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Uniqueness of the Infinite Loop Space Structures on Connective Fibre Spaces of BSO

By

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§ 1. Introduction

In [1] J. F. Adams and S. B. Priddy showed that after localization at any prime p the infinite loop space structure on the space BSO is essentially unique. Let $BO(d, \infty)$ be the $(d-1)$-connected fibre space of BO. Then the purpose of the present paper is to show after localization at any prime p the infinite loop space structure on the space $BO(d, \infty)$ is essentially unique if $d \geq 2$. If the word 'localization' is replaced by 'completion', the result continues to hold.

Let $\boldsymbol{K_R}$ be the spectrum which represents classical (periodic) real K-theory. Let d be a fixed integer; let $\boldsymbol{bo}(d, \infty)$ be the spectrum obtained from $\boldsymbol{K_R}$ by killing the homotopy groups in degree $<d$, while retaining the homotopy groups in degree $\geq d$. Then $\boldsymbol{bo}(d, \infty)$ represents $(d-1)$-connected real K-theory; similarly $\boldsymbol{K_C}$ and $\boldsymbol{bu}(d, \infty)$ in the complex case. Let Λ be either the ring $\boldsymbol{Z}_p$ of p-adic integers or the ring $\boldsymbol{Z}_{(p)}$ of integers localized at p. We can introduce coefficient Λ into any spectrum X by setting

$$\boldsymbol{X}_\Lambda = \boldsymbol{M\Lambda} \wedge \boldsymbol{X}$$

where $\boldsymbol{M\Lambda}$ is the Moore spectrum for the group Λ. We write $\boldsymbol{F}_p$ for the field with p elements and A_p for the mod p Steenrod algebra. Let us arrange for $\boldsymbol{bo}(d, \infty)_\Lambda$ and $\boldsymbol{bu}(d, \infty)_\Lambda$ to be Ω-spectra. Note that the equivalence $X_0 \simeq \Omega X_1$ determines an H-space structure on X_0. Then the main purpose of the present paper is to show the following theorem:

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Theorem 1. 1. *Let $\boldsymbol{X}=\{X_i\}$ be a connected Ω-spectrum. Suppose given a homotopy equivalence of spaces*

$$X_0 \simeq BO(d, \infty)_A ,$$

where $BO(d, \infty)_A$ is the 0-th term of the Ω-spectrum $\boldsymbol{bo}(d, \infty)_A$. If $d \geq 2$, then there is an equivalence of spectra

$$\boldsymbol{X} \simeq \boldsymbol{bo}(d, \infty)_A .$$

The paper is organized as follows:

In Section 2 an Ω-spectrum $\boldsymbol{\Sigma}^n\boldsymbol{X}$ which represents the n-fold suspension of $\boldsymbol{X}$ is defined. In the next section the uniqueness of the infinite loop space structures of the space ΩSp_A or the H-spaces $\Omega\mathrm{Spin}_A$ and SO_A (definitions are given in Section 3) is proved. In Section 4 the main theorem is proved.

Throughout this paper we use the following notation: for a space X, $X(n, m)$ denotes that term in the Postnikov system of X whose homotopy groups π_r are the same as those of X for $n \leq r \leq m$. We use the notation $\boldsymbol{X}(n, m)$ for spectra analogous to that which we use for spaces. The symbol $\underset{H}{\simeq}$ means H-equivalence.

§ 2. Suspension of $\boldsymbol{\Omega}$-Spectra

Let $\boldsymbol{X}=\{X_i, \varepsilon_i: X_i \rightarrow \Omega X_{i+1}\}$ be an Ω-spectrum and n an integer. Define Y_i and λ_i as follows:

$$Y_i = \begin{cases} X_{i+n} & i \geq -n \\ \Omega^{-n-i} X_0 & i < -n , \end{cases}$$

$$\lambda_i = \begin{cases} \varepsilon_{i+n}: X_{i+n} \rightarrow \Omega X_{i+n+1} & i \geq -n \\ \Omega^{-n-i} 1_{X_0}: \Omega^{-n-i} X_0 \rightarrow \Omega\Omega^{-n-i-1} X_0 \ (=\Omega^{-n-i} X_0) & i < -n . \end{cases}$$

Then clearly $\boldsymbol{\Sigma}^n\boldsymbol{X}=\{Y_i, \lambda_i: Y_i \rightarrow \Omega Y_{i+1}\}$ is an Ω-spectrum. Moreover $\boldsymbol{\Sigma}^n\boldsymbol{X}$ represents the n-fold suspension of $\boldsymbol{X}$. The following is easily proved:

Lemma 2. 1. (1) *Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be Ω-spectra. Then $\boldsymbol{X}$ is equi-*

valent to X' *if and only if* $\boldsymbol{\Sigma}^n X$ *is equivalent to* $\boldsymbol{\Sigma}^n X'$ *for some* n.

