality provides an L_∞-quasi-isomorphism between the differential graded Lie algebra (DGLA) of multivector fields and the multidifferential operators, resp., the invariant versions. One open question and our main motivation is to investigate the compatibility of deformation quantization and phase space reduction in the Poisson setting.

In the classical setting, one considers here the Marsden–Weinstein reduction [22]. Suppose that a Lie group G acts by Poisson diffeomorphisms on the Poisson manifold M and that it allows an Ad^*-equivariant momentum map $J: M \to \mathfrak{g}^*$ with $0 \in \mathfrak{g}^*$ as regular value, where $\mathfrak{g}$ is the Lie algebra of G. Then $C = J^{-1}(\{0\})$ is a closed embedded submanifold of M and the reduced manifold $M_{\mathrm{red}} = C/\mathrm{G}$ is again a Poisson manifold if the action on C is proper and free. Reduction theory is very important and it is still a very active field of research. Among the others, we mention the categorical reformulation performed in [9].

2020 Mathematics Subject Classification. Primary 53D20; Secondary 17B55, 16W60, 53D55.
Keywords. Reduction, multivector fields, L_∞-morphism.

In the setting of deformation quantization, a quantum reduction scheme has been introduced in [4]; see also [16] for a slightly different formulation, which allows the study of the compatibility between the reduction scheme and the properties of the star product, as in [14]. One crucial ingredient are quantum momentum maps (see [32]), and pairs consisting of star products with compatible quantum momentum maps are called *equivariant star products*. For symplectic manifolds, these equivariant star products have recently been classified and it has been shown that quantization commutes with reduction; see [27–29]. More precisely, equivariant star products on M are classified by certain elements in the cohomology of the Cartan model for equivariant de Rham cohomology [15], and the characteristic classes of the equivariant star product and the reduced star product are related by pull-backs.

In the more general setting of Poisson manifolds, star products are classified by Maurer–Cartan elements in the DGLA of multivector fields, i.e., by formal Poisson structures. Unfortunately, in this case there is no pull-back available and one has to use different techniques. Motivated by the aim of reducing the formality, we want to describe the reductions in terms of L_∞-morphisms. In particular, in this paper we construct such a reduction for the classical side, i.e., for the *equivariant multivector fields* $T_{\mathfrak{g}}(M)$, a certain DGLA whose Maurer–Cartan elements are invariant Poisson structures with equivariant momentum maps. Assuming for simplicity that $M = C \times \mathfrak{g}^*$, which always holds locally in suitable situations, we can perform a Taylor expansion around C, obtaining a new DGLA $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$. On $C \times \mathfrak{g}^*$, we have the canonical momentum map J given by the projection on $\mathfrak{g}^*$ and the canonical linear Poisson structure π_{KKS} induced by the action Lie algebroid. They give a new DGLA structure on $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ with differential $[\pi_{\mathrm{KKS}} - J, \cdot\,]$ and we show that this DGLA is quasi-isomorphic to the multivector fields on M_{red}, as desired. One has an L_∞-quasi-isomorphism between these two DGLAs (see Theorem 4.21):

Theorem. *There exists an L_∞-quasi-isomorphism* $\widetilde{\mathrm{T}}_{\mathrm{red}}\colon T_{\mathrm{Tay}}(C \times \mathfrak{g}^*) \to T_{\mathrm{poly}}(M_{\mathrm{red}})$.

The morphism $\widetilde{\mathrm{T}}_{\mathrm{red}}$ is obtained by inverting a certain inclusion i of DGLAs. In order to give a more explicit formula, we look at general deformation retracts: let (A, d_A) and (B, d_B) be two DGLAs and assume that we have

$$(A, \mathrm{d}_A) \underset{p}{\overset{i}{\rightleftarrows}} (B, \mathrm{d}_B) \circlearrowleft h, \tag{1.1}$$

where i and p are quasi-isomorphisms of cochain complexes with homotopy h, and where $p \circ i = \mathrm{id}_A$ and $h^2 = h \circ i = p \circ h = 0$. Using for a coalgebra morphism $F\colon \mathrm{S}(B[1]) \to \mathrm{S}(A[1])$ the notation

$$L_{\infty,k+1}(F) = Q^1_{A,2} \circ F^2_{k+1} - F^1_k \circ Q^k_{B,k+1},$$

where $Q^k_{A,k+1}$ and $Q^k_{B,k+1}$ are the extensions of the Lie brackets to the symmetric algebras, and extending h in an appropriate way to H_k on $\mathrm{S}^k(B[1])$, we prove in Propositions 3.2 and 3.3 the following:

Proposition. *Given a deformation retract as in (1.1),*

(i) *if i is a DGLA morphism, then $P\colon \mathrm{S}^\bullet(B[1]) \to \mathrm{S}^\bullet(A[1])$ with structure maps $P_1^1 = p$ and $P_{k+1}^1 = L_{\infty,k+1}(P) \circ H_{k+1}$ for $k \geq 1$ yields an L_∞-quasi-isomorphism that is quasi-inverse to i;*

(ii) *if p is a DGLA morphism, then $I\colon \mathrm{S}^\bullet(A[1]) \to \mathrm{S}^\bullet(B[1])$ with structure maps $I_1^1 = i$ and $I_k^1 = h \circ L_{\infty,k}(I)$ for $k \geq 2$ is an L_∞-quasi-isomorphism that is quasi-inverse to p.*

These explicit formulas allow us to give a more precise description of $\widetilde{\mathrm{T}}_{\mathrm{red}}$ and its L_∞-quasi-inverse. Moreover, they allow us to globalize the result (compare Theorem 5.1):

Theorem. *There exists a curved L_∞-morphism*

$$\mathrm{T}_{\mathrm{red}}\colon \big(T_{\mathfrak{g}}(M), \lambda, -[J, \cdot\,], [\,\cdot\,,\cdot\,]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, 0, [\,\cdot\,,\cdot\,]\big), \tag{1.2}$$

where the curvature $\lambda = e^i \otimes (e_i)_M$ is given by the fundamental vector fields of the G-*action.*

We call $\mathrm{T}_{\mathrm{red}}$ *reduction L_∞-morphism* and we extend the statements to the setting of formal power series in $\hbar$. After rescaling the involved curvatures and differentials appropriately, $\mathrm{T}_{\mathrm{red}}$ gives, in particular, a way to associate formal Maurer–Cartan elements. In $T_{\mathfrak{g}}(M)[[\hbar]]$ resp. $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$ with rescaled structures, formal Maurer–Cartan elements can be interpreted as formal Poisson structures $\pi_\hbar$ with formal momentum map $J_\hbar = J + \hbar J'$. Thus, we have the following properties:

- the Poisson bracket $\{\,\cdot\,,\cdot\,\}_\hbar$ induced by $\pi_\hbar$ is G-invariant;
- the fundamental vector fields are given by $\xi_M = \{\,\cdot\,, J_\hbar(\xi)\}_\hbar \in \Gamma^\infty(TM)$;
- $\{J_\hbar(\xi), J_\hbar(\eta)\}_\hbar = J_\hbar([\xi,\eta])$.

Comparing the orders of $\hbar$ directly shows that the lowest order is a well-defined Poisson structure on M and that J is an equivariant momentum map with respect to it; moreover, $\mathrm{T}_{\mathrm{red}}$ maps such an object to a formal Poisson structure on M_{red}.

Note that there is also another reduction scheme for such formal Poisson structures with formal momentum maps, obtained by adapting the Koszul part of the classical Becchi–Rouet–Stora–Tyutin (BRST) reduction [19, 30] to the formal setting. This can be done by using the homological perturbation lemma as in [8], and it corresponds to the classical analogue of the reduction scheme for star products from [4, 16]. Finally, we show in Theorem 5.4 the following:

Theorem. *The reduction of formal equivariant Poisson structures with formal momentum maps via the reduction L_∞-morphism coincides with the reduction of formal Poisson structures via the formal Koszul complex.*

The paper is organized as follows: in Section 2, we recall the basic notions of (curved) L_∞-algebras, L_∞-morphisms, and twists. In Section 3, we consider general deformation retracts of DGLAs and prove the explicit formulas for the extensions of the inclusion resp.

projection to L_∞-morphisms needed to describe $\widetilde{\mathrm{T}}_{\mathrm{red}}$ in Section 4. Here we also construct the reduction scheme for the Taylor expansion, both in the classical and the formal setting. Finally, in Section 5, we construct the global reduction L_∞-morphism and compare the reduction via $\mathrm{T}_{\mathrm{red}}$ with the classical Marsden–Weinstein reduction and with the reduction of formal Poisson structures via the formal Koszul complex as explained in Appendix A.

2. Preliminaries

In this section, we recall the notions of (curved) L_∞-algebras, L_∞-morphisms, and their twists by Maurer–Cartan elements to fix the notation. Proofs and further details can be found in [10, 11, 13].

We denote by $V^\bullet$ a graded vector space over a field $\mathbb{K}$ of characteristic 0 and define the *shifted* vector space $V[k]^\bullet$ by

$$V[k]^\ell = V^{\ell+k}.$$

A degree $+1$ coderivation Q on the coaugmented counital conilpotent cocommutative coalgebra $S^c(\mathfrak{L})$ cofreely cogenerated by the graded vector space $\mathfrak{L}[1]^\bullet$ over $\mathbb{K}$ is called an L_∞-*structure* on the graded vector space $\mathfrak{L}$ if $Q^2 = 0$. The (universal) coalgebra $S^c(\mathfrak{L})$ can be realized as the symmetrized deconcatenation coproduct on the space $\bigoplus_{n\geq 0} \mathrm{S}^n\,\mathfrak{L}[1]$, where $\mathrm{S}^n\,\mathfrak{L}[1]$ is the space of coinvariants for the usual (graded) action of S_n (the symmetric group in n letters) on $\otimes^n(\mathfrak{L}[1])$; see, e.g., [13]. Any degree $+1$ coderivation Q on $S^c(\mathfrak{L})$ is uniquely determined by the components

$$Q_n \colon \mathrm{S}^n\left(\mathfrak{L}[1]\right) \to \mathfrak{L}[2] \tag{2.1}$$

through the formula

$$Q(\gamma_1 \vee \cdots \vee \gamma_n) = \sum_{k=0}^{n} \sum_{\sigma\in \mathrm{Sh}(k,n-k)} \varepsilon(\sigma) Q_k(\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k)}) \vee \gamma_{\sigma(k+1)} \vee \cdots \vee \gamma_{\sigma(n)}. \tag{2.2}$$

Here $\mathrm{Sh}(k,n-k)$ denotes that the set of $(k,n-k)$ shuffles in S_n, $\varepsilon(\sigma) = \varepsilon(\sigma,\gamma_1,\ldots,\gamma_n)$ is a sign given by the rule $\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(n)} = \varepsilon(\sigma)\gamma_1 \vee \cdots \vee \gamma_n$ and we use the conventions that $\mathrm{Sh}(n,0) = \mathrm{Sh}(0,n) = \{\mathrm{id}\}$ and that the empty product equals the unit. Note, in particular, that we also consider a term Q_0 and thus we are actually considering curved L_∞-algebras (which will be convenient in the following). Sometimes, we also write $Q_k = Q_k^1$ and following [7] we denote by Q_n^i the component of $Q_n^i \colon \mathrm{S}^n\,\mathfrak{L}[1] \to \mathrm{S}^i\,\mathfrak{L}[2]$ of Q. It is given by

$$Q_n^i(x_1 \vee \cdots \vee x_n) = \sum_{\sigma\in \mathrm{Sh}(n+1-i,i-1)} \varepsilon(\sigma) Q_{n+1-i}^1(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n+1-i)}) \vee x_{\sigma(n+2-i)} \vee \cdots \vee x_{\sigma(n)}, \tag{2.3}$$

where Q_{n+1-i}^1 are the usual structure maps.

Example 2.1 (Curved Lie algebra). A basic example of an L_∞-algebra is that of a (curved) Lie algebra $(\mathfrak{L}, R, \mathrm{d}, [\cdot\,,\,\cdot])$ by setting $Q_0(1) = -R$, $Q_1 = -\mathrm{d}$, $Q_2(\gamma \vee \mu) = -(-1)^{|\gamma|}[\gamma, \mu]$, and $Q_i = 0$ for all $i \geq 3$. Note that we denoted the degree in $\mathfrak{L}[1]$ by $|\cdot|$.

Let us consider two L_∞-algebras: $(\mathfrak{L}, Q)$ and $(\widetilde{\mathfrak{L}}, \widetilde{Q})$. A degree 0 counital coalgebra morphism

$$F\colon S^c(\mathfrak{L}) \to S^c(\widetilde{\mathfrak{L}})$$

such that $FQ = \widetilde{Q}F$ is said to be an L_∞*-morphism*. A coalgebra morphism F from $S^c(\mathfrak{L})$ to $S^c(\widetilde{\mathfrak{L}})$ such that $F(1) = 1$ is uniquely determined by its components (also called the *Taylor coefficients*)

$$F_n\colon \mathrm{S}^n\big(\mathfrak{L}[1]\big) \to \widetilde{\mathfrak{L}}[1],$$

where $n \geq 1$. Namely, we set $F(1) = 1$ and use the formula

$$\begin{aligned} F(\gamma_1 \vee \cdots \vee \gamma_n) &= \sum_{p\geq 1} \sum_{\substack{k_1,\ldots,k_p\geq 1\\ k_1+\cdots+k_p=n}} \sum_{\sigma\in \mathrm{Sh}(k_1,\ldots,k_p)} \frac{\varepsilon(\sigma)}{p!} F_{k_1}(\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k_1)}) \\ &\qquad \vee \cdots \vee F_{k_p}(\gamma_{\sigma(n-k_p+1)} \vee \cdots \vee \gamma_{\sigma(n)}), \end{aligned} \tag{2.4}$$

where $\mathrm{Sh}(k_1, \ldots, k_p)$ denotes the set of $(k_1, \ldots, k_p)$-shuffles in S_n (again we set $\mathrm{Sh}(n) = \{\mathrm{id}\}$). We also write $F_k = F_k^1$ and similarly to (2.3) we get coefficients $F_n^j\colon \mathrm{S}^n\,\mathfrak{L}[1] \to \mathrm{S}^j\,\widetilde{\mathfrak{L}}[1]$ of F by taking the corresponding terms in [12, equation (2.15)]. Note that F_n^j depends only on $F_k^1 = F_k$ for $k \leq n - j + 1$. Given an L_∞-morphism F of (non-curved) L_∞-algebras $(\mathfrak{L}, Q)$ and $(\widetilde{\mathfrak{L}}, \widetilde{Q})$, we obtain the map of complexes

$$F_1\colon (\mathfrak{L}, Q_1) \to (\widetilde{\mathfrak{L}}, \widetilde{Q}_1).$$

In this case, the L_∞-morphism F is called an L_∞*-quasi-isomorphism* if F_1 is a quasi-isomorphism of complexes. Given a DGLA $(\mathfrak{L}, \mathrm{d}, [\cdot, \cdot])$ and an element $\pi \in \mathfrak{L}[1]^0$, we can obtain a curved Lie algebra by defining a new differential $\mathrm{d} + [\pi, \cdot\,]$ and considering the curvature $R^\pi = \mathrm{d}\pi + \frac{1}{2}[\pi, \pi]$. In fact, the same procedure can be applied to a curved Lie algebra $(\mathfrak{L}, R, \mathrm{d}, [\cdot\,,\,\cdot])$ to obtain the twisted curved Lie algebra $(\mathfrak{L}, R^\pi, \mathrm{d} + [\pi, \cdot\,], [\cdot\,,\,\cdot])$, where

$$R^\pi := R + \mathrm{d}\pi + \frac{1}{2}[\pi, \pi].$$

The element π is called a *Maurer–Cartan element* if it satisfies the equation

$$R + \mathrm{d}\pi + \frac{1}{2}[\pi, \pi] = 0.$$

Finally, it is important to recall that given a DGLA morphism, or more generally an L_∞-morphism, $F\colon \mathfrak{L} \to \widetilde{\mathfrak{L}}$, one may associate to any (curved) Maurer–Cartan element $\pi \in \mathfrak{L}[1]^0$ a (curved) Maurer–Cartan element

$$\pi_F := \sum_{n\geq 1} \frac{1}{n!} F_n(\pi \vee \cdots \vee \pi) \in \widetilde{\mathfrak{L}}[1]^0.$$

In order to make sense of these infinite sums, we consider complete filtered L_∞-algebras and we demand that Maurer–Cartan elements be in a positive filtration; see [10, 13] for details on such filtrations.

3. An explicit formula for the L_∞-projection and -inclusion

From the general theory of L_∞-algebras, one knows that L_∞-quasi-isomorphisms always admit L_∞-quasi-inverses. Moreover, it is well known that, given a homotopy retract, one can transfer L_∞-structures. Explicitly, given two cochain complexes (A, d_A) and (B, d_B) with

$$(A, \mathrm{d}_A) \underset{p}{\overset{i}{\rightleftarrows}} (B, \mathrm{d}_B) \circlearrowleft h, \tag{3.1}$$

where $h \circ \mathrm{d}_B + \mathrm{d}_B \circ h = \mathrm{id} - i \circ p$ and where i is a quasi-isomorphism, the homotopy transfer theorem in [21, Section 10.3] states that if there exists an L_∞-structure on B, then one can transfer it to A in such a way that i extends to an L_∞-quasi-isomorphism.

Let us consider the special case of deformation retracts for DGLAs. More explicitly, let A, B be two DGLAs. A deformation retract of (A, d_A) is given by the diagram

$$(A, \mathrm{d}_A) \underset{p}{\overset{i}{\rightleftarrows}} (B, \mathrm{d}_B) \circlearrowleft h, \tag{3.2}$$

where i and p are quasi-isomorphisms of cochain complexes with homotopy h, i.e., $h\mathrm{d}_B + \mathrm{d}_B h = \mathrm{id}_B - ip$, as well as

$$p \circ i = \mathrm{id}_A, \quad h^2 = 0, \quad h \circ i = 0, \quad \text{and} \quad p \circ h = 0.$$

In addition, we assume that i is a DGLA morphism. As already mentioned, the homotopy transfer theorem and the invertibility of L_∞-quasi-isomorphisms imply that p extends to an L_∞-quasi-isomorphism denoted by P; see, e.g., [21, Proposition 10.3.9]. In the following, we give a more explicit description of P inspired by the symmetric tensor trick [3, 17]. The DGLA structures yield the codifferentials Q_A on $\mathrm{S}(A[1])$ and Q_B on $\mathrm{S}(B[1])$, and the map h extends to a homotopy $H_n \colon \mathrm{S}^n(B[1]) \to \mathrm{S}^n(B[1])[-1]$ with respect to

$$Q^n_{B,n} \colon \mathrm{S}^n\left(B[1]\right) \to \mathrm{S}^n\left(B[1]\right)[1];$$

see, e.g., [21, p. 383] for the construction on the tensor algebra, which adapted to our setting works roughly like: we define the operator

$$K_n \colon \mathrm{S}^n\left(B[1]\right) \to \mathrm{S}^n\left(B[1]\right)$$

by

$$K_n(x_1 \vee \cdots \vee x_n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \frac{\varepsilon(\sigma)}{n-i} ipX_{\sigma(1)} \vee \cdots \vee ipX_{\sigma(i)} \vee X_{\sigma(i+1)} \vee X_{\sigma(n)}.$$

Note that here we sum over the whole symmetric group and not the shuffles, since in this case the formulas are easier. We extend $-h$ to a coderivation to $\mathrm{S}(B[1])$, i.e.,

$$\widetilde{H}_n(x_1 \vee \cdots \vee x_n) := - \sum_{\sigma \in \mathrm{Sh}(1,n-1)} \varepsilon(\sigma) h x_{\sigma(1)} \vee x_{\sigma(2)} \vee \cdots \vee x_{\sigma(n)}$$

and define

$$H_n = K_n \circ \widetilde{H}_n = \widetilde{H}_n \circ K_n.$$

Since i and p are chain maps, we have

$$K_n \circ Q^n_{B,n} = Q^n_{B,n} \circ K_n,$$

where $Q^n_{B,n}$ is the extension of the differential $Q^1_{B,1} = -\mathrm{d}_B$ to $\mathrm{S}^n(B[1])$ as coderivation. Hence we have

$$Q^n_{B,n} H_n + H_n Q^n_{B,n} = (n \cdot \mathrm{id} - ip) \circ K_n,$$

where ip is extended as a coderivation to $\mathrm{S}(B[1])$. A combinatorial and not very enlightening computation shows that finally

$$Q^n_{B,n} H_n + H_n Q^n_{B,n} = \mathrm{id} - (ip)^{\vee n}. \tag{3.3}$$

Suppose that we have constructed a morphism of coalgebras P with structure maps

$$P^1_k \colon \mathrm{S}^k \big(B[1]\big) \to A[1]$$

that is an L_∞-morphism up to order k, i.e.,

$$\sum_{\ell=1}^{m} P^1_\ell \circ Q^\ell_{B,m} = \sum_{\ell=1}^{m} Q^1_{A,\ell} \circ P^\ell_m$$

for all $m \leq k$. Then we have the following statement.

Lemma 3.1. *Let $P \colon \mathrm{S}(B[1]) \to \mathrm{S}(A[1])$ be an L_∞-morphism up to order $k \geq 1$. Then*

$$L_{\infty,k+1} = \sum_{\ell=2}^{k+1} Q^1_{A,\ell} \circ P^\ell_{k+1} - \sum_{\ell=1}^{k} P^1_\ell \circ Q^\ell_{B,k+1} = Q^1_{A,2} \circ P^2_{k+1} - P^1_k \circ Q^k_{B,k+1} \tag{3.4}$$

satisfies

$$L_{\infty,k+1} \circ Q^{k+1}_{B,k+1} = -Q^1_{A,1} \circ L_{\infty,k+1}. \tag{3.5}$$

Proof. The statement follows from a straightforward computation. For convenience, we omit the index of the differential:

$$L_{\infty,k+1} Q^{k+1}_{k+1} = \sum_{\ell=2}^{k+1} Q^1_\ell (P \circ Q)^\ell_{k+1} - \sum_{\ell=2}^{k+1} \sum_{i=1}^{k} Q^1_\ell P^\ell_i Q^i_{k+1} + \sum_{\ell=1}^{k} \sum_{i=1}^{k} P^1_\ell Q^\ell_i Q^i_{k+1}$$

$$= \sum_{\ell=2}^{k+1} Q_\ell^1 (Q \circ P)_{k+1}^\ell - \sum_{\ell=2}^{k+1} \sum_{i=1}^{k} Q_\ell^1 P_i^\ell Q_{k+1}^i + \sum_{\ell=1}^{k} \sum_{i=1}^{k} Q_\ell^1 P_i^\ell Q_{k+1}^i$$

$$= -Q_1^1 (Q \circ P)_{k+1}^1 + Q_1^1 \sum_{i=1}^{k} P_i^1 Q_{k+1}^i = -Q_1^1 L_{\infty,k+1},$$

where the last equality follows from $Q_1^1 Q_1^1 = 0$. ∎

This allows us to obtain the L_∞-quasi-inverse of i, denoted by P, in (3.2) recursively:

Proposition 3.2. *Defining $P_1^1 = p$ and $P_{k+1}^1 = L_{\infty,k+1} \circ H_{k+1}$ for $k \geq 1$ yields an L_∞-quasi-isomorphism*

$$P\colon \mathrm{S}\left(B[1]\right) \to \mathrm{S}\left(A[1]\right)$$

that is quasi-inverse to i.

Proof. We observe $P_{k+1}^1(ix_1 \vee \cdots \vee ix_{k+1}) = 0$ for all $k \geq 1$ and $x_i \in A$, which directly follows from $h \circ i = 0$ and thus $H_{k+1} \circ i^{\vee(k+1)} = 0$. In addition, one also has for all $k \geq 1$ the identity $L_{\infty,k+1}(ix_1, \ldots, ix_{k+1}) = 0$, which follows from the definition of $L_{\infty,k+1}$ and the fact that i is a morphism of DGLAs. We know that P is an L_∞-morphism up to order one. Suppose that we already know that it is an L_∞-morphism up to order $k \geq 1$, then this implies that

$$\begin{aligned} P_{k+1}^1 \circ Q_{k+1}^{k+1} &= L_{\infty,k+1} \circ H_{k+1} \circ Q_{k+1}^{k+1} \\ &= L_{\infty,k+1} - L_{\infty,k+1} \circ Q_{k+1}^{k+1} \circ H_{k+1} - L_{\infty,k+1} \circ (i \circ p)^{\vee(k+1)} \\ &= L_{\infty,k+1} + Q_1^1 \circ P_{k+1}^1 \end{aligned}$$

by the above lemma, and therefore

$$P_{k+1}^1 \circ Q_{k+1}^{k+1} - Q_1^1 \circ P_{k+1}^1 = L_{\infty,k+1}.$$

Hence P is an L_∞-morphism up to order $k+1$ and the statement follows inductively. ∎

Let us now assume that $p\colon B \to A$ in the deformation retract (3.2) is a DGLA morphism and that i is just a chain map. Then we can analogously give a formula for the extension I of i to an L_∞-quasi-isomorphism.

Proposition 3.3. *The coalgebra map $I\colon \mathrm{S}^\bullet(A[1]) \to \mathrm{S}^\bullet(B[1])$ recursively defined by the maps $I_1^1 = i$ and $I_k^1 = h \circ L_{\infty,k}$ for $k \geq 2$ is an L_∞-quasi inverse of p. Since $h^2 = 0 = h \circ i$, one even has $I_k^1 = h \circ Q_2^1 \circ I_k^2$.*

Proof. We proceed by induction: assume that I is an L_∞-morphism up to order k, then we have

$$\begin{aligned} I_{k+1}^1 Q_{A,k+1}^{k+1} - Q_{B,1}^1 I_{k+1}^1 &= -Q_{B,1}^1 \circ h \circ L_{\infty,k+1} + h \circ L_{\infty,k+1} \circ Q_{A,k+1}^{k+1} \\ &= -Q_{B,1}^1 \circ h \circ L_{\infty,k+1} - h \circ Q_{B,1}^1 \circ L_{\infty,k+1} \\ &= (\mathrm{id} - i \circ p) L_{\infty,k+1}. \end{aligned}$$

We used that $Q^1_{B,1} = -\mathrm{d}_B$ and the homotopy equation of h. Moreover, since p is a DGLA morphism and $p \circ h = 0$, we have that $p \circ L_{\infty,k+1} = 0$ for $k \geq 0$. Since I is an L_∞-morphism up to order one, i.e., a chain map, the claim is proven. ∎

4. Reduction of multivector fields

In the following, we want to use the above language and considerations to formulate a reduction scheme for multivector fields. We first introduce a new complex of multivector field which contains the data of Hamiltonian actions in the case of Lie group actions $\Phi: G \times M \to M$.

Definition 4.1 (Equivariant multivectors). The DGLA of equivariant multivector fields is given by the complex $T^\bullet_\mathfrak{g}(M)$ defined by

$$T^k_\mathfrak{g}(M) = \bigoplus_{2i+j=k} \big(\mathrm{S}^i\, \mathfrak{g}^* \otimes \Gamma^\infty(\Lambda^{j+1}TM)\big)^{\mathrm{G}} = \bigoplus_{2i+j=k} \big(\mathrm{S}^i\, \mathfrak{g}^* \otimes T^j_{\mathrm{poly}}(M)\big)^{\mathrm{G}},$$

together with the trivial differential and the Lie bracket

$$[\alpha \otimes X, \beta \otimes Y]_\mathfrak{g} = \alpha \vee \beta \otimes [X, Y]$$

for any $\alpha \otimes X, \beta \otimes Y \in T^\bullet_\mathfrak{g}(M)$.

Here $[\,\cdot\,,\,\cdot\,]$ refers to the usual Schouten–Nijenhuis bracket on $T_{\mathrm{poly}}(M)$. Notice that invariance with respect to the group action means invariance under the transformations $\mathrm{Ad}^*_g \otimes \Phi^*_g$ for all $g \in G$. We can equivalently interpret this complex in terms of polynomial maps $\mathfrak{g} \to T^j_{\mathrm{poly}}(M)$ which are equivariant with respect to adjoint and push-forward action. Using this point of view, the bracket can be rewritten as

$$[X, Y]_\mathfrak{g}(\xi) = \big[X(\xi), Y(\xi)\big]. \tag{4.1}$$

Furthermore, we introduce the canonical linear map

$$\lambda: \mathfrak{g} \ni \xi \mapsto \xi_M \in T^0_{\mathrm{poly}} M, \tag{4.2}$$

where ξ_M denotes the fundamental vector field corresponding to the action Φ. It is easy to see that λ is central and as a consequence we can turn $T^\bullet_\mathfrak{g} M$ into a *curved* Lie algebra with curvature λ. Now let (M, π) be a Poisson manifold and denote by $\{\,\cdot\,,\,\cdot\,\}$ the corresponding Poisson bracket. Recall that an (equivariant) momentum map for the action Φ is a map $J: \mathfrak{g} \to \mathscr{C}^\infty(M)$ such that

$$\xi_M = \{\,\cdot\,, J_\xi\} \quad \text{and} \quad J_{[\xi,\eta]} = \{J_\xi, J_\eta\}. \tag{4.3}$$

An action Φ admitting a momentum map is what we called *Hamiltonian*. In the following, we prove a characterization of Hamiltonian actions in terms of equivariant multivectors.

Lemma 4.2. *The curved Maurer–Cartan elements of $T_{\mathfrak{g}}^{\bullet}(M)$ are equivalent to pairs (π, J), where π is a G-invariant Poisson structure and J is a momentum map*

$$J : \mathfrak{g} \to T_{\mathrm{poly}}^{-1}(M).$$

Proof. The curved Maurer–Cartan equation reads

$$\lambda + \frac{1}{2}[\Pi, \Pi]_{\mathfrak{g}} = 0$$

for $\Pi \in T_{\mathfrak{g}}^{1}(M)$. If we decompose $\Pi = \pi - J \in (T_{\mathrm{poly}}^{1}(M))^{\mathrm{G}} \oplus (\mathfrak{g}^* \otimes T_{\mathrm{poly}}^{-1}(M))^{\mathrm{G}}$, it is easy to see that the curved Maurer–Cartan equation together with the invariance of the elements is equivalent to the conditions (4.3) defining the momentum map. ∎

As in the Marsden–Weinstein reduction procedure, we fix a constraint surface $C \subseteq M$, by choosing an equivariant map $J : M \to \mathfrak{g}^*$ and setting $C = J^{-1}(\{0\})$. Here we always assume that $0 \in \mathfrak{g}^*$ is a regular value of the momentum map, making C a closed embedded submanifold of M. Note that G acts canonically of C, since J is equivariant. From now on we also require the action Φ to be proper around C and free on C.

To implement this choice in our algebraic setting, we consider from now on the curved DGLA

$$\big(T_{\mathfrak{g}}^{\bullet}(M), \lambda, -[J, \cdot\,], [\,\cdot\,, \cdot\,]\big). \tag{4.4}$$

Note that this is in fact a curved Lie algebra since $[J, J] = 0 = [\lambda, \cdot\,]$. We have to pass to the formal setting in order to see why this curved Lie algebra is actually interesting. Note that one advantage of the setting of formal power series is that we immediately get a complete filtration by the $\hbar$-degrees, i.e., by setting

$$\mathcal{F}^k T_{\mathfrak{g}}^{\bullet}(M)[[\hbar]] = \hbar^k T_{\mathfrak{g}}^{\bullet}(M)[[\hbar]].$$

In particular, if we consider formal Maurer–Cartan elements $\hbar(\pi - J') \in \hbar T_{\mathfrak{g}}^{1}(M)[[\hbar]]$, then the twisting procedures and infinite sums from Section 2 are all well defined. However, we have to rescale in this case the curvature, whence we consider the curved Lie algebra

$$\big(T_{\mathfrak{g}}^{\bullet}(M)[[\hbar]], \hbar\lambda, -[J, \cdot\,], [\,\cdot\,, \cdot\,]\big). \tag{4.5}$$

Here the differential, i.e., the classical momentum map, is not rescaled by some order of $\hbar$. This is due to the fact that we only want to consider formal Poisson structures with formal momentum maps that deform the classical J. Allowing general formal momentum maps would lead to formal Poisson structures with different reduced manifolds, but we want to fix C and M_{red}. This becomes clear in the following lemma:

Lemma 4.3. *The formal curved Maurer–Cartan elements of the curved DGLA*

$$\big(T_{\mathfrak{g}}^{\bullet}(M)[[\hbar]], \hbar\lambda, -[J, \cdot\,], [\,\cdot\,, \cdot\,]\big)$$

are equivalent to pairs $\hbar(\pi, J')$, where π is a G-invariant formal Poisson structure with formal moment map $J + \hbar J' : \mathfrak{g} \to T_{\mathrm{poly}}^{-1}(M)[[\hbar]]$.

Proof. The proof follows directly by Lemma 4.2 by counting $\hbar$-degrees. ■

The rest of this paper is devoted to the construction of a curved L_∞-morphism

$$\mathrm{T}_{\mathrm{red}} \colon \big(T_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, -[J,\cdot], [\cdot,\cdot]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, 0, [\cdot,\cdot]\big)$$

with $M_{\mathrm{red}} := C/G$. This morphism is frequently referred to as *reduction morphism*.

4.1. The Taylor series expansion around C

The main goal of this section is the study of a partial Taylor series expansion of the multi-vector fields on M around C. Let us assume $M = C \times \mathfrak{g}^*$. This is not a strong assumption as we know from [4, Lemma 3] that, if G acts properly on an open neighborhood of C, we can always find a G-invariant open neighborhood $M_{\mathrm{nice}} \subseteq M$ of C, such that there exists a G-equivariant diffeomorphism $M_{\mathrm{nice}} \cong U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^*$. Here the Lie group G acts on $C \times \mathfrak{g}^*$ as

$$\Phi_g = \Phi_g^C \times \mathrm{Ad}^*_{g^{-1}},$$

where Φ^C is the induced action on C. Note that in this setting the momentum map on U_{nice} is simply given by the projection to $\mathfrak{g}^*$. The idea of a Taylor expansion uses the fact that we have the isomorphism

$$T^k_{\mathrm{poly}}(C \times \mathfrak{g}^*) \cong \bigoplus_{i+j=k} \mathscr{C}^\infty(C \times \mathfrak{g}^*) \otimes_{\mathscr{C}^\infty(C)} \big(\Lambda^i \mathfrak{g}^* \otimes T^j_{\mathrm{poly}}(C)\big).$$

First, we define

$$T_{\mathfrak{g}^*} \colon \mathscr{C}^\infty(C \times \mathfrak{g}^*) \ni f \mapsto \sum_{I \in \mathbb{N}_0^n}^{\infty} \frac{1}{I!} e_I \otimes \iota^* \frac{\partial}{\partial \alpha_I} f \in \prod_i \big(\mathrm{S}^i\, \mathfrak{g} \otimes \mathscr{C}^\infty(C)\big),$$

where $\alpha_i e^i$ are coordinates on $\mathfrak{g}^*$ and ι^* is the restriction to C.

