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Computing the spectral action for fuzzy geometries: from random noncommutative geometry to bi-tracial multimatrix models

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Abstract. A fuzzy geometry is a certain type of spectral triple whose Dirac operator crucially turns out to be a finite matrix. This notion incorporates familiar examples like fuzzy spheres and fuzzy tori. In the framework of random noncommutative geometry, we use Barrett's characterization of Dirac operators of fuzzy geometries in order to systematically compute the spectral action $S(D) = \operatorname{Tr} f(D)$ for $2n$-dimensional fuzzy geometries. In contrast to the original Chamseddine–Connes spectral action, we take a polynomial f with $f(x) \to \infty$ as $|x| \to \infty$ in order to obtain a well-defined path integral that can be stated as a random matrix model with action of the type $S(D) = N \cdot \operatorname{tr} F + \sum_i \operatorname{tr} A_i \cdot \operatorname{tr} B_i$, being F, A_i and B_i noncommutative polynomials in 2^{2n-1} complex $N \times N$ matrices that parametrize the Dirac operator D. For arbitrary signature—thus for any admissible KO-dimension—formulas for 2-dimensional fuzzy geometries are given up to a sextic polynomial, and up to a quartic polynomial for 4-dimensional ones, with focus on the octo-matrix models for Lorentzian and Riemannian signatures. The noncommutative polynomials F, A_i and B_i are obtained via chord diagrams and satisfy: independence of N; self-adjointness of the main polynomial F (modulo cyclic reordering of each monomial); also up to cyclicity, either self-adjointness or anti-self-adjointness of A_i and B_i simultaneously, for fixed i. Collectively, this favors a free probabilistic perspective for the large-N limit we elaborate on.

1. Introduction

In some occasions, the core concept of a novel research avenue can be traced back to a defiant attitude towards a no-go theorem. However uncommon this is, some prolific theories that arose from a slight perturbation of the original assumptions, aiming at an escape from the no-go, have shaped the modern landscape of mathematical physics. Arguably, the best-known story fitting this description is supersymmetry.

Another illustration is found in noncommutative geometry (NCG) applications to particle physics: In an attempt to unify all fundamental interactions, the proposal of trading *gravitation coupled to matter on a usual spacetime manifold* M by *pure gravitation on an extended space* $M \times F$ is bound to fail—as a well-known symmetry argument

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(amidst other objections) shows—as far as spacetime is extended by an ordinary manifold F. Rebelling against the no-go result, while not giving up a gravitational unification approach, shows the way out of the realm of commutative spaces (manifolds) after restating the symmetries in an algebraic fashion. For the precise argument we refer to [25, Sec. 9.9] and for the details on the obstruction to [32,56,70].

Following the path towards a noncommutative description of the 'internal space' F (initially a two-point space), Connes was able to incorporate the Higgs field on a geometrically equal footing with gauge fields, simultaneously avoiding the Kaluza–Klein tower that an augmentation of spacetime by an ordinary space F would cause. As a matter of fact, not only the Higgs sector but the whole classical action of the Standard Model of particle physics has been geometrically derived [4, 17] from the *Chamseddine–Connes spectral action*[1] [16]. The three-decade-old history of the impact of Connes' groundbreaking idea on the physics beyond the Standard Model is told in [19] (to whose comprehensive references one could add later works [9,10,13,55]); see his own review [24] for the impact of the spectral formalism on mathematics.

On top of the very active quest for the noncommutative internal space F that corresponds to a chosen field theory, it is pertinent to point out that such theory is classical and that quantum field theory tools (for instance, the renormalization group) are adapted to it. The proposals on presenting noncommutative geometries in an inherently quantum setting are diverse: A spin network approach led to the concept of *gauge networks*, along with a blueprint for spin foams in NCG, as a quanta of NCG [54]; therein, from the spectral action (for Dirac operators) on gauge networks, the Wilson action for Higgs-gauge lattice theories and the Kogut–Susskind Hamiltonian (for a 3-dimensional lattice) were derived, as an interesting result of the interplay among lattice gauge theory, spin networks and NCG. Also, significant progress on the matter of fermionic second quantization of the spectral action, relating it to the von Neumann entropy, has been proposed in [18]; and a bosonic second quantization was undertaken more recently in [28]. The context of this paper is a different, random geometrical approach motivated by the path-integral quantization of noncommutative geometries

$$\mathcal{Z} = \int_{\mathcal{M}} \mathrm{e}^{-\operatorname{Tr} f(D)} \mathrm{d}D, \tag{1.1}$$

where $S(D) = \operatorname{Tr} f(D)$ is the (bosonic) spectral action. The integration is over the space $\mathcal{M}$ of geometries encoded by Dirac operators D on a Hilbert space that, in commutative geometry, corresponds to the square integrable spinors $\mathcal{H}_M$ (well defining this $\mathcal{Z}$ is a fairly simplified version of the actual open problem stated in [25, Ch. 18.4]). The meaning of this partition function $\mathcal{Z}$ is not clear for Dirac operators corresponding to an ordinary

[1] To be precise, in this article 'spectral action' means 'bosonic spectral action'. The derivation of the Standard Model requires also a fermionic spectral action $\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$ where $\widetilde{\psi}$ is a matrix (see [30]) of classical fermions. See [52] for a physics review and [71] and [25, Sections 9–18] for detailed mathematical exposition.

spacetime M. In order to get a finite-rank Dirac operator one can, on the one hand, truncate the algebra $C^\infty(M)$ and the Hilbert space $\mathcal{H}_M$ in order to get a well-defined measure $\mathrm{d}D$ on the space of geometries $\mathcal{M}$, now parametrized by finite, albeit large, matrices. On the other hand, one does not want to fall in the class of *lattice geometries* [46] nor *finite geometries* [51].

Fuzzy geometries are finite-dimensional geometries that escape the classification of finite geometries given in [51], depicted in terms of the Krajewski diagrams, and [61]. In fact, fuzzy geometries retain also a (finite-dimensional) model of the spinor space that is not present in a finite geometry. Moreover, in contradistinction to lattices, fuzzy geometries are genuinely—and not only in spirit—noncommutative. In particular, the path-integral quantization of fuzzy geometries differs also from the approach in [46] for lattice geometries.

Of course, fuzziness is not new [53] and can be understood as limited spatial resolution on spaces. The prototype is the space spanned by finitely many spherical harmonics approximating the algebra of functions on the sphere S^2. This picture is in line with models of quantum gravity, since classical spacetime is expected to break down at scales below Planck length [29].

Although the three components of a *spectral triple* have sometimes been evoked in the study of fuzzy spaces [26] and their Dirac operators on some fuzzy spaces are well studied (e.g. the Grosse–Prešnajder Dirac operator [41]), a novelty in [5] is their systematic spectral triple formulation; for instance, fuzzy tori, elsewhere addressed (e.g. [27,68]), acquire a spectral triple [7]. Spectral triples are data that algebraically generalize spin manifolds. More precisely, when the spectral triple is commutative (i.e. the algebraic structure that generalizes the algebra of coordinates is commutative, with additional assumptions we omit) a strong theorem is the ability to construct, out of it, an oriented, smooth manifold, with its metric and spin^c structure. This has been proven by Connes [23] taking some elements from previous constructs by Rennie–Várilly [67].

This paper computes the spectral action for fuzzy geometries. Compared with the smooth case, our methods are simpler. For an ordinary manifold M or an almost commutative space $M \times F$ (being F a finite geometry [71, Sec. 8]), one commonly relies on a heat kernel expansion

$$\mathrm{Tr}(\mathrm{e}^{-tD^2}) \sim \sum_{n\geq 0} t^{\frac{n-\dim(M)}{2}} a_n(D^2), \qquad (t^+ \to 0),$$

which allows, for f of the Laplace–Stieltjes transform type $f(x) = \int_{\mathbb{R}^+} \mathrm{e}^{-tx^2} \mathrm{d}\nu(t)$, to determine the spectral action $\mathrm{Tr}\, f(D/\Lambda)$ in terms of the Seeley–DeWitt coefficients $a_{2n}(D^2)$ [36], being $t = \Lambda^{-1}$ the inverse of the cutoff Λ; see also [31]. The elements of Gilkey's theory are not used here. Crucially, f is instead assumed to be a polynomial (with $f(x) \to \infty$ for $|x| \to \infty$), which enables one to directly compute traces of powers of the Dirac operator. This alteration of the Chamseddine–Connes spectral action—in which f is typically a symmetric bump function around the origin—comes from a convergence requirement for the path-integral (1.1), as initiated by Barrett–Glaser in [8] (a polynomial spectral action itself is already considered in [54], though, for gauge networks arising from embedded quivers in a spin manifold).

1.1. Recent and parallel progress in random NCG

'Random NCG' is a short name for the construction of probability measures on families of finite-dimensional[2] Dirac operators, which is in line with equation (1.1). Since this partition function resembles the canonical ensemble in statistical physics, 'random NCG' is called 'Dirac ensembles' elsewhere [49]. Also the terminology 'dynamical fuzzy spectral triples' is used in [35], where the Batalin–Vilkovisky formalism is addressed for this type of models.

Initially, the motivation of [8] was to access information about fuzzy geometries by looking at the statistics of the eigenvalues of D using Markov chain Monte Carlo simulations[3]. This and a posterior study [6] deliver evidence for a phase transition to a 2-dimensional behavior (also of significance in quantum gravity [14]). Glaser explored the phase transition of the fuzzy-sphere—like $(1,3)$ case—that is of KO-dimension 2, as satisfied by the Grosse–Prešnajder operator—together with that of $(1,1)$ and $(2,0)$ geometries of KO-dimensions 0 and 6, respectively. For the $(1,3)$ geometry, the spectral action used in the numerical simulations of [37] was obtained inside MCMCv4, a computer code aimed at simulating random fuzzy geometries; the formula (in C++ language) for the spectral action can be found in the file `Dirac.cpp` of [38].

In [44], algebraic relations among the moments $\mathbb{E}[\frac{1}{N}\operatorname{Tr}_N(K_1^a K_2^b K_1^c K_2^d)]$ (for $a, b, c, d \in \mathbb{Z}_{>0}$) of the $N \times N$ matrices K_1, K_2 that parametrize the Dirac operator in dimension 2 (see Section 4 for details) are found, using the loop equations. The phase diagram (spanned by the second moment and by the coupling constant) of a quadratic-quartic theory is explored. In [48] and [44] results are reported in agreement with computer simulations [37].

By using the chord-diagrams techniques of the present paper, a Yang–Mills(–Higgs) gauge theory was obtained in [65] from the spectral action on fuzzy geometries (in the sense that the sectors of the Yang–Mills–Higgs theory on a smooth manifold are identifiable with those of [65].

The partition function (1.1) should be interpreted as 'convergent integral'. If this is rather grasped as a 'formal integral', we have a tool to generate *random maps* of certain kind (edge-colored [66] maps which might be *stuffed* [11], as the renormalization flow yields an effective action with more traces). The correlation functions obey Topological Recursion [3]. A recent review of it is [45].

[2] Most properly, as we will see, the terminology should be 'random finite-dimensional spectral triples'. Notice however, that not all finite-dimensional spectral triples are considered in this class.

[3] Formulas for the spectral action for geometries of signature $(0,3)$, which lead to a tetra-matrix model were presented in [8, App. A.6] (strictly seen, these lead to an octo-matrix model, but a simplification is allowed by the fact that the product of all gamma matrices is a scalar). Concerning this paper, our solely analytic approach to spectral action computations yields, out of a single general proof, a formula for any admissible KO-dimension, as it will become apparent in Proposition 5.4.

Aiming at the search of fixed points of the renormalization flow (which might be candidate for a transition to a manifold phase) the Functional Renormalization Group is developed from scratch in [64] for the matrix models that are motivated by random NCG.

1.2. Organization of this paper

Finally, the paper is organized as follows: the next section, based on [5], introduces spectral triples and fuzzy geometries in a self-contained way. The definition is slightly technical, but the essence of a fuzzy geometry can be understood from its matrix algebra, its Hilbert space $\mathscr{H}$ and Barrett's characterization of Dirac operators (Sections 2.3 and 2.4). In Section 3, we compute the spectral action in a general setting. A convenient graphical description of 'trace identities' for gamma matrices (due to the Clifford module structure of $\mathscr{H}$, Section 3.1) is provided in terms of *chord diagrams*, which later serve as organizational tool in the computation of $\operatorname{Tr}(D^m)$, $m \in \mathbb{N}$. As the main results in Sections 4 and 5, we derive formulas for the spectral action for 2- and 4-dimensional fuzzy geometries, respectively. In the latter case, we elaborate on the Riemannian and Lorentzian cases, being these the first reported (analytic) derivations for the spectral action of d-dimensional fuzzy geometries with general Dirac operators in $d > 3$.

In Section 6, we restate our results, aiming at free probabilistic tools towards the large-N limit (being N the matrix size in Barrett's parametrization of the Dirac operator). In order to define noncommutative (NC) distributions, one often departs from a self-adjoint NC polynomial. It turns out that only a weaker concept ('cyclic self-adjointness') defined here is satisfied by the main NC polynomial P in

$$\operatorname{Tr} f(D) = N \cdot \operatorname{Tr}_N P + \sum_i \operatorname{Tr}_N \Phi_i \operatorname{Tr}_N \Psi_i;$$

the other NC polynomials Φ_i and Ψ_i (for fixed i) either satisfy this very condition, or they are both cyclically anti-self-adjoint. The trace Tr_N cannot tell apart these conditions from the actual self-adjointness of an NC polynomial.

The conclusions and the outlook are presented in the last two sections. Even though the text is self-contained, Supplementary Material is offered that might be of help for the full understanding of this paper. It is available online at the article's web page. Its content is the following:

- Section I provides in detail computations of chord diagrams.
- Some properties of gamma matrices in general signature that were useful for the main text appear in Section II.
- Section III gives the spectral action for Riemannian and Lorentzian signatures explicitly (for a quadratic-quartic potential)
- Some proofs omitted in the main text are located in Section IV.
- The definition of cyclic self-adjointness is given in Section V.
- Section VI relates a double-trace matrix model to expectations taken with respect to an auxiliary (single-trace) matrix model.

2. Fuzzy geometries as spectral triples

The formalism of spectral triples in noncommutative geometry can be very intricate and its full machinery will not be used here. We refer to [71] for more details on the usage of spectral triples in high energy physics.

The essential structure is the *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a unital, involutive algebra of bounded operators on a Hilbert space $\mathcal{H}$. The *Dirac operator* D is a self-adjoint operator on $\mathcal{H}$ with compact resolvent and such that $[D, a]$ is bounded for all $a \in \mathcal{A}$. On $\mathcal{H}$, the algebraic behavior between of the Dirac operator and the algebra $\mathcal{A}$—and later also among D and some additional operators on $\mathcal{H}$—encodes geometrical properties. For instance, the geodesic distance d_g between two points x and y of a Riemannian (spin) manifold (M, g), can be recovered from

$$d_g(x, y) = \sup_{a\in\mathcal{A}}\{|a(x) - a(y)| \mid a \in \mathcal{A} \text{ and } \|[D, a]\| \leq 1\},$$

being $\mathcal{A}$ the algebra of functions on M and D the canonical Dirac operator [22, Sec. VI.1].

Precisely, those additional operators lead to the concept of *real, even* spectral triple, which allows to build physical models. Next definition, taken from [5], is given here by completeness, since fuzzy geometries are a specific type of real (in this paper all of them even) spectral triples.

Definition 2.1. A *real, even spectral triple* of KO-dimension $s \in \mathbb{Z}/8\mathbb{Z}$ consists in the following objects and relations:

(i) an algebra $\mathcal{A}$ with involution $*$,

(ii) a Hilbert space $\mathcal{H}$ together with a faithful, $*$-algebra representation $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$,

(iii) an anti-linear unitarity (called *real structure*) $J : \mathcal{H} \to \mathcal{H}$, $\langle Jv, Jw\rangle = \langle w, v\rangle$, being $\langle\,\bullet\,,\,\bullet\,\rangle$ the inner product of $\mathcal{H}$,

(iv) a self-adjoint operator $\gamma : \mathcal{H} \to \mathcal{H}$ commuting with the representation ρ and satisfying $\gamma^2 = 1$ (called *chirality*),

(v) for each $a, b \in \mathcal{A}$, $[\rho(a), J\rho(b)J^{-1}] = 0$,

(vi) a self-adjoint operator D on $\mathcal{H}$ that satisfies

$$\big[[D, \rho(a)], J\rho(b)J^{-1}\big] = 0, \qquad a, b \in \mathcal{A},$$

(vii) the relations

$$J^2 = \epsilon, \tag{2.1a}$$

$$JD = \epsilon' DJ, \tag{2.1b}$$

$$J\gamma = \epsilon''\gamma J, \tag{2.1c}$$

with the signs $\epsilon, \epsilon', \epsilon''$ determined by s according to the following table:

s	0	1	2	3	4	5	6	7
ϵ	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ'	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ''	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$

A *fermion space* of KO-dimension s is a collection of objects $(\mathcal{A}, \mathcal{H}, J, \gamma)$ satisfying axioms (i) through (v) and (vii), except for equation (2.1b).

2.1. Gamma matrices and Clifford modules

Given a *signature* $(p,q) \in \mathbb{Z}^2_{\geq 0}$, a *spinor (vector) space* V is a representation c of the Clifford algebra[4] $\mathcal{C}\ell(p,q)$. Thus, elements of the basis e^a and $e^{\dot{a}}$ of $\mathbb{R}^{p,q}$, $a = 1,\dots,p$ and $\dot{a} = 1,\dots,q$, become endomorphisms $c(e^a) = \gamma^a$, $c(e^{\dot{a}}) = \gamma^{\dot{a}}$ of V. If $d = q + p$ is even, V is assumed to be irreducible, whereas only the eigenspaces $V^+, V^- \subset V$ of γ are, if $q + p$ is odd. The size of these square matrices (the Dirac *gamma matrices*) is $2^{\lfloor p+q \rfloor}$.

It follows from the relations of the Clifford algebra that

$$\gamma^\mu \gamma^\nu = \gamma^{[\mu}\gamma^{\nu]} + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \gamma^{[\mu}\gamma^{\nu]} + g^{\mu\nu} 1_V,$$

which can be used to iteratively compute products of gamma matrices in terms of $g^{\mu\nu}$ and their anti-symmetrization. Taking their trace Tr_V (contained in the spectral action) gets rid of the latter, so we are left with $\dim V \cdot g^{\mu_1\nu_1} g^{\mu_2\nu_2} \cdots$. A product of an odd number of gamma matrices is traceless; the trace of a product of $2n$ gamma matrices can be expressed as a sum of over $(2n-1)!!$ products of n bilinears $g^{\mu\nu}$ that will be represented diagrammatically.

2.2. Fuzzy geometries

Section 2.2 is based on [5]. A fuzzy geometry can be thought of as a finite-dimensional approximation to a smooth geometry. A simple matrix algebra $M_N(\mathbb{C})$ conveys information about the resolution of a space (an inverse power of N, e.g. $\sim 1/\sqrt{N}$ for the fuzzy sphere $\mathbb{S}^4_N$ [69]) where the noncommutativity effects are no longer negligible. To do geometry on a matrix algebra one needs additional information, which, in the case of fuzzy geometries, is in line with the spectral formalism of NCG.

Definition 2.2 (Paraphrased from [5]). A *fuzzy geometry* of *signature* $(p,q) \in \mathbb{Z}^2_{\geq 0}$ is given by

- a simple matrix algebra $\mathcal{A}$ with coefficients in $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$; in the latter case, $M_{N/2}(\mathbb{H}) \subset M_N(\mathbb{C})$, otherwise $\mathcal{A}$ is $M_N(\mathbb{R})$ or $M_N(\mathbb{C})$—in this paper we take always $\mathcal{A} = M_N(\mathbb{C})$;

[4] We recall that the Clifford algebra $\mathcal{C}\ell(p,q)$ is the tensor algebra of $\mathbb{R}^{p+q}$ modulo the relation $2g(v,w) = v \otimes w + w \otimes v$ for each $u, w \in \mathbb{R}^{p+q}$, being $g = \mathrm{diag}(+,\dots,+,-,\dots,-)$ the quadratic form with p positive and q negative signs, and $\mathbb{C}\ell(p,q)$ is the complexification of $\mathcal{C}\ell(p,q)$.

