Supplementary Material to “Computing the spectral action for fuzzy geometries: from random noncommutative geometry to bi-tracial multimatrix models”

This Supplementary Material contains the following sections:

- Section I presents in detail how to obtain from chord diagrams noncommutative polynomials that contribute to the spectral action
- Section II gives and proves properties of the gamma matrices in general signature
- Section III departs from the main result and displays for Riemannian and Lorenzian signatures explicitly the spectral action for a quadratic-quartic potential in the Dirac operator
- Section IV gives all the proofs that were skipped in the main text by sake of conciseness
- Section V is the definition of cyclically self-adjoint polynomial without reference to a matrix realization
- Section VI, finally, presents a double-trace matrix model restated as differential operators on (single-trace) auxiliary matrix model.

I. Full computation of one chord diagram

We perform some explicit computations left out in the proof of Proposition 4.1. Since $t = 2n = 6$ will be constant in this section, we drop the subindices n in $\mathfrak{a}_n$, $\mathfrak{b}_n$ and $\mathfrak{s}_n$. Exclusively in this section, we abbreviate the traces as follows:

$$|\mu_i\mu_j \ldots \mu_m| := \mathrm{Tr}_N\left(K_{\mu_i}K_{\mu_j}\cdots K_{\mu_m}\right) \qquad \mu_i, \mu_j, \ldots, \mu_m \in \{1,2\}\,.$$

Then the action functional $\mathfrak{a}(\chi)$ of a chord diagram χ of six points is given by

$$\mathfrak{a}(\chi) = \sum_{\mu_1\mu_2\ldots\mu_6} (-1)^{\mathrm{cr}(\chi)}\Big(\prod_{w\sim_\chi v} g^{\mu_v\mu_w}\Big)\Big(\sum_{\Upsilon\in\mathscr{P}_6}\Big[\prod_{i\in\Upsilon} e_{\mu_i}\Big]\cdot|\mu(\Upsilon^c)|\cdot|\mu(\Upsilon)|\Big)\,,$$

that is,

$$\begin{aligned}\sum_{\mu_1\mu_2\ldots\mu_6} (-1)^{\mathrm{cr}(\chi)} \prod_{w\sim_\chi v} g^{\mu_v\mu_w}\Bigg[&N\Big(|\mu_1\mu_2\ldots\mu_6| + e|\mu_6\mu_5\ldots\mu_1|\Big)\\ &+\sum_i e_i\Big(|\mu_1\mu_2\ldots\widehat{\mu_i}\ldots\mu_6| + e|\mu_6\mu_5\ldots\widehat{\mu_i}\ldots\mu_1|\Big)\cdot|\mu_i|\\ &+\sum_{i<j} e_ie_j\Big(|\mu_1\mu_2\ldots\widehat{\mu_i}\ldots\widehat{\mu_j}\ldots\mu_6| + e|\mu_6\mu_5\ldots\widehat{\mu_j}\ldots\widehat{\mu_i}\ldots\mu_1|\Big)\cdot|\mu_i\mu_j|\\ &+\sum_{i<j<k} e_ie_je_k\Big(|\mu_1\mu_2\ldots\widehat{\mu_i}\ldots\widehat{\mu_j}\ldots\widehat{\mu_k}\ldots\mu_6|\cdot|\mu_i\mu_j\mu_k|\Big)\Bigg]\,,\end{aligned}$$

where $e = e(\mu_1,\ldots,\mu_6) = e_{\mu_1}\cdots e_{\mu_6}$. We just conveniently listed the terms corresponding to Υ and Υ^c together, in the first line displaying those with $\#\Upsilon = 0$ and

$\#\Upsilon = 6$ ('trivial partitions'); in the second $\#\Upsilon = 1$ or 5; on the third line $\#\Upsilon = 2$ or 4; the fourth line corresponds to the $\#\Upsilon = 3$ cases. We also used the fact that e_μ is a sign $\pm$, and that $e \cdot e_{\mu_i} e_{\mu_j} \cdots e_{\mu_v}$ equals the product of the e_{μ_r}'s with $r \neq i, j, \ldots, v$, i.e. precisely those not appearing in $e_{\mu_i} e_{\mu_j} \cdots e_{\mu_v}$. But since in the non-vanishing terms e implies a repetition of indices by pairs, $e \equiv 1$ for non-vanishing terms. Then we gain a factor 2 for those terms (i.e. except for traces of three matrices) and $\mathfrak{a}(\chi)$ is therefore given by

$$\sum_{\mu_1\mu_2\ldots\mu_6} \bigg((-1)^{\mathrm{cr}(\chi)} \prod_{w \sim_\chi v} g^{\mu_v \mu_v} \bigg) \cdot \bigg\{ \sum_{\mu_1,\mu_2,\ldots,\mu_6} 2N |\mu_1\mu_2\ldots\mu_6| \tag{I.1a}$$

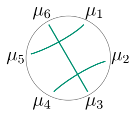
$$+ \sum_i 2e_i |\mu_1\mu_2\ldots\widehat{\mu_i}\ldots\mu_6| \cdot |\mu_i| \tag{I.1b}$$

$$+ \sum_{i<j} 2e_i e_j |\mu_1\mu_2\ldots\widehat{\mu_i}\ldots\widehat{\mu_j}\ldots\mu_6| \cdot |\mu_i\mu_j| \tag{I.1c}$$

$$+ \sum_{i<j<k} e_i e_j e_k |\mu_1\mu_2\ldots\widehat{\mu_i}\ldots\widehat{\mu_j}\ldots\widehat{\mu_k}\ldots\mu_6| \cdot |\mu_i\mu_j\mu_k| \bigg\} . \tag{I.1d}$$

We thus compute the first diagram of 6-points by giving line by line last expression. We perform first the computation for the third line (I.1c) (since this is the longest) explicitly, which can be expanded as

$$2\sum_{\boldsymbol{\mu}}$$



$$\begin{aligned} \times \Big[& e_{\mu_1}e_{\mu_2}|\mu_1\mu_2| \cdot |\mu_3\mu_4\mu_5\mu_6| + e_{\mu_1}e_{\mu_3}|\mu_1\mu_3| \cdot |\mu_2\mu_4\mu_5\mu_6| \\ & + e_{\mu_1}e_{\mu_4}|\mu_1\mu_4| \cdot |\mu_2\mu_3\mu_5\mu_6| + e_{\mu_1}e_{\mu_5}|\mu_1\mu_5| \cdot |\mu_2\mu_3\mu_4\mu_6| \\ & + e_{\mu_1}e_{\mu_6}|\mu_1\mu_6| \cdot |\mu_2\mu_3\mu_4\mu_5| + e_{\mu_2}e_{\mu_3}|\mu_2\mu_3| \cdot |\mu_1\mu_4\mu_5\mu_6| \\ & + e_{\mu_2}e_{\mu_4}|\mu_2\mu_4| \cdot |\mu_1\mu_3\mu_5\mu_6| + e_{\mu_2}e_{\mu_5}|\mu_2\mu_5| \cdot |\mu_1\mu_3\mu_4\mu_6| \\ & + e_{\mu_2}e_{\mu_6}|\mu_2\mu_6| \cdot |\mu_1\mu_3\mu_4\mu_5| + e_{\mu_3}e_{\mu_4}|\mu_3\mu_4| \cdot |\mu_1\mu_2\mu_5\mu_6| \\ & + e_{\mu_3}e_{\mu_5}|\mu_3\mu_5| \cdot |\mu_1\mu_2\mu_4\mu_6| + e_{\mu_3}e_{\mu_6}|\mu_3\mu_6| \cdot |\mu_1\mu_2\mu_4\mu_5| \\ & + e_{\mu_4}e_{\mu_5}|\mu_4\mu_5| \cdot |\mu_1\mu_2\mu_3\mu_6| + e_{\mu_4}e_{\mu_6}|\mu_4\mu_6| \cdot |\mu_1\mu_2\mu_3\mu_5| \\ & + e_{\mu_5}e_{\mu_6}|\mu_5\mu_6| \cdot |\mu_1\mu_2\mu_3\mu_4| \Big] . \end{aligned} \tag{I.2}$$

The diagram's meaning is the sign and product of $g^{\mu_v \nu_w}$'s before the braces in eq. (I.1). After contraction with the term in square brackets in (I.2) one gets

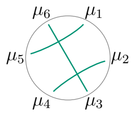
$$\begin{aligned} 2\sum_{\mu,\nu,\rho} (e_\mu e_\nu e_\rho) \Big\{ & e_\rho|\mu\nu| \cdot |\rho\nu\mu\rho| + e_\rho|\mu\nu| \cdot |\rho\rho\mu\nu| + e_\rho|\mu\nu| \cdot |\nu\rho\mu\rho| \\ & + e_\mu e_\rho e_\nu |\mu^2| \cdot |\nu\rho\nu\rho| + e_\rho|\mu\nu| \cdot |\rho\nu\rho\mu| + e_\rho|\mu\nu| \cdot |\rho\mu\rho\nu| \\ & + e_\mu e_\rho e_\nu |\mu^2| \cdot |\nu\rho\nu\rho| + e_\rho|\mu\nu| \cdot |\nu\rho\mu\rho| + e_\rho|\mu\nu| \cdot |\rho\nu\mu\rho| \\ & + e_\rho|\mu\nu| \cdot |\rho\nu\rho\mu| + e_\rho|\mu\nu| \cdot |\nu\rho^2\mu| + e_\mu e_\rho e_\nu |\mu^2| \cdot |\nu\rho^2\nu| \\ & + e_\rho|\mu\nu| \cdot |\nu\mu\rho^2| + e_\rho|\mu\nu| \cdot |\rho\mu\nu\rho| + e_\rho|\mu\nu| \cdot |\mu\rho\nu\rho| \Big\} , \end{aligned} \tag{I.3}$$

where the signs $e_\mu e_\nu e_\rho$ are due to $g^{\lambda\sigma} = e_\lambda \delta^{\lambda\sigma}$ (no sum). Following the notation of eq. (4.8), using the cyclicity of the trace and renaming indices, this expression can be written as

$$2\sum_{\mu,\nu,\rho} (6R_{\mu\nu\rho} + 6S_{\mu\nu\rho} + 2T_{\mu\nu\rho} + U_{\mu\nu\rho}) . \tag{I.1c$'$}$$

