On anchored Lie algebras and the Connes–Moscovici bialgebroid construction

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Abstract. We show how the Connes–Moscovici bialgebroid construction naturally provides universal objects for Lie algebras acting on non-commutative algebras. As a supplement, we see how these objects interact with the study of flat bimodule connections.

Introduction

Given a Hopf algebra H (possibly with bijective antipode S) over a field $\Bbbk$ and a left H-module algebra A, one can turn the vector space $A \otimes H \otimes A$ into a left bialgebroid $\mathcal{H} := A \odot H \odot A$ over A in a natural way. This procedure has been introduced independently, and under different forms, by Connes and Moscovici [10] in their study of the index theory of transversely elliptic operators, and by Kadison [27] in connection with his work on (pseudo-)Galois extensions. Later, Panaite and Van Oystaeyen proved in [39] that the two constructions are in fact equivalent (isomorphic as A-bialgebroids) and that, as algebras, they are particular instances of the L-R-smash product introduced in [40]. Nevertheless, following [5], we refer to the bialgebroid $A \odot H \odot A$ as the *Connes–Moscovici bialgebroid*.

Following the foregoing and, in particular, in view of the results in [39], two observations were made, that triggered the present investigation,

(i) that whenever a Lie algebra L acts by derivations on an associative algebra A (for the sake of simplicity, let us call it an *A-anchored Lie algebra*), then A becomes naturally an $U_{\Bbbk}(L)$-module algebra and

(ii) that the associated Connes–Moscovici bialgebroid construction satisfies a universal property (both as A^e-ring and as A-bialgebroid, see [39, Proposition 3.1 and Theorem 3.2]) which suggests the possibility that $A \odot U_{\Bbbk}(L) \odot A$ plays for an A-anchored Lie algebra L the same role played by the universal enveloping algebra for a Lie algebra.

Among anchored Lie algebras we find the well-known Lie–Rinehart algebras, which are in particular Lie algebras acting on commutative algebras. As it can be inferred from the substantial literature on the topic, Lie–Rinehart algebras are a deeply investigated area, in particular for its connections with differential geometry (the global sections of a Lie algebroid $\mathcal{L} \to \mathcal{M}$ form a Lie–Rinehart algebra over $\mathcal{C}^\infty(\mathcal{M})$). Rinehart himself gave an explicit construction of the universal enveloping algebra $U(R, L)$ of a Lie–Rinehart algebra in [44] and proved a Poincaré–Birkhoff–Witt theorem for the latter. Other equivalent constructions are provided in [16, §3.2], [23, page 64], [48, §18]. The universal property of $U(R, L)$ as an algebra is spelled out in [23, page 64] and [32, page 174] (where it is attributed to Feld'man). Its universal property as an A-bialgebroid is codified in the Cartier–Milnor–Moore Theorem for $U(R, L)$ proved in [35, §3], where Moerdijk and Mrčun show that the construction of the universal enveloping algebra provides a left adjoint to the functor associating any cocommutative bialgebroid with its Lie–Rinehart algebra of primitive elements and they find natural conditions under which this adjunction becomes an equivalence (as it has been done in [34] for cocommutative bialgebras and Lie algebras). Further algebraic and categorical properties and applications are investigated in [2, 17, 23–25].

However, there are many important examples of Lie algebras acting by derivations on associative algebras which are not necessarily commutative (actually, any Lie algebra acts by derivations on its universal enveloping algebra and any associative algebra acts by inner derivations on itself). Furthermore, while the space of primitive elements of a bialgebra is always a Lie algebra and a primitively generated bialgebra is always cocommutative, the space of primitive elements of a bialgebroid is not, in general, a Lie–Rinehart algebra and not every primitively generated bialgebroid is necessarily cocommutative. A third observation that stood up for the present investigation is that, instead, the space of primitive elements of a bialgebroid is always a Lie algebra acting by derivations on the base algebra.

These facts, together with the two foregoing observations (i) and (ii), called for the study of Lie algebras L acting on non-commutative algebras A in their own right and, in particular, for the study of the associated Connes–Moscovici bialgebroid $A \odot U_{\Bbbk}(L) \odot A$, as it has been done for Lie–Rinehart algebras and their universal enveloping algebras.

In the present paper, we are mainly concerned with two universal properties of $\mathcal{B}_L := A \odot U_{\Bbbk}(L) \odot A$, as an A^e-ring and as an A-bialgebroid, which reflect the two well-known universal properties of universal enveloping algebras reported above. The first one (Theorem 2.9) exhibits $A \odot U_{\Bbbk}(L) \odot A$ as the universal A^e-ring associated with the A-anchored Lie algebra L, similarly to what happens for $U(R, L)$ in [23, page 64]. Namely, for any A^e-ring $\phi_A : A^e \to \mathcal{R}$ and any $\Bbbk$-Lie algebra morphism $\phi_L : L \to \mathcal{L}(\mathcal{R})$ such that

$$\big[\phi_L(X), \phi_A(a \otimes b^o)\big] = \phi_A(X \cdot (a \otimes b^o))$$

for all $a, b \in A$ and all $X \in L$, there exists a unique morphism of A^e-rings $\Phi : \mathcal{B}_L \to \mathcal{R}$ extending ϕ_L. This naturally affects the study of the representations of L (see Corollary 2.10). The second universal property (Proposition 3.4) exhibits $A \odot U_{\Bbbk}(L) \odot A$ as

the universal A-bialgebroid associated with the A-anchored Lie algebra L, similarly to what happens for $U(R, L)$ in [35, Theorem 3.1(i)]. Namely, for any A-bialgebroid $\mathcal{B}$ and any morphism of $\Bbbk$-Lie algebras $\phi_L : L \to \mathcal{B}$ which lands into the space $\mathsf{Prim}(\mathcal{B})$ of primitive elements of $\mathcal{B}$ and that is compatible with the anchors, there exists a unique morphism of A-bialgebroids $\Phi : \mathcal{B}_L \to \mathcal{B}$ that extends ϕ_L.

Parallel to the new insights into the structure of bialgebroids over non-commutative base, which have been the author's original motivation for the present study and that are mainly collected in Sections 3 and 4, the representation theory of anchored Lie algebras provide also an alternative, derivation-based approach to non-commutative differential calculus over associative algebras, as shown briefly in Section 5. In this, the universal property of $A \odot U_{\Bbbk}(L) \odot A$ as an A^e-ring and its bialgebroid structure play a crucial role in deducing categorical properties of the associated flat bimodule connections.

Concretely, after a first section devoted to recalling some definitions and some preliminary results, we introduce A-anchored Lie algebras in Section 2.1 and we prove that the Connes–Moscovici bialgebroid associated to an A-anchored Lie algebra satisfies the stated universal property as A^e-ring in Section 2.2 (Theorem 2.9). In Section 3.1, we detail how taking the space of primitives of an A-bialgebroid induces a functor from the category of A-bialgebroids to the category of A-anchored Lie algebras and in Section 3.2 we show that the Connes–Moscovici construction provides a natural left adjoint to this latter functor (Theorem 3.6) and, at the same time, we prove the second universal property of $A \odot U_{\Bbbk}(L) \odot A$ (Proposition 3.4). At this point, by finding inspiration from the Milnor–Moore and the Moerdijk–Mrčun theorems, we look for intrinsic conditions on a bialgebroid that allow us to recognize it as a $A \odot U_{\Bbbk}(L) \odot A$ for a certain A-anchored Lie algebra. Section 4 is devoted to finding a first answer (Theorem 4.14). After studying in more detail the space of primitives of a Connes–Moscovici bialgebroid in Section 4.1, we tackle the question in the general framework in Section 4.2 and in the particular case of bialgebroids over a commutative base in Section 4.3. Finally, we conclude with some final remarks about future lines of investigation in Section 4.4. As a supplement, in Section 5 we show how representations of an A-anchored Lie algebra L are naturally equivalent to certain flat bimodule connections on the first-order differential calculus $\mathsf{Hom}_{\Bbbk}(L, A)$ and we comment on how this research direction can be extended further and connected with the differential calculi studied by Dubois-Violette and collaborators (see, e.g., [12–14]).

Notation

All over the paper, we assume a certain familiarity of the reader with the language of monoidal categories and of (co)monoids therein (see, for example, [30, VII]).

We work over a ground field $\Bbbk$ of characteristic 0. All vector spaces are assumed to be over $\Bbbk$. The unadorned tensor product $\otimes$ stands for $\otimes_{\Bbbk}$. All (co)algebras and bialgebras are intended to be $\Bbbk$-(co)algebras and $\Bbbk$-bialgebras, that is to say, (co)algebras and bialgebras in the symmetric monoidal category of vector spaces $(\mathsf{Vect}_{\Bbbk}, \otimes, \Bbbk)$. Every (co)module has an underlying vector space structure. Identity morphisms Id_V are often denoted by V.

In order to avoid confusion between indexes of elements and coproducts or coactions, we will adopt the following variant of the Heyneman–Sweedler's Sigma Notation. For c in a coalgebra C, m in a left C-comodule M and n in a right C-comodule N we write

$$\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}, \qquad \rho_M(m) = \sum m_{(-1)} \otimes m_{(0)}, \qquad \rho_N(n) = \sum n_{(0)} \otimes n_{(1)}.$$

Given an algebra A, we denote by A^o its opposite algebra. We freely use the canonical isomorphism between the category of left A-modules $_A\mathsf{Mod}$ and that of right A^o-modules Mod_{A^o}. We also set $A^e := A \otimes A^o$ and we identify the category of left A^e-modules $_{A^e}\mathsf{Mod}$ with that of A-bimodules $_A\mathsf{Mod}_A$. Recall that any morphism of algebras $\eta : A^e \to R$ leads to two commuting algebra maps $s : A \to R$, $a \mapsto a \otimes 1^o$, and $t : A^o \to R$, $a^o \mapsto 1 \otimes a^o$, (i.e. $s(a)t(b^o) = t(b^o)s(a)$ for all $a, b \in A$) and conversely. Given two R-bimodules M and N, this gives rise to several A-module structures on M and N and it leads to several ways of considering the tensor product over A between the underlying A-bimodules. In the present paper, we focus on the A-bimodule structure induced by A^e acting on the left via η and which we denote by $_sM_{t^o}$. We usually consider this bimodule structure when taking tensor products. If we want to stress the fact that M is considered as a left A^e-module, we may also write $_\eta M$. Therefore, given two R-bimodules M and N, we consider the tensor product A-bimodule

$$M \otimes_A N := {}_sM_{t^o} \otimes_A {}_sN_{t^o} = \frac{M \otimes N}{\langle t(a^o)m \otimes n - m \otimes s(a)n \mid m \in M, n \in N, a \in A \rangle}. \quad (1)$$

Inside $M \otimes_A N$, we will also consider the distinguished subspace

$$M \times_A N := \left\{ \sum_i m_i \otimes_A n_i \in M \otimes_A N \mid \sum_i m_i t(a^o) \otimes_A n_i = \sum_i m_i \otimes_A n_i s(a) \right\}, \quad (2)$$

which is often called *Takeuchi–Sweedler's ×-product*. It is an A-subbimodule with actions

$$a \cdot \left(\sum_i m_i \otimes_A n_i \right) = \sum_i s(a)m_i \otimes_A n_i$$

and

$$\left(\sum_i m_i \otimes_A n_i \right) \cdot a = \sum_i m_i \otimes_A t(a^o)n_i$$

for all $\sum_i m_i \otimes_A n_i \in M \times_A N$ and $a \in A$ (see [48, Definition 2.1] and [49, page 460]). In particular, the following relations hold for all $m \in M$, $n \in N$, $\sum_i m_i \otimes_A n_i \in M \times_A N$ and for all $a \in A$:

$$t(a^o)m \otimes_A n \overset{(1)}{=} m \otimes_A s(a)n \qquad \text{and} \qquad \sum_i m_i t(a^o) \otimes_A n_i \overset{(2)}{=} \sum_i m_i \otimes_A n_i s(a). \quad (3)$$

For the sake of clarity, it will be useful to set

$$a \rhd m \lhd b = s(a)t(b^o)m = \eta(a \otimes b^o)m, \quad b \blacktriangleright m \blacktriangleleft a = ms(a)t(b^o) = m\eta(a \otimes b^o) \quad (4)$$

for $m \in {}_\eta M_\eta$ and $a, b \in A$.

1. Preliminaries

We begin by collecting some facts about bimodules, corings and bialgebroids that will be needed in the sequel. The aim is to keep the exposition self-contained. Many results and definitions we will present herein hold in a more general context and under less restrictive hypotheses, but we preferred to limit ourselves to the essentials.

Given a (preferably, non-commutative) $\Bbbk$-algebra A, the category of A-bimodules forms a non-strict monoidal category $(_A\mathsf{Mod}_A, \otimes_A, A, \mathfrak{a}, \mathfrak{l}, \mathfrak{r})$. Nevertheless, all over the paper we will behave as if the structural natural isomorphisms

$$\mathfrak{a}_{M,N,P} : (M \otimes_A N) \otimes_A P \to M \otimes_A (N \otimes_A P),$$
$$(m \otimes_A n) \otimes_A p \mapsto m \otimes_A (n \otimes_A p),$$
$$\mathfrak{l}_M : A \otimes_A M \to M, \quad a \otimes_A m \mapsto a \cdot m,$$
$$\mathfrak{r}_M : M \otimes_A A \to M, \quad m \otimes_A a \mapsto m \cdot a,$$

were "the identities", that is, as if $_A\mathsf{Mod}_A$ was a strict monoidal category.

1.1. Graded and filtered A-bimodules

As far as we are concerned, we assume A to be filtered over $\mathbb{Z}$ with filtration

$$F_n(A) = 0 \quad \text{for all } n < 0 \qquad \text{and} \qquad F_n(A) = A \quad \text{for all } n \geq 0$$

and we assume it to be graded over $\mathbb{Z}$ with grading

$$A_0 = A \qquad \text{and} \qquad A_n = 0 \quad \text{for all } n \neq 0.$$

By a *graded A-bimodule* we mean an A-bimodule M with a family of A-subbimodules $\{M_n \mid n \in \mathbb{Z}\}$ such that $M = \bigoplus_{n \in \mathbb{Z}} M_n$. By a *filtered A-bimodule* we mean an A-bimodule M with a chain of A-subbimodules $\{F_n(M) \mid n \in \mathbb{Z}\}$, that is $F_p(M) \subseteq F_q(M)$ if $p \leq q$. The filtration is said to be *exhaustive* if $M = \bigcup_{n \in \mathbb{Z}} F_n(M)$. Given a filtered A-bimodule M, one can consider the *associated graded* bimodule $\mathrm{gr}(M)$ defined by

$$\mathrm{gr}_n(M) := \frac{F_n(M)}{F_{n-1}(M)} \qquad \text{and} \qquad \mathrm{gr}(M) := \bigoplus_{n \in \mathbb{Z}} \mathrm{gr}_n(M).$$

In what follows we will be interested in *positively* filtered and graded bimodules, that is, those for which the negative terms are 0.

Given two filtered bimodules M, N, we can perform their tensor product $M \otimes_A N$ and this is still a filtered bimodule: the k-th term of the filtration on $M \otimes_A N$ is the A-subbimodule generated by the elements $m \otimes_A n$ such that $m \in F_s(M)$, $n \in F_t(N)$ and $s + t = k$; analogously for two graded A-bimodules. With these tensor products, we have that the categories $(_A\mathsf{FMod}_A, \otimes_A, A)$ and $(_A\mathsf{GMod}_A, \otimes_A, A)$ of filtered and graded bimodules, respectively, are monoidal categories (it follows, for instance, from [36, Chapter A,

Proposition I.2.14 and Chapter D, Lemma VIII.1]). Morphisms of filtered (respectively, graded) bimodules are A-bilinear maps that respect the filtration (respectively, grading).

The result we are principally interested in is that the construction of the graded bimodule associated to a filtered bimodule is functorial (see, for example, [36, chapter D, §III]). Moreover, the natural surjection

$$\varphi_{M,N} : \mathrm{gr}(M) \otimes_A \mathrm{gr}(N) \to \mathrm{gr}(M \otimes_A N), \tag{5}$$

uniquely determined by

$$(m + F_{s-1}(M)) \otimes_A (n + F_{t-1}(N)) \mapsto (m \otimes_A n) + F_{s+t-1}(M \otimes_A N)$$

for $m \in F_s(M)$ and $n \in F_t(N)$ (see [36, page 318]), and the isomorphism $\varphi_0 : A \cong \mathrm{gr}(A)$ endow the functor

$$\mathrm{gr}(-) : {}_A\mathsf{FMod}_A \to {}_A\mathsf{GMod}_A, \quad M \mapsto \mathrm{gr}(M),$$

with a structure of lax monoidal functor (see [1, Definition 3.1]). For further details about filtered and graded bimodules, we refer the reader to [36].

1.2. A-corings

Now, we recall that an A-coring is a monoid in the monoidal category of A-bimodules $({}_A\mathsf{Mod}_A, \otimes_A, A)$. More concretely, an A-coring is an A-bimodule $\mathcal{C}$ endowed with a comultiplication $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and a counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that

$$(\Delta_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}} = (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} \quad \text{and} \quad (\varepsilon_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}} = \mathcal{C} = (\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}. \tag{6}$$

For the general theory of corings and their comodules, we refer to [8].

Later on, we will be particularly interested in (exhaustively) filtered A-corings such that the associated graded components are projective as A-bimodules. These are A-corings $\mathcal{C}$ endowed with an increasing filtration $\{F_n(\mathcal{C}) \mid n \in \mathbb{N}\}$ as A-bimodules such that $\mathcal{C} = \bigcup_n F_n(\mathcal{C})$, $\mathrm{gr}_n(\mathcal{C}) = F_n(\mathcal{C})/F_{n-1}(\mathcal{C})$ is a projective A-bimodule and

$$\Delta_{\mathcal{C}}(F_n(\mathcal{C})) \subseteq \sum_{i+j=n} F_i(\mathcal{C}) \otimes_A F_j(\mathcal{C}) \tag{7}$$

for all $n \geq 0$ (i.e., $\Delta_{\mathcal{C}}$ is a morphism of filtered A-bimodules). We will refer to these A-corings as *graded projective filtered A-corings*. By convention, we put $F_{-1}(\mathcal{C}) = 0$. Notice that the inclusion (7) makes sense in view of the following well-known result for filtered (bi)modules (see, for instance, [16, Lemma B.1]).

Lemma 1.1. *Let $\mathcal{C}$ be an A-bimodule with filtration $\{F_n(\mathcal{C}) \mid n \in \mathbb{N}\}$ such that $\mathrm{gr}_n(\mathcal{C})$ is a projective A-bimodule for all $n \geq 0$. Then*

$$F_n(\mathcal{C}) \cong \bigoplus_{k=0}^{n} \frac{F_k(\mathcal{C})}{F_{k-1}(\mathcal{C})} \tag{8}$$

for all $n \geq 0$ and so $F_n(\mathcal{C})$ is a projective A-bimodule. Moreover, the canonical map

$$\mathrm{gr}(\mathcal{C}) \otimes_A \mathrm{gr}(\mathcal{C}) \to \mathrm{gr}(\mathcal{C} \otimes_A \mathcal{C}),$$
$$\big(\xi + F_h(\mathcal{C})\big) \otimes_A \big(\xi' + F_k(\mathcal{C})\big) \mapsto \xi \otimes_A \xi' + F_{h+k+1}(\mathcal{C} \otimes_A \mathcal{C}),$$

from (5) *is an isomorphism of A-bimodules. If, in addition, the filtration $\{F_n(\mathcal{C}) \mid n \in \mathbb{N}\}$ is exhaustive, then $\mathcal{C} \cong \mathrm{gr}(\mathcal{C})$ as A-bimodules and, in particular, $\mathcal{C}$ is a projective A-bimodule.*

Proof. By definition and projectivity of $\mathrm{gr}_n(\mathcal{C})$, we have a split short exact sequence

$$0 \longrightarrow F_{n-1}(\mathcal{C}) \longrightarrow F_n(\mathcal{C}) \longrightarrow \mathrm{gr}_n(\mathcal{C}) \longrightarrow 0$$

of A-bimodules, which implies that, as A-bimodules,

$$F_n(\mathcal{C}) \cong F_{n-1}(\mathcal{C}) \oplus \mathrm{gr}_n(\mathcal{C}).$$

By proceeding recursively, one reaches (8). The second claim follows from [31, Theorem C.24]. About the last claim in the statement, saying that the filtration is exhaustive means that $\mathcal{C} \cong \varinjlim_n (F_n(\mathcal{C}))$ as A-bimodules. Since (8) means that $F_n(\mathcal{C}) \cong F_n(\mathrm{gr}(\mathcal{C}))$ as A-bimodules, we have that $\mathcal{C} \cong \varinjlim_n (F_n(\mathcal{C})) \cong \varinjlim_n (F_n(\mathrm{gr}(\mathcal{C}))) \cong \mathrm{gr}(\mathcal{C})$ as claimed. ∎

Analogously to the theory of filtered coalgebras (see, for example, [47, §11.1]), the graded A-bimodule $\mathrm{gr}(\mathcal{C})$ associated to a graded projective filtered A-coring $\mathcal{C}$ becomes a graded A-coring in a natural way, as the subsequent Proposition 1.2 formalizes.

For the sake of clarity, by a *graded A-coring* we mean an A-coring $\mathcal{D}$ endowed with an $\mathbb{N}$-grading $\{\mathcal{D}_n \mid n \in \mathbb{N}\}$ as A-bimodule such that every $\mathcal{D}_n$ is projective as A-bimodule,

$$\Delta_{\mathcal{D}}(\mathcal{D}_n) \subseteq \bigoplus_{i+j=n} \mathcal{D}_i \otimes_A \mathcal{D}_j \quad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad \varepsilon_{\mathcal{D}}(\mathcal{D}_n) = 0 \quad \text{for all } n \geq 1.$$

It can be seen as a comonoid in the monoidal category of graded A-bimodules. Notice that $\Delta_{\mathcal{D}}$ is uniquely determined by the A-bilinear maps

$$\Delta_{\mathcal{D}}^{[n]} : \mathcal{D}_n \to \bigoplus_{i+j=n} \mathcal{D}_i \otimes_A \mathcal{D}_j$$

obtained by (co)restriction of $\Delta_{\mathcal{D}}$ to the graded components of $\mathcal{D}$ and $\mathcal{D} \otimes_A \mathcal{D}$ and which, in turn, are uniquely determined by the A-bilinear maps

$$\Delta_{\mathcal{D}}^{[h,k]} := \left(\mathcal{D}_n \xrightarrow{\Delta_{\mathcal{D}}^{[n]}} \bigoplus_{i+j=n} \mathcal{D}_i \otimes_A \mathcal{D}_j \xrightarrow{p_{h,k}^{\mathcal{D}}} \mathcal{D}_h \otimes_A \mathcal{D}_k \right) \tag{9}$$

for all $n \geq 0$ and for all $h + k = n$. Following [3, Definition 2.2], we say that the graded A-coring $\mathcal{D}$ is *strongly graded* whenever $\Delta_{\mathcal{D}}^{[h,k]}$ is injective for all $h, k \in \mathbb{N}$.

Proposition 1.2. *Let $\mathcal{C}$ be a graded projective filtered A-coring. Then the A-coring structure on $\mathcal{C}$ induces an A-coring structure on $\mathrm{gr}(\mathcal{C})$. Moreover, for any morphism of graded projective filtered A-corings $f : \mathcal{C} \to \mathcal{B}$, the induced graded A-bilinear morphism $\mathrm{gr}(f) : \mathrm{gr}(\mathcal{C}) \to \mathrm{gr}(\mathcal{B})$ is a morphism of graded A-corings.*

Proof. The first claim follows from the functoriality of $\mathrm{gr}(-)$ and from the fact that $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \to A$ are filtered morphisms of A-bimodules. In fact, they induce graded morphisms of A-bimodules

$$\mathrm{gr}(\mathcal{C}) \xrightarrow{\mathrm{gr}(\Delta_{\mathcal{C}})} \mathrm{gr}(\mathcal{C} \otimes_A \mathcal{C}) \cong \mathrm{gr}(\mathcal{C}) \otimes_A \mathrm{gr}(\mathcal{C}) \qquad \text{and} \qquad \mathrm{gr}(\mathcal{C}) \xrightarrow{\mathrm{gr}(\varepsilon_{\mathcal{C}})} \mathrm{gr}(A) \cong A,$$

which provides an A-coring structure on $\mathrm{gr}(\mathcal{C})$, since $\mathrm{gr}(-)$ is lax monoidal. Concerning the second claim, it is enough to apply $\mathrm{gr}(-)$ to the diagrams expressing the comultiplicativity and counitality of f. ∎

Remark 1.3. For any morphism $f : \mathcal{C} \to \mathcal{B}$ of filtered A-corings, we have that

$$
\begin{array}{ccc}
\mathrm{gr}_n(\mathcal{C}) & \xrightarrow{\Delta^{[n]}_{\mathrm{gr}(\mathcal{C})}} & \bigoplus_{i+j=n} \mathrm{gr}_i(\mathcal{C}) \otimes_A \mathrm{gr}_j(\mathcal{C}) \\
{\scriptstyle \mathrm{gr}_n(f)} \downarrow & & \downarrow {\scriptstyle \bigoplus_{i+j=n} \mathrm{gr}_i(f) \otimes_A \mathrm{gr}_j(f)} \\
\mathrm{gr}_n(\mathcal{B}) & \xrightarrow[\Delta^{[n]}_{\mathrm{gr}(\mathcal{B})}]{} & \bigoplus_{i+j=n} \mathrm{gr}_i(\mathcal{B}) \otimes_A \mathrm{gr}_j(\mathcal{B})
\end{array}
\tag{10}
$$

commutes for all $n \geq 0$.

Corollary 1.4 (of Proposition 1.2). *The assignment $\mathcal{C} \mapsto \mathrm{gr}(\mathcal{C})$ induces a functor $\mathrm{gr}(-)$ from the category of graded projective filtered A-corings to the category of graded A-corings.*

1.3. A-bialgebroids

Next, we recall the definition of a left bialgebroid. It can be considered as a revised version of the notion of a $\times_A$-bialgebra as it appears in [45, Definition 4.3]. However, we prefer to mimic [29] as presented in [7, Definition 2.2] (in light of [7, Theorem 3.1], this is something we may do).

Definition 1.5. A *left bialgebroid* is the datum consisting of

(B1) a pair $(A, \mathcal{B})$ of $\Bbbk$-algebras, called *the base algebra* and *the total algebra* respectively,

(B2) a $\Bbbk$-algebra map $\eta_{\mathcal{B}} : A^e \to \mathcal{B}$, inducing a *source* $s_{\mathcal{B}} : A \to \mathcal{B}$ and a *target* $t_{\mathcal{B}} : A^o \to \mathcal{B}$ which are $\Bbbk$-algebra maps, and making of $\mathcal{B}$ an A^e-bimodule,

(B3) an A-coring structure $(\mathcal{B}, \Delta_{\mathcal{B}}, \varepsilon_{\mathcal{B}})$ on the A-bimodule ${}_\eta \mathcal{B} = {}_s \mathcal{B}_{t^o}$,

satisfying

(B4) $\Delta_{\mathcal{B}}$ takes values into $\mathcal{B} \times_A \mathcal{B}$ and $\Delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \times_A \mathcal{B}$ is a morphism of $\Bbbk$-algebras, where the algebra structure on $\mathcal{B} \times_A \mathcal{B}$ is given by the component-wise product

$$(\xi \otimes_A \zeta)(\xi' \otimes_A \zeta') = \xi\xi' \otimes_A \zeta\zeta', \tag{11}$$

for all $\xi, \xi', \zeta, \zeta' \in \mathcal{B}$,

(B5) $\varepsilon_{\mathcal{B}}\big(\xi s_{\mathcal{B}}(\varepsilon(\xi'))\big) = \varepsilon_{\mathcal{B}}(\xi\xi') = \varepsilon_{\mathcal{B}}\big(\xi t_{\mathcal{B}}(\varepsilon_{\mathcal{B}}(\xi')^o)\big)$ for all $\xi, \xi' \in \mathcal{B}$,

(B6) $\varepsilon_{\mathcal{B}}(1_{\mathcal{B}}) = 1_A$.

A $\Bbbk$-linear map $\varepsilon_{\mathcal{B}} : \mathcal{B} \to A$ which is left A^e-linear and satisfies (B5) and (B6) is called a *left character* on the A^e-ring $\mathcal{B}$ (see [5, Lemma 2.5 and following]).

A morphism of left bialgebroids from $(A, \mathcal{B})$ to $(A', \mathcal{B}')$ is a pair of $\Bbbk$-algebra morphisms $\phi_0 : A \to A'$ and $\phi_1 : \mathcal{B} \to \mathcal{B}'$ such that

$$\phi_1 \circ s_{\mathcal{B}} = s_{\mathcal{B}'} \circ \phi_0, \qquad \phi_1 \circ t_{\mathcal{B}} = t_{\mathcal{B}'} \circ \phi_0, \qquad \varepsilon_{\mathcal{B}'} \circ \phi_1 = \phi_0 \circ \varepsilon_{\mathcal{B}}$$

and

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\phi_1} & \mathcal{B}' \\
\Delta_{\mathcal{B}} \downarrow & & \downarrow \Delta_{\mathcal{B}'} \\
\mathcal{B} \otimes_A \mathcal{B} & \xrightarrow{\phi_1 \otimes_A \phi_1} \mathcal{B}' \otimes_A \mathcal{B}' \twoheadrightarrow & \mathcal{B}' \otimes_{A'} \mathcal{B}'
\end{array}
$$

commutes. In this paper, we will focus on left bialgebroids over a fixed base algebra A, that we call *left A-bialgebroids*. A morphism of left A-bialgebroids between $\mathcal{B}$ and $\mathcal{B}'$ is then an algebra map $\phi : \mathcal{B} \to \mathcal{B}'$ such that

$$\phi \circ s_{\mathcal{B}} = s_{\mathcal{B}'}, \qquad \phi \circ t_{\mathcal{B}} = t_{\mathcal{B}'}, \qquad \varepsilon_{\mathcal{B}'} \circ \phi = \varepsilon_{\mathcal{B}}, \qquad (\phi \otimes_A \phi) \circ \Delta_{\mathcal{B}} = \Delta_{\mathcal{B}'} \circ \phi.$$

The category of left A-bialgebroids will be denoted by Bialgd_A.

We will often omit to specify the A-bialgebroid $\mathcal{B}$ in writing the comultiplication $\Delta_{\mathcal{B}}$ or the counit $\varepsilon_{\mathcal{B}}$, when it is clear from the context.

Remark 1.6. Let us make explicit some of the relations involved in the definition of a left bialgebroid and some of their consequences. In terms of elements of A and $\mathcal{B}$, and by resorting to Sweedler Sigma Notation, relations (6) become

$$
\begin{aligned}
\sum \xi_{(1)(1)} \otimes_A \xi_{(1)(2)} \otimes_A \xi_{(2)} &= \sum \xi_{(1)} \otimes_A \xi_{(2)(1)} \otimes_A \xi_{(2)(2)}, \\
\sum s\big(\varepsilon(\xi_{(1)})\big)\xi_{(2)} &= \xi = \sum t\big(\varepsilon(\xi_{(2)})^o\big)\xi_{(1)}
\end{aligned}
\tag{12}
$$

for all $\xi \in \mathcal{B}$. Moreover, for all $a, b \in A$ the A-bilinearity of Δ forces

$$\sum \big(s(a)t(b^o)\xi\big)_{(1)} \otimes_A \big(s(a)t(b^o)\xi\big)_{(2)} = \sum s(a)\xi_{(1)} \otimes_A t(b^o)\xi_{(2)}.$$

In particular,

$$\Delta(s(a)) = s(a) \otimes_A 1_{\mathcal{B}} \qquad \text{and} \qquad \Delta(t(a^o)) = 1_{\mathcal{B}} \otimes_A t(a^o) \tag{13}$$

for all $a \in A$. As a consequence, the multiplicativity of Δ forces

$$
\begin{aligned}
\Delta(\xi s(a)) = \Delta(\xi)\Delta(s(a)) &= \Big(\sum \xi_{(1)} \otimes_A \xi_{(2)} \Big)(s(a) \otimes_A 1_{\mathcal{B}}) \\
&= \sum \xi_{(1)} s(a) \otimes_A \xi_{(2)}, \\
\Delta(\xi t(a^o)) = \Delta(\xi)\Delta(t(a^o)) &= \Big(\sum \xi_{(1)} \otimes_A \xi_{(2)} \Big)(1_{\mathcal{B}} \otimes_A t(a^o)) \\
&= \sum \xi_{(1)} \otimes_A \xi_{(2)} t(a^o),
\end{aligned}
\tag{14}
$$

for all $\xi \in \mathcal{B}$. The A-bilinearity of ε becomes

$$
\varepsilon\big(s(a)t(b^o)\xi\big) = a\varepsilon(\xi)b
\tag{15}
$$

and since it also preserves the unit, we have that

$$
\varepsilon(s(a)) = \varepsilon(s(a)1_{\mathcal{B}}) = a\varepsilon(1_{\mathcal{B}}) = a = \varepsilon(1_{\mathcal{B}})a = \varepsilon(t(a^o)).
\tag{16}
$$

Therefore, in light of the character condition on ε,

$$
\varepsilon(\xi s(a)) \overset{\text{(B5)}}{=} \varepsilon\big(\xi t\big(\varepsilon(s(a))^o\big)\big) \overset{\text{(16)}}{=} \varepsilon(\xi t(a^o)).
\tag{17}
$$

One may also define, in a symmetric way, the notion of a *right bialgebroid* (see [28, §2]). Henceforth, however, all the bialgebroids we consider will be left bialgebroids, whence we will often omit to specify the adjective "left".

Example 1.7. Let us give some examples.

(a) Any bialgebra over a field $\Bbbk$ is a $\Bbbk$-bialgebroid.

(b) On the algebra $A \otimes A^o$ we may consider the morphisms

$$
\begin{aligned}
&s : A \to A \otimes A^o, \quad a \mapsto a \otimes 1^o, \\
&t : A^o \to A \otimes A^o, \quad b^o \mapsto 1 \otimes b^o, \\
&\Delta : A \otimes A^o \to (A \otimes A^o) \otimes_A (A \otimes A^o), \quad a \otimes b^o \mapsto (a \otimes 1^o) \otimes_A (1 \otimes b^o), \\
&\varepsilon : A \otimes A^o \to A, \quad a \otimes b^o \mapsto ab.
\end{aligned}
$$

These make of $A \otimes A^o$ an A-bialgebroid (see [29, Example 3.1]).

(c) Assume that A is a finite-dimensional algebra over $\Bbbk$ and set $\mathcal{B} := \mathsf{End}_{\Bbbk}(A)$. Consider $s : A \to \mathcal{B}$ given by left multiplication and $t : A \to \mathcal{B}$ given by right multiplication. The morphism

$$
\mathcal{B} \otimes \mathcal{B} \to \mathsf{Hom}_{\Bbbk}(A \otimes A, A), \qquad f \otimes g \mapsto [a \otimes b \mapsto f(a)g(b)]
$$

induces an isomorphism $\mathcal{B} \otimes_A \mathcal{B} \cong \mathsf{Hom}_{\Bbbk}(A \otimes A, A)$. In view of this, one can endow $\mathcal{B}$ with a structure of A-bialgebroid with source s, target t, Δ given by

$$
\Delta(f)(a \otimes b) = f(ab)
$$

(up to the latter isomorphism) and ε by evaluation at 1_A (see [29, page 56]).

(d) Let $R \to S$ be a depth two ring extension (see [28, Definition 3.1]) and set $A := C_S(R)$, the centralizer of R in S. Then the ring of endomorphisms $\mathcal{B} := \mathsf{End}_R(S)$ of S as an R-bimodule satisfies

$$\mathcal{B} \otimes_A \mathcal{B} \cong {}_R\mathsf{Hom}_R(S \otimes_R S, S), \qquad f \otimes_A g \mapsto [s \otimes_R s' \mapsto f(s)g(s')],$$

as above (see [28, Proposition 3.11]) and we may endow it with an A-bialgebroid structure exactly as in (c) ([28, Theorem 4.1]).

(e) Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra and let A be a braided commutative algebra in ${}^H_H \mathcal{YD}$. This means that A is at the same time a left H-module algebra (that is, an algebra in the monoidal category $({}_H\mathsf{Mod}, \otimes, \Bbbk)$ of left H-modules) with action $H \otimes A \to A, h \otimes a \mapsto h \cdot a$, satisfying

$$h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b) \qquad \text{and} \qquad h \cdot 1_A = \varepsilon(h)1_A$$

for all $h \in H$, $a, b \in A$, and a left H-comodule algebra with coaction $\rho : A \to H \otimes A$, satisfying

$$\rho(ab) = \sum a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)} \qquad \text{and} \qquad \rho(1_A) = 1_H \otimes 1_A$$

for all $a, b \in A$. Furthermore, these structures are required to satisfy

$$\sum h_{(1)}a_{(-1)} \otimes h_{(2)} \cdot a_{(0)} = \sum (h_{(1)} \cdot a)_{(-1)}h_{(2)} \otimes (h_{(1)} \cdot a)_{(0)}$$

and $\sum (a_{(-1)} \cdot b)a_{(0)} = ab$ for all $a, b \in A$ and $h \in H$ (the latter expresses the braided commutativity). Under these conditions, the smash product algebra $H\#A$ is an A-bialgebroid with

$$(h \otimes a)(h' \otimes b) = \sum h_{(1)}h' \otimes (h_{(2)} \cdot b)a,$$
$$1_{H\#A} = 1_H \otimes 1_A,$$
$$s(a) = \sum a_{(-1)} \otimes a_{(0)},$$
$$t(a^o) = 1_H \otimes a,$$
$$\Delta(h \otimes a) = \sum (h_{(1)} \otimes 1_A) \otimes_A (h_{(2)} \otimes a),$$
$$\varepsilon(h \otimes a) = \varepsilon(h)a$$

for all $a, b \in A$, $h, h' \in H$. This is a left-left symmetrical version of [5, Example 3.4.7] and [7, Theorem 4.1].

(f) *Connes–Moscovici bialgebroid* (see [10, 39] and [5, Example 3.4.6]). Let H be a Hopf algebra (in fact, a bialgebra would suffice) and let A be an H-module algebra. The vector space $\mathcal{B} := A \otimes H \otimes A$ becomes an algebra via

$$(a \otimes h \otimes b)(a' \otimes h' \otimes b') = \sum a(h_{(1)} \cdot a') \otimes h_{(2)}h' \otimes (h_{(3)} \cdot b')b \qquad (18)$$

and unit $1_A \otimes 1_H \otimes 1_A$. It can be endowed with an A-bialgebroid structure as follows:

$$s_{\mathcal{B}}(a) := a \otimes 1_H \otimes 1_A,$$
$$t_{\mathcal{B}}(a^o) := 1_A \otimes 1_H \otimes a,$$
$$\Delta_{\mathcal{B}}(a \otimes h \otimes b) := \sum (a \otimes h_{(1)} \otimes 1_A) \otimes_A (1_A \otimes h_{(2)} \otimes b),$$
$$\varepsilon_{\mathcal{B}}(a \otimes h \otimes b) := a\varepsilon_H(h)b.$$

Following [39], we will denote this bialgebroid by $A \odot H \odot A$, avoiding the use of symbols like # or $\ltimes, \rtimes$ in order to avoid confusion with two-sided smash or crossed products in the sense of [19]. Notice that for all $a, a', b, b' \in A$ and $h \in H$ we have

$$a \triangleright (a' \otimes h \otimes b') \triangleleft b \overset{(4)}{=} (a \otimes 1_H \otimes 1_A)(1_A \otimes 1_H \otimes b)(a' \otimes h \otimes b')$$
$$\overset{(18)}{=} aa' \otimes h \otimes b'b,$$
$$a \blacktriangleright (a' \otimes h \otimes b') \blacktriangleleft b \overset{(4)}{=} (a' \otimes h \otimes b')(b \otimes 1_H \otimes 1_A)(1_A \otimes 1_H \otimes a)$$
$$\overset{(18)}{=} \sum a'(h_{(1)} \cdot b) \otimes h_{(2)} \otimes (h_{(3)} \cdot a)b'.$$

Remark 1.8. It is known that an A^e-ring is a left bialgebroid if and only if the category $_{\mathcal{B}}\mathsf{Mod}$ of left $\mathcal{B}$-modules is monoidal in such a way that the forgetful functor $_{\mathcal{B}}\mathsf{Mod} \to {}_{A^e}\mathsf{Mod}$ is a monoidal functor (see, for example, [45, Theorem 5.1]). In particular, the tensor product of two $\mathcal{B}$-modules M and N is $M \otimes_A N$ with diagonal action $\xi \cdot (m \otimes_A n) := \sum(\xi_{(1)} \cdot m) \otimes_A (\xi_{(2)} \cdot n)$ and A is a $\mathcal{B}$-module with

$$\xi \cdot a := \varepsilon(\xi t(a^o)) \overset{(17)}{=} \varepsilon(\xi s(a)). \tag{19}$$

It is, in fact, a left $\mathcal{B}$-module algebra (see [5, §3.7.1] for a right-handed analogue). In particular, for all $a, b \in A$ and all $\xi \in \mathcal{B}$ we have

$$\sum(\xi_{(1)} \cdot a)(\xi_{(2)} \cdot b) = \xi \cdot ab \qquad \text{and} \qquad \xi \cdot 1_A = \varepsilon(\xi). \tag{20}$$

2. The Connes–Moscovici bialgebroid as universal A^e-ring

In this section, we introduce A-anchored Lie algebras and we show that the Connes–Moscovici bialgebroid $A \odot U_{\Bbbk}(L) \odot A$ naturally associated to an A-anchored Lie algebra satisfies a universal property as A^e-ring. In particular, unless stated otherwise, we assume to work over a fixed base algebra A, possibly non-commutative. We conclude the section with an extension of the PBW theorem to bialgebroids of the form $A \odot U_{\Bbbk}(L) \odot A$.

2.1. *A*-anchored Lie algebras

Definition 2.1. An *A-anchored Lie algebra* is an ordinary Lie algebra L over $\Bbbk$ together with a Lie algebra morphism $\omega : L \to \mathsf{Der}_{\Bbbk}(A)$, called the *anchor*. We will often write $X \cdot a$ for $\omega(X)(a)$. A morphism of A-anchored Lie algebras between (L, ω) and (L', ω') is a Lie algebra morphism $f : L \to L'$ such that $\omega' \circ f = \omega$. The category of A-anchored Lie algebras and their morphisms will be denoted by $\mathsf{AnchLie}_A$. As a matter of notation, we may write simply (A, L, ω) to mean the A-anchored Lie algebra (L, ω).

Remark 2.2. The reader needs to be warned that the terminology "A-anchored" used here is inspired from the literature, but it neither strictly coincides with the classical notion of A-anchored module, nor it properly extends it. In fact, in the literature, an "A-anchored module" [42, §1] (also called "A-module with arrow" [41, §3] or "A-module fléché" [43, §1]) is an A-module M over a *commutative* algebra A together with an *A-linear map* $M \to \mathsf{Der}_{\Bbbk}(A)$. Since, in the present framework, A is assumed to be preferably non-commutative, the vector space $\mathsf{Der}_{\Bbbk}(A)$ does not carry any natural A-module structure and hence we do not have any reasonable way to speak about an A-linear anchor. In spite of this, in order to limit the proliferation of different terminology in the field and trusting that the non-commutative context will help in distinguishing between the two notions, we decided to adopt the term "A-anchored" in this framework as well.

Example 2.3. Let us give a few important examples.

(a)   The Lie algebra $\mathsf{Der}_{\Bbbk}(A)$ with the identity map is an A-anchored Lie algebra.

(b)   Any Lie algebra L is a $U_{\Bbbk}(L)$-anchored Lie algebra.

(c)   Any $\Bbbk$-algebra A, with the associated Lie algebra structure $\mathscr{L}(A) = (A, [-, -]_A)$ given by the commutator bracket, is an A-anchored Lie algebra with anchor induced by the adjoint representation. Namely,

$$\mathscr{L}(A) \to \mathsf{Der}_{\Bbbk}(A), \qquad a \mapsto \big[b \mapsto [a, b]_A \big].$$

(d)   A *Lie–Rinehart algebra* over a commutative algebra R (called in this way in honour of G. S. Rinehart, who studied them in [44] under the name of (K, R)-Lie algebras) is a Lie algebra L endowed with a (left) R-module structure $R \otimes L \to L$, $r \otimes X \mapsto r \cdot X$, and with a Lie algebra morphism $\omega : L \to \mathsf{Der}_{\Bbbk}(R)$ such that, for all $r \in R$ and $X, Y \in L$,

$$\omega(r \cdot X) = r \cdot \omega(X) \qquad \text{and} \qquad [X, r \cdot Y] = r \cdot [X, Y] + \omega(X)(r) \cdot Y.$$

Clearly, any Lie–Rinehart algebra over R is an R-anchored Lie algebra.

(e)   Let $\mathscr{B}$ be an A-bialgebroid and consider the vector space of *primitive elements*

$$\mathsf{Prim}(\mathscr{B}) := \big\{ X \in \mathscr{B} \mid \Delta(X) = X \otimes_A 1 + 1 \otimes_A X \big\}.$$

This is a Lie algebra with the commutator bracket. Assume that $X \in \mathsf{Prim}(\mathscr{B})$. In

light of Equation (20), X acts on A by derivations, which means that the assignment

$$\omega_{\mathscr{B}} : \mathsf{Prim}(\mathscr{B}) \to \mathsf{Der}_{\Bbbk}(A), \qquad X \mapsto X \cdot (-),$$

is well defined. Moreover,

$$
\begin{aligned}
\omega_{\mathscr{B}}(XY - YX)(a) &= (XY) \cdot a - (YX) \cdot a = X \cdot (Y \cdot a) - Y \cdot (X \cdot a) \\
&= \omega_{\mathscr{B}}(X)(\omega_{\mathscr{B}}(Y)(a)) - \omega_{\mathscr{B}}(Y)(\omega_{\mathscr{B}}(X)(a)) \\
&= [\omega_{\mathscr{B}}(X), \omega_{\mathscr{B}}(Y)](a)
\end{aligned}
$$

implies that $\omega_{\mathscr{B}}$ is a morphism of Lie algebras and hence it is an anchor for $\mathsf{Prim}(\mathscr{B})$. As a matter of notation, we write $\theta_{\mathscr{B}} : \mathsf{Prim}(\mathscr{B}) \to \mathscr{B}$ for the canonical inclusion.

Definition 2.4. Assume that (L, ω) is an A-anchored Lie algebra.

(L1) An *A-anchored Lie ideal* (L', ω') in (L, ω) is a Lie ideal L' in L together with an anchor ω' such that the inclusion $L' \subseteq L$ is a morphism in $\mathsf{AnchLie}_A$. Equivalently, it is vector subspace $L' \subseteq L$ such that $[X, X']_L \in L'$ for all $X' \in L'$, $X \in L$ with anchor ω' given by the restriction of ω.

(L2) An *A-anchored Lie subalgebra* (L'', ω'') in (L, ω) is a Lie subalgebra L'' of L together with an anchor ω'' such that the inclusion $L'' \subseteq L$ is a morphism in $\mathsf{AnchLie}_A$. Equivalently, it is a vector subspace $L'' \subseteq L$ such that $[X'', Y'']_L \in L''$ for all $X'', Y'' \in L''$ with anchor ω'' given by the restriction of ω.

(L3) If we have two A-anchored Lie algebras (L', ω') and (L'', ω'') and a Lie algebra morphism $\delta : L'' \to \mathsf{Der}_{\Bbbk}(L')$ such that

$$[\omega''(X''), \omega'(X')] = \omega'\big(\delta(X'')(X')\big) \tag{21}$$

in $\mathsf{Der}_{\Bbbk}(A)$ for all $X' \in L'$ and all $X'' \in L''$, then we define the *semi-direct product* of (L', ω') and (L'', ω'') to be the $\Bbbk$-vector space $L'' \times L'$ with Lie bracket

$$
\begin{aligned}
\big[(X'', X'), &(Y'', Y')\big] \\
&:= \big([X'', Y'']_{L''}, \delta(X'')(Y') - \delta(Y'')(X') + [X', Y']_{L'}\big) \tag{22}
\end{aligned}
$$

and anchor

$$L \to \mathsf{Der}_{\Bbbk}(A), \qquad (X'', X') \mapsto \omega''(X'') + \omega'(X'). \tag{23}$$

We denote it by $(L'' \ltimes_{\delta} L', \omega_{\delta})$.

For the sake of brevity, from now on we will only speak about ideals and subalgebras without reporting the syntagma "A-anchored Lie" in front. Definition 2.4(L3) is consistent in view of the following results.

Lemma 2.5. *The semi-direct product (L, ω) of two A-anchored Lie algebras (L', ω') and (L'', ω'') is an A-anchored Lie algebra.*

Proof. The fact that the semi-direct product is a Lie algebra follows from the fact that, as Lie algebras, it is the semi-direct product of L' and L''. Thus, we only need to check that ω_δ is a Lie algebra morphism. To this aim, we compute directly

$$
\begin{aligned}
\big[\omega_\delta\big((X'', X')\big), &\omega_\delta\big((Y'', Y')\big)\big] \\
&\overset{(23)}{=} \big[\omega''(X'') + \omega'(X'), \omega''(Y'') + \omega'(Y')\big] \\
&= \big[\omega''(X''), \omega''(Y'')\big] + \big[\omega'(X'), \omega''(Y'')\big] + \big[\omega''(X''), \omega'(Y')\big] \\
&\quad + \big[\omega'(X'), \omega'(Y')\big] \\
&\overset{(21)}{=} \omega''\big([X'', Y'']_{L''}\big) + \omega'\big(\delta(X'')(Y')\big) - \omega'\big(\delta(Y'')(X')\big) + \omega'\big([X', Y']_{L'}\big) \\
&\overset{(23)}{=} \omega_\delta\big(\big([X'', Y'']_{L''}, \delta(X'')(Y') - \delta(Y'')(X') + [X', Y']_{L'}\big)\big) \\
&\overset{(22)}{=} \omega_\delta\big(\big[(X'', X'), (Y'', Y')\big]\big)
\end{aligned}
$$

for all $X', Y' \in L'$ and all $X'', Y'' \in L''$. ∎

The following lemma should not be surprising.

Lemma 2.6. *Let (L, ω) be an A-anchored Lie algebra and let (L', ω') and (L'', ω'') be subalgebras of (L, ω). Then there exists a semi-direct product $(L'' \ltimes_\delta L', \omega_\delta)$ of (L', ω') and (L'', ω'') such that $L'' \ltimes_\delta L' \to L, (X'', X') \mapsto X'' + X'$, is an isomorphism of A-anchored Lie algebras if and only if*

- *(L', ω') is an ideal in (L, ω);*
- *$L = L'' + L'$ as $\Bbbk$-vector spaces;*
- *$L' \cap L'' = 0$ (that is, $L = L'' \oplus L'$).*

If this is the case, then $\delta : L'' \to \mathsf{Der}_\Bbbk(L')$ is given by $X'' \mapsto [X'', -]_L$.

Proof. In one direction, notice that (L', ω') is an ideal in $(L'' \ltimes_\delta L', \omega_\delta)$ via the canonical morphism $L' \to L'' \oplus L', X' \mapsto (0, X')$, and that (L'', ω'') is a subalgebra of $(L'' \ltimes_\delta L', \omega_\delta)$ via the canonical morphism $L'' \to L'' \oplus L', X'' \mapsto (X'', 0)$. Furthermore, one recovers δ as

$$
\big[(X'', 0), (0, X')\big] = \big(0, \delta(X'')(X')\big)
$$

for all $X' \in L', X'' \in L''$. In the other direction, assume that (L', ω') is an ideal in (L, ω) and that $L = L'' \oplus L'$ as $\Bbbk$-vector spaces. Consider further the assignment $\delta : L'' \to \mathsf{Der}_\Bbbk(L')$ given by $X'' \mapsto [X'', -]_L$ (which is well defined because L' is an ideal). It satisfies

$$
\begin{aligned}
\big[\omega''(X''), \omega'(X')\big] &= \omega(X'') \circ \omega(X') - \omega(X') \circ \omega(X'') \\
&= \omega\big([X'', X']_L\big) = \omega\big(\delta(X'')(X')\big) = \omega'\big(\delta(X'')(X')\big)
\end{aligned}
$$

for all $X' \in L'$, $X'' \in L''$, which is (21), and so we may perform the semi-direct product $(L'' \ltimes_\delta L', \omega_\delta)$. Then

$$
\begin{aligned}
\left[X'' + X', Y'' + Y'\right]_L &= [X'', Y'']_L + [X', Y'']_L + [X'', Y']_L + [X', Y']_L \\
&= [X'', Y'']_{L''} + \big(\delta(X'')(Y') - \delta(Y'')(X') + [X', Y']_{L'}\big),
\end{aligned}
$$

so that $L'' \oplus L' \cong L$ is a morphism of Lie algebras and moreover

$$
\omega(X'' + X') = \omega(X'') + \omega(X') = \omega''(X'') + \omega'(X'),
$$

whence it is of A-anchored Lie algebras, too. ∎

Remark 2.7. The reader has to be warned that, despite the definition of ideal and of semi-direct product in Definition 2.4 has been inspired by the subsequent Lemma 4.1 and Proposition 4.3 and by the results in Section 4.3, they may turn out to be improper terminologies in the future. In fact, it is not true in general that the quotient of an A-anchored Lie algebra by an ideal is an A-anchored Lie algebra (unless the ideal has the zero anchor) or that (L, ω) is a semi-direct product of (L', ω') and (L'', ω'') if and only if there is a short exact sequence of A-anchored Lie algebras

$$
0 \to L' \xrightarrow{f} L \xrightarrow{g} L'' \to 0 \tag{24}
$$

such that g admits a section σ which is a morphism of A-anchored Lie algebras. On the one hand, the canonical projection $L'' \ltimes_\delta L' \to L''$ is not a morphism of A-anchored Lie algebras because it is not compatible with the anchors. On the other hand, in order to have that $\omega'' \circ g = \omega$ and that $\omega \circ f = \omega'$, we should have had that

$$
\omega' = \omega \circ f = \omega'' \circ g \circ f = 0,
$$

which is not the case in general.

What one may observe is that (L, ω) is a semi-direct product of (L', ω') and (L'', ω'') if and only if there is a short exact sequence of Lie algebras (24) such that g admits a section σ and both f and σ are morphisms of A-anchored Lie algebras (but g is not, in general).

2.2. A universal A^e-ring construction

Assume that we are given an A-anchored Lie algebra (L, ω). Recall that we may consider the universal enveloping algebra $U_{\Bbbk}(L)$ of L and that there is a canonical injective $\Bbbk$-linear map

$$
j_L : L \to U_{\Bbbk}(L), \qquad X \mapsto x, \tag{25}
$$

which allows us to identify X with its image x in $U_{\Bbbk}(L)$. The anchor ω makes of A a left representation of L with L acting as derivations, that is, we have a Lie algebra morphism

$$
L \xrightarrow{\omega} \mathsf{Der}_{\Bbbk}(A) \subseteq \mathscr{L}(\mathsf{End}_{\Bbbk}(A)) \tag{26}
$$

where we have that $\mathcal{L}(\mathsf{End}_{\Bbbk}(A))$ is the Lie algebra associated to the associative algebra $(\mathsf{End}_{\Bbbk}(A), \circ, \mathrm{id}_A)$. By the universal property of the universal enveloping algebra, there is a unique algebra morphism

$$\Omega : U_{\Bbbk}(L) \to \mathsf{End}_{\Bbbk}(A)$$

which extends ω.

Lemma 2.8. *The base algebra A is naturally an $U_{\Bbbk}(L)$-module algebra.*

Proof. See, for instance, [11, Example 6.1.13(3)]. Explicitly,

$$x \cdot a = \omega(X)(a), \qquad x \cdot 1_A \overset{(26)}{=} 0 \qquad \text{and} \qquad u \cdot a = \Omega(u)(a) \tag{27}$$

for all $X \in L$, $u \in U_{\Bbbk}(L)$ and $a \in A$. ∎

As a consequence of Lemma 2.8, we may consider the Connes–Moscovici A-bialge-broid $A \odot U_{\Bbbk}(L) \odot A$. For the sake of simplicity, we will often denote it by $\mathcal{B}_L$. As an A^e-ring, it comes endowed with a Lie algebra morphism

$$J_L : L \to \mathcal{L}(\mathcal{B}_L), \qquad X \mapsto 1 \otimes x \otimes 1,$$

which satisfies

$$\left[J_L(X), \eta_{\mathcal{B}_L}(a \otimes b^o) \right]_{\mathcal{B}_L} = \eta_{\mathcal{B}_L}(X \cdot (a \otimes b^o)) \tag{28}$$

for all $a, b \in A$ and all $X \in L$, where

$$X \cdot (a \otimes b^o) = (X \cdot a) \otimes b^o + a \otimes (X \cdot b)^o$$

is the L-module structure on the tensor product of two L-modules. Equivalently, $\eta_{\mathcal{B}_L} : A^e \to \mathcal{B}_L$ is a morphism of L-modules, where $\mathcal{B}_L$ has the L-module structure induced by J_L.

The A^e-ring $\mathcal{B}_L$ with J_L is universal among pairs $(\mathcal{R}, \phi_L)$ satisfying these properties.

Theorem 2.9. *Given an A^e-ring $\mathcal{R}$ with $\Bbbk$-algebra morphism $\phi_A : A^e \to \mathcal{R}$ and given a Lie algebra morphism $\phi_L : L \to \mathcal{L}(\mathcal{R})$ such that*

$$\left[\phi_L(X), \phi_A(a \otimes b^o) \right]_{\mathcal{R}} = \phi_A(X \cdot (a \otimes b^o)), \tag{29}$$

for all $a, b \in A$ and all $X \in L$, there exists a unique morphism of A^e-rings $\Phi : \mathcal{B}_L \to \mathcal{R}$ such that $\Phi \circ J_L = \phi_L$. It is explicitly given by

$$\Phi : \mathcal{B}_L \to \mathcal{R}, \qquad a \otimes u \otimes b \mapsto \phi_A(a \otimes b^o)\phi'(u), \tag{30}$$

where $\phi' : U_{\Bbbk}(L) \to \mathcal{R}$ is the unique morphism of $\Bbbk$-algebras such that $\phi' \circ j_L = \phi_L$.

Proof. By the universal property of the universal enveloping $\Bbbk$-algebra $U_\Bbbk(L)$, there exists a unique morphism of $\Bbbk$-algebras $\phi' : U_\Bbbk(L) \to \mathcal{R}$ such that $\phi' \circ j_L = \phi_L$. Now, set $U := U_\Bbbk(L)$ and consider the $\Bbbk$-linear map

$$\Phi : A \odot U \odot A \to \mathcal{R}, \qquad a \otimes u \otimes b \mapsto \phi_A(a \otimes b^o)\phi'(u)$$

of (30). It follows immediately from the definition that

$$\Phi \circ \eta_{\mathcal{B}_L} = \phi_A \qquad \text{and} \qquad \Phi \circ J_L = \phi_L.$$

Now, a straightforward check using (29) and induction on a PBW basis of U shows that

$$\phi'(u)\phi_A(a \otimes b^o) = \sum \phi_A\big((u_{(1)} \cdot a) \otimes (u_{(3)} \cdot b)^o\big)\phi'(u_{(2)}) \tag{31}$$

for all $u \in U$ and $a, b \in A$. In view of this, we have

$$\begin{aligned}
\Phi\big(&(a \otimes u \otimes b)(a' \otimes v \otimes b')\big) \\
&\overset{(18)}{=} \sum \Phi\big(a(u_{(1)} \cdot a') \otimes u_{(2)}v \otimes (u_{(3)} \cdot b')b\big) \\
&\overset{(30)}{=} \sum \phi_A\big(a(u_{(1)} \cdot a') \otimes ((u_{(3)} \cdot b')b)^o\big)\phi'(u_{(2)}v) \\
&= \sum \phi_A(a \otimes b^o)\phi_A\big((u_{(1)} \cdot a') \otimes (u_{(3)} \cdot b')^o\big)\phi'(u_{(2)})\phi'(v) \\
&\overset{(31)}{=} \phi_A(a \otimes b^o)\phi'(u)\phi_A(a' \otimes b'^o)\phi'(v) \\
&= \Phi(a \otimes u \otimes b)\Phi(a' \otimes v \otimes b')
\end{aligned}$$

for all $u, v \in U$, $a, a', b, b' \in A$. Thus, Φ is a morphism of A^e-rings and it is clearly the unique one satisfying $\Phi \circ J_L = \phi_L$. ∎

Corollary 2.10. *Given an A-anchored Lie algebra (L, ω), any representation $\rho : L \to \mathsf{End}_\Bbbk(M)$ of L into an A-bimodule M satisfying the Leibniz condition*

$$\rho(X)(a \cdot m \cdot b) = a \cdot \rho(X)(m) \cdot b + \omega(X)(a) \cdot m \cdot b + a \cdot m \cdot \omega(X)(b) \tag{32}$$

for all $a, b \in A$, $X \in L$ and $m \in M$, makes of M a left $A \odot U_\Bbbk(L) \odot A$-module, and conversely.

Proof. If M is an A-bimodule, then the assignments

$$s_E : A \to \mathsf{End}_\Bbbk(M), \ a \mapsto [m \mapsto a \cdot m],$$

and

$$t_E : A^o \to \mathsf{End}_\Bbbk(M), \ b^o \mapsto [m \mapsto m \cdot b],$$

make of $\mathsf{End}_\Bbbk(M)$ an A^e-ring with $\phi_A : A^e \to \mathsf{End}_\Bbbk(M), a \otimes b^o \mapsto s_E(a)t_E(b^o)$. If we consider the Lie algebra morphism $\phi_L := \rho$, then equation (32) is exactly condition (29)

and hence there is a unique morphism of A^e-rings $R : A \odot U_{\Bbbk}(L) \odot A \to \mathsf{End}_{\Bbbk}(M)$ such that $R \circ J_L = \rho$. The other way around, if we have a morphism of A^e-rings $R : A \odot U_{\Bbbk}(L) \odot A \to \mathsf{End}_{\Bbbk}(M)$ and we compose it with J_L we get a Lie algebra morphism $\phi_L : L \to \mathsf{End}_{\Bbbk}(M)$ such that

$$
\begin{aligned}
\phi_L&(X)(a \cdot m \cdot b) \\
&= (R(1_A \otimes x \otimes 1_A) \circ \phi_A(a \otimes b^o))(m) \\
&= R((1_A \otimes x \otimes 1_A)(a \otimes 1_U \otimes b))(m) \\
&\overset{(18)}{=} R\big((a \otimes x \otimes b) + (\omega(X)(a) \otimes 1_U \otimes b) + (a \otimes 1_U \otimes \omega(X)(b))\big)(m) \\
&= a \cdot \phi_L(X)(m) \cdot b + \omega(X)(a) \cdot m \cdot b + a \cdot m \cdot \omega(X)(b).
\end{aligned}
$$
∎

Remark 2.11. Observe that the algebra maps $\phi' : U_{\Bbbk}(L) \to \mathcal{R}$ and $\phi_A : A^e \to \mathcal{R}$ satisfy (31) if and only if they satisfy

$$
\sum \phi'(u_{(1)})\phi_A\big(a \otimes (S(u_{(2)}) \cdot b)^o\big) = \sum \phi_A(u_{(1)} \cdot a \otimes b^o)\phi'(u_{(2)})
$$

for all $u \in U$, $a, b \in A$. This implies that the A^e-ring morphism $\Phi : A \odot U_{\Bbbk}(L) \odot A \to \mathcal{R}$ in Theorem 2.9 coincides, up to the isomorphism of A-bialgebroids from [39, Theorem 2.5], with the algebra map ω of [39, Proposition 3.1].

Proposition 2.12. *Let $\mathbf{B} := \{X_\alpha \mid \alpha \in S\}$ be a $\Bbbk$-basis for L, where S is an ordered set of indexes. Then $\mathcal{B}_L$ is a free left A^e-module with basis given by $1_A \otimes 1_U \otimes 1_A$ and $1_A \otimes x_{\alpha_1} \cdots x_{\alpha_n} \otimes 1_A$ where the $x_{\alpha_1} \cdots x_{\alpha_n}$'s are the cosets of the standard monomials $X_{\alpha_1} \cdots X_{\alpha_n}$ in the basis $\mathbf{B}$ (see [26, §V.2]).*

Proof. It follows from the PBW theorem (see, for instance, [26, Theorem V.2.3]) and the definition of the left A^e-module structure on $\mathcal{B}_L$. ∎

3. The Connes–Moscovici bialgebroid as universal enveloping bialgebroid

Our next aim is to prove that the Connes–Moscovici bialgebroid $\mathcal{B}_L = A \odot U_{\Bbbk}(L) \odot A$ satisfies a universal property as A-bialgebroid as well, in the form of an adjunction between the category of A-anchored Lie algebras and the category of A-bialgebroids.

3.1. The primitive functor

In light of Example 2.3(e), we may consider the assignment

$$
\mathbb{P} : \mathsf{Bialgd}_A \to \mathsf{AnchLie}_A
$$

given on objects by $\mathbb{P}(\mathcal{B}) = (\mathrm{Prim}(\mathcal{B}), \omega_{\mathcal{B}})$ and on morphisms by simply (co)restricting any $\phi : \mathcal{B} \to \mathcal{B}'$ to the primitive elements, that is $\phi \circ \theta_{\mathcal{B}} = \theta_{\mathcal{B}'} \circ \mathbb{P}(\phi)$. The latter gives

a well-defined morphism of A-anchored Lie algebras because it is compatible with the commutator bracket and

$$\omega_{\mathcal{B}'}(\phi(X))(a) = \varepsilon_{\mathcal{B}'}(\phi(X)t_{\mathcal{B}'}(a^o)) = \varepsilon_{\mathcal{B}'}(\phi(X)\phi(t_{\mathcal{B}}(a^o)))$$
$$= \varepsilon_{\mathcal{B}'}(\phi(Xt_{\mathcal{B}}(a^o))) = \varepsilon_{\mathcal{B}}(Xt_{\mathcal{B}}(a^o)) = \omega_{\mathcal{B}}(X)(a)$$

for all $X \in \mathsf{Prim}(\mathcal{B})$ and $a \in A$. Summing up, we have the following result.

Proposition 3.1. *There is a well-defined functor $\mathbb{P} : \mathsf{Bialgd}_A \to \mathsf{AnchLie}_A$ which assigns to every A-bialgebroid $\mathcal{B}$ its Lie algebra of primitive elements $\mathsf{Prim}(\mathcal{B})$ with anchor $\omega_{\mathcal{B}} : \mathsf{Prim}(\mathcal{B}) \to \mathsf{Der}_{\Bbbk}(A)$ sending X to*

$$\omega_{\mathcal{B}}(X) : A \to A, \qquad a \mapsto \omega_{\mathcal{B}}(X)(a) = X \cdot a. \tag{33}$$

Next lemma states a property of the primitive elements of an A-bialgebroid that we already observed for $\mathcal{B}_L$ in (28) and that will be useful to prove the universal property in the forthcoming section.

Lemma 3.2. *For $\mathcal{B}$ an A-bialgebroid, every primitive element $X \in \mathsf{Prim}(\mathcal{B})$ satisfies*

$$Xt(a^o) - t(a^o)X = t\big(\varepsilon\big(Xt(a^o)\big)^o\big) \quad and \quad Xs(a) - s(a)X = s\big(\varepsilon\big(Xs(a)\big)\big) \tag{34}$$

for all $a \in A$. In particular,

$$\big[X, \eta_{\mathcal{B}}(a \otimes b^o)\big]_{\mathcal{B}} = \eta_{\mathcal{B}}\big(X \cdot (a \otimes b^o)\big) \tag{35}$$

for all $X \in \mathsf{Prim}(\mathcal{B})$ and all $a, b \in A$.

Proof. In view of the definition of $\mathcal{B} \otimes_A \mathcal{B}$ and of $\mathcal{B} \times_A \mathcal{B}$ we have that

$$Xt(a^o) \otimes_A 1_{\mathcal{B}} + t(a^o) \otimes_A X \overset{(3)}{=} X \otimes_A s(a) + 1_{\mathcal{B}} \otimes_A Xs(a).$$

By resorting to the left-hand side identity in (3), this relation can be written equivalently as

$$Xt(a^o) \otimes_A 1_{\mathcal{B}} + t(a^o) \otimes_A X = t(a^o)X \otimes_A 1_{\mathcal{B}} + 1_{\mathcal{B}} \otimes_A Xs(a) \tag{36}$$

or

$$Xt(a^o) \otimes_A 1_{\mathcal{B}} + 1_{\mathcal{B}} \otimes_A s(a)X = X \otimes_A s(a) + 1_{\mathcal{B}} \otimes_A Xs(a). \tag{37}$$

By applying $\mathcal{B} \otimes_A \varepsilon$ to both sides of (36) and by recalling that $\varepsilon(X) = 0$, we get

$$Xt(a^o) = t(a^o)X + t\big(\varepsilon\big(Xs(a)\big)^o\big) \overset{(17)}{=} t(a^o)X + t\big(\varepsilon\big(Xt(a^o)\big)^o\big).$$

If we apply instead $\varepsilon \otimes_A \mathcal{B}$ to (37) then we get

$$s\big(\varepsilon\big(Xt(a^o)\big)\big) + s(a)X = Xs(a),$$

which gives the other relation in (34). ∎

3.2. An adjunction between $\mathsf{AnchLie}_A$ and Bialgd_A

We show now how the Connes–Moscovici bialgebroid construction provides a left adjoint to the functor $\mathbb{P}$ in a way that mimics the well-known "universal enveloping algebra/space of primitives" adjunction

$$
\begin{array}{c}
\mathsf{Bialg}_{\Bbbk} \\
U_{\Bbbk}(-) \; \Big\uparrow \; \dashv \; \Big\downarrow \; \mathsf{Prim}(-) \\
\mathsf{Lie}_{\Bbbk}.
\end{array}
\tag{38}
$$

Remark 3.3. Despite being well-known, it seems that no "classical" reference explicitly reports the adjunction (38) in the form we stated it here. Nevertheless, it is straightforward to check that the involved functors are well defined (they are, in fact, slight adjustments of the functors considered in [34, page 239]) and that they form an adjoint pair. The unit $L \to \mathsf{Prim}(U_{\Bbbk}(L))$ (induced by the canonical map j_L of (25)) and the counit $U_{\Bbbk}(\mathsf{Prim}(B)) \to B$ (the unique algebra morphism extending the Lie algebra inclusion $\mathsf{Prim}(B) \subseteq \mathscr{L}(B)$) are the obvious natural morphism which are proved to be bijective in [34, Theorem 5.18].

Proposition 3.4. *Let (L, ω) be an A-anchored Lie algebra. Given an A-bialgebroid $\mathscr{B}$ and given a Lie algebra morphism $\phi_L : L \to \mathscr{L}(\mathscr{B})$ such that $\phi_L(L) \subseteq \mathsf{Prim}(\mathscr{B})$ and*

$$
\phi_L(X) \cdot a = \omega(X)(a)
\tag{39}
$$

for all $a \in A$ and for all $X \in L$, there exists a unique morphism of A-bialgebroids $\Phi : \mathscr{B}_L \to \mathscr{B}$ such that $\Phi \circ J_L = \phi_L$ and it is explicitly given by (30).

Proof. Set $U := U_{\Bbbk}(L)$ and let $\phi' : U \to \mathscr{B}$ be the unique $\Bbbk$-algebra map extending ϕ_L. In view of (35) and (39), $\eta_{\mathscr{B}}$ and ϕ_L satisfy (29). Thus, by Theorem 2.9, there exists a unique morphism of A^e-rings $\Phi : \mathscr{B}_L \to \mathscr{B}$ satisfying $\Phi \circ J_L = \phi_L$. Moreover, $\eta_{\mathscr{B}} : A^e \to \mathscr{B}$ is always a morphism of A-bialgebroids, where A^e has the A-bialgebroid structure from Example 1.7(b). Therefore, in view of Remark 2.11 and of [39, Theorem 3.2], if we show that $(1_A, \phi') : (\Bbbk, U) \to (A, \mathscr{B})$ is a morphism of bialgebroids, then we can conclude that Φ is a morphism of A-bialgebroids and finish the proof. Equivalently, we need to check that $\phi' : U \to \mathscr{B}$ satisfies

$$
\sum \phi'(u_{(1)}) \otimes_A \phi'(u_{(2)}) = \sum \phi'(u)_{(1)} \otimes_A \phi'(u)_{(2)}
\tag{40}
$$

and

$$
\varepsilon_{\mathscr{B}}(\phi'(u)) = \varepsilon_U(u) 1_A
\tag{41}
$$

for all $u \in U$. Since, in view of the PBW theorem, U admits a $\Bbbk$-basis of the form

$$
\{1_U\} \cup \{x_1 \cdots x_n \mid n \geq 1, X_1, \ldots, X_n \in L\},
$$

it is enough to check (40) and (41) on the elements of this basis. A direct computation shows that $\varepsilon_{\mathcal{B}}(\phi'(1_U)) = \varepsilon_{\mathcal{B}}(1_{\mathcal{B}}) = 1_A = \varepsilon_U(1_U)1_A$ and that

$$\varepsilon_{\mathcal{B}}\big(\phi'(x_1\cdots x_n)\big) = \varepsilon_{\mathcal{B}}\big(\phi'(x_1\cdots x_{n-1})\phi'(x_n)\big) \overset{(B5)}{=} \varepsilon_{\mathcal{B}}\big(\phi'(x_1\cdots x_{n-1})s(\varepsilon_{\mathcal{B}}(\phi'(x_n)))\big)$$
$$= \varepsilon_{\mathcal{B}}\big(\phi'(x_1\cdots x_{n-1})s(\varepsilon_{\mathcal{B}}(\phi_L(X_n)))\big) = 0 = \varepsilon_U(x_1\cdots x_n)1_A$$

for all $n \geq 1$. Therefore, relation (41) holds. Concerning (40), we notice first of all that

$$\phi'(1_U) \otimes_A \phi'(1_U) = 1_{\mathcal{B}} \otimes_A 1_{\mathcal{B}} = \sum \phi'(1_U)_{(1)} \otimes_A \phi'(1_U)_{(2)},$$

which shows that it is satisfied for $u = 1_U$, and we prove by induction on $n \geq 1$ that it also holds for $u = x_1\cdots x_n$, where $X_1,\ldots,X_n \in L$. For $n = 1$ we have

$$\sum \phi'(x_{(1)}) \otimes_A \phi'(x_{(2)}) = \phi'(x) \otimes_A \phi'(1_U) + \phi'(1_U) \otimes_A \phi'(x)$$
$$= \phi_L(X) \otimes_A 1_{\mathcal{B}} + 1_{\mathcal{B}} \otimes_A \phi_L(X)$$
$$= \sum \phi'(x)_{(1)} \otimes_A \phi'(x)_{(2)}.$$

Assume now that (40) holds for $n \geq 1$, that is, that we have

$$\sum \phi'((x_1\cdots x_n)_{(1)}) \otimes_A \phi'((x_1\cdots x_n)_{(2)}) = \sum \phi'(x_1\cdots x_n)_{(1)} \otimes_A \phi'(x_1\cdots x_n)_{(2)} \tag{42}$$

for all $X_1,\ldots,X_n \in L$ and hence, in particular,

$$\sum \phi'((x_1\cdots x_n)_{(1)}) \otimes_A \phi'((x_1\cdots x_n)_{(2)}) \in \mathcal{B} \times_A \mathcal{B}.$$

Let us compute

$$\sum \phi'((x_1 x_2\cdots x_{n+1})_{(1)}) \otimes_A \phi'((x_1 x_2\cdots x_{n+1})_{(2)})$$
$$= \sum \phi'((x_1)_{(1)}(x_2\cdots x_{n+1})_{(1)}) \otimes_A \phi'((x_1)_{(2)}(x_2\cdots x_{n+1})_{(2)})$$
$$= \sum \phi'((x_1)_{(1)})\phi'((x_2\cdots x_{n+1})_{(1)}) \otimes_A \phi'((x_1)_{(2)})\phi'((x_2\cdots x_{n+1})_{(2)})$$
$$\overset{(11)}{=} \Big(\sum \phi'((x_1)_{(1)}) \otimes_A \phi'((x_1)_{(2)})\Big)$$
$$\cdot \Big(\sum \phi'((x_2\cdots x_{n+1})_{(1)}) \otimes_A \phi'((x_2\cdots x_{n+1})_{(2)})\Big)$$
$$\overset{(42)}{=} (\phi_L(X_1) \otimes_A 1_{\mathcal{B}} + 1_{\mathcal{B}} \otimes_A \phi_L(X_1))\Big(\sum \phi'(x_2\cdots x_{n+1})_{(1)} \otimes_A \phi'(x_2\cdots x_{n+1})_{(2)}\Big)$$
$$\overset{(11)}{=} \sum \phi'(x_1)_{(1)}\phi'(x_2\cdots x_{n+1})_{(1)} \otimes_A \phi'(x_1)_{(2)}\phi'(x_2\cdots x_{n+1})_{(2)}$$
$$= \sum (\phi'(x_1)\phi'(x_2\cdots x_{n+1}))_{(1)} \otimes_A (\phi'(x_1)\phi'(x_2\cdots x_{n+1}))_{(2)}$$
$$= \sum \phi'(x_1 x_2\cdots x_{n+1})_{(1)} \otimes_A \phi'(x_1 x_2\cdots x_{n+1})_{(2)},$$

so that it holds for $n + 1$ and we may conclude that it holds for every n by induction. ∎

Corollary 3.5. *Let (L, ω) be an A-anchored Lie algebra. Given an A-bialgebroid $\mathcal{B}$ and given a morphism of A-anchored Lie algebras $\phi_L : (L, \omega) \to (\mathrm{Prim}(\mathcal{B}), \omega_{\mathcal{B}})$, there exists a unique morphism of A-bialgebroids $\Phi : \mathcal{B}_L \to \mathcal{B}$ such that $\Phi \circ J_L = \theta_{\mathcal{B}} \circ \phi_L$, where $\theta_{\mathcal{B}} : \mathrm{Prim}(\mathcal{B}) \to \mathcal{B}$ is the canonical inclusion.*

Proof. The morphism of Lie algebras $(\theta_{\mathcal{B}} \circ \phi_L) : L \to \mathcal{L}(\mathcal{B})$ satisfies (39) if and only if ϕ_L is of A-anchored Lie algebras. ∎

Theorem 3.6. *The assignment*

$$\mathbb{B} : \mathsf{AnchLie}_A \to \mathsf{Bialgd}_A, \qquad (L, \omega) \mapsto A \odot U_{\Bbbk}(L) \odot A$$

induces a well-defined functor which is left adjoint to the functor

$$\mathbb{P} : \mathsf{Bialgd}_A \to \mathsf{AnchLie}_A, \qquad \mathcal{B} \mapsto (\mathrm{Prim}(\mathcal{B}), \omega_{\mathcal{B}}),$$

where $\omega_{\mathcal{B}} : \mathrm{Prim}(\mathcal{B}) \to \mathrm{Der}_{\Bbbk}(A)$ is the anchor of equation (33). Write

$$\vartheta_{\mathcal{B}} : U_{\Bbbk}(\mathrm{Prim}(\mathcal{B})) \to \mathcal{B}$$

for the unique algebra map extending the inclusion $\theta_{\mathcal{B}} : \mathrm{Prim}(\mathcal{B}) \subseteq \mathcal{B}$. Then the unit and the counit of this adjunction are given by

$$\begin{aligned} \gamma_L : (L, \omega) &\to (\mathrm{Prim}(A \odot U_{\Bbbk}(L) \odot A), \omega_{A \odot U_{\Bbbk}(L) \odot A}), \\ X &\mapsto J_L(X) = 1_A \otimes x \otimes 1_A, \end{aligned} \tag{43}$$

and

$$\epsilon_{\mathcal{B}} : A \odot U_{\Bbbk}(\mathrm{Prim}(\mathcal{B})) \odot A \to \mathcal{B}, \qquad a \otimes u \otimes b \mapsto s_{\mathcal{B}}(a) t_{\mathcal{B}}(b^o) \vartheta_{\mathcal{B}}(u),$$

respectively, where $x = j_L(X)$, as usual. Furthermore, every component of the unit is a monomorphism and hence $\mathbb{B}$ is faithful.

Proof. We need to see how $\mathbb{B}$ operates on morphisms. Let $f : (L, \omega) \to (L', \omega')$ be a morphism of A-anchored Lie algebras. In view of the fact that

$$\begin{aligned} \omega_{\mathcal{B}_L}(\gamma_L(X))(a) &\overset{(43)}{=} \omega_{\mathcal{B}_L}(1_A \otimes x \otimes 1_A)(a) \overset{(33)}{=} (1_A \otimes x \otimes 1_A) \boldsymbol{\cdot} a \\ &\overset{(19)}{=} \varepsilon_{\mathcal{B}_L}\big((1_A \otimes x \otimes 1_A)(a \otimes 1_U \otimes 1_A)\big) \\ &\overset{(18)}{=} \varepsilon_{\mathcal{B}_L}\big((x \cdot a) \otimes 1_U \otimes 1_A + a \otimes x \otimes 1_A + a \otimes 1_U \otimes (x \cdot 1_A)\big) \\ &\overset{(27)}{=} x \cdot a + a \varepsilon_U(x) \overset{(27)}{=} \omega(X)(a) \end{aligned}$$

for all $X \in L$ and $a \in A$, the morphism $\gamma_L : L \to \mathrm{Prim}(\mathcal{B}_L)$ induced by J_L (namely, we have $J_L = \theta_{\mathcal{B}_L} \circ \gamma_L$) is a morphism of A-anchored Lie algebras for every L in $\mathsf{AnchLie}_A$ and hence, by Corollary 3.5, there exists a unique morphism of A-bialgebroids

$\Phi : \mathcal{B}_L \to \mathcal{B}_{L'}$ such that $\Phi \circ J_L = J_{L'} \circ f = \theta_{\mathcal{B}_{L'}} \circ \gamma_{L'} \circ f$ and it is explicitly given by (30). We set $\mathbb{B}(f) = \Phi$.

In order to conclude, consider the natural assignment

$$\mathsf{Bialgd}_A(A \odot U_{\Bbbk}(L) \odot A, \mathcal{B}) \to \mathsf{AnchLie}_A(L, \mathsf{Prim}(\mathcal{B})),$$
$$\Psi \mapsto \mathsf{Prim}(\Psi) \circ \gamma_L. \tag{44}$$

Corollary 3.5 states that for every ϕ_L in $\mathsf{AnchLie}_A(L, \mathsf{Prim}(\mathcal{B}))$ there exists a unique Φ in $\mathsf{Bialgd}_A(A \odot U_{\Bbbk}(L) \odot A, \mathcal{B})$ such that $\Phi \circ J_L = \theta_{\mathcal{B}} \circ \phi_L$, which implies that (44) is bijective. ∎

Remark 3.7. In the context of the proof above, let $F : U \to U'$ be the unique $\Bbbk$-algebra morphism satisfying $F \circ j_L = j_{L'} \circ f$. Since

$$\psi : U \to \mathcal{B}_{L'}, \qquad u \mapsto 1_A \otimes F(u) \otimes 1_A,$$

is a $\Bbbk$-algebra morphism such that $\psi \circ j_L = J_{L'} \circ f$, the unique morphism of A-bialgebroids $\mathbb{B}(f) : \mathcal{B}_L \to \mathcal{B}_{L'}$ induced by $f : L \to L'$ has the form

$$\mathbb{B}(f)(a \otimes u \otimes b) = a \otimes F(u) \otimes b$$

for all $a, b \in A$ and $u \in U$. Thus, if $f : (L, \omega) \to (L', \omega')$ is a morphism of A-anchored Lie algebras, we will also write $A \odot U(f) \odot A$ to denote the A-bialgebroid morphism $\mathbb{B}(f)$, where $U(f) : U_{\Bbbk}(L) \to U_{\Bbbk}(L')$ is the unique $\Bbbk$-algebra morphism such that

$$U(f) \circ j_L = j_{L'} \circ f.$$

Example 3.8. Let A be a finite-dimensional algebra and let $\mathcal{B} = \mathsf{End}_{\Bbbk}(A)$ with the A-bialgebroid structure of Example 1.7(c). It is clear that $\mathsf{Prim}(\mathcal{B}) = \mathsf{Der}_{\Bbbk}(A)$. Then, the image of the associated Connes–Moscovici bialgebroid $A \odot U_{\Bbbk}(\mathsf{Der}_{\Bbbk}(A)) \odot A$ in $\mathsf{End}_{\Bbbk}(A)$ via $\epsilon_{\mathcal{B}}$ is the A^e-subring of $\mathsf{End}_{\Bbbk}(A)$ generated by A^e and $\mathsf{Der}_{\Bbbk}(A)$. In this sense, it can be interpreted as the *derivation A^e-ring* of A in the sense of [33, Chapter 15, §1.4].

In particular, the counit of the adjunction is not surjective in general. We will see with Corollary 4.4 why also the unit is not surjective.

4. An intrinsic description of $A \odot U_{\Bbbk}(L) \odot A$

Inspired by the results of Milnor–Moore and Moerdijk–Mrčun, which give an intrinsic description of those bialgebras/bialgebroids that are universal enveloping algebras of Lie algebras/Lie–Rinehart algebras, we look for necessary and sufficient conditions on an A-bialgebroid $\mathcal{B}$ in order to claim that it is a Connes–Moscovici bialgebroid $A \odot U_{\Bbbk}(L) \odot A$ for some A-anchored Lie algebra (L, ω).

4.1. The primitives of the Connes–Moscovici bialgebroid

To begin with, we need a more detailed analysis of the space of primitives of $\mathcal{B}_L$.

Lemma 4.1. *Let $\mathcal{B}$ be an A-bialgebroid. Then the $\Bbbk$-vector subspace*

$$\langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle := \{ s_{\mathcal{B}}(a) - t_{\mathcal{B}}(a^o) \mid a \in A \} \subseteq \mathcal{B}$$

is an ideal in $\mathrm{Prim}(\mathcal{B})$. *The Lie bracket is explicitly given by*

$$\big[s_{\mathcal{B}}(a) - t_{\mathcal{B}}(a^o), s_{\mathcal{B}}(b) - t_{\mathcal{B}}(b^o) \big] = s_{\mathcal{B}}\big([a,b]_A\big) - t_{\mathcal{B}}\big([a,b]_A^o\big)$$

and the anchor by

$$\omega'_{\mathcal{B}} : \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle \to \mathrm{Der}_{\Bbbk}(A), \qquad s_{\mathcal{B}}(a) - t_{\mathcal{B}}(a^o) \mapsto [a,-]_A,$$

for all $a, b \in A$.

Proof. Set $s := s_{\mathcal{B}}$ and $t := t_{\mathcal{B}}$. The fact that $\langle s - t \rangle$ is contained in $\mathrm{Prim}(\mathcal{B})$ follows from

$$\Delta_{\mathcal{B}}(s(a) - t(a^o)) \overset{(13)}{=} s(a) \otimes_A 1_{\mathcal{B}} - 1_{\mathcal{B}} \otimes_A t(a^o)$$
$$\overset{(3)}{=} \big(s(a) - t(a^o)\big) \otimes_A 1_{\mathcal{B}} - 1_{\mathcal{B}} \otimes_A \big(s(a) - t(a^o)\big).$$

The fact that it is a Lie ideal in $\mathrm{Prim}(\mathcal{B})$ with respect to the commutator bracket follows because

$$\big[X, s(a) - t(a^o)\big]_{\mathcal{B}} = X s(a) - X t(a^o) - s(a) X + t(a^o) X$$
$$\overset{(34)}{=} s\big(\varepsilon(X s(a))\big) - t\big(\varepsilon(X t(a^o))^o\big)$$
$$\overset{(B5)}{=} s\big(\varepsilon(X s(a))\big) - t\big(\varepsilon(X s(a))^o\big) \in \langle s - t \rangle.$$

Now, a direct computation shows that

$$\big[s(a) - t(a^o), s(b) - t(b^o)\big]_{\mathcal{B}} = s\big([a,b]_A\big) - t\big([a,b]_A^o\big)$$

for all $a, b \in A$ as claimed. Furthermore,

$$\omega_{\mathcal{B}}\big(s(a) - t(a^o)\big)(b) \overset{(33)}{=} \big(s(a) - t(a^o)\big) \cdot b$$
$$\overset{(19)}{=} \varepsilon_{\mathcal{B}}(s(a)s(b)) - \varepsilon_{\mathcal{B}}(t(a^o)t(b^o)) = [a,b]_A$$

for all $a, b \in A$ and hence the proof is concluded. ∎

Lemma 4.1 makes it clear why the counit $\epsilon_{\mathcal{B}} : \mathcal{B}_{\mathrm{Prim}(\mathcal{B})} \to \mathcal{B}$ of Theorem 3.6 is not injective in general. Namely, every element of the form

$$a \otimes 1_U \otimes 1_A - 1_A \otimes 1_U \otimes a - 1_A \otimes (s_{\mathcal{B}}(a) - t_{\mathcal{B}}(a^o)) \otimes 1_A$$

for $a \in A$ lives in the kernel of $\epsilon_{\mathcal{B}}$.

Example 4.2. Consider the A-bialgebroid $\mathcal{B} = \mathsf{End}_{\Bbbk}(A)$ from Example 1.7(c). As we have seen in Example 3.8, the primitive elements of $\mathcal{B}$ are the derivations of A. It is easy to see that $\langle s - t \rangle$ are exactly the inner derivations.

For H a Hopf algebra and A an H-module algebra, we will often write

$$1_A \otimes \mathsf{Prim}(H) \otimes 1_A$$

for $\{1_A \otimes h \otimes 1_A \mid h \in \mathsf{Prim}(H)\}$.

Proposition 4.3. *For any Hopf algebra H and any H-module algebra A, we have that $1_A \otimes \mathsf{Prim}(H) \otimes 1_A$ and $\langle s - t \rangle$ are subalgebras of $\mathsf{Prim}(A \odot H \odot A)$. Moreover, we have an isomorphism of A-anchored Lie algebras*

$$\left(1_A \otimes \mathsf{Prim}(H) \otimes 1_A \right) \ltimes_\delta \langle s - t \rangle \cong \mathsf{Prim}(A \odot H \odot A)$$

given by the vector space sum.

Proof. The fact that $\langle s - t \rangle$ is an ideal in $\mathsf{Prim}(A \odot H \odot A)$ has been established in Lemma 4.1 for a general A-bialgebroid. The fact that $1_A \otimes \mathsf{Prim}(H) \otimes 1_A$ is a subalgebra is a straightforward computation. Moreover, notice that if

$$1_A \otimes X \otimes 1_A = a \otimes 1_H \otimes 1_A - 1_A \otimes 1_H \otimes a$$

for some $X \in \mathsf{Prim}(H)$ and some $a \in A$, then

$$0 = (A \otimes \varepsilon_H \otimes A)(1_A \otimes X \otimes 1_A) = (A \otimes \varepsilon_H \otimes A)(a \otimes 1_H \otimes 1_A - 1_A \otimes 1_H \otimes a)$$
$$= a \otimes 1_A - 1_A \otimes a$$

which implies that

$$a \otimes 1_H \otimes 1_A - 1_A \otimes 1_H \otimes a = \eta(a \otimes 1_A^o - 1_A \otimes a^o)(1_A \otimes 1_H \otimes 1_A) = 0.$$

In view of Lemma 2.6, we are left to check that

$$\mathsf{Prim}(A \odot H \odot A) \subseteq (1_A \otimes \mathsf{Prim}(H) \otimes 1_A) + \langle s - t \rangle. \tag{45}$$

Let us consider a primitive element $\xi \in A \odot H \odot A$. Fix a basis $\{e_i \mid i \in S\}$ for A as a vector space, where S is some set of indexes with a distinguished index 0 and $e_0 = 1_A$. Write

$$\xi = \sum_{i,j} e_i \otimes h_{ij} \otimes e_j$$

where almost all the h_{ij} are 0. Consider also the dual elements $\{e_i^* \mid i \in S\}$ of the e_i's. Since ξ is primitive, the following relation holds

$$\sum_{i,j} (e_i \otimes (h_{ij})_{(1)} \otimes 1_A) \otimes_A (1_A \otimes (h_{ij})_{(2)} \otimes e_j)$$

$$= \sum_{i,j} (e_i \otimes h_{ij} \otimes e_j) \otimes_A (1_A \otimes 1_H \otimes 1_A)$$

$$+ (1_A \otimes 1_H \otimes 1_A) \otimes_A (e_i \otimes h_{ij} \otimes e_j). \tag{46}$$

For $k \neq 0 \neq l$, let us apply the $\Bbbk$-linear morphism $(e_k^* \otimes H \otimes A) \otimes_A (A \otimes H \otimes e_l^*)$ to both sides of the identity (46). We find out that

$$\sum ((h_{kl})_{(1)} \otimes 1_A) \otimes_A (1_A \otimes (h_{kl})_{(2)}) = 0$$

from which it follows that $h_{kl} = 0$, by applying $(H \otimes A) \otimes_A (A \otimes \varepsilon_H)$ to both sides again. Therefore,

$$\xi = 1_A \otimes h_{00} \otimes 1_A + \sum_{i \neq 0} e_i \otimes h_{i0} \otimes 1_A + \sum_{i \neq 0} 1_A \otimes h_{0i} \otimes e_i.$$

Consider again the identity (46), that now rewrites

$$\sum (1_A \otimes (h_{00})_{(1)} \otimes 1_A) \otimes_A (1_A \otimes (h_{00})_{(2)} \otimes 1_A)$$
$$+ \sum_{i \neq 0} (e_i \otimes (h_{i0})_{(1)} \otimes 1_A) \otimes_A (1_A \otimes (h_{i0})_{(2)} \otimes 1_A)$$
$$+ \sum_{i \neq 0} (1_A \otimes (h_{0i})_{(1)} \otimes 1_A) \otimes_A (1_A \otimes (h_{0i})_{(2)} \otimes e_i)$$
$$= (1_A \otimes h_{00} \otimes 1_A) \otimes_A (1_A \otimes 1_H \otimes 1_A) + (1_A \otimes 1_H \otimes 1_A) \otimes_A (1_A \otimes h_{00} \otimes 1_A)$$
$$+ \sum_{i \neq 0} (e_i \otimes h_{i0} \otimes 1_A) \otimes_A (1_A \otimes 1_H \otimes 1_A)$$
$$+ (1_A \otimes 1_H \otimes 1_A) \otimes_A (e_i \otimes h_{i0} \otimes 1_A)$$
$$+ \sum_{i \neq 0} (1_A \otimes h_{0i} \otimes e_i) \otimes_A (1_A \otimes 1_H \otimes 1_A)$$
$$+ (1_A \otimes 1_H \otimes 1_A) \otimes_A (1_A \otimes h_{0i} \otimes e_i).$$

If we apply $(1_A^* \otimes H \otimes A) \otimes_A (A \otimes H \otimes 1_A^*)$ then we get that

$$\sum ((h_{00})_{(1)} \otimes 1_A) \otimes_A (1_A \otimes (h_{00})_{(2)})$$
$$= (h_{00} \otimes 1_A) \otimes_A (1_A \otimes 1_H) + (1_H \otimes 1_A) \otimes_A (1_A \otimes h_{00})$$
$$+ \sum_{i \neq 0} (1_H \otimes 1_A) \otimes_A (e_i \otimes h_{i0}) + (h_{0i} \otimes e_i) \otimes_A (1_A \otimes 1_H). \qquad (47)$$

By resorting to the $\Bbbk$-linear isomorphism $(H \otimes A) \otimes_A (A \otimes H) \cong H \otimes A \otimes H$, the equality (47) becomes

$$\sum (h_{00})_{(1)} \otimes 1_A \otimes (h_{00})_{(2)} = h_{00} \otimes 1_A \otimes 1_H + 1_H \otimes 1_A \otimes h_{00}$$
$$+ \sum_{i \neq 0} h_{0i} \otimes e_i \otimes 1_H + 1_H \otimes e_i \otimes h_{i0}. \qquad (48)$$

By applying $H \otimes 1_A^* \otimes H$ to both sides of the identity (48) we get that

$$\sum (h_{00})_{(1)} \otimes (h_{00})_{(2)} = h_{00} \otimes 1_H + 1_H \otimes h_{00},$$

whence h_{00} is primitive in H, and by applying $H \otimes e_k^* \otimes H$ for all $k \neq 0$ we get

$$1_H \otimes h_{k0} + h_{0k} \otimes 1_H = 0. \tag{49}$$

By applying further $H \otimes \varepsilon_H$ we find that $h_{0k} = -\varepsilon(h_{k0})1_H$ and hence from (49) we deduce that

$$0 = 1_H \otimes h_{k0} - \varepsilon(h_{k0})1_H \otimes 1_H = 1_H \otimes (h_{k0} - \varepsilon(h_{k0})1_H),$$

which in turn entails that $h_{k0} - \varepsilon(h_{k0})1_H = 0$ by applying $\varepsilon_H \otimes H$ to both sides, that is, $h_{k0} = \varepsilon(h_{k0})1_H$ for all $k \neq 0$. Summing up,

$$\xi = 1_A \otimes h_{00} \otimes 1_A + \sum_{k \neq 0} \varepsilon(h_{k0})(e_k \otimes 1_H \otimes 1_A - 1_A \otimes 1_H \otimes e_k)$$

which proves that the inclusion (45) holds. ∎

Corollary 4.4. *For any A-anchored Lie algebra (L, ω) we have*

$$\mathsf{Prim}(A \odot U_{\Bbbk}(L) \odot A) \cong L \ltimes_\delta \langle s_{\mathcal{B}_L} - t_{\mathcal{B}_L} \rangle.$$

Proof. It follows from [34, Theorem 5.18] that $\mathsf{Prim}(U_{\Bbbk}(L)) = L$. Moreover, it is clear that $L \cong 1_A \otimes L \otimes 1_A$ as A-anchored Lie algebras. ∎

It is evident from Corollary 4.4 why, in general, the unit $\gamma_L : L \to \mathsf{Prim}(\mathcal{B}_L)$ from Theorem 3.6 cannot be surjective.

4.2. Primitively generated bialgebroids

Let $\mathcal{B}$ be an A-bialgebroid and consider the A^e-bimodule

$$M_{\mathcal{B}} := A^e \otimes \mathsf{Prim}(\mathcal{B}) \otimes A^e.$$

There is a canonical A^e-bilinear map $\varphi_{\mathcal{B}} : M_{\mathcal{B}} \to \mathcal{B}$ given by

$$A^e \otimes \mathsf{Prim}(\mathcal{B}) \otimes A^e \xrightarrow{\eta_{\mathcal{B}} \otimes \theta_{\mathcal{B}} \otimes \eta_{\mathcal{B}}} \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \xrightarrow{m_2} \mathcal{B},$$

where $\theta_{\mathcal{B}} : \mathsf{Prim}(\mathcal{B}) \to \mathcal{B}$ is the inclusion and

$$m_2 : \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}, \qquad x \otimes y \otimes z \mapsto xyz.$$

Therefore, by the universal property of the tensor A^e-ring $T_{A^e}(M_{\mathcal{B}})$ (considered as the free A^e-ring on the A^e-bimodule $M_{\mathcal{B}}$; see [30, Theorem VII.3.2] or [37, Proposition 1.4.1]), there exists a unique morphism of A^e-rings

$$\Phi_{\mathcal{B}} : T_{A^e}(M_{\mathcal{B}}) \to \mathcal{B} \tag{50}$$

that extends $\varphi_{\mathcal{B}}$. We set $\varsigma_{\mathcal{B}} : M_{\mathcal{B}} \to T_{A^e}(M_{\mathcal{B}})$ for the canonical inclusion.

Since $T_{A^e}(M_{\mathcal{B}})$ is a graded A^e-ring with grading given by

$$T_{A^e}(M_{\mathcal{B}})_0 := A^e \qquad \text{and} \qquad T_{A^e}(M_{\mathcal{B}})_n := M_{\mathcal{B}} \otimes_{A^e} \cdots \otimes_{A^e} M_{\mathcal{B}} \quad (n \text{ times}),$$

for all $n \geq 1$, $\mathcal{B}$ inherits a natural filtration as A^e-ring given by

$$F_n(\mathcal{B}) := \Phi_{\mathcal{B}}\left(\bigoplus_{k=0}^{n} T_{A^e}(M_{\mathcal{B}})_k\right). \tag{51}$$

We set by definition $F_{-1}(\mathcal{B}) := 0$, as usual. Notice that $F_1(\mathcal{B})$ is the A^e-subbimodule of $\mathcal{B}$ generated by A^e and $\mathrm{Prim}(\mathcal{B})$, thus we call $\{F_n(\mathcal{B}) \mid n \in \mathbb{N}\}$ the *primitive filtration*.

Example 4.5. Let (L, ω) be an A-anchored Lie algebra and consider the Connes–Moscovici bialgebroid $\mathcal{B}_L = A \odot U_{\mathbb{k}}(L) \odot A$. Recall that $U_{\mathbb{k}}(L)$ is a filtered $\mathbb{k}$-algebra with filtration induced by the canonical projection $T_{\mathbb{k}}(L) \to U_{\mathbb{k}}(L)$. Then

$$F_n(A \odot U_{\mathbb{k}}(L) \odot A) = A \otimes F_n(U_{\mathbb{k}}(L)) \otimes A. \tag{52}$$

Definition 4.6. We say that an A-bialgebroid $\mathcal{B}$ is *primitively generated* if

$$\mathcal{B} = \bigcup_{n \geq 0} F_n(\mathcal{B}).$$

Remark 4.7. Definition 4.6 is given in the same spirit of [34, page 239]. In particular, the Connes–Moscovici bialgebroid $A \odot U_{\mathbb{k}}(L) \odot A$ of an A-anchored Lie algebra (L, ω) is primitively generated. Notice also that $\mathcal{B}$ is primitively generated if and only if $\Phi_{\mathcal{B}}$ of (50) is surjective.

Proposition 4.8. *The canonical morphism $\Phi_{\mathcal{B}}$ is natural in $\mathcal{B} \in \mathsf{Bialgd}_A$. Namely, every morphism $\phi : \mathcal{B} \to \mathcal{B}'$ of A-bialgebroids induces a morphism $T_\phi : T_{A^e}(M_{\mathcal{B}}) \to T_{A^e}(M_{\mathcal{B}'})$ of graded A^e-rings in a functorial way and the following diagram commutes*

$$\begin{array}{ccc} T_{A^e}(M_{\mathcal{B}}) & \xrightarrow{\quad T_\phi \quad} & T_{A^e}(M_{\mathcal{B}'}) \\ \Phi_{\mathcal{B}} \downarrow & & \downarrow \Phi_{\mathcal{B}'} \\ \mathcal{B} & \xrightarrow{\quad \phi \quad} & \mathcal{B}'. \end{array} \tag{53}$$

Proof. By the universal property of the tensor A^e-ring and a standard argument, any morphism $\phi : \mathcal{B} \to \mathcal{B}'$ of A-bialgebroids induces a unique morphism $T_\phi : T_{A^e}(M_{\mathcal{B}}) \to T_{A^e}(M_{\mathcal{B}'})$ of graded A^e-rings extending $A^e \otimes \mathbb{P}(\phi) \otimes A^e$. It satisfies

$$\phi \circ \Phi_{\mathcal{B}} = \Phi_{\mathcal{B}'} \circ T_\phi. \qquad \blacksquare$$

Corollary 4.9. *Any morphism $\phi : \mathcal{B} \to \mathcal{B}'$ of A-bialgebroids is filtered with respect to the primitive filtration, that is to say, $\phi(F_n(\mathcal{B})) \subseteq F_n(\mathcal{B}')$ for all $n \geq 0$. In particular, the component of the counit $\epsilon_{\mathcal{B}}$ corresponding to an A-bialgebroid $\mathcal{B}$ is a filtered morphism. Furthermore, $\epsilon_{\mathcal{B}}$ is surjective if and only if $\mathcal{B}$ is primitively generated.*

Proof. The fact that any morphism of A-bialgebroids is filtered follows from the commutativity of (53) and the definition of the primitive filtration. Now, set $U := U_\Bbbk(\mathsf{Prim}(\mathcal{B}))$ and $L := \mathsf{Prim}(\mathcal{B})$. Since, by Proposition 4.3,

$$\mathsf{Prim}(A \odot U \odot A) = (1_A \otimes L \otimes 1_A) \oplus \langle s_{\mathcal{B}_L} - t_{\mathcal{B}_L}\rangle$$

and since $\mathbb{P}(\epsilon_\mathcal{B})(1_A \otimes X \otimes 1_A) = X$ for all $X \in \mathsf{Prim}(\mathcal{B})$, it is clear that $\mathbb{P}(\epsilon_\mathcal{B})$ is surjective. Thus, $T_{\epsilon_\mathcal{B}} : T_{A^e}(M_{\mathcal{B}_L}) \to T_{A^e}(M_\mathcal{B})$ is surjective as well and we know from Example 4.5 that $\Phi_{\mathcal{B}_L}$ is surjective. Since, by naturality of $\Phi_\mathcal{B}$, the following diagram commutes

$$\begin{array}{ccc} T_{A^e}(M_{\mathcal{B}_L}) & \xrightarrow{T_{\epsilon_\mathcal{B}}} & T_{A^e}(M_\mathcal{B}) \\ \Phi_{\mathcal{B}_L}\downarrow & & \downarrow\Phi_\mathcal{B} \\ \mathcal{B}_L & \xrightarrow{\epsilon_\mathcal{B}} & \mathcal{B}, \end{array}$$

$\epsilon_\mathcal{B}$ is surjective if and only if $\Phi_\mathcal{B}$ is. ∎

Example 4.10. Let $A = \mathsf{Mat}_n(\Bbbk)$ be the algebra of $n \times n$ matrices with coefficients in $\Bbbk$, for $n \geq 2$. Consider again the bialgebroid $\mathcal{B} = \mathsf{End}_\Bbbk(A)$ of Example 1.7(c), its space of primitive elements $\mathsf{Prim}(\mathcal{B}) = \mathsf{Der}_\Bbbk(A)$ and the associated Connes–Moscovici bialgebroid $A \odot U_\Bbbk(\mathsf{Der}_\Bbbk(A)) \odot A$ as in Example 3.8. Every $f \in \mathsf{End}_\Bbbk(A)$ is uniquely determined by the images $f(E_{i,j}) = \sum_{h,k} f^{i,j}_{h,k} E_{h,k}$ for all i, j and it satisfies

$$f(M) = \sum_{h,k}\left(\sum_{i,j} m_{i,j} f^{i,j}_{h,k}\right)E_{h,k} = \sum_{h,k}\sum_{i,j} f^{i,j}_{h,k} E_{h,i} M E_{j,k}$$
$$= \eta_\mathcal{B}\left(\sum_{i,j,h,k} f^{i,j}_{h,k} E_{h,i} \otimes E_{j,k}\right)(M),$$

whence $\epsilon_\mathcal{B}$ is surjective and $\mathcal{B}$ is primitively generated.

Let $\mathcal{B}$ be an A-bialgebroid. Recall that, given the filtered left A^e-module $\mathcal{B}$ with filtration $\{F_n(\mathcal{B}) \mid n \geq 0\}$ as in (51), we can consider the associated graded left A^e-module $\mathsf{gr}(\mathcal{B})$ as in Section 1.1. Since $\epsilon_\mathcal{B}$ is a morphism of filtered A^e-modules, it induces a left A^e-linear homomorphism

$$\mathsf{gr}(\epsilon_\mathcal{B}) : \mathsf{gr}(A \odot U(\mathsf{Prim}(\mathcal{B})) \odot A) \to \mathsf{gr}(\mathcal{B}).$$

The following lemma, which should be well known, is implicitly needed in the proof of Theorem 4.13 below. Its statement resembles closely [35, Remark 2.4]. Its proof can be deduced from the results in [16, Appendix B] and it follows closely the argument reported in [47, page 229] for coalgebras over a field, but we sketch it here for the sake of the reader.

Lemma 4.11. *Let $\mathcal{B}$ be a primitively generated A-bialgebroid. Then $({}_s\mathcal{B}_{t^o}, \Delta_{\mathcal{B}}, \varepsilon_{\mathcal{B}})$ is a filtered A-coring with the primitive filtration. In particular, if $\mathrm{gr}_n(\mathcal{B})$ is a projective left A^e-module for all $n \geq 0$, then $F_n(\mathcal{B})$ is a projective left A^e-module for all $n \geq 0$, the map*

$$\mathrm{gr}(\mathcal{B}) \otimes_A \mathrm{gr}(\mathcal{B}) \to \mathrm{gr}(\mathcal{B} \otimes_A \mathcal{B}),$$
$$(x + F_h(\mathcal{B})) \otimes_A (y + F_k(\mathcal{B})) \mapsto x \otimes_A y + F_{h+k+1}(\mathcal{B} \otimes_A \mathcal{B}),$$

is an isomorphism of left A^e-modules and the A-coring structure on $\mathcal{B}$ induces an A-coring structure on $\mathrm{gr}(\mathcal{B})$. Furthermore, any morphism $\phi : \mathcal{B} \to \mathcal{B}'$ of primitively generated A-bialