On the lifting property for C^*-algebras

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Abstract. We characterize the lifting property (LP) of a separable C^*-algebra A by a property of its maximal tensor product with other C^*-algebras, namely we prove that A has the LP if and only if for any family $\{D_i \mid i \in I\}$ of C^*-algebras the canonical map $\ell_\infty(\{D_i\}) \otimes_{\max} A \to \ell_\infty(\{D_i \otimes_{\max} A\})$ is isometric. Equivalently, this holds if and only if $M \otimes_{\max} A = M \otimes_{\mathrm{nor}} A$ for any von Neumann algebra M.

1. Introduction

A separable C^*-algebra A has the lifting property (LP in short) if any contractive completely positive (c.c.p. in short) map $u : A \to C/\mathcal{I}$ into a quotient C^*-algebra admits a c.c.p. lifting $\hat{u} : A \to C$. In [9], Choi and Effros proved that all (separable) nuclear C^*-algebras have the LP. Later on in [19] Kirchberg proved that the full C^*-algebra of the free group $\mathbb{F}_n$ with $n > 1$ (or $n = \infty$) generators, which is notoriously non-nuclear, also has the LP. It follows that separable unital C^*-algebras with the LP are just the quotients of $C^*(\mathbb{F}_\infty)$ for which the quotient map admits a unital completely positive (u.c.p. in short) lifting. More generally, as observed by Boca in [7], it follows from the latter fact that the LP is stable by unital (maximal) free products. Indeed, it is an immediate consequence of Boca's theorem in [6] (see [11] for a recent simpler proof) that the free product of a family of unital ∗-homomorphisms that are liftable by u.c.p. maps is also liftable by a u.c.p. map. However, it is well known that the reduced C^*-algebra of $\mathbb{F}_n$ fails the LP.

Our main result is a very simple (functorial) characterization of the LP in terms of maximal tensor products (namely (1.4) below) that seems to have been overlooked by previous authors. In [18], Kirchberg gave a tensor product characterization of the local version of the LP called the LLP. He showed that a C^*-algebra A has the LLP if and only if $B(\ell_2) \otimes_{\max} A = B(\ell_2) \otimes_{\min} A$. He then went on to conjecture that for separable C^*-algebras the LLP implies the LP, and he showed that a negative solution would imply a negative answer for the Connes embedding problem (see Remark 1.17 for more information). We hope that our new criterion for the LP will help to answer this question whether the LLP implies the LP. More specifically, we believe that a modification of the construction in [28] might lead to a counterexample.

Let $(D_i)_{i \in I}$ be a family of C^*-algebras. We will denote by $\ell_\infty(\{D_i \mid \in I\})$ or simply by $\ell_\infty(\{D_i\})$ the C^*-algebra formed of the families $d = (d_i)_{i \in I}$ in $\prod_{i \in I} D_i$ such that $\sup_{i \in I} \|d_i\| < \infty$, equipped with the norm $d \mapsto \sup_{i \in I} \|d_i\|$. Consider the following property of a C^*-algebra A:

For any family $(D_i)_{i \in I}$ of C^*-algebras and any $t \in \ell_\infty(\{D_i\}) \otimes A$ we have $\qquad$ (1.1)

$$\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} A} \leq \sup_{i \in I} \|t_i\|_{D_i \otimes_{\max} A}, \qquad (1.2)$$

where $t_i = (p_i \otimes \mathrm{Id}_A)(t)$ with $p_i : \ell_\infty(\{D_i\}) \to D_i$ denoting the i-th coordinate projection.

Of course, for any $i \in I$ we have $\|p_i \otimes \mathrm{Id}_A : \ell_\infty(\{D_i\}) \otimes_{\max} A \to D_i \otimes_{\max} A\| \leq 1$, and hence we have a natural contractive $*$-homomorphism

$$\ell_\infty(\{D_i\}) \otimes_{\max} A \to \ell_\infty(\{D_i \otimes_{\max} A\}). \qquad (1.3)$$

Thus, (1.2) means that we have a natural isometric embedding

$$\ell_\infty(\{D_i\}) \otimes_{\max} A \subset \ell_\infty(\{D_i \otimes_{\max} A\}). \qquad (1.4)$$

More precisely $\ell_\infty(\{D_i\}) \otimes_{\max} A$ can be identified with the closure of $\ell_\infty(\{D_i\}) \otimes A$ (algebraic tensor product) in $\ell_\infty(\{D_i \otimes_{\max} A\})$.

Let us denote

$$\mathcal{C} = C^*(\mathbb{F}_\infty),$$

the full (or "maximal") C^*-algebra of the free group $\mathbb{F}_\infty$ with countably infinitely many generators.

Using the description of the norm in $D \otimes_{\max} \ell_1^n$ (for an arbitrary C^*-algebra D), with $\ell_1^n \subset \mathcal{C}$ identified as usual with the span of n free generators, it is easy to check that $\mathcal{C}$ has this property (1.1) (see Lemma 6.1 below). In fact, as a consequence, any unital separable A with LP has this property. Our main result is that conversely this characterizes the LP.

The fact that (1.1) implies the LP contains many previously known lifting theorems, for instance the Choi–Effros one. It also gives a new proof of the LP for $C^*(\mathbb{F}_\infty)$.

The key step will be a new form of the reflexivity principle. Consider for $E \subset A$ finite dimensional the normed space $MB(E, C)$ defined below in Section 2 as formed of the max $\to$ max-tensorizing maps into another C^*-algebra C. We will show that if (and only if) A has property (1.1) then the natural map

$$MB(E, C^{**}) \subset MB(E, C)^{**}$$

is contractive for any finite dimensional $E \subset A$. As a consequence it follows that any $u \in MB(E, C)$ admits an extension in $MB(A, C)$ with the same MB-norm up to $\varepsilon > 0$. From this extension property we deduce a lifting one: if A assumed unital and separable satisfies (1.1), any unital c.p. map $u : A \to C/\mathcal{I}$ ($C/\mathcal{I}$ any quotient) admits a unital c.p. lifting. In other words, this says that (1.1) implies the LP.

Remark 1.1. Let $(D_i)_{i \in I}$ be a family of C^*-algebras. Consider the algebraic tensor product $\ell_\infty(\{D_i\}) \otimes A$. We will use the natural embedding

$$\ell_\infty(\{D_i\}) \otimes A \subset \prod_{i \in I}(D_i \otimes A).$$

Any $t \in \ell_\infty(\{D_i\}) \otimes A$ can be identified with a bounded family (t_i) with $t_i \in D_i \otimes E$ for some f.d. subspace $E \subset A$. Thus, the correspondence $t \mapsto (t_i)$ gives us a canonical *linear* embedding

$$\ell_\infty(\{D_i\}) \otimes A \subset \ell_\infty(\{D_i \otimes_{\max} A\}). \tag{1.5}$$

The property in (1.1) is equivalent to the assertion that this map is an isometric embedding when $\ell_\infty(\{D_i\}) \otimes A$ is equipped with the maximal C^*-norm.

More precisely, for any f.d. subspace $E \subset A$ we have a canonical linear isomorphism

$$\ell_\infty(\{D_i\}) \otimes E \simeq \ell_\infty(\{D_i \otimes E\}).$$

By convention, for any C^*-algebra D let us denote by $D \otimes_{\max} E$ the normed space obtained by equipping $D \otimes E$ with the norm induced on it by $D \otimes_{\max} A$. Then the property in (1.1) is equivalent to the assertion that for any f.d. $E \subset A$ we have an isometric isomorphism

$$\ell_\infty(\{D_i\}) \otimes_{\max} E \simeq \ell_\infty(\{D_i \otimes_{\max} E\}). \tag{1.6}$$

Notation. Let D, A be C^*-algebras and let $E \subset A$ be a subspace. We will denote

$$(D \otimes E)^+_{\max} = (D \otimes_{\max} A)_+ \cap (D \otimes E).$$

Theorem 1.2. *Let A be a separable C^*-algebra. The following are equivalent:*

(i)   *The algebra A has the lifting property (LP).*

(ii) *For any family $(D_i)_{i \in I}$ of C^*-algebras and any $t \in \ell_\infty(\{D_i\}) \otimes A$ we have*

$$\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} A} \leq \sup_{i \in I} \|t_i\|_{D_i \otimes_{\max} A}.$$

In other words, the natural $*$-homomorphism (1.3) is isometric.

(ii)$_+$ *For any family $(D_i)_{i \in I}$ of C^*-algebras and any $t \in \ell_\infty(\{D_i\}) \otimes A$, the following implication holds:*

$$\forall i \in I \quad t_i \in (D_i \otimes A)^+_{\max} \Rightarrow t \in (\ell_\infty(\{D_i\}) \otimes A)^+_{\max}.$$

We prove Theorem 1.2 in Section 6.

Remark 1.3 (On the nuclear case). If one replaces everywhere max by min in (ii) in Theorem 1.2, then the property clearly holds for *all* C^*-algebras. Thus, Theorem 1.2 implies as a corollary the Choi–Effros lifting theorem from [9], which asserts that nuclear C^*-algebras have the LP.

Remark 1.4. It is easy to see that if A is the direct sum of finitely many C^*-algebras satisfying (1.1) then A also does.

Remark 1.5. Let A be a C^*-algebra. It is classical (see e.g. [29, Prop. 7.19]) that for any (self-adjoint, closed and two sided) ideal $\mathcal{I} \subset A$ and any C^*-algebra D we have an isometric embedding $D \otimes_{\max} \mathcal{I} \subset D \otimes_{\max} A$. This shows that if A satisfies (1.1) then any ideal $\mathcal{I}$ in A also does.

Similarly, for any ideal $\mathcal{I} \subset D$ we have an isometric embedding $\mathcal{I} \otimes_{\max} A \subset D \otimes_{\max} A$. Thus, if (1.1) holds for a given family (D_i) it also holds for any family $(\mathcal{I}_i)$ where each $\mathcal{I}_i$ is an ideal in D_i. Since any D is an ideal in its unitization, one deduces from this that if (1.1) holds for any family (D_i) of *unital C^*-algebras*, then it holds for any family. One also sees that A satisfies (1.1) if and only if its unitization satisfies it.

Remark 1.6. We will use the fact due to Kirchberg [18] that for any $t \in D_i \otimes A$ there is a separable C^*-subalgebra $\Delta_i \subset D_i$ such that $t \in \Delta_i \otimes A$ and

$$\|t\|_{\Delta_i \otimes_{\max} A} = \|t\|_{D_i \otimes_{\max} A}.$$

Indeed, by [29, Lem. 7.23] for any $\varepsilon > 0$ there is a separable $D_i^\varepsilon \subset D_i$ such that $t \in D_i^\varepsilon \otimes A$ and $\|t\|_{D_i^\varepsilon \otimes_{\max} A} \leq (1 + \varepsilon)\|t\|_{D_i \otimes_{\max} A}$. This implies that the C^*-algebra Δ_i generated by $\{D_i^\varepsilon \mid \varepsilon = 1/n, n \geq 1\}$ has the announced property. Using this fact in property (1.1) we may assume that all the D_i's are separable (and unital by the preceding remark).

Remark 1.7. Let $C/\mathcal{I}$ be a quotient C^*-algebra and let A be another C^*-algebra. It is well known (see e.g. [29, Prop. 7.15]) that

$$(C/\mathcal{I}) \otimes_{\max} A = (C \otimes_{\max} A)/(\mathcal{I} \otimes_{\max} A). \tag{1.7}$$

Equivalently, $A \otimes_{\max} (C/\mathcal{I}) = (A \otimes_{\max} C)/(A \otimes_{\max} \mathcal{I})$. Moreover, for any f.d. subspace $E \subset A$ we have (for a detailed proof see e.g. [29, Lem. 4.26])

$$(C/\mathcal{I}) \otimes_{\max} E = (C \otimes_{\max} E)/(\mathcal{I} \otimes_{\max} E). \tag{1.8}$$

Moreover, the closed unit ball of $(C \otimes_{\max} E)/(\mathcal{I} \otimes_{\max} E)$ coincides with the image under the quotient map of the closed unit ball of $C \otimes_{\max} E$. This known fact can be checked just like for the min-norm in [29, Lem. 7.44].

Since any separable unital C^*-algebra D can be viewed as a quotient of $\mathcal{C}$, so that, say $D = \mathcal{C}/\mathcal{I}$, we have an isomorphism

$$D \otimes_{\max} A = (\mathcal{C} \otimes_{\max} A)/(\mathcal{I} \otimes_{\max} A). \tag{1.9}$$

Using Remarks 1.6 and 1.7, one obtains:

Proposition 1.8. *To verify property* (1.1) *we may assume without loss of generality that* $D_i = \mathcal{C}$ *for any* $i \in I$.

Lemma 1.9. *Let $(D_i)_{i \in I}$ be a family of C^*-algebras. Then for any C^*-algebra C we have natural isometric embeddings*

$$\ell_\infty(\{D_i \otimes_{\max} C\}) \subset \ell_\infty(\{D_i^{**} \otimes_{\max} C\}), \qquad (1.10)$$

$$\ell_\infty(\{D_i\}) \otimes_{\max} C \subset \ell_\infty(\{D_i^{**}\}) \otimes_{\max} C. \qquad (1.11)$$

Proof. We will use the classical fact that (1.11) (or (1.10)) holds when $(D_i)_{i \in I}$ is reduced to a single element (see Remark 1.15 or [29, Prop. 7.26]), which means that we have for each $i \in I$ a natural isometric embedding $D_i \otimes_{\max} C \subset D_i^{**} \otimes_{\max} C$. From this, (1.10) is immediate.

To check (1.11), since any unital separable C is a quotient of $\mathcal{C}$, we may assume by Remark 1.7 that $C = \mathcal{C}$ (see [29, Thm. 7.29] for details). Now, since $\mathcal{C}$ satisfies (1.1) (or equivalently the LP) we have isometric embeddings

$$\ell_\infty(\{D_i\}) \otimes_{\max} \mathcal{C} \subset \ell_\infty(\{D_i \otimes_{\max} \mathcal{C}\})$$

and

$$\ell_\infty(\{D_i^{**}\}) \otimes_{\max} \mathcal{C} \subset \ell_\infty(\{D_i^{**} \otimes_{\max} \mathcal{C}\}).$$

Thus, (1.11) follows from (1.10) for $C = \mathcal{C}$, and hence in general. ∎

Proposition 1.10. *To verify property (1.1) we may assume without loss of generality that D_i is the bidual of a C^*-algebra for any $i \in I$. A fortiori we may assume that D_i is a von Neumann algebra for any $i \in I$.*

Proof. This is an immediate consequence of Lemma 1.9. ∎

In [19], where Kirchberg shows that the C^*-algebra $\mathcal{C} = C^*(\mathbb{F}_\infty)$ has the LP, he also states that if a C^*-algebra C has the LP then for any von Neumann algebra M the nor-norm coincides on $M \otimes C$ (or on $C \otimes M$) with the max-norm. We will show that the converse also holds. The nor-norm of an element $t \in M \otimes C$ (C any C^*-algebra) was defined by Effros and Lance [13] as

$$\|t\|_{\mathrm{nor}} = \sup\{\|\sigma \cdot \pi(t)\|\}$$

where the sup runs over all H and all commuting pairs of representations $\sigma : M \to B(H)$, $\pi : C \to B(H)$ with the restriction that σ is normal on M. Here, $\sigma \cdot \pi : M \otimes C \to B(H)$ is the $*$-homomorphism defined by $\sigma \cdot \pi(m \otimes c) = \sigma(m)\pi(c)$ ($m \in M, c \in C$). The norm $\|\ \|_{\mathrm{nor}}$ is a C^*-norm and $M \otimes_{\mathrm{nor}} C$ is defined as the completion of $M \otimes C$ relative to this norm. One can formulate a similar definition for $C \otimes_{\mathrm{nor}} M$. See [29, p. 162] for more on this. We will invoke the following elementary fact (we include its proof for lack of a reference).

Lemma 1.11. *Let D be another C^*-algebra. Then for any c.c.p. map $u : C \to D$ the mapping $\mathrm{Id}_M \otimes u : M \otimes C \to M \otimes D$ extends to a contractive (and c.p.) map from $M \otimes_{\mathrm{nor}} C$ to $M \otimes_{\mathrm{nor}} D$.*

Proof. Let $S_{\max}$ (resp. S_{nor}) denote the set of states on $M \otimes_{\max} C$ (resp. $M \otimes_{\text{nor}} C$). An element of $S_{\max}$ can be identified with a bilinear form $f : M \times C \to \mathbb{C}$ of norm ≤ 1 that is c.p. in the sense that $\sum f(x_{ij}, y_{ij}) \geq 0$ for all n and all $[x_{ij}] \in M_n(M)_+$ and $[y_{ij}] \in M_n(C)_+$. The set S_{nor} corresponds to the forms $f \in S_{\max}$ that are normal in the first variable. See [13] or [29, §4.5] for more information. Let $u : C \to D$ be a c.c.p. map. Clearly, for any $f \in S_{\text{nor}}$ the form $(x, y) \mapsto f(x, u(y))$ is still in S_{nor}. Since $\|t\|_{\text{nor}} \approx \sup_{f \in S_{\text{nor}}} |f(t)|$ for any $t \in M \otimes C$, the mapping $\text{Id}_M \otimes u$ (uniquely) extends to a bounded linear map $u_M : M \otimes_{\text{nor}} C \to M \otimes_{\text{nor}} D$. Since u is c.p. we have $u_M(t^*t) \in (M \otimes C)_+ := \text{span}\{s^*s \mid s \in M \otimes C\}$ for any $t \in M \otimes C$, and hence by density $u_M(t^*t) \in (M \otimes_{\text{nor}} C)_+$ for any $t \in M \otimes_{\text{nor}} C$. This means that u_M is positive. Replacing M by $M_n(M)$ shows that u_M is c.p. Since $\|t\|_{\text{nor}} = \sup_{f \in S_{\text{nor}}} |f(t)|$ when $t \geq 0$, we have $\|u_M\| \leq 1$. ∎

We will invoke the following simple elementary fact.

Lemma 1.12. *Let A be a C^*-algebra. For any family $(D_i)_{i \in I}$ of C^*-algebras we have a natural isometric embedding*

$$\ell_\infty(\{D_i^{**}\}) \otimes_{\text{nor}} A \subset \ell_\infty(\{D_i^{**} \otimes_{\text{nor}} A\}).$$

Proof. Let $D = c_0(\{D_i\})$ so that $D^{**} = \ell_\infty(\{D_i^{**}\})$. Consider $t \in D^{**} \otimes A$. Let $\pi : A \to B(H)$, $\sigma : D^{**} \to \pi(A)'$ be commuting non-degenerate $*$-homomorphisms with σ normal. Thus, σ is the canonical weak* to weak* continuous extension of $\sigma_{|D} : D \to \pi(A)'$. It will be notationally convenient to view D_i as a subalgebra of D. We observe that $\sigma_{|D}$ is the direct sum of representations of the D_i's. Let $\sigma_i : D_i \to \pi(A)'$ be defined by $\sigma_i(x) = \sigma(x)$ (recall $D_i \subset D$). There is an orthogonal decomposition of H of the form $\text{Id}_H = \sum_{i \in I} p_i$ with $p_i \in \sigma(D) \subset \pi(A)'$ such that for any $c = (c_i) \in D$ we have $\sigma(c) = \text{norm sense} \sum_{i \in I} \sigma_i(c_i)$ and also $\sigma_i(x) = p_i \sigma_i(x) = \sigma_i(x) p_i$ for any $x \in D_i$. Let $\ddot{\sigma}_i : D_i^{**} \to \pi(A)'$ be the weak* to weak* continuous extension of σ_i. Since σ is normal, we have then for any $c'' = (c_i'') \in \ell_\infty(\{D_i^{**}\})$

$$\sigma(c'') = \text{weak* sense} \sum_{i \in I} \ddot{\sigma}_i(c_i'').$$

This means $\sigma \simeq \oplus \ddot{\sigma}_i$. Therefore, for any $t \in D^{**} \otimes A = \ell_\infty(\{D_i^{**}\}) \otimes A$ we have

$$\|\sigma \cdot \pi(t)\| = \sup_{i \in I} \|\ddot{\sigma}_i \cdot \pi(t_i)\| \leq \sup_{i \in I} \|t_i\|_{D_i^{**} \otimes_{\text{nor}} A}.$$

Taking the sup over all the above specified pairs (σ, π), we obtain

$$\|t\|_{\text{nor}} \leq \sup_{i \in I} \|t_i\|_{D_i^{**} \otimes_{\text{nor}} A},$$

whence (since the converse is obvious) a natural isometric embedding

$$D^{**} \otimes_{\text{nor}} A \subset \ell_\infty(\{D_i^{**} \otimes_{\text{nor}} A\}).$$

This completes the proof. ∎

The next statement is now an easy consequence of Theorem 1.2.

Theorem 1.13. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *The algebra A has the lifting property (LP).*

(i)′ *The algebra A satisfies* (1.1).

(ii) *For any von Neumann algebra M we have*

$$M \otimes_{\mathrm{nor}} A = M \otimes_{\mathrm{max}} A \quad \textit{(or equivalently } A \otimes_{\mathrm{nor}} M = A \otimes_{\mathrm{max}} M \textit{).} \quad (1.12)$$

(ii)′ *For any C^*-algebra D we have*

$$D^{**} \otimes_{\mathrm{nor}} A = D^{**} \otimes_{\mathrm{max}} A \quad \textit{(or equivalently } A \otimes_{\mathrm{nor}} D^{**} = A \otimes_{\mathrm{max}} D^{**} \textit{).}$$

Proof. The equivalence (i) ⇔ (i)′ duplicates for convenience part of Theorem 1.2.

(i) ⇒ (ii) boils down to Kirchberg's result from [19] that $\mathcal{C}$ satisfies (ii), for which a simpler proof was given in [26] (see also [29, Thm. 9.10]). Once this is known, if A has the LP then A itself satisfies (ii). Indeed, we may assume $A = \mathcal{C}/\mathcal{I}$ and by Lemma 1.11 if $r : A \to \mathcal{C}$ is a c.c.p. lifting then $r_M : M \otimes_{\mathrm{nor}} A$ to $M \otimes_{\mathrm{nor}} \mathcal{C} = M \otimes_{\mathrm{max}} \mathcal{C}$ is contractive, from which (1.12) follows. A priori this uses the separability of A but we will give a direct proof of (i)′ ⇒ (ii) valid in the non-separable case in Section 8.

(ii) ⇒ (ii)′ is trivial. Assume (ii)′ (with A possibly non-separable). By Lemma 1.12 we have a natural isometric embedding

$$\ell_\infty(\{D_i^{**}\}) \otimes_{\mathrm{max}} A \subset \ell_\infty(\{D_i^{**} \otimes_{\mathrm{max}} A\}).$$

By (1.10) and (1.11) we must have (1.1), which means that (i)′ holds. Thus, (ii)′ ⇒ (i)′. ∎

We will prove a variant of the preceding theorem in terms of ultraproducts in Section 7.

Remark 1.14. It is known (see [29, Thm. 8.22]) that we always have an isometric natural embedding

$$D^{**} \otimes_{\mathrm{bin}} A^{**} \subset (D \otimes_{\mathrm{max}} A)^{**}.$$

Therefore, for arbitrary D and A, we have an isometric natural embedding

$$D^{**} \otimes_{\mathrm{nor}} A \subset (D \otimes_{\mathrm{max}} A)^{**}. \quad (1.13)$$

Thus, (ii)′ in Theorem 1.13 can be reformulated as saying that for any D we have an isometric natural embedding

$$D^{**} \otimes_{\mathrm{max}} A \subset (D \otimes_{\mathrm{max}} A)^{**}. \quad (1.14)$$

This is the analogue for the max-tensor product of Archbold and Batty's property C' from [4], which is closely related to the local reflexivity of [12] (see [27, p. 310] or [29, Rem. 8.34] for more information on this topic). However, what matters here is the injectivity of the ∗-homomorphism in (1.14). Its continuity is guaranteed by (1.13). In sharp contrast for property C' continuity is what matters while injectivity is automatic since the min-tensor product is injective.

Remark 1.15. Let A be a C^*-algebra. We already used in Lemma 1.9 the fact that for any other C^*-algebra D we have an *isometric* natural morphism

$$D \otimes_{\max} A \to D \otimes_{\max} A^{**}.$$

Similarly, for any von Neumann algebra M, the analogous morphism

$$M \otimes_{\text{nor}} A \to M \otimes_{\text{nor}} A^{**}$$

is *isometric*. This follows from the basic observation that if $\sigma : M \to B(H)$ and $\pi : A \to B(H)$ are representations with commuting ranges, then the canonical normal extension $\ddot{\pi}$ of π to A^{**} takes values in $\sigma(M)'$.

Remark 1.16 (On the LLP). Following Kirchberg [18], a unital C^*-algebra A is said to have the local lifting property (LLP) if for any unital c.p. map $u : A \to C/\mathcal{I}$ ($C/\mathcal{I}$ being any quotient of a unital C^*-algebra C) the map u is "locally liftable" in the following sense: for any f.d. operator system $E \subset A$ the restriction $u_{|E} : E \to C/\mathcal{I}$ admits a u.c.p. lifting $u^E : E \to C$. The *crucial difference* between "local and global" is that a priori $u_{|E}$ *does not* extend to a u.c.p. map on the whole of A. If A is not unital we say that it has the LLP if its unitization does. Equivalently, a general C^*-algebra A has the LLP if for any c.c.p. map $u : A \to C/\mathcal{I}$ ($C/\mathcal{I}$ being any quotient of a C^*-algebra C) and any f.d. subspace $E \subset A$ the restriction $u_{|E} : E \to C/\mathcal{I}$ admits a lifting $u^E : E \to C$ such that $\|u^E\|_{\text{cb}} \le 1$ (see [29, Thm. 9.38]).

Kirchberg [18] proved that A has the LLP if and only if $B(\ell_2) \otimes_{\min} A = B(\ell_2) \otimes_{\max} A$ or equivalently if and only if $\ell_\infty(\{M_n \mid n \ge 1\}) \otimes_{\min} A = \ell_\infty(\{M_n \mid n \ge 1\}) \otimes_{\max} A$. Using this it is easy to check, taking $\{D_i\} = \{M_n \mid n \ge 1\}$, that (1.1) implies the LLP.

Remark 1.17 (Global versus local). Clearly the LP implies the LLP. Kirchberg observed in [18] that if his conjecture[1] that $\mathcal{C} \otimes_{\min} A = \mathcal{C} \otimes_{\max} A$ for any A with LLP (or equivalently just for $A = \mathcal{C}$) is correct then conversely the LLP implies the LP for separable C^*-algebras.

Note that if $\mathcal{C} \otimes_{\min} \mathcal{C} = \mathcal{C} \otimes_{\max} \mathcal{C}$, then since $\mathcal{C}$ satisfies (1.1) we have $\ell_\infty(\mathcal{C}) \otimes_{\min} \mathcal{C} = \ell_\infty(\mathcal{C}) \otimes_{\max} \mathcal{C}$, and hence for any A with LLP also

$$\ell_\infty(\mathcal{C}) \otimes_{\min} A = \ell_\infty(\mathcal{C}) \otimes_{\max} A. \tag{1.15}$$

Now (1.15) obviously implies (recalling Proposition 1.8) the property in (1.1), which by Theorem 1.2 implies the LP in the separable case, whence another viewpoint on Kirchberg's observation.

Kirchberg showed in [18] that a C^*-algebra D has the weak expectation property (WEP) (for which we refer the reader to [29, p. 188]) if and only if $D \otimes_{\max} \mathcal{C} = D \otimes_{\min} \mathcal{C}$.

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[1]According to a recent paper entitled MIP* = RE posted on arXiv in January 2020 by Ji, Natarajan, Vidick, Wright, and Yuen this conjecture is not correct.

Thus, his conjecture is equivalent to the assertion that $\mathcal{C}$ has the WEP or that any C^*-algebra D is a quotient of a WEP C^*-algebra. Such a D is called QWEP. He also showed that it is equivalent to the Connes embedding problem (see [29, p. 291]).

Note that if all the D_i are WEP, the fact that $\mathcal{C}$ satisfies (1.1) tells us that

$$\ell_\infty(\{D_i\}) \otimes_{\max} \mathcal{C} \to \ell_\infty(\{D_i \otimes_{\min} \mathcal{C}\})$$

is isometric, and hence that $\ell_\infty(\{D_i\}) \otimes_{\max} \mathcal{C} = \ell_\infty(\{D_i\}) \otimes_{\min} \mathcal{C}$. Thus, it follows that $\ell_\infty(\{D_i\})$ is WEP.

Yet another way to look at the problem whether the LLP implies the LP for separable C^*-algebras is through the observation (pointed out by a referee) that A has the LLP if and only if $\ell_\infty(\{D_i\}) \otimes_{\max} A \to \ell_\infty(\{D_i \otimes_{\max} A\})$ is isometric (or equivalently injective) for any family (D_i) of QWEP C^*-algebras. This is easily checked using (1.9). If Kirchberg's conjecture was correct, the LLP of A would imply the same for arbitrary D_i and hence by Theorem 1.2 this would imply that A has the LP in the separable case.

Remark 1.18 (Counterexamples to LP). For the full C^*-algebra $C^*(G)$ of a discrete group G, the LP holds both when G is amenable and when G is a free group. It is thus not easy to find counterexamples, but the existence of $C^*(G)$'s failing LP has been proved using property (T) groups by Ozawa in [24]. Later on, Thom [32] gave an explicit example of G for which $C^*(G)$ fails the LLP. More recently, Ioana, Spaas and Wiersma [16] proved that $SL_n(\mathbb{Z})$ for $n \geq 3$ and many other similar property (T) groups fail the LLP. See Section 10 for more on this theme.

Abbreviations and notation. As is customary we abbreviate completely positive by c.p., completely bounded by c.b., unital completely positive by u.c.p. and contractive completely positive by c.c.p. Analogously we will abbreviate maximally bounded, maximally positive and unital maximally positive respectively by m.b., m.p. and u.m.p. We also use f.d. for finite dimensional. We denote by B_E the closed unit ball of a normed space E, and by $\mathrm{Id}_E : E \to E$ the identity operator on E. We denote by $E \otimes F$ the *algebraic* tensor product of two vector spaces. Lastly, by an ideal in a C^*-algebra we implicitly mean a two sided, closed and self-adjoint ideal.

2. Maximally bounded and maximally positive maps

Let $E \subset A$ be an operator space sitting in a C^*-algebra A. Let D be another C^*-algebra. Recall we denote (abusively) by $D \otimes_{\max} E$ the closure of $D \otimes E$ in $D \otimes_{\max} A$, and we denote by $\| \ \|_{\max}$ the norm induced on $D \otimes_{\max} E$ by $D \otimes_{\max} A$. We define similarly $E \otimes_{\max} D$. We should emphasize that $D \otimes_{\max} E$ (or $E \otimes_{\max} D$) depends on A and on the embedding $E \subset A$, but there will be no risk of confusion. Of course we could also define $E \otimes_{\max} F \subset A \otimes_{\max} D$ for a subspace $F \subset D$ but we will not go that far. Let C be another C^*-algebra.

We will denote by $MB(E, C)$ the set of maps $u : E \to C$ such that for any C^*-algebra D, the map $\mathrm{Id}_D \otimes u : D \otimes E \to D \otimes C$ extends to a bounded mapping $u_D : D \otimes_{\max} E \to D \otimes_{\max} C$, and moreover such that the bound on $\|u_D\|$ is uniform over all D's. We call such maps "maximally bounded" or "max-bounded" or simply m.b. for short. (They are called max $\to$ max-tensorizing in [29].) We denote

$$\|u\|_{\mathrm{mb}} = \sup\{\|u_D : D \otimes_{\max} E \to D \otimes_{\max} C\|\},$$

where the sup runs over all possible D's. We have clearly $\|u_D\|_{\mathrm{mb}} \le \|u\|_{\mathrm{mb}}$.

Moreover, if A is unital, assuming that E is an operator system let us denote by $MP(E, C)$ the subset formed of the maps u such that $\mathrm{Id}_D \otimes u : D \otimes E \to D \otimes C$ is positive for any D, by which we mean that

$$[\mathrm{Id}_D \otimes u]((D \otimes E)_{\max}^+) \subset (D \otimes C)_{\max}^+.$$

Equivalently, u_D is positive for all D's. We call such maps "maximally positive" (m.p. in short). Replacing D by $M_n(D)$ shows that the latter maps will be automatically c.p.

In passing, recall that for any c.p. $u : E \to C$ we have $\|u\|_{\mathrm{cb}} = \|u\| = \|u(1)\|$, and hence $\|u\|_{\mathrm{mb}} = \|u(1)\|$ for any m.p. u. Obviously, we have $\|u\|_{\mathrm{mb}} = 1$ if $E = A$ and u is a $*$-homomorphism.

Clearly (taking $D = M_n$), we have $MB(E, C) \subset CB(E, C)$ and

$$\forall u \in MB(E, C) \quad \|u\|_{\mathrm{cb}} \le \|u\|_{\mathrm{mb}}. \tag{2.1}$$

Proposition 2.1. *A map $u : E \to C$ belongs to $MB(E, C)$ if and only if $\mathrm{Id}_{\mathcal{C}} \otimes u$ defines a bounded map from $\mathcal{C} \otimes_{\max} E$ to $\mathcal{C} \otimes_{\max} C$ and we have*

$$\|u\|_{\mathrm{mb}} = \|\mathrm{Id}_{\mathcal{C}} \otimes u : \mathcal{C} \otimes_{\max} E \to \mathcal{C} \otimes_{\max} C\|.$$

Proof. This follows easily from (1.9). ∎

Corollary 2.2. *If C has the WEP then any c.b. map $u : E \to C$ is in $MB(E, C)$ and $\|u\|_{\mathrm{cb}} = \|u\|_{\mathrm{mb}}$.*

Proof. By the WEP of C, we have $\mathcal{C} \otimes_{\max} C = \mathcal{C} \otimes_{\min} C$ (see [8, p. 380] or [29, p. 188]), and hence

$$\begin{aligned}
\|\mathrm{Id}_{\mathcal{C}} \otimes u : \mathcal{C} \otimes_{\max} E \to \mathcal{C} \otimes_{\max} C\| &= \|\mathrm{Id}_{\mathcal{C}} \otimes u : \mathcal{C} \otimes_{\max} E \to \mathcal{C} \otimes_{\min} C\| \\
&\le \|\mathrm{Id}_{\mathcal{C}} \otimes u : \mathcal{C} \otimes_{\min} E \to \mathcal{C} \otimes_{\min} C\| \\
&\le \|u\|_{\mathrm{cb}}. \quad \blacksquare
\end{aligned}$$

Remark 2.3. Let $u \in MB(E, C)$. Let $F \subset C$ be such that $u(E) \subset F$ and let B be another C^*-algebra. Then for any $v \in MB(F, B)$ the composition $vu : E \to B$ is in $MB(E, B)$ and $\|vu\|_{\mathrm{mb}} \le \|v\|_{\mathrm{mb}}\|u\|_{\mathrm{mb}}$.

Remark 2.4. When $E = A$ any c.p. map $u : A \rightarrow C$ is in $MB(A, C)$ and $\|u\|_{\mathrm{mb}} = \|u\|$ (see e.g. [27, p. 229] or [29, Cor. 7.8]). However, this is no longer true in general when E is merely an operator system in A. *This distinction is important for the present work.*

Remark 2.5. We will use the following well-known elementary fact. Let A, C be unital C^*-algebras. Let $E \subset A$ be an operator system. Let $u : E \rightarrow C$ be a unital c.b. map. Then u is c.p. if and only if $\|u\|_{\mathrm{cb}} = 1$ (see e.g. [25], [27, p. 24] or [29, Thm. 1.35]).

Remark 2.6. In the same situation, any *unital* map $u \in MB(E, C)$ such that $\|u\|_{\mathrm{mb}} = 1$ must be c.p. Indeed, (2.1) implies $\|u\|_{\mathrm{cb}} = \|u(1)\| = 1$, and u is c.p. by the preceding remark. More precisely, the same reasoning applied to the maps u_D (with D unital) shows that u is m.p.

Remark 2.7. Let $u : E \rightarrow C$ with E an operator system. Let $u_* : E \rightarrow C$ be defined by $u_*(x) = (u(x^*))^*$ ($x \in E$), so that $\|u_*\| = \|u\|$. We claim $\|u_*\|_{\mathrm{mb}} = \|u\|_{\mathrm{mb}}$ for any $u \in MB(E, C)$. Indeed, this is easy to check using $(u_D)_* = (u_*)_D$. A map $u : E \rightarrow C$ is called "self-adjoint" if $u = u_*$.

The following important result was shown to the author by Kirchberg with permission to include it in [27]. It also appears in [29, Thm. 7.6].

Theorem 2.8 ([20]). *Let A, C be C^*-algebras. Let $i_C : C \rightarrow C^{**}$ denote the inclusion map. A map $u : A \rightarrow C$ is in $MB(A, C)$ if and only if $i_C u : A \rightarrow C^{**}$ is decomposable. Moreover, we have*

$$\|u\|_{\mathrm{mb}} = \|i_C u\|_{\mathrm{dec}}.$$

Recall that a map $u : A \rightarrow C$ is called decomposable if it is a linear combination of c.p. maps. See [14] or [29, §6] for more on decomposable maps and the definition of the dec-norm.

Elaborating on Kirchberg's argument, we included in [27] and later again in [29, Thm. 7.4] the following variant as an extension theorem.

Theorem 2.9. *Let A, C be C^*-algebras. Let $E \subset A$ be a subspace. Consider a map $u : E \rightarrow C$ in $MB(E, C)$. Then*

$$\|u\|_{\mathrm{mb}} = \inf\|\tilde{u}\|_{\mathrm{dec}},$$

where the infimum runs over all maps $\tilde{u} : A \rightarrow C^{**}$ such that $\tilde{u}_{|E} = i_C u$. Moreover, the latter infimum is attained.

Remark 2.10. A fortiori, since $\|i_C u\|_{\mathrm{dec}} \leq \|u\|_{\mathrm{dec}}$, we have $\|u\|_{\mathrm{mb}} \leq \|u\|_{\mathrm{dec}}$. Moreover, for any $\tilde{u} : A \rightarrow C^{**}$ we have

$$\|\tilde{u}\|_{\mathrm{mb}} = \|\tilde{u}\|_{\mathrm{dec}}. \tag{2.2}$$

Indeed, since there is a c.c.p. projection $P : (C^{**})^{**} \rightarrow C^{**}$, we have

$$\|\tilde{u}\|_{\mathrm{mb}} = \|i_{C^{**}}\tilde{u}\|_{\mathrm{dec}} \geq \|P i_{C^{**}}\tilde{u}\|_{\mathrm{dec}} = \|\tilde{u}\|_{\mathrm{dec}}.$$

Remark 2.11. For any C^*-algebras C, D the natural inclusion $D \otimes_{\max} C \subset D \otimes_{\max} C^{**}$ is isometric (see Remark 1.15). Therefore, for any $u : E \to C$ we have

$$\|u\|_{\mathrm{mb}} = \|i_C u\|_{\mathrm{mb}},$$

where $i_C : C \to C^{**}$ is the canonical inclusion. Moreover, if there is a projection $P : C^{**} \to C$ with $\|P\|_{\mathrm{dec}} = 1$ (for instance if P is c.c.p.) then $\|u\|_{\mathrm{dec}} = \|i_C u\|_{\mathrm{dec}}$.

Remark 2.12 (A refinement of Theorem 2.9). Let $E \subset A$ and let $M \subset B(H)$ be a von Neumann algebra. Consider a map $u : E \to M$. Let us denote by $\hat{u} : M' \otimes E \to B(H)$ the linear map such that $\hat{u}(x' \otimes a) = x' u(a)$ $(x' \in M', a \in E)$. Then

$$\|u\|_{\mathrm{mb}} = \|\hat{u} : M' \otimes_{\max} E \to B(H)\|_{\mathrm{cb}} = \inf\{\|\tilde{u}\|_{\mathrm{dec}} \mid \tilde{u} : A \to M, \tilde{u}_{|E} = u\}.$$

If either M has infinite multiplicity or M' has a cyclic vector, then

$$\|\hat{u} : M' \otimes_{\max} E \to B(H)\|_{\mathrm{cb}} = \|\hat{u} : M' \otimes_{\max} E \to B(H)\|.$$

This is as in [29, Thm. 6.20 with Cor. 6.21 and Cor. 6.23].

Remark 2.13. In the situation of the preceding remark, if E is an operator system and if $\hat{u} : M' \otimes_{\max} E \to B(H)$ is c.p. (in particular if $u \in MP(E, M)$) then u admits a c.p. extension $\tilde{u} : A \to M$ with $\|\tilde{u}\| = \|u\|$ (in particular $\tilde{u} \in MP(A, M)$). This follows from Arveson's extension theorem for c.p. maps and the argument called "the trick" in [8, p. 87].

Let $E \subset A$ and C be as in Theorem 2.9. To state our results in full generality, we introduce the space $SB(E, C)$ for which the unit ball is formed of all $u : E \to C$ satisfying an operator analogue of (1.1).

Definition 2.14. We call strongly maximally bounded the maps $u : E \to C$ such that, for any family $(D_i)_{i \in I}$, the map $\mathrm{Id} \otimes u$ defines a uniformly bounded map from $\ell_\infty(\{D_i\}) \otimes E$ equipped with the norm induced by $\ell_\infty(\{D_i \otimes_{\max} E\})$ to $\ell_\infty(\{D_i\}) \otimes C$ equipped with the norm induced by $\ell_\infty(\{D_i\}) \otimes_{\max} C$. We define

$$\|u\|_{\mathrm{sb}} = \sup\{\|(\mathrm{Id} \otimes u)(t)\|_{\ell_\infty(\{D_i\}) \otimes_{\max} C} \mid (t_i)_{i \in I} \in \ell_\infty(\{D_i\}) \otimes E,$$
$$\sup_{i \in I} \|t_i\|_{D_i \otimes_{\max} C} \leq 1\}, \qquad (2.3)$$

where the sup runs over all possible families $(D_i)_{i \in I}$ of C^*-algebras.

Definition 2.15. Let A, C be C^*-algebras. Let $E \subset A$ be a subspace. A linear map $u : E \to C$ will be called maximally isometric (or max-isometric in short) if for any C^*-algebra D the associated map $\mathrm{Id}_D \otimes u : D \otimes E \to D \otimes C$ extends to an isometric map $D \otimes_{\max} E \to D \otimes_{\max} C$. Equivalently, this means that $E \otimes_{\max} D \to C \otimes_{\max} D$ is isometric for all C^*-algebras D.

These maps were called max-injective in [29]. We adopt here "maximally isometric" to emphasize the analogy with completely isometric.

Remark 2.16. For example, for any C^*-algebra C the canonical inclusion $i_C : C \to C^{**}$ is max-isometric (see Remark 1.15 or [29, Cor. 7.27]). Thus, for any D we have an isometric $*$-homomorphism $D \otimes_{\max} C \to D \otimes_{\max} C^{**}$, which implies

$$(D \otimes C) \cap (D \otimes_{\max} C^{**})_+ \subset (D \otimes_{\max} C)_+. \tag{2.4}$$

Moreover, the inclusion $\mathcal{I} \to C$ of an ideal, in particular the inclusion $C \to C_1$ of C into its unitization, is max-isometric (see e.g. [29, Prop. 7.19]).

Remark 2.17. We have $\|\mathrm{Id}_A\|_{SB(A,A)} = 1$ if (and only if) A satisfies property (1.1). In that case, by Remark 2.6, for any $u \in CP(A, C)$ (where C is an arbitrary C^*-algebra) we have

$$\|u\|_{SB(A,C)} = \|u\|. \tag{2.5}$$

Remark 2.18. We will use the elementary fact (see [29, Cor. 7.16] for a proof) that $D \otimes_{\max} E \to D \otimes_{\max} C$ is isometric for all D (i.e. it is max-isometric) if and only if this holds for $D = \mathcal{C}$. In particular, checking D separable is enough.

As we will soon show (see Theorem 5.3), maximally bounded maps admit maximally bounded liftings when (1.1) holds. To tackle u.c.p. liftings we will need a bit more work. The following perturbation lemma is the m.p. analogue of [12, Thm. 2.5].

Lemma 2.19. *Let $E \subset A$ be an n-dimensional operator system, C a unital C^*-algebra. Let $0 < \varepsilon < 1/2n$. For any self-adjoint unital map $u : E \to C$ with $\|u\|_{\mathrm{mb}} \leq 1 + \varepsilon$, there is a unital maximally positive (u.m.p. in short) map $v : E \to C$ such that $\|u - v\|_{\mathrm{mb}} \leq 8n\varepsilon$. Here, u self-adjoint means that $u(x) = u(x^*)^*$ for any $x \in E$.*

Sketch. The proof is essentially the same as that of the corresponding statement for the c.b. norm and c.p. maps appearing in [12, Thm. 2.5] and reproduced in [29, Thm. 2.28]. However, we have to use the "m.p. order" instead of the c.p. one. The only notable difference is that one should use Theorem 2.9 instead of the injectivity of $B(H)$. Using the latter we can find a self-adjoint extension $\widetilde{u} : A \to C^{**}$ with $\|\widetilde{u}\|_{\mathrm{dec}} \leq 1 + \varepsilon$. We can write $\widetilde{u} = v_1 - v_2$ with v_1 and v_2 in $MP(A, C^{**})$ such that $\|v_1 + v_2\| \leq 1 + \varepsilon$. Since $u(1) = 1$ we have $\|1 + 2v_2(1)\| = \|(v_1 + v_2)(1)\| \leq 1 + \varepsilon$, and hence $\|v_2(1)\| \leq \varepsilon/2$. Arguing as in [12, Thm. 2.5] or [29, Thm. 2.28] there is $f \in E^*$ with $\|f\| \leq 2n\varepsilon$ such that the mapping $w_2 : x \mapsto f(x)1 - v_2(x)$ is in $MP(E, C^{**})$. We then set $v_0 = v_{1|E} + w_2 = u + f(\cdot)1$. Note that $v_0(E) \subset C$. A priori $v_0 \in MP(E, C^{**})$, but we actually have $v_0 \in MP(E, C)$ by (2.4). Moreover, $\|v_0 - u\|_{\mathrm{mb}} \leq 2n\varepsilon$. In particular, $\|v_0(1) - 1\| \leq 2n\varepsilon < 1$, which shows that $v_0(1)$ is invertible and close to 1 when ε is small. The rest of the proof is as in [12, Thm. 2.5] or [29, Thm. 2.28]. ∎

The following variant is immediate:

Lemma 2.20. *Let $E \subset A$ be an n-dimensional operator system, C a unital C^*-algebra. For any self-adjoint map $u : E \to C$ such that $\|u(1) - 1\| < \varepsilon$ and $\|u\|_{\mathrm{mb}} \leq 1 + \varepsilon$, there*

is a unital map $v \in MP(E, C)$ such that $\|u - v\|_{\mathrm{mb}} \le f_E(\varepsilon)$, where f_E is a function of $\varepsilon \in (0, 1)$ such that $\lim_{\varepsilon \to 0} f_E(\varepsilon) = 0$.

Sketch. Let $u'(\cdot) = u(1)^{-1/2} u(\cdot) u(1)^{-1/2}$. Then $\|u - u'\| \le f'(\varepsilon)$ with $\lim_{\varepsilon \to 0} f'(\varepsilon) = 0$. We may apply Lemma 2.19 to u'. ∎

3. Arveson's principle

To tackle global lifting problems a principle due to Arveson has proved very useful. It asserts roughly that in the separable case pointwise limits of suitably liftable maps are liftable. Its general form can be stated as follows. Let E be a separable operator space, C a unital C^*-algebra, $\mathcal{I} \subset C$ an ideal.

A bounded subset of $\mathcal{F} \subset B(E, C)$ will be called admissible if for any pair f, g in $\mathcal{F}$ and any $\sigma \in C_+$ with $\|\sigma\| \le 1$ the mapping

$$x \mapsto \sigma^{1/2} f(x) \sigma^{1/2} + (1 - \sigma)^{1/2} g(x)(1 - \sigma)^{1/2}$$

belongs to $\mathcal{F}$. This implies that $\mathcal{F}$ is convex.

Let $q : C \to C/\mathcal{I}$ denote the quotient map and let

$$q(\mathcal{F}) = \{qf \mid f \in \mathcal{F}\}.$$

Then Arveson's principle (see [5, p. 351]) can be stated like this:

Theorem 3.1 (Arveson's principle). *Assume E separable and $\mathcal{F}$ admissible. For the topology of pointwise convergence on E we have*

$$\overline{q(\mathcal{F})} = q(\overline{\mathcal{F}}).$$

Actually, we do not even need to assume $\mathcal{F}$ bounded if we restrict to the pointwise convergence on a countable subset of E.

One can verify this assertion by examining the presentations [10, p. 266] or [27, p. 46 and p. 425] or [29, Thm. 9.46].

The classical admissible classes are contractions, complete contractions and, when E is an operator system, positive contractions or completely positive (c.p. in short) contractions. In the unital case, unital positive or unital completely positive (u.c.p. in short) maps form admissible classes. Let $f, g \in B_{MB(E,C)}$. Then it is easy to see on one hand that the map $x \mapsto (f(x), g(x))$ is in $B_{MB(E,C \oplus C)}$. On the other hand, the map $\psi : C \oplus C \to C$ defined by $v(a, b) = \sigma^{1/2} a \sigma^{1/2} + (1 - \sigma)^{1/2} b(1 - \sigma)^{1/2}$ is unital c.p. and hence in $B_{MB(C \oplus C, C)}$ (see Remark 2.4). This shows (with Remark 2.3) that the unit ball of $MB(E, C)$ is admissible.

In the rest of this section, we record a few elementary facts.

Lemma 3.2. *Let A be a unital C^*-algebra. Let $E \subset A$ be an operator system. Consider a unital map $u : E \to C/\mathcal{I}$ ($C/\mathcal{I}$ any quotient C^*-algebra). Let $v \in MB(E, C)$ (resp. $v \in SB(E, C)$, resp. $v \in CB(E, C)$) be a lifting of u. Then for any $\varepsilon > 0$ there is a unital lifting $v' \in MB(E, C)$ (resp. $v' \in SB(E, C)$, resp. $v' \in CB(E, C)$) with $\|v'\|_{\mathrm{mb}} \le (1 + \varepsilon)\|v\|_{\mathrm{mb}}$ (resp. $\|v'\|_{\mathrm{sb}} \le (1 + \varepsilon)\|v\|_{\mathrm{sb}}$, resp. $\|v'\|_{\mathrm{cb}} \le (1 + \varepsilon)\|v\|_{\mathrm{cb}}$). If v is merely assumed bounded, we also have $\|v'\| \le (1 + \varepsilon)\|v\|$.*

Proof. Let (σ_α) be a quasicentral approximate unit in $\mathcal{I}$ in the sense of [5] (see also e.g. [10] or [29, p. 454]). Let $T_\alpha : C \oplus \mathbb{C} \to C$ be the unital c.p. map defined by

$$T_\alpha(x, c) = \left((1 - \sigma_\alpha)^{1/2} \sigma_\alpha^{1/2}\right) \begin{pmatrix} x & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} (1 - \sigma_\alpha)^{1/2} \\ \sigma_\alpha^{1/2} \end{pmatrix}$$

$$= (1 - \sigma_\alpha)^{1/2} x (1 - \sigma_\alpha)^{1/2} + c\sigma_\alpha.$$

Since $v(1) - 1 \in \mathcal{I}$ we have

$$\|(1 - \sigma_\alpha)^{1/2}[v(1) - 1](1 - \sigma_\alpha)^{1/2}\| \to 0. \tag{3.1}$$

Let $f : E \to \mathbb{C}$ be a state, i.e. a positive linear form such that $f(1) = 1$. Note that $\|f\|_{\mathrm{sb}} = \|f\|_{\mathrm{mb}} = \|f\| = 1$. Also $\|v\| \ge \|u\| \ge 1$. The map $\psi : E \to C \oplus \mathbb{C}$ defined by $\psi(x) = (v(x), f(x))$ clearly satisfies $\|\psi\|_{\mathrm{mb}} \le \|v\|_{\mathrm{mb}}$. Let $T'_\alpha(x) = T_\alpha \psi(x)$. Then $\|T'_\alpha\|_{\mathrm{mb}} \le \|v\|_{\mathrm{mb}}$ (see Remarks 2.3 and 2.4), $T'_\alpha : E \to C$ lifts u and by (3.1) we have $\|T'_\alpha(1) - 1\| \to 0$, so that going far enough in the net we can ensure that $T'_\alpha(1)$ is invertible in C. We then define $v' : E \to C$ by $v'(x) = T'_\alpha(1)^{-1} T'_\alpha(x)$ (we could use $x \mapsto T'_\alpha(1)^{-1/2} T'_\alpha(x) T'_\alpha(1)^{-1/2}$). Now v' is unital and

$$\|v'\|_{\mathrm{mb}} \le \|T'_\alpha(1)^{-1}\| \|T'_\alpha\|_{\mathrm{mb}} \le \|T'_\alpha(1)^{-1}\| \|v\|_{\mathrm{mb}}.$$

Choosing α "large" enough so that $\|T'_\alpha(1)^{-1}\| < 1 + \varepsilon$, we obtain the desired bound for the m.b.-case. The other cases are identical. ∎

Let I be a directed set. Assume $x_i \in B(H)$ for all $i \in I$ and $x \in B(H)$. Recall that (by definition) x_i tends to x for the strong* operator topology (in short $x = \text{sot*-}\lim x_i$) if $x_i h \to xh$ and $x_i^* h \to x^* h$ for any $h \in H$.