Similarly, for the terms obeying $\#\Upsilon$ or $\#\Upsilon^c = 1$, i.e. line (I.1b), one has

$$2\sum_{\boldsymbol{\mu}}$$



$$\times \Big\{ e_{\mu_1}|\mu_1|\cdot|\mu_2\mu_3\mu_4\mu_5\mu_6| + e_{\mu_2}|\mu_2|\cdot|\mu_1\mu_3\mu_4\mu_5\mu_6| \\ + e_{\mu_3}|\mu_3|\cdot|\mu_1\mu_2\mu_4\mu_5\mu_6| + e_{\mu_4}|\mu_4|\cdot|\mu_1\mu_2\mu_3\mu_5\mu_6| \\ + e_{\mu_5}|\mu_5|\cdot|\mu_1\mu_2\mu_3\mu_4\mu_6| + e_{\mu_6}|\mu_6|\cdot|\mu_1\mu_2\mu_3\mu_4\mu_5| \Big\},$$

which amounts to

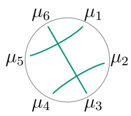
$$2\sum_{\mu,\nu,\rho} e_\nu e_\rho |\mu| \cdot \Big\{ |\nu\rho\nu\mu\rho| + |\nu\mu\rho\nu\rho| + |\nu\rho\mu\nu\rho| + |\nu\mu\rho\nu\rho| + |\mu\nu\rho\nu\rho| + |\nu\rho\mu\rho\nu| \Big\},$$

or, relabeling, to

$$2\sum_{\mu,\nu,\rho} (4O_{\mu\nu\rho} + 2P_{\mu\nu\rho}). \tag{I.1b$'$}$$

The terms with $\#\Upsilon = 3$ remain to be computed:

$$\sum_{\boldsymbol{\mu}}$$



$$\times \Big\{ (e_{\mu_1}e_{\mu_2}e_{\mu_3} + e_{\mu_4}e_{\mu_5}e_{\mu_6})|\mu_1\mu_2\mu_3|\cdot|\mu_4\mu_5\mu_6| \\ +(e_{\mu_1}e_{\mu_2}e_{\mu_4} + e_{\mu_3}e_{\mu_5}e_{\mu_6})|\mu_1\mu_2\mu_4|\cdot|\mu_3\mu_5\mu_6| \\ +(e_{\mu_1}e_{\mu_2}e_{\mu_5} + e_{\mu_3}e_{\mu_4}e_{\mu_6})|\mu_1\mu_2\mu_5|\cdot|\mu_3\mu_4\mu_6| \\ +(e_{\mu_1}e_{\mu_2}e_{\mu_6} + e_{\mu_3}e_{\mu_4}e_{\mu_5})|\mu_1\mu_2\mu_6|\cdot|\mu_3\mu_4\mu_5| \\ +(e_{\mu_1}e_{\mu_3}e_{\mu_4} + e_{\mu_2}e_{\mu_5}e_{\mu_6})|\mu_1\mu_3\mu_4|\cdot|\mu_2\mu_5\mu_6| \\ +(e_{\mu_1}e_{\mu_3}e_{\mu_5} + e_{\mu_2}e_{\mu_4}e_{\mu_6})|\mu_1\mu_3\mu_5|\cdot|\mu_2\mu_4\mu_6| \\ +(e_{\mu_1}e_{\mu_3}e_{\mu_6} + e_{\mu_2}e_{\mu_4}e_{\mu_5})|\mu_1\mu_3\mu_6|\cdot|\mu_2\mu_4\mu_5| \\ +(e_{\mu_1}e_{\mu_4}e_{\mu_5} + e_{\mu_2}e_{\mu_3}e_{\mu_6})|\mu_1\mu_4\mu_5|\cdot|\mu_2\mu_3\mu_6| \\ +(e_{\mu_1}e_{\mu_4}e_{\mu_6} + e_{\mu_2}e_{\mu_3}e_{\mu_5})|\mu_1\mu_4\mu_6|\cdot|\mu_2\mu_3\mu_5| \\ +(e_{\mu_1}e_{\mu_5}e_{\mu_6} + e_{\mu_2}e_{\mu_3}e_{\mu_4})|\mu_1\mu_5\mu_6|\cdot|\mu_2\mu_3\mu_4| \Big\}.$$

Although $\mathrm{Tr}_N(M_1M_2M_3) = \mathrm{Tr}_N(M_3M_2M_1)$ is false for general matrices M_1, M_2 and M_3 (e.g. for $M_j = \sigma_j$, the Pauli matrices), having at our disposal only two matrices, K_1 and K_2, the relation $\mathrm{Tr}_N(K_\mu K_\nu K_\rho) = \mathrm{Tr}_N(K_\rho K_\nu K_\mu)$ does hold. This fact was used to obtain the last equation. Contracting with the diagram, as we already did for other partitions, one gets

$$\sum_{\mu,\nu,\rho} 8V_{\mu\nu\rho} + 12W_{\mu\nu\rho}. \tag{I.1d$'$}$$

By collecting the terms from the three equations with primed tags, the bi-trace term for the I-diagrams one obtains

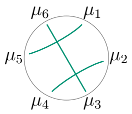
$$\mathfrak{b}(\mathrm{I}) = +2\sum_{\mu,\nu,\rho}\Big(4O_{\mu\nu\rho} + 2P_{\mu\nu\rho} + 6R_{\mu\nu\rho} + 6S_{\mu\nu\rho} \\ + 2T_{\mu\nu\rho} + U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho}\Big),$$

which is a claim amid the proof of Proposition 4.1.

Notice that these integer coefficients add up to 62, and so will these (denoted $p_\chi, q_\chi, \ldots, w_\chi$ in the main text) in absolute value for a general diagram χ. There are two missing terms to get the needed $2^6 = \#\mathscr{P}_6$ terms. These are the trivial cases $\Upsilon, \Upsilon^c = \emptyset$, which can be readily computed.

For the I-diagram,

$$\mathfrak{s}(\mathrm{I}) = 2N \cdot \sum_{\boldsymbol{\mu}}$$



$$\times\, |\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6|$$

$$\begin{aligned} &= 2N \cdot \sum_{\mu,\nu,\rho} e_\mu e_\nu e_\rho |\mu\nu\rho\nu\mu\rho| \\ &= 2N \cdot \mathrm{Tr}_N \Big\{ e_1 K_1^6 + 2e_2 (K_1K_2)^2 K_1^2 + e_1 (K_2^2 K_1)^2 \\ &\qquad + e_2 K_2^6 + 2e_1 (K_2K_1)^2 K_2^2 + e_2 (K_1^2K_2)^2 \Big\} . \end{aligned}$$

The single-trace action $\mathcal{S}_6$ in Proposition 4.1 is then obtained by summing over all 6-point chord diagrams $\sum_\chi \mathfrak{s}(\chi)$, whose values are found by similar computations.

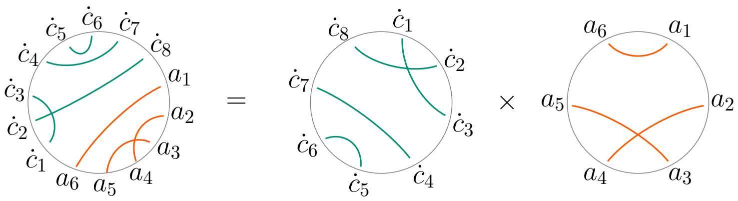


FIGURE I.1. Splitting of a chord diagram for indices of a mixed signature. One of the diagrams appearing in the computation of $\langle a_1 \cdots a_6 \dot c_1 \cdots \dot c_8 \rangle$ 14 points, all of which split into two (of 8 and 6 points). The equality of diagrams means equality of the product of the bilinears $g^{a_i a_j}$ and $g^{\dot c_l \dot c_k}$ determined by the depicted chords and the signs for simple crossing

Remark I.1. For a mixed signature, $q, p > 0$, any non-vanishing $\langle \mu_1 \ldots \mu_{2n} \rangle$ has the form (up to a reordering sign) $\langle a_1 \ldots a_{2r} \dot c_1 \ldots \dot c_{2u} \rangle$ with $r + u = n$. Since $g^{a\dot c}$ vanishes, any chord diagram χ in the sum of eq. (3.2) splits into a pair (σ, ρ) of smaller chord diagrams, of $2r$ and $2u$ points, whose chords do not cross (see Fig. I.1), so $\mathrm{cr}(\chi) = \mathrm{cr}(\sigma) + \mathrm{cr}(\rho)$. Therefore

$$\begin{aligned} \langle a_1 \ldots a_{2r} \dot c_1 \ldots \dot c_{2u} \rangle &= \sum_{\substack{2n\text{-pt chord} \\ \text{diagrams } \chi}} (-1)^{\mathrm{cr}(\chi)} \prod_{\substack{i,j \\ i \sim_\chi j}} g^{a_i a_j} \times \prod_{\substack{u,v \\ u \sim_\chi v}} g^{\dot c_u \dot c_v} \\ &= \sum_{\substack{(2r,2u)\text{-pt chord} \\ \text{diagrams } (\rho,\sigma)}} (-1)^{\mathrm{cr}(\sigma)} \prod_{\substack{i,j \\ i \sim_\rho j}} g^{a_i a_j} \times (-1)^{\mathrm{cr}(\rho)} \prod_{\substack{u,v \\ u \sim_\sigma v}} g^{\dot c_u \dot c_v} \\ &= \langle a_1 \ldots a_{2r} \rangle \langle \dot c_1 \ldots \dot c_{2u} \rangle . \end{aligned} \tag{I.4}$$

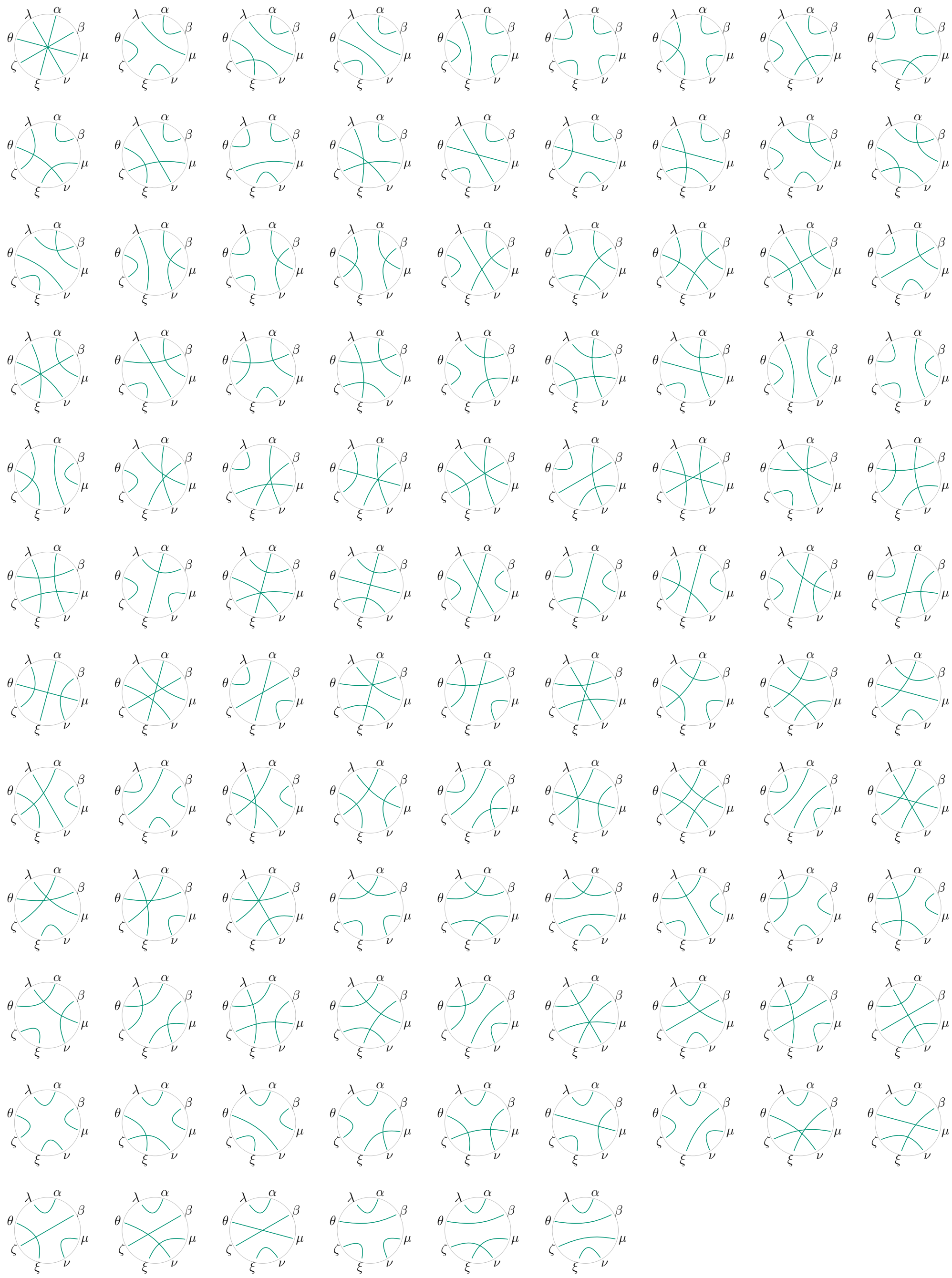


FIGURE I.2. The $7!! = 105$ chord diagrams with eight points. These assist to compute $\mathrm{Tr}_V(\gamma^\alpha\gamma^\beta\cdots\gamma^\theta\gamma^\lambda)$ in any dimension with diagonal metric of any signature. The sign of a diagram χ is $(-1)^{\#\{\text{simple crossings of }\chi\}}$. Thus, the 'pizza-cut' diagram in the upper left corner that appears as a summand in the normalized trace $\langle\alpha\beta\mu\nu\xi\zeta\theta\lambda\rangle$ evaluates to $(-1)^{1+2+3}g^{\xi\alpha}g^{\zeta\beta}g^{\theta\mu}g^{\lambda\nu}$, where $\langle\cdots\rangle = (1/\dim V)\mathrm{Tr}_V(\cdots)$

For the metric $g^{\mu\nu} = \mathrm{diag}(+,\dots,+,-,\dots,-)$ the two factors are

$$\langle a_1 \dots a_{2r}\rangle = \sum_{\substack{2r\text{-pt chord}\\ \text{diagrams } \rho}} (-1)^{\mathrm{cr}(\sigma)} \prod_{\substack{i,j\\ i\sim_\rho j}} \delta^{a_i a_j}\,, \tag{I.5a}$$

$$\langle \dot c_1 \dots \dot c_{2u}\rangle = (-1)^u \sum_{\substack{2u\text{-pt chord}\\ \text{diagrams } \sigma}} (-1)^{\mathrm{cr}(\rho)} \prod_{\substack{w,v\\ w\sim_\sigma v}} \delta^{\dot c_w \dot c_v}\,. \tag{I.5b}$$

II. Some properties of gamma matrices

In order to deal with d-dimensional matrix geometries we prove some of the properties of the corresponding gamma matrices.

First, notice that in any signature for each multi-index $I = (\mu_1,\dots,\mu_r) \in \Lambda_d$ one has

$$\gamma^{\mu_r}\cdots\gamma^{\mu_1} = (-1)^{r(r-1)/2}\gamma^{\mu_1}\cdots\gamma^{\mu_r} = (-1)^{\lfloor r/2\rfloor}\gamma^{\mu_1}\cdots\gamma^{\mu_r}\,. \tag{II.1}$$

This can be proven by induction on the number $r-1$ of products. For $r=2$, this is just $\{\gamma^{\mu_1},\gamma^{\mu_2}\}=0$, which holds since the indices are different. Suppose that eq. (II.1) holds for an $r\in\mathbb{N}$. Then if $(\mu_1,\dots,\mu_{r+1})\in\Lambda_d$, one has

$$\begin{aligned}\gamma^{\mu_{r+1}}\gamma^{\mu_r}\cdots\gamma^{\mu_2}\gamma^{\mu_1} &= (-1)^r(\gamma^{\mu_r}\cdots\gamma^{\mu_2}\gamma^{\mu_1})\gamma^{\mu_{r+1}}\\ &= (-1)^{r+r(r-1)/2}\gamma^{\mu_1}\cdots\gamma^{\mu_{r+1}}\\ &= (-1)^{r(r+1)/2}\gamma^{\mu_1}\cdots\gamma^{\mu_{r+1}} = (-1)^{\lfloor (r+1)/2\rfloor}\gamma^{\mu_1}\cdots\gamma^{\mu_{r+1}}\,.\end{aligned}$$

Now let us fix a signature (p,q). We dot the spacial indices $\dot c = 1,\dots,q$ and leave the temporal in usual Roman lowercase, $a=1,\dots,p$. Given a mult-index $I = (a_1\dots,a_t,\dot c_1,\dots,\dot c_u)\in\Lambda_d$ (so $t\le p$ and $u\le q$) it will be useful to know whether Γ^I is hermitian or anti-hermitian. We compute its hermitian conjugate $(\Gamma^I)^*$:

$$\begin{aligned}(\gamma^{a_1}\dots\gamma^{a_t}\gamma^{\dot c_1}\dots\gamma^{\dot c_u})^* &= (\gamma^{\dot c_u})^*\dots(\gamma^{\dot c_1})^*(\gamma^{a_t})^*\dots(\gamma^{a_1})^*\\ &= (-1)^u\gamma^{\dot c_u}\dots\gamma^{\dot c_1}\gamma^{a_t}\dots\gamma^{a_1}\\ &= (-1)^{u+\lfloor (u+t)/2\rfloor}\gamma^{a_1}\dots\gamma^{a_t}\gamma^{\dot c_1}\dots\gamma^{\dot c_u}\,.\end{aligned}$$

With the conventions set in eq. (2.2), have the following

- In Riemannian signature $(0,d)$ a product of u gamma matrices associated to $I\in\Lambda_d$ is (anti-)hermitian if $u(u+1)/2$ is even (odd).
- In $(d,0)$-signature a product of u gamma matrices associated to $I\in\Lambda_d$ is hermitian if $t(t-1)/2$ is even, and anti-hermitian if it is odd.