- a hermitian $\mathbb{C}\ell(p,q)$-module V with a *chirality* γ. That is a linear map $\gamma : V \to V$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$;
- a Hilbert space $\mathcal{H} = V \otimes M_N(\mathbb{C})$ with inner product
$$\langle v \otimes R, w \otimes S\rangle = (v,w)\,\mathrm{Tr}(R^* S)$$
for each $R, S \in M_N(\mathbb{C})$, being $(\bullet\,,\,\bullet)$ the inner product of V;
- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}$, $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$;
- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

$s \equiv q - p \pmod 8$	0	1	2	3	4	5	6	7
ϵ	+	+	−	−	−	−	+	+
ϵ'	+	−	+	+	+	−	+	+
ϵ''	+	+	−	+	+	+	−	+

- a *real structure* $J = C \otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on V satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon' \gamma^\mu C$ for all gamma matrices $\mu = 1, \dots, p+q$;
- a self-adjoint operator D on $\mathcal{H}$ satisfying the *order-one condition*
$$\big[[D, \rho(a)], J\rho(b)J^{-1}\big] = 0, \qquad \text{for all } a, b \in \mathcal{A};$$
- a chirality $\Gamma = \gamma \otimes 1_{\mathcal{A}}$ for $\mathcal{H}$, where γ is the chirality of V. These signs impose on the operators the following conditions:
$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \qquad J\Gamma = \epsilon''\Gamma J.$$

For s odd, Γ can be thought of as the identity $1_{\mathcal{H}}$. The number $d = p + q$ is the *dimension* of the spectral triple and $s = q - p$ is its KO-*dimension*.

We pick gamma matrices that satisfy

$$(\gamma^\mu)^2 = +1, \qquad \mu = 1, \dots, p, \qquad \gamma^\mu \text{ hermitian}, \tag{2.2a}$$
$$(\gamma^\mu)^2 = -1, \qquad \mu = p+1, \dots, p+q, \qquad \gamma^\mu \text{ anti-hermitian}, \tag{2.2b}$$

in terms of which the chirality for V is given by $\gamma = (-\mathrm{i})^{s(s-1)/2}\gamma^1 \cdots \gamma^{p+q}$. For mixed signatures it will be convenient to separate spatial from time like indices, and denote by lowercase Roman the former ($a = 1, \dots, p$) and by dotted indices[5] ($\dot{c} = p+1, \dots, p+q$) the latter. The gamma matrices γ^a are hermitian matrices squaring to $+1$, and $\gamma^{\dot{c}}$'s denote here the anti-hermitian matrices squaring to -1. Greek indices are spacetime indices $\alpha, \beta, \mu, \nu, \dots \in \Delta_d := \{1, 2, \dots, d\}$.

We let the gamma matrices generate $\Omega := \langle \gamma^1, \dots, \gamma^d\rangle_{\mathbb{R}}$ as algebra; this splits as $\Omega = \Omega^+ \oplus \Omega^-$, where Ω^+ contains products of even number of gamma matrices and Ω^- an odd number of them.

[5] Dotted indices are here unrelated to their usual interpretation in the theory of spinors. Also for the Lorentzian signature, the 0, 1, 2, 3 numeration (without any dots) is used.

2.3. General Dirac operator

Using the spectral triple axioms for fuzzy geometries, their Dirac operators can be characterized as self-adjoint operators of the form [5, Sec. 5.1]

$$D(v \otimes R) = \sum_I \omega^I v \otimes (K_I R + \epsilon' R K_I), \qquad v \in V, R \in M_N(\mathbb{C}), \tag{2.3}$$

with $\{\omega^I\}_I$ a linearly independent set and I an abstract index to be clarified now. For $r \in \mathbb{N}_{\leq d}$, let Λ_d^r be the set of r-tuples of increasingly ordered spacetime indices $\mu_i \in \Delta_d$, i.e., $\Lambda_d^r = \{(\mu_1, \dots, \mu_r) \mid \mu_i < \mu_j, \text{ if } i < j\}$. We let $\Lambda_d = \dot\bigcup_r \Lambda_d^r$, whose odd part is denoted by Λ_d^-,

$$\begin{aligned} \Lambda_d^- &= \{(\mu_1, \dots, \mu_r) \mid \text{for some odd } r,\ 1 \leq r \leq d\ \ \&\ \mu_i < \mu_j \text{ if } i < j\} \\ &= \{1, \dots, d\} \cup \{(\mu_1, \mu_2, \mu_3) \mid 1 \leq \mu_1 < \mu_2 < \mu_3 \leq d\} \\ &\quad \cup \{(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \mid 1 \leq \mu_1 < \mu_2 < \dots < \mu_5 \leq d\} \cup \dots \end{aligned}$$

The most general Dirac operator in dimension d writes in terms of products Γ^I of gamma matrices that correspond to indices I in these sets, each bearing a matrix coefficient k_I,

$$D^{(p,q)} = \begin{cases} \sum_{I \in \Lambda_d^-} \Gamma^I \otimes k_I & \text{for } d = p + q \text{ even}, \\ \sum_{I \in \Lambda_d} \Gamma^I \otimes k_I & \text{for } d = p + q \text{ odd}. \end{cases} \tag{2.4}$$

We elaborate on each of the tensor-factors, Γ^I and k_I. First, Γ^I is the ordered product of gamma matrices with all single indices appearing in I,

$$\Gamma^I = \gamma^{\mu_1} \cdots \gamma^{\mu_r},$$

for $I = (\mu_1, \dots, \mu_r) \in \Lambda_d^r$. This can be thought of as each gamma matrix γ^μ corresponding to a one-form $\mathrm{d}x^\mu$ (in fact, via Clifford multiplication for canonical spectral triples) and Λ_d as the basis elements of the exterior algebra. The set $\Lambda_d = \dot\bigcup_r \Lambda_d^r$ can thus be seen as an abstract backbone of the de Rham algebra $\Omega_{\mathrm{dR}}^* = \bigoplus_r \Omega_{\mathrm{dR}}^r$ and Λ_d^r of the r-forms Ω_{dR}^r. There are $\#(\Lambda_d^r) = \binom{d}{r}$ independent r-tuple products of gamma matrices. We now separate the cases according to the parity of s (or of d). Second, $k_I(R) = (K_I R + \epsilon' R K_I)$ is an operator on $M_N(\mathbb{C}) \ni R$, which needs a dimension-dependent characterization.

2.4. Characterization of the Dirac operator in even dimensions

We constrain the discussion to even (KO-)dimension. In Definition 2.2, the table implies $\epsilon' = 1$; on top of this, the self-adjointness of D implies that for each I, both ω^I and $R \mapsto (K_I R + R K_I^*)$ are either hermitian or both anti-hermitian. In terms of the matrices K_I, this condition reads $K_I^* = +K_I$ or $K_I^* = -K_I$, respectively. In the first case, we write $K_I = H_I$, in the latter $K_I = L_I$. One can thus split the sum in equation (2.3) as

$$D(v \otimes R) = \sum_I \omega^I v \otimes (H_I R + R H_I) + \sum_I \omega^I v \otimes (L_I R - R L_I). \tag{2.5}$$

Additionally, since $d = 2p - s$ is even, $\gamma\gamma^a + \gamma^a\gamma = 0$. Hence, the same anti-commutation relation $\gamma\omega + \omega\gamma = 0$ holds for each $\omega \in \Omega^-$. This leads to the splitting

$$D = \sum_{I \in \Lambda_d^-} \tau^I \otimes \{H_I, \bullet\} + \sum_{I \in \Lambda_d^-} \alpha^I \otimes [L_I, \bullet], \tag{2.6}$$

where each α^I and τ^I is an odd product of gamma matrices, and Λ_d^- is the set of multi-indices of an odd number of indices $\mu \in \{1, \dots, d\}$. In summary,

$$(\tau^I)^* = \tau^I \in \Omega^-, \qquad (\alpha^I)^* = -\alpha^I \in \Omega^-, \qquad H_I^* = H_I, \qquad L_I^* = -L_I.$$

We generally treat commutators and anti-commutators as (noncommuting) letters $k_I = \{K_I, \bullet\}_\pm$, for each $I \in \Lambda_d$. The sign $e_I = \pm$ determines the type of the letter for k_I, being the latter defined by the rule

$$\text{if } e_I = \begin{cases} +1 & \text{then } K_I = H_I, \quad \text{therefore } k_I = h_I, \\ -1 & \text{then } K_I = L_I, \quad \text{therefore } k_I = l_I, \end{cases} \tag{2.7}$$

so $k_I(R) = K_I R + e_I R K_I$, for $R \in M_N(\mathbb{C})$. Explicitly, one has

$$\begin{aligned} D^{(p,q)} = \sum_\mu \gamma^\mu \otimes k_\mu + \sum_{\mu,\nu,\rho} \gamma^\mu\gamma^\nu\gamma^\rho \otimes k_{\mu\nu\rho} + \dots \\ + \sum_{\widehat{\mu\nu\rho}} \Gamma^{\widehat{\mu\nu\rho}} \otimes k_{\widehat{\mu\nu\rho}} + \sum_{\hat\mu} \Gamma^{\hat\mu} \otimes k_{\hat\mu}, \end{aligned}$$

which runs through $\sum_{\boldsymbol{\mu}} \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{d/2}} \otimes k_{\mu_1\mu_2\cdots\mu_{d/2}}$ if the 4 divides d, or through

$$\sum_{\boldsymbol{\mu}} \Gamma^{\mu_1\cdots\mu_{d/2-1}} \otimes k_{\mu_1\mu_2\cdots\mu_{d/2-1}} + \sum_{\boldsymbol{\mu}} \Gamma^{\mu_1\cdots\mu_{d/2+1}} \otimes k_{\mu_1\mu_2\cdots\mu_{d/2-1}}$$

if d is even but not divisible by 4. Hatted indices are, as usual, those excluded from $\{1, \dots, d\}$,

$$\widehat{\mu\nu\cdots\rho} = (1, 2, \dots, \mu-1, \mu+1, \dots, \nu-1, \nu+1, \dots, \rho-1, \rho+1, \dots, d). \tag{2.8}$$

In order for a Dirac operator to be self-adjoint, k_I is constrained by the parity of $r = r(I)$, being $|I| = 2r - 1$, and by the number $u(I)$ of spatial gamma matrices in the product Γ^I. In a mixed signature setting, $p, q > 0$, an arbitrary $I \in \Lambda_d^-$ has the form $I = (a_1, \dots, a_t, \dot c_1, \dots, \dot c_u)$ for $0 \le t \le p$ and $0 \le u \le q$, and so the corresponding matrix satisfies

$$(\Gamma^I)^* = (-1)^{u + \lfloor (u+t)/2 \rfloor} \Gamma^I = (-1)^{u+r-1} \Gamma^I. \tag{2.9}$$

The first equality is shown in detail in Supplementary Material, Section II. The second is just due to $(-1)^{\lfloor (u+t)/2 \rfloor} = (-1)^{\lfloor (2r-1)/2 \rfloor} = (-1)^{r-1}$. This decides whether k_I should be an 'h_I-operator' or an 'l_I-operator' (see equation (2.7)), which is summarized in Table 1.

$u(I)$	$r(I)$	k_I
even	odd	h_I
odd	odd	l_I
even	even	l_I
odd	even	h_I

Table 1. For $I = (a_1, \dots, a_u, \dot{c}_1, \dots, \dot{c}_t) \in \Lambda_d^{2r-1}$ a hermitian matrix H_I or an anti-hermitian matrix L_I parametrizes k_I according to the shown operators $h_I = \{H_I, \bullet\}$ or $l_I = [L_I, \bullet]$

For indices running where the dimension bounds allow, one has

$$\begin{aligned} D^{(p,q)} &= \sum_{a=1}^{p} \gamma^a \otimes h_a + \sum_{\dot{c}=p+1}^{p+q} \gamma^{\dot{c}} \otimes l_{\dot{c}} + \sum_{a,b,c} \gamma^a \gamma^b \gamma^c \otimes l_{abc} \\ &+ \sum_{a,b,\dot{c}} \gamma^a \gamma^b \gamma^{\dot{c}} \otimes h_{ab\dot{c}} + \sum_{a,\dot{b},\dot{c}} \gamma^a \gamma^{\dot{b}} \gamma^{\dot{c}} \otimes l_{a\dot{b}\dot{c}} + \sum_{\dot{a},\dot{b},\dot{c}} \gamma^{\dot{a}} \gamma^{\dot{b}} \gamma^{\dot{c}} \otimes l_{a\dot{b}\dot{c}} + \dots \\ &+ \begin{cases} \sum_a \Gamma^{\hat{a}} \otimes h_{\hat{a}} + \sum_{\dot{c}} \Gamma^{\hat{\dot{c}}} \otimes l_{\hat{\dot{c}}} & \text{if } q \text{ and } d/2 \text{ have same parity,} \\ \sum_a \Gamma^{\hat{a}} \otimes l_{\hat{a}} + \sum_{\dot{c}} \Gamma^{\hat{\dot{c}}} \otimes h_{\hat{\dot{c}}} & \text{if } q \text{ and } d/2 \text{ have opposite parity.} \end{cases} \end{aligned}$$

The last term is a product of $d-1 = p+q-1$ matrices. This expression is again determined by observing that the operator $k_{\hat{\mu}}$ is self-adjoint if $(-1)^{u+d/2}$ equals $+1$ and otherwise anti-hermitian, being u the number of spatial gamma matrices in $\Gamma^{\hat{\mu}} = \gamma^1 \cdots \widehat{\gamma^{\mu}} \cdots \gamma^d$. We proceed to give some examples.

Example 2.3 (Fuzzy $d = 2$ geometries). The next operators appear in [5]:

- *Type* $(0,2)$. Then $s = d = 2$, so $\epsilon' = 1$. The gamma matrices are anti-hermitian and satisfy $(\gamma^i)^2 = -1$. The Dirac operator is
$$D^{(0,2)} = \gamma^{\dot{1}} \otimes [L_1, \bullet] + \gamma^{\dot{2}} \otimes [L_2, \bullet].$$
- *Type* $(1,1)$. Then $d = 2$, $s = 0$, so $\epsilon' = 1$. The Dirac operator is
$$D^{(1,1)} = \gamma^1 \otimes \{H, \bullet\} + \gamma^{\dot{2}} \otimes [L, \bullet].$$
- *Type* $(2,0)$. Then $d = 2$, $s = 6$, so $\epsilon' = 1$. The gamma matrices are hermitian and satisfy $(\gamma^i)^2 = +1$. The Dirac operator is
$$D^{(2,0)} = \gamma^1 \otimes \{H_1, \bullet\} + \gamma^2 \otimes \{H_2, \bullet\}.$$

Example 2.4 (Some fuzzy $d = 4$ geometries). For realistic models the most important 4-fuzzy geometries have signatures $(0,4)$ and $(1,3)$ corresponding to the Riemannian and Lorentzian cases. We derive the first one in detail in order to arrive at the result in [5, Ex. 10]. The rest follows from considering equation (2.9).

- *Type* $(0,4)$, $s=4$, *Riemannian*. Notice that the gamma matrices are all anti-hermitian and square to -1 in this case. Therefore, products of three gamma matrices are self-adjoint: $(\gamma^{\dot a}\gamma^{\dot b}\gamma^{\dot c})^* = (-)^3\gamma^{\dot c}\gamma^{\dot b}\gamma^{\dot a} = \gamma^{\dot a}\gamma^{\dot b}\gamma^{\dot c}$. The accompanying operators have the form $\{H_{\dot a\dot b\dot c}, \bullet\}$ for $H_{\dot a\dot b\dot c}$ self-adjoint,

$$D^{(0,4)} = \sum_{\dot a}\gamma^{\dot a}\otimes[L_{\dot a},\bullet] + \sum_{\dot a<\dot b<\dot c}\gamma^{\dot a}\gamma^{\dot b}\gamma^{\dot c}\otimes\{H_{\dot a\dot b\dot c},\bullet\} \tag{2.10}$$

- *Type* $(1,3)$, $s=2$, *Lorentzian*. Call γ^0 the only gamma matrix that squares to $+1$, and denote the rest by $\gamma^{\dot c}$, $\dot c = 1,2,3$. Then

$$\begin{aligned} D^{(1,3)} &= \gamma^0\otimes\{H,\bullet\} + \gamma^{\dot c}\otimes[L_{\dot c},\bullet] \\ &\quad + \sum_{\dot a<\dot c}\gamma^0\gamma^{\dot a}\gamma^{\dot c}\otimes[L_{\dot a\dot c},\bullet] + \gamma^{\dot 1}\gamma^{\dot 2}\gamma^{\dot 3}\otimes\{\widetilde H,\bullet\}. \end{aligned} \tag{2.11}$$

  (For the Lorentzian signature, the dotted-index convention is redundant with the usual $0,1,2,3$ spacetime indices; then we henceforward drop it.)

- *Type* $(4,0)$, $s=-4=4 \pmod 8$. The opposite case to 'Riemannian': now all gamma matrices are hermitian, square to $+1$, and triple products $\Gamma^{\hat a}$ are skew-hermitian,

$$D^{(4,0)} = \sum_a\gamma^a\otimes\{H_a,\bullet\} + \sum_{a<b<c}\gamma^a\gamma^b\gamma^c\otimes[L_{abc},\bullet]. \tag{2.12}$$

Fuzzy geometries with odd s allow elements of Ω^+ also to parametrize Dirac operators,

$$\begin{aligned} D^{(p,q)} &= \sum_\mu\gamma^\mu\otimes k_\mu + \sum_{\mu_1,\mu_2}\gamma^{\mu_1}\gamma^{\mu_2}\otimes k_{\mu_1\mu_2} + \sum_{\mu,\nu,\rho}\gamma^\mu\gamma^\nu\gamma^\rho\otimes k_{\mu\nu\rho} + \dots \\ &\quad + \sum_{\widehat{\mu\nu\rho}}\Gamma^{\widehat{\mu\nu\rho}}\otimes k_{\widehat{\mu\nu\rho}} + \sum_{\widehat{\mu\nu}}\Gamma^{\widehat{\mu\nu}}\otimes k_{\widehat{\mu\nu}} + \sum_{\hat\mu}\Gamma^{\hat\mu}\otimes k_{\hat\mu}. \end{aligned} \tag{2.13}$$

Examples of D for a $d=3$ geometry are given in [5] and are not treated here.

2.5. Random fuzzy geometries

Given a fermion space of fixed signature (p,q), that is to say, a list $(\mathcal{A},\mathcal{H},\bullet,J,\Gamma)$ satisfying the listed properties in Definition 2.1 ignoring those concerning D, we consider the space $\mathcal{M}\equiv\mathcal{M}(\mathcal{A},\mathcal{H},J,\Gamma,p,q)$ of all possible Dirac operators D that make of $(\mathcal{A},\mathcal{H},D,J,\Gamma)$ a real even spectral triple of signature $(p,q)\in\mathbb{Z}^2_{\geq 0}$.

The symmetries of a spectral triple are encoded in $\mathrm{Aut}(\mathcal{A})$, $\mathrm{Inn}(\mathcal{A})$ and $\mathrm{Out}(\mathcal{A})$, none of which implies the Dirac operator. This can be compared with the classical situation, in which fixing the data $(\mathcal{A},\mathcal{H},J,\Gamma)$ can be interpreted as imposing symmetries on the system and subsequently finding compatible geometries, encoded in $D\in\mathcal{M}$, typically via the extremization $\delta S(D_0)=0$ of an action functional $S(D)$ that eventually selects a unique classical solution $D_0 \in \mathcal{M}$. The random noncommutative setting that appears in [8], on the other hand, considers 'off-shell' geometries. These can be stated as the following matrix integral

$$\mathcal{Z}^{(p,q)} = \int_{\mathcal{M}} \mathrm{e}^{-S(D)} \mathrm{d}D, \qquad S(D) = \operatorname{Tr} f(D), \tag{2.14}$$

being $f(x)$ an ordinary[6] polynomial of real coefficients and no constant term. We next compute the spectral action $\operatorname{Tr} f(D)$.