Let $M \subset B(H)$ be a von Neumann subalgebra of $B(H)$. Assume that there is a C^*-subalgebra $C \subset M$ such that $C'' = M$. It is well known that the unit ball of M is the closure of that of C for the strong* topology (see [31, Thm. 4.8. p. 82]). This implies the following well-known and elementary fact representing M as a quotient of a natural C^*-subalgebra of $\ell_\infty(\{C\})$.

Lemma 3.3. *In the preceding situation, for a suitable directed index set I, with which we set $C_i = C$ for all $i \in I$, there is a C^*-subalgebra $L \subset \ell_\infty(\{C_i \mid i \in I\})$ and a surjective $*$-homomorphism $Q : L \to M$ such that for any $(x_i) \in L$ we have $Q((x_i)) = \text{sot*-}\lim x_i$. If C is unital, we can get L and Q unital as well.*

Proof. Choose a directed set I such that for any $x \in B_M$ there is $(x_i)_{i \in I} \in \prod_{i \in I} B_{C_i}$ (recall $C_i = C$ for all $i \in I$) such that $x = \text{sot*-}\lim x_i$. For instance, the set of neighborhoods of 0 for the sot*-topology can play this role. Let $L \subset \ell_\infty(\{C_i\})$ denote the unital C^*-subalgebra formed of the elements $(x_i)_{i \in I}$ in L such that sot*-$\lim x_i$ exists. We then set $Q(x) = \text{sot*-}\lim x_i$ for any $x = (x_i) \in L$. The last assertion is obvious. ∎

Let $X \subset A$ be a separable subspace of a C^*-algebra A. For any C^*-algebra C we give ourselves an admissible class of mappings $\mathcal{F}(X, C)$ that we always assume closed for pointwise convergence. We will say that $X \subset A$ has the $\mathcal{F}$-lifting property if for any $u \in \mathcal{F}(X, C/\mathcal{I})$ there is a lifting $\hat{u}$ in $\mathcal{F}(X, C)$.

Remark 3.4. In the sequel, we will assume that $\mathcal{F}$ is admissible and in addition that for any $u \in \mathcal{F}(X, C)$ and any *-homomorphism $\pi : C \to D$ between C^*-algebras $\pi \circ u \in \mathcal{F}(X, D)$. When $\mathcal{F}$ is formed of unital maps (X is then an operator system and C a unital C^*-algebra), we assume moreover that $\pi \circ u \in \mathcal{F}(X, D)$ for any u.c.p. map $\pi : C \to D$.

Remark 3.5 (Examples). Examples of such classes are those formed of maps that are contractive, positive and contractive, n-contractive, n-positive and n-contractive, completely contractive or c.c.p. When dealing with positive or c.p. maps, we assume that X is an operator system. Then we may intersect the latter classes with that of unital ones.

To tackle liftings within biduals we will use the following classical fact (for which a proof can be found e.g. in [29, p. 465]): for any C^*-algebra C and any ideal $\mathcal{I} \subset C$ with which we can form the quotient C^*-algebra $C/\mathcal{I}$, we have a canonical isomorphism:

$$C^{**} \simeq (C/\mathcal{I})^{**} \oplus \mathcal{I}^{**} \tag{3.2}$$

and hence an isomorphism

$$(C/\mathcal{I})^{**} \simeq C^{**}/\mathcal{I}^{**}.$$

The next two statements (which follow easily from Arveson's principle) show that the global (resp. local) lifting property is equivalent to some sort of "global (resp. local) reflexivity principle".
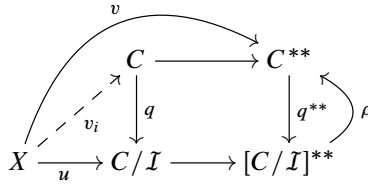
Theorem 3.6. *Let $X \subset A$ and $\mathcal{F}$ be as in Remark 3.4. Then the following are equivalent:*

(i) *The space X has the $\mathcal{F}$-LP.*

(ii) *For any von Neumann algebra M, any $C \subset M$ such that $C'' = M$ and any $u \in \mathcal{F}(X, M)$ there is a net of maps $u_i \in \mathcal{F}(X, C)$ tending pointwise sot* to u.*

(iii) *For any C and any $u \in \mathcal{F}(X, C^{**})$ there is a net of maps $u_i \in \mathcal{F}(X, C)$ tending pointwise to u for the weak* (i.e. $\sigma(C^{**}, C^*)$) topology.*

Proof. Assume (i). We apply Lemma 3.3. We have a C^*-subalgebra $L \subset \ell_\infty(\{C_i\})$ and a surjective *-homomorphism $Q : L \to M$ such that for any $(x_i) \in L$ we have $Q((x_i)) = \text{sot*-}\lim x_i$ (and a fortiori $Q((x_i)) = \text{weak*-}\lim x_i$). Let $u \in \mathcal{F}(X, M)$ and let $\hat{u} \in \mathcal{F}(X, L)$ be a lifting for Q. Then $(u_i)_{i \in I}$ such that $\hat{u} = (u_i)_{i \in I}$ gives us (ii).

(ii) $\Rightarrow$ (iii) is obvious.

Assume (iii). Let $u \in \mathcal{F}(X, C/\mathcal{I})$. We will use (3.2). The proof can be read on the following diagram:

$$
\begin{array}{ccc}
 & C & \longrightarrow & C^{**} \\
 & \nearrow \ \downarrow q & & \downarrow q^{**} \quad \rho \\
X & \xrightarrow{\ u\ } C/\mathcal{I} & \longrightarrow & [C/\mathcal{I}]^{**}
\end{array}
$$

By (3.2) we have a lifting $\rho : (C/\mathcal{I})^{**} \simeq C^{**}/\mathcal{I}^{**} \to C^{**}$ which is a $*$-homomorphism. Let f be a state on $(C/\mathcal{I})^{**}$ and let p denote the unit in $\mathcal{I}^{**} \subset C^{**}$. Then $\rho_1 : C^{**}/\mathcal{I}^{**} \to C^{**}$ defined by $\rho_1 = \rho(\cdot) + f(\cdot)p$ is a u.c.p. lifting (replacing ρ by ρ_1 is needed only when $\mathcal{F}$ is formed of unital maps). Therefore, recalling Remark 3.4, we can find a map $v \in \mathcal{F}(X, C^{**})$ such that $q^{**}v = i_{C/\mathcal{I}}u$ (where as usual $i_D : D \to D^{**}$ the canonical inclusion). By (iii) there is a net (v_i) in $\mathcal{F}(X, C)$ tending weak* to v. Then $qv_i = q^{**}v_i$ tends pointwise weak* to $q^{**}v = u$. This means $qv_i \to u$ pointwise for the weak topology of $C/\mathcal{I}$. We can then invoke Mazur's classical theorem to obtain (after passing to suitable convex combinations) a net such that $qv_i \to u$ pointwise in norm (see e.g. [29, Rem. A.10] for details). By Arveson's principle, u admits a lifting in $\mathcal{F}$, so we obtain (i). ∎

Let $X \subset A$ be as above. In addition for any C and any f.d. subspace $E_0 \subset X$ we assume given another f.d. $E \supset E_0$ and a class $\mathcal{F}(E, C)$ satisfying the same two assumptions as $\mathcal{F}(X, C)$. We also assume that $u \in \mathcal{F}(X, C)$ implies $u_{|E} \in \mathcal{F}(E, C)$. We will say that $u : X \to C/\mathcal{I}$ is locally $\mathcal{F}$-liftable if for any f.d. $E_0 \subset X$ there is a f.d. $E \supset E_0$ and a map $u^E \in \mathcal{F}(E, C)$ lifting the restriction $u_{|E}$. We will say that X has the $\mathcal{F}$-LLP if for any C, any $u \in \mathcal{F}(X, C/\mathcal{I})$ is locally $\mathcal{F}$-liftable. The introduction of E in place of E_0 is a convenient way to include e.g. the class $\mathcal{F}$ formed of unital c.p. maps on f.d. operator systems. In the latter case, E can be any f.d. operator system containing E_0.

Theorem 3.7. *With the above assumptions and those of Theorem* 3.6*, the following are equivalent:*

(i)   *The space X has the $\mathcal{F}$-LLP.*

(ii)   *For any C, any $u \in \mathcal{F}(X, C^{**})$ and any f.d. $E_0 \subset X$, there is a f.d. $E \supset E_0$ and a net of maps $u_i^E \in \mathcal{F}(E, C)$ tending pointwise weak* to $u_{|E}$.*

Proof. The proof of Theorem 3.6 can be easily adapted to prove this. ∎

Since the work of T.B. Andersen and Ando [1, 2] it has been known that if a separable Banach space X has the metric approximation property then any contractive $u : X \to C/\mathcal{I}$ admits a contractive lifting. Although we will not use it, we conclude our general discussion by reformulating this result in our framework. Let us say that $u : X \to C/\mathcal{I}$ is $\mathcal{F}$-approximable if there is a net of finite rank maps $u_\alpha \in \mathcal{F}(X, C/\mathcal{I})$ tending pointwise

to u. For the examples of $\mathcal{F}$ considered in Remark 3.5 it is obviously equivalent to say that the identity map of $C/\mathcal{I}$ is locally $\mathcal{F}$-liftable or to say that for any X any finite rank map in $\mathcal{F}(X, C/\mathcal{I})$ is locally $\mathcal{F}$-liftable.

With this terminology, Ando's well-known result [2] can nowadays be reformulated like this:

Proposition 3.8. *Assume that any finite rank map in $\mathcal{F}(X, C/\mathcal{I})$ is liftable in $\mathcal{F}(X, C)$. If $u : X \to C/\mathcal{I}$ is $\mathcal{F}$-approximable, then u admits a (global) lifting in $\mathcal{F}(X, C)$.*

Proof. This follows from Arveson's principle. ∎

Let us denote by $\mathcal{F}_{\mathrm{ccp}}$ (resp. $\mathcal{F}_{\mathrm{ucp}}$) the class of c.c.p. (resp. u.c.p.) maps. Then the LP is just the $\mathcal{F}_{\mathrm{ccp}}$-LP. In the unital case, we show below that the $\mathcal{F}_{\mathrm{ccp}}$-LP and the $\mathcal{F}_{\mathrm{ucp}}$-LP are equivalent. Of course in the latter case we restrict to lifting quotients of unital C^*-algebras.

The following statement about the unitization process is a well-known consequence of the works of Choi–Effros and Kirchberg [9, 19].

Proposition 3.9. *Let A be a separable C^*-algebra and let A_1 be its unitization. The following properties of A are equivalent:*

(i)   *The lifting property LP, meaning the $\mathcal{F}_{\mathrm{ccp}}$-LP.*

(ii)   *The unitization A_1 has the $\mathcal{F}_{\mathrm{ucp}}$-LP.*

Moreover, when A is unital (i) *and* (ii) *are equivalent to*

(iii) *The C^*-algebra A has the $\mathcal{F}_{\mathrm{ucp}}$-LP.*

Proof. Assume (i). Let $q : C \to C/\mathcal{I}$ be the quotient map. Assuming C unital, let $u : A_1 \to C/\mathcal{I}$ be a u.c.p. map. By (i) there is $w \in CP(A, C)$ with $\|w\| \leq 1$ such that $qw = u_{|A} : A \to C/\mathcal{I}$. Let $w_1 : A_1 \to C_1$ be the unital extension of w. By [9, Lemma 3.9], w_1 is c.p. Since C is unital there is a unital $*$-homomorphism $\pi : C_1 \to C$ extending Id_C. Let $\widehat{u} = \pi w_1 : A_1 \to C$. Then $\widehat{u}$ is a u.c.p. map such that $q\widehat{u}_{|A} = u_{|A}$ and $q\widehat{u}(1) = 1$. It follows that $q\widehat{u} = u$, whence (ii).

Assume (ii). Let $u : A \to C/\mathcal{I}$ be a c.c.p. map. Let $u_1 : A_1 \to (C/\mathcal{I})_1 \simeq C_1/\mathcal{I}$ be the unital map extending u. By [9, Lemma 3.9] again, u_1 is c.p. By (ii) there is a unital c.p. map $\widehat{u_1} : A_1 \to C_1$ lifting u_1. Let $q : C \to C/\mathcal{I}$ and $Q : C_1 \to C_1/\mathcal{I} \simeq (C/\mathcal{I})_1$ be the quotient maps. We have $Q\widehat{u_1} = u_1$ and $u_{1|A} = u$. A moment of thought shows that Q is the unital extension of q, so that $Q^{-1}(C/\mathcal{I}) = C$. Thus $Q\widehat{u_1}(a) = u_1(a) = u(a) \in C/\mathcal{I}$ for any $a \in A$ implies that $\widehat{u_1}(a) \in C$ for any $a \in A$. We conclude that $\widehat{u_1}_{|A} : A \to C$ is a c.p. lifting of u with norm ≤ 1, whence (i).

Assume (i) with A unital. Let $u : A \to C/\mathcal{I}$ be a u.c.p. map. Let $v : A \to C$ be a c.c.p. lifting of u. To show (iii) let f be a state on A and let $\widehat{u} = v + (1 - v(1))f$. Then $\widehat{u}$ is a u.c.p. lifting. This shows (i) $\Rightarrow$ (iii).

Conversely, assume (iii). To show (i) let $u : A \to C/\mathcal{I}$ be a c.c.p. map. Let $u' : A \to C_1/\mathcal{I}$ be the map u composed with the inclusion of C into its unitization $C_1/\mathcal{I}$. Note that since u' takes its values in $C/\mathcal{I}$, any lifting $\widehat{u}$ of u' must take its values in C, and hence

be a lifting of u. Therefore, it suffices to show that u' admits a c.c.p. lifting. Assume first that $u'(1)$ is invertible in C_1. We will argue as for Lemma 3.2. Define $w \in CP(A, C_1/\mathcal{I})$ by $w(x) = u'(1)^{-1/2} u'(x) u'(1)^{-1/2}$. Then w is unital. Let $\hat{w} : A \to C_1$ be a u.c.p. lifting of w. Let $z \in C_1$ be a lifting of $u'(1)$ with $\|z\| = \|u'(1)\| \leq 1$. Then the mapping $\hat{u} : A \to C_1$ defined by $\hat{u}(\cdot) = z^{1/2} \hat{w}(\cdot) z^{1/2}$ is a c.c.p. lifting of u'. Let f be a state on A. To complete the proof note that for any $\varepsilon > 0$ the map $u_\varepsilon \in CP(A, C_1/\mathcal{I})$ defined by $u_\varepsilon(\cdot) = u'(\cdot) + \varepsilon f(\cdot)$ is a c.p. perturbation of u' with $u_\varepsilon(1)$ invertible. By what precedes, the maps $(1 + \varepsilon)^{-1} u_\varepsilon$ admit c.c.p. liftings. By Arveson's principle u' also does. This shows (iii) $\Rightarrow$ (i). ∎

Remark 3.10. If A is unital (i)–(iii) in Proposition 3.9 are also equivalent to:

   (iv)   Any unital *-homomorphism $u : A \to \mathcal{C}/\mathcal{I}$ into a quotient of $\mathcal{C}$ admits a u.c.p. lifting.

Indeed, (iii) $\Rightarrow$ (iv) is trivial, and to show (iv) $\Rightarrow$ (iii) one can realize A as a quotient of $\mathcal{C}$. Then (iv) implies that the identity of A factors via u.c.p. maps through $\mathcal{C}$, so that (iii) follows from Kirchberg's theorem that $A = \mathcal{C}$ satisfies (iii) or (ii).

   However, we deliberately avoid using the equivalence with (iv) to justify our claim that Theorem 1.2 yields a new proof of the latter theorem of Kirchberg.

Remark 3.11. In [18], Kirchberg defines the LP for A by property (ii) in Proposition 3.9. We prefer to use the equivalent definition in (i) that avoids the unitization.

   See [15] for a discussion of lifting properties in the more general context of M-ideals.

4. A "new" extension theorem

We start by a new version of the "local reflexivity principle" (see particularly (4.3)). This is the analogue of [12, Lemma 5.2] for the maximal tensor product.

Theorem 4.1. *Assume that A satisfies* (1.1). *Let $E \subset A$ be any f.d. subspace. Then for any C^*-algebra C we have a contractive inclusion*

$$MB(E, C^{**}) \to MB(E, C)^{**}. \tag{4.1}$$

In other words any u in the unit ball of $MB(E, C^{**})$ is the weak* limit of a net (u_i) in the unit ball of $MB(E, C)$.

Proof. This will follow from the bipolar theorem. We first need to identify the dual of $MB(E, C)$. As a vector space $MB(E, C) \simeq C \otimes E^*$ and hence $MB(E, C)^* \simeq C^* \otimes E$ (or say $(C^*)^{\dim(E)}$). We equip $C^* \otimes E$ with the norm α defined as follows. Let $K \subset MB(E, C)^*$ denote the set of those $f \in MB(E, C)^*$ for which there is a C^*-algebra D, a functional w in the unit ball of $(D \otimes_{\max} C)^*$ and $t \in B_{D \otimes_{\max} E}$, so that

$$\forall u \in MB(E, C) \quad f(u) = \langle w, [\mathrm{Id}_D \otimes u](t) \rangle.$$

We could rephrase this in tensor product language: note that we have a natural bilinear map

$$(D \otimes_{\max} C)^* \times (D \otimes_{\max} E) \to C^* \otimes E$$

associated to the duality $D^* \times D \to \mathbb{C}$, then K can be identified with the union (over all D's) of the images of the product of the two unit balls under this bilinear map.

We will show that K is the unit ball of $MB(E, C)^*$. Let D_1, D_2 be C^*-algebras. Let $D = D_1 \oplus D_2$ with the usual C^*-norm. Using the easily checked identities (here the direct sum is in the ℓ_∞-sense)

$$D \otimes_{\max} E = (D_1 \otimes_{\max} E) \oplus (D_2 \otimes_{\max} E),$$

and

$$D \otimes_{\max} C = (D_1 \otimes_{\max} C) \oplus (D_2 \otimes_{\max} C),$$

and hence $(D \otimes_{\max} C)^* = (D_1 \otimes_{\max} C)^* \oplus_1 (D_2 \otimes_{\max} C)^*$ (direct sum in the ℓ_1-sense), it is easy to check that K is convex and hence that K is the unit ball of some norm α on $MB(E, C)^*$.

Our main point is the claim that K is weak* closed. To prove this, let (f_i) be a net in K converging weak* to some $f \in MB(E, C)^*$. Let D_i be C^*-algebras, $w_i \in B_{(D_i \otimes_{\max} C)^*}$ and $t_i \in B_{D_i \otimes_{\max} E}$ such that we have

$$\forall u \in MB(E, C) \quad f_i(u) = \langle w_i, [\mathrm{Id}_{D_i} \otimes u](t_i) \rangle.$$

Let $D = \ell_\infty(\{D_i\})$ and let $t \in D \otimes E$ be associated to (t_i). By (1.2) we know that $\|t\|_{\max} \le 1$. Let $p_i : D \to D_i$ denote the canonical coordinate projection, and let $v_i \in (D \otimes_{\max} C)^*$ be the functional defined by $v_i(x) = w_i([p_i \otimes \mathrm{Id}_C](x))$. Clearly $v_i \in B_{(D \otimes_{\max} C)^*}$ and $f_i(u) = \langle v_i, [\mathrm{Id}_D \otimes u](t) \rangle$. Let w be the weak* limit of (v_i). By weak* compactness, $w \in B_{(D \otimes_{\max} C)^*}$. Then $f(u) = \lim f_i(u) = \langle w, [\mathrm{Id}_D \otimes u](t) \rangle$. Thus we conclude $f \in K$, which proves our claim.

By the very definition of $\|u\|_{MB(E,C)}$ we have

$$\|u\|_{\mathrm{mb}} = \sup\{|f(u)| \mid f \in K\}.$$

This implies that the unit ball of the dual of $MB(E, C)$ is the bipolar of K, which is equal to its weak* closure. By what precedes, the latter coincides with K. Thus, the gauge of K is the announced dual norm $\alpha = \| \ \|_{MB^*}$.

Let $u'' \in MB(E, C^{**})$ with $\|u''\|_{\mathrm{mb}} \le 1$. By the bipolar theorem, to complete the proof it suffices to show that u'' belongs to the bipolar of K, or equivalently that $|f(u'')| \le 1$ for any $f \in K$. To show this, consider $f \in K$ taking $u \in MB(E, C)$ to $f(u) = \langle w, [\mathrm{Id}_D \otimes u](t) \rangle$ with $w \in B_{(D \otimes_{\max} C)^*}$ and $t \in B_{D \otimes_{\max} E}$. Observe that $[\mathrm{Id}_D \otimes u''](t) \in D \otimes C^{**} \subset (D \otimes_{\max} C)^{**}$. Recall that $MB(E, C^{**}) \simeq MB(E, C)^{**} \simeq (C^{**})^{\dim(E)}$ as vector spaces. Thus, we may view $f \in MB(E, C)^*$ as a weak* continuous functional on $MB(E, C^{**})$ to define $f(u'')$. We claim that

$$f(u'') = \langle w, [\mathrm{Id}_D \otimes u''](t) \rangle, \tag{4.2}$$

where the duality is relative to the pair $\langle (D \otimes_{\max} C)^*, (D \otimes_{\max} C)^{**} \rangle$. From this claim the conclusion is immediate. Indeed, we have $\| [\mathrm{Id}_D \otimes u''](t) \|_{D \otimes_{\max} C^{**}} \leq \| u'' \|_{\mathrm{mb}} \leq 1$, and by the maximality of the max-norm on $D \otimes C^{**}$ we have a fortiori

$$\| [\mathrm{Id}_D \otimes u''](t) \|_{(D \otimes_{\max} C)^{**}} \leq \| [\mathrm{Id}_D \otimes u''](t) \|_{D \otimes_{\max} C^{**}} \leq 1.$$

Therefore $| f(u'') | = | \langle w, [\mathrm{Id}_D \otimes u''](t) \rangle | \leq \| w \|_{(D \otimes_{\max} C)^*} \leq 1$, which completes the proof modulo our claim (4.2).

To prove the claim, note that identity (4.2) holds for any $u \in MB(E, C)$. Thus, it suffices to prove that the right-hand side of (4.2) is a weak* continuous function of u'' (which is obvious for the left-hand side). To check this, one way is to note that $t \in D \otimes E$ can be written as a finite sum $t = \sum d_k \otimes e_k$ ($d_k \in D, e_k \in E$) and if we denote by $\dot{w} : D \to C^*$ the linear map associated to w we have

$$\langle w, [\mathrm{Id}_D \otimes u''](t) \rangle = \sum_k \langle w, [d_k \otimes u''(e_k)] \rangle = \sum_k \langle \dot{w}(d_k), u''(e_k) \rangle,$$

and since $\dot{w}(d_k) \in C^*$ the weak* continuity as a function of u'' is obvious, completing the proof. ∎

Remark 4.2. The preceding proof actually shows that, without any assumption on A or C, we have a contractive inclusion

$$SB(E, C^{**}) \to SB(E, C)^{**}. \tag{4.3}$$

Of course, if A satisfies (1.1) then $SB(E, C) = MB(E, C)$ isometrically and we recover Theorem 4.1.

Remark 4.3. The converse inclusion to (4.1) holds in general: we claim that we have a contractive inclusion

$$MB(E, C)^{**} \to MB(E, C^{**}). \tag{4.4}$$

Indeed, let $u_i : E \to C$ be a net with $\| u_i \|_{\mathrm{mb}} \leq 1$ tending weak* to $u : E \to C^{**}$. Let $t \in \mathcal{C} \otimes E$ with $\| t \|_{\max} \leq 1$, say with $t \in F \otimes E$ with $F \subset \mathcal{C}$ f.d. Then $[\mathrm{Id}_{\mathcal{C}} \otimes u](t) \in B_{[F \otimes_{\max} C]^{**}}$. Note that since F is f.d., we have

$$[F \otimes_{\max} C]^{**} \subset [\mathcal{C} \otimes_{\max} C]^{**} \cap [F \otimes C^{**}].$$

We have clearly an isometric inclusion $[F \otimes_{\max} C]^{**} \subset [\mathcal{C} \otimes_{\max} C]^{**}$. Moreover, (see [29, Thm. 8.22]) we have an isometric embedding

$$\mathcal{C}^{**} \otimes_{\mathrm{bin}} C^{**} \subset [\mathcal{C} \otimes_{\max} C]^{**}.$$

Since $F \subset \mathcal{C}$, the norm induced by $\mathcal{C}^{**} \otimes_{\mathrm{bin}} C^{**}$ on $F \otimes C^{**}$ coincides with that induced by $\mathcal{C} \otimes_{\mathrm{nor}} C^{**}$. By Kirchberg's theorem to the effect that (1.12) holds when $A = \mathcal{C}$ (see [29, Thm. 9.10]), we have $\mathcal{C} \otimes_{\mathrm{nor}} C^{**} = \mathcal{C} \otimes_{\max} C^{**}$. Thus we find $\| u \|_{MB(E, C^{**})} \leq 1$. This proves the claim.

Remark 4.4. We will use an elementary perturbation argument as follows. Let A, C be C^*-algebras and let $E \subset A$ be a f.d. subspace. Let $v : E \to C$. For any $\delta > 0$ there is $\varepsilon > 0$ (possibly depending on E) satisfying the following: for any map $v' : A \to C$ such that $\|v'_{|E} - v\| \le \varepsilon$, there is a map $v'' : A \to C$ such that

$$v''_{|E} = v \quad \text{and} \quad \|v'' - v'\|_{\mathrm{mb}} \le \delta, \quad \text{and hence} \quad \|v''\|_{\mathrm{mb}} \le \|v'\|_{\mathrm{mb}} + \delta.$$
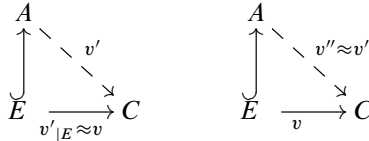
Let $\Delta = v - v'_{|E}$. Let $\|\Delta\|_N$ denote the nuclear norm (in the Banach space sense) of $\Delta :$ $E \to C$. By definition (here E and C are Banach spaces with E f.d.), this is the infimum of $\sum_1^d \|f_j\|_{E^*}\|c_j\|_C$ over all the possible representations of Δ as $\Delta(x) = \sum_1^d f_j(x)c_j$ $(x \in E)$. Let $k_E = \|\mathrm{Id}_E\|_N$. It is immediate that for any $\Delta : E \to C$ we have $\|\Delta\|_N \le k_E\|\Delta\|$. By Hahn–Banach, Δ admits an extension $\widetilde{\Delta} : A \to E$ with
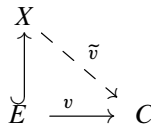
$$\|\widetilde{\Delta}\|_N \le \|\Delta\|_N \le k_E\|\Delta\| \le \varepsilon k_E.$$

Let $v'' = v' + \widetilde{\Delta}$. Then $v''_{|E} = v$ and

$$\|v'' - v'\|_{\mathrm{mb}} \le \|v'' - v'\|_N = \|\widetilde{\Delta}\|_N \le \varepsilon k_E.$$

Whence the announced result with $\delta = \varepsilon k_E$.

Theorem 4.5. *Assume that A satisfies* (1.1) *(or merely the conclusion of Theorem* 4.1). *Let C be a C^*-algebra. Let $E \subset X$ be a f.d. subspace of a separable subspace $X \subset A$. Then for any $\varepsilon > 0$, any map $v : E \to C$ admits an extension $\widetilde{v} : X \to C$ such that $\|\widetilde{v}\|_{\mathrm{mb}} \le (1 + \varepsilon)\|v\|_{\mathrm{mb}}$.*

Proof. Let $E_1 \supset E$ be any finite dimensional superspace with $E_1 \subset X$. Our first goal will be to show that for any $\varepsilon > 0$ there is an extension $w : E_1 \to C$ such that $w_{|E} = v$ with $\|w\|_{\mathrm{mb}} \le \|v\|_{\mathrm{mb}} + \varepsilon$.

By Theorem 2.9 and by (2.2), there is a decomposable map $\widetilde{v} : A \to C^{**}$ such that

$$\|\widetilde{v}\|_{D(A,C^{**})} = \|\widetilde{v}\|_{MB(A,C^{**})} \le \|v\|_{\mathrm{mb}},$$

which is an extension of v in the sense that $\widetilde{v}_{|E} = i_C v$ where $i_C : C \to C^{**}$ is the canonical inclusion. Let $v_1 : E_1 \to C^{**}$ be the restriction of $\widetilde{v}$ such that $v_1 = \widetilde{v}_{|E_1}$. A fortiori, $\|v_1 : E_1 \to C^{**}\|_{\mathrm{mb}} \le \|v\|_{\mathrm{mb}}$. By Theorem 4.1 there is a net of maps $v^i : E_1 \to C$ with

$\|v^i\|_{\mathrm{mb}} \leq \|v\|_{\mathrm{mb}}$ that tend weak* to v_1. It follows that the restrictions $v^i_{|E}$ tend weak* to $v_{1|E} = \widetilde{v}_{|E} = i_C v$. This means that $v^i_{|E}(x)$ tends weakly in C (i.e. for $\sigma(C, C^*)$) to $v(x)$ for any $x \in E$. By a well-known application of Mazur's theorem (see e.g. [29, Rem. A.10] for details), passing to suitable convex combinations of the v^i's, we can get a similar net such that in addition $v^i_{|E}$ tends pointwise in norm to v. Since E is f.d. this implies $\|v^i_{|E} - v\| \to 0$. By perturbation (see Remark 4.4), for any $\varepsilon > 0$ we can find $w : E_1 \to C$ such that $w_{|E} = v$ with $\|w\|_{\mathrm{mb}} \leq \|v\|_{\mathrm{mb}} + \varepsilon$, so we reach our first goal.

Lastly, we use the separability of X to form an increasing sequence (E_n) of f.d. subspaces of X with dense union such that $E_0 = E$. Let $w_0 = v : E_0 \to C$. By induction, we can find a sequence of maps $w_n : E_n \to C$ such that $w_{n+1|E_n} = w_n$ and $\|w_{n+1}\|_{\mathrm{mb}} \leq \|w_n\|_{\mathrm{mb}} + 2^{-n-1}\delta$. By density this extends to a linear operator $w : X \to C$ such that $\|w\|_{\mathrm{mb}} \leq \|w_0\|_{\mathrm{mb}} + \delta = \|v\|_{\mathrm{mb}} + \delta$. This completes the proof. ∎

Remark 4.6. Actually, it suffices by Remark 4.2 to assume that $X \subset A$ satisfies (1.1) (in place of A) for the preceding proof to be valid. We rephrase this in the next statement.

Theorem 4.7. *Let $E \subset X$ be a f.d. subspace of a separable subspace $X \subset A$ of a C^*-algebra. For any $\varepsilon > 0$, any map $v : E \to C$ into a C^*-algebra C admits an extension $\widetilde{v} : X \to C$ such that $\|\widetilde{v}\|_{\mathrm{sb}} \leq (1 + \varepsilon)\|v\|_{\mathrm{sb}}$.*

Remark 4.8. Let A, C be unital C^*-algebras. Let $E \subset A$ be an operator system and let $u : E \to C$ be a unital map. Recall (see Remark 2.6) that u is m.p. if and only if $\|u\|_{\mathrm{mb}} = 1$. We will denote by $\mathscr{F}_{\mathrm{ump}}(E, C)$ the class of u.m.p. maps $u : E \to C$. This is clearly an admissible class and it is pointwise closed.

We will need the following consequence of Theorem 4.1 for u.m.p. maps:

Corollary 4.9. *Let A, C be unital C^*-algebras. Assume that A satisfies the conclusion of Theorem 4.1. Let $E \subset A$ be a f.d. operator system. Let $u : E \to C^{**}$ be a u.m.p. map. There is a net of u.m.p. maps $u_i : E \to C$ tending pointwise weak* to u.*

Proof. By Theorem 4.1 there is a net of maps $v_i : E \to C$ with $\|v_i\|_{\mathrm{mb}} \leq 1$ tending pointwise weak* to u. Replacing v_i by $(v_i + (v_i)_*)/2$, we may assume each v_i self-adjoint (see Remark 2.7). Moreover, since $u(1) = 1$, we may observe that $v_i(1) - 1 \to 0$ in the weak topology (i.e. $\sigma(C, C^*)$) of C. Passing to convex combinations, we may assume (by Mazur's theorem) that $\|v_i(1) - 1\| \to 0$. By Lemma 2.20 there is a unital map $u_i \in MP(E, C)$ such that $\|u_i - v_i\| \to 0$. Since $v_i \to u$ pointwise weak*, we also have $u_i \to u$ pointwise weak*. ∎

Theorem 4.10. *Let A, C be unital C^*-algebras. Assume that A satisfies the conclusion of Theorem 4.1. Let $E \subset X$ be a f.d. operator subsystem of a separable operator system $X \subset A$. Let $u : E \to C$ be a u.m.p. map. Then for any $\varepsilon > 0$ there is a u.m.p. map $v : X \to C$ such that $\|v_{|E} - u\| < \varepsilon$.*

Proof. We may assume that we have an increasing sequence of f.d. operator systems (E_n) with $E_0 = E$ such that $\overline{\bigcup E_n} = X$. We give ourselves $\varepsilon_n > 0$ such that $\sum_{n \geq 1} \varepsilon_n < \varepsilon$. The plan is to construct a sequence of u.m.p. maps $v_n : E_n \to C$ such that $v_0 = u$ and $\|v_{n+1|E_n} - v_n\| < \varepsilon_{n+1}$ for all $n \geq 0$. Then the map v defined by $v(x) = \lim v_n(x)$ for $x \in \bigcup E_n$ extends by density to a u.m.p. map $v : X \to C$ such that $\|v_{|E} - u\| \leq \sum_{n \geq 1} \varepsilon_n < \varepsilon$.

The sequence (v_n) will be constructed by induction starting from $v_0 = u$. Assume that we have constructed $v_0, \ldots, v_n$ ($n \geq 0$) satisfying the announced properties and let us produce v_{n+1}. Since $\|v_n\|_{\text{mb}} = 1$ by Theorem 2.9 and (2.2) there is a (still unital) map $\tilde{v}_n : A \to C^{**}$ such that $\tilde{v}_{n|E_n} = i_C v_n$ with $\|\tilde{v}_n\|_{\text{mb}} = 1$. By Remark 2.6, the map $\tilde{v}_n$ is u.m.p. (see also Remark 2.13). A fortiori, $\tilde{v}_{n|E_{n+1}} : E_{n+1} \to C^{**}$ is u.m.p. By Corollary 4.9 there is a net (u_i) of u.m.p. maps from E_{n+1} to C such that $u_i - \tilde{v}_{n|E_{n+1}}$ tends weak* to 0. Since $E_n \subset E_{n+1}$, the restrictions $(u_i - \tilde{v}_n)_{|E_n}$ also tend weak* to 0. Thus, $u_{i|E_n} - v_n$ tends weak* to 0 when $i \to \infty$. Since $u_{i|E_n} - v_n$ takes values in C, this tends to 0 weakly in C. Passing to convex combinations of the maps u_i we may assume that $\lim_i \|u_{i|E_n} - v_n\| = 0$. Choosing i large enough, we have $\|u_{i|E_n} - v_n\| < \varepsilon_{n+1}$, so we may set $v_{n+1} = u_i$. This completes the induction step. ∎

5. A "new" lifting theorem

We start by what is now a special case of Theorem 3.6. Let $C/\mathcal{I}$ be a quotient C^*-algebra with quotient map $q : C \to C/\mathcal{I}$.

Lemma 5.1. *Let $E \subset A$ a f.d. subspace. Then any $u : E \to C/\mathcal{I}$ admits a lifting $\hat{u} : E \to C$ such that*

$$\|\hat{u}\|_{\text{sb}} = \|u\|_{\text{sb}}.$$

Proof. By (iii) $\Rightarrow$ (i) in Theorem 3.6 it suffices to show that the classes

$$\mathcal{F}(E, C) = \{u : E \to C \mid \|u\|_{\text{sb}} \leq 1\}$$

satisfy assertion (iii) in Theorem 3.6 with $X = E$. This is precisely what (4.3) says. ∎

Remark 5.2. In the situation of Lemma 5.1, consider $u : E \to C/\mathcal{I}$. We claim that if the map $MB(E, C^{**}) \to MB(E, C)^{**}$ is contractive then u admits a lifting $\hat{u} : E \to C$ such that

$$\|\hat{u}\|_{\text{mb}} = \|u\|_{\text{mb}}.$$

Indeed, just as in the preceding proof, we may apply (iii) $\Rightarrow$ (i) in Theorem 3.6 to the class

$$\mathcal{F}(E, C) = \{u : E \to C \mid \|u\|_{\text{mb}} \leq 1\}.$$

Theorem 5.3. *Let $X \subset A$ be a separable subspace of a C^*-algebra A. Then any $u \in SB(X, C/\mathcal{I})$ admits a lifting $\hat{u} : X \to C$ with $\|\hat{u}\|_{SB(X,C)} = \|u\|_{SB(X,C/\mathcal{I})}$. If A satisfies property (1.1), then any $u \in MB(X, C/\mathcal{I})$ admits a lifting $\hat{u} : X \to C$ with $\|\hat{u}\|_{MB(X,C)} = \|u\|_{MB(X,C/\mathcal{I})}$.*

Proof. Let $u \in SB(X, C/\mathcal{I})$. Let $E \subset X$ be a f.d. subspace. By Lemma 5.1, there is $u^E : E \to C$ lifting $u_{|E}$ with

$$\|u^E\|_{SB(E,C)} \leq \|u_{|E}\|_{SB(E,C/\mathcal{I})} \leq \|u\|_{SB(X,C/\mathcal{I})}.$$

Let $\varepsilon > 0$. By Theorem 4.5, the map u^E admits an extension $\widetilde{u^E} : X \to C$ such that

$$\|\widetilde{u^E}\|_{SB(X,C)} \leq (1+\varepsilon)\|u^E\|_{SB(E,C)} \leq (1+\varepsilon)\|u\|_{SB(X,C/\mathcal{I})}.$$

Since for any $x \in X$ we have $x \in F$ for some f.d. $F \subset X$, we have $q\widetilde{u^E}(x) = u(x)$ whenever $E \supset F$. Therefore, the map $u : X \to C/\mathcal{I}$ is in the pointwise closure of the liftable maps $\{(1+\varepsilon)^{-1}q\widetilde{u^E} \mid E \subset X\}$. By normalization, the liftings $\{(1+\varepsilon)^{-1}\widetilde{u^E} \mid E \subset X\}$ are in the unit ball of $SB(X, C)$ and the latter is an admissible class. Thus, the conclusion of the first part follows from Arveson's principle. The second part is obvious since (1.1) implies that $MB(E, C) = SB(E, C)$ isometrically for any C. ∎

Let A, C be unital C^*-algebras. Let $E \subset A$ be an operator system. Further, recall that $\mathcal{F}_{\text{ump}}(E, C)$ denotes the class of unital maps in $MP(E, C)$ (see Remarks 2.6 or 4.8).

Theorem 5.4. *Assume that A satisfies the conclusion of Theorem 4.1. Let $X \subset A$ be a separable operator system. Any $u \in \mathcal{F}_{\text{ump}}(X, C^{**})$ is the pointwise weak* limit of a net in $\mathcal{F}_{\text{ump}}(X, C)$.*

Proof. Let $E \subset X$ be a f.d. operator system. By Corollary 4.9 there is a net $u_i \in \mathcal{F}_{\text{ump}}(E, C)$ such that $u_i \to u_{|E}$ pointwise weak*. By Theorem 4.10 there are maps $v_i \in \mathcal{F}_{\text{ump}}(X, C)$ such that $\|v_{i|E} - u_i\| \to 0$, and hence $(v_i - u)_{|E} \to 0$ pointwise weak*. This implies that u is in the pointwise weak* closure of $\mathcal{F}_{\text{ump}}(X, C)$. ∎

By Theorem 3.6 we may state:

Corollary 5.5. *If A satisfies (1.1) any separable operator system $X \subset A$ has the $\mathcal{F}_{\text{ump}}$-LP.*

Corollary 5.6. *Assume A, C unital and A separable. If A satisfies (1.1) or merely the conclusion of Theorem 4.1, any unital $u \in CP(A, C/\mathcal{I})$ admits a unital lifting $\hat{u} \in CP(A, C)$.*

Proof. By Remarks 2.4 and 2.6, a unital map $u : A \to C$ is c.p. if and only if $u \in \mathcal{F}_{\text{ump}}(A, C)$. ∎

6. Proof of Theorem 1.2

We first note that $\mathcal{C} = C^*(\mathbb{F}_\infty)$ satisfies property (1.1).

Lemma 6.1. *For any free group $\mathbb{F}$ (in particular for $\mathbb{F} = \mathbb{F}_\infty$) the C^*-algebra $C^*(\mathbb{F})$ satisfies (1.1).*

Proof. We outline the argument already included in [29, Thm. 9.11]. By the main idea in [26] it suffices to check (1.2) for any $t \in \ell_\infty(\{D_i\}) \otimes E$ when E is the span of the unit and a finite number of the free unitary generators, say $U_1, \ldots, U_n$ of $C^*(\mathbb{F})$ (and actually $n = 2$ suffices). Let $U_0 = 1$ for convenience. It is known (see [27, pp. 257–258] or [29, pp. 130–131]) that for any C^*-algebra D and any $(x_j)_{0 \leq j \leq n} \in D^{n+1}$, with respect to the inclusion $E \subset \mathcal{C}$, we have

$$\Big\| \sum x_j \otimes U_j \Big\|_{D \otimes_{\max} E} = \inf \Big\{ \Big\| \sum a_j^* a_j \Big\|^{1/2} \Big\| \sum b_j^* b_j \Big\|^{1/2} \Big\}$$

where the infimum runs over all possible decompositions $x_j = a_j^* b_j$ in D. Using this characterization, it is immediate that $\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E} \leq \sup_{i \in I} \|t_i\|_{D_i \otimes_{\max} E}$. ∎

Proof of Theorem 1.2. Assume (i). We may assume that A is unital so that $A = \mathcal{C}/\mathcal{I}$ for some $\mathcal{I}$ and there is a unital c.p. lifting $r : A \to \mathcal{C}$. Then, using

$$\|(\mathrm{Id}_{D_i} \otimes r)(t_i)\|_{\max} \leq \|t_i\|_{\max}$$

for any $t_i \in D_i \otimes E$ (see Remark 2.4), we easily pass from "$\mathcal{C}$ satisfies (ii)'" (which is Lemma 6.1) to "A satisfies (ii)'". This proves (i) $\Rightarrow$ (ii).

Conversely, assume (ii). By Corollary 5.6, A has the $\mathcal{F}_{\mathrm{ucp}}$-LP, and hence (i) holds by Proposition 3.9.

To prove (ii) $\Rightarrow$ (ii)$_+$, we use another fact. For any $x = x^*$ with $\|x\| \leq 1$ in C where C is any unital C^*-algebra, we have

$$x \in C_+ \Leftrightarrow \|1 - x\| \leq 1.$$

Using this, one easily checks (ii) $\Rightarrow$ (ii)$_+$. Now assume (ii)$_+$. Using $\|x\| = \|x^*x\|^{1/2}$ one can reduce checking (ii) when all t_i's are self-adjoint. Then using the same fact we obtain (ii)$_+$ $\Rightarrow$ (ii).

Note: an alternate route is to use

$$\|x\| \leq 1 \Leftrightarrow \left\| \begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix} \right\|_{M_2(C)} \leq 1.$$

Using this with $M_2(D_i)$ in place of D_i, one easily checks (ii)$_+$ $\Rightarrow$ (ii). ∎

Actually, the preceding proof can be adapted to use only the conclusion of Theorem 4.1 to obtain the LP, therefore we have the following theorem.

Theorem 6.2. *A separable C^*-algebra A has the LP (or equivalently the property in* (1.1)*) if and only if the natural map $MB(E, C^{**}) \to MB(E, C)^{**}$ is contractive for all f.d. subspaces $E \subset A$ and all C^*-algebras C. In fact, it suffices to have this in the case $C = \mathcal{C}$.*

Proof. Assume A unital for simplicity. By Theorem 4.1 it suffices to show the "if" part. The latter is contained in Corollary 5.6. To justify the last assertion, we reduce the LP to the case when C is separable, and view C as a quotient of $\mathcal{C}$. ∎

7. LP and ultraproducts

Let $\mathcal{U}$ be an ultrafilter on a set I. Recall that the ultraproduct $(D_i)_{\mathcal{U}}$ of a family $(D_i)_{i \in I}$ of C^*-algebras is defined as the quotient

$$\ell_\infty(\{D_i\})/c_0^{\mathcal{U}}(\{D_i\}),$$

where

$$c_0^{\mathcal{U}}(\{D_i\}) = \{x \in \ell_\infty(\{D_i\}) \mid \lim_{\mathcal{U}} \|x_i\|_{D_i} = 0\}.$$

It is usually denoted as $\prod_{i \in I} D_i / \mathcal{U}$. For short we will also denote it just by $(D_i)_{\mathcal{U}}$.

Recall that, for each $i \in I$, we have trivially a contractive morphism $\ell_\infty(\{D_i\}) \to D_i$ and hence also $\ell_\infty(\{D_i\}) \otimes_{\max} A \to D_i \otimes_{\max} A$, whence a contractive morphism

$$\Psi : \ell_\infty(\{D_i\}) \otimes_{\max} A \to \ell_\infty(\{D_i \otimes_{\max} A\}), \tag{7.1}$$

which, after passing to the quotient gives us a contractive morphism

$$\ell_\infty(\{D_i\}) \otimes_{\max} A \to (D_i \otimes_{\max} A)_{\mathcal{U}}.$$

The latter morphism obviously vanishes on $c_0^{\mathcal{U}}(\{D_i\}) \otimes A$, and hence also on its closure in $\ell_\infty(\{D_i\}) \otimes_{\max} A$, i.e. on $c_0^{\mathcal{U}}(\{D_i\}) \otimes_{\max} A$. Thus we obtain a natural contractive morphism

$$(\ell_\infty(\{D_i\}) \otimes_{\max} A)/(c_0^{\mathcal{U}}(\{D_i\}) \otimes_{\max} A) \to (D_i \otimes_{\max} A)_{\mathcal{U}}.$$

By (1.7) we have for any C^*-algebra A

$$(D_i)_{\mathcal{U}} \otimes_{\max} A = (\ell_\infty(\{D_i\}) \otimes_{\max} A)/(c_0^{\mathcal{U}}(\{D_i\}) \otimes_{\max} A),$$

whence a natural contractive morphism

$$\Phi : (D_i)_{\mathcal{U}} \otimes_{\max} A \to (D_i \otimes_{\max} A)_{\mathcal{U}}. \tag{7.2}$$

In this section, we will show that (7.2) is isometric for any $(D_i)_{\mathcal{U}}$ if and only if A has the LP.

Theorem 7.1. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *The algebra A satisfies (1.1) (i.e. A has the LP).*

(ii) *For any family $(D_i)_{i \in I}$ of C^*-algebras and any ultrafilter on I we have*

$$\forall t \in \Big[\prod_{i \in I} D_i / \mathcal{U}\Big] \otimes A, \quad \|t\|_{\max} = \lim_{\mathcal{U}} \|t_i\|_{D_i \otimes_{\max} A}.$$

In other words, we have a natural isometric embedding

$$\Big[\prod_{i \in I} D_i / \mathcal{U}\Big] \otimes_{\max} A \subset \prod_{i \in I} [D_i \otimes_{\max} A]/\mathcal{U}.$$

Proof. Assume (i). If A satisfies (1.1), Ψ in (7.1) is isometric. We claim that (7.2) must also be isometric, or equivalently, injective. Indeed, let $z \in (D_i)_{\mathcal{U}} \otimes_{\max} A$ such that $\Phi(z) = 0$. Let $z' \in \ell_\infty(\{D_i\}) \otimes_{\max} A$ lifting z. Then $\Psi(z') \in c_0^{\mathcal{U}}(\{D_i \otimes_{\max} A\})$. This means that for any $\varepsilon > 0$ there is a set $\alpha \in \mathcal{U}$ such that $\sup_{i \in \alpha} \|\Psi(z')_i\|_{D_i \otimes_{\max} A} < \varepsilon$. Let $P_\alpha : \ell_\infty(\{D_i\}) \to \ell_\infty(\{D_i\})$ be the projection onto α defined for any $x \in \ell_\infty(\{D_i\})$ by $P_\alpha(x)_i = x_i$ if $i \in \alpha$ and $P_\alpha(x)_i = 0$ otherwise. Since (7.1) is isometric we have

$$\|[P_\alpha \otimes \mathrm{Id}_A](z')\| = \|\Psi([P_\alpha \otimes \mathrm{Id}_A](z'))\| = \sup_{i \in \alpha} \|\Psi(z')_i\|_{D_i \otimes_{\max} A} \leq \varepsilon.$$

Denoting by $q : \ell_\infty(\{D_i\}) \otimes_{\max} A \to (D_i)_{\mathcal{U}} \otimes_{\max} A$ the quotient map, since $q(z') = z$, this implies

$$\|z\| = \|q(z')\| = \|q([P_\alpha \otimes \mathrm{Id}_A](z'))\| \leq \|[P_\alpha \otimes \mathrm{Id}_A](z')\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $z = 0$, proving our claim. This proves (i) $\Rightarrow$ (ii).

Assume (ii). Consider the set $\hat{I}$ formed of all the finite subsets of I, viewed as directed with respect to the inclusion order. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\hat{I}$ refining this net. Let $d = (d_i)_{i \in I} \in \ell_\infty(\{D_i \mid i \in I\})$. For any $J \in \hat{I}$ we set $D_J = \ell_\infty(\{D_i \mid i \in J\})$. Using natural coordinate projections we have a natural morphism $d = (d_i)_{i \in I} \mapsto (d_J)_{J \in \hat{I}} \in \ell_\infty(\{D_J \mid J \in \hat{I}\})$. Clearly,

$$\|(d_J)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|d_J\| = \|d\|_{\ell_\infty(\{D_i \mid i \in I\})}.$$

Let $\psi_1 : \ell_\infty(\{D_i \mid i \in I\}) \to (D_J)_{\mathcal{U}}$ be the corresponding isometric $*$-homomorphism. We claim that we have a c.p. contraction $\psi_2 : (D_J)_{\mathcal{U}} \to \ell_\infty(\{D_i^{**} \mid i \in I\})$ such that the composition $\psi_2 \psi_1$ coincides with the inclusion $\ell_\infty(\{D_i \mid i \in I\}) \subset \ell_\infty(\{D_i^{**} \mid i \in I\})$. Indeed, consider $d = (d_J)_{\mathcal{U}} \in (D_J)_{\mathcal{U}}$. For any $J \in \hat{I}$ we may write $d_J = (d_J(i))_{i \in J} \in \ell_\infty(\{D_i \mid i \in J\})$. Fix $i \in I$. We will define $\psi_2(d)$ by its i-th coordinate $\psi_{2i}(d)$ taking values in D_i^{**}. Note that $i \in J$ for all J far enough in the net. We then set

$$\psi_{2i}(d) = \text{weak*} \lim_{\mathcal{U}} d_J(i).$$

It is then easy to check our claim.

By (1.11) the composition $\psi_2 \psi_1$ is max-isometric, and $\|\psi_2\|_{\mathrm{mb}} \leq 1$ by Remark 2.4, therefore ψ_1 is max-isometric. In particular, (taking $C = A$) $\psi_1 \otimes \mathrm{Id}_A$ defines an isometric embedding

$$\ell_\infty(\{D_i \mid i \in I\}) \otimes_{\max} A \subset (D_J)_{\mathcal{U}} \otimes_{\max} A.$$

By (ii) we have an isometric embedding $(D_J)_{\mathcal{U}} \otimes_{\max} A \subset \prod_{J \in \hat{I}}[D_J \otimes_{\max} A]/\mathcal{U}$. It follows that for any $t = (t_i) \in \ell_\infty(\{D_i \mid i \in I\}) \otimes_{\max} A$ we may write

$$\|t\|_{\ell_\infty(\{D_i \mid i \in I\}) \otimes_{\max} A} = \lim_{\mathcal{U}} \|t_J\|_{D_J \otimes_{\max} A},$$

and since $\|t_J\|_{D_J \otimes_{\max} A} = \sup_{i \in J} \|t_i\|_{D_i \otimes_{\max} A}$ for any *finite* J, the inequality in (1.1) follows. This proves (ii) $\Rightarrow$ (i). ∎

8. The non-separable case

Let us say that a (not necessarily separable) C^*-algebra has the LP if for any separable subspace E there is a separable C^*-subalgebra A_s with the LP that contains E. For example, all nuclear C^*-algebras and also $C^*(\mathbb{F})$ for any free group $\mathbb{F}$ have the LP.

To tackle the non-separable case the following statement will be useful.

Proposition 8.1. *Let A be a C^*-algebra. Let $E \subset A$ be a separable subspace. There is a separable C^*-subalgebra A_s satisfying $E \subset A_s \subset A$ such that, for any C^*-algebra D, the $*$-homomorphism $D \otimes_{\max} A_s \to D \otimes_{\max} A$ is isometric. In other words, the inclusion $A_s \to A$ is max-isometric.*

Proof. By Remark 2.18 it suffices to prove this for $D = \mathbb{C}$. The latter case is proved in detail in [29, Prop. 7.24] (except that the factors are flipped). ∎

Remark 8.2. Let $A_s \subset A$ be a C^*-subalgebra. Assume that the inclusion $i : A_s \to A$ is max-isometric. We claim that for any von Neumann algebra M the map $\mathrm{Id}_M \otimes i : M \otimes A_s \to M \otimes A$ extends to an isometric morphism $M \otimes_{\mathrm{nor}} A_s \to M \otimes_{\mathrm{nor}} A$.

Indeed, our assumption implies that there is a c.c.p. map (a so called "weak expectation") $T : A \to A_s^{**}$ such that $T_{|A_s} : A_s \to A_s^{**}$ coincides with the canonical inclusion; see [8, p. 88] or [29, Thm. 7.29] for a detailed proof. By Lemma 1.11 and Remark 1.15 the claim follows.

Remark 8.3. If $A_s \to A$ is max-isometric, for any $E \subset A_s$ the spaces $MB^{E \subset A_s}(E, C)$ and $MB^{E \subset A}(E, C)$ associated to the respective embeddings are (isometrically) identical for any C. By Theorem 6.2, this shows that A has the LP if and only if $\|MB(E, C^{**}) \to MB(E, C)^{**}\| = 1$ for any f.d. subspace $E \subset A$. Moreover, if A satisfies (1.1) then A_s also does.

More precisely:

Corollary 8.4. *A C^*-algebra A satisfies the property in (1.1) if and only if for any separable subspace $E \subset A$ there is a separable C^*-subalgebra A_s satisfying (1.1) such that $E \subset A_s \subset A$.*

Proof. Let $t \in \ell_\infty(\{D_i\}) \otimes E$ with $E \subset A_s \subset A$ as in the proposition. Then t satisfies (1.2) with respect to A if and only if it does with respect to A_s. ∎

It is now easy to extend Theorems 1.2 and 1.13 to the non-separable case:

Theorem 8.5. *A (not necessarily separable) C^*-algebra has the LP if and only if it satisfies the property (1.1). Moreover, Theorem 1.13 remains valid in the non-separable case.*

Proof. The first part follows from the separable case by Corollary 8.4. For the second part, assume that A satisfies (1.1). Let $t \in M \otimes E$ with E f.d. Let A_s be as in Proposition 8.1. Then $\|t\|_{M \otimes_{\max} A} = \|t\|_{M \otimes_{\max} A_s}$. By Remark 8.2 we also have $\|t\|_{M \otimes_{\mathrm{nor}} A} = \|t\|_{M \otimes_{\mathrm{nor}} A_s}$.

By the separable case of Theorem 1.13 we know that $\|t\|_{M \otimes_{\mathrm{nor}} A_s} = \|t\|_{M \otimes_{\mathrm{max}} A_s}$. Therefore, $\|t\|_{M \otimes_{\mathrm{nor}} A} = \|t\|_{M \otimes_{\mathrm{max}} A}$. This shows that (i)$'$ $\Rightarrow$ (ii) in Theorem 1.13 remains valid in the non-separable case. The other implications have already been proved there. ∎

Remark 8.6. We will not enumerate all the other non-separable variants of the equivalent forms of the LP, except for the following one, that is in some sense the weakest lifting requirement sufficient for the LP: a C^*-algebra A has the LP if and only if for any separable operator system $E \subset A$ and any $*$-homomorphism $u : A \to C/\mathcal{I}$, the restriction $u_{|E}$ admits a c.c.p. lifting. Indeed, if the latter holds and if A is assumed unital for simplicity, writing $A = C^*(\mathbb{F})/\mathcal{I}$ for some large enough free group $\mathbb{F}$, it is easy to deduce the desired LP for A from that of $C^*(\mathbb{F})$. This proves the "if" part. The converse is clear, say, by Proposition 3.9 applied to some separable $A_s \subset A$.

9. On the OLP

Let X be an operator space. The associated universal unital C^*-algebra $C_u^*\langle X \rangle$ is characterized by the following property: it contains X completely isometrically, is generated by X and the unit and any complete contraction $u : X \to C$ into a unital C^*-algebra (uniquely) extends to a unital $*$-homomorphism from $C_u^*\langle X \rangle$ to C (see e.g. [29, Thm. 2.25]).

Following [22], we say that X has the OLP if $C_u^*\langle X \rangle$ has the LP. By the universal property of $C_u^*\langle X \rangle$, it is easy to check that X has the OLP if and only if any $u \in CB(X, C/\mathcal{I})$ into an arbitrary quotient C^*-algebra admits a lifting $\hat{u} \in CB(X, C)$ such that $\|\hat{u}\|_{\mathrm{cb}} = \|u\|_{\mathrm{cb}}$. See [22, Thm. 2.12] for various examples of X with the OLP.

Note that, with respect to the inclusion $X \subset A = C_u^*\langle X \rangle$, the norm induced on $D \otimes X$ by the max-norm on $D \otimes A$ can be identified with the so called δ-norm (see [27, p. 240]). We merely recall that the associated completion $D \otimes_\delta X$ can be identified as an operator space with the quotient $[D \otimes_h X \otimes_h D]/\ker(Q)$ where $Q : D \otimes_h X \otimes_h D \to D \otimes_\delta X$ is associated to the product map $D \otimes D \to D$. We have then isometrically when we view X as sitting in $A = C_u^*\langle X \rangle$,

$$D \otimes_\delta X = D \otimes_{\mathrm{max}} X. \tag{9.1}$$

This implies that we have also isometrically for any C^*-algebra C,

$$MB(X, C) = CB(X, C). \tag{9.2}$$

Consider $E \subset X$ f.d. Let C be a C^*-algebra. Recall (see Theorem 2.9) that the unit ball of the space $MB(E, C)$ is formed of the maps $u : E \to C$ that extend to some $\dot{u} : A \to C^{**}$ with $\|\dot{u}\|_{\mathrm{dec}} \leq 1$. Let $\tilde{u} = \dot{u}_{|X} : X \to C^{**}$. Then $\|\tilde{u}\|_{\mathrm{cb}} \leq 1$, so u extends to a complete contraction $\tilde{u} : X \to C^{**}$. Conversely, if u extends to a complete contraction $\tilde{u} : X \to C^{**}$, then $\tilde{u}$ extends to a $*$-homomorphism $\pi : A \to C^{**}$, and this implies $\|u\|_{MB(E,C)} \leq 1$.

Therefore, in this case

$$\|u\|_{MB(E,C)} = \|u\|_{\text{ext}(E,C)}, \tag{9.3}$$

where (recall $i_C : C \to C^{**}$ denotes the canonical inclusion)

$$\|u\|_{\text{ext}(E,C)} = \inf\{\|\tilde{u}\|_{\text{cb}} \mid \tilde{u} : X \to C^{**}, \tilde{u}_{|E} = i_C u\}.$$

Whence the following characterization of the OLP. The equivalence (i) ⇔ (iii) already appears in [22, Prop. 2.9].

Theorem 9.1. *Let X be a separable operator space. The following are equivalent:*

(i)   *X has the OLP.*

(ii)   *For any C^*-algebra C and any f.d. subspace $E \subset X$ we have an isometric identity*

$$\text{ext}(E, C^{**}) = \text{ext}(E, C)^{**}.$$

(iii) *For any C^*-algebra C, every complete contraction $u : X \to C^{**}$ is the point-wise-weak* limit of a net of complete contractions $u_i : X \to C$.*

(iv)   *For any family $(D_i)_{i \in I}$ of C^*-algebras, the natural mapping*

$$\ell_\infty(\{D_i\}) \otimes_\delta X \to \ell_\infty(\{D_i \otimes_\delta X\})$$

is isometric.

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 4.1 and (9.3). (ii) ⇔ (iii) is easy. (i) ⇔ (iii) follows from Theorem 3.6 with $\mathcal{F}$ the class of complete contractions. (i) $\Rightarrow$ (iv) is clear by (9.1) since (by Theorem 1.2) the LP for $C_u^*\langle X\rangle$ implies that it satisfies the property in (1.1).

Assume (iv). Then by (9.1) and (9.2) we have $SB(X, C) = MB(X, C) = CB(X, C)$ isometrically for any C. By Theorem 5.3, (i) holds.   ∎

In particular, we may apply this when X is a maximal operator space. Then $\|\tilde{u}\|_{\text{cb}} = \|\tilde{u}\|$ for any $\tilde{u} : X \to C$ (or any $\tilde{u} : X \to C^{**}$). This case is closely related to several results of Ozawa in his PhD thesis and in [22]; see also Oikhberg [21]. The property (ii) in Theorem 9.1 is reminiscent of Johnson and Oikhberg extendable local reflexivity from [17]. It gives some information on the existence of bounded liftings in the Banach space setting.

10. Illustration: Property (T) groups

To illustrate the focus our paper gives to the property in (1.1) and its variants, we turn to Kazhdan's property (T), following [16,22,32]. The proof of Theorem 10.1 below is merely a reformulation of an argument from [16, Thm. G], but appealing to (1.1) is perhaps a bit quicker.

Theorem 10.1 ([16]). *Let G be a discrete group with property (T). Let (N_n) be an increasing sequence of normal subgroups of G and let $N_\infty = \bigcup N_n$. If $C^*(G/N_\infty)$ has the LP then $N_\infty = N_n$ for all large enough n.*

Proof. Let $S \subset G$ be a finite unital generating subset. Let $G_n = G/N_n$ and $G_\infty = G/N_\infty$. Let $q_n : G \to G_n$ and $q_\infty : G \to G_\infty$ denote the quotient maps. Let $\mathcal{M}_n = \lambda_{G_n}(G_n)''$ denote the von Neumann algebra of G_n, and $\mathcal{M}_\mathcal{U}$ its ultraproduct with respect to a nontrivial ultrafilter $\mathcal{U}$. We will define unitary representations π_n and $\pi_\mathcal{U}$ on G. We set $\pi_n(t) = \lambda_{G_n}(q_n(t))$ for any $t \in G$. Let $Q_\mathcal{U} : \ell_\infty(\{\mathcal{M}_n\}) \to \mathcal{M}_\mathcal{U}$ be the quotient map. We set $\pi_\mathcal{U}(t) = Q_\mathcal{U}((\pi_n(t)))$. Since the kernel of $G \ni t \mapsto \pi_\mathcal{U}(t)$ contains N_∞, it defines a unitary representation on G/N_∞ and hence we have

$$|S| = \left\| \sum_S \pi_\mathcal{U}(s) \otimes \overline{\pi_\mathcal{U}(s)} \right\|_{\mathcal{M}_\mathcal{U} \otimes_{\max} \overline{\mathcal{M}_\mathcal{U}}}$$

$$\leq \left\| \sum_S \pi_\mathcal{U}(s) \otimes \overline{U_{G/N_\infty}(q_\infty(s))} \right\|_{\mathcal{M}_\mathcal{U} \otimes_{\max} \overline{C^*(G/N_\infty)}}.$$

Let $A = C^*(G/N_\infty)$. Using (1.1) for A we find

$$|S| \leq \lim_{n,\mathcal{U}} \left\| \sum_S \pi_n(s) \otimes \overline{U_{G/N_\infty}(q_\infty(s))} \right\|_{\mathcal{M}_n \otimes_{\max} \overline{A}}. \tag{10.1}$$

Also note that the LP of A implies

$$\mathcal{M}_n \otimes_{\max} \overline{A} = \mathcal{M}_n \otimes_{\mathrm{nor}} \overline{A}.$$

For some Hilbert space H_n there is a faithful representation $\sigma_n : \mathcal{M}_n \otimes_{\mathrm{nor}} \overline{A} \to B(H_n)$ that is normal in the first variable. By a well-known argument, (10.1) implies that the sequence $\{\pi_n \otimes \overline{U_{G/N_\infty} \circ q_\infty}\}$, viewed as unitary representations of G in $B(H_n)$, admits (asymptotically) almost invariant vectors. By property (T) for all n large enough, $\pi_n \otimes \overline{U_{G/N_\infty} \circ q_\infty}$ must admit an invariant unit vector in H_n. This means there is a unit vector $\xi \in H_n$ such that

$$\forall t \in G \quad \sigma_n\big(\pi_n(t) \otimes \overline{U_{G/N_\infty} \circ q_\infty}(t)\big)(\xi) = \xi.$$

Recall $\pi_n = \lambda_{G_n} \circ q_n$. Let

$$\forall g \in G_n \quad f(g) = \langle \xi, \sigma_n(\lambda_{G_n}(g) \otimes 1)(\xi) \rangle.$$

Then f is a normal state on $\mathcal{M}_n$ such that $f(g) = 1$ whenever $g \in N_\infty/N_n \subset G_n$. As is well known, we may rewrite f as $f(g) = \langle \eta, \lambda_{G_n}(g)\eta \rangle$ for some unit vector $\eta \in \ell_2(G_n)$. Now, $f(g) = 1$ whenever $g \in N_\infty/N_n \subset G_n$ means that $\lambda_{G_n}(g)\eta = \eta$ whenever $g \in N_\infty/N_n \subset G_n$. From this follows that N_∞/N_n must be finite and hence N_∞ is the union of finitely many translates of N_n by points, say $t_1, \ldots, t_k$ in N_∞. Choosing $m \geq n$ so that $t_1, \ldots, t_k \in N_m$ we conclude that $N_m = N_\infty$. ∎

Corollary 10.2 ([16]). *Let Γ be a discrete group with (T). If $C^*(\Gamma)$ has the LP then Γ is finitely presented.*

Proof. We quickly repeat the proof in [16]. By a result of Shalom [30], any property (T) discrete group Γ is a quotient of a finitely presented group G with property (T). Enumerating the (a priori countably many) relations that determine Γ, we can find a sequence N_n and N_∞ as in Theorem 10.1 with G/N_n finitely presented such that $\Gamma = G/N_\infty$. By Theorem 10.1, if $C^*(\Gamma)$ has the LP then $\Gamma = G/N_n$ for some n and hence Γ is finitely presented. ∎

Remark 10.3. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. If we do not assume the sequence (N_n) nested, let N_∞ be defined by

$$t \in N_\infty \Leftrightarrow \lim_{\mathcal{U}} 1_{t \in N_n} = 1.$$

Then the same argument shows that $|q_n(\ker(q_\infty))| < \infty$ for all n "large enough" along $\mathcal{U}$.

As pointed out in [16] there is a continuum of property (T) groups that are not finitely presented, namely those considered earlier by Ozawa in [24]. By the preceding corollary, $C^*(G)$ fails the LP for all those Gs.

11. Some uniform estimates derived from (1.1)

In this section, our goal is to show that property (1.1) has surprisingly strong "local" consequences. The results are motivated by the construction in [28] and our hope to find a separable A with LLP but failing LP (see Remark 1.17).

Definition 11.1. Let B, C be C^*-algebras. Let $\mathcal{E} \subset B$ be a self-adjoint subspace and let $\varepsilon > 0$. A linear map $\psi : \mathcal{E} \to C$ will be called an ε-morphism if

    (i) $\|\psi\| \leq 1 + \varepsilon$,

    (ii)   for any $x, y \in \mathcal{E}$ with $xy \in \mathcal{E}$ we have $\|\psi(xy) - \psi(x)\psi(y)\| \leq \varepsilon\|x\|\|y\|$,

    (iii) for any $x \in \mathcal{E}$ we have $\|\psi(x^*) - \psi(x)^*\| \leq \varepsilon\|x\|$.

Lemma 11.2. *Let A be any unital C^*-algebra and let D be another C^*-algebra. Let $t \in D \otimes A$.*

    (i)   *Then for any $\delta > 0$ there is $\varepsilon > 0$ and a f.d. operator system $\mathcal{D} \subset A$ such that $t \in D \otimes \mathcal{D}$ and for any C^*-algebra C and any $v \in CP_\varepsilon(\mathcal{D}, C)$ where*

$$CP_\varepsilon(\mathcal{D}, C) = \{v : \mathcal{D} \to C, \exists v' \in CP(\mathcal{D}, C) \text{ with } \|v'\| \leq 1 \text{ and } \|v - v'\| \leq \varepsilon\},$$

       we have

$$\|(\mathrm{Id}_D \otimes v)(t)\|_{\max} \leq (1 + \delta)\|t\|_{\max}. \tag{11.1}$$

(ii)   *Assume $t \in (D \otimes A)^+_{\max}$. Then for any $\delta > 0$ there is $\varepsilon > 0$ and a f.d. operator system $\mathcal{D} \subset A$ such that $t \in D \otimes \mathcal{D}$ and for any C^*-algebra C and any $v \in CP_\varepsilon(\mathcal{D}, C)$ we have*

$$d((\mathrm{Id}_D \otimes v)(t), (D \otimes_{\max} C)_+) \leq \delta, \tag{11.2}$$

where the distance d is meant in $D \otimes_{\max} C$.

(iii) *For any $\delta > 0$ there is $\varepsilon > 0$ and a f.d. subspace $\mathcal{E} \subset A$ such that $t \in D \otimes \mathcal{E}$ and for any ε-morphism $\psi : \mathcal{E} \to C$ (C any other C^*-algebra) we have*

$$\|(\mathrm{Id}_D \otimes \psi)(t)\|_{D \otimes_{\max} C} \leq (1 + \delta)\|t\|_{D \otimes_{\max} A}.$$

Proof. By Remark 1.5 we may assume D unital. Let $t \in D \otimes A$. Let $E \subset A$ f.d. such that $t \in D \otimes E$. First note that (see Remark 1.6 or Proposition 8.1) there is a separable C^*-subalgebra B with $E \subset B \subset A$ such that $\|t\|_{D \otimes_{\max} B} = \|t\|_{D \otimes_{\max} A}$. Moreover, assuming $\|t\|_{D \otimes_{\max} A} = 1$ (for simplicity), if $t \in (D \otimes A)^+_{\max} = (D \otimes A) \cap (D \otimes_{\max} A)_+$ (which means that t is of finite rank with $t = t^*$ and $\|1 - t\|_{D \otimes_{\max} A} \leq 1$) there is a B such that in addition $t \in (D \otimes B)^+_{\max}$. It follows that we may assume A separable without loss of generality. Then let (E_n) be an increasing sequence of f.d. subspaces (or operator systems if we wish) such that $E \subset E_0$ and $A = \bigcup E_n$.

(i) We may assume by homogeneity that $\|t\|_{\max} = 1$. We will work with the sequence $(E_n, 1/n)$. The idea is that this sequence "tends" to $(A, 0)$. For any $n \in \mathbb{N}$ we define δ_n by

$$1 + \delta_n = \sup\|(\mathrm{Id}_D \otimes v)(t)\|_{\max},$$

where the sup runs over all C's and $v \in CP_{1/n}(E_n, C)$. Note that this is finite because the rank of t is so. For any n there is C_n and $v_n \in CP_{1/n}(E_n, C_n)$ such that

$$\|(\mathrm{Id}_D \otimes v_n)(t)\|_{\max} \geq 1 + \delta_n - 1/n. \tag{11.3}$$

Let $L = \ell_\infty(\{C_n\})$ and $\mathcal{I} = c_0(\{C_n\})$. Consider the quotient space $\mathcal{L} = L/\mathcal{I}$ so that for any family $x = (x_n) \in \ell_\infty(\{C_n\})$ the image of x under the quotient map $Q : \ell_\infty(\{C_n\}) \to \mathcal{L}$ satisfies

$$\|Q(x)\| = \limsup\|x_n\|.$$

Note that by (1.7) we have

$$D \otimes_{\max} \mathcal{L} = [D \otimes_{\max} L]/[D \otimes_{\max} \mathcal{I}]. \tag{11.4}$$

The net (v_n) defines a c.p. map v from A to $\mathcal{L}$ with $\|v\| \leq 1$. It follows (see Remark 2.4) that $\|(\mathrm{Id}_D \otimes v)(t)\|_{D \otimes_{\max} \mathcal{L}} \leq 1$. By (1.7) we have

$$\|(\mathrm{Id}_D \otimes v)(t)\|_{D \otimes_{\max} \mathcal{L}} = \|(\mathrm{Id}_D \otimes v)(t)\|_{(D \otimes_{\max} L)/(D \otimes_{\max} \mathcal{I})}.$$

We have a natural morphism $D \otimes_{\max} L \to D \otimes_{\max} C_n$ for each n and hence a natural morphism $D \otimes_{\max} L \to \ell_\infty(\{D \otimes_{\max} C_n\})$, taking $D \otimes_{\max} \mathcal{I}$ to $c_0(\{D \otimes_{\max} C_n\})$. This implies

$$\limsup_n \|(\mathrm{Id}_D \otimes v_n)(t)\|_{D \otimes_{\max} C_n} \leq \|(\mathrm{Id}_D \otimes v)(t)\|_{D \otimes_{\max} \mathcal{L}} \leq 1.$$

Therefore, by (11.3),
$$\limsup_n \delta_n \leq 0.$$

Thus, for all n large enough we have $\delta_n < \delta$. This proves (i).

(ii) For any n, let δ_n denote the supremum of the left-hand side of (11.2) over all C's and all $v \in CP_{1/n}(E_n, C)$. We introduce v_n and $v \in CP(A, \mathcal{L})$ by proceeding as in the preceding argument. Now, $(\mathrm{Id}_D \otimes v)(t) \in (D \otimes \mathcal{L})^+_{\max}$, which, in view of (11.4), implies
$$\limsup d((\mathrm{Id}_D \otimes v_n)(t), (D \otimes_{\max} C_n)_+) = 0.$$

We conclude as for part (i).

Lastly, (iii) is proved similarly to (i). ∎

Lemma 11.3. *Let A be any unital C^*-algebra satisfying* (1.1). *Let $E \subset A$ be a f.d. sub-space. Then for any $\delta > 0$ there is $\varepsilon > 0$ and a f.d. operator system $\mathcal{D} \subset A$ containing E such that the following holds.*

(i)   *For any C^*-algebras C, D and any $v \in CP_\varepsilon(\mathcal{D}, C)$ (in particular for any $v \in CP(\mathcal{D}, C)$ with $\|v\| \leq 1$) we have*

$$\forall t \in D \otimes E \quad \|(\mathrm{Id}_D \otimes v)(t)\|_{\max} \leq (1 + \delta)\|t\|_{\max}. \qquad (11.5)$$

In other words,
$$\|v_{|E}\|_{MB(E,C)} \leq 1 + \delta. \qquad (11.6)$$

(ii)   *For any C^*-algebras C, D and any $v \in CP_\varepsilon(\mathcal{D}, C)$ we have*

$$\forall t \in (D \otimes C)^+_{\max} \quad d((\mathrm{Id}_D \otimes v)(t), (D \otimes_{\max} C)_+) \leq \delta\|t\|_{\max}, \qquad (11.7)$$

where the distance d is meant in $D \otimes_{\max} C$.

(iii)   *For any C^*-algebras C, D and ε-morphism $\psi : \mathcal{D} \to C$ we have*

$$\|(\mathrm{Id}_D \otimes \psi)(t)\|_{D \otimes_{\max} C} \leq (1 + \delta)\|t\|_{D \otimes_{\max} A}.$$

Proof. It is easy to see (by Remark 1.6 or Proposition 8.1) that we may restrict to separable D's. We then assemble the set I as the disjoint union of unit balls of $D \otimes_{\max} E$, when D is an arbitrary separable C^*-algebra (say viewed as quotient of $\mathcal{C}$). Let $t_i \in D_i \otimes E$ be the element corresponding to $i \in I$. By (1.2) we know that $\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} A} \leq 1$, and hence $\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E} \leq 1$. Let $D = \ell_\infty(\{D_i\})$ so that $t \in D \otimes E$. Let $\varepsilon > 0$ and $\mathcal{D} \supset E$ be associated to t as in Lemma 11.2. For any $v \in CP_\varepsilon(\mathcal{D}, C)$ we have

$$\|(\mathrm{Id}_D \otimes v)(t)\|_{\max} \leq (1 + \delta)\|t\|_{\max} \leq 1 + \delta.$$

A fortiori, using the canonical morphisms $D \to D_i$,

$$\sup_{i \in I}\|(\mathrm{Id}_{D_i} \otimes v)(t_i)\|_{\max} \leq 1 + \delta,$$

equivalently we conclude $\|v\|_{MB(E,C)} \leq 1 + \delta$.

(ii) and (iii) are proved similarly. A priori they lead to distinct $\mathcal{D}$'s but since replacing $\mathcal{D}$ by a larger f.d. one preserves the 3 properties, we may obtain all 3 for a common $\mathcal{D}$. ∎

Remark 11.4. In the converse direction, assume that for any E and $\delta > 0$ there is $\mathcal{D} \supset E$ such that, for any C, any $v \in CP(\mathcal{D}, C)$ satisfies (11.6). If A (separable unital for simplicity) has the LLP then A satisfies (1.1). Indeed, assuming the LLP, any quotient morphism $q : \mathcal{C} \to A$ will admit unital c.p. local liftings on arbitrarily large f.d. operator systems $\mathcal{D}$ in A. By (11.6), we have local liftings $u^E : E \to \mathcal{C}$ with $\|u^E\|_{\mathrm{mb}} \leq 1 + \delta$ on arbitrarily large f.d. subspaces E in A. From the latter, it is now easy to transplant (1.1) from $\mathcal{C}$ to A (as in the proof of (i) $\Rightarrow$ (ii) in Theorem 1.2). Thus, the property described in part (i) of Lemma 11.3 can be interpreted as describing what is missing in the LLP to get the LP.

Our last result is a reformulation of the equivalence of the LP and the property in (1.1). Under this light, the LP appears as a very strong property.

Theorem 11.5. *Let A be a C^*-algebra. The following are equivalent:*

(i)   *The algebra A satisfies* (1.1) *(i.e. it has the lifting property LP).*

(ii)   *For any f.d. $E \subset A$ there is $P = P_E \in \mathcal{C} \otimes E$ with $\|P\|_{\mathcal{C} \otimes_{\max} A} \leq 1$ such that for any unital separable C^*-algebra D and any $t \in B_{D \otimes_{\max} E}$ there is a unital $*$-homomorphism[2] $q^D : \mathcal{C} \to D$ such that*

$$[q^D \otimes \mathrm{Id}_A](P) = t.$$

(ii)′   *For any f.d. $E \subset A$ there is a unital separable C^*-algebra C and $P = P_E \in C \otimes E$ with $\|P\|_{C \otimes_{\max} A} \leq 1$ such that the same as* (ii) *holds with C in place of $\mathcal{C}$.*

(iii)   *For any f.d. $E \subset A$ and $\varepsilon > 0$ there is $P = P_E \in \mathcal{C} \otimes E$ with $\|P\|_{\mathcal{C} \otimes_{\max} A} \leq 1$ such that for any unital separable C^*-algebra D and any $t \in B_{D \otimes_{\max} E}$ there is a unital $*$-homomorphism $q^D : \mathcal{C} \to D$ such that*

$$\|[q^D \otimes \mathrm{Id}_A](P) - t\|_{\max} < \varepsilon.$$

(iii)′   *For any f.d. $E \subset A$ and $\varepsilon > 0$ there is a unital separable C^*-algebra C and $P \in C \otimes E$ such that the same as* (iii) *holds with C in place of $\mathcal{C}$.*

(iv)   *Same as* (iii)′ *restricted to $D = \mathcal{C}$.*

Proof. Assume (i). We view the set of all possible D's as the set of quotients of $\mathcal{C}$. Let I be the disjoint union of the sets $B_{D \otimes_{\max} E}$. Let $t = (t_i)_{i \in I}$ be the family of all possible $t \in B_{D \otimes_{\max} E}$. By (1.2), we have

$$\|t\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E} \leq 1.$$

Let $q : C^*(\mathbb{F}) \to \ell_\infty(\{D_i\})$ be a surjective $*$-homomorphism. Let $t' \in C^*(\mathbb{F}) \otimes E$ be a lifting of t such that $\|t'\|_{C^*(\mathbb{F}) \otimes_{\max} E} \leq 1$. There is a copy of $\mathbb{F}_\infty$, say $\mathbb{F}' \subset \mathbb{F}$ such that

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[2]Here, actually, we could use instead of a $*$-homomorphism a u.c.p. map or a q with $\|q\|_{\mathrm{mb}} \leq 1$.

$t' \in C^*(\mathbb{F}') \otimes E$. Then $C^*(\mathbb{F}') \subset C^*(\mathbb{F})$. Let $C = C^*(\mathbb{F}')$. We can take for q^D the restriction of q_i to $C^*(\mathbb{F}') \simeq \mathcal{C}$, and we set $P = t'$ viewed as sitting in $\mathcal{C} \otimes E$. This gives us (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (ii)$'$ is trivial.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iii)$'$ $\Rightarrow$ (iv) are trivial.

Assume (iii)$'$. Let $Q : \mathcal{C} \to C$ be a quotient unital morphism. Let $P \in C \otimes E$ be as in (iii)$'$. Let $P' \in \mathcal{C} \otimes E$ with $\|P'\|_{\max} \leq 1$ be such that $[Q \otimes \mathrm{Id}_E](P') = P$ (this exists by the assertion following (1.8)). Then P' has the property required in (iii). Thus (iii)$'$ $\Rightarrow$ (iii). Similarly (ii)$'$ $\Rightarrow$ (ii).

The implication (iv) $\Rightarrow$ (iii)$'$ is easy to check similarly using the fact that any D is a quotient of $\mathcal{C}$. We skip the details. It remains to show (iii) $\Rightarrow$ (i).

Assume (iii). We will show (1.2). We need a preliminary elementary observation. Let (e_j) be a normalized algebraic basis of E. Then it is easy to see that there is a constant c (depending on E and (e_j)) such that for any D and any $d_j, d_j' \in D$ we have

$$\sup_j \|d_j - d_j'\| \leq c \left\| \sum (d_j - d_j') \otimes e_j \right\|_{D \otimes_{\max} E}. \tag{11.8}$$

Now, let $t_i \in B_{D_i \otimes_{\max} E}$. Let $P \in B_{\mathcal{C} \otimes_{\max} E}$ be as in (iii). For some quotient morphism $q^i : \mathcal{C} \to D_i$ we have

$$\|t_i - [q^i \otimes \mathrm{Id}_A](P)\|_{\mathcal{C} \otimes_{\max} E} \leq \varepsilon.$$

Let $P_i = P$ and $\mathcal{C}_i = \mathcal{C}$ for all $i \in I$. Clearly (since $P \mapsto (P_i)$ is a $*$-homomorphism on $\mathcal{C} \otimes A$),

$$\|(P_i)\|_{\ell_\infty(\{\mathcal{C}_i\}) \otimes_{\max} E} \leq \|P\|_{\mathcal{C} \otimes_{\max} E} \leq 1.$$

Note $(q^i) : \ell_\infty(\{\mathcal{C}_i\}) \to \ell_\infty(\{D_i\})$ is a $*$-homomorphism. Let $(t_i') = [q^i \otimes \mathrm{Id}_A](P_i)$. We have

$$\|(t_i')\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E} \leq 1 \quad \text{and} \quad \sup_i \|t_i - t_i'\|_{\mathcal{C} \otimes_{\max} E} \leq \varepsilon.$$

By the triangle inequality and (11.8) we have $\|(t_i) - (t_i')\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E} \leq c \dim(E)\varepsilon$. Therefore, $\|(t_i)\|_{\ell_\infty(\{D_i\}) \otimes_{\max} E} \leq 1 + c \dim(E)\varepsilon$. Since $\varepsilon > 0$ is arbitrary for any fixed E, this means that (1.2) holds whence (i) (and the LP by Theorem 1.2). ∎

In passing we note:

Corollary 11.6. *For the property in* (1.1) *it suffices to check the case when* $I = \mathbb{N}$ *(and* $D_i = \mathcal{C}$ *for all* $i \in I$*).*

Proof. We already observed that $D_i = \mathcal{C}$ for all $i \in I$ suffices (see Proposition 1.8). Assume that (1.2) holds for $I = \mathbb{N}$ and $D_i = \mathcal{C}$. In the proof of (i) $\Rightarrow$ (ii) in Theorem 11.5 we may use for the set I the disjoint union of dense sequences in $B_{D_k \otimes_{\max} E}$ with $k \in \mathbb{N}$ and $D_k = \mathcal{C}$. Then I is countable and with the latter I the proof of (i) $\Rightarrow$ (ii) in Theorem 11.5 gives us (iii) in Theorem 11.5. By (iii) $\Rightarrow$ (i) in Theorem 11.5, the corollary follows. ∎

Remark 11.7. Let $\mathcal{P}$ denote the (non-closed) $*$-algebra generated by the free unitary generators of $\mathcal{C}$. Let $P \in \mathcal{C} \otimes E$ be as in (ii) in Theorem 11.5. Note for any $\varepsilon > 0$ there is $Q \in \mathcal{P} \otimes E$ with $\|Q - P\|_{\wedge} < \varepsilon$ (where $\| \ \|_{\wedge}$ denotes the projective norm), and hence also $\|q^D(Q) - t\|_{\wedge} < \varepsilon$. After a suitable normalization we may also assume $\|Q\|_{\max} < 1$. Then Q has a factorization in products in $\bigcup M_n(\mathcal{C})$ and $\bigcup M_n(A)$ and a further Blecher–Paulsen type factorization in the style described in [27, §26]. Thus we find a *common* pattern in the approximate factorization of *all* the elements of $B_{D \otimes_{\max} E}$.

The simplest illustration of this phenomenon is the case when $A = \mathcal{C}$ and E is the span of the unitary generators (U_i). In that case we have $t = \sum d_i \otimes U_i \in B_{D \otimes_{\max} E}$ if and only if for any $\varepsilon > 0$ there is a factorization of the form $d_i = a_i b_i^*$ with $a_i, b_i \in D$ such that $\|\sum a_i a_i^*\|^{1/2} = \|\sum b_i b_i^*\|^{1/2} < 1 + \varepsilon$ (see [29, pp. 130–131]). Equivalently, this holds if and only if for any $\varepsilon > 0$ there are $a_i, b_i \in D$ such that $\|\sum a_i a_i^*\|^{1/2} = \|\sum b_i b_i^*\|^{1/2} = 1$ and $\sum_i \|d_i - a_i b_i^*\| < \varepsilon$. Let us assume that the latter holds. Consider first the column operator space C_n, then its universal C^*-algebra $C_u^*\langle C_n \rangle$, and lastly the free product $C = C_u^*\langle C_n \rangle * C_u^*\langle C_n \rangle$. Let $e_{i1}^{(1)} \in C$ and $e_{i1}^{(2)} \in C$ denote the natural basis of the column space C_n of each free factor. Let $P_E = \sum e_{i1}^{(1)} e_{i1}^{(2)*} \otimes U_i \in C \otimes E$. By our assumption on (d_i), there is a $*$-homomorphism $q^D : C \otimes E$ such that $q^D(e_{i1}^{(1)}) = a_i$ and $q^D(e_{i1}^{(2)}) = b_i$ and hence $\|q^D(P_E) - \sum d_i \otimes U_i\| \leq \varepsilon$. This illustrates property (iii)$'$ from Theorem 11.5. Note in passing that since the column space C_n has the OLP, the free product $C = C_u^*\langle C_n \rangle * C_u^*\langle C_n \rangle$ has the LP. The latter algebra can be substituted with $\mathcal{C}$ in many questions involving tensor products.

Using the preceding remark, one can give an independent proof of Lemma 11.3.

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