In the main text, it will be useful to know that

$$e_\mu e_{\hat\mu} = (-1)^{q+1} \qquad\qquad d=4,\ \text{with signature } (p,q)\,. \tag{II.2}$$

This follows from $(\Gamma^{\hat\mu})^* = e_{\hat\mu}\Gamma^{\hat\mu}$, from $e_1e_2e_3e_4 = (-1)^q$ and from

$$(\Gamma^{\hat\mu})^* = (\gamma^4)^*\cdots\widehat{(\gamma^\mu)^*}\cdots(\gamma^1)^* = -e_1e_2e_3e_4e_\mu\gamma^1\cdots\widehat{\gamma^\mu}\cdots\gamma^4 = -e_1e_2e_3e_4e_\mu\Gamma^{\hat\mu}\,.$$

III. SPECTRAL ACTION FOR RIEMANNIAN AND LORENTZIAN GEOMETRIES

Before writing down the action functionals for Riemannian and Lorentzian geometries, it will be helpful to restate eqs. (5.13) and (5.14) via

$$-\sum_{\alpha,\beta,\mu,\nu}\delta_{\alpha\beta\mu\nu}e_\alpha e_\beta \mathrm{Tr}_N\left[(K_\mu X_\nu)^2+2K_\mu^2X_\nu^2\right] = (-1)^{1+q}\sum_{\mu\neq\nu}2e_\mu e_\nu \mathrm{Tr}_N\left[(K_\mu X_\nu)^2+2K_\mu^2X_\nu^2\right]$$

and by writing out ('cycl.' next means equality after cyclic reordering)

$$\begin{aligned}8(-1)^{q+1}\sum_{\mu,\boldsymbol{\nu}}(-1)^{|\sigma(\boldsymbol{\nu},\mu)|}\delta_{\mu\nu_1\nu_2\nu_3}e_\mu(X_\mu K_{\nu_1}K_{\nu_2}K_{\nu_3}+K_\mu X_{\nu_1}X_{\nu_2}X_{\nu_3}) \\ \overset{\text{cycl.}}{\equiv} -8(-1)^q\Big[e_1X_1\Big(K_2[K_3,K_4]+K_3[K_4,K_2]+K_4[K_2,K_3]\Big) \\ +e_2X_2\Big(K_1[K_3,K_4]+K_3[K_4,K_1]+K_4[K_1,K_3]\Big) \\ +e_3X_3\Big(K_1[K_2,K_4]+K_2[K_4,K_1]+K_4[K_1,K_2]\Big) \\ +e_4X_4\Big(K_1[K_2,K_3]+K_2[K_3,K_1]+K_3[K_1,K_2]\Big)\Big] \\ -8(-1)^q\Big[e_1K_1\Big(X_2[X_3,X_4]+X_3[X_4,X_2]+X_4[X_2,X_3]\Big) \\ +e_2K_2\Big(X_1[X_3,X_4]+X_3[X_4,X_1]+X_4[X_1,X_3]\Big) \\ +e_3K_3\Big(X_1[X_2,X_4]+X_2[X_4,X_1]+X_4[X_1,X_2]\Big) \\ +e_4K_4\Big(X_1[X_2,X_3]+X_2[X_3,X_1]+X_3[X_1,X_2]\Big)\Big],\end{aligned} \tag{III.1}$$

as well as

$$\begin{aligned}-8\sum_{\nu,\boldsymbol{\mu}}(-1)^{|\sigma(\boldsymbol{\mu},\nu)|}\delta_{\nu\mu_1\mu_2\mu_3}\cdot\Big\{-\mathrm{Tr}_N X_\nu\cdot\mathrm{Tr}_N(K_{\mu_1}K_{\mu_2}K_{\mu_3}) \\ +e_{\mu_2}e_{\mu_3}\Big(\mathrm{Tr}_N K_{\mu_1}\cdot\mathrm{Tr}_N(X_\nu K_{\mu_2}K_{\mu_3})+\mathrm{Tr}_N X_{\mu_1}\cdot\mathrm{Tr}_N(K_\nu X_{\mu_2}X_{\mu_3})\Big) \\ +(-1)^q\mathrm{Tr}_N K_\nu\cdot\mathrm{Tr}_N(X_{\mu_1}X_{\mu_2}X_{\mu_3})\Big\}\end{aligned} \tag{III.2}$$

$$\begin{aligned}=+24\mathrm{Tr}_N X_1\cdot\mathrm{Tr}_N\Big(K_2[K_3,K_4]\Big)+24\mathrm{Tr}_N X_2\cdot\mathrm{Tr}_N\Big(K_1[K_3,K_4]\Big) \\ +24\mathrm{Tr}_N X_3\cdot\mathrm{Tr}_N\Big(K_1[K_2,K_4]\Big)+24\mathrm{Tr}_N X_4\cdot\mathrm{Tr}_N\Big(K_1[K_2,K_3]\Big) \\ -8\mathrm{Tr}_N K_1\cdot\mathrm{Tr}_N\Big(e_3e_4[K_3,K_4]X_2+e_2e_4[K_2,K_4]X_3+e_2e_3[K_2,K_3]X_4\Big)\end{aligned}$$

$$- 8\mathrm{Tr}_N K_2 \cdot \mathrm{Tr}_N \Big(e_3 e_4 [K_3, K_4] X_1 + e_1 e_4 [K_4, K_1] X_3 + e_1 e_3 [K_3, K_1] X_4 \Big)$$

$$- 8\mathrm{Tr}_N K_3 \cdot \mathrm{Tr}_N \Big(e_2 e_4 [K_2, K_4] X_1 + e_1 e_4 [K_4, K_1] X_2 + e_1 e_2 [K_1, K_2] X_4 \Big)$$

$$- 8\mathrm{Tr}_N K_4 \cdot \mathrm{Tr}_N \Big(e_2 e_3 [K_2, K_3] X_1 + e_1 e_3 [K_1, K_3] X_2 + e_1 e_2 [K_1, K_2] X_3 \Big)$$

$$+ (-1)^{1+q} \Big\{ 24\mathrm{Tr}_N K_1 \cdot \mathrm{Tr}_N \Big(X_2 [X_3, X_4] \Big) + 24\mathrm{Tr}_N K_2 \cdot \mathrm{Tr}_N \Big(X_1 [X_3, X_4] \Big)$$

$$+ 24\mathrm{Tr}_N K_3 \cdot \mathrm{Tr}_N \Big(X_1 [X_2, X_4] \Big) + 24\mathrm{Tr}_N K_4 \cdot \mathrm{Tr}_N \Big(X_1 [X_2, X_3] \Big) \Big\}$$

$$- 8\mathrm{Tr}_N X_1 \cdot \mathrm{Tr}_N \Big(e_3 e_4 [X_3, X_4] K_2 + e_2 e_4 [X_2, X_4] K_3 + e_2 e_3 [X_2, X_3] K_4 \Big)$$

$$- 8\mathrm{Tr}_N X_2 \cdot \mathrm{Tr}_N \Big(e_3 e_4 [X_3, X_4] K_1 + e_1 e_4 [X_4, X_1] K_3 + e_1 e_3 [X_3, X_1] K_4 \Big)$$

$$- 8\mathrm{Tr}_N X_3 \cdot \mathrm{Tr}_N \Big(e_2 e_4 [X_2, X_4] K_1 + e_1 e_4 [X_4, X_1] K_2 + e_1 e_2 [X_1, X_2] K_4 \Big)$$

$$- 8\mathrm{Tr}_N X_4 \cdot \mathrm{Tr}_N \Big(e_2 e_3 [X_2, X_3] K_1 + e_1 e_3 [X_1, X_3] K_2 + e_1 e_2 [X_1, X_2] K_3 \Big) .$$

From these expressions the Riemannian and Lorentzian cases are next derived.

III.1. **Riemannian fuzzy geometries.** The metric $g = \mathrm{diag}(-1,-1,-1,-1)$ implies $e_\mu = -1$ for each $\mu \in \Delta_4$ and $q = 4$. The Dirac operator $D = D(\mathbf{L}, \tilde{\mathbf{H}})$ (Ex. 2.4) is parametrized by four anti-hermitian matrices $K_\mu = L_\mu$ (where $[L_\mu, \,\cdot\,]$ corresponds to the derivatives ∂_μ in the smooth case) and four hermitian matrices $X_\mu = \tilde{H}_\mu$ (corresponding to the spin connection ω_μ in the smooth spin geometry case represented by $[\tilde{H}_\mu, \,\cdot\,]$ here). In Example 2.4 above these have been called $\tilde{H}_1 = H_{234}, \ldots, \tilde{H}_4 = H_{123}$. The bi-tracial octo-matrix model has the following quadratic part

$$\frac{1}{8}\mathrm{Tr}\Big([D^{\mathrm{Riemann}}]^2 \Big) = N \sum_{\mu=1}^{4} \mathrm{Tr}[\tilde{H}_\mu^2 - L_\mu^2] + (\mathrm{Tr}_N \tilde{H}_\mu)^2 + (\mathrm{Tr}_N L_\mu)^2 \,, \qquad \text{(III.3)}$$

which directly follows from eq. (5.2). The quartic part is more complicated:

$$\frac{1}{4}\mathrm{Tr}\Big([D^{\mathrm{Riemann}}]^4 \Big) = N \mathcal{S}_4^{\mathrm{Riemann}} + \mathcal{B}_4^{\mathrm{Riemann}}$$