Lemma 4.4. *The map $T_{\mathfrak{g}^*}$ is equivariant, i.e.,*

$$T_{\mathfrak{g}^*} \circ \Phi_{g,*} = (\mathrm{Ad}_g \otimes \Phi^C_{g,*}) \circ T_{\mathfrak{g}^*}. \tag{4.6}$$

Proof. We just observe that

$$\frac{\partial}{\partial \alpha_i} \circ \Phi_g^* = (\mathrm{Ad}_{g^{-1}})^i_j \cdot (\Phi_g^C)^* \circ \frac{\partial}{\partial \alpha_j}$$

for $\mathrm{Ad}_g\, e_i = (\mathrm{Ad}_g)^j_i e_j$. Hence we have

$$T_{\mathfrak{g}^*}(\Phi_{g,*} f) = \sum_{I \in \mathbb{N}_0^n}^{\infty} \frac{1}{I!} e_I \otimes \iota^* \frac{\partial}{\partial \alpha_I} \Phi_{g,*} f = (\mathrm{Ad}_g \otimes \Phi^C_{g,*}) \circ T_{\mathfrak{g}^*} f$$

by shifting the components $(\mathrm{Ad}_{g^{-1}})^i_j = (\mathrm{Ad}_g)^j_i$ to the symmetric powers of $\mathfrak{g}$. ■

Remark 4.5. It is now clear that this map can be restricted to invariant functions in order to obtain invariant elements in $\prod_i (\mathrm{S}^i\, \mathfrak{g} \otimes \mathscr{C}^\infty(C))$. Moreover, with a slight adaption of the proof of the Borel lemma (see, e.g., [25, Theorem 1.3]), one can show that the map $T_{\mathfrak{g}^*}$ is surjective. The more remarkable fact is that the properness of the action ensures that the map

$$T_{\mathfrak{g}^*}\colon \mathscr{C}^\infty(M \times \mathfrak{g}^*)^{\mathrm{G}} \ni f \mapsto \sum_{I \in \mathbb{N}_0^n}^{\infty} \frac{1}{I!} e_I \otimes \iota^* \frac{\partial}{\partial \alpha_I} f \in \prod_i (\mathrm{S}^i\, \mathfrak{g} \otimes \mathscr{C}^\infty(C))^{\mathrm{G}}$$

is surjective. We omit this proof as we do not use it here and it is just an adaption of the corresponding statement in [24].

We extend this map to $T^\bullet_{\mathrm{poly}}(C \times \mathfrak{g}^*)$ via

$$\begin{aligned} T_{\mathfrak{g}^*}\colon T^k_{\mathrm{poly}}(C \times \mathfrak{g}^*) &\to \prod_i \big(\mathrm{S}^i\, \mathfrak{g} \otimes \Lambda \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C)\big) \\ (f \otimes \xi \otimes X) &\mapsto \sum_{I \in \mathbb{N}_0^n}^{\infty} \frac{1}{I!} e_I \otimes \xi \otimes \iota^* \frac{\partial}{\partial \alpha_I} f \cdot X \end{aligned}$$

and using Lemma 4.4, we see that also this map can be restricted to the equivariant multivector fields:

Definition 4.6 (Taylor expansion around C). The map

$$T_{\mathfrak{g}^*}\colon \big(\mathrm{S}\, \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C \times \mathfrak{g}^*)\big)^{\mathrm{G}} \to \Big(\mathrm{S}\, \mathfrak{g}^* \otimes \prod_{i=0}^{\infty} \big(\mathrm{S}^i\, \mathfrak{g} \otimes \Lambda \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C)\big)\Big)^{\mathrm{G}} \tag{4.7}$$

is called the Taylor expansion around C and we write

$$T_{\mathrm{Tay}}(C \times \mathfrak{g}^*) := \Big(\mathrm{S}\, \mathfrak{g}^* \otimes \prod_{i=0}^{\infty} \big(\mathrm{S}^i\, \mathfrak{g} \otimes \Lambda \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C)\big)\Big)^{\mathrm{G}}.$$

Having in mind that the vector space $\prod_{i=0}^{\infty} (\mathrm{S}^i\, \mathfrak{g} \otimes \Lambda \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}$ just consists of the Taylor expansions, it is not surprising that it also inherits the structure of a DGLA: for $P, Q \in \prod_i \mathrm{S}^i\, \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$, the brackets are given by

$$[P, Q] = 0, \quad [P \otimes \xi, Q] = P \vee \mathrm{i_s}(\xi) Q,$$
$$[P \otimes \xi, Q \otimes \eta] = P \vee \mathrm{i_s}(\xi) Q \otimes \eta - Q \vee \mathrm{i_s}(\eta) P \otimes \xi,$$

where $\mathrm{i_s}$ denotes the symmetric left insertion, and they are extended as a Gerstenhaber bracket with respect to the graded commutative product

$$(P \otimes \xi) \cdot (Q \otimes \eta) := P \vee Q \otimes \xi \wedge \eta.$$

We combine it with the usual DGLA structure on $T_{\mathrm{poly}}(C)$ and extend it as in the case of $T^\bullet_{\mathfrak{g}}(M)$ trivially to all of $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$. Summarizing, we have a DGLA structure on the Taylor expansion around C with zero differential.

Lemma 4.7. *The Taylor expansion*

$$T_{\mathfrak{g}^*}\colon T_{\mathfrak{g}}(M) \to T_{\mathrm{Tay}}(C \times \mathfrak{g}^*) \tag{4.8}$$

is a DGLA morphism.

Proof. This is an easy verification on generators. ■

As a next step, we want to include the curvature $\lambda \in T^2_{\mathfrak{g}}(M)$ from Section 4. Recall that

$$\lambda = e^i \otimes (e_i)_M \in T^2_{\mathfrak{g}}(M) = \big(\mathfrak{g}^* \otimes T^0_{\mathrm{poly}}(M)\big)^{\mathrm{G}}.$$

Using our assumption that $M = C \times \mathfrak{g}^*$ and that G acts as the product of the action on C and the coadjoint action, we see that

$$(e_i)_M = (e_i)_C + \alpha_k f^k_{ji} \frac{\partial}{\partial \alpha_j}, \tag{4.9}$$

where $(e_i)_C$ denotes the fundamental vector field of the action on C and where f^k_{ji} are the structure constants of $\mathfrak{g}$. This means in particular that

$$T_{\mathfrak{g}^*}(\lambda) = e^i \otimes 1 \otimes 1 \otimes (e_i)_C + f^j_{ji} e^i \otimes e_k \otimes e^j \otimes 1 \in T_{\mathrm{Tay}}(C \times \mathfrak{g}^*).$$

With a slight abuse of notation, we write λ instead of $T_{\mathfrak{g}^*}(\lambda)$. The same argument leads to the observation that

$$T_{\mathfrak{g}^*}(J) = e^i \otimes e_i \otimes 1 \otimes 1,$$

where we also write J instead of $T_{\mathfrak{g}^*}(J)$ in the sequel.

Corollary 4.8. *The map*

$$T_{\mathfrak{g}^*}\colon \big(T_{\mathfrak{g}}(M), \lambda, -[J, \cdot], [\cdot, \cdot]\big) \to \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot, \cdot]\big) \tag{4.10}$$

is a morphism of curved DGLAs.

One main advantage of the Taylor expansion $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ consists in the fact that we have a canonical element

$$\pi_{\mathrm{KKS}} := 1 \otimes \left(\frac{1}{2} f^k_{ij} e_k \otimes e^i \wedge e^j \otimes 1 - 1 \otimes e^i \otimes (e_i)_C \right),$$

which is not available in $T_{\mathfrak{g}}(M)$. Note that π_{KKS} encodes the action on C and the Lie algebra structure on $\mathfrak{g}$.

Remark 4.9 (Action Lie algebroid). The bundle $C \times \mathfrak{g} \to C$ can be equipped with the structure of a Lie algebroid since $\mathfrak{g}$ acts on C by the fundamental vector fields. The bracket of this *action Lie algebroid* is given by

$$[\xi, \eta]_{C \times \mathfrak{g}}(p) = \big[\xi(p), \eta(p)\big] - (\mathscr{L}_{\xi_C} \eta)(p) + (\mathscr{L}_{\eta_C} \xi)(p) \tag{4.11}$$

for $\xi, \eta \in \mathscr{C}^\infty(C, \mathfrak{g})$. The anchor is given by $\rho(p, \xi) = -\xi_C|_p$. In particular, one can check that π_{KKS} is the negative of the linear Poisson structure on its dual $C \times \mathfrak{g}^*$ in the convention of [26].

The canonical π_{KKS} is of big importance since it is part of some kind of normal form for every invariant Poisson structure on $C \times \mathfrak{g}^*$ with moment map J. In the Taylor expansion, this becomes more clear in the following lemma:

Lemma 4.10. *Let*

$$\pi \in \left(\prod_{i=0}^{\infty} \big(\mathrm{S}^i\, \mathfrak{g} \otimes \Lambda \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C) \big) \right)^{\mathrm{G}} \subseteq T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$$

be a curved Maurer–Cartan element, then

$$\pi = \pi_{\mathrm{KKS}} + \pi_C \tag{4.12}$$

with $\pi_C \in (\prod_{i=0}^{\infty} \mathrm{S}^i\, \mathfrak{g} \otimes T^1_{\mathrm{poly}}(C))^{\mathrm{G}}$.

Proof. By (4.9) we have for $\xi \in \mathfrak{g}$, $c \in C$ and $\alpha = \alpha_i e^i \in \mathfrak{g}^*$

$$\xi_M|_{(c,\alpha)} = -\big(\mathrm{i}\big(\mathrm{d}J(\xi)\big)\pi\big)|_{(c,\alpha)} = \xi_C|_c + \xi_{\mathfrak{g}^*}\big|_\alpha = \xi_C|_c - f^i_{j\ell}\,\mathrm{i}(e_i)\alpha e^j(\xi) \frac{\partial}{\partial \alpha_\ell}.$$

This implies directly

$$\pi = \pi_C + (e_i)_C \wedge \frac{\partial}{\partial \alpha_i} + \frac{1}{2}\alpha_k f^k_{ij} \frac{\partial}{\partial \alpha_i} \wedge \frac{\partial}{\partial \alpha_j},$$

where $\pi_C \in (\prod_{i=0}^{\infty} \mathrm{S}^i\, \mathfrak{g} \otimes T^1_{\mathrm{poly}}(C))^{\mathrm{G}}$ is tangent to C, but can possibly depend on all of $M = C \times \mathfrak{g}^*$. In the Taylor expansion, $\frac{\partial}{\partial \alpha_\ell}$ corresponds to e^ℓ and the lemma is shown. ∎

Comparing now the terms in $[\pi, \pi] = 0$ with the same $\mathfrak{g}^*$ and C degrees gives hints concerning the coefficient function of π_C that can also depend on $\mathfrak{g}^*$. In particular, the terms in $\Gamma^\infty(\Lambda^3 TC)$ are given by

$$[\pi_C, \pi_C] + 2(e_i)_C \wedge \left[\frac{\partial}{\partial \alpha_i}, \pi_C \right] = 0. \tag{4.13}$$

To conclude this section, we define for later use the operator

$$\partial := \mathrm{id} \otimes \mathrm{i_s}(e^i) \otimes \mathrm{id} \otimes (e_i)_C \wedge . \tag{4.14}$$

Note that we assume the Koszul sign rule; i.e., applying ∂ to $\xi \otimes P \otimes \alpha \otimes X$, we get a sign $(-1)^{|\alpha|}$. We directly see that $\partial^2 = 0$ and equation (4.13) can be written as

$$\frac{1}{2}[\pi_C, \pi_C] + (e_i)_C \wedge \mathrm{i_s}(e^i)\pi_C = \frac{1}{2}[\pi_C, \pi_C] + \partial \pi_C = 0. \tag{4.15}$$

4.2. The Cartan model of multivector fields

In the case of symplectic manifolds, it has been shown in [27] that quantization and reduction commute by exploiting the diagram

$$\big((\mathrm{S}\,\mathfrak{g}^* \otimes \Omega(M))^{\mathrm{G}}, \mathrm{d}_\mathfrak{g}\big) \xrightarrow{\iota^*} \big((\mathrm{S}\,\mathfrak{g}^* \otimes \Omega(C))^{\mathrm{G}}, \mathrm{d}_\mathfrak{g}\big) \xleftarrow{p^*} \big(\Omega^\bullet(M_{\mathrm{red}}), \mathrm{d}\big) \tag{4.16}$$

for $M \xleftarrow{\iota} C \xrightarrow{p} M_{\mathrm{red}}$. Here $(\mathrm{S}\,\mathfrak{g}^* \otimes \Omega(C))^{\mathrm{G}}$ is the so-called Cartan model for equivariant de Rham cohomology [15, 23, 28] that we want to recall briefly:

Remark 4.11 (Cartan model for equivariant de Rham cohomology). Let M be a smooth manifold with a Lie group action $\Phi\colon G \times M \to M$. The *Cartan complex of* G*-equivariant differential forms* is defined by

$$\Omega^k_{\mathrm{G}}(M) = \Big(\bigoplus_{2i+j=k} \big(\mathrm{S}^i\,\mathfrak{g}^* \otimes \Omega^j(M)\big)^{\mathrm{G}}, \mathrm{d}_\mathfrak{g} = \mathrm{d} + \mathrm{i}_\bullet\Big). \tag{4.17}$$

Here d denotes the usual de Rham differential, the invariants are taken with respect to the product action

$$g \triangleright (p \otimes \alpha) = (\mathrm{Ad}^*_{g^{-1}}\, p) \otimes \Phi^*_{g^{-1}}\alpha, \tag{4.18}$$

and one has $\mathrm{i}_\bullet = e^i \vee \otimes \mathrm{i_a}((e_i)_M)$. In the special case of a principal G-bundle $M = C$, one can show that

$$\mathrm{H}\big(\Omega^\bullet_{\mathrm{G}}(C), \mathrm{d}_\mathfrak{g}\big) \cong \mathrm{H}_{\mathrm{dR}}(C/\mathrm{G}); \tag{4.19}$$

see, e.g., [27, Corollary 3.5] and the references therein. In particular, the map p^* from (4.16) is a quasi-isomorphism.

We aim to transfer this construction and in particular (4.19) to the setting of Poisson manifolds by using the above observation as a guideline. For this reason, we introduce our notion for the *Cartan model of equivariant multivector fields* and compute its relation with $T_{\mathrm{poly}}(M_{\mathrm{red}})$ and with the Taylor expansion of multivector fields around C from the previous section. We start with the following observation:

Proposition 4.12. *The cohomology of* $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), -[J, \cdot\,], [\,\cdot\,,\cdot\,])$ *is given by the Lie algebra* $((\prod_{i=0}^\infty (\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)))^{\mathrm{G}}, [\,\cdot\,,\cdot\,])$. *Therefore, the canonical inclusion*

$$\iota\colon \Big(\Big(\prod_{i=0}^\infty \big(\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)\big)\Big)^{\mathrm{G}}, 0, [\,\cdot\,,\cdot\,]\Big) \to \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [-J, \cdot\,], [\,\cdot\,,\cdot\,]\big) \tag{4.20}$$

becomes a quasi-isomorphism of DGLAs.

Proof. The map $h = \mathrm{i_s}(e_\ell) \otimes \mathrm{id} \otimes e^\ell \wedge \otimes \mathrm{id}$ satisfies

$$\begin{aligned} &-[J, \cdot\,] \circ h(\xi \otimes P \otimes \alpha \otimes X) - h \circ [J, \cdot\,](\xi \otimes P \otimes \alpha \otimes X) \\ &\quad = \big(\deg(\alpha) + \deg(\xi)\big)(\xi \otimes P \otimes \alpha \otimes X) \end{aligned}$$

and the statement follows. ∎

Note that the cohomology $(\prod_{i=0}^{\infty}(\mathrm{S}^i\,\mathfrak{g}\otimes T_{\mathrm{poly}}(C)))^{\mathrm{G}}$ can be equipped with a non-trivial, but canonical differential.

Proposition 4.13. *The differential* ∂ *defined in* (4.14) *turns*

$$\left(\left(\prod_{i=0}^{\infty}\left(\mathrm{S}^i\,\mathfrak{g}\otimes T_{\mathrm{poly}}(C)\right)\right)^{\mathrm{G}}, [\,\cdot\,,\,\cdot\,]\right)$$

into a DGLA.