3. Computing the spectral action

In the spectral action (2.14) the trace is taken on the Hilbert space $\mathscr{H}$. We do not label it but, to avoid confusion, we label traces on other spaces: the trace Tr_V is that of the spinor space V, the trace of operators on the matrix space $M_N(\mathbb{C})$ is denoted by $\operatorname{Tr}_{M_N(\mathbb{C})}$, and Tr_N stands for the trace on $\mathbb{C}^N$.

A homogeneous element spanning the Dirac operator $D = \sum_I \omega_I \otimes k_I$ contains a first factor ω_I, consisting of products of gamma matrices, and a second factor k_I determined by a matrix that is either hermitian or anti-hermitian [5]. We describe each factor and then give a general formula to compute the spectral action.

3.1. Traces of gamma matrices

We now rewrite the quantity

$$\langle \mu_1 \cdots \mu_{2n} \rangle := \frac{1}{\dim V} \operatorname{Tr}_V (\gamma^{\mu_1} \cdots \gamma^{\mu_{2n}}) \tag{3.1}$$

in terms of *chord diagrams* of $2n$ points[7], to wit n (disjoint) pairings among $2n$ cyclically ordered points. These are typically placed on a circle in whose interior the pairings are represented by chords that might cross. One finds

$$\langle \mu_1 \cdots \mu_{2n} \rangle = \sum_{\substack{2n\text{-pt chord} \\ \text{diagrams}\,\chi}} (-1)^{\#\{\text{crossings of chords in } \chi\}} \prod_{\substack{i,j=1, \\ i \sim_\chi j}}^{2n} g^{\mu_i \mu_j} \tag{3.2}$$

where $\sim_\chi$ means that the point i is joined with j in the chord diagram χ. We denote the total number of crossings of chords by $\operatorname{cr}(\chi)$. We count only simple crossings; for instance, the sign of the 8-pt chord diagram with longest chords in the upper left corner of Figure I.2 in the Supplementary Material is $(-1)^6$.

[6] In contrast to noncommutative polynomials mentioned below.

[7] In a more involved context, these are called 'chord diagrams with one backbone' [2].

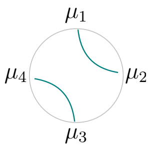
It will be convenient to denote by CD_{2n} the *set of* $2n$*-pt chord diagrams* and to associate a tensor $\chi^{\mu_1\cdots\mu_{2n}}$ with $\chi \in \mathrm{CD}_{2n}$ and an index set $\mu_1, \ldots, \mu_{2n} \in \Delta_d$,

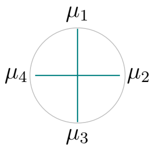
$$\chi^{\mu_1\cdots\mu_{2n}} = (-1)^{\#\{\text{crossings of chords in } \chi\}} \prod_{\substack{i,j=1,\\ i\sim_\chi j}}^{2n} g^{\mu_i\mu_j}. \tag{3.3}$$

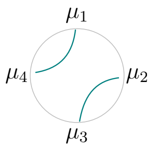
This χ-tensor is a version of the chord diagram χ whose i-th point on the circle is decorated with the spacetime index μ_i; thus $\chi^{\mu_1\cdots\mu_{2n}}$ depends on the dimension, although it is not explicitly so denoted. All known identities for traces of gamma matrices can be stated in terms of these tensors, for instance

$$\mathrm{Tr}_{\mathbb{C}^4}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = 4(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3})$$

in four dimensions: If θ, ξ, ζ denote the three 4-pt chord diagrams, one can rewrite in terms of their corresponding tensors

$\theta^{\mu_1\mu_2\mu_3\mu_4} =$ 

, $\xi^{\mu_1\mu_2\mu_3\mu_4} =$ 

, $\zeta^{\mu_1\mu_2\mu_3\mu_4} =$ 

, (3.4)

the aforementioned trace identity as

$$\mathrm{Tr}_V(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = \dim V(\theta^{\mu_1\mu_2\mu_3\mu_4} + \xi^{\mu_1\mu_2\mu_3\mu_4} + \zeta^{\mu_1\mu_2\mu_3\mu_4}).$$

For small n, this seems to be a heavy notation, which, however, will pay off for higher values (the double factorial growth $\#\mathrm{CD}_{2n} = (2n-1)!!$ notwithstanding) since by cyclic symmetry one ends up computing few diagrams.

3.2. Traces of random matrices

The aim of this subsection is to compute traces of words of the form[8] $\mathrm{Tr}_{M_N(\mathbb{C})}(k_{I_1}\cdots k_{I_{2t}})$ using the isomorphism $M_N(\mathbb{C}) = \mathbf{N} \otimes \bar{\mathbf{N}}$ (being $\mathbf{N}$ the fundamental representation) at the level of the operators. By [8],

$$k_I = K_I \otimes 1_N + e_I \cdot (1_N \otimes K_I^T), \qquad e_I = \pm.$$

The sign e_I is determined by Table 1.

[8]Choosing the basis of $M_N(\mathbb{C})$ that consists of the matrices ϵ_{ij} $(i, j = 1, \ldots, N)$ with only non-zero entry equal 1 at the (i, j)-entry, the trace $\mathrm{Tr}_{M_N(\mathbb{C})}$ of an operator $A : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ reads $\mathrm{Tr}_{M_N(\mathbb{C})}(A) = \sum_{i,j} \epsilon_{ij}^*[A(\epsilon_{ij})]$, where $\{\epsilon_{ij}^*\}_{i,j=1,\ldots,N}$ the dual basis. This basis is convenient to re-express the operators $l, h : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ given by $l : m \mapsto [L, m]$ or $h : m \mapsto \{H, m\}$ (for fixed complex N by N matrices $L^* = -L$, $H^* = H$) in terms $\mathrm{Tr}_N\, L$ and $\mathrm{Tr}_N\, H$, respectively.

Proposition 3.1. *For any $r \in \mathbb{N}$,*

$$\mathrm{Tr}_{M_N(\mathbb{C})}(k_{I_1} \cdots k_{I_r}) = \sum_{\Upsilon \in \mathcal{P}_r} \mathrm{sgn}(I_\Upsilon) \cdot \mathrm{Tr}_N \big(K_{I_{\Upsilon^c}}\big) \cdot \mathrm{Tr}_N \big[(K^T)_{I_\Upsilon}\big], \tag{3.5}$$

where

- Tr_N *and* $\mathrm{Tr}_{M_N(\mathbb{C})}$ *are the traces on* $\mathrm{End}(\mathbf{N})$ *and* $\mathrm{End}(M_N(\mathbb{C}))$*, respectively,*
- $\mathcal{P}_r$ *is the power set* $2^{\{1,\dots,r\}}$ *of* $\{1,\dots,r\}$*, thus* $\Upsilon^c = \{1,\dots,r\} \setminus \Upsilon$ *for* $\Upsilon \in \mathcal{P}_r$*,*
- $\mathrm{sgn}(I_\Upsilon)$ *is* $(-1)^{\#\{\text{commutators appearing in all the } k_{I_j} \text{ with } j \in \Upsilon\}}$*, that is*

$$\mathrm{sgn}(I_\Upsilon) = \Big(\prod_{i \in \Upsilon} e_{I_i}\Big) \in \{-1, +1\},$$

- *and, finally, the cyclic order* $(\dots \to r \to 1 \to 2 \to 3 \to \dots)$ *on the set* $\{1,\dots,r\}$*, which can be read off from the trace in the left-hand side of equation* (3.5)*, induces a cyclic order on a given subset* $\Xi = \{b_1,\dots,b_\xi\} \in \mathcal{P}_r$*. Respecting this order, define*
  - ◇ $K_{I_\Xi} = K_{I_{b_1}} K_{I_{b_2}} \cdots K_{I_{b_\xi}}$ *and*
  - ◇ $(K^T)_{I_\Xi} = K^T_{I_{b_1}} K^T_{I_{b_2}} \cdots K^T_{I_{b_\xi}} = (K_{I_{b_\xi}} \cdots K_{I_{b_2}} K_{I_{b_1}})^T$.

Proof. By induction on the number $r-1$ of products, we prove first that

$$k_{I_1} \cdots k_{I_r} = \sum_{\Upsilon \in \mathcal{P}_r} \prod_{i \in \Upsilon} \mathrm{sgn}(I_\Upsilon) K_{I_{\Upsilon^c}} \otimes (K^T)_{I_\Upsilon}.$$

The statement holds for $r = 2$, by direct computation; we now prove that the statement being true for r implies its veracity for $r+1$. In the first line of the right-hand side of the next equation we use the assumption and then directly compute

$$\begin{aligned}
(k_{I_1} \cdots k_{I_r}) k_{I_{r+1}} &= \prod_{w=1}^{r} \big[K_{I_w} \otimes 1_N + e_{I_w} \cdot (1_N \otimes K^T)_{I_w}\big] \\
&\quad \cdot \big(K_{I_{r+1}} \otimes 1_N + e_{I_{r+1}} \cdot (1_N \otimes K^T)_{I_{r+1}}\big) \\
&= \Big(\sum_{\Upsilon \in \mathcal{P}_r} \Big(\prod_{i \in \Upsilon} e_{I_i}\Big) K_{I_{\Upsilon^c}} \otimes (K^T)_{I_\Upsilon}\Big) \\
&\quad \cdot \big(K_{I_{r+1}} \otimes 1_N + e_{I_{r+1}} \cdot (1_N \otimes K^T)_{I_{r+1}}\big) \\
&= \sum_{\Upsilon \in \mathcal{P}_r} \Big(\prod_{i \in \Upsilon} e_{I_i}\Big) K_{I_{\Upsilon^c}} K_{I_{r+1}} \otimes (K^T)_{I_\Upsilon} \\
&\quad + \sum_{\Upsilon \in \mathcal{P}_r} \Big(\prod_{i \in \Upsilon} e_{I_i}\Big) e_{I_{r+1}} K_{I_{\Upsilon^c}} \otimes (K^T)_{I_\Upsilon} K^T_{I_{r+1}}) \\
&= \sum_{\Theta \in \mathcal{P}_{r+1}} \Big(\prod_{i \in \Theta} e_{I_i}\Big) \cdot K_{I_{\Theta^c}} \otimes (K^T)_{I_\Theta}.
\end{aligned}$$

To the last equality, one arrives by considering that any set $\Theta \in \mathcal{P}_{1+r}$ either contains $r+1$ (thus $\Theta = \Upsilon \cup \{r+1\}$ for some $\Upsilon \in \mathcal{P}_r$) or does not ($\Theta = \Upsilon \in \mathcal{P}_r$). These two sets are listed in the sum after the third equal sign (concretely, the second term and the first one, respectively). Then, it only remains to take traces

$$\begin{aligned}\mathrm{Tr}_{M_N(\mathbb{C})}(k_{I_1}\cdots k_{I_r}) &= \sum_{\Upsilon\in\mathcal{P}_r} \mathrm{sgn}(I_\Upsilon)\cdot \mathrm{Tr}_{\mathbf{N}\otimes\bar{\mathbf{N}}}\left(K_{I_{\Upsilon^c}} \otimes (K^T)_{I_\Upsilon}\right) \\ &= \sum_{\Upsilon\in\mathcal{P}_r} \mathrm{sgn}(I_\Upsilon)\cdot \mathrm{Tr}_N\left[K_{I_{\Upsilon^c}}\right]\cdot \mathrm{Tr}_N\left[(K^T)_{I_\Upsilon}\right]. \qquad \blacksquare\end{aligned}$$

3.3. The general structure of Tr D^m

From the analysis of the gamma matrices one infers that for a polynomial $f(x) = \sum f_t x^t$ the spectral action selects only the even coefficients $\mathrm{Tr}\, f(D) = \sum f_{2t}\, \mathrm{Tr}(D^{2t})$. In order to compute the spectral action of any matrix geometry we only need to know the traces of the even powers, which we now proceed to compute.

Proposition 3.2. *Given a collection of multi-indices $I_i \in \Lambda_d^-$, let $2n$ denote the total number of indices, $2n = 2n(I_1, \ldots, I_{2t}) := |I_1| + \ldots + |I_{2t}|$. The even powers of the Dirac operator satisfy*

$$\begin{aligned}\frac{1}{\dim V}\mathrm{Tr}(D^{2t}) = \sum_{I_1,\ldots,I_{2t}\in\Lambda_d} \Bigg\{ &\sum_{\chi\in\mathrm{CD}_{2n}} \chi^{I_1\cdots I_{2t}} \\ &\times\Bigg[\sum_{\Upsilon\in\mathcal{P}_{2t}} \mathrm{sgn}(I_\Upsilon)\cdot \mathrm{Tr}_N(K_{I_{\Upsilon^c}})\cdot \mathrm{Tr}_N((K^T)_{I_\Upsilon})\Bigg]\Bigg\},\end{aligned} \tag{3.6}$$

in whose terms the spectral action $S(D) = \mathrm{Tr}\, f(D) = \sum_t f_{2t}\,\mathrm{Tr}(D^{2t})$ can be completely evaluated.

Proof. For $t \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{\dim V}\mathrm{Tr}(D^{2t}) &= \frac{1}{\dim V}\mathrm{Tr}\Big[\Big(\sum_{I\in\Lambda_d}\Gamma^I\otimes k_I\Big)^{2t}\Big] \\ &= \sum_{I_1,\ldots,I_{2t}\in\Lambda} \frac{1}{\dim V}\mathrm{Tr}_V(\Gamma^{I_1}\cdots\Gamma^{I_{2t}})\,\mathrm{Tr}_N(k_{I_1}\cdots k_{I_{2t}}) \\ &= \sum_{I_1,\ldots,I_{2t}\in\Lambda} \langle I_1\cdots I_{2t}\rangle\,\mathrm{Tr}_{M_N(\mathbb{C})}(k_{I_1}\cdots k_{I_{2t}}).\end{aligned} \tag{3.7}$$

One uses then equation (3.2) and Proposition 3.1 with the notation of equation (3.3). $\blacksquare$

Notice that since the indices μ_i of a multi-index $I = (\mu_1 \cdots \mu_{|I|}) \in \Lambda_d$ are pairwise different, the traces of the gamma matrices greatly simplify. This also ensures that there are no contractions between indices of the same k-operator, say $g^{\mu\nu}k_{\mu\nu\ldots}$ (k's with repeated indices do not exist).

In even dimension d, the Dirac operator is spanned by the number $\kappa(d)$ of independent odd products of gamma matrices. This equals

$$\kappa(d) = \#(\Lambda_d^-) = \binom{d}{1} + \binom{d}{3} + \cdots + \binom{d}{d-1}$$

which can be rearranged (using Pascal's identity) as

$$\kappa(d) = \binom{d-1}{0} + \binom{d-1}{1} + \cdots + \binom{d-1}{d-2} + \binom{d-1}{d-1} = 2^{d-1}.$$

The Dirac operator has then as many 'matrix coefficients' and is therefore parametrized by (what will turn out to be a subspace of) $M_N(\mathbb{C})^{\oplus\kappa(d)}$. In this manner, the 'random spectral action' (2.14) becomes a $\kappa(d)$-tuple matrix model.

Definition 3.3. Given integers $t, n \in \mathbb{N}$ (interpreted as in Proposition 3.2) and a chord diagram $\chi \in \mathrm{CD}_{2n}$, its *action (functional)* $\mathfrak{a}_n(\chi)$ is a $\mathbb{C}$-valued functional on the matrix space $M_N(\mathbb{C})^{\oplus\kappa(d)}$ defined by

$$\mathfrak{a}_n(\chi)[\boldsymbol{K}] = \sum_{\substack{I_1,\ldots,I_{2t}\in\Lambda_d^-,\\ 2n=\sum_i |I_i|}} \chi^{I_1\cdots I_{2t}} \Big[\sum_{\Upsilon\in\mathcal{P}_{2t}} \mathrm{sgn}(I_\Upsilon)\cdot \mathrm{Tr}_N(K_{I_{\Upsilon^c}})\cdot \mathrm{Tr}_N((K^T)_{I_\Upsilon}) \Big] \tag{3.8}$$

for $\boldsymbol{K} = \{K_{I_i} \in M_N(\mathbb{C}) \mid I_i \in \Lambda_d^-\} \in M_N(\mathbb{C})^{\oplus\kappa(d)}$. We often shall omit the dependence on the matrices and write only $\mathfrak{a}_n(\chi)$. We define the *bi-trace functional* as a sum over the non-trivial subsets Υ in equation (3.8)

$$\mathfrak{b}_n(\chi)[\boldsymbol{K}] = \sum_{\substack{I_1,\ldots,I_{2t}\in\Lambda_d^-,\\ 2n=\sum_i |I_i|}} \chi^{I_1\cdots I_{2t}} \Big[\sum_{\substack{\Upsilon\in\mathcal{P}_{2t},\\ \Upsilon,\Upsilon^c\neq\emptyset}} \mathrm{sgn}(I_\Upsilon)\cdot \mathrm{Tr}_N(K_{I_{\Upsilon^c}})\cdot \mathrm{Tr}_N((K^T)_{I_\Upsilon}) \Big] \tag{3.9}$$

and the *single trace functional* $\mathfrak{s}_n(\chi)$ via $\mathfrak{a}_n(\chi) = N\cdot\mathfrak{s}_n(\chi) + \mathfrak{b}_n(\chi)$. The factor N ensures that $\mathfrak{s}_n$ does not depend on N (cf. Section 3.4).

The restriction $2n = \sum_i |I_i|$ allows one to exchange the sums over the multi-indices I and the chord diagrams in equation (3.6). Then one can restate Proposition 3.2 as $(1/\dim V)\,\mathrm{Tr}(D^{2t}) = N\mathcal{S}_{2t} + \mathcal{B}_{2t}$, where

$$\mathcal{S}_{2t} = \sum_{n=t}^{t\cdot(d-1)} \sum_{\chi\in\mathrm{CD}_{2n}} \mathfrak{s}_n(\chi), \tag{3.10}$$

$$\mathcal{B}_{2t} = \sum_{n=t}^{t\cdot(d-1)} \sum_{\chi\in\mathrm{CD}_{2n}} \mathfrak{b}_n(\chi). \tag{3.11}$$

The parameter n lists all the numbers of points $2n = 2t, 2(t+1), \ldots, 2t\cdot(d-1)$ that chord diagrams contributing to $\mathrm{Tr}(D^{2t})$ can have in dimension d.

In view of equations (3.10) and (3.11), all boils down to computing the single trace $\mathfrak{s}_n(\chi)$ and multiple trace part $\mathfrak{b}_n(\chi)$ of chord diagrams. We begin with the former.

3.4. Single trace matrix model in the spectral action—manifest $\mathcal{O}(N)$

For geometries with even KO-dimension $s = q - p$, the spectral action's *manifest* leading order in N can be found from the last proposition (see Section 6). Aiming at their large-N limit we state the following.

Corollary 3.4. *Let $d = q + p$ (thus s) be even. The spectral action (2.14) for the fuzzy (p, q)-geometry with Dirac operator $D = D^{(p,q)}$ satisfies, for any polynomial $f(x) = \sum_{1\le r\le m} f_r x^r$, the following:*

$$\frac{1}{\dim V}\operatorname{Tr} f(D) = \frac{1}{\dim V}\sum_{1\le 2t\le m} f_{2t}\operatorname{Tr}\big([D^{(p,q)}]^{2t}\big) = N\sum_{1\le 2t\le m} f_{2t}\,\mathcal{S}_{2t} + \mathcal{B}$$

where $2n(I_1,\dots,I_{2t}) = \sum_i |I_i|$. Here, $\mathcal{B}$ stands for products of two traces, whose coefficients are all independent of N, with $\mathcal{S}_{2t}$ given by equation (3.10) and for $\chi \in \mathrm{CD}_{2n}$,

$$\mathfrak{s}_n(\chi) = \sum_{\substack{I_1,\dots,I_{2t}\in\Lambda_d^-,\\ 2n=\sum_i |I_i|}} \chi^{I_1\dots I_{2t}}\Big\{\operatorname{Tr}_N(K_{I_1}K_{I_2}\cdots K_{I_{2t}}) + (e_{I_1}\cdots e_{I_{2t}})\operatorname{Tr}_N(K_{I_{2t}}K_{I_{2t-1}}\cdots K_{I_1})\Big\}. \tag{3.12}$$

Proof. For any given collection of multi-indices $I_1,\dots,I_{2t}$ one selects in Proposition 3.2 the two sets $\Upsilon = \emptyset$ and $\Upsilon = \{1,\dots,2t\}$ which correspond with the first and second summands between curly brackets. The overall N-factor corresponds to $\operatorname{Tr}_N(1_N)$. Clearly any other subset Υ has no factor of N since is of the form

$$\operatorname{Tr}_N\Big(\overrightarrow{\prod_{i\in\{1,\dots,2t\}\setminus\Upsilon}} K_{I_i}\Big)\times\operatorname{Tr}_N\Big(\overleftarrow{\prod_{i\in\Upsilon}} K_{I_i}\Big),$$

where none of the products is empty. The arrows indicate the order in which the product is performed ($\rightarrow$ preserves it and $\leftarrow$ inverts it, but this is irrelevant to the point of this corollary). Therefore no trace of 1_N appears. One easily arrives then to equation (3.12) by excluding from $\mathfrak{a}_n(\chi)$ all the non-trivial sets, that is $\Upsilon, \Upsilon^c \neq \emptyset$. ∎

3.5. Formula for Tr D^2 in any dimension and signature

We evaluate in this section $\operatorname{Tr}(D^2)$ for Dirac operators D of a fuzzy geometry in any signature (p, q).

Proposition 3.5. *The Dirac operator of a fuzzy geometry of signature (p, q) satisfies for odd $d = p + q$*

$$\frac{1}{\dim V}\operatorname{Tr}\big([D^{(p,q)}]^2\big) = 2\sum_{I\in\Lambda_d}(-1)^{u(I)+\binom{|I|}{2}}\big[N\cdot\operatorname{Tr}_N(K_I^2) + e_I(\operatorname{Tr}_N K_I)^2\big],$$

being $u(I)$ *the number of spatial indices in* I. *If* d *is even, then the sum is only over* $I \in \Lambda_d^-$, *and the expression reads*

$$\frac{1}{\dim V} \operatorname{Tr}\big([D^{(p,q)}]^2\big) = 2 \sum_{I \in \Lambda_d^-} (-1)^{u(I)+r(I)-1} \big[N \cdot \operatorname{Tr}_N(K_I^2) + e_I (\operatorname{Tr}_N K_I)^2\big],$$

with $|I| = 2r(I) - 1$.

Proof. In order to use equation (3.7), notice that $\langle I_1 I_2 \rangle \neq 0$ implies that $I_1, I_2 \in \Lambda_d$ have the same cardinality. If that were not the case (without loss of generality $|I_1| > |I_2|$), since any non-zero term from $\langle I_1 I_2 \rangle$ arises from a contraction of indices (cf. equation (3.2)), a different number of indices would imply that there is a chord connecting two indices μ_i, μ_j of $I_1 = (\mu_1, \dots, \mu_r)$. Since $I_1 \in \Lambda_d$, those indices are different, so $g^{\mu_i \mu_j} = 0$. Thus, only chord diagrams for pairings $g^{\mu\nu}$ of indices with $\mu \in I_1$ and $\nu \in I_2$ survive. Since the indices of I_1 and I_2 are strictly increasing, both ordered sets have to agree. This means that we only have to care about evaluating $\langle I I' \rangle$, with $I' = (\mu_1', \dots, \mu_w')$ being a copy of $I = (\mu_1, \dots, \mu_w)$, i.e. $\mu_i' = \mu_i$. Since this last equality is the only possible index repetition

$$\langle \mu_1, \dots, \mu_w \mu_1', \dots, \mu_w' \rangle = \sum_{\substack{2w\text{-pt chord} \\ \text{diagrams } \chi}} (-1)^{\operatorname{cr}(\chi)} \prod_{\substack{i,j, \\ i \sim j}} g^{\mu_i \mu_j'} \delta_{ij} = (-1)^{\operatorname{cr}(\pi)} \prod_{\mu=1}^{w} g^{\mu\mu}$$

where π is the diagram with longest chords, that is joining antipodal points. The number of crossings $\operatorname{cr}(\pi)$ is $\binom{w}{2}$. An additional sign $(-1)^u$ comes from $\prod_{i=1}^{w} g^{\mu_i \mu_i}$, being $u \leq q$ the number of spatial indices in I, yielding

$$\langle I_1 I_2 \rangle = \delta_{I_2}^{I_1} (-1)^{u(I_1) + \binom{w}{2}}. \tag{3.13}$$

From equation (3.7) with $t = 1$ one has

$$\begin{aligned}
\frac{1}{\dim V} \operatorname{Tr}\big([D^{(p,q)}]^2\big) &= \sum_{I_1 I_2 \in \Lambda} \langle I_1 I_2 \rangle \operatorname{Tr}_{M_N(\mathbb{C})}(k_{I_1} k_{I_2}) \\
&= \sum_{I \in \Lambda} (-1)^{u(I) + \binom{|I|}{2}} \operatorname{Tr}_{\mathbf{N} \otimes \bar{\mathbf{N}}} \big[\big(K_I \otimes 1_N + e_I \otimes K_I^T\big)^2\big] \\
&= \sum_{I \in \Lambda} (-1)^{u(I) + \binom{|I|}{2}} \big[\operatorname{Tr}_N(K_I) \operatorname{Tr}_N(1_N) + \operatorname{Tr}_N(1_N) \operatorname{Tr}_N(K_I^T) \\
&\qquad + 2 e_I \operatorname{Tr}_N(K_I) \operatorname{Tr}_N(K_I^T)\big].
\end{aligned}$$

In the second equality we used equation (3.13). The third one follows from Proposition 3.1. For $p + q$ even, the sum runs only over $I \in \Lambda_d^-$. In the sign appearing in equation (3.13), $\binom{|I|}{2}$ could then be replaced by $r(I) - 1$ with $2r(I) - 1 = |I|$, for $\binom{|I|}{2} \equiv r - 1 \pmod 2$. ∎

4. Two-dimensional fuzzy geometries in general signature

We compute traces of D^2, D^4, D^6 for 2-dimensional fuzzy geometries general signatures. Concretely, for $d = 2$ the spinor space is $V = \mathbb{C}^2$.

4.1. Quadratic term

For a metric $g = \operatorname{diag}(e_1, e_2)$ notice that $e_\mu = (-1)^{u(\mu)}$. Therefore, by Proposition 3.5 one gets

$$\begin{aligned} \frac{1}{4} \operatorname{Tr}\big[(D^{(p,q)})^2\big] &= \sum_\mu (-1)^{u(\mu)} N \cdot \operatorname{Tr}_N(K_\mu^2) + \sum_\mu [\operatorname{Tr}_N(K_\mu)]^2 \\ &= N \sum_{\mu,\nu} g^{\mu\nu} \operatorname{Tr}_N(K_\mu K_\nu) + \sum_\mu [\operatorname{Tr}_N(K_\mu)]^2, \end{aligned} \tag{4.1}$$

where $u(\mu) = 0$ if μ is time-like and if its spatial, $u(\mu) = 1$. Case by case,

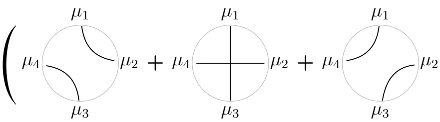
$$\begin{cases} \sum_{a=1}^2 \big(-N \cdot \operatorname{Tr}_N(L_{\dot a}^2) + [\operatorname{Tr}_N(L_{\dot a})]^2\big) & \text{for } (p,q) = (0,2), \\ N \cdot \operatorname{Tr}_N(H^2 - L^2) + [\operatorname{Tr}_N(H)]^2 + [\operatorname{Tr}_N(L)]^2 & \text{for } (p,q) = (1,1), \\ \sum_{a=1}^2 \big(+N \cdot \operatorname{Tr}_N(H_a^2) + [\operatorname{Tr}_N(H_a)]^2\big) & \text{for } (p,q) = (2,0), \end{cases}$$

reproducing some of the formulas reported in [8, App. A].

4.2. Quartic term

In [8, App. A] also the quartic term for $d = 2$ was computed. We recompute for a general $d = 2$ geometry of arbitrary signature aiming at illustrating the chord diagrams at work. Since $d = 2$, multi-indices $\mu \in \Lambda_2^-$ are just spacetime indices $\mu = 1, 2$. Hence, after Proposition 3.2,

$$\frac{1}{2} \operatorname{Tr}(D^4) = \sum_{\mu_1,\ldots,\mu_4 \in \Lambda_2^-} \Big(\text{[chord diagrams]} \Big)$$



$$\begin{aligned} \times \Big\{ & N \cdot \operatorname{Tr}_N(K_{\mu_1} K_{\mu_2} K_{\mu_3} K_{\mu_4}) \\ & + \sum_{i=1}^4 e_{\mu_i} \operatorname{Tr}_N(K_{\mu_1} \cdots \widehat{K_{\mu_i}} \cdots K_{\mu_4}) \operatorname{Tr}_N(K_{\mu_i}) \\ & + \sum_{1 \le i < j \le 4} e_{\mu_i} e_{\mu_i} \big[\operatorname{Tr}_N(K_{\mu_v} K_{\mu_w}) \operatorname{Tr}_N(K_{\mu_i} K_{\mu_j}) \big] \\ & + \sum_{i=1}^4 e \cdot e_{\mu_i} \cdot \operatorname{Tr}_N(K_{\mu_i}) \operatorname{Tr}_N(K_{\mu_4} \cdots \widehat{K_{\mu_i}} \cdots K_{\mu_1}) \\ & + e \cdot N \cdot \operatorname{Tr}_N(K_{\mu_4} K_{\mu_3} K_{\mu_2} K_{\mu_1}) \Big\}, \end{aligned} \tag{4.2}$$

with $e = e_{\mu_1}e_{\mu_2}e_{\mu_3}e_{\mu_4}$ and $\{i, j, v, w\} = \Delta_4$. In the first line, the value of the chord diagrams is $g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}$, the signs $e_\mu = \pm$ appearing in $g = \mathrm{diag}(e_1, e_2)$ being determined by (p, q). Summing over all indices, one gets

$$\begin{aligned}\frac{1}{4}\,\mathrm{Tr}\left([D^{(p,q)}]^4\right) = N\big[&\,\mathrm{Tr}_N(K_1^4) + \mathrm{Tr}_N(K_2^4)\\ &+ 4e_1e_2\,\mathrm{Tr}_N(K_1^2K_2^2) - 2e_1e_2\,\mathrm{Tr}_N(K_1K_2K_1K_2)\big]\\ &+ 4\Big\{[\mathrm{Tr}_N(K_1K_2)]^2 + \sum_{\mu=1,2}\mathrm{Tr}_N\,K_\mu\cdot\mathrm{Tr}_N\left[K_\nu(e_1K_1^2 + e_2K_2^2)\right]\Big\}\\ &+ 3\sum_{\mu=1,2}\left[\mathrm{Tr}_N(K_\mu^2)\right]^2 + 2e_1e_2\,\mathrm{Tr}_N(K_1^2)\cdot\mathrm{Tr}_N(K_2^2).\end{aligned} \tag{4.3}$$

One gets directly the results of [8, App. A.3, A.4, A.5] by setting

$$K_1 = \begin{cases} H_1 & \text{for } (p,q) = (2,0) \text{ and } (1,1),\\ L_{\dot 1} & \text{for } (p,q) = (0,2),\end{cases} \tag{4.4}$$

and

$$K_2 = \begin{cases} H_2 & \text{for } (p,q) = (2,0),\\ L_{\dot 2} & \text{for } (p,q) = (1,1) \text{ and } (0,2).\end{cases} \tag{4.5}$$

The conventions for which these hold are $(\gamma^\nu)^* = e_\nu\gamma^\nu$ (no sum, $\nu = 1, 2$).

4.3. Sextic term

We now compute the sixth-order term.

Proposition 4.1. *Let $g = \mathrm{diag}(e_1, e_2)$ denote the quadratic form associated to the signature (p, q) of a 2-dimensional fuzzy geometry with Dirac operator D. Then*

$$\frac{1}{2}\,\mathrm{Tr}(D^6) = N\cdot\mathcal{S}_6[K_1, K_2] + \mathcal{B}_6[K_1, K_2],$$

where the single-trace part is given by

$$\begin{aligned}\mathcal{S}_6[K_1,K_2] = 2\cdot\mathrm{Tr}_N\,\{&e_1K_1^6 + 6e_2K_1^4K_2^2 - 6e_2K_1^2(K_1K_2)^2 + 3e_2(K_1^2K_2)^2\\ &+ e_2K_2^6 + 6e_1K_2^4K_1^2 - 6e_1K_2^2(K_2K_1)^2 + 3e_1(K_2^2K_1)^2\}\end{aligned}$$

and the bi-trace part is

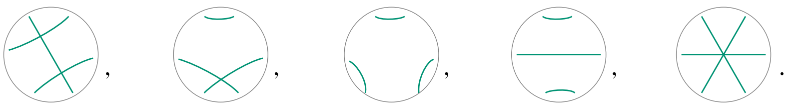
$$\begin{aligned}\mathcal{B}_6[K_1,K_2] = 6\,\mathrm{Tr}_N(K_1)\Big\{&2\,\mathrm{Tr}_N(K_1^5) + 2\,\mathrm{Tr}_N(K_1K_2^4)\\ &+ 6e_1e_2\,\mathrm{Tr}_N(K_1^3K_2^2) - 2e_1e_2\,\mathrm{Tr}_N(K_1^2K_2K_1K_2)]\Big\}\\ + 6\,\mathrm{Tr}_N(K_2)\Big\{&2\,\mathrm{Tr}_N(K_2^5) + 2\,\mathrm{Tr}_N(K_2K_1^4)\\ &+ 6e_1e_2\,\mathrm{Tr}_N(K_2^3K_1^2) - 2e_1e_2\,\mathrm{Tr}_N(K_2^2K_1K_2K_1)]\Big\}\\ + 48\,\mathrm{Tr}_N(K_1K_2)\cdot\big[&e_1\,\mathrm{Tr}_N(K_1^3K_2) + e_2\,\mathrm{Tr}_N(K_2^3K_1)\big]\end{aligned}$$

$$
\begin{aligned}
&+ 6\,\mathrm{Tr}_N(K_1^2)\cdot\Big\{e_2\big[8\,\mathrm{Tr}_N(K_1^2K_2^2) - 2\,\mathrm{Tr}_N(K_2K_1K_2K_1)\big] \\
&\qquad\qquad + e_1\big[5\,\mathrm{Tr}_N(K_1^4) + \mathrm{Tr}_N(K_2^4)\big]\Big\} \\
&+ 6\,\mathrm{Tr}_N(K_2^2)\cdot\Big\{e_1\big[8\,\mathrm{Tr}_N(K_1^2K_2^2) - 2\,\mathrm{Tr}_N(K_1K_2K_1K_2)\big] \\
&\qquad\qquad + e_2\big[5\,\mathrm{Tr}_N(K_2^4) + \mathrm{Tr}_N(K_1^4)\big]\Big\} \\
&+ 4\big(5[\mathrm{Tr}_N(K_1^3)]^2 + 6e_1e_2\,\mathrm{Tr}_N(K_1K_2^2)\,\mathrm{Tr}_N(K_1^3) + 9[\mathrm{Tr}_N\,K_1^2K_2]^2 \\
&\qquad + 5[\mathrm{Tr}_N(K_2^3)]^2 + 6e_1e_2\,\mathrm{Tr}_N(K_1^2K_2)\,\mathrm{Tr}_N(K_2^3) + 9[\mathrm{Tr}_N\,K_1K_2^2]^2\big).
\end{aligned}
$$

Proof. The part $\langle\mu_1\cdots\mu_6\rangle$ concerning the chord diagrams evaluates to

$$
\begin{aligned}
&(-1)^0g^{\mu_1\mu_2}g^{\mu_3\mu_4}g^{\mu_5\mu_6} + (-1)^1g^{\mu_1\mu_2}g^{\mu_3\mu_5}g^{\mu_4\mu_6} + (-1)^0g^{\mu_1\mu_2}g^{\mu_3\mu_6}g^{\mu_4\mu_5} \\
&\quad + (-1)^1g^{\mu_1\mu_3}g^{\mu_2\mu_4}g^{\mu_5\mu_6} + (-1)^2g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6} + (-1)^1g^{\mu_1\mu_3}g^{\mu_2\mu_6}g^{\mu_4\mu_5} \\
&\quad + (-1)^0g^{\mu_1\mu_4}g^{\mu_2\mu_3}g^{\mu_5\mu_6} + (-1)^3g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6} + (-1)^2g^{\mu_1\mu_4}g^{\mu_2\mu_6}g^{\mu_3\mu_5} \\
&\quad + (-1)^1g^{\mu_1\mu_5}g^{\mu_2\mu_3}g^{\mu_4\mu_6} + (-1)^2g^{\mu_1\mu_5}g^{\mu_2\mu_4}g^{\mu_3\mu_6} + (-1)^1g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_3\mu_4} \\
&\quad + (-1)^0g^{\mu_6\mu_1}g^{\mu_2\mu_3}g^{\mu_4\mu_5} + (-1)^1g^{\mu_6\mu_1}g^{\mu_2\mu_4}g^{\mu_3\mu_5} + (-1)^0g^{\mu_6\mu_1}g^{\mu_2\mu_5}g^{\mu_3\mu_4}
\end{aligned}
\tag{4.6}
$$

but it is actually useful to depict these terms as in Figure 1, for then, due to the cyclicity of Tr_N, one can compute by classes (modulo $\pi\mathbb{Z}_6/3$-rotations) of diagrams. To each class, a Roman number is assigned,



I, II, III, IV, V. (4.7)

One can relabel the μ_j-indices to obtain

$$
\frac{1}{\dim V}\,\mathrm{Tr}(D^6) = \sum_{\chi\in\mathrm{CD}_6}\mathfrak{a}(\chi) = 3\mathfrak{a}(\mathrm{I}) + 6\mathfrak{a}(\mathrm{II}) + 2\mathfrak{a}(\mathrm{III}) + 3\mathfrak{a}(\mathrm{IV}) + \mathfrak{a}(\mathrm{V}),
$$

the factors being the multiplicity of each diagram class. The single-trace part $\mathcal{S}_6$ can be computed for each diagram directly (in the Supplementary Material one of these is shown). We simplified the notation: $\mathfrak{a}_3$ as $\mathfrak{a}$, and similarly we shall write $\mathfrak{b}$ for $\mathfrak{b}_3$, since therein only 6-pt diagrams appear (a power $2t \geq 2$ of the Dirac operator determines the number of points of the chord diagrams only for dimensions $d \leq 2$).

We now compute the bi-trace term. Defining

$$O_{\mu\nu\rho} = e_\nu e_\rho\cdot\mathrm{Tr}_N(K_\mu)\cdot\mathrm{Tr}_N(K_\mu K_\rho K_\nu K_\rho K_\nu), \tag{4.8a}$$

$$P_{\mu\nu\rho} = e_\nu e_\rho\cdot\mathrm{Tr}_N(K_\mu)\cdot\mathrm{Tr}_N(K_\mu K_\rho K_\nu^2 K_\rho), \tag{4.8b}$$

$$Q_{\mu\nu\rho} = e_\nu e_\rho\cdot\mathrm{Tr}_N(K_\mu)\cdot\mathrm{Tr}_N(K_\mu K_\rho^2 K_\nu^2), \tag{4.8c}$$

$$R_{\mu\nu\rho} = e_\rho\cdot\mathrm{Tr}_N(K_\mu K_\nu)\cdot\mathrm{Tr}_N(K_\mu K_\nu K_\rho^2), \tag{4.8d}$$

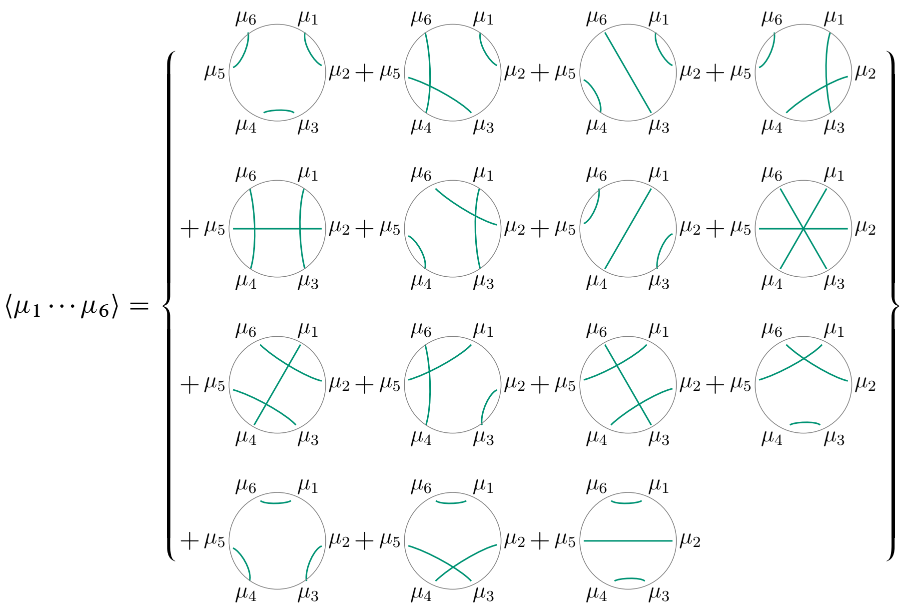


Figure 1. On the proof of Proposition 4.1.

$$S_{\mu\nu\rho} = e_\rho \cdot \mathrm{Tr}_N(K_\mu K_\nu) \cdot \mathrm{Tr}_N(K_\mu K_\rho K_\nu K_\rho), \tag{4.8e}$$

$$T_{\mu\nu\rho} = e_\mu e_\nu e_\rho \cdot \mathrm{Tr}_N(K_\mu^2) \cdot \mathrm{Tr}_N(K_\nu K_\rho K_\nu K_\rho), \tag{4.8f}$$

$$U_{\mu\nu\rho} = e_\mu e_\nu e_\rho \cdot \mathrm{Tr}_N(K_\mu^2) \cdot \mathrm{Tr}_N(K_\nu^2 K_\rho^2), \tag{4.8g}$$

$$V_{\mu\nu\rho} = [\mathrm{Tr}_N(K_\mu K_\nu K_\rho)]^2, \tag{4.8h}$$

$$W_{\mu\nu\rho} = e_\nu e_\rho \cdot \mathrm{Tr}_N(K_\mu K_\nu^2) \cdot \mathrm{Tr}_N(K_\mu K_\rho^2), \tag{4.8i}$$

we can find by direct computation, that for any of the 6-pt chord diagrams χ there are integers $p_\chi, q_\chi, \ldots, v_\chi, w_\chi$ such that

$$\mathfrak{b}(\chi) = \sum_{\mu,\nu,\rho} o_\chi O_{\mu\nu\rho} + p_\chi P_{\mu\nu\rho} + q_\chi Q_{\mu\nu\rho} \tag{4.9a}$$

$$+ r_\chi R_{\mu\nu\rho} + s_\chi S_{\mu\nu\rho} + t_\chi T_{\mu\nu\rho} + u_\chi U_{\mu\nu\rho} \tag{4.9b}$$

$$+ v_\chi V_{\mu\nu\rho} + w_\chi W_{\mu\nu\rho}. \tag{4.9c}$$

The terms O, P, Q come from the $(1,5)$ partition of 6, i.e.

$$\mathrm{Tr}_N(1 \text{ matrix}) \times \mathrm{Tr}_N(5 \text{ matrices});$$

R, S, T, U terms come from the $(2,4)$ partition and W, V from the $(3,3)$ partition of 6. This claim is verified by direct computation; the proof for $\mathfrak{b}(\mathrm{I})$ is presented in the

Supplementary Material and the rest is similarly obtained by

$$\mathfrak{b}(\mathrm{I}) = +2 \sum_{\mu,\nu,\rho} \big(4O_{\mu\nu\rho} + 2P_{\mu\nu\rho} + 6R_{\mu\nu\rho} + 6S_{\mu\nu\rho} + 2T_{\mu\nu\rho} + U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho}\big), \tag{4.10a}$$

$$\mathfrak{b}(\mathrm{II}) = -2 \sum_{\mu,\nu,\rho} \big(2O_{\mu\nu\rho} + 2P_{\mu\nu\rho} + 2Q_{\mu\nu\rho} + 8R_{\mu\nu\rho} + 4S_{\mu\nu\rho} + T_{\mu\nu\rho} + 2U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho}\big), \tag{4.10b}$$

$$\mathfrak{b}(\mathrm{III}) = +2 \sum_{\mu,\nu,\rho} \big(6Q_{\mu\nu\rho} + 12R_{\mu\nu\rho} + 3U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho}\big), \tag{4.10c}$$

$$\mathfrak{b}(\mathrm{IV}) = +2 \sum_{\mu,\nu,\rho} \big(2P_{\mu\nu\rho} + 4Q_{\mu\nu\rho} + 8R_{\mu\nu\rho} + 4S_{\mu\nu\rho} + 3U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho}\big), \tag{4.10d}$$

$$\mathfrak{b}(\mathrm{V}) = -2 \sum_{\mu,\nu,\rho} \big(6O_{\mu\nu\rho} + 6R_{\mu\nu\rho} + 6S_{\mu\nu\rho} + 3T_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho}\big). \tag{4.10e}$$

One then performs the sums explicitly and arrives at the claim for $\mathscr{B} = \sum_\chi \mathfrak{b}(\chi)$. ■

5. Four-dimensional geometries in general signature

We compute now the spectral action for 4-dimensional fuzzy geometries.

5.1. The term Tr D^2

For any four-dimensional geometry $p + q = 4$ of signature (p, q) there are eight matrices, K_1 ,K_2, K_3, K_4, X_1, X_2, X_3 and $X_4 \in M_N(\mathbb{C})$, parametrizing the Dirac operator

$$D^{(p,q)} = \sum_{\mu=1}^{4} \gamma^\mu \otimes k_\mu + \Gamma^{\hat{\mu}} \otimes x_\mu. \tag{5.1}$$

Here, the lower case operators on $M_N(\mathbb{C})$ are related to said matrices by

$$k_\mu = \{K_\mu, \bullet\}_{e_\mu} \qquad \text{and} \qquad x_\mu = \{X_\mu, \bullet\}_{e_{\hat{\mu}}}$$

where given a sign $\varepsilon = \pm$, the braces $\{A, B\}_\varepsilon = AB + \varepsilon BA$ represent a commutator or an anti-commutator. As before, $\Gamma^{\hat{1}} = \gamma^2\gamma^3\gamma^4$, $\Gamma^{\hat{2}} = \gamma^1\gamma^3\gamma^4$, etc., but in favor of a lighter notation we have replaced $K_{\hat{\mu}}$ by X_μ. The metric here is $g = \mathrm{diag}(e_1, e_2, e_3, e_4)$ and the spinor space is $V = \mathbb{C}^4$.

The numbers $u(\mu)$ and $u(\hat{\nu})$ of spatial subindices of each (multi-)index, μ and $\hat{\nu}$, can be written in terms of the signs e_μ and $e_{\hat{\nu}}$ that define the (anti-)hermiticity conditions—namely $(\gamma^\mu)^* = e_\mu\gamma^\mu$ and $(\gamma^{\hat{\nu}})^* = e_{\hat{\nu}}\gamma^{\hat{\nu}}$. First, trivially, $e_\mu = (-1)^{u(\mu)}$. On the other hand, since $u(\nu) + u(\hat{\nu})$ is the total number q of spatial indices, one has, by the Supplementary Material, Section II,

$$e_{\hat{\nu}} = (-1)^{u(\hat{\nu})+\lfloor 3/2\rfloor} = (-1)^{q+1+u(\nu)} = e_\nu(-1)^{q+1}.$$

Since the spinor space is four-dimensional, by Proposition 3.5 one has

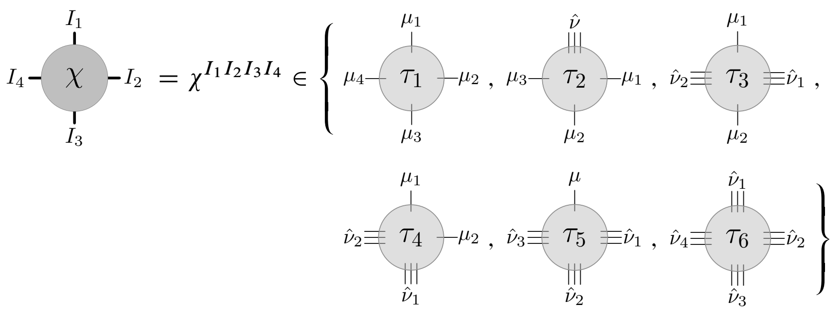
$$
\begin{aligned}
\frac{1}{2\cdot 4}\operatorname{Tr}\big[(D^{(p,q)})^2\big] &= \sum_{\mu=1}^{4}(-1)^{u(\mu)+\lfloor 1/2\rfloor}\big[N\cdot \operatorname{Tr}_N(K_\mu^2)+e_\mu(\operatorname{Tr}_N K_\mu)^2\big] \\
&\quad + \sum_{\nu=1}^{4}(-1)^{u(\hat\nu)+\lfloor 3/2\rfloor}\big[N\cdot \operatorname{Tr}_N(X_\nu^2)+e_{\hat\nu}(\operatorname{Tr}_N X_\nu)^2\big] \\
&= \sum_{\mu=1}^{4} e_\mu N\cdot \operatorname{Tr}_N\big[K_\mu^2+(-1)^{q+1}X_\mu^2\big] \\
&\quad + (\operatorname{Tr}_N K_\mu)^2+(\operatorname{Tr}_N X_\mu)^2. \qquad (5.2)
\end{aligned}
$$

In Section III in the Supplementary Material we specialize equation (5.2) to fuzzy Riemannian and Lorentzian geometries. Before, it will be useful to obtain the quartic term in order to integrate it with the quadratic one.

5.2. The term Tr D^4

To access $\operatorname{Tr}(D^4)$ we now detect the non-vanishing chord diagrams.

5.2.1. Non-vanishing chord diagrams. In four dimensions, chord diagrams of various number of points ($2n = 4, 6, 8, 10, 12$) have to be computed to access $\operatorname{Tr}(D^4)$. Next proposition helps to see the only non-trivial diagrams and requires some new notation. With each multi-index I_i running over eight values $I_i = \mu, \hat\nu$ ($\mu, \nu = \Delta_4$), the 8^4 decorations for the tensor $\chi^{I_1 I_2 I_3 I_4}$ fall into the following τ-types:



(5.3)

The leftmost diagram χ is of generic type. On the other hand, not only do the diagrams in the list indicate the number of points (the total number of bars transversal to the circle), they also state how these are grouped: normal indices $\mu_i = 1, \dots, 4$ being a single line and multiple $\hat\nu_i$ a triple line. Although they are in fact ordinary chord diagrams, they cannot have contractions between the grouped lines due to the strict increasing ordering of their indices.

If a diagram χ accepts a decoration of the type τ_i in the left-hand side of (5.3), up to rotation, we symbolically write $\chi \in \tau_i$. In the τ-types of the right-hand side, however, I_1 corresponds strictly to the upper index of the respective diagram in the list, I_2 to the rightmost, and so on clockwise. One can sum over the τ_i classes—since we are interested in products of $\chi^{I_1\cdots I_4}$ with traces (which are cyclic) and products of two traces (which are summed over all the subsets of $\{1,2,3,4\}$, see equation (3.8) for details)—and in order to do so, one has to include symmetry factors, namely $\{1,4,2,4,4,1\}$ in that order. All the chord diagrams contributing to $\mathrm{Tr}(D^4)$ in $d=4$ are then covered by

$$\Big(\sum_{\chi\in\tau_4} + 4\sum_{\chi\in\tau_2} + 2\sum_{\chi\in\tau_3} + 4\sum_{\chi\in\tau_4} + 4\sum_{\chi\in\tau_5} + \sum_{\chi\in\tau_6}\Big). \tag{5.4}$$

A cross check is that the symmetry factors add up to 16 and, since each (multi-)index in the list (5.3) can take four values, the number of all diagram index decorations is $4^4 \times 16 = 8^4$. Which of them survives is shown next.

Proposition 5.1. *Let $g = \mathrm{diag}(e_1, e_2, e_3, e_4)$ denote the quadratic form given by the signature (p, q). For any $\mu, \nu, \mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2, \nu_3, \nu_4 = 1, \ldots, 4$ the following holds for each one of the diagrams χ of the type τ_i—defined by equation (5.3)—indicated to the right of each equation:*

$$\chi^{\mu_1\mu_2\mu_3\mu_4} = e_{\mu_1}e_{\mu_3}\delta^{\mu_2}_{\mu_1}\delta^{\mu_4}_{\mu_3} - e_{\mu_1}e_{\mu_2}\delta^{\mu_3}_{\mu_1}\delta^{\mu_4}_{\mu_2} + e_{\mu_1}e_{\mu_3}\delta^{\mu_4}_{\mu_1}\delta^{\mu_3}_{\mu_2}, \tag{τ_1}$$

$$\chi^{\hat\nu\mu_1\mu_2\mu_3} = (-1)^{|\sigma|+1}e_{\mu_1}e_{\mu_2}e_{\mu_3}\delta_{\nu\mu_1\mu_2\mu_3}, \tag{τ_2}$$

where $\sigma = \sigma(\boldsymbol{\mu}, \nu) := \binom{\alpha_1\alpha_2\alpha_3}{\mu_1\mu_2\mu_3} \in \mathrm{Sym}\{\alpha_1, \alpha_2, \alpha_3\}$, with $\hat\nu = (\alpha_1, \alpha_2, \alpha_3)$ ordered as $\alpha_1 < \alpha_2 < \alpha_3$. Also

$$\delta_{\alpha\mu\nu\rho} = \begin{cases} 1 & \text{when } \{\alpha,\mu,\nu,\rho\} = \Delta_4, \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

(i.e. $\delta_{\alpha\mu\nu\rho}$ is the Levi-Civita symbol in absolute value). Whenever not all the four indices $\mu_1, \mu_2, \nu_1, \nu_2$ agree,

$$\chi^{\mu_1\hat\nu_1\mu_2\hat\nu_2} = -(-1)^{\mu_1+\mu_2}e_1e_2e_3e_4(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2}) \mp e_{\mu_1}\Big(\prod_{\alpha\neq\nu_1} e_\alpha\Big)\delta^{\mu_2}_{\mu_1}\delta^{\nu_2}_{\nu_1}, \tag{τ_3}$$

$$\chi^{\mu_1\mu_2\hat\nu_1\hat\nu_2} = +(-1)^{\mu_1+\mu_2}e_1e_2e_3e_4(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2}) \mp e_{\mu_1}\Big(\prod_{\alpha\neq\nu_1} e_\alpha\Big)\delta^{\mu_2}_{\mu_1}\delta^{\nu_2}_{\nu_1}, \tag{τ_4}$$

(see below for the sign choice). Otherwise these two diagrams satisfy $\chi^{\mu\hat\mu\mu\hat\mu} = e_1e_2e_3e_4$ and $\chi^{\mu\mu\hat\mu\hat\mu} = -e_1e_2e_3e_4$. Moreover, letting $\sigma = \sigma(\boldsymbol{\nu}, \mu) = \binom{\theta_1\theta_2\theta_3}{\nu_1\nu_2\nu_3} \in \mathrm{Sym}\{\theta_1, \theta_2, \theta_3\}$, with $\hat\nu = (\theta_1, \theta_2, \theta_3)$ ordered as $\theta_1 < \theta_2 < \theta_3$, one has

$$\chi^{\mu\hat\nu_1\hat\nu_2\hat\nu_3} = (-1)^{|\sigma|+1}e_{\nu_1}e_{\nu_2}e_{\nu_3}\delta_{\mu\nu_1\nu_2\nu_3}, \tag{τ_5}$$

and, finally, if $\chi \in \tau_6$

$$\chi^{\hat\nu_1\hat\nu_2\hat\nu_3\hat\nu_4} = \pm\big[e_{\nu_1}e_{\nu_3}\delta^{\nu_2}_{\nu_1}\delta^{\nu_4}_{\nu_3} - e_{\nu_1}e_{\nu_2}\delta^{\nu_3}_{\nu_1}\delta^{\nu_4}_{\nu_2} + e_{\nu_1}e_{\nu_3}\delta^{\nu_4}_{\nu_1}\delta^{\nu_3}_{\nu_2}\big]. \tag{τ_6}$$

The upper signs in equations (τ_3), (τ_4) and (τ_6) are taken if χ has minimal crossings.

Proof. See Supplementary Material. ∎

The minimality condition on the crossings, assumed for the $\tau_{3,4,6}$ classes, is meant to shorten the proof. Exactly for those classes, the spacetime indices do not necessarily determine a unique diagram by assuming that it does not vanish. This requirement can be left out, and in that case equation (τ_6) should have a global sign $\pm$ that depends on the diagram; in the $\tau_{3,4}$ cases (τ_3) and (τ_4) the term $e_{\mu_1}\big(\prod_{\alpha\neq\nu_1} e_\alpha\big)\delta^{\mu_2}_{\mu_1}\delta^{\nu_2}_{\nu_1}$ would undergo a diagram-dependent sign change. However, as we will see, these will be 'effectively' replaced by the minimal-crossing diagram, so the simplified claim suffices.

This is based on the following Lemma, where we assume that any of the chord crossings of each diagram is transversal.

Lemma 5.2 ([1]). *For any $n \in \mathbb{N}$, the sum of the crossing parities of all diagrams with n-chords is* 1,

$$\sum_{\chi\in \mathrm{CD}_{2n}} (-1)^{\#\{\textit{crossings of } \chi\}} = 1. \tag{5.6}$$

The proof by Aizenman–Warzel given in [1, Lemma 4.4], can be restated in terms of chord diagrams:

Proof. By induction in n. Since CD_2 consists of a single chord, there are no crossings and the sum (5.6) equals indeed $(-1)^0$. We take (5.6) as induction hypothesis and prove $\sum_{\chi\in \mathrm{CD}_{2n+2}}(-1)^{\#\{\text{crossings of } \chi\}} = 1$. For $a \in \{2,3,\ldots,2(n+1)\}$, denote by $\chi_{\hat a} \in \mathrm{CD}_n$ the diagram obtained from χ after removing the chord $(1a)$; see Figure 2. We can thus split CD_{2n+2} in $2n+1$ copies $\{\mathrm{CD}_{2n;a}\}_a$ of CD_{2n}, where $\mathrm{CD}_{2n;a}$ is the image of $\chi \mapsto \chi_{\hat a}$. Then

$$\sum_{\chi\in \mathrm{CD}_{2n+2}} (-1)^{\mathrm{cr}(\chi)} = \sum_{a=2}^{2n+2} \sum_{\chi_a\in \mathrm{CD}_{2n;a}} (-1)^{\mathrm{cr}(\chi_{\hat a})}(-1)^{\#\{\text{crossings between } \chi_{\hat a} \text{ and } (1a)\}} \tag{5.7}$$

where we recall that $\mathrm{cr}(\xi) = \#\{\text{crossings of (all chords of) } \xi\}$ for any chord diagram ξ. To determine the second sign, notice that the crossings between the chord diagram χ and the chord $(1a)$ have all parity $(-1)^a$. Therefore,

$$\begin{aligned}\sum_{\chi\in \mathrm{CD}_{2n+2}} (-1)^{\mathrm{cr}(\chi)} &= \sum_{a=2}^{2n+2} (-1)^a \sum_{\chi_a\in \mathrm{CD}_{2n;a}} (-1)^{\mathrm{cr}(\chi_{\hat a})} \\ &= \sum_{a=2}^{2n+2} (-1)^a = 1 - 1 + 1 - \cdots + 1 = 1.\end{aligned}$$

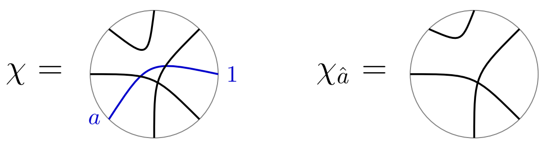


Figure 2. The parity of the crossings respects $(-1)^a$ (independent from the particular χ).