having single-trace action

$$\begin{aligned} \mathcal{S}_4^{\mathrm{Riemann}} = \mathrm{Tr}_N \Big\{ & 2 \sum_\mu (L_\mu^4 + \tilde{H}_\mu^4) \\ & + 4 \sum_{\mu<\nu} (2L_\mu^2 L_\nu^2 + 2\tilde{H}_\mu^2 \tilde{H}_\nu^2 - L_\mu L_\nu L_\mu L_\nu - \tilde{H}_\mu \tilde{H}_\nu \tilde{H}_\mu \tilde{H}_\nu) \\ & - \sum_{\mu\neq\nu} \Big[2(L_\mu \tilde{H}_\nu)^2 + 4L_\mu^2 \tilde{H}_\nu^2 \Big] + \sum_\mu \Big[2(L_\mu \tilde{H}_\mu)^2 - 4L_\mu^2 \tilde{H}_\mu^2 \Big] \end{aligned}$$

$$
\begin{aligned}
&+8\Big[\tilde{H}_1\Big(L_2[L_3,L_4]+L_3[L_4,L_2]+L_4[L_2,L_3]\Big)\\
&\quad+\tilde{H}_2\Big(L_1[L_3,L_4]+L_3[L_4,L_1]+L_4[L_1,L_3]\Big)\\
&\quad+\tilde{H}_3\Big(L_1[L_2,L_4]+L_2[L_4,L_1]+L_4[L_1,L_2]\Big)\\
&\quad+\tilde{H}_4\Big(L_1[L_2,L_3]+L_2[L_3,L_1]+L_3[L_1,L_2]\Big)\Big]\\
&+8\Big[L_1\Big(\tilde{H}_2[\tilde{H}_3,\tilde{H}_4]+\tilde{H}_3[\tilde{H}_4,\tilde{H}_2]+\tilde{H}_4[\tilde{H}_2,\tilde{H}_3]\Big)\\
&\quad+L_2\Big(\tilde{H}_1[\tilde{H}_3,\tilde{H}_4]+\tilde{H}_3[\tilde{H}_4,\tilde{H}_1]+\tilde{H}_4[\tilde{H}_1,\tilde{H}_3]\Big)\\
&\quad+L_3\Big(\tilde{H}_1[\tilde{H}_2,\tilde{H}_4]+\tilde{H}_2[\tilde{H}_4,\tilde{H}_1]+\tilde{H}_4[\tilde{H}_1,\tilde{H}_2]\Big)\\
&\quad+L_4\Big(\tilde{H}_1[\tilde{H}_2,\tilde{H}_3]+\tilde{H}_2[\tilde{H}_3,\tilde{H}_1]+\tilde{H}_3[\tilde{H}_1,\tilde{H}_2]\Big)\Big]\Big\}\,, \qquad \text{(III.4)}
\end{aligned}
$$

and bi-tracial action

$$
\begin{aligned}
\mathcal{B}_4^{\text{Riemann}} &= 8\sum_{\mu,\nu}\operatorname{Tr}_N\tilde{H}_\mu\cdot\operatorname{Tr}_N(\tilde{H}_\mu\tilde{H}_\nu^2)-\operatorname{Tr}_N(L_\mu)\cdot\operatorname{Tr}_N(L_\mu L_\nu^2) \qquad \text{(III.5)}\\
&+\sum_{\mu,\nu=1}^{4}\Big\{2\operatorname{Tr}_N(\tilde{H}_\mu^2)\cdot\operatorname{Tr}_N(\tilde{H}_\nu^2)+4\Big[\operatorname{Tr}_N(\tilde{H}_\mu\tilde{H}_\nu)\Big]^2\Big\}\\
&+\sum_{\mu,\nu=1}^{4}\Big\{2\operatorname{Tr}_N(L_\mu^2)\cdot\operatorname{Tr}_N(L_\nu^2)+4\Big[\operatorname{Tr}_N(L_\mu L_\nu)\Big]^2\Big\}\\
&+4\sum_{\mu=1}^{4}\Big\{2\operatorname{Tr}_N L_\mu\cdot\operatorname{Tr}_N(L_\mu\tilde{H}_\mu^2)-2\operatorname{Tr}_N\tilde{H}_\mu\cdot\operatorname{Tr}_N(\tilde{H}_\mu L_\mu^2)\\
&\qquad-\operatorname{Tr}_N(\tilde{H}_\mu^2)\cdot\operatorname{Tr}_N(L_\mu^2)+2\Big[\operatorname{Tr}_N(L_\mu\tilde{H}_\mu)\Big]^2\Big\}\\
&+24\sum_{\mu\neq\nu=1}^{4}\operatorname{Tr}_N L_\mu\cdot\operatorname{Tr}_N\Big(L_\mu X_\nu^2\Big)-\operatorname{Tr}_N\tilde{H}_\nu\cdot\operatorname{Tr}_N\Big(L_\mu^2\tilde{H}_\nu\Big)\\
&+12\sum_{\mu\neq\nu}\Big\{2\Big[\operatorname{Tr}_N(L_\mu\tilde{H}_\nu)\Big]^2-\operatorname{Tr}_N(L_\mu^2)\cdot\operatorname{Tr}_N(\tilde{H}_\nu^2)\Big\}\\
&+24\operatorname{Tr}_N\tilde{H}_1\cdot\operatorname{Tr}_N\Big(L_2[L_3,L_4]\Big)+24\operatorname{Tr}_N\tilde{H}_2\cdot\operatorname{Tr}_N\Big(L_1[L_3,L_4]\Big)\\
&+24\operatorname{Tr}_N\tilde{H}_3\cdot\operatorname{Tr}_N\Big(L_1[L_2,L_4]\Big)+24\operatorname{Tr}_N\tilde{H}_4\cdot\operatorname{Tr}_N\Big(L_1[L_2,L_3]\Big)\\
&-8\operatorname{Tr}_N L_1\cdot\operatorname{Tr}_N\Big([L_3,L_4]\tilde{H}_2+[L_2,L_4]\tilde{H}_3+[L_2,L_3]\tilde{H}_4\Big)
\end{aligned}
$$

$$- 8\mathrm{Tr}_N\, L_2 \cdot \mathrm{Tr}_N \Big([L_3, L_4]\tilde{H}_1 + [L_4, L_1]\tilde{H}_3 + [L_3, L_1]\tilde{H}_4 \Big)$$

$$- 8\mathrm{Tr}_N\, L_3 \cdot \mathrm{Tr}_N \Big([L_2, L_4]\tilde{H}_1 + [L_4, L_1]\tilde{H}_2 + [L_1, L_2]\tilde{H}_4 \Big)$$

$$- 8\mathrm{Tr}_N\, L_4 \cdot \mathrm{Tr}_N \Big([L_2, L_3]\tilde{H}_1 + [L_1, L_3]\tilde{H}_2 + [L_1, L_2]\tilde{H}_3 \Big)$$

$$- 24\mathrm{Tr}_N\, L_1 \cdot \mathrm{Tr}_N \Big(\tilde{H}_2[\tilde{H}_3, \tilde{H}_4] \Big) - 24\mathrm{Tr}_N\, L_2 \cdot \mathrm{Tr}_N \Big(\tilde{H}_1[\tilde{H}_3, \tilde{H}_4] \Big)$$

$$- 24\mathrm{Tr}_N\, L_3 \cdot \mathrm{Tr}_N \Big(\tilde{H}_1[\tilde{H}_2, \tilde{H}_4] \Big) - 24\mathrm{Tr}_N\, L_4 \cdot \mathrm{Tr}_N \Big(\tilde{H}_1[\tilde{H}_2, \tilde{H}_3] \Big)$$

$$- 8\mathrm{Tr}_N\, \tilde{H}_1 \cdot \mathrm{Tr}_N \Big([\tilde{H}_3, \tilde{H}_4]L_2 + [\tilde{H}_2, \tilde{H}_4]L_3 + [\tilde{H}_2, \tilde{H}_3]L_4 \Big)$$

$$- 8\mathrm{Tr}_N\, \tilde{H}_2 \cdot \mathrm{Tr}_N \Big([\tilde{H}_3, \tilde{H}_4]L_1 + [\tilde{H}_4, \tilde{H}_1]L_3 + [\tilde{H}_3, \tilde{H}_1]L_4 \Big)$$

$$- 8\mathrm{Tr}_N\, \tilde{H}_3 \cdot \mathrm{Tr}_N \Big([\tilde{H}_2, \tilde{H}_4]L_1 + [\tilde{H}_4, \tilde{H}_1]L_2 + [\tilde{H}_1, \tilde{H}_2]L_4 \Big)$$

$$- 8\mathrm{Tr}_N\, \tilde{H}_4 \cdot \mathrm{Tr}_N \Big([\tilde{H}_2, \tilde{H}_3]L_1 + [\tilde{H}_1, \tilde{H}_3]L_2 + [\tilde{H}_1, \tilde{H}_2]L_3 \Big) .$$

III.2. **Lorentzian fuzzy geometries.** Here we keep the usual conventions: the index 0 for time and (undotted) Latin spatial indices $a = 1, 2, 3$. In the Lorentzian setting $g = \mathrm{diag}(+1, -1, -1, -1)$, so $q = 3$, $e_0 = +1$ and $e_a = -1$ for each spatial a. A parametrization of the Dirac operator of the form $D = D(H, L_a, Q, R_a)$ by six anti-hermitian matrices $K_a = L_a$, $X_a = R_a$ and two hermitian matrices $K_0 = H$ and $X_0 = Q$ follows. As before, we give first the quadratic part and then the quartic. The former follows from eq. (5.2),

$$\frac{1}{8}\mathrm{Tr}\, D^2 = N\mathrm{Tr}_N \Big\{ H^2 + Q^2 - \textstyle\sum_a (L_a^2 + R_a^2) \Big\} + (\mathrm{Tr}_N\, H)^2 + (\mathrm{Tr}_N\, Q)^2 + \sum_a (\mathrm{Tr}_N\, L_a)^2 + (\mathrm{Tr}_N\, R_a)^2 . \qquad \text{(III.6)}$$

Using eqs. (III.1) and (III.2) to rewrite Proposition 5.4, one gets

$$\begin{aligned} \mathcal{S}_4^{\mathrm{Lorentz}} = \mathrm{Tr}_N \Big\{ & 2H^4 + 2Q^4 + \sum_a \Big(L_a^4 + R_a^4 \Big) - \sum_a \Big[2(L_a R_a)^2 + 4L_a^2 \Big] \\ & + \sum_a \Big[-8H^2 L_a^2 - 8Q^2 R_a^2 + 4(HL_a)^2 + 4(QR_a)^2 \Big] \\ & + \sum_{a<c} \Big[8L_a^2 L_c^2 + 8R_a^2 R_c^2 - 4(L_a L_c)^2 - 4(R_a R_c)^2 \Big] \\ & - \sum_a \Big[2(HR_a)^2 + 4H^2 R_a^2 + 2(L_a Q)^2 + 4L_a^2 Q^2 \Big] \\ & + \sum_{a\neq c} 2(L_a R_c)^2 + 4L_a^2 R_c^2 - 2(HQ)^2 + 4H^2 Q^2 \end{aligned}$$

$$
\begin{aligned}
&+8\Big[Q\Big(L_1[L_2,L_3]+L_2[L_3,L_1]+L_3[L_1,L_2]\Big)\\
&\quad -R_1\Big(H[L_2,L_3]+L_2[L_3,H]+L_3[H,L_2]\Big)\\
&\quad -R_2\Big(H[L_1,L_3]+L_1[L_3,H]+L_3[H,L_1]\Big)\\
&\quad -R_3\Big(H[L_1,L_2]+L_1[L_2,H]+L_2[H,L_1]\Big)\Big]\\
&+8\Big[H\Big(R_1[R_2,R_3]+R_2[R_3,R_1]+R_3[R_1,R_2]\Big)\\
&\quad -L_1\Big(Q[R_2,R_3]+R_2[R_3,Q]+R_3[Q,R_2]\Big)\\
&\quad -L_2\Big(Q[R_1,R_3]+R_1[R_3,Q]+R_3[Q,R_1]\Big)\\
&\quad -L_3\Big(Q[R_1,R_2]+R_1[R_2,Q]+R_2[Q,R_1]\Big)\Big]\Big\}\,, \qquad \text{(III.7)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{B}_4^{\text{Lorentz}} &= 8\mathrm{Tr}_N\,Q\cdot\mathrm{Tr}_N\Big\{Q^3-\textstyle\sum_a(QR_a^2+3L_aQ^2\Big)+QH^2 \qquad \text{(III.8)}\\
&\qquad +3L_1[L_2,L_3]+[R_3,R_2]L_1+[R_3,R_1]L_2+[R_2,R_2]L_3\Big\}\\
&+8\mathrm{Tr}_N\,H\cdot\mathrm{Tr}_N\Big\{H^3-\textstyle\sum_a(HL_a^2-3HR_a^2\Big)+HQ^2\\
&\qquad +3R_1[R_2,R_3]+[L_3,L_2]R_1+[L_3,L_1]R_2+[L_2,L_1]R_3)\Big\}\\
&+8\sum_a\mathrm{Tr}_N\,R_a\cdot\mathrm{Tr}_N\Big\{R_aH^2-R_a\textstyle\sum_c R_c^2-L_aR_a^2\\
&\qquad +3H^2R_a-3\textstyle\sum_{c(c\neq a)}R_a^2L_c\Big\}\\
&-8\mathrm{Tr}_N\,R_1\cdot\mathrm{Tr}_N\Big([R_2,R_3]H-[R_3,Q]L_2-[R_2,Q]L_3+3H[L_3,L_2]\Big)\\
&-8\mathrm{Tr}_N\,R_2\cdot\mathrm{Tr}_N\Big([R_1,R_3]H-[R_3,Q]L_1-[Q,R_1]L_3+3H[L_3,L_1]\Big)\\
&-8\mathrm{Tr}_N\,R_3\cdot\mathrm{Tr}_N\Big([R_1,R_2]H-[Q,R_2]L_1-[Q,R_1]L_2+3H[L_2,L_1]\Big)\\
&+8\sum_a\mathrm{Tr}_N\,L_a\cdot\mathrm{Tr}_N\Big\{L_aH^2-L_a\textstyle\sum_c L_c^2-R_aL_a^2\\
&\qquad +3L_aQ^2-3\textstyle\sum_{c(c\neq a)}L_aR_c^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- 8\mathrm{Tr}_N\, L_1 \cdot \mathrm{Tr}_N \Big([L_2, L_3]Q - [L_3, H]R_2 - [L_2, H]R_3 + 3Q[R_3, R_2] \Big) \\
&- 8\mathrm{Tr}_N\, L_2 \cdot \mathrm{Tr}_N \Big([L_1, L_3]Q - [L_3, H]R_1 - [H, L_1]R_3 + 3Q[R_3, R_1] \Big) \\
&- 8\mathrm{Tr}_N\, L_3 \cdot \mathrm{Tr}_N \Big([L_1, L_2]Q - [H, L_2]R_1 - [H, L_1]R_2 + 3Q[R_2, R_1] \Big) \\
&+ 6\Big[\mathrm{Tr}_N\, Q^2\Big]^2 + \sum_a \Big\{ 4\mathrm{Tr}_N\,(R_a^2) \cdot \mathrm{Tr}_N\,(Q^2) - 8\Big[\mathrm{Tr}_N\,(QR_a)\Big]^2 \Big\} \\
&+ \sum_{a,c} \Big(2\mathrm{Tr}_N\,(R_a^2) \cdot \mathrm{Tr}_N\,(R_c^2) + 4\Big[\mathrm{Tr}\,(R_aR_c)\Big]^2 \Big) \\
&+ 6\Big[\mathrm{Tr}_N\, H^2\Big]^2 + \sum_a \Big\{ 4\mathrm{Tr}_N\,(L_a^2) \cdot \mathrm{Tr}_N\,(H^2) - 8\Big[\mathrm{Tr}_N\,(HL_a)\Big]^2 \Big\} \\
&+ \sum_{a,c} \Big(2\mathrm{Tr}_N\,(L_a^2) \cdot \mathrm{Tr}_N\,(L_c^2) + 4\Big[\mathrm{Tr}\,(L_aL_c)\Big]^2 \Big) \\
&+ 4\mathrm{Tr}_N\,(Q^2) \cdot \mathrm{Tr}_N\,(H^2) + 8\Big[\mathrm{Tr}_N\,(HQ)\Big]^2 \\
&+ 4\sum_a \mathrm{Tr}_N\,(R_a^2) \cdot \mathrm{Tr}_N\,(L_a^2) + 8\Big[\mathrm{Tr}_N\,(L_aR_a)\Big]^2 \\
&+ 24\sum_a \Big\{ \Big[\mathrm{Tr}_N\,(HR_a)\Big]^2 + \Big[\mathrm{Tr}_N\,(QL_a)\Big]^2 \Big\} \\
&+ 12\sum_{a\neq c} 2\Big[\mathrm{Tr}_N\,(L_aR_c)\Big]^2 + \mathrm{Tr}_N\,(L_a^2) \cdot \mathrm{Tr}_N\,(R_c^2) \\
&- 12\sum_a \Big\{ \mathrm{Tr}_N\,(L_a^2) \cdot \mathrm{Tr}_N\,(Q^2) + \mathrm{Tr}_N\,(R_a^2) \cdot \mathrm{Tr}_N\,(H^2) \Big\}\,.
\end{aligned}
$$

IV. PROOFS SKIPPED IN THE MAIN ARTICLE

Proof of Proposition 5.1. The proof is by direct, even if in cases lengthy, computation. One first seeks the conditions one has to impose on the indices for a diagram not to vanish, typically in terms of Kronecker deltas, and then one computes their coefficients in terms of the quadratic form $g = \mathrm{diag}(e_1, e_2, e_3, e_4)$. We order the proof by similarity of the statements:

- *Type* τ_1. The τ_1-type diagram is well-known, for it is the only one here without any single multi-index (see the end of Sec. 3.1).
- *Type* τ_6. Notice that at least two pairings of the ν_l's are needed for the diagram not to vanish: $\nu_i = \nu_j =: \nu$ and $\nu_t = \nu_r =: \mu$ with $\{i, j, t, r\} = \Delta_4$. Therefore, the Kronecker deltas are placed precisely as for the τ_1 type. The computation of their e-factors is a matter of counting: for each chord joining two points labeled with, say, α there is an e_α factor. There are 6 such chords, labeled by $\{e_\alpha\}_{\alpha\neq\nu} \cup \{e_\rho\}_{\rho\neq\mu}$, for in $\hat\nu$ all the indices $\alpha \neq \nu$ appear and similarly for $\hat\mu$. Thus, if $\nu \neq \alpha \neq \mu$, e_α appears twice, so $e_\alpha^2 = 1$. The two remaining chord-labels are those appearing either in $\{e_\alpha\}_{\alpha\neq\nu}$ or in $\{e_\rho\}_{\rho\neq\mu}$. Thus the factor is $e_\mu e_\nu$ and we only have to compute the sign: the $\hat\mu\hat\nu\hat\mu\hat\nu$ configuration with minimal crossings has sign $(-1)^5$. For $\hat\mu\hat\mu\hat\nu\hat\nu$ and $\hat\mu\hat\nu\hat\nu\hat\mu$ the crossings yield a positive sign $(-1)^6$.

- *Type* τ_2. Since $\hat{\nu} \in \Lambda^-_{d=4}$, all the three indices α_i in $\hat{\nu} = (\alpha_1, \alpha_2, \alpha_3)$ different. For this diagram not to vanish, the set equality $\{\alpha_1, \alpha_2, \alpha_3\} = \{\mu_1, \mu_2, \mu_3\}$ should hold, i.e. a permutation $\sigma \in \{\mu_1, \mu_2, \mu_3\}$ with $\alpha_{\sigma(i)} = \mu_i$ is needed. This says first, that ν cannot be any of μ_i (whence the $\delta_{\nu\mu_1\mu_2\mu_3}$) and second, that each of the three chords yields a factor e_{μ_i} with a sign $(-1)^{|\sigma|+1}$. The extra minus is due to the convention to place the indices, e.g. for $\hat{4}123$, the numbers 123123 are put cyclicly; this permutation σ is the identity, which nevertheless looks like the 'V diagram' in eq. (4.7).
- *Type* τ_5. Suppose that two indices of a non-vanishing diagram agree. Then either $\mu = \nu_i$ or (wlog) $\nu_1 = \nu_2$. In the first case notice that in μ and $\hat{\nu}_i$ the indices $1, 2, 3, 4$ all appear listed. This implies for the remaining two multi-indices have to be of the form $\hat{\nu}_l = (* * *)$ and $\hat{\nu}_m = (1 * 4)$ or $\hat{\nu}_l = (1 * *)$ and $\hat{\nu}_m = (* * 4)$ where $\{i, l, m\} = \{1, 2, 3\}$ and $* \in \Delta_4$.
  - ◇ In the first case, $\hat{\nu}_l = (* * *)$ and $\hat{\nu}_m = (1 * 4)$ the numbers $\rho, \rho, 2, 3$ (for some $\rho \in \Delta_4$) should fill the placeholders $*$. Then ρ has to appear in both $\hat{\nu}_l$ and $\hat{\nu}_m$, but no value of ρ fulfills this if the increasing ordering is to be preserved, hence we are only left with next case
  - ◇ If $\hat{\nu}_l = (1 * *)$ and $\hat{\nu}_m = (* * 4)$, say $\hat{\nu}_l = (1wx)$ and $\hat{\nu}_m = (yz4)$, then $\{w, x\}$ and $\{y, z\}$ are the sets $\{2, \rho\}$ and $\{3, \rho\}$ (not necessarily in this order), for some ρ. Clearly, ρ cannot be either 1 or 4 since each appears once in one multi-index. But $\rho = 2, 3$ would force also a repetition of indices in at least one multi-index, which contradicts $\hat{\nu}_m, \hat{\nu}_l \in \Lambda^-_4$.

  This contradiction implies that if μ equals some ν_i then the diagram vanishes. By a similar analysis one sees that a repetition $\nu_i = \nu_j$ implies also that the diagram is zero. Hence the diagram is a multiple of $\delta_{\mu\nu_1\nu_2\nu_3}$. Thereafter it is easy to compute the e-coefficients following a similar argument to the given for the type τ_2 diagram to arrive at the sign $(-1)^{|\lambda|+1}$ for λ a permutation of $\{\nu_1, \nu_2, \nu_3\}$.
- *Type* τ_3. If no indices coincide then one gets two different numbers $i, j \in \Delta_4$ appearing exactly once in the list $\mu_1, \hat{\nu}_1, \mu_2, \hat{\nu}_2$. Since these cannot be matched by a chord, a non-zero diagram requires repetitions.
  - ◇ If $\mu_1 = \mu_2$ then $\nu_1 = \nu_2$. Since by hypothesis the four cannot agree the minimal crossings for this configuration is seen to be one, so the sign is (-1). The e-factors are: e_{μ_1} for the chord between μ_1 and μ_2, the product of three $\prod_{\alpha\neq\nu_1} e_\alpha$, for the three chords between $\hat{\nu}_1$ and $\hat{\nu}_2$. This accounts for $-e_{\mu_1}\bigg(\prod_{\alpha\neq\nu} e_\alpha\bigg)\delta^{\mu_2}_{\mu_1}\delta^{\nu_2}_{\nu_1}$.
  - ◇ If $\mu_1 = \nu_1$, then again $\mu_2 = \nu_2$ in order for the indices listed in $\mu_1, \hat{\nu}_1, \mu_2, \hat{\nu}_2$ to appear precisely twice. Since $\mu_1 = \nu_1$ implies that μ_1 does not appear in $\hat{\nu}_1$, there is one chord (thus a factor e_α) for each $\alpha \in \Delta_4$. After straightforward (albeit neither brief nor very illuminating) computation one finds the sign $(-1)^{\mu_1+\mu_2+1}$. All in all, one gets $(-1)^{\mu_1+\mu_2+1}e_1e_2e_3e_4\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}$.
  - ◇ If $\mu_1 = \nu_2$, then again $\mu_2 = \nu_1$. But this is the same as the last point with $\nu_1 \leftrightarrow \nu_2$. This accounts for $(-1)^{\mu_1+\mu_2}e_1e_2e_3e_4\delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2}$.
- *Type* τ_4. Mutatis mutandis from the type τ_3. □

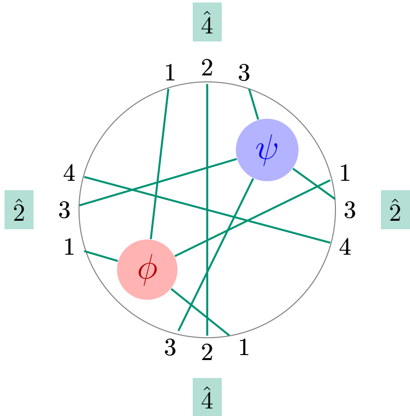


FIGURE IV.3. The diagram $\chi^{\hat{2}\hat{4}\hat{2}\hat{4}}(\phi,\psi)$ shows the nested structure of two τ_1-type diagrams, ϕ and ψ, in a τ_6-type referred to in Remark IV.1

Remark IV.1. We just used the 'minimal' number of crossings for diagrams with a more than two-fold index repetition. For instance, for the four point diagram evaluated in $\chi^{1111}=1$ there might be one crossing or no crossings, but crucially two diagrams have no crossing so $\sum_\chi \chi^{1111}\times$ traces $=(1-1+1)\times$ traces. This reappears in the computation of twelve-point diagrams in a nested fashion, as shown in Figure IV.3. If we pick $\hat{2}\hat{4}\hat{2}\hat{4}$ as configuration of the indices, then imposing $\chi^{\hat{2}\hat{4}\hat{2}\hat{4}}\neq 0$ does not determine $\chi\in \mathrm{CD}_6$: the lines joining the two 2-indices and the two 4-indices diagonally are mandatory, but for the four 1-indices and four 3-indices one can choose at the blobs tagged with ϕ,ψ one of three possibilities (shown in the diagrams of eq. (3.4) as θ,ξ,ζ). Then there are 9 possible sign values. Again, it is essential that there are 5 positive and 4 negative global signs in $\{\chi^{\hat{2}\hat{4}\hat{2}\hat{4}}(\phi,\psi)\}_{\psi,\phi\in\{\theta,\xi,\zeta\}}$ and the sum $\sum_\chi \chi^{\hat{2}\hat{4}\hat{2}\hat{4}}(\text{traces})$ can be replaced by the diagram with minimal crossings (of global positive sign).

Proof of Claim 5.3. To obtain these relations one needs Proposition 5.1. The first and last cases are obvious, since $\chi^{I_1I_2I_3I_4}\neq 0$ requires in each case a repetition $e^2_{\mu_i}e^2_{\mu_j}=1$ or $e^2_{\hat{\nu}_i}e^2_{\hat{\nu}_j}=1$.

For the second, $e_{\hat{\nu}}e_{\mu_1}e_{\mu_2}e_{\mu_3}=(-1)^{u(\hat{\nu})+\lfloor 3/2\rfloor+\sum_i u(\mu_i)}$, by Appendix II. The non-vanishing of $\chi^{\hat{\nu}\mu_1\mu_2\mu_3}$ implies that $\hat{\nu}$ is the multi-index containing μ_1,μ_2,μ_3, so $u(\mu_1)+u(\mu_2)+u(\mu_3)=u(\hat{\nu})$ and eq. (5.9) follows.

For the third identity, if $\chi^{\mu_1\mu_2\hat{\nu}_1\hat{\nu}_2}$ (and thus $\chi^{\mu_1\hat{\nu}_1\mu_2\hat{\nu}_2}$) does not vanish, then it is either of the form $\chi^{\mu\mu\hat{\mu}\hat{\mu}}$, $\chi^{\mu\mu\hat{\nu}\hat{\nu}}$ or $\chi^{\mu\nu\hat{\mu}\hat{\nu}}$ ($\mu\neq\nu$). Only for the latter one needs a non-trivial check:

$$\begin{aligned} e_\mu e_\nu e_{\hat{\mu}} e_{\hat{\nu}} &= e_\mu\cdot(-1)^{1+u(\Delta_4-\{\mu\})}e_\nu\cdot(-1)^{1+u(\Delta_4-\{\nu\})} \\ &= e_\mu e_\nu (-1)^{2u(\Delta_4-\{u,v\})}(-1)^{u(\mu)+u(\nu)}, \end{aligned}$$

and since $e_\mu=(-1)^{u(\mu)}$, $e_\mu e_\nu e_{\hat{\mu}}e_{\hat{\nu}}=1$. In either case, eq. (5.10) follows.

We are left with the fourth identity. By assumption all the indices $\nu_j \neq \nu_i \neq \mu$ if $i \neq j$. Then by eq. (2.9)

$$\begin{aligned} e_\mu e_{\hat\nu_1} e_{\hat\nu_2} e_{\hat\nu_3} &= e_\mu \cdot (-1)^{3\times\lfloor 3/2\rfloor + u(\Delta_4\backslash\{\nu_1\}) + u(\Delta_4\backslash\{\nu_2\}) + u(\Delta_4\backslash\{\nu_3\})} \\ &= -e_\mu (-1)^{3u(\mu)} (-1)^{2u(\nu_1)} (-1)^{2u(\nu_2)} (-1)^{2u(\nu_3)} = -1 \end{aligned}$$

From the first to the second line we used $\Delta_4 - \{\nu_1\} = \{\mu, \nu_2, \nu_3\}$, and similar relations. □

Proof of Corollary 6.2. Up to an irrelevant (as to assess cyclic self-adjointness) ambiguity in a global factor for Φ_i and Ψ_i, all the NC polynomials can be read off from eqs. (4.1), (4.3) and Proposition 4.1 for the $d = 2$ case. For $d = 4$, the result follows by inspection of each term, which is immediate since formulae (III.8), (III.5), (III.3) and (III.6) are given in terms of commutators. Then one uses that $[h, l]^* = [h, l]$, $[h_1, h_2]^* = -[h_1, h_2]$ and $[l_1, l_2]^* = -[l_1, l_2]$.

The only non-obvious part is dealing with expressions like

$$\mathcal{P} = Q\{L_1[L_2, L_3] + L_2[L_3, L_1] + L_3[L_1, L_2]\}\,,$$

which appears in $\mathcal{S}_4^{\text{Lorentz}}$ according to eq. (III.7). However, if P is the NC polynomial given in eq. (6.4) then $\mathcal{P}$ equals $P(Q, L_1, L_2, L_3)$, hence it is cyclic self-adjoint by Example 6.1. Also for the NC polynomial Ψ defined there, $-8\text{Tr}_N L_1 \cdot \text{Tr}_N (\Psi(Q, L_2, L_3))$ appears in the expression given by eq. (III.8) for $\mathcal{B}_4^{\text{Lorentz}}$, being both $\Phi(L_1) = L_1$ and Ψ cyclically anti-self-adjoint. □

V. Definition of 'cyclically self-adjointness'

Definition V.1. Given variables $z_1, \ldots, z_\kappa$, each of which satisfies either formal self-adjointness (i.e. for an involution $*$, $z_i^* = +z_i$ holds, in whose case we let $z_i =: h_i$) or formal anti-self-adjointness ($z_i^* = -z_i$; and if so, write $z_i =: l_i$), a noncommutative (NC) polynomial $P \in \mathbb{R}\langle z_1, \ldots, z_\kappa\rangle$ is said to be *cyclically self-adjoint* if the following conditions hold:

- for each word w (or monomial) of P there exists a word w' in P such that

  $$[w(z_1, \ldots, z_\kappa)]^* = +(\sigma \cdot w')(z_1, \ldots, z_\kappa) \text{ holds for some } \sigma \in \mathbb{Z}/|w'|\mathbb{Z}\,, \quad \text{(V.1)}$$

  being
  - ⋄ $|w|$ the length of the word w (or order of the monomial w) and
  - ⋄ $\sigma \cdot w'$ the action of $\mathbb{Z}/|w'|\mathbb{Z}$ on the word w' by cyclic permutation of its letters.
- The map defined by $w \mapsto w'$ is a bijection in the set of the words of P.

Similarly, a polynomial $G \in \mathbb{R}\langle z_1, \ldots, z_\kappa\rangle$ is *cyclic anti-self-adjoint* if for each of its words w if there exist a $\sigma \in \mathbb{Z}/|w'|\mathbb{Z}$ for which the condition

$$[w(z_1, \ldots, z_\kappa)]^* = -(\sigma \cdot w')(z_1, \ldots, z_\kappa) \quad \text{(V.2)}$$

holds, and if, additionally, the map that results from this condition, $w \mapsto w'$, is a bijection in the set of words of G.

VI. AN AUXILIARY MODEL FOR THE $d = 1$ CASE

One could reformulate the partition functions of $d = 1$-models in terms of auxiliary models that do not contain multi-traces, if these are interpreted at least as formal integrals. We pick for concreteness the signature $(p, q) = (1, 0)$ and the polynomial $f(x) = (x^2 + \lambda x^4)/2$ for the spectral action $\mathrm{Tr}\, f(D)$. We explain why the ordinary matrix model given by $\mathcal{Z}^{\mathrm{aux}}_{(1,0)} = \int_{\mathbb{H}_N} \mathrm{e}^{\gamma \mathrm{Tr}\, H - \alpha \mathrm{Tr}\,(H^2) + \beta \mathrm{Tr}\,(H^3) - N\lambda \mathrm{Tr}\,(H^4)} \mathrm{d}H$ over the hermitian $N \times N$ matrices $\mathbb{H}_N$, allows to restate the quartic-quadratic (1,0)-type Barrett-Glaser model with partition function

$$\mathcal{Z}^{\mathrm{BG}}_{(1,0)} = \int_{\mathcal{M}} \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,(D^2 + \lambda D^4)} \mathrm{d}D \tag{VI.1}$$

as formally equivalent to the functional

$$\langle \exp\{-(3\lambda \mathrm{Tr}\, H^2 + 4\lambda \mathrm{Tr}\, H \cdot \mathrm{Tr}\, H^3 + (\mathrm{Tr}\, H)^2)\}\rangle_{\mathrm{aux},0}\ , \tag{VI.2}$$

where the expectation value of an observable Φ is taken with respect to the auxiliary model

$$\langle \Phi \rangle_{\mathrm{aux}} = \frac{1}{\mathcal{Z}^{\mathrm{aux}}_{(1,0)}} \int_{\mathbb{H}_N} \Phi(H) \mathrm{e}^{-\mathscr{S}(H)} \mathrm{d}H\,,$$

being $\mathscr{S}(H) = \alpha \mathrm{Tr}\,(H^2) + N\lambda \mathrm{Tr}\,(H^4) + \gamma \mathrm{Tr}\, H + \beta \mathrm{Tr}\,(H^3)$. The zero subindex 'aux,0' means evaluation in the parameters

$$\alpha = N, \gamma = \beta = 0\,. \tag{VI.3}$$

Indeed, one can use the explicit form of the Dirac operator $D = \{H, \,\cdot\,\}$ to rewrite the integral in terms of the matrix H. One gets

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}\,(D^2 + \lambda D^4) &= N[\mathrm{Tr}\,(H^2) + \lambda \mathrm{Tr}\,(H^4)] \\ &\quad + 3\lambda[\mathrm{Tr}\,(H^2)]^2 + 4\lambda \mathrm{Tr}\, H \cdot \mathrm{Tr}\, H^3 + [\mathrm{Tr}\,(H)]^2 \\ &= N[\mathrm{Tr}\,(H^2) + \lambda \mathrm{Tr}\,(H^4)] + \mathfrak{b}[H]\,. \end{aligned} \tag{VI.4}$$

The second line of eq. (VI.4) contains the bi-tracial terms; this term will be denoted by $\mathfrak{b}[H]$. Inserting last equations into (VI.1)

$$\mathcal{Z}^{\mathrm{BG}}_{(1,0)} = \int_{\mathbb{H}_N} \mathrm{e}^{-N\mathrm{Tr}\,(H^2 + \lambda H^4)} \mathrm{e}^{-\mathfrak{b}[H]} \mathrm{d}H$$

Since $\mathscr{S}(H)|_{\alpha=N,\gamma=\beta=0} = N[\mathrm{Tr}\,(H^2) + \lambda \mathrm{Tr}\,(H^4)]$, one can replace the first exponential by $\mathrm{e}^{-\mathscr{S}(H)}$ and evaluate the parameters as in eq. (VI.3):

$$\mathcal{Z}^{\mathrm{BG}}_{(1,0)} = \int_{\mathbb{H}_N} \Big[\mathrm{e}^{-\mathscr{S}(H)}\Big]_0 \mathrm{e}^{-\mathfrak{b}[H]} \mathrm{d}H\,.$$

If one knows the partition function $\mathcal{Z}^{\mathrm{aux}}_{(1,0)}$, one can compute the model in question by taking out $\mathrm{e}^{-\mathfrak{b}(H)}$ from the integral and accordingly substituting the traces by the appropriate derivatives:

$$\mathcal{Z}^{\mathrm{BG}}_{(1,0)} = \Big[\mathrm{e}^{-\mathfrak{b}_\partial} \int_{\mathbb{H}_N} \mathrm{e}^{-\mathscr{S}(H)} \mathrm{d}H\Big]_0 \qquad \text{where } \mathfrak{b}_\partial = 3\lambda \partial^2_\alpha + 4\lambda \partial_\beta \partial_\gamma + \partial_\alpha\,. \tag{VI.5}$$

That is $\mathcal{Z}^{\mathrm{BG}}_{(1,0)} = [\mathrm{e}^{-\mathfrak{b}_\partial} \mathcal{Z}^{\mathrm{aux}}_{(1,0)}]_0$, which also proves eq. (VI.2). This motivates to look for similar methods in order restate, for $d \geq 2$, the bi-tracial part of the models addressed here as single-trace auxiliary multi-matrix models.