Proof. A straightforward computation shows that

$$\begin{aligned}
\partial[\xi\otimes X, \eta\otimes Y] &= \mathrm{i_s}(e^i)(\xi\vee\eta)\otimes(e_i)_C\wedge[X,Y],\\
\left[\partial(\xi\otimes X), \eta\otimes Y\right] &= (-1)^k\mathrm{i_s}(e^i)(\xi)\vee\eta\otimes X\wedge[(e_i)_C, Y]\\
&\quad + \mathrm{i_s}(e^i)(\xi)\vee\eta\otimes(e_i)_C\wedge[X,Y],\\
\left[\xi\otimes X, \partial(\eta\otimes Y)\right] &= (-1)^{k-1}\xi\vee\mathrm{i_s}(e^i)(\eta)\otimes(e_i)_C\wedge[X,Y]\\
&\quad - \xi\vee\mathrm{i_s}(e^i)(\eta)\otimes[(e_i)_C, X]\wedge Y,
\end{aligned}$$

where $X\in T^{k-1}_{\mathrm{poly}}(C)$. Using the G-invariance, we get

$$\xi\otimes\left[(e_i)_C, X\right] = f_{ij}^k e_k\vee\mathrm{i_s}(e^j)\xi\otimes X,\quad \eta\otimes\left[(e_i)_C, Y\right] = f_{ij}^k e_k\vee\mathrm{i_s}(e^j)\eta\otimes Y.$$

Summarizing, this yields

$$\partial[\xi\otimes X, \eta\otimes Y] = \left[\partial(\xi\otimes X), \eta\otimes Y\right] + (-1)^{k-1}\left[\xi\otimes X, \partial(\eta\otimes Y)\right]$$

and the proposition is shown. ■

This motivates the following definition.

Definition 4.14 (Cartan model). Let G be a Lie group acting on a manifold C. The DGLA defined by

$$\left(\prod_{i=0}^{\infty}\left(\mathrm{S}^i\,\mathfrak{g}\otimes T_{\mathrm{poly}}(C)\right)^{\mathrm{G}}, \partial, [\,\cdot\,,\,\cdot\,]\right) \tag{4.21}$$

is called the *Cartan model* and is denoted by $T_{\mathrm{Cart}}(C)$.

Seen as a $\mathscr{C}^\infty(C)$-module, $\prod_{i=0}^{\infty}(\mathrm{S}^i\,\mathfrak{g}\otimes T_{\mathrm{poly}}(C))$ is the dual of $\mathrm{S}\,\mathfrak{g}^*\otimes\Omega(C)$, whose invariants are the space underlying the Cartan model for the equivariant de Rham cohomology from Remark 4.11. Even the differential $\partial = \mathrm{i_s}(e^i)\otimes(e_i)_C\wedge$ is dual to the insertion $\mathrm{i}_\bullet = e^i\vee\otimes\mathrm{i_a}((e_i)_C)$. In the case of forms, we saw in (4.19) that the equivariant cohomology of the principal fiber bundle C is isomorphic to the de Rham cohomology of the reduced manifold, whereas in our setting we want to show that we get the multivector fields on M_{red} as cohomology. Note that we have a canonical DGLA map

$$p\colon \left(T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\,\cdot\,]\right) \to \left(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, [\,\cdot\,,\,\cdot\,]\right),$$

which is just given by the projection to the symmetric degree 0 followed by the projection to M_{red}. It is well defined since invariant multivector fields are projectable.

Proposition 4.15. *The DGLA map*

$$p\colon \left(\left(\prod_{i=0}^{\infty}\big(\mathrm{S}^i\,\mathfrak{g}\otimes T_{\mathrm{poly}}(C)\big)\right)^{\mathrm{G}}, \partial, [\,\cdot\,,\,\cdot\,]\right) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, [\,\cdot\,,\,\cdot\,]\big) \tag{4.22}$$

is a quasi-isomorphism.

Proof. Consider the principal bundle $\mathrm{pr}\colon C \to M_{\mathrm{red}}$ and choose a principal bundle connection $\omega = \omega^i \otimes e_i \in \Omega^1(C)\otimes\mathfrak{g}$, i.e., an equivariant horizontal lift inducing

$$TC = \mathrm{Ver}(C)\oplus\mathrm{Hor}(C) = \ker T\,\mathrm{pr}\oplus\ker\omega,$$

where $\ker\omega \cong \mathrm{pr}^* TM_{\mathrm{red}}$. Then we can construct a homotopy for ∂ by $h = e_i \vee \otimes \mathrm{i_a}(\omega^i)$. Since $\omega^i((e_j)_M) = \delta^i_j$, it satisfies

$$h\partial + \partial h = (\deg_{\mathfrak{g}} + \deg_{\mathrm{ver}})\mathrm{id}.$$

With the vertical degree, we mean the degree in the splitting

$$\Lambda^k TC = \bigoplus_{i+j=k} \Lambda^i\,\mathrm{Ver}(C)\otimes\Lambda^j\,\mathrm{Hor}(C). \qquad \blacksquare$$

In other words, the above proposition yields for every principal connection $\omega \in \Omega^1(C)\otimes\mathfrak{g}$ the deformation retract

$$T_{\mathrm{poly}}(M_{\mathrm{red}}) \underset{p}{\overset{i}{\rightleftarrows}} \left(\prod_{i=0}^{\infty}\big(\mathrm{S}^i\,\mathfrak{g}\otimes T_{\mathrm{poly}}(C)\big)\right)^{\mathrm{G}} \circlearrowleft h, \tag{4.23}$$

where i denotes the horizontal lift with respect to the connection ω and the homotopy h is given on all homogeneous elements by

$$h(\xi\otimes X) = \begin{cases} \frac{1}{\deg(\xi)+\deg_{\mathrm{ver}}(X)}\, e_i \vee \xi \otimes \mathrm{i_a}(\omega^i)X & \text{if } \deg(\xi)+\deg_{\mathrm{ver}}(X) \neq 0, \\ 0 & \text{else.} \end{cases}$$

Indeed, the algebraic relations of a deformation retract between i, p, and h are easily seen to be verified. Recall that additionally p is a DGLA morphism, which puts us exactly in the situation of Proposition 3.3. So before we continue to put the Cartan model in the context of reduction, we give an explicit formula for a quasi-inverse of p.

Proposition 4.16. *For a fixed principal fiber connection $\omega \in \Omega^1(C) \otimes \mathfrak{g}$ with curvature $\Omega \in \Omega^2(C) \otimes \mathfrak{g}$, one obtains an L_∞-quasi-inverse of p*

$$i_\infty : \mathrm{S}\left(T_{\mathrm{poly}}(M_{\mathrm{red}})[1]\right) \to \mathrm{S}\left(\left(\prod_{i=0}^{\infty}\left(\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)\right)\right)^{\mathrm{G}}[1]\right) \tag{4.24}$$

given by

$$i_\infty = \mathrm{e}^{\Omega} \circ (\,\cdot\,)^{\mathrm{hor}}, \tag{4.25}$$

where one extends $(\,\cdot\,)^{\mathrm{hor}}$ as a coalgebra morphism and Ω as a coderivation of degree 0. *In particular,*

$$i_{\infty,1}(X) = X^{\mathrm{hor}} \quad \textit{and} \quad i_{\infty,2}(X,Y) = (-1)^{|X|} e_i \otimes \Omega^i(X^{\mathrm{hor}}, Y^{\mathrm{hor}}) \tag{4.26}$$

for a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$.

Proof. Let us fix a principal connection $\omega \in \Omega^1(C) \otimes \mathfrak{g}$ and denote by h the corresponding homotopy and by Ω its curvature. Due to that fact that equation (4.23) is a deformation retract and p is a DGLA morphism, we are exactly in the situation of Proposition 3.3 and the statement becomes a purely computational issue; so let us start with some book-keeping. Throughout the proof, we will make use of the following equation for $X \in \Gamma^\infty(\Lambda^k TC)$, $Y \in \Gamma^\infty(\Lambda^\ell TC)$, and $\alpha \in \Omega^1(C)$:

$$\mathrm{d}\alpha(X,Y) = \left[\mathrm{i_a}(\alpha)X, Y\right] - (-1)^k\left[X, \mathrm{i_a}(\alpha)Y\right] - \mathrm{i_a}(\alpha)[X,Y], \tag{*}$$

where for the left-hand side, we define

$$\mathrm{d}\alpha(X,Y) = (\mathrm{d}\alpha)_{ij}\,\mathrm{i_a}(\mathrm{d}x^i)X \wedge \mathrm{i_a}(\mathrm{d}x^j)Y$$

in a coordinate patch. The validity of equation (*) for one-forms of the type $\alpha = f\,\mathrm{d}g$ follows by the usual Schouten calculus. By $\mathbb{R}$-linearity of equation (*), its validity follows for general one-forms in every coordinate patch and hence also globally. Let us define, using the curvature Ω, the map

$$\Omega : \mathrm{S}^2\left(\prod_i \left(\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)\right)^{\mathrm{G}}[1]\right) \to \prod_i \left(\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)\right)^{\mathrm{G}}[1]$$

defined on homogeneous and factorizing elements $P_j \otimes X_j \in \prod_i (\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}[1]$, $j = 1,2$ by

$$\Omega(P_1 \otimes X_1 \vee P_2 \otimes X_2) = (-1)^{|X_1|} e_i \vee P_1 \vee P_2 \otimes \Omega^i(X_1, X_2).$$

This map is well defined, i.e., in fact graded symmetric, and of degree 0. With a slight abuse of notation, we denote also by

$$\Omega : \mathrm{S}^\bullet\left(\prod_i \left(\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)\right)^{\mathrm{G}}[1]\right) \to \mathrm{S}^{\bullet-1}\left(\prod_i \left(\mathrm{S}^i\,\mathfrak{g} \otimes T_{\mathrm{poly}}(C)\right)^{\mathrm{G}}[1]\right)$$

its extension as a coderivation of degree 0, i.e.,

$$\Omega(X_1 \vee \cdots \vee X_k) = \sum_{\sigma \in \mathrm{Sh}(2,k-2)} \varepsilon(\sigma)\Omega(X_{\sigma(1)} \vee X_{\sigma(2)}) \vee X_{\sigma(3)} \vee \cdots \vee X_{\sigma(k)}$$

for $X_j \in \prod_i (\mathrm{S}^i\, \mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}[1]$. Note that for every

$$k \in \mathbb{N} \quad \text{and} \quad X_j \in \prod_i \big(\mathrm{S}^i\, \mathfrak{g} \otimes T_{\mathrm{poly}}(C)\big)^{\mathrm{G}}[1],$$

we have that

$$\Omega^k(X_1 \vee \cdots \vee X_k) = 0$$

since Ω decreases the symmetric degree by one and hence the expression

$$\mathrm{e}^{\Omega} := \sum_k \frac{1}{k!}\Omega^k$$

is a well-defined map. Since Ω is a coderivation of degree 0, it is even a coalgebra morphism. Its components are given by

$$(\mathrm{e}^{\Omega})^{\ell}_k = \frac{1}{(k-\ell)!}\Omega^{k-\ell},$$

which can be seen again by counting symmetric degrees. This shows, in particular, that

$$(\mathrm{e}^{\Omega} \circ (\,\cdot\,)^{\mathrm{hor}})^1_1 = (\,\cdot\,)^{\mathrm{hor}}.$$

We proceed now inductively, so let us assume that $\mathrm{e}^{\Omega} \circ (\,\cdot\,)^{\mathrm{hor}}$ coincides with i_∞ from Proposition 3.3 up to order k. For $X_j \in T_{\mathrm{poly}}(M_{\mathrm{red}})[1]$, $j = 1, \ldots, k+1$, we have

$$\begin{aligned}
&i_{\infty,k+1}(X_1 \vee \cdots \vee X_{k+1}) \\
&\quad = h \circ Q^1_2 \circ i^2_{\infty,k+1}(X_1 \vee \cdots \vee X_{k+1}) \\
&\quad = \sum_{j=1}^{k} \sum_{\sigma \in \mathrm{Sh}(j,k+1-j)} \frac{\varepsilon(\sigma)}{2} h \circ Q^1_2 \\
&\qquad \times \big(i^1_{\infty,j}(X_{\sigma(1)} \vee \cdots \vee X_{\sigma(j)}) \vee i^1_{\infty,k-j+1}(X_{\sigma(j+1)} \vee \cdots \vee X_{\sigma(k+1)})\big) \\
&\quad = \sum_{j=1}^{k} \sum_{\sigma \in \mathrm{Sh}(j,k+1-j)} \frac{\varepsilon(\sigma)}{2} h \circ Q^1_2 \\
&\qquad \times \bigg(\frac{\Omega^{j-1}}{(j-1)!}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}) \vee \frac{\Omega^{k-j}}{(k-j)!}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\bigg).
\end{aligned}$$

Let us now take a look at

$$
\begin{aligned}
&h \circ Q_2^1\big(\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}) \vee \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big)\\
&= (-1)^{1+\sum_{k=1}^{j}|X_{\sigma(j)}|} h\big[\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}), \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big]\\
&= \frac{(-1)^{1+\sum_{k=1}^{j}|X_{\sigma(j)}|}}{k} e_i\\
&\quad \otimes \mathrm{i}_{\mathrm{a}}(\omega^i)\big[\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}), \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big]\\
&\overset{(*)}{=} \frac{(-1)^{\sum_{k=1}^{j}|X_{\sigma(j)}|}}{k} e_i\\
&\quad \otimes \mathrm{d}\omega^i\big(\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}), \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big).
\end{aligned}
$$

The factor $\frac{1}{k}$ appears, since Ω^k raises the symmetric degree in $\mathfrak{g}$ by k and hence the commutator has $k-1$ symmetric degrees in $\mathfrak{g}$ degrees and at most one vertical degree, since both of the entries are horizontal multivector fields. Moreover, since $\mathrm{i}_{\mathrm{a}}(\omega^i)$ annihilates the terms which do not have a vertical degree, the formula is valid. Note that by definition of the curvature of ω, we have $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ or for a chosen basis $\Omega^i = \mathrm{d}\omega^i + \frac{1}{2} f^i_{kl}\omega^k \wedge \omega^l$. Since ω^i vanishes on horizontal lifts, we can write

$$
\begin{aligned}
&h \circ Q_2^1\big(\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}) \vee \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big)\\
&= \frac{(-1)^{\sum_{k=1}^{j}|X_{\sigma(j)}|}}{k} e_i\\
&\quad \otimes \Omega^i\big(\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}), \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big)\\
&= \frac{1}{k}\Omega\big(\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}), \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&i_{\infty,k+1}(X_1 \vee \cdots \vee X_{k+1})\\
&= \sum_{j=1}^{k} \sum_{\sigma \in \mathrm{Sh}(j,k+1-j)} \frac{\varepsilon(\sigma)}{2k}\Omega\\
&\quad \times \left(\frac{\Omega^{j-1}}{(j-1)!}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}) \vee \frac{\Omega^{k-j}}{(k-j)!}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\right)\\
&= \frac{1}{k!}\Omega^k(X^{\mathrm{hor}}_1 \vee \cdots \vee X^{\mathrm{hor}}_{k+1}).
\end{aligned}
$$

The last equality follows from the observation that

$$
\begin{aligned}
&\Omega^k(X^{\mathrm{hor}}_1 \vee \cdots \vee X^{\mathrm{hor}}_{k+1})\\
&= \Omega\big(\Omega^{k-1}(X^{\mathrm{hor}}_1 \vee \cdots \vee X^{\mathrm{hor}}_{k+1})\big)\\
&= \Omega\big((k-1)!(\mathrm{e}^{\Omega})^2_{k+1}(X^{\mathrm{hor}}_1 \vee \cdots \vee X^{\mathrm{hor}}_{k+1})\big)
\end{aligned}
$$

$$= \sum_{j=1}^{k} \sum_{\sigma \in \mathrm{Sh}(j,k+1-j)} \frac{\varepsilon(\sigma)}{2} \frac{(k-1)!}{(j-1)!(k-j)!}$$
$$\times \Omega\big(\Omega^{j-1}(X^{\mathrm{hor}}_{\sigma(1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(j)}) \vee \Omega^{k-j}(X^{\mathrm{hor}}_{\sigma(j+1)} \vee \cdots \vee X^{\mathrm{hor}}_{\sigma(k+1)})\big),$$

and the proof is completed. ∎

Corollary 4.17. *The induced map at the level of Maurer–Cartan elements*

$$p\colon MC\big(T_{\mathrm{Cart}}(C)\big) \to MC\big(T_{\mathrm{poly}}(M_{\mathrm{red}})\big)$$

is surjective.

Proof. Let $\pi \in T^1_{\mathrm{poly}}(M_{\mathrm{red}})$ be a Maurer–Cartan element, i.e., a Poisson structure. We define

$$\Pi = \sum_{k \geq 1} \frac{1}{k!} i_{\infty,k}(\pi^{\vee k}).$$

This series is actually well defined in $T_{\mathrm{Cart}}(C)$, since we have

$$\Pi = \sum_{k \geq 1} \frac{1}{k!} \frac{1}{(k-1)!} \Omega^{k-1}\big((\pi^{\mathrm{hor}})^{\vee k}\big)$$

using the explicit for of i_∞ as in Proposition 4.16. But

$$\Omega^{k-1}\big((\pi^{\mathrm{hor}})^{\vee k}\big) \in \big(\mathrm{S}^{k-1}\,\mathfrak{g} \otimes T^1_{\mathrm{poly}}(C)\big)^{\mathrm{G}},$$

whence $\Pi \in MC(T_{\mathrm{Cart}}(C))$ is well defined. The identity $p(\Pi) = \pi$ is then clear using again the explicit form. ∎

Remark 4.18. In particular, the above proposition shows not only that if C admits a flat connection, then i_∞ has $i_1 = (\,\cdot\,)^{\mathrm{hor}}$ as only a structure map, but also how to correct the horizontal lift in order to obtain an L_∞-quasi-ismorphism.

Having seen the importance of the ad-hoc defined differential ∂ on $T_{\mathrm{Cart}}(C)$, we lean now again towards $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ and try to find an extension of the differential $-[J,\,\cdot\,]$ in order to make the inclusion $\iota\colon T_{\mathrm{Cart}}(C) \to T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ a quasi-isomorphism with respect to ∂. As a first step we have the following proposition.

Proposition 4.19. *The map* $[\pi_{\mathrm{KKS}}, \cdot]$ *is a well-defined differential on* $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ *that is explicitly given by*

$$[\pi_{\mathrm{KKS}}, \xi \otimes P \otimes \alpha \otimes X] = \xi \otimes \delta_{\mathrm{CE}}(P \otimes \alpha \otimes X) + \partial(\xi \otimes P \otimes \alpha \otimes X), \tag{4.27}$$

where δ_{CE} *denotes the Chevalley-Eilenberg differential. Moreover, the canonical inclusion*

$$\iota\colon \big(T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\,\cdot\,]\big) \to \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J,\,\cdot\,], [\,\cdot\,,\,\cdot\,]\big)$$

becomes a DGLA morphism.

Proof. Since the bracket does not depend on the $\mathrm{S}\,\mathfrak{g}^*$-part we restrict ourselves to $P \otimes \alpha \otimes X$. Let us compute

$$\begin{aligned}
&\left[\frac{1}{2} f_{ij}^k e_k \otimes e^i \wedge e^j, P \otimes \alpha \otimes X\right] \\
&\quad = \frac{1}{2} f_{ij}^k \big(e_k \otimes [e^i \wedge e^j, P \otimes \alpha] \otimes X + [e_k, P \otimes \alpha] \wedge e^i \wedge e^j \otimes X\big) \\
&\quad = f_{ij}^k e_k \vee \mathrm{i_s}(e^j) P \otimes e^i \wedge \alpha \otimes X - \frac{1}{2} f_{ij}^k P \otimes e^i \wedge e^j \wedge \mathrm{i_a}(e_k) \alpha \otimes X
\end{aligned}$$

and

$$\begin{aligned}
&\big[- e^i \wedge (e_i)_C, P \otimes \alpha \otimes X\big] \\
&\quad = -P \otimes e^i \wedge \alpha \otimes \mathscr{L}_{(e_i)_C} X - (-1)^{|\alpha|+|X|} \mathrm{i_s}(e^i) P \otimes \alpha \otimes X \wedge (e_i)_C,
\end{aligned}$$

where $|X|$ denotes the multivector field degree and $|\alpha|$ the exterior degree. Putting this together, we directly get (4.27). Since π_{KKS} is a Poisson structure, we directly see that it squares to zero. Moreover, $[\pi_{\mathrm{KKS}}, \cdot\,]$ boils down to ∂ when restricted to elements in the image of the canonical inclusion ι, i.e., in $(1 \otimes \prod_{i=0}^{\infty} (\mathrm{S}^i\, \mathfrak{g} \otimes 1 \otimes T_{\mathrm{poly}}(C)))^{\mathrm{G}}$. ■

Alternatively, the identity

$$[\pi_{\mathrm{KKS}}, J] = \lambda \tag{4.28}$$

implies that the canonical π_{KKS} defines a curved Maurer–Cartan element in the curved DGLA $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot\,], [\,\cdot\,,\cdot\,])$. Therefore, twisting by π_{KKS} yields a Lie algebra differential on $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ with curvature zero. The next step is, of course, to check if ι is still a quasi-isomorphism.

Proposition 4.20. *The inclusion*

$$\iota\colon \big(T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\cdot\,]\big) \to \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,,\cdot\,]\big) \tag{4.29}$$

is a quasi-isomorphism of DGLAs.

Proof. Let us compute the cohomology of $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ by interpreting it as a double complex. The two differentials are $[-J, \cdot\,]$ and $[\pi_{\mathrm{KKS}}, \cdot\,]$ and as bigrading we set

$$C^{p,q} = \left(\mathrm{S}^q\, \mathfrak{g}^* \otimes \prod_{i=0}^{\infty} \big(\mathrm{S}^i\, \mathfrak{g} \otimes \big(\Lambda \mathfrak{g}^* \otimes T_{\mathrm{poly}}(C)\big)^{p-q}\big)\right)^{\mathrm{G}}.$$

One can directly see that the differentials are compatible with the bigrading in the sense that

$$[-J, \cdot\,]\colon C^{p,q} \to C^{p,q+1} \quad \text{and} \quad [\pi_{\mathrm{KKS}}, \cdot\,]\colon C^{p,q} \to C^{p+1,q}.$$

By Proposition 4.12, the cohomology of $[-J, \cdot\,]$ is given by $(\prod_{i=0}^{\infty} (\mathrm{S}^i\, \mathfrak{g} \otimes T_{\mathrm{poly}}(C)))^{\mathrm{G}}$, on which the horizontal differential $[\pi_{\mathrm{KKS}}, \cdot\,]$ is just ∂. Thus ι is an isomorphism on the first sheet and thus on the cohomology. ■

The above results show that the Cartan model is an intertwiner of $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ and $T_{\mathrm{poly}}(M_{\mathrm{red}})$, which can be summarized in the diagram

$$\begin{array}{c} \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,,\,\cdot\,]\big) \\ \iota \big\uparrow \\ \big(T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\,\cdot\,]\big) \\ i_\infty \rightsquigarrow \quad \big\downarrow p \\ \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, [\,\cdot\,,\,\cdot\,]\big). \end{array} \tag{4.30}$$

So far we have shown that both ι and p are DGLA morphisms and also quasi-isomorphisms. For convenience, we included the L_∞-quasi-inverse i_∞ of p. From this diagram and the fact that every L_∞-quasi-isomorphism is quasi-invertible, we have the following:

Theorem 4.21. *There exists an L_∞-quasi-isomorphism*

$$\big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,,\,\cdot\,]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, [\,\cdot\,,\,\cdot\,]\big).$$

Note that the KKS Poisson structure is not defined on M, but just in an open neighborhood of C. Recall that we aim to find a curved L_∞-morphism

$$\mathrm{T}_{\mathrm{red}}\colon \big(T_{\mathfrak{g}}(M), \lambda, -[J, \cdot\,], [\,\cdot\,,\,\cdot\,]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, 0, [\,\cdot\,,\,\cdot\,]\big)$$

and its formal correspondence. To achieve this, we proceed in the following way: we construct a (non-curved) quasi-inverse of ι in Diagram (4.30) denoted by P and then twist it by $-\pi_{\mathrm{KKS}}$ in order to find a curved morphism

$$P^{-\pi_{\mathrm{KKS}}}\colon \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot\,], [\,\cdot\,,\,\cdot\,]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), \lambda_{\mathrm{red}}, [\pi_{\mathrm{KKS,red}}, \cdot\,], [\,\cdot\,,\,\cdot\,]\big)$$

for

$$\lambda_{\mathrm{red}} := \sum_{k \geq 0} \frac{(-1)^k}{k!} P_{1+k}(\lambda \vee \pi_{\mathrm{KKS}}^{\vee k}) \quad \text{and} \quad \pi_{\mathrm{KKS,red}} := \sum_{k \geq 0} \frac{(-1)^k}{k!} P_k(\pi_{\mathrm{KKS}}^{\vee k}).$$

There are now two issues with this approach:

- since we did not introduce a complete filtration on the involved DGLAs, we have to check by hand that both of the series actually converge in a suitable sense;
- this is actually not what we want, since our target, i.e., $T_{\mathrm{poly}}(M_{\mathrm{red}})$, has to have zero curvature and zero differential.

These two problems are solved in Section 4.4, where we construct a quasi-inverse of ι such that $\lambda_{\mathrm{red}} = \pi_{\mathrm{KKS,red}} = 0$ and we show that the series are well defined. But at first we need to extend our considerations to the formal setting, where we have a complete filtration by degrees of $\hbar$.

4.3. Formal equivariant multivector fields and their reduction

We want to consider the formal analogue of the equivariant multivector fields on M from equation (4.4). Since we are only interested in formal Maurer–Cartan elements, we have to rescale the curvature by $\hbar$ as in (4.5); i.e., we consider the curved DGLA

$$\big(\big(\mathrm{S}\,\mathfrak{g}^* \otimes T_{\mathrm{poly}}(M)\big)^{\mathrm{G}}[[\hbar]], \hbar\lambda, -[J, \cdot\,], [\,\cdot\,, \cdot\,]\big).$$

A formal curved Maurer–Cartan element $\hbar(\pi - J') \in \hbar(\mathrm{S}\,\mathfrak{g}^* \otimes T_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]]$ corresponds to an invariant formal Poisson structure π with formal momentum map $J + \hbar J'$.

The Taylor series expansion discussed in Section 4.1 allows us to interpret the element $\hbar\pi_{\mathrm{KKS}}$ as a formal curved Maurer–Cartan element. Thus we can perform the twisting procedure, yielding the flat DGLA

$$\big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\hbar\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,, \cdot\,]\big).$$

For a formal Maurer–Cartan element $\hbar(\pi - J')$, one can check that $\pi_{\mathrm{KKS}} + \pi$ is a G-invariant formal Poisson structure with formal momentum map $J + \hbar J'$ as desired and again $\pi = \pi_C + \mathcal{O}(\hbar)$. Moreover, the Cartan model for the multivector fields reads in the formal setting:

$$\big(T_{\mathrm{Cart}}(C)[[\hbar]], \hbar\partial, [\,\cdot\,, \cdot\,]\big)$$

and the bracket on $T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]$ is simply extended $\hbar$-bilinearly. Summarizing, we have the following claim.

Theorem 4.22. *In the diagram*

$$\begin{array}{c}\big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\hbar\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,, \cdot\,]\big)\\ \iota\big\uparrow \\ \big(T_{\mathrm{Cart}}(C)[[\hbar]], \hbar\partial, [\,\cdot\,, \cdot\,]\big)\\ p\big\downarrow \\ \big(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, [\,\cdot\,, \cdot\,]\big),\end{array}$$

both maps are DGLA morphisms and ι is still a quasi-isomorphism of DGLAs.

Proof. The proof essentially follows from the above considerations. More explicitely, the inclusion of the Cartan model into $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$ is a quasi-isomorphism of DGLAs since the bracket with $[-J, \cdot\,]$ is not scaled by $\hbar$ and $[\hbar\pi_{\mathrm{KKS}}, \cdot\,]$ is just $\hbar\partial$ in the cohomology of $[-J, \cdot\,]$. In other words, the argument from Proposition 4.20 applies. ∎

Note that here we only prove the fact that the L_∞-quasi-inverse of ι exists. In Section 4.4, we give an explicit formula for this map.

Remark 4.23 (Laurent series). We observe that the map p in the above theorem is *not* a quasi-isomorphism due to the scaling problem by $\hbar$. Concerning the projection from the Cartan model to M_{red}, we still have the map h satisfying $\hbar\partial h + h\hbar\partial = \hbar(\deg_{\mathfrak{g}} + \deg_{\mathrm{ver}})\mathrm{id}$, as in Proposition 4.15. However, since we are not allowed to divide by $\hbar$, the projection p in the formal setting is no longer a quasi-isomorphism. We remark that, if we consider instead the Laurent series in $\hbar$ in all the complexes, e.g., $T_{\mathrm{poly}}(M_{\mathrm{red}})[\hbar^{-1},\hbar]]$, then it remains a quasi-isomorphism.

Moreover, we know from [18, Theorem 4.6] that L_∞-quasi-isomorphisms induce bijections on the equivalence classes of formal Maurer–Cartan elements. In our setting, this yields the following corollary.

Corollary 4.24. *Every formal Maurer–Cartan element $\hbar(\pi - J')$ in $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$ is equivalent to a formal Maurer–Cartan element*

$$\hbar\pi_C \in T_{\mathrm{Cart}}(C)^1[[\hbar]] \subset T^1_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]].$$

In other words, the above Corollary states that every formal Poisson structure $\pi_{\mathrm{KKS}} + \pi$ with formal momentum map $J + \hbar J'$ is equivalent to a formal Poisson structure $\pi_{\mathrm{KKS}} + \pi_C$ with undeformed momentum map J. Finally, we can construct an explicit equivalence transformation from a generic Maurer–Cartan element $\hbar(\pi - J')$ to one with $J' = 0$. Set $X^1_\hbar = \hbar J'_i e^i$ and $J'^2_i = \exp(X^1_\hbar)(J_i) - J_i - \hbar J'_i$. One can recursively define for $k \geq 1$

$$X^{k+1}_\hbar = -J'^{k+1}_i e^i := -\big(\exp(X^k_\hbar)\cdots\exp(X^1_\hbar)(J_i) - J_i - \hbar J'_i\big)e^i. \tag{4.31}$$

Proposition 4.25. *Let $\hbar(\pi - J')$ be a formal Maurer–Cartan element in $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$. Then*

$$X^\infty_\hbar = \log\Big(\lim_{k\to\infty} \exp(X^k_\hbar)\cdots\exp(X^1_\hbar)\Big) \tag{4.32}$$

satisfies $\exp(X^\infty_\hbar)(J_i) = J_i + \hbar J'_i$ *and hence* $\hbar\exp(-X^\infty_\hbar)(\pi_{\mathrm{KKS}} + \pi) - \hbar\pi_{\mathrm{KKS}}$ *is a formal Maurer–Cartan element in* $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$ *equivalent to* $\hbar(\pi - J')$.

Proof. Note that $X^1_\hbar \in \mathcal{O}(\hbar)$ and inductively one gets

$$\begin{aligned} J_i + \hbar J'_i + J'^{k+1}_i &= \exp(X^k_\hbar)\exp(X^{k-1}_\hbar)\cdots\exp(X^1_\hbar)(J_i) = \exp(X^k_\hbar)(J_i + \hbar J'_i + J'^k_i) \\ &= J_i + \hbar J'_i + J'^k_i + X^k_\hbar(J_i) + \mathcal{O}(\hbar^{k+1}). \end{aligned}$$

Hence $J'^{k+1}_i \in \mathcal{O}(\hbar^{k+1})$ as well as $X^{k+1}_\hbar \in \mathcal{O}(\hbar^{k+1})$. In particular, $X^\infty_\hbar$ is well defined and satisfies

$$\exp(X^\infty_\hbar)(J_i) = J_i + \hbar J'_i + \lim_{k\to\infty} J'^k_i = J_i + \hbar J'_i$$

in the $\hbar$-adic topology. The gauge equivalence $\exp(-X^\infty_\hbar)$, therefore, maps $\hbar(\pi - J')$ to

$$\begin{aligned} \exp(-X^\infty_\hbar) \triangleright \hbar(\pi - J') &= \exp(-X^\infty_\hbar)\big(\hbar\pi_{\mathrm{KKS}} - J + \hbar(\pi - J')\big) - (\hbar\pi_{\mathrm{KKS}} - J) \\ &= \hbar\exp(-X^\infty_\hbar)(\pi_{\mathrm{KKS}} + \pi) - \hbar\pi_{\mathrm{KKS}}; \end{aligned}$$

compare [31, Proposition 6.2.34] for a formula of the gauge action. ∎

4.4. L_∞-quasi-inverse of ι

Finally, we want to find an explicit description of the L_∞-quasi-inverse of ι, i.e., an L_∞-quasi-isomorphism

$$P\colon \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J, \cdot], [\cdot, \cdot]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, [\cdot, \cdot]\big).$$

One can check that the homotopy h of $[-J, \cdot]$ from Proposition 4.12 does not commute with $[\pi_{\mathrm{KKS}}, \cdot]$. The idea is to start with $[-J, \cdot]$ as differential on the Taylor decomposition and zero differential on the Cartan model, construct the L_∞-quasi-isomorphism P in this case, and then investigate the compatibility with $[\pi_{\mathrm{KKS}}, \cdot]$.

Let us focus on the deformation retract of DGLAs

$$\big(T_{\mathrm{Cart}}(C), 0\big) \underset{p}{\overset{i}{\rightleftarrows}} \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [-J, \cdot]\big) \circlearrowleft h \tag{4.33}$$

and apply the construction from Section 3. By Proposition 3.2, we have an L_∞-quasi-isomorphism P given by $P_1 = p$ and

$$P_n = P_n^1 = (R_2^1 P_n^2 - P_{n-1}^1 Q_n^{n-1}) \circ H_n, \tag{4.34}$$

where Q and R denote the L_∞-structure on $\mathrm{S}(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[1])$ and on $\mathrm{S}(T_{\mathrm{Cart}}(C)[1])$, respectively. Moreover, H_n is the extension of

$$h(\xi \otimes P \otimes \alpha \otimes X) = \begin{cases} \frac{-1}{\deg_{S\mathfrak{g}^*} \xi + \deg_{\Lambda \mathfrak{g}^*} \alpha} \mathrm{i_s}(e_\ell)\xi \otimes P \otimes e^\ell \wedge \alpha \otimes X & \text{if } \deg_{S\mathfrak{g}^*} \xi + \deg_{\Lambda\mathfrak{g}^*} \alpha \neq 0, \\ 0 & \text{else,} \end{cases}$$

since $Q_1^1 = [J, \cdot]$; compare Proposition 4.12.

Lemma 4.26. *For $n = 2$, one has*

$$P_2(X_1 \vee X_2) = -p\big((-1)^{|X_1|}[hX_1, X_2] - [X_1, hX_2]\big) \tag{4.35}$$

for all homogeneous $X_1, X_2 \in T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[1]$.

Proof. One has $P_2^2 \circ H_2 = 0$. Furthermore, for $Q_2^1(X_1, Y_1) = -(-1)^{|X_1|}[X_1, X_2]$ with $|X_1|$ denoting the shifted degree in $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[1]$, we have with the formula for H_2, see [21, p. 383],

$$\begin{aligned} &P_2(X_1 \vee X_2) \\ &\quad = -p \circ Q_2^1 \circ H_2(X_1 \vee X_2) \\ &\quad = -\frac{p}{2}\big(-(-1)^{|X_1|+1}[hX_1, X_2 + ipX_2] + (-1)^{|X_1|+|X_1|+1}[X_1 + ipX_1, hX_2]\big) \\ &\quad = -p\big((-1)^{|X_1|}[hX_1, X_2] - [X_1, hX_2]\big). \end{aligned}$$

The last step is easily seen for homogeneous elements by counting the $\mathfrak{g}^*$-degrees. In fact, if $X_2 = ipX_2$, then $hX_2 = 0$ and the statement holds. If $ipX_2 = 0$, then $p([hX_1, X_2]) = 0$ since the bracket contains at least one $\mathfrak{g}^*$-component that is annihilated by p. The same holds for $1 \leftrightarrow 2$. ∎

As a next step, we want to obtain an L_∞-morphism between $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J, \cdot\,])$, and $(T_{\mathrm{Cart}}(C), \partial)$. Let us first observe that P_n contains $n-1$ brackets and $n-1$ applications of h, increasing the $\Lambda^\bullet \mathfrak{g}^*$-degrees. This implies that the P_n are non-zero only if all n arguments have no $\Lambda \mathfrak{g}^*$-contribution and the sum of the $\mathrm{S}\,\mathfrak{g}^*$-degrees is $n-1$. As a consequence, all $n-1$ brackets consist of pairings between $\Lambda \mathfrak{g}^*$-components coming from h and the $\prod \mathrm{S}\,\mathfrak{g}$-components, whereas the $T_{\mathrm{poly}}(C)$-components are just wedged together. Moreover, the first term in (4.34) does not contribute since the bracket R_2^1 is here in C-direction and we have

$$P_n = P_n^1 = -P_{n-1}^1 \circ Q_n^{n-1} \circ H_n. \tag{4.36}$$

Therefore, to prove the compatibility of P with the differentials $[\pi_{\mathrm{KKS}}, \cdot\,]$ and ∂, we only have to show that

$$-\partial P_n^1 = P_n^1 \circ (Q^\pi)_n^n,$$

where $(Q^\pi)_n^n$ is the extension of $-[\pi_{\mathrm{KKS}}, \cdot\,]$. By the proof of Proposition 4.19 and the above arguments, the only part with a non-trivial contribution is the extension of $-\partial = -\mathrm{id} \otimes \mathrm{i}_{\mathrm{s}}(e^i) \otimes \mathrm{id} \otimes (e_i)_C \wedge$.

Proposition 4.27. *The map P from (4.36) is an L_∞-quasi-isomorphism from the Taylor series expansion $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\pi_{\mathrm{KKS}} - J, \cdot\,])$ to $(T_{\mathrm{Cart}}(C), \partial)$ and an L_∞-quasi-inverse to the inclusion ι from Proposition 4.20. The same holds in the formal setting with the rescaled differentials $[\hbar\pi_{\mathrm{KKS}} - J, \cdot\,]$ and $\hbar\partial$.*

Proof. By the above reasoning all brackets consist of pairings in $\mathfrak{g}^*$-direction and the $T_{\mathrm{poly}}(C)$-components are just wedged together, so ∂ satisfies a Leibniz rule. Let us show the statement inductively. For $n = 1$, it is obvious. In addition, we know that $[h, \partial] = 0$ and thus

$$\begin{aligned} P_{n-1}^1 \circ Q_n^{n-1} \circ H_n \circ (Q^\pi)_n^n &= P_{n-1}^1 \circ Q_n^{n-1} \circ H_n \circ (-\partial)_n^n \\ &= -P_{n-1}^1 \circ Q_n^{n-1} \circ (-\partial)_n^n \circ H_n \\ &= -P_{n-1}^1 \circ Q_n^{n-1} \circ (Q^\pi)_n^n \circ H_n \end{aligned}$$

as the only part of Q^π that contributes is $-\partial$. Moreover, we have

$$(Q^\pi)_n^n Q_{n+1}^n = -Q_{n+1}^n (Q^\pi)_{n+1}^{n+1}$$

since $(Q^\pi)_1^1 = [-\pi_{\mathrm{KKS}}, \cdot\,]$ is a derivation. Then (4.36) gives

$$-\partial P_{n+1}^1 = \partial P_n^1 Q_{n+1}^n H_{n+1} = -P_n (Q^\pi)_n^n Q_{n+1}^n H_{n+1} = -P_n Q_{n+1}^n H_{n+1} (Q^\pi)_{n+1}^{n+1}$$

and the statement follows by induction. ∎

Remark 4.28. Note that here we cannot use the usual twisting procedure since we have no complete filtration compatible to P such that π_{KKS} is of degree one. Of course, this is to be expected since the differential on $T_{\mathrm{Cart}}(C)$ is not an inner one.

We can also show that P is compatible with the curvature, which is easier to show in the formal setting.

Proposition 4.29. *The map P from (4.36) is an L_∞-morphism between the curved Taylor expansion $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], \hbar\lambda, [-J, \cdot], [\cdot, \cdot])$ and $(T_{\mathrm{Cart}}(C)[[\hbar]], 0, \hbar\partial, [\cdot, \cdot])$.*

Proof. We can twist P from Proposition 4.27 with $-\hbar\pi_{\mathrm{KKS}}$ as in [13, Lemma 2.7]. Then we obtain an L_∞-morphism from the Taylor expansion $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], \hbar\lambda, [-J, \cdot])$ to $(T_{\mathrm{Cart}}(C), 0, \hbar\partial)$. This is clear since the new codifferential on $\mathrm{S}(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]][1])$ is given by

$$Q'_0 = Q_1(-\hbar\pi_{\mathrm{KKS}}) + \frac{1}{2}Q_2\big(-\hbar\pi_{\mathrm{KKS}}, -\hbar\pi_{\mathrm{KKS}}\big) = [\hbar\pi_{\mathrm{KKS}} - J, \hbar\pi_{\mathrm{KKS}}] = -\hbar\lambda,$$
$$Q'_1(X) = Q_1(X) + Q_2(-\hbar\pi_{\mathrm{KKS}}, X) = [-\hbar\pi_{\mathrm{KKS}} + J + \hbar\pi_{\mathrm{KKS}}, X].$$

Since π_{KKS} contains a $\Lambda\mathfrak{g}^*$-degree, the twisting does not change the L_∞-structure on the Cartan model and the twisted morphism is just given by P. ■

Note that, in this case, P is no longer a quasi-isomorphism, and that the result also holds in the classical setting:

Corollary 4.30. *The map P from (4.36) is also an L_∞-morphism between the curved DGLAs $(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, [-J, \cdot])$ and $(T_{\mathrm{Cart}}(C), 0, \partial)$.*

Proof. Since the morphism P is $\hbar$-linear, we can compute explicitly that the Taylor coefficients of P are compatible with the above curved DGLA structures. By the construction of P, we know that

$$R_2^1 P_n^2 = P_n^1 Q_n^n + P_{n-1}^1 Q_n^{n-1},$$

where R_2^1 is the bracket on the Cartan model and Q_1^1 is the extension of $[J, \cdot]$. Moreover, we have by Proposition 4.29 that

$$\hbar R_1^1 P_n^1 + R_2^1 P_n^2 = P_{n+1}^1(\hbar Q_0 \vee \cdot) + P_n^1 Q_n^n + P_{n-1}^1 Q_n^{n-1},$$

where $R_1^1 = -\partial$ and $Q_0 = -\lambda$. This gives

$$\hbar R_1^1 P_n^1 = P_{n+1}^1(\hbar Q_0 \vee \cdot) \Rightarrow R_1^1 P_n^1 = P_{n+1}^1(Q_0 \vee \cdot)$$

and the statement is shown. ■

Remark 4.31. This can also be directly shown for the classical setting. Indeed, we do not have the complete filtration, but by the explicit forms of P and π_{KKS}, all the appearing series in the twisting procedure are still well defined.

5. The reduction L_∞-morphism and reduction of formal Poisson structures

Let us now merge together all the results we obtained in the previous sections in order to finalize the construction of the reduction scheme. Let a Lie group action $\Phi\colon G \times M \to M$ on a general manifold M and an equivariant map $J\colon M \to \mathfrak{g}^*$ with regular value 0 interpreted as an element $J \in (\mathfrak{g}^* \otimes \mathscr{C}^\infty(M))^{\mathrm{G}}$ be given. In (4.4), we defined the curved DGLA

$$\big(T_\mathfrak{g}(M), \lambda, -[J, \cdot], [\cdot\,, \cdot]\big),$$

and we want to obtain an L_∞-morphism to $T_{\mathrm{poly}}(M_{\mathrm{red}})$ with zero differential in order to reduce, in particular, formal Poisson structures.

5.1. The reduction L_∞-morphism

Under the above assumptions that the action is proper in an open neighborhood of the constraint surface $C := J^{-1}(\{0\})$, we find an open G-invariant neighborhood $C \subseteq M_{\mathrm{nice}} \cong U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^*$, such that the momentum map on U_{nice} is just the projection on the second factor and such the group acts as the product of the action on C and the coadjoint action. This yields the curved DGLA morphism

$$\cdot|_{U_{\mathrm{nice}}}\colon \big(T_\mathfrak{g}(M), \lambda, -[J, \cdot], [\cdot\,, \cdot]\big) \to \big(T_\mathfrak{g}(U_{\mathrm{nice}}), \lambda|_{U_{\mathrm{nice}}}, -[J|_{U_{\mathrm{nice}}}, \cdot], [\cdot\,, \cdot]\big)$$

which is just the restriction to the invariant open subset M_{nice} concatenated with the extension of the G-equivariant diffeomorphism to U_{nice}. Moreover, we know from [4, Lemma 3] that U_{nice} is an open neighborhood of $C \times \{0\}$ such that $U_{\mathrm{nice}} \cap (\{p\} \times \mathfrak{g}^*)$ is star-shaped around $\{p\} \times \{0\}$ for all $p \in C$, hence we also have the Taylor expansion as in equation (4.7). It is a morphism of curved DGLAs

$$T_{\mathfrak{g}^*}\colon \big(T_\mathfrak{g}(U_{\mathrm{nice}}), \lambda|_{U_{\mathrm{nice}}}, -[J|_{U_{\mathrm{nice}}}, \cdot], [\cdot\,, \cdot]\big) \to \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot\,, \cdot]\big).$$

With Proposition 4.29 and Corollary 4.30, we obtain, furthermore, a curved L_∞-morphism

$$P\colon \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot\,, \cdot]\big) \to \big(T_{\mathrm{Cart}}(C), 0, \partial, [\cdot\,, \cdot]\big)$$

and finally we have the projection

$$p\colon \big(T_{\mathrm{Cart}}(C), 0, \partial, [\cdot\,, \cdot]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, 0, [\cdot\,, \cdot]\big)$$

from equation (4.23) that is a DGLA morphism and hence also a morphism of (curved) L_∞-algebras.

Theorem 5.1. *The composition of the above morphisms results in a curved L_∞-morphism*

$$\mathrm{T}_{\mathrm{red}}\colon \big(T_\mathfrak{g}(M), \lambda, -[J, \cdot], [\cdot\,, \cdot]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0, 0, [\cdot\,, \cdot]\big), \tag{5.1}$$

called reduction L_∞-morphism. Considering the setting of formal power series in $\hbar$, we can extend $\mathrm{T}_{\mathrm{red}}$ $\hbar$-linearly and obtain

$$\mathrm{T}_{\mathrm{red}}\colon \big(T_\mathfrak{g}(M)[[\hbar]], \hbar\lambda, -[J, \cdot], [\cdot\,, \cdot]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, 0, [\cdot\,, \cdot]\big).$$

5.2. Reduction of formal Poisson structures

As mentioned above, a formal curved Maurer–Cartan element $\hbar(\pi - J') \in \hbar T_{\mathfrak{g}}(M)[[\hbar]]$ is an invariant formal Poisson structure $\hbar\pi$ with a formal moment map $J + \hbar J'$. By $\mathrm{T}_{\mathrm{red}}$ we obtain, therefore, a formal Maurer–Cartan element

$$\hbar\pi_{\mathrm{red}} = \sum_{k\geq 1} \frac{1}{k!} \mathrm{T}_{\mathrm{red},k}\big(\hbar(\pi - J')^{\vee k}\big) \tag{5.2}$$

in $T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]$ which corresponds to a formal Poisson structure π_{red} on M_{red}.

In order to show that this morphism gives indeed a non-trivial reduction scheme for formal Poisson structures, we show at first that we recover the Marsden–Weinstein reduction. This classical setting is included in our formulation by considering special curved formal Maurer–Cartan elements $\hbar\pi \in \hbar T_{\mathfrak{g}}(M)[[\hbar]]$, where in fact $\pi \in T^1_{\mathrm{poly}}(M)$ does not depend on $\hbar$, i.e., is a classical G-invariant Poisson structure with a momentum map J.

Proposition 5.2. *The reduction procedure of Marsden–Weinstein coincides with the one via* $\mathrm{T}_{\mathrm{red}}$ *from Theorem 5.1 for Maurer–Cartan elements of the form* $\hbar\pi \in \hbar T_{\mathfrak{g}}(M)[[\hbar]]$ *with* $\pi \in T^1_{\mathrm{poly}}(M)$.

Proof. By Lemma 4.10, we know that $\hbar\pi$ takes in the Taylor expansion the form $\hbar\pi_{\mathrm{KKS}} + \hbar\pi_C$, where $\pi_C = \prod_i \pi_C^i$ with $\pi_C^i \in \mathrm{S}^i\,\mathfrak{g} \otimes T^1_{\mathrm{poly}}(C)$. Then the application of $p \circ P$ yields a Maurer–Cartan element $\hbar\pi_{\mathrm{red}}$ in the reduced DGLA $(T_{\mathrm{poly}}(M_{\mathrm{red}}[[\hbar]]), 0, [\,\cdot\,,\,\cdot\,])$ via

$$\hbar\pi_{\mathrm{red}} = \sum_{k\geq 1} \frac{1}{k!} p \circ P_k\big(\hbar(\pi_{\mathrm{KKS}} + \pi_C), \ldots, \hbar(\pi_{\mathrm{KKS}} + \pi_C)\big) = p(\hbar\pi_C^0),$$

so this series is indeed well defined. This Maurer–Cartan element corresponds to a classical Poisson structure π_{red} with

$$p^*\pi_{\mathrm{red}}(\mathrm{d}\phi, \mathrm{d}\psi) = \pi_C^0(\mathrm{d}p^*\phi, \mathrm{d}p^*\psi) = \iota^*(\pi_{\mathrm{KKS}} + \pi_C)(\mathrm{d}\,\mathrm{prol}\, p^*\phi, \mathrm{d}\,\mathrm{prol}\, p^*\psi)$$

for $\phi, \psi \in \mathscr{C}^\infty(M_{\mathrm{red}})$, where $\mathrm{prol}\colon \mathscr{C}^\infty(C) \to \mathscr{C}^\infty(C) \otimes \prod_i \mathrm{S}^i\,\mathfrak{g}$ is the canonical prolongation. But this is just the usual reduced Poisson structure from Marsden–Weinstein reduction. ∎

Now we want to show that our construction is indeed a non-trivial extension of the classical Marsden–Weinstein reduction to the formal setting. For simplicity, let us consider for a moment just a part of $\mathrm{T}_{\mathrm{red}}$, namely the map

$$\widetilde{\mathrm{T}}_{\mathrm{red}} = p \circ P\colon \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], \hbar\lambda, -[J,\,\cdot\,], [\,\cdot\,,\,\cdot\,]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, 0, [\,\cdot\,,\,\cdot\,]\big).$$

Lemma 5.3. *The induced map at the level of Maurer–Cartan elements*

$$\begin{aligned} \widetilde{\mathrm{T}}_{\mathrm{red}}\colon MC\big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]\big) &\to MC\big(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]\big) \\ \hbar(\pi - J') &\mapsto \sum_{k\geq 1} \frac{1}{k!} \widetilde{\mathrm{T}}_{\mathrm{red},k}\big(\hbar(\pi - J')^{\vee k}\big) \end{aligned} \tag{5.3}$$

is a surjection.

Proof. Let $\hbar\pi_{\mathrm{red}} \in MC(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]])$, then we know from Corollary 4.17 that

$$\Pi = \sum_{k\geq 1} \frac{1}{k!}\iota_{\infty,k}\big((\pi_{\mathrm{red}})^{\vee k}\big)$$

is a well-defined Maurer–Cartan element in $T_{\mathrm{Cart}}(C)[[\hbar]]$ with differential ∂. Thus $\hbar\Pi$ is a Maurer–Cartan element with respect to the differential $\hbar\partial$ and it satisfies $p(\hbar\Pi) = \hbar\pi_{\mathrm{red}}$. Using Proposition 4.19, we see that $\hbar(\pi_{\mathrm{KKS}} + \Pi) \in MC(T_{\mathrm{Tay}}(C \times \mathfrak{g})[[\hbar]])$ and

$$\sum_{k\geq 1} \frac{1}{k!}\widetilde{\mathrm{T}}_{\mathrm{red},k}\Big(\big(\hbar(\pi_{\mathrm{KKS}} + \Pi)\big)^{\vee k}\Big) = p(\hbar\Pi) = \hbar\pi_{\mathrm{red}}$$

as desired. ∎

5.3. Comparison of the reduction procedures

We conclude with a comparison of the different reduction procedures. More explicitly, we want to compare the reduction via $\mathrm{T}_{\mathrm{red}}$ from Theorem 5.1 with the reduction of formal Poisson structures via the formal Koszul complex; see Appendix A.

In the setting of curved DGLAs or curved L_∞-algebras, it is more tricky to talk about equivalent Maurer–Cartan elements. Thus we switch to the description of our reduction in terms of flat DGLAs as in Theorem 4.22. Here we need π_{KKS} which is not available in the general setting, so from now on we restrict ourselfes to the Taylor expansion

$$\big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*), [\hbar\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,,\cdot\,]\big).$$

Consider a formal Poisson structure $\pi_\hbar = \sum_{r=0}^{\infty} \hbar^r\pi_r \in \Gamma^\infty(\Lambda^2 TM)[[\hbar]]$ with a formal equivariant momentum map $J_\hbar = J + \hbar J' \colon \mathfrak{g} \to \mathscr{C}^\infty(M)[[\hbar]]$. By Proposition A.3, one gets an induced formal Poisson bracket on $M_{\mathrm{red}} = J^{-1}(\{0\})/G$ via

$$\pi^*\{u, v\}_{\mathrm{red}} = \boldsymbol{\iota}^*\big\{[\mathrm{prol}\,\pi^* u], [\mathrm{prol}\,\pi^* v]\big\}_\hbar,$$

where the deformed restriction map is given by

$$\boldsymbol{\iota}^* = \iota^*\big(\mathrm{id} + \mathrm{i_a}(\hbar J')h_0\big)^{-1} = \iota^* \sum_{k=0}^{\infty}\big(-\mathrm{i_a}(\hbar J')h_0\big)^k; \tag{5.4}$$

compare Proposition A.3. We directly see that the reduction procedure works analogously for $\pi_\hbar \in T^1_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$.

Theorem 5.4. *The reduction of formal equivariant Poisson structures with formal momentum maps via*

$$\widetilde{\mathrm{T}}_{\mathrm{red}} = p \circ P \colon \big(T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\hbar\pi_{\mathrm{KKS}} - J, \cdot\,], [\,\cdot\,,\cdot\,]\big) \to \big(T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, [\,\cdot\,,\cdot\,]\big)$$

coincides with the reduction of formal Poisson structures via the formal Koszul complex from Proposition A.3.

Proof. We show at first that the reduction procedures coincide on Maurer–Cartan elements of the form $\hbar\pi_C$, i.e., where the quantum momentum map is just the classical momentum map. Note that by Corollary 4.24 every formal Maurer–Cartan element $\hbar(\pi' - J')$ is equivalent to such an $\hbar\pi_C$. Writing again

$$\pi_C^i \in \big(\mathrm{S}^i\,\mathfrak{g} \otimes T^1_{\mathrm{poly}}(C)\big)[[\hbar]],$$

the reduced Poisson structure via $\widetilde{\mathrm{T}}_{\mathrm{red}}$ is easy to describe, namely by

$$\begin{aligned}\hbar\pi_{\mathrm{red}} &= \sum_{k=1}^{\infty} \frac{1}{k!}\widetilde{\mathrm{T}}^1_{\mathrm{red},k}(\hbar\pi_C \vee \cdots \vee \hbar\pi_C) \\ &= \sum_{k=1}^{\infty} \frac{\hbar^k}{k!} p \circ P_k(\pi_C, \ldots, \pi_C) = p(\hbar\pi_C^0).\end{aligned}$$

In the reduction via the formal Koszul complex, one has $\boldsymbol{\iota}^* = \iota^*$ and thus the reduced formal Poisson structures coincide by the same reasons as in the classical setting of Proposition 5.2.

The idea is now to use the explicit equivalence from Proposition 4.25. Let $\hbar(\pi - J')$ be a formal Maurer–Cartan element in $T_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]]$ and let $X_\hbar^\infty$ be the equivalence between the formal Maurer–Cartan elements $(\pi_{\mathrm{KKS}} + \pi, J + \hbar J')$ and $(\pi_{\mathrm{KKS}} + \pi_C, J)$. The reduction via the formal Koszul complex maps both Poisson structures to the same formal Poisson structure on M_{red}. This follows from formula (A.13) for the equivalence between the reduced Poisson structures since $X_\hbar^\infty$ differentiates only in the direction of $\mathfrak{g}^*$. We only have to show that $\widetilde{\mathrm{T}}_{\mathrm{red}}$ also maps both to the same one. But $X_\hbar^\infty$ induces the following equivalence on the level of the reduced manifold:

$$p \circ P^1\big(X_\hbar^\infty \vee \exp\big(\exp(X_\hbar^\infty) \triangleright \hbar(\pi - J')\big)\big) = 0,$$

see, e.g., [7, Proposition 4.9], whence both reduced structures are again equal. This proves the theorem. ∎

A. BRST-like reduction of formal Poisson structures

In this appendix, we want to recall a reduction scheme for formal Poisson structures similarly to the reduction of star products in [16] resp. to the BRST reduction as formulated in [4]. It is obtained by extending the Koszul part of the classical BRST reduction as in [19, 30] to the formal setting. This can be achieved by the homological perturbation lemma; see [8, Theorem 2.4] and [28, Chapter 2.4] for versions adapted to our setting.

A.1. Homological perturbation lemma

Definition A.1 (Homotopy equivalence data). A *homotopy equivalence data* (HE data) consists of two chain complexes (C, d_C) and (D, d_D) over a commutative ring R together with two quasi-isomorphisms

$$p\colon C \to D \quad \text{and} \quad i\colon D \to C \tag{A.1}$$

and a chain homotopy

$$h \colon D \to D \quad \text{with} \quad \mathrm{id}_D - pi = \mathrm{d}_D h + h \mathrm{d}_D \tag{A.2}$$

between pi and id_D.

For a shorter notation, we will denote such an HE data by

$$p \colon (C, \mathrm{d}_C) \rightleftarrows (D, \mathrm{d}_D) \colon i, h.$$

Moreover, we say that a graded map $B \colon D_\bullet \to D_{\bullet-1}$ with $(\mathrm{d}_D + B)^2 = 0$ is a *perturbation* of the HE data. The perturbation is called *small* if $\mathrm{id}_D + Bh$ is invertible, and the homological perturbation lemma states that in this case the perturbed HE data is again an HE data; see [8, Theorem 2.4] for a proof.

Proposition A.2 (Homological perturbation lemma). *Let*

$$p \colon (C, \mathrm{d}_C) \rightleftarrows (D, \mathrm{d}_D) \colon i, h$$

be an HE data and let B be small perturbation of d_D. *Then the perturbed data*

$$P \colon (C, \hat{\mathrm{d}}_C) \rightleftarrows (D, \hat{\mathrm{d}}_D) \colon I, H \tag{A.3}$$

with

$$\begin{aligned} A &= (\mathrm{id}_D + Bh)^{-1} B, \quad \hat{\mathrm{d}}_D = \mathrm{d}_D + B, \quad \hat{\mathrm{d}}_C = \mathrm{d}_C + iAp, \\ P &= p - hAp, \quad I = i - iAh, \quad H = h - hAh, \end{aligned} \tag{A.4}$$

is again an HE data.

We will even encounter a simpler situation, namely that the complex C is concentrated in degree 0 and $D_n = 0$ for $n < 0$:

$$\begin{array}{ccccccc} 0 & \longleftarrow & D_0 & \underset{h_0}{\overset{\mathrm{d}_{D,1}}{\rightleftarrows}} & D_1 & \underset{h_1}{\overset{\mathrm{d}_{D,2}}{\rightleftarrows}} & \cdots \\ & & i \downarrow \uparrow p & & & & \\ 0 & \longleftarrow & C_0 & \longleftarrow & 0 & & \end{array} \tag{A.5}$$

In this case, the perturbed HE data corresponding to a small perturbation B according to (A.4) is given by

$$P = p, \quad I = i - i(\mathrm{id}_D + B_1 h_0)^{-1} B_1 h_0, \quad H = h - h(\mathrm{id}_D + Bh)^{-1} Bh$$

and, using the geometric power series, this can be simplified to

$$P = p, \quad I = i(\mathrm{id}_D + B_1 h_0)^{-1}, \quad H = h(\mathrm{id}_D + Bh)^{-1}. \tag{A.6}$$

Here we denote by $B_1 \colon D_1 \to D_0$ the degree one component of B, analogously for h.

A.2. Formal Koszul complex

We start with the classical Koszul complex $\Lambda^\bullet\mathfrak{g} \otimes \mathscr{C}^\infty(M)$ that can be interpreted as the smooth functions on M with values in the complexified Grassmann algebra of $\mathfrak{g}$. The Koszul differential ∂ is given by

$$\partial\colon \Lambda^q\mathfrak{g} \otimes \mathscr{C}^\infty(M) \to \Lambda^{q-1}\mathfrak{g} \otimes \mathscr{C}^\infty(M), \quad a \mapsto \mathrm{i}(J_0)a = J_{0,i}\,\mathrm{i}_{\mathrm{a}}(e^i)a, \tag{A.7}$$

where i denotes the left insertion and $J_0 = J_{0,i}e^i$ the decomposition of J_0 with respect to a basis $e^1, \dots, e^n$ of $\mathfrak{g}^*$. Then $\partial^2 = 0$ follows immediately with the commutativity of the pointwise product in $\mathscr{C}^\infty(M)$. The differential ∂ is also a derivation with respect to the associative and super-commutative product on the Koszul complex, consisting of the $\wedge$-product on $\Lambda^\bullet\mathfrak{g}$ tensored with the pointwise product on the functions. Moreover, it is invariant with respect to the induced $\mathfrak{g}$-representation

$$\mathfrak{g} \ni \xi \mapsto \rho(\xi) = \mathrm{ad}(\xi) \otimes \mathrm{id} - \mathrm{id} \otimes \mathscr{L}_{\xi_M} \in \mathrm{End}\left(\Lambda^\bullet\mathfrak{g} \otimes \mathscr{C}^\infty(M)\right) \tag{A.8}$$

as we have

$$\begin{aligned}
&\partial\rho(e_a)(x \otimes f)\\
&\quad = f^k_{aj} e_k \wedge \mathrm{i}(e^j) \wedge \mathrm{i}(e^i)x \otimes J_{0,i} f + f^i_{aj}\mathrm{i}(e^j)x \otimes J_{0,i} f + \mathrm{i}(e^i)x \otimes J_{0,i}\{J_{0,a}, f\}_0\\
&\quad = \rho(e_a)\partial(x \otimes f)
\end{aligned}$$

for all $x \in \Lambda^\bullet\mathfrak{g}$ and $f \in \mathscr{C}^\infty(M)$.

One can show that the Koszul complex is acyclic in positive degree with homology $\mathscr{C}^\infty(C)$ in order zero, and that one has a G-equivariant homotopy

$$h_i\colon \Lambda^i\mathfrak{g} \otimes \mathscr{C}^\infty(M) \to \Lambda^{i+1}\mathfrak{g} \otimes \mathscr{C}^\infty(M) \tag{A.9}$$

given on $C \subset U_{\mathrm{nice}} \subset C \times \mathfrak{g}^*$ by

$$h_k(x)(c,\mu) = e_i \wedge \int_0^1 t^k \frac{\partial x}{\partial \mu_i}(c, t\mu)\mathrm{d}t, \quad \text{with } \partial h_0 = \mathrm{id}_0 - \mathrm{prol}\,\iota^*$$

and $h_0 \circ \mathrm{prol} = 0$, where $x \in \Lambda^k\mathfrak{g} \otimes \mathscr{C}^\infty(C \times \mathfrak{g}^*)$ and $(c,\mu) \in C \times \mathfrak{g}^*$; see [4, Lemma 6] and [16] for the notation U_{nice}. In other words, this means that

$$\mathrm{prol}\colon \left(\mathscr{C}^\infty(C), 0\right) \rightleftarrows \left(\Lambda^\bullet\mathfrak{g} \otimes \mathscr{C}^\infty(M), \partial\right)\colon \iota^*, h$$

is an HE data of the special type of (A.5); i.e., we have the diagram

$$\begin{array}{ccccccc}
0 & \longleftarrow & \mathscr{C}^\infty(M) & \underset{h_0}{\overset{\partial_1}{\rightleftarrows}} & \Lambda^1\mathfrak{g} \otimes \mathscr{C}^\infty(M) & \underset{h_1}{\overset{\partial_2}{\rightleftarrows}} & \cdots\\
 & & {\scriptstyle\iota^*}\big\downarrow\big\uparrow{\scriptstyle\mathrm{prol}} & & & & \\
0 & \longleftarrow & \mathscr{C}^\infty(C) & \longleftarrow & 0 & &
\end{array}$$

Let now $\pi_\hbar$ be an invariant formal Poisson structure with a formal equivariant momentum map $J_\hbar$. In order to take care of the formal momentum map, we extend the Koszul complex $\hbar$-linearly and gain the HE data

$$\mathrm{prol}\colon \big(\mathscr{C}^\infty(C)[[\hbar]], 0\big) \rightleftarrows \big(\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)[[\hbar]], \partial\big)\colon \iota^*, h.$$

Since the formal momentum map $J_\hbar$ is a deformation of J_0 in the sense that the difference $J_\hbar - J_0 = J'\colon \mathfrak{g} \to \hbar\mathscr{C}^\infty(M)[[\hbar]]$ starts in order one of $\hbar$, the formal differential $\partial_\hbar = \mathrm{i}(J_\hbar) = \partial + B$ with $B = \mathrm{i}(J')$ on $\Lambda^\bullet\mathfrak{g} \otimes \mathscr{C}^\infty(M)[[\hbar]]$ is a small perturbation in the sense of the homological perturbation lemma (Lemma A.2). Indeed, $\partial_\hbar^2 = 0$ follows for the same reasons as $\partial^2 = 0$, and $\mathrm{id} + Bh$ is invertible as formal power series since Bh stars in order one of $\hbar$. Consequently, the corresponding perturbed HE data of the *formal Koszul complex*

$$\mathbf{prol}\colon \big(\mathscr{C}^\infty(C)[[\lambda]], 0\big) \rightleftarrows \big(\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)[[\lambda]], \partial_\hbar\big)\colon \boldsymbol{\iota}^*, \boldsymbol{h}$$

is given by

$$\mathbf{prol} = \mathrm{prol}, \quad \boldsymbol{\iota}^* = \iota^*(\mathrm{id} + B_1 h_0)^{-1}, \quad \boldsymbol{h} = h(\mathrm{id} + Bh)^{-1}; \tag{A.10}$$

compare (A.6). In particular, we have $\boldsymbol{\iota}^*\partial_\hbar = 0$,

$$\mathrm{id}_{\Lambda^\bullet\mathfrak{g}\otimes\mathscr{C}^\infty(M)[[\hbar]]} - \mathrm{prol}\,\boldsymbol{\iota}^* = \partial_\hbar \boldsymbol{h} + \boldsymbol{h}\partial_\hbar \tag{A.11}$$

as well as $\boldsymbol{\iota}^*\,\mathrm{prol} = \mathrm{id}_{\mathscr{C}^\infty(C)[[\hbar]]}$ because of $h_0\,\mathrm{prol} = 0$. Moreover, $\partial_\hbar$ is still a $\mathfrak{g}$-equivariant derivation of the algebra structure. Therefore, also $\boldsymbol{\iota}^*$ and $\boldsymbol{h}$ are $\mathfrak{g}$-equivariant as all involved maps are.

We denote the image of the deformed Koszul differential by

$$\mathcal{J}_\hbar = \mathrm{im}\,\partial_\hbar\big|_{\Lambda^1\mathfrak{g}\otimes\mathscr{C}^\infty(M)[[\hbar]]} = \langle J_{\hbar,i}\rangle_i.$$

Since $\mathrm{prol}\,\boldsymbol{\iota}^*$ is a projection with kernel $\mathcal{J}_\hbar$, compare (A.11), we get with the injectivity of prol

$$\mathcal{J}_\hbar = \ker \boldsymbol{\iota}^*\big|_{\mathscr{C}^\infty(M)[[\hbar]]}.$$

As $\partial_\hbar$ is $\mathscr{C}^\infty(M)[[\hbar]]$-linear, $\mathcal{J}_\hbar$ is an ideal in $\mathscr{C}^\infty(M)[[\hbar]]$ with respect to the pointwise product. Moreover, $\mathcal{J}_\hbar$ is a Poisson subalgebra of $(\mathscr{C}^\infty(M)[[\hbar]], \{\,\cdot\,,\,\cdot\,\}_\hbar)$ because of

$$\begin{aligned}
&\boldsymbol{\iota}^*\{f, g\}_\hbar \\
&= \boldsymbol{\iota}^*\big(f^i g^j \{J_{\hbar,i}, J_{\hbar,j}\}_\hbar + f^i J_{\hbar,j}\{J_{\hbar,i}, g^j\}_\hbar + J_{\hbar,i} g^j \{f^i, J_{\hbar,j}\}_\hbar + J_{\hbar,i} J_{\hbar,j}\{f^i, g^j\}_\hbar\big) \\
&= 0
\end{aligned}$$

for $f = f^i J_{\hbar,i}$, $g = g^j J_{\hbar,j} \in \mathcal{J}_\hbar$. As usual, one can consider the Poisson normalizer

$$\mathcal{B}_\hbar = \big\{ f \in \mathscr{C}^\infty(M)[[\hbar]] \mid \{f, \mathcal{J}_\hbar\} \subset \mathcal{J}_\hbar \big\}$$

the biggest Poisson subalgebra containing $\mathcal{J}_\hbar$ as a Poisson ideal. Then we know that the quotient is a Poisson algebra and we even have the following proposition.