In the line change we used the induction hypothesis $\sum_{\chi_a \in \mathrm{CD}_{2n;a}} (-1)^{\mathrm{cr}(\chi_{\hat{a}})}$ for each copy $\mathrm{CD}_{2n;a}$ of CD_{2n}. ∎

As a last piece of preparation, we need to determine the signs $e_{I_1}e_{I_2}e_{I_3}e_{I_4}$ for each τ_i-type. These turn out to be constant and fully determined by the τ_i-type.

Claim 5.3. *Assuming that for $I_1, I_2, I_3, I_4 \in \Lambda^-_{d=4}$ the tensor $\chi^{I_1 I_2 I_3 I_4}$ does not vanish, then $e_{I_1}e_{I_2}e_{I_3}e_{I_4}$ reads in each case*

$$e_{\mu_1}e_{\mu_2}e_{\mu_3}e_{\mu_4} \equiv +1, \tag{5.8}$$

$$e_{\hat{\nu}}e_{\mu_1}e_{\mu_2}e_{\mu_3} \equiv -1, \tag{5.9}$$

$$e_{\mu_1}e_{\mu_2}e_{\hat{\nu}_1}e_{\hat{\nu}_2} \equiv +1, \tag{5.10}$$

$$e_{\mu}e_{\hat{\nu}_1}e_{\hat{\nu}_2}e_{\hat{\nu}_3} \equiv -1, \tag{5.11}$$

$$e_{\hat{\nu}_1}e_{\hat{\nu}_2}e_{\hat{\nu}_3}e_{\hat{\nu}_4} \equiv +1. \tag{5.12}$$

Proof. See Supplementary Material. ∎

5.2.2. Main claim. With help of these two results, we state the main one. We recall that the definition of the permutation $\sigma(\boldsymbol{\nu}, \mu)$, appearing next, is given in equation (5.5).

Proposition 5.4. *For a 4-dimensional fuzzy geometry of signature (p, q), the purely quartic spectral action $\frac{1}{4}\operatorname{Tr}(D^4) = N\mathcal{S}_4 + \mathcal{B}_4$ is given by*

$$\begin{aligned}
\mathcal{S}_4 = \operatorname{Tr}_N \Big\{ & 2\sum_{\mu} K_\mu^4 + X_\mu^4 \\
& + 4\sum_{\mu<\nu} e_\mu e_\nu \big(2K_\mu^2 K_\nu^2 + 2X_\mu^2 X_\nu^2 - K_\mu K_\nu K_\mu K_\nu - X_\mu X_\nu X_\mu X_\nu\big) \\
& - \sum_{\alpha,\beta,\mu,\nu} \delta_{\alpha\beta\mu\nu} e_\alpha e_\beta \big[(K_\mu X_\nu)^2 + 2K_\mu^2 X_\nu^2\big] + 2(-1)^q \sum_\mu \big[(K_\mu X_\mu)^2 - 2K_\mu^2 X_\mu^2\big] \\
& + 8(-1)^{q+1} \sum_{\mu,\boldsymbol{\nu}} (-1)^{|\sigma(\boldsymbol{\nu},\mu)|} \delta_{\mu\nu_1\nu_2\nu_3} e_\mu \big(X_\mu K_{\nu_1} K_{\nu_2} K_{\nu_3} + K_\mu X_{\nu_1} X_{\nu_2} X_{\nu_3}\big) \Big\},
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
\mathcal{B}_4 = {} & 8\sum_{\mu,\nu} (-1)^{q+1} e_\nu \operatorname{Tr}_N X_\mu \cdot \operatorname{Tr}_N(X_\mu X_\nu^2) + e_\nu \operatorname{Tr}_N(K_\mu) \cdot \operatorname{Tr}_N(K_\mu K_\nu^2) \\
& + \sum_{\mu,\nu=1}^{4} \Big\{ 2\operatorname{Tr}_N(X_\mu^2) \cdot \operatorname{Tr}_N(X_\nu^2) + 4e_\mu e_\nu \big[\operatorname{Tr}_N(X_\mu X_\nu)\big]^2 \Big\} \\
& + \sum_{\mu,\nu=1}^{4} \Big\{ 2\operatorname{Tr}_N(K_\mu^2) \cdot \operatorname{Tr}_N(K_\nu^2) + 4e_\mu e_\nu \big[\operatorname{Tr}_N(K_\mu K_\nu)\big]^2 \Big\}
\end{aligned}$$

$$
\begin{aligned}
&+4\sum_{\mu=1}^{4}\Big\{2(-1)^{1+q}e_\mu \operatorname{Tr}_N(K_\mu)\cdot\operatorname{Tr}_N(K_\mu X_\mu^2)+2e_\mu\operatorname{Tr}_N(X_\mu)\cdot\operatorname{Tr}_N(X_\mu K_\mu^2)\\
&\qquad\quad+(-1)^{1+q}\operatorname{Tr}_N(X_\mu^2)\cdot\operatorname{Tr}_N(K_\mu^2)+2\big[\operatorname{Tr}_N(K_\mu X_\mu)\big]^2\Big\}\\
&-8\sum_{\nu,\boldsymbol{\mu}=1}^{4}(-1)^{|\sigma(\boldsymbol{\mu},\nu)|}\delta_{\nu\mu_1\mu_2\mu_3}\cdot\big\{-\operatorname{Tr}_N(X_\nu)\cdot\operatorname{Tr}_N(K_{\mu_1}K_{\mu_2}K_{\mu_3})\\
&\qquad\quad+e_{\mu_2}e_{\mu_3}\big(\operatorname{Tr}_N K_{\mu_1}\cdot\operatorname{Tr}_N(X_\nu K_{\mu_2}K_{\mu_3})\big)\\
&\qquad\quad+\operatorname{Tr}_N X_{\mu_1}\cdot\operatorname{Tr}_N(K_\nu X_{\mu_2}X_{\mu_3})+(-1)^q\operatorname{Tr}_N K_\nu\cdot\operatorname{Tr}_N(X_{\mu_1}X_{\mu_2}X_{\mu_3})\big\}\\
&+24\sum_{\mu\neq\nu=1}^{4}(-1)^{1+q}e_\nu\operatorname{Tr}_N(K_\mu)\cdot\operatorname{Tr}_N\left(K_\mu X_\nu^2\right)+e_\mu\operatorname{Tr}_N(X_\nu)\cdot\operatorname{Tr}_N\left(K_\mu^2 X_\nu\right)\\
&+12\sum_{\mu\neq\nu}\Big\{2\big[\operatorname{Tr}_N(K_\mu X_\nu)\big]^2+e_\mu e_\nu(-1)^{q+1}\operatorname{Tr}_N(K_\mu^2)\cdot\operatorname{Tr}_N(X_\nu^2)\Big\}. \qquad (5.14)
\end{aligned}
$$

The eight matrices K_μ, X_μ satisfy the following (anti-)hermiticity conditions:

$$
K_\mu^* = e_\mu K_\mu \qquad \text{and} \qquad X_\mu^* = e_\mu(-1)^{q+1}X_\mu \qquad \text{for any } \mu\in\Delta_4, \qquad (5.15)
$$

where each $e_\mu \in \{+1,-1\}$ is determined by $g = \operatorname{diag}(e_1,e_2,e_3,e_4)$.

Proof. We first find $\mathcal{S}_4 = \sum_{n=2}^{6}\mathfrak{s}_n(\chi)$ using equations (5.3) and (5.4). By direct computation

$$
\begin{aligned}
\sum_{\chi\in\mathrm{CD}_2}\mathfrak{s}_2(\chi) &= 2\sum_\mu \operatorname{Tr}_N(K_\mu^4)+8\sum_{\mu<\nu}e_\mu e_\nu\operatorname{Tr}_N(K_\mu^2K_\nu^2)\\
&\quad-4\sum_{\mu<\nu}e_\mu e_\nu\operatorname{Tr}_N(K_\mu K_\nu K_\mu K_\nu). \qquad (5.16)
\end{aligned}
$$

In view of (5.8) and (5.12) and the similarity of the τ_1 and τ_6 type diagrams, one gets the same result by replacing K_μ by $K_{\hat\mu} = X_\mu$, namely

$$
\begin{aligned}
\sum_{\chi\in\mathrm{CD}_6}\mathfrak{s}_6(\chi) &= 2\sum_\mu \operatorname{Tr}_N(X_\mu^4)+8\sum_{\mu<\nu}e_\mu e_\nu\operatorname{Tr}_N(X_\mu^2X_\nu^2)\\
&\quad-4\sum_{\mu<\nu}e_\mu e_\nu\operatorname{Tr}_N(X_\mu X_\nu X_\mu X_\nu). \qquad (5.17)
\end{aligned}
$$

Next, using equation (5.9), the 6-pt diagrams are evaluated,

$$
\begin{aligned}
\sum_{\chi\in\mathrm{CD}_3}\mathfrak{s}_3(\chi) &= 4\sum_{\nu,\boldsymbol{\mu}}\operatorname{Tr}_N\big[\delta_{\nu\mu_1\mu_2\mu_3}e_{\mu_1}e_{\mu_2}e_{\mu_3}(-1)^{1+|\sigma_{\boldsymbol{\mu}}|}\\
&\qquad\qquad\cdot(X_\nu K_{\mu_1}K_{\mu_2}K_{\mu_3}-K_{\mu_3}K_{\mu_2}K_{\mu_1}X_\nu)\big]\\
&= -8\sum_{\nu,\boldsymbol{\mu}}\operatorname{Tr}_N\big[\delta_{\nu\mu_1\mu_2\mu_3}e_{\mu_1}e_{\mu_2}e_{\mu_3}(-1)^{|\sigma_{\boldsymbol{\mu}}|}X_\nu K_{\mu_1}K_{\mu_2}K_{\mu_3}\big]\\
&= 8(-1)^{1+q}\sum_{\nu,\boldsymbol{\mu}}\operatorname{Tr}_N\big[\delta_{\nu\mu_1\mu_2\mu_3}e_\nu(-1)^{|\sigma_{\boldsymbol{\mu}}|}X_\nu K_{\mu_1}K_{\mu_2}K_{\mu_3}\big]. \qquad (5.18)
\end{aligned}
$$

Here, again using the duality between τ_2 and τ_5 evident in Proposition 5.1 and Claim 5.3, the $\mathfrak{s}_5$ term can be computed by swapping each K_μ matrix with the $K_{\hat{\mu}}$ matrix,

$$\sum_{\chi\in \mathrm{CD}_5} \mathfrak{s}_5(\chi) = \sum_{\chi\in \mathrm{CD}_3} \mathfrak{s}_3(\chi)\Big|_{K_\nu \leftrightarrow X_\nu \text{ for all } \nu=1,2,3,4} \tag{5.19}$$

(but there is no sign swap $e_\mu \leftrightarrow e_{\hat{\mu}}$).

Finally, we split the sum $\sum_{\chi\in \mathrm{CD}_4} = 2\sum_{\chi\in\tau_3} + 4\sum_{\chi\in\tau_4}$ in order to compute the term $\mathfrak{s}_4$. The calculation simplifies using equation (5.10) and noticing that (for $\mu_1 = \mu_2 = \nu_1 = \nu_2$ being false), one has

$$\begin{aligned}
&\sum_{\boldsymbol{\nu},\boldsymbol{\mu}} (-1)^{\mu_1+\mu_2} (\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2}) \big[\mathrm{Tr}_N(K_{\mu_1}X_{\nu_1}K_{\mu_2}X_{\nu_2}) \\
&\qquad + e_{\mu_1}e_{\mu_2}e_{\hat{\nu}_1}e_{\hat{\nu}_2} \mathrm{Tr}_N(X_{\nu_2}K_{\mu_2}X_{\nu_1}K_{\mu_1})\big] \\
&\quad = \sum_{\mu\neq\nu} (-1)^{\mu+\nu} \mathrm{Tr}_N \big\{ K_\mu X_\mu K_\nu X_\nu + X_\nu K_\nu X_\mu K_\mu \\
&\qquad - K_\mu X_\nu K_\nu X_\mu - X_\mu K_\nu X_\nu K_\mu \big\} = 0,
\end{aligned} \tag{5.20}$$

using the cyclicity of the trace. Therefore, in both equations (τ_3) and (τ_4) the only contribution to $\mathfrak{s}_4$ comes from the term $e_{\mu_1}\big(\prod_{\alpha\neq\nu} e_\alpha\big)\delta^{\mu_2}_{\mu_1}\delta^{\nu_2}_{\nu_1}$ (which require $\mu_i \neq \nu_i$) and from the terms $\chi^{\mu\hat{\mu}\mu\hat{\mu}}$ and $\chi^{\mu\mu\hat{\mu}\hat{\mu}}$. These terms appear, respectively, in the first and second lines of

$$\begin{aligned}
\sum_{\chi\in \mathrm{CD}_4} \mathfrak{s}_4(\chi) = &- \sum_{\alpha,\beta,\mu,\nu} \delta_{\alpha\beta\mu\nu} e_\alpha e_\beta \mathrm{Tr}_N \big[(K_\mu X_\nu)^2 + 2K_\mu^2 X_\nu^2\big] \\
&+ 2e_1e_2e_3e_4 \sum_\mu \mathrm{Tr}_N \big[(K_\mu X_\mu)^2 - 2K_\mu^2 X_\mu^2\big].
\end{aligned} \tag{5.21}$$

Expressing this via the delta $\delta_{\alpha\beta\mu\nu}$ is motivated by

$$e_\mu\Big(\prod_{\rho\neq\nu} e_\rho\Big) = \prod_{\substack{\rho\neq\mu,\\ \rho\neq\nu}} e_\rho.$$

We now compute in steps the bi-tracial functional

$$\begin{aligned}
\mathscr{B}_4 = \sum_{I\in(\Lambda_4^-)^{\times 4}} \Bigg\{ &\sum_{\chi\in \mathrm{CD}_{2n(I)}} \chi^{I_1I_2I_3I_4} \times \Big[\sum_{i=1}^4 e_{I_i} \mathrm{Tr}_N(K_{I_1}\cdots \widehat{K_{I_i}} \cdots K_{I_4}) \cdot \mathrm{Tr}_N K_{I_i} \\
&+ \sum_{1\leq i<j\leq 4} \sum_{v,w\neq i,j} e_{I_i}e_{I_j} \big(\mathrm{Tr}_N(K_{I_v}K_{I_w}) \mathrm{Tr}_N(K_{I_i}K_{I_j})\big) \\
&+ \sum_{i=1}^4 \Big(\prod_{j\neq i} e_{I_j}\Big) \mathrm{Tr}_N(K_{I_i}) \cdot \mathrm{Tr}_N(K_{I_4}\cdots \widehat{K_{I_i}} \cdots K_{I_1})\Big]\Bigg\}.
\end{aligned} \tag{5.22}$$

The contribution to $\mathscr{B}_4$ arising from the term in the square brackets in the first, second and third lines are referred to as the $(1,3)$, $(2,2)$ and $(3,1)$ partitions, respectively. For a fixed number $2r$ of points, these are denoted by $\sum_\chi \mathfrak{b}_r^\pi(\chi)$, for $\pi \in \{(1,3),(2,2),(3,1)\}$. In view of the partial duality established in Proposition 5.1, we obtain the contributions to $\mathscr{B}_4$ by similarity; thus we first compute 12-pt and 4-pt diagrams together and later 6-pt and 10-pt diagrams. This duality would be perfect if both replacements $K_\nu \leftrightarrow K_{\hat{\nu}}(= X_\nu)$ and $e_\nu \leftrightarrow e_{\hat{\nu}}$ would swap the equations $(\tau_1 \leftrightarrow \tau_6)$ and $(\tau_2 \leftrightarrow \tau_5)$ in Proposition 5.1. However, $e_\nu \leftrightarrow e_{\hat{\nu}}$ is not needed for the swapping to hold.

We begin with the 12-pt diagrams for the $(1,3)$ and $(3,1)$ partitions. As consequence of Claim 5.3,

$$\begin{aligned}\sum_{\chi\in\tau_6} \mathfrak{b}_6^{(1,3)}(\chi) + \mathfrak{b}_6^{(3,1)}(\chi) &= \sum_{\nu_1,\dots,\nu_4=1}^{4} \chi^{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3\hat{\nu}_4}\big[e_{\hat{\nu}_1} \operatorname{Tr}_N X_{\nu_1} \cdot \operatorname{Tr}_N(X_{\nu_2}\{X_{\nu_3}, X_{\nu_4}\}) \\ &\qquad + \text{cyclic}\big] \\ &= 4\sum_{\mu,\nu} e_\mu e_\nu e_{\hat{\mu}} \operatorname{Tr}_N X_\mu \cdot \operatorname{Tr}_N(X_\mu\{X_\nu, X_\nu\}) \\ &= 8\sum_{\mu,\nu}(-1)^{q+1} e_\nu \operatorname{Tr}_N X_\mu \cdot \operatorname{Tr}_N(X_\mu X_\nu^2) \end{aligned} \tag{5.23}$$

after some simplification; the last equality follows from equation (II.2) from the Supplementary Material. The $(2,2)$ partition evaluates similarly to

$$\begin{aligned}\sum_{\chi\in\tau_6} \mathfrak{b}_6^{(2,2)}(\chi) &= 2\sum_{\mu,\nu}\{e_\mu e_\nu e_{\hat{\mu}}^2 \operatorname{Tr}(X_\mu^2)\cdot \operatorname{Tr}(X_\nu^2) + 2e_\mu e_\nu e_{\hat{\mu}} e_{\hat{\nu}} \operatorname{Tr}(X_\mu X_\nu)^2\} \\ &= \sum_{\mu,\nu=1}^{4} \Big\{2\operatorname{Tr}_N(X_\mu^2)\cdot\operatorname{Tr}_N(X_\nu^2) + 4e_\mu e_\nu\big[\operatorname{Tr}_N(X_\mu X_\nu)\big]^2\Big\}, \end{aligned} \tag{5.24}$$

since $e_\mu e_\nu e_{\hat{\mu}} e_{\hat{\nu}} = (-1)^{2(1+q)}$ by equation (II.2) from the Supplementary Material. One then computes $\sum_{\chi\in\tau_6} \mathfrak{b}_6(\chi)$ by summing equations (5.24) and (5.23).

The 4-pt diagrams contain ordinary indices and their computation is not illuminating. Since it moreover resembles that for the 12-pt diagrams we omit it and present the result

$$\begin{aligned}\sum_{\chi\in\tau_1} \mathfrak{b}_2(\chi) &= \sum_{\mu,\nu=1}^{4} 8e_\nu \operatorname{Tr}_N(K_\mu)\cdot\operatorname{Tr}_N(K_\mu K_\nu^2) \\ &\quad + \sum_{\mu,\nu=1}^{4}\Big\{2\operatorname{Tr}_N(K_\mu^2)\cdot\operatorname{Tr}_N(K_\nu^2) + 4e_\mu e_\nu\big[\operatorname{Tr}_N(K_\mu K_\nu)\big]^2\Big\}. \end{aligned} \tag{5.25}$$

We now present the computation of 6-pt and 10-pt diagrams. Again, we remark that the terms corresponding to the $(1,3)$ and $(3,1)$ partitions agree, $\sum_\chi \mathfrak{b}_3^{(1,3)}(\chi) = \sum_\chi \mathfrak{b}_3^{(3,1)}(\chi)$.

In order to see this, first we notice that $\sum_\chi \mathfrak{b}_3^{(1,3)}(\chi)$ equals

$$4\sum_{\nu,\boldsymbol{\mu}}(-1)^{1+|\sigma|}\delta_{\nu\mu_1\mu_2\mu_3}\cdot\big\{-\mathrm{Tr}_N\, X_\nu\cdot\mathrm{Tr}_N(K_{\mu_1}K_{\mu_2}K_{\mu_3}) + e_{\mu_2}e_{\mu_3}\,\mathrm{Tr}_N\, K_{\mu_1}\cdot\mathrm{Tr}_N(X_\nu K_{\mu_2}K_{\mu_3}) + e_{\mu_1}e_{\mu_3}\,\mathrm{Tr}_N\, K_{\mu_2}\cdot\mathrm{Tr}_N(X_\nu K_{\mu_1}K_{\mu_3}) + e_{\mu_1}e_{\mu_2}\,\mathrm{Tr}_N\, K_{\mu_3}\cdot\mathrm{Tr}_N(X_\nu K_{\mu_1}K_{\mu_2})\big\},$$

due to equation (τ_2) and $e_{\hat\nu}e_{\mu_1}e_{\mu_2}e_{\mu_3}=-1$ (see Claim 5.3). But also departing from $\sum_\chi \mathfrak{b}_3^{(3,1)}(\chi)$, using (5.9) to convert the triple signs to a single one (e.g. $e_{\hat\nu}e_{\mu_1}e_{\mu_3}=-e_{\mu_2}$), renaming indices (which gets rid of the minus sign via the skew-symmetric factor $(-1)^{1+|\sigma|}$) one gets to the same expression. Thus, $\sum_\chi \mathfrak{b}_3^{(1,3)}(\chi)+\sum_\chi \mathfrak{b}_3^{(3,1)}(\chi)$ equals

$$8\sum_{\nu,\boldsymbol{\mu}}(-1)^{1+|\sigma|}\cdot\big\{-\mathrm{Tr}_N\, X_\nu\cdot\mathrm{Tr}_N(K_{\mu_1}K_{\mu_2}K_{\mu_3})+e_{\mu_2}e_{\mu_3}\,\mathrm{Tr}_N\, K_{\mu_1}\cdot\mathrm{Tr}_N(X_\nu K_{\mu_2}K_{\mu_3})\big\}.$$

Using the skew-symmetry of $(-1)^{1+|\sigma|}$ and the cyclicity of the trace, one proves easily that the $(2,2)$ partition $\mathfrak{b}_3^{(2,2)}$ vanishes, and so does in fact $\mathfrak{b}_5^{(2,2)}$. Thus, the only contributions from 10-pt diagrams are the partitions $(1,3)$ and $(3,1)$ which can be computed similarly as for the 6-pt contributions, by a similar token. Thus

$$\begin{aligned}\sum_{\chi\in\mathrm{CD}_{10}}\mathfrak{b}_5(\chi) &= 2\sum_{\chi\in\tau_5}\mathfrak{b}_5^{(1,3)}(\chi)\\ &= -8\sum_{\mu,\nu_1,\nu_2,\nu_3}\delta_{\mu\nu_1\nu_2\nu_3}(-1)^{|\lambda_{\boldsymbol{\nu}}|}e_{\nu_1}e_{\nu_2}e_{\nu_3}\\ &\quad\times\big[e_\mu\,\mathrm{Tr}_N\, K_\mu\cdot\mathrm{Tr}_N(X_{\nu_1}X_{\nu_2}X_{\nu_3})+e_{\nu_1}\,\mathrm{Tr}_N\, X_{\nu_1}\cdot\mathrm{Tr}_N(K_\mu X_{\nu_2}X_{\nu_3})\\ &\quad+e_{\nu_2}\,\mathrm{Tr}_N\, X_{\nu_2}\cdot\mathrm{Tr}_N(K_\mu X_{\nu_1}X_{\nu_3})+e_{\nu_3}\,\mathrm{Tr}_N\, X_{\nu_3}\cdot\mathrm{Tr}_N(K_\mu X_{\nu_1}X_{\nu_2})\big].\end{aligned}$$

By performing the sum of the terms in the last line one sees that they cancel out due to the skew-symmetry of $(-1)^{|\lambda_{\boldsymbol{\nu}}|}$. The only contribution comes therefore from the two first terms in the square brackets, which are directly seen to yield

$$\sum_{\chi\in\mathrm{CD}_{10}}\mathfrak{b}_5(\chi)=-8\sum_{\mu,\nu_1,\nu_2,\nu_3}\delta_{\mu\nu_1\nu_2\nu_3}(-1)^{|\lambda_{\boldsymbol{\nu}}|}\times\big[(-1)^q\,\mathrm{Tr}_N\, K_\mu\cdot\mathrm{Tr}_N(X_{\nu_1}X_{\nu_2}X_{\nu_3}) + e_{\nu_2}e_{\nu_3}\,\mathrm{Tr}_N\, X_{\nu_1}\cdot\mathrm{Tr}_N(K_\mu X_{\nu_2}X_{\nu_3})\big].$$

Concerning the 8-pt diagrams,

$$\sum_{\chi\in\mathrm{CD}_8}\mathfrak{b}_4(\chi)=\sum_{\substack{\mu_1,\mu_2,\nu_1,\nu_2,\\ \text{not all equal}}}\big(2\chi^{\mu_1\hat\nu_1\mu_2\hat\nu_2}+4\chi^{\mu_1\mu_2\hat\nu_1\hat\nu_2}\big)\{\text{non-trivial partitions}\} + \sum_\mu(2\chi^{\mu\hat\mu\mu\hat\mu}+4\chi^{\mu\mu\hat\mu\hat\mu})\{\text{non-trivial partitions}\}. \quad (5.26)$$

The sum over the 8-pt chord diagrams is splitted in the τ_3 and τ_4 types with their symmetry factors; in each line these are, respectively, the two summands in parenthesis. Here 'non-trivial partitions' in curly brackets refers to (1, 3), (2, 2) and (3, 1). We call the second line $\mathbf{\Delta}$, for which straightforward computation yields

$$\mathbf{\Delta} = 4(-1)^{1+q} \sum_{\mu=1}^{4} \Big\{2e_\mu \operatorname{Tr}_N(K_\mu) \cdot \operatorname{Tr}_N(K_\mu X_\mu^2) + 2e_{\hat\mu} \operatorname{Tr}_N(X_\mu) \cdot \operatorname{Tr}_N(X_\mu K_\mu^2) + \operatorname{Tr}_N(X_\mu^2) \cdot \operatorname{Tr}_N(K_\mu^2) + 2(-1)^{1+q}\big[\operatorname{Tr}_N(K_\mu X_\mu)\big]^2\Big\} \tag{5.27}$$

by rewriting $e_1e_2e_3e_4 = (-1)^q$. We now compute the first line of equation (5.26) considering first only the τ_3 diagrams (the τ_4-type is addressed later). The sum (1, 3) + (3, 1) of partitions can be straightforwardly obtained,

$$\begin{aligned}
&2 \sum_{\substack{\mu_1,\mu_2,\nu_1,\nu_2,\\ \text{not all equal}}} \chi^{\mu_1\hat\nu_1\mu_2\hat\nu_2}\{(1,3) + (3,1) \text{ partitions}\}\\
&= 2 \sum_{\substack{\mu_1,\mu_2,\nu_1,\nu_2,\\ \text{not all equal}}} \Big[-(-1)^{\mu_1+\mu_2} e_1e_2e_3e_4(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2}) - e_{\mu_1}\Big(\prod_{\alpha\neq\nu_1} e_\alpha\Big)\delta^{\mu_2}_{\mu_1}\delta^{\nu_2}_{\nu_1}\Big]\\
&\quad\times\big[e_{\mu_1}\operatorname{Tr}_N(K_{\mu_1})\cdot\operatorname{Tr}_N\big(X_{\nu_1}\{K_{\mu_2},X_{\mu_2}\}\big) + e_{\mu_2}\operatorname{Tr}_N(K_{\mu_2})\cdot\operatorname{Tr}_N\big(K_{\mu_1}\{X_{\nu_1},X_{\mu_2}\}\big)\\
&\quad+ e_{\nu_1}\operatorname{Tr}_N(X_{\nu_1})\cdot\operatorname{Tr}_N\big(K_{\mu_1}\{K_{\mu_2},X_{\nu_2}\}\big) + e_{\nu_2}\operatorname{Tr}_N(X_{\nu_2})\cdot\operatorname{Tr}_N\big(K_{\mu_1}\{K_{\mu_2},X_{\nu_1}\}\big)\big]\\
&= -8\sum_{\mu\neq\nu}\Big(\prod_{\alpha\neq\nu} e_\alpha\Big)\big(\operatorname{Tr}_N(K_\mu)\cdot\operatorname{Tr}_N(K_\mu X_\nu^2) + e_\mu e_{\hat\nu}\operatorname{Tr}_N(X_\nu)\cdot\operatorname{Tr}_N\big(K_\mu^2 X_\nu\big)\big)\\
&= 8\sum_{\mu\neq\nu}(-1)^{1+q} e_\nu \operatorname{Tr}_N(K_\mu)\cdot\operatorname{Tr}_N\big(K_\mu X_\nu^2\big) + e_\mu\operatorname{Tr}_N(X_\nu)\cdot\operatorname{Tr}_N\big(K_\mu^2X_\nu\big).
\end{aligned} \tag{5.28}$$

In the first equality we just used the expression for $\chi^{\mu_1\hat\nu_1\mu_2\hat\nu_2}$. In order to obtain the second one, it can be shown that the terms proportional to $(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2})$ cancel out. Using equation (II.2) from the Supplementary Material one simplifies the signs to obtain the last equality. The condition of $\mu \neq \nu$ in the sum of the last equations reflects only the fact that the four indices cannot coincide (cf. assumptions in Proposition 5.1). The remaining partition reads

$$\begin{aligned}
&2 \sum_{\substack{\mu_1,\mu_2,\nu_1,\nu_2,\\ \text{not all equal}}} \chi^{\mu_1\hat\nu_1\mu_2\hat\nu_2}\times\{(2,2) \text{ partition}\}\\
&= -4\sum_{\mu\neq\nu}\Big(\prod_{\alpha\neq\nu} e_\alpha\Big)\big[2e_{\hat\nu}\operatorname{Tr}_N(K_\mu X_\nu)^2 + e_\mu\operatorname{Tr}_N(K_\mu^2)\cdot\operatorname{Tr}_N(X_\nu^2)\big]\\
&= +4\sum_{\mu\neq\nu}\big\{2\big[\operatorname{Tr}_N(K_\mu X_\nu)\big]^2 + e_\mu e_\nu(-1)^{q+1}\operatorname{Tr}_N(K_\mu^2)\cdot\operatorname{Tr}_N(X_\nu^2)\big\}.
\end{aligned} \tag{5.29}$$

Using a similar approach (which would be redundant here), one can similarly show that the contribution of the τ_4-diagrams is precisely twice that of τ_3, obtaining in total $3 \times$ [equations (5.28) + (5.29)] for the 8-pt diagrams. The claim follows from

$$\mathcal{S}_4 = \sum_{r=2}^{6} \sum_{\chi \in \mathrm{CD}_{2r}} \mathfrak{s}_r(\chi) \qquad \text{and} \qquad \mathcal{B}_4 = \sum_{\pi} \sum_{r=2}^{6} \sum_{\chi \in \mathrm{CD}_{2r}} \mathfrak{b}_r^{\pi}(\chi) \tag{5.30}$$

where π runs over the non-trivial partitions $\pi \in \{(1,3),(2,2),(3,1)\}$. ∎

Remark 5.5. Each anti-hermitian parametrizing matrix L can be replaced by a traceless one $L' = L - (\mathrm{Tr}_N\, L/N) \cdot 1_N$, since L appears in D only via anti-commutators; since $\mathrm{Tr}_N\, L$ is purely imaginary, L' is also anti-hermitian.

6. Large-N limit via free probability?

Random matrix theory provides an important class of *noncommutative probability spaces* that can be studied with free probability [43,57,58]. (The potentials for such random matrices have a single trace, but this is not essential: other models whose potentials feature two or more traces—called *trace polynomials*—are elsewhere considered in related contexts [12,15,47] in probability.) One could start with noncommutative self-adjoint polynomials $P \in \mathbb{R}\langle x_1, \dots, x_\kappa \rangle$, to wit, $P(x_1, \dots, x_\kappa) = P(x_1, \dots, x_\kappa)^*$, if each of the noncommutative variables x_i satisfies formal self-adjointness $x_i^* = x_i$. For instance, the following polynomials are self-adjoint:

$$P_2(x_1, \dots, x_\kappa) = x_1^2 + \dots + x_\kappa^2 + \frac{\lambda}{2} \sum_{i \neq j} x_i x_j, \tag{6.1a}$$

$$P_4(x_1, \dots, x_\kappa) = x_1^4 + \dots + x_\kappa^4 + \frac{1}{2} \sum_{i \neq j} \big(\lambda_1 x_i x_j x_i x_j + \lambda_2 x_i^2 x_j^2\big), \tag{6.1b}$$

being λ_i, λ real coupling constants. One can instead evaluate P in square matrices of size, say, N and define

$$\begin{aligned} \mathrm{d}\nu_N(\mathbb{X}^{(N)}) = \mathrm{d}\nu_N(X_1^{(N)}, \dots, X_\kappa^{(N)}) = C_N \cdot \mathrm{e}^{-N^2 \mathrm{Tr}_N [P(X_1^{(N)}, \dots, X_\kappa^{(N)})]} \\ \cdot \mathrm{d}\Lambda(X_1^{(N)}) \cdots \mathrm{d}\Lambda(X_\kappa^{(N)}) \end{aligned} \tag{6.2}$$

being C_N a normalization constant and Λ the Lebesgue measure

$$\mathrm{d}\Lambda(Y) = \prod_i \mathrm{d}Y_{ii} \prod_{i<j} \Re(\mathrm{d}Y_{ij}) \Im(\mathrm{d}Y_{ij}), \qquad Y \in M_N(\mathbb{C}).$$

The distributions φ_N defined by

$$\varphi_N(X_{j_1}^{(N)}, \dots, X_{j_k}^{(N)}) = \int \mathrm{Tr}_N(X_{j_1}^{(N)} \cdots X_{j_1}^{(N)}) \mathrm{d}\nu_N(\mathbb{X}^{(N)}) \tag{6.3}$$

are of interest in free probability. As shown in Sections 3, 4 and 5, we have developed a geometrically interesting way to produce noncommutative polynomials. Although these are not directly self-adjoint, self-adjointness is not essential in order for one to ponder the possible convergence of the measures they define. As far as the trace does not detect it, the weaker notion of *cyclic self-adjointness* suffices. This requirement is satisfied by polynomials in $P \in \mathbb{C}\langle Z_1, \ldots, Z_n\rangle$ that fulfill $\mathrm{Tr}_N[P^*(Z_1, \ldots, Z_n)] = \mathrm{Tr}_N[P(Z_1, \ldots, Z_n)]$, where $Z_1, \ldots, Z_n$ are either hermitian or anti-hermitian $N \times N$ matrices. In the Supplementary Material we provide a definition of this relaxed kind of self-adjointness (Definition V.1) that does not make reference to a matrix realization, but that concept will be clear from the next examples.

Example 6.1. Consider the formal adjoint P^* of the NC polynomial P given by

$$P(h, l_1, l_2, l_3) = l_1\{h(l_2l_3 - l_3l_2) + l_2(l_3h - hl_3) + l_3(hl_2 - l_2h)\}.$$

Recalling that the h's (resp. the l's) are hermitian (resp. anti-hermitian), one obtains

$$\begin{aligned}[P(h, l_1, l_2, l_3)]^* &= (h[l_2, l_3] + l_2[l_3, h] + l_3[h, l_2])^* l_1^* \\ &= \{(l_2l_3 - l_3l_2)h + (l_3h - hl_3)l_2 + (hl_2 - l_2h)l_3\}l_1 \\ &= (h[l_2, l_3] + l_2[l_3, h] + l_3[h, l_2])l_1, \end{aligned} \tag{6.4}$$

where $[\bullet, \bullet]$ is the commutator in $\mathbb{R}\langle h, l_1, l_2, l_3\rangle$. Clearly $P^* \neq P$, but up to the cyclic permutation $\sigma \in \mathbb{Z}/4\mathbb{Z}$ defined by bringing the letter l_1 from the last to the first position of each word, a bijection of words in P is established. Hence P is cyclic self-adjoint. On the other hand, take $\Psi(h, l_2, l_3) = (l_2l_3 - l_3l_2)h$. By a similar token, one sees that Ψ is cyclically anti-self-adjoint.

It would not be surprising that the spectral action $\mathrm{Tr}\, f(D)$ for fuzzy geometries (since it has to be real) in any dimension and allowed KO-dimension leads for any ordinary polynomial f to the type of NC polynomials we just introduced. Preliminarily, we verify this statement only for the explicit computations we performed in this article:

Corollary 6.2. *For the cases*

- *$d = 2$, in arbitrary signature and being f a sextic polynomial; and*
- *Riemannian and Lorentzian signatures ($d = 4$) being f quartic polynomial,*

the spectral action $\mathrm{Tr}\, f(D) = \dim V(N \cdot \mathcal{S}_f + \mathcal{B}_f)$ *for fuzzy geometries has the form*

$$\mathcal{S}_f = \mathrm{Tr}_N\, P \qquad \textit{and} \qquad \mathcal{B}_f = \sum_i \mathrm{Tr}_N\, \Phi_i \cdot \mathrm{Tr}_N\, \Psi_i, \tag{6.5}$$

where $P, \Phi_i, \Psi_i \in \mathbb{R}\langle x_1, \ldots, z_{\kappa(d)}\rangle$ with $\kappa(d) = 2^{d-1}$ are NC polynomials such that

- *P is cyclically self-adjoint*
- *and Φ_i and Ψ_i are both either cyclically self-adjoint or both cyclically anti-self-adjoint.*

7. Conclusions

We computed the spectral action $\operatorname{Tr} f(D)$ for fuzzy geometries of even dimension d, whose 'quantization' was stated as a 2^{d-1}-matrix model with action

$$\operatorname{Tr} f(D) = \dim V(N \cdot \mathcal{S}_f + \mathcal{B}_f)$$

being the single trace $\mathcal{S}_f$ and bi-tracial parts $\mathcal{B}_f$ of the form

$$\mathcal{S}_f = \operatorname{Tr}_N F, \qquad \mathcal{B}_f = \textstyle\sum_i (\operatorname{Tr}_N \otimes \operatorname{Tr}_N)\{\Phi_i \otimes \Psi_i\}.$$

With the aid of chord diagrams that encode non-vanishing traces on the spinor space V, we organized the obtention of (finitely many) noncommutative polynomials F, Φ_i and Ψ_i in 2^{d-1} hermitian or anti-hermitian matrices in $M_N(\mathbb{C})$. These polynomials are defined up to cyclic permutation of their words and have integer coefficients that are independent of N. We commented on a free probabilistic perspective towards the large-N limit of the spectral action and adapted the concept of self-adjoint noncommutative polynomial to a more relaxed one (cyclic self-adjointness) that is satisfied by F. Furthermore, for fixed i, either both Φ_i and Ψ_i are cyclic self-adjoint or both are cyclically anti-self-adjoint.

On the one hand, we elaborated on 2-dimensional fuzzy geometries in arbitrary signature (p, q). When quantized (or randomized), the corresponding partition function is

$$\mathcal{Z}^{(p,q)} = \int_{\mathcal{M}^{p,q}} \mathrm{e}^{-S(D)} \mathrm{d}K_1 \mathrm{d}K_2, \qquad D = D(K_1, K_2), \quad p + q = 2. \tag{7.1}$$

The space $\mathcal{M}^{p,q}$ of Dirac operators is $\mathcal{M}^{p,q} = (\mathbb{H}_N)^{\times p} \times \mathfrak{su}(N)^{\times q}$, but this simple parametrization does not generally hold for $d > 2$. Here $\mathfrak{su}(N) = \operatorname{Lie}(\mathrm{SU}(N))$ stands for the (Lie algebra of) traceless $N \times N$ skew-hermitian matrices and $\mathbb{H}_N$ for hermitian matrices. Concretely, in Section 4 formulas for $S(D) = \operatorname{Tr}(D^2 + \lambda_4 D^4 + \lambda_6 D^6)$ are deduced, but the present method enables to obtain $S(D) = \operatorname{Tr} f(D)$ for a polynomial f. This first result is an extension (by the sextic term) of the spectral action presented by Barrett–Glaser [8] up to quartic polynomials in $d = 2$.

On the other hand, the novelties (to the best of our knowledge) are the analytic derivations we provided for Riemannian and Lorentzian fuzzy geometries—and in fact, in arbitrary signature in 4 dimensions—as well as a systematic approach that maps random fuzzy geometries to multi-matrix bi-tracial models. For one thing, this sheds some light on arbitrary-dimensional geometries and, for the other thing, on extensions to (quantum) models including bosonic fields. For the quadratic-quartic spectral action $S(D) = \operatorname{Tr}(D^2 + \lambda_4 D^4)$ computed in Section 5 one could study the octo-matrix model

$$\mathcal{Z}^{(p,q)} = \int_{\mathcal{M}^{p,q}} \mathrm{e}^{-S(D)} \prod_{\mu=1}^{4} \mathrm{d}K_\mu \mathrm{d}X_\mu, \qquad D = D(K_\mu, X_\mu), \quad p + q = 4, \tag{7.2}$$

being, in particular,

$$\mathcal{M}^{p,q} = \begin{cases} \mathbb{H}_N^{\times 4} \times \mathfrak{su}(N)^{\times 4} & p = 0, q = 4 \text{ (Riemannian)}, \\ \mathbb{H}_N^{\times 2} \times \mathfrak{su}(N)^{\times 6} & p = 1, q = 3 \text{ (Lorentzian)}. \end{cases} \tag{7.3}$$

For the rest of the signatures, $\mathcal{M}^{p,q}$ can be readily obtained with the aid of equation (II.2) from the Supplementary Material as described for the Lorentzian and Riemannian cases. As a closing point, it is pertinent to remark that determining whether the Dirac operator of a fuzzy geometry is a truncation of a spin^c geometry is a subtle problem addressed in [39] from the viewpoint of the Heisenberg uncertainty principle.

8. Outlook

We present a miscellanea of shortly described topics for further work:

- *Gauge theory.* The NCG-framework pays off in high energy physics precisely for gauge-Higgs theories. A natural step would be to come back to this initial motivation and to define *almost commutative fuzzy geometries* (ongoing project) in order to derive from them the Yang–Mills–Higgs theory on a fuzzy base. The classical action of that model was obtained in [65], which remains to be quantized in full detail. A possibility is to follow the BV-formalism [35] for the dynamic fuzzy geometry (without matter).
- *Analytic approach.* A non-perturbative approach to matrix models, which led to the solvability of all quartic matrix models [40] (after key progress in [59]) consists in exploiting the $\mathrm{U}(N)$-Ward–Takahashi identities in order to descend the tower of the Schwinger–Dyson (or loop) equations (SDE). This was initially formulated for a quartic analogue of Kontsevich's model [50], but the Grosse–Wulkenhaar approach (SDE + Ward Identity [42]) showed also applicability to tensor field theory [60, 63], and seems to be flexible. A first use of the SDE equations of Dirac ensembles is [44].
- *Topological Recursion.* Probably the analytic approach would lead to a (or multiple) Topological Recursion (TR), as it appeared in [40]. Alternatively, one could build upon the direct TR-approach [3]. Namely, the blobbed [11] Topological Recursion [20, 33, 34] has been lately applied [3] to general multi-trace models that encompass the 1-dimensional version of the models derived here. An extension of their TR to dimension $d \geq 2$ would be interesting.
- *Combinatorics.* Finally, chord diagrams are combinatorically interesting by themselves. For instance, together with decorated versions known as Jacobi and Gauß diagrams, they are used in algebraic knot theory [21, Sections 3.4 and 4] in order to describe Vassiliev invariants. Those appearing here are related to the Penner matrix model [62]. One can still explore their generating function [2] in relation to the matrix model with action

  $$S(X) = \mathrm{Tr}_N[X^2/2 - st \cdot (1 - tX)^{-1}],$$

  for $X \in \mathbb{H}_N$. The free energies $\mathcal{F}_g$ of this Andersen–Chekhov–Penner–Reidys–Sułkowski (ACPRS) model $\mathcal{Z}^{\mathrm{ACPRS}} = \mathrm{e}^{\sum_g N^{2-2g}\mathcal{F}_g}$ generate numbers that are moreover important in computational biology, as they encode topologically non-trivial complexes of interacting RNA molecules. These numbers are related to the isomorphism classes of chord diagrams with a certain number of cuts in the circle, leaving segments ('backbones', cf. [2]) but also a connected diagram. For the ACPRS-model there is also a Topological Recursion ([2]).

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References

[1] M. Aizenman and S. Warzel, Kac-Ward formula and its extension to order-disorder correlators through a graph zeta function. *J. Stat. Phys.* **173** (2018), no. 6, 1755–1778 Zbl 1405.82006 MR 3881004

[2] J. E. Andersen, L. O. Chekhov, R. C. Penner, C. M. Reidys, and P. Sułkowski, Topological recursion for chord diagrams, RNA complexes, and cells in moduli spaces. *Nucl. Phys., B* **866** (2013), no. 3, 414–443 Zbl 1262.81122 MR 2982079

[3] S. Azarfar and M. Khalkhali, Random finite noncommutative geometries and topological recursion. 2019, arXiv:1906.09362

[4] J. W. Barrett, Lorentzian version of the noncommutative geometry of the Standard Model of particle physics. *J. Math. Phys.* **48** (2007), no. 1, article ID 012303 Zbl 1121.81122 MR 2292605

[5] J. W. Barrett, Matrix geometries and fuzzy spaces as finite spectral triples. *J. Math. Phys.* **56** (2015), no. 8, article ID 082301 Zbl 1364.81157 MR 3455331

[6] J. W. Barrett, P. Druce, and L. Glaser, Spectral estimators for finite non-commutative geometries. *J. Phys. A* **52** (2019), no. 27, article ID 275203 MR 3962351

[7] J. W. Barrett and J. Gaunt, Finite spectral triples for the fuzzy torus. 2019, arXiv:1908.06796

[8] J. W. Barrett and L. Glaser, Monte Carlo simulations of random non-commutative geometries. *J. Phys. A* **49** (2016), no. 24, article ID 245001 Zbl 1344.81111 MR 3512080

[9] F. Besnard, A $U(1)_{B-L}$-extension of the standard model from noncommutative geometry. *J. Math. Phys.* **62** (2021), no. 1, article ID 012301 Zbl 1456.81471 MR 4204326

[10] A. Bochniak and A. Sitarz, Spectral geometry for the standard model without fermion doubling. *Phys. Rev. D* **101** (2020), no. 7, article ID 075038 MR 4100346

[11] G. Boro, Blobbed topological recursion. *Theor. Math. Phys.* **185** (2015), no. 3, 1729–1740. Translation from: *Teor. Mat. Fiz.* **185** (2015), no.3, 423–437 Zbl 1338.81273 MR 3438628

[12] G. Borot, A. Guionnet, and K. K. Kozlowski, Large-N asymptotic expansion for mean field models with Coulomb gas interaction. *Int. Math. Res. Not. IMRN* (2015), no. 20, 10451–10524 Zbl 1332.82069 MR 3455872

[13] L. Boyle and S. Farnsworth, The standard model, the Pati-Salam model, and 'Jordan geometry'. *New J. Phys.* **22** (2020), no. July, article ID 073023 MR 4162821

[14] S. Carlip, Dimension and dimensional reduction in quantum gravity. *Classical Quantum Gravity* **34** (2017), no. 19, article ID 193001 Zbl 1373.83002 MR 3696572

[15] G. Cébron, Free convolution operators and free Hall transform. *J. Funct. Anal.* **265** (2013), no. 11, 2645–2708 Zbl 1294.46058 MR 3096986

[16] A. H. Chamseddine and A. Connes, The spectral action principle. *Comm. Math. Phys.* **186** (1997), no. 3, 731–750 Zbl 0894.58007 MR 1463819

[17] A. H. Chamseddine, A. Connes, and M. Marcolli, Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.* **11** (2007), no. 6, 991–1089 Zbl 1140.81022 MR 2368941

[18] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom, Entropy and the spectral action. *Comm. Math. Phys.* **373** (2020), no. 2, 457–471 Zbl 1444.46049 MR 4056640

[19] A. H. Chamseddine and W. D. van Suijlekom, A survey of spectral models of gravity coupled to matter. In *Advances in noncommutative geometry—on the occasion of Alain Connes' 70th birthday*, edited by A. H. Chamseddine et al., pp. 1–51, Springer, Cham, 2019 Zbl 1443.58006 MR 4300551

[20] L. Chekhov, B. Eynard, and N. Orantin, Free energy topological expansion for the 2-matrix model. *J. High Energy Phys.* (2006), no. 12, article ID 053 Zbl 1226.81250 MR 2276699

[21] S. Chmutov, S. Duzhin, and J. Mostovoy, *Introduction to Vassiliev knot invariants*. Cambridge Univ. Press, Cambridge, 2012 Zbl 1245.57003 MR 2962302

[22] A. Connes, *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[23] A. Connes, On the spectral characterization of manifolds. *J. Noncommut. Geom.* **7** (2013), no. 1, 1–82 Zbl 1287.58004 MR 3032810

[24] A. Connes, Noncommutative geometry, the spectral standpoint. In *New spaces in physics—formal and conceptual reflections*, pp. 23–84, Cambridge Univ. Press, Cambridge, 2021 Zbl 07425826 MR 4273605

[25] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*. Amer. Math. Soc. Colloq. Publ. 55, American Mathematical Society, Providence, RI; Hindustan Book Agency, New Delhi, 2008 Zbl 1159.58004 MR 2371808

[26] B. P. Dolan, I. Huet, S. Murray, and D. O'Connor, A universal Dirac operator and noncommutative spin bundles over fuzzy complex projective spaces. *J. High Energy Phys.* (2008), no. 3, 029, 21 MR 2391091

[27] B. P. Dolan and D. O'Connor, A fuzzy three sphere and fuzzy tori. *J. High Energy Phys.* (2003), no. 10, 060, 16 MR 2030601

[28] R. Dong, M. Khalkhali, and W. D. van Suijlekom, Second quantization and the spectral action. *J. Geom. Phys.* **167** (2021), Paper No. 104285 Zbl 1481.53107 MR 4271792

[29] S. Doplicher, K. Fredenhagen, and J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields. *Comm. Math. Phys.* **172** (1995), no. 1, 187–220 Zbl 0847.53051 MR 1346376

[30] L. Dąbrowski, F. D'Andrea, and A. Sitarz, The standard model in noncommutative geometry: fundamental fermions as internal forms. *Lett. Math. Phys.* **108** (2018), no. 5, 1323–1340 Zbl 1395.58007 MR 3785732

[31] M. Eckstein and B. Iochum, *Spectral action in noncommutative geometry*. SpringerBriefs Math. Phys. 27, Springer, Cham, 2018 Zbl 1416.81008 MR 3887635

[32] D. B. A. Epstein, The simplicity of certain groups of homeomorphisms. *Compos. Math.* **22** (1970), 165–173 Zbl 0205.28201 MR 267589

[33] B. Eynard, An overview of the topological recursion. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pp. 1063–1085, Kyung Moon Sa, Seoul, 2014 Zbl 1373.14029 MR 3729064

[34] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion. *Commun. Number Theory Phys.* **1** (2007), no. 2, 347–452 Zbl 1161.14026 MR 2346575

[35] J. Gaunt, H. Nguyen, and A. Schenkel, BV quantization of dynamical fuzzy spectral triples. 2022, arXiv:2203.04817

[36] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*. Second edn., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995 Zbl 0856.58001 MR 1396308

[37] L. Glaser, Scaling behaviour in random non-commutative geometries. *J. Phys. A* **50** (2017), no. 27, article ID 275201 Zbl 1377.46050 MR 3664361

[38] L. Glaser, Code for simulating random fuzzy spaces. Accessed 10 January 2020, https://github.com/LisaGlaser/MCMCv4

[39] L. Glaser and A. B. Stern, Understanding truncated non-commutative geometries through computer simulations. *J. Math. Phys.* **61** (2020), no. 3, article ID 033507 Zbl 1439.81064 MR 4076626

[40] H. Grosse, A. Hock, and R. Wulkenhaar, Solution of all quartic matrix models. 2019, arXiv:1906.04600

[41] H. Grosse and P. Prešnajder, The Dirac operator on the fuzzy sphere. *Lett. Math. Phys.* **33** (1995), no. 2, 171–181 Zbl 0840.58011 MR 1316346

[42] H. Grosse and R. Wulkenhaar, Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory. *Comm. Math. Phys.* **329** (2014), no. 3, 1069–1130 Zbl 1305.81129 MR 3212880

[43] A. Guionnet and J. Novak, Asymptotics of unitary multimatrix models: the Schwinger-Dyson lattice and topological recursion. *J. Funct. Anal.* **268** (2015), no. 10, 2851–2905 Zbl 1335.46057 MR 3331788

[44] H. Hessam, M. Khalkhali, and N. Pagliaroli, Bootstrapping Dirac ensembles. *J. Phys. A* **55** (2022), no. 33, Paper No. 335204 MR 4474311

[45] H. Hessam, M. Khalkhali, N. Pagliaroli, and L. S. Verhoeven, From noncommutative geometry to random matrix theory. *J. Phys. A* **55** (2022), no. 41, Paper No. 413002 MR 4492421

[46] A. Holfter and M. Paschke, Moduli spaces of discrete gravity. I. A few points. *J. Geom. Phys.* **47** (2003), no. 2-3, 101–127 Zbl 1041.81073 MR 1991469

[47] D. Jekel, W. Li, and D. Shlyakhtenko, Tracial smooth functions of non-commuting variables and the free Wasserstein manifold. *Diss. Math.* **580** (2022), 1–150 Zbl 07560209 MR 4451910

[48] M. Khalkhali and N. Pagliaroli, Phase transition in random noncommutative geometries. *J. Phys. A* **54** (2021), no. 3, Paper No. 035202 MR 4209133

[49] M. Khalkhali and N. Pagliaroli, Spectral statistics of Dirac ensembles. *J. Math. Phys.* **63** (2022), no. 5, Paper No. 053504 MR 4423391

[50] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.* **147** (1992), no. 1, 1–23 Zbl 0756.35081 MR 1171758

[51] T. Krajewski, Classification of finite spectral triples. *J. Geom. Phys.* **28** (1998), no. 1-2, 1–30 Zbl 0932.58006 MR 1653118

[52] F. Lizzi, Noncommutative Geometry and Particle Physics. In: Corfu Summer Institute 2017 "Schools and Workshops on Elementary Particle Physics and Gravity" (CORFU2017). Proceedings of Science 318, SISSA, Trieste, 2018

[53] J. Madore, The fuzzy sphere. *Classical Quantum Gravity* **9** (1992), no. 1, 69–87 Zbl 0742.53039 MR 1146044

[54] M. Marcolli and W. D. van Suijlekom, Gauge networks in noncommutative geometry. *J. Geom. Phys.* **75** (2014), 71–91 Zbl 1315.81098 MR 3126935

[55] P. Martinetti and D. Singh, Lorentzian fermionic action by twisting Euclidean spectral triples. *J. Noncommut. Geom.* **16** (2022), no. 2, 513–559 Zbl 07597893 MR 4478263

[56] J. N. Mather, Commutators of diffeomorphisms. II. *Comment. Math. Helv.* **50** (1975), 33–40 Zbl 0299.58008 MR 375382

[57] J. A. Mingo and R. Speicher, *Free probability and random matrices*. Fields Inst. Monogr. 35, Springer, New York Zbl 1387.60005 MR 3585560

[58] A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*. Lond. Math. Soc. Lect. Note Ser. 335, Cambridge University Press, Cambridge, 2006 Zbl 1133.60003 MR 2266879

[59] E. Panzer and R. Wulkenhaar, Lambert-W solves the noncommutative Φ^4-model. *Comm. Math. Phys.* **374** (2020), no. 3, 1935–1961 Zbl 1436.81091 MR 4076091

[60] R. Pascalie, C. I. Pérez-Sánchez, and R. Wulkenhaar, Correlation functions of $\mathrm{U}(N)$-tensor models and their Schwinger-Dyson equations. *Ann. Inst. Henri Poincaré D* **8** (2021), no. 3, 377–458 Zbl 1483.81097 MR 4321221

[61] M. Paschke and A. Sitarz, Discrete spectral triples and their symmetries. *J. Math. Phys.* **39** (1998), no. 11, 6191–6205 Zbl 0934.58006 MR 1653113

[62] R. C. Penner, Perturbative series and the moduli space of Riemann surfaces. *J. Differ. Geom.* **27** (1988), no. 1, 35–53 Zbl 0608.30046 MR 918455

[63] C. I. Pérez-Sánchez, The full Ward-Takahashi identity for colored tensor models. *Comm. Math. Phys.* **358** (2018), no. 2, 589–632 Zbl 1387.83034 MR 3774432

[64] C. I. Pérez-Sánchez, On multimatrix models motivated by random noncommutative geometry I: the functional renormalization group as a flow in the free algebra. *Ann. Henri Poincaré* **22** (2021), no. 9, 3095–3148 Zbl 07405136 MR 4297108

[65] C. I. Perez-Sanchez, On multimatrix models motivated by random noncommutative geometry II: A Yang-Mills-Higgs matrix model. *Ann. Henri Poincaré* **23** (2022), no. 6, 1979–2023 Zbl 07541495 MR 4420568

[66] C. I. Pérez-Sánchez, A ribbon graph derivation of the algebra of functional renormalization for random multi-matrices with multi-trace interactions. *Lett. Math. Phys.* **112** (2022), no. 3, Paper No. 58 Zbl 07544195 MR 4441533

[67] A. Rennie and J. C. Varilly, Reconstruction of manifolds in noncommutative geometry. 2006, arXiv:math/0610418

[68] P. Schreivogl and H. Steinacker, Generalized fuzzy torus and its modular properties. *SIGMA, Symmetry Integrability Geom. Methods Appl.* **9** (2013), Paper 060, 23 Zbl 1287.81099 MR 3141528

[69] M. Sperling and H. C. Steinacker, Higher spin gauge theory on fuzzy S_N^4. *J. Phys. A* **51** (2018), no. 7, 075201 Zbl 1390.81350 MR 3755601

[70] W. Thurston, Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.* **80** (1974), 304–307 Zbl 0295.57014 MR 339267

[71] W. D. van Suijlekom, *Noncommutative geometry and particle physics*. Math. Phys. Stud., Springer, Dordrecht, 2015 Zbl 1305.81007 MR 3237670

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