(2) $H^*(\boldsymbol{\Sigma}^n X; \boldsymbol{F}_p)$ *is isomorphic to* $\Sigma^n H^*(X; \boldsymbol{F}_p)$ *as a module over* A_p *where the graded module* $\Sigma^n M$ *is defined by regarding* M *so that an element of degree* k *in* M *appears as an element of degree* $k+n$ *in* $\Sigma^n M$.

(3) $\boldsymbol{\Sigma}^n(\boldsymbol{\Sigma}^m X) = \boldsymbol{\Sigma}^{n+m} X$.

§ 3. Some Postnikov Invariants

In this section $p=2$ and so $A=A_2$ and $\Lambda=\boldsymbol{Z}_{(2)}$ or $\boldsymbol{Z}_2$. A generator of $H^n(\boldsymbol{EM}(\Lambda, n); \Lambda)$ (resp. $H^n(\boldsymbol{EM}(\boldsymbol{Z}/2, n); \Lambda)$) is denoted by u_n (resp. v_n) and the mod 2 reduction of u_n (resp. v_n) is denoted by u'_n (resp. v'_n).

First we prove the following:

Lemma 3. 1. *Let* $\boldsymbol{M}$ *be a connected* Ω*-spectrum such that*

$$M_0 \simeq \Omega^6 BO(8, \infty)_\Lambda (= \Omega Sp_\Lambda),$$

then there is an equivalence of spectra

$$\boldsymbol{M} \simeq \boldsymbol{\Sigma}^{-6} \boldsymbol{bo}(8, \infty)_\Lambda .$$

Proof. Consider the connected spectrum $M' = \boldsymbol{\Sigma}^{-2} M(3, \infty)$, then $M'_0 \underset{H}{\simeq} BO_\Lambda$. So $M' \simeq \boldsymbol{bo}_\Lambda$ by Theorem 1. 2 of Adams-Priddy [1]. In particular

$$H^*(\boldsymbol{M}'; \boldsymbol{F}_2) = H^*(\boldsymbol{bo}; \boldsymbol{F}_2) = \Sigma^1(A/(ASq^2)).$$

Consider the two stage Postnikov system

$$\boldsymbol{M}(2, 3) \to \boldsymbol{EM}(\Lambda, 2) \xrightarrow{k} \boldsymbol{EM}(\boldsymbol{Z}/2, 4).$$

Then $k \in H^4(\boldsymbol{EM}(\Lambda, 2); \boldsymbol{F}_2) = \boldsymbol{Z}/2$, which is generated by $Sq^2 u'_2$. If $k=0$, then $M(2,3)_0 \simeq K(\Lambda, 2) \times K(\boldsymbol{Z}/2, 3)$ and so $H^3(M(2,3)_0; \boldsymbol{F}_2) \neq 0$. On the other hand since the natural map $H^3(M(2,3)_0; \boldsymbol{F}_2) \to H^3(M_0; \boldsymbol{F}_2)$ is a monomorphism and $H^3(M_0; \boldsymbol{F}_2) = H^3(\Omega Sp; \boldsymbol{F}_2) = 0$, it follows that $H^3(M(2,3)_0; \boldsymbol{F}_2) = 0$. So $k \neq 0$. Then using the A-module exact sequence

$$0 \to \Sigma^4(A/(ASq^2)) \xrightarrow[\cdot Sq^2]{} \Sigma^2(A/(ASq^1)) \to \Sigma^2(A/(ASq^1+ASq^2)) \to 0,$$

we have

$$H^*(\boldsymbol{M}; \boldsymbol{F}_2) = \Sigma^2(A/(ASq^1+ASq^2)).$$

So by Theorem 1.1 of Adams-Priddy [1], we have $\boldsymbol{\Sigma}^6\boldsymbol{M} \simeq \boldsymbol{bo}(8, \infty)_A$ and so $\boldsymbol{M} \simeq \boldsymbol{\Sigma}^{-6}\boldsymbol{bo}(8, \infty)_A$.

Next we prove the following:

Lemma 3.2. *Let* $\boldsymbol{N}$ *be a connected* Ω*-spectrum such that*

$$N_0 \underset{H}{\simeq} \Omega^2 BO(4, \infty)_A \,(= \Omega Spin_A),$$

then there is an equivalence of spectra

$$\boldsymbol{N} \simeq \boldsymbol{\Sigma}^{-2}\boldsymbol{bo}(4, \infty)_A.$$

Proof. Consider the connected Ω-spectrum $\boldsymbol{N}' = \boldsymbol{\Sigma}^{-4}\boldsymbol{N}(6, \infty)$, then $N_0' \simeq \Omega^6 BO(8, \infty)_A$. So $\boldsymbol{N}' \simeq \boldsymbol{\Sigma}^{-6}\boldsymbol{bo}(8, \infty)_A$ by Lemma 3.1. In particular,

$$H^*(\boldsymbol{N}'; \boldsymbol{F}_2) = \Sigma^2(A/(ASq^1+ASq^2)).$$

Consider the two stage Postnikov system

$$\boldsymbol{N}(2, 6) \to \boldsymbol{EM}(\Lambda, 2) \xrightarrow{k'} \boldsymbol{EM}(\Lambda, 7).$$

Then $k' \in H^7(\boldsymbol{EM}(\Lambda, 2); \Lambda) = \boldsymbol{Z}/2$. Note that since $H^7(\boldsymbol{EM}(\Lambda, 2); \boldsymbol{F}_2)$ $(= \boldsymbol{Z}/2)$ is generated by $Sq^2Sq^3u_2'$, $k' \neq 0$ if and only if $k'^*(u_7') = Sq^2Sq^3u_2'$. If $k' = 0$, then as an infinite loop space $N(2, 6)_0 \simeq K(\Lambda, 2) \times K(\Lambda, 6)$. So as an algebra over the Dyer-Lashof algebra,

$$H_*(N(2, 6)_0; \boldsymbol{F}_2) = H_*(K(\Lambda, 2); \boldsymbol{F}_2) \otimes H_*(K(\Lambda, 6); \boldsymbol{F}_2).$$

In particular $Q^4x_2' = 0$, where x_2' is a generator of $H_2(H(2, 6)_0; \boldsymbol{F}_2)$. On the other hand the natural map $H_*(N_0; \boldsymbol{F}_2) \to H_*(N(2, 6)_0; \boldsymbol{F}_2)$ is isomorphic for $* \leq 6$ and commutes with the Dyer-Lashof operations. But by Theorem 6.9 of Nagata [2], $Q^4x_2 \neq 0$, where x_2 is a generator of $H_2(N_0; \boldsymbol{F}_2)$. So $k' \neq 0$. Then using the A-module exact sequence

$$0 \to \Sigma^7(A/ASq^1+ASq^2)) \xrightarrow[\cdot Sq^2Sq^3]{} \Sigma^2(A/(ASq^1))$$
$$\to \Sigma^2(A/(ASq^1+ASq^2Sq^3)) \to 0$$

we have

$$H^*(N; F_2) = \Sigma^2(A/(ASq^1 + ASq^2Sq^3)).$$

So by Theorem 1.1 of Adams-Priddy [1], we have $\boldsymbol{\Sigma}^2 N \simeq \boldsymbol{bo}(4, \infty)_A$ and so $N \simeq \boldsymbol{\Sigma}^{-2}\boldsymbol{bo}(4, \infty)_A$.