Proposition A.3. *There exists a unique formal Poisson structure π_{red} on M_{red} such that*

$$\mathscr{B}_\hbar/\mathscr{J}_\hbar \ni [f] \mapsto \boldsymbol{\iota}^* f \in \pi^*\mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$$

is an isomorphism of Poisson algebras with inverse $\pi^* u \mapsto [\operatorname{prol} \pi^* u]$.

Proof. We have for $u \in \mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$, $j = j^k J_{\hbar,k} \in \mathscr{J}_\hbar$, and $f \in \mathscr{B}_\hbar$

$$\begin{aligned}\boldsymbol{\iota}^*\{\operatorname{prol}\pi^*u, j\}_\hbar &= \boldsymbol{\iota}^*\big(j^k\{\operatorname{prol}\pi^*u, J_{\hbar,k}\}_\hbar + J_{\hbar,k}\{\operatorname{prol}\pi^*u, j^k\}_\hbar\big)\\ &= \boldsymbol{\iota}^*(j^k \mathscr{L}_{(e_k)_M} \operatorname{prol}\pi^*u) = 0\end{aligned}$$

as well as

$$\mathscr{L}_{(e_i)_C}\boldsymbol{\iota}^* f = \boldsymbol{\iota}^*\mathscr{L}_{(e_i)_M} f = \boldsymbol{\iota}^*\{f, J_{\hbar,i}\}_\hbar = 0,$$

thus the maps are both well defined. The fact that the maps are mutually inverse is clear since

$$\boldsymbol{\iota}^* \operatorname{prol} = \mathrm{id} \quad \text{and} \quad \mathrm{id} - \operatorname{prol}\boldsymbol{\iota}^* = \partial_\hbar \boldsymbol{h} \in \mathscr{J}_\hbar.$$

The compatibility with the pointwise product follows from the explicit form $\boldsymbol{\iota}^* = \iota^* \circ \sum_k(-B_1 h_0)^k$ and the fact that

$$h_0(f \operatorname{prol}\phi) = \operatorname{prol}\phi \cdot h_0 f,$$

which directly yields

$$\boldsymbol{\iota}^*\big([fg]\big) = \boldsymbol{\iota}^*\big([f \operatorname{prol}\boldsymbol{\iota}^* g]\big) = \boldsymbol{\iota}^* f \cdot \boldsymbol{\iota}^* g.$$

The compatibility of prol in the setting $M = M_{\mathrm{nice}}$ in the notation of [16] is clear since it is just a pull-back. In addition, we get a unique induced formal Poisson structure on M_{red} via

$$\pi^*\{u, v\}_{\mathrm{red}} = \boldsymbol{\iota}^*\big\{[\operatorname{prol}\pi^*u], [\operatorname{prol}\pi^*v]\big\}_\hbar.$$

Antisymmetry is clear and also the Jacobi identity follows directly, where we omit the sign for the equivalence classes:

$$\begin{aligned}&\pi^*\big\{u, \{v, w\}_{\mathrm{red}}\big\}_{\mathrm{red}}\\ &= \boldsymbol{\iota}^*\big\{\operatorname{prol}\pi^*u, \operatorname{prol}\boldsymbol{\iota}^*\{\operatorname{prol}\pi^*v, \operatorname{prol}\pi^*w\}_\hbar\big\}_\hbar\\ &= \boldsymbol{\iota}^*\big(\big\{\{\operatorname{prol}\pi^*u, \operatorname{prol}\pi^*v\}_\hbar, \operatorname{prol}\pi^*w\big\}_\hbar + \big\{\operatorname{prol}\pi^*v, \{\operatorname{prol}\pi^*u, \operatorname{prol}\pi^*w\}_\hbar\big\}_\hbar\big)\\ &= \pi^*\big(\big\{\{u, v\}_{\mathrm{red}}, w\big\}_{\mathrm{red}} + \big\{v, \{u, w\}_{\mathrm{red}}\big\}_{\mathrm{red}}\big).\end{aligned}$$

Concerning the Leibniz identity, we get

$$\begin{aligned}&\pi^*\{u, vw\}_{\mathrm{red}}\\ &= \boldsymbol{\iota}^*\big\{\operatorname{prol}\pi^*u, \operatorname{prol}(\pi^*v)\operatorname{prol}(\pi^*w)\big\}_\hbar\\ &= \boldsymbol{\iota}^*\big(\big\{\operatorname{prol}\pi^*u, \operatorname{prol}(\pi^*v)\big\}_\hbar \operatorname{prol}(\pi^*w) + \operatorname{prol}(\pi^*v)\big\{\operatorname{prol}\pi^*u, \operatorname{prol}(\pi^*w)\big\}_\hbar\big)\\ &= \pi^*\big(v\{u, w\}_{\mathrm{red}} + \{u, v\}_{\mathrm{red}} w\big),\end{aligned}$$

since $\boldsymbol{\iota}^*(f \operatorname{prol}\phi) = \boldsymbol{\iota}^*(f)\phi$. ∎

Now we want to show that the reduction procedure is compatible with equivalences, i.e., that equivalent formal Poisson structures with formal momentum maps are reduced to equivalent reduced Poisson structures.

Proposition A.4. *Let* $T = \exp(X_\hbar)\colon (\pi_\hbar, J_\hbar) \to (\pi'_\hbar, J'_\hbar)$ *be an equivalence of formal invariant Poisson structures with momentum maps, i.e.,* $X_\hbar \in \hbar\Gamma^\infty(TM)[[\hbar]]$ *such that*

$$T\pi_\hbar = \pi'_\hbar \quad \text{and} \quad T \circ J_\hbar = J'_\hbar. \tag{A.12}$$

Then one has even $X_\hbar \in \hbar\Gamma^\infty(TM)^{\mathrm{G}}[[\hbar]]$ *and*

$$T_{\mathrm{red}} = (\pi^*)^{-1} \circ \iota^{*\prime} \circ T \circ \mathrm{prol} \circ \pi^* \tag{A.13}$$

is an equivalence between the reduced formal Poisson structures π_{red} *and* π'_{red}.

Proof. The proof is analogue to the case of star products in [28, Lemma 4.3.1]. At first, as in [31, Proposition 6.2.20], one can show that $T\pi_\hbar = \pi'_\hbar$ is equivalent to

$$T\{f, g\}_\hbar = \{Tf, Tg\}'_\hbar.$$

But then (A.12) implies that

$$\mathscr{L}_{\xi_M} Tf = \left\{Tf, J'_\hbar(\xi)\right\}'_\hbar = T\left\{f, J_\hbar(\xi)\right\}_\hbar = T\mathscr{L}_{\xi_M} f.$$

In particular, this yields $[\xi_M, X_\hbar] = 0$ and thus the invariance of $X_\hbar$. In addition, recall from Proposition A.3 that we have an isomorphism of Poisson algebras

$$\left(\mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]], \pi_{\mathrm{red}}\right) \cong \frac{\mathscr{B}_\hbar}{\mathscr{J}_\hbar}.$$

By [31, Proposition 6.2.7], we know that T is an automorphism with respect to the pointwise product, thus we see directly from the definition of the deformed Koszul differential that

$$T \circ \partial_\hbar = \partial'_\hbar \circ T \Rightarrow T\colon \mathscr{J}_\hbar \xrightarrow{\cong} \mathscr{J}'_\hbar.$$

Analogously, we have for $j' \in \mathscr{J}'_\hbar$ with $j = T^{-1}j' \in \mathscr{J}_\hbar$ and $f \in \mathscr{B}_\hbar$

$$\{Tf, j'\}'_\hbar = T\{f, j\}_\hbar \in T\mathscr{J}_\hbar = \mathscr{J}_\hbar{}' \Rightarrow T\colon \mathscr{B}_\hbar \xrightarrow{\cong} \mathscr{B}'_\hbar.$$

Thus T_{red} establishes an isomorphism of the spaces $\mathscr{B}_\hbar/\mathscr{J}_\hbar$ and $\mathscr{B}'_\hbar/\mathscr{J}'_\hbar$. It remains to check the compatibility with the Poisson bracket:

$$\pi^* T_{\mathrm{red}}\{u, v\}_{\mathrm{red}} = \iota^{*\prime} T \,\mathrm{prol}\, \iota^*\{\mathrm{prol}\,\pi^* u, \mathrm{prol}\,\pi^* v\}_\hbar = \iota^{*\prime} T\{\mathrm{prol}\,\pi^* u, \mathrm{prol}\,\pi^* v\}_\hbar$$

since T maps the kernel of ι^* into the kernel of $\iota^{*\prime}$. On the other hand, we get

$$\begin{aligned} \pi^*\{T_{\mathrm{red}} u, T_{\mathrm{red}} v\}'_{\mathrm{red}} &= \iota^{*\prime}\{\mathrm{prol}\,\iota^{*\prime} T \,\mathrm{prol}\,\pi^* u, \mathrm{prol}\,\iota^{*\prime} T\,\mathrm{prol}\,\pi^* v\}'_\hbar \\ &= \iota^{*\prime}\{T\,\mathrm{prol}\,\pi^* u, T\,\mathrm{prol}\,\pi^* v\}'_\hbar \end{aligned}$$

since we take on the right-hand side the bracket in $\mathcal{B}'_\hbar/\mathcal{J}'_\hbar$, where $[\mathrm{prol}\,\iota^{*\prime} f] = [f]$. Thus the compatibility with the brackets is shown. It remains to show that T_{red} is of the form $T_{\mathrm{red}} = \exp(X_{\mathrm{red},\hbar})$ for some vector field $X_{\mathrm{red},\hbar} \in \hbar\Gamma^\infty(TM_{\mathrm{red}})[[\hbar]]$. Since $T = \exp(X_\hbar)$ we know that T_{red} is a formal power series of $\mathbb{C}[[\hbar]]$-linear operators starting with $\mathrm{id} + \hbar(\cdots)$. We can write $T_{\mathrm{red}} = \exp(\hbar D)$ via

$$\hbar D = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s}(T - \mathrm{id})^s.$$

Again by [31, Proposition 6.2.7] it suffices to show that $T_{\mathrm{red}}(uv) = T_{\mathrm{red}}(u)T_{\mathrm{red}}(v)$, which directly implies that $T_{\mathrm{red}} = \exp(X_{\mathrm{red}})$ for some vector field $X_{\mathrm{red}} \in \hbar\Gamma^\infty(TM_{\mathrm{red}})[[\hbar]]$. But this is clear since each of the involved maps in the definition of T_{red} is compatible with the pointwise product: the maps prol, π^*, and $(\pi^*)^{-1}$ since they, resp. their inverses, are pull-backs, the map T since $T = \exp(X_\hbar)$, and ι^* by Proposition A.3. ∎

Acknowledgments. The authors are grateful to Alejandro Cabrera and Stefan Waldmann for the helpful comments. Furthermore, we thank James Stasheff for his feedback which helped us to improve the first version of this paper.

Funding. This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA – INdAM). The third author is supported by the DFG research training group "gk1821: Cohomological Methods in Geometry".

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization. I. Deformations of symplectic structures. *Ann. Physics* **111** (1978), no. 1, 61–110 Zbl 0377.53024 MR 496157

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Deformation theory and quantization. II. Physical applications. *Ann. Physics* **111** (1978), no. 1, 111–151 Zbl 0377.53025 MR 496158

[3] A. Berglund, Homological perturbation theory for algebras over operads. *Algebr. Geom. Topol.* **14** (2014), no. 5, 2511–2548 Zbl 1305.18030 MR 3276839

[4] M. Bordemann, H.-C. Herbig, and S. Waldmann, BRST cohomology and phase space reduction in deformation quantization. *Comm. Math. Phys.* **210** (2000), no. 1, 107–144 Zbl 0961.53046 MR 1748172

[5] D. Calaque, Formality for Lie algebroids. *Comm. Math. Phys.* **257** (2005), no. 3, 563–578 Zbl 1079.53138 MR 2164943

[6] D. Calaque, V. Dolgushev, and G. Halbout, Formality theorems for Hochschild chains in the Lie algebroid setting. *J. Reine Angew. Math.* **612** (2007), 81–127 Zbl 1141.53084 MR 2364075

[7] A. Canonaco, L_∞-algebras and quasi-isomorphisms. In *Algebraic Geometry Seminars, 1998–1999 (Italian) (Pisa)*, pp. 67–86, Scuola Norm. Sup., Pisa, 1999 Zbl 1058.53511 MR 1754795

[8] M. Crainic, On the perturbation lemma, and deformations. 2004, preprint arXiv:0403266
[9] M. Dippell, C. Esposito, and S. Waldmann, Coisotropic triples, reduction and classical limit. *Doc. Math.* **24** (2019), 1811–1853 Zbl 1427.53105 MR 4033826
[10] V. Dolgushev, Covariant and equivariant formality theorems. *Adv. Math.* **191** (2005), no. 1, 147–177 Zbl 1116.53065 MR 2102846
[11] V. Dolgushev, *A proof of Tsygan's formality conjecture for an arbitrary smooth manifold*. Ph.D. thesis, Massachusetts Institute of Technology, 2005
[12] V. Dolgushev, A formality theorem for Hochschild chains. *Adv. Math.* **200** (2006), no. 1, 51–101 Zbl 1106.53054 MR 2199629
[13] C. Esposito and N. de Kleijn, L_∞-resolutions and twisting in the curved context. *Rev. Mat. Iberoam.* **37** (2021), no. 4, 1581–1598 Zbl 07367563 MR 4269410
[14] C. Esposito, A. Kraft, and S. Waldmann, BRST reduction of quantum algebras with *-involutions. *Comm. Math. Phys.* **378** (2020), no. 2, 1391–1416 Zbl 1462.81114 MR 4134949
[15] V. W. Guillemin and S. Sternberg, *Supersymmetry and Equivariant de Rham Theory*. Mathematics Past and Present, Springer, Berlin, 1999 Zbl 0934.55007 MR 1689252
[16] S. Gutt and S. Waldmann, Involutions and representations for reduced quantum algebras. *Adv. Math.* **224** (2010), no. 6, 2583–2644 Zbl 1236.53070 MR 2652217
[17] J. Huebschmann, The Lie algebra perturbation lemma. In *Higher Structures in Geometry and Physics*, pp. 159–179, Progr. Math. 287, Springer, New York, 2011 Zbl 1279.01003 MR 2762544
[18] M. Kontsevich, Deformation quantization of Poisson manifolds. *Lett. Math. Phys.* **66** (2003), no. 3, 157–216 Zbl 1058.53065 MR 2062626
[19] B. Kostant and S. Sternberg, Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Ann. Physics* **176** (1987), no. 1, 49–113 Zbl 0642.17003 MR 893479
[20] H.-Y. Liao, M. Stiénon, and P. Xu, Formality and Kontsevich-Duflo type theorems for Lie pairs. *Adv. Math.* **352** (2019), 406–482 Zbl 1417.58012 MR 3964152
[21] J.-L. Loday and B. Vallette, *Algebraic Operads*. Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012 Zbl 1260.18001 MR 2954392
[22] J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.* **5** (1974), no. 1, 121–130 Zbl 0327.58005 MR 402819
[23] E. Meinrenken, Equivariant cohomology and the Cartan model. In *Encyclopedia of Mathematical Physics*, edited by J.-P. Françoise, G. L. Naber, and T. S. Tsun, pp. 242–250, Academic Press, Oxford, 2006
[24] L. Miaskiwskyi, Invariant Hochschild cohomology of smooth functions. *J. Lie Theory* **31** (2021), no. 2, 557–574 Zbl 07372721 MR 4225043
[25] I. Moerdijk and G. E. Reyes, *Models For Smooth Infinitesimal Analysis*. Springer, New York, 1991 Zbl 0715.18001 MR 1083355
[26] N. Neumaier and S. Waldmann, Deformation quantization of Poisson structures associated to Lie algebroids. *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009), Paper No. 074 Zbl 1189.53086 MR 2529173
[27] T. Reichert, Characteristic classes of star products on Marsden-Weinstein reduced symplectic manifolds. *Lett. Math. Phys.* **107** (2017), no. 4, 643–658 Zbl 1365.53081 MR 3623275
[28] T. Reichert, *Classification and reduction of equivariant star products on symplectic manifolds*. Ph.D. thesis, Universität Würzburg, 2017
[29] T. Reichert and S. Waldmann, Classification of equivariant star products on symplectic manifolds. *Lett. Math. Phys.* **106** (2016), no. 5, 675–692 Zbl 1341.53123 MR 3490952

[30] J. Stasheff, Homological reduction of constrained Poisson algebras. *J. Differential Geom.* **45** (1997), no. 1, 221–240 Zbl 0874.58020 MR 1443334

[31] S. Waldmann, *Poisson-Geometrie und Deformationsquantisierung. Eine Einführung*. Springer, Berlin, 2007 Zbl 1139.53001

[32] P. Xu, Fedosov $*$-products and quantum momentum maps. *Comm. Math. Phys.* **197** (1998), no. 1, 167–197 Zbl 0939.37048 MR 1646487

Received 30 July 2020.

Chiara Esposito
Dipartimento di Matematica, Università degli Studi di Salerno, Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy; chesposito@unisa.it

Andreas Kraft
Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland; akraft@impan.pl

Jonas Schnitzer
Department of Mathematics, University of Freiburg, Ernst-Zermelo-Straße 1, 79104 Freiburg, Germany; jonas.schnitzer@math.uni-freiburg.de