As a corollary of the above lemmas, we can easily show

Corollary 3.3. *Let X be a connected Ω-spectrum such that*

$$X_0 \simeq BO(4, \infty)_A \ (resp.\ X_0 \simeq BO(8, \infty)_A),$$

then there is an equivalence of spectra

$$X \simeq \boldsymbol{bo}(4, \infty)_A \ (resp.\ X \simeq \boldsymbol{bo}(8, \infty)_A).$$

Proof. If $X_0 \simeq BO(4, \infty)_A$, then $(\Sigma^{-2}X)_0 \underset{H}{\simeq} \Omega^2 BO(4, \infty)_A$. So by Lemma 3.2, $\boldsymbol{\Sigma}^{-2}X \simeq \boldsymbol{\Sigma}^{-2}\boldsymbol{bo}(4, \infty)_A$. The case $X_0 \simeq BO(8, \infty)_A$ is similar.

Using Theorem 6.9 of Nagata [2], we can prove the following by a quite similar method to Lemma 3.2:

Lemma 3.4. *Let X be a connected Ω-spectrum such that*

$$\chi_0 \underset{H}{\simeq} \Omega BO(2, \infty)_A (= \Omega BSO_A = SO_A),$$

then there is an equivalence of spectra

$$X \simeq \boldsymbol{\Sigma}^{-1}\boldsymbol{bo}(2, \infty)_A.$$

§ 4. Proof of the Main Theorem

In this section the main theorem, Theorem 1.1 is proved. By the Bott periodicity theorem, $\Omega^8 BO(n+8, \infty) \simeq BO(n, \infty)$. Then we can easily show

Lemma 4.1. *Let n, n' and n'' be non-negative integers such that $n = 8n' + n''$. Then $\boldsymbol{\Sigma}^{-8n'}\boldsymbol{bo}(n, \infty) \simeq \boldsymbol{bo}(n'', \infty)$.*

First we assume that p is an odd prime. Then $\Omega^4 BO(5, \infty)_A \simeq$

$BO(1, \infty)_A$ and

$$BO(1, \infty)_A = BO(2, \infty)_A = BO(3, \infty)_A = BO(4, \infty)_A .$$

Put $d = 4m + m'$ $(1 \leq m' \leq 4)$. Let $\boldsymbol{X}$ be a connected Ω-spectrum such that $X_0 \simeq BO(d, \infty)_A$. Put $\boldsymbol{X}' = \boldsymbol{\Sigma}^{-4m}\boldsymbol{X}$. Then $X'_0 \simeq BO(m', \infty)_A \simeq BO_A$. So by Theorem 1.2 of Adams-Priddy [1], we have $\boldsymbol{X}' \simeq \boldsymbol{bo}_A (\simeq \boldsymbol{bo}(m', \infty)_A$. Then $\boldsymbol{X} \simeq \boldsymbol{\Sigma}^{4m}\boldsymbol{bo}(m', \infty)_A \simeq \boldsymbol{bo}(d, \infty)_A$.

Next we assume $p=2$. Put $d = 8n + n'$ $(1 \leq n' \leq 8)$. Note that $BO(1, \infty) = BO$, $BO(2, \infty) = BSO$, $BO(3, \infty) = BO(4, \infty) = BSpin$ and $BO(5, \infty) = BO(6, \infty) = BO(7, \infty) = BO(8, \infty)$. So Theorem 1.1 is true for $d \leq 8$ by Theorem 1.2 of [1] and Corollary 3.3. We may assume that $n \geq 1$. Put $\boldsymbol{X}' = \boldsymbol{\Sigma}^{-8n}\boldsymbol{X}$, then $X'_0 \underset{H}{\simeq} BO(n', \infty)_A$. By Theorem 1.1 of [1] and Corollary 3.3, we have $\boldsymbol{X}' \simeq \boldsymbol{bo}(n', \infty)_A$. Then by Lemma 4.1, we have

$$\boldsymbol{X} \simeq \boldsymbol{\Sigma}^{8n}\boldsymbol{bo}(n', \infty)_A \simeq \boldsymbol{bo}(d, \infty)_A .$$

For $\boldsymbol{K_C}$ we have

Theorem 4.2. *Let $\boldsymbol{X}$ be a connected Ω-spectrum. Suppose given a homotopy equivalence of spaces*

$$X_0 \simeq BU(d, \infty)_A .$$

If $d \geq 3$, then there is an equivalence of spectra

$$\boldsymbol{X} \simeq \boldsymbol{bu}(d, \infty)_A .$$

References

[1] Adams, J. F., and Priddy, S., Uniqueness of *BSO*, *Math. Proc. Camb. Phil. Soc.*, **80** (1976), 475-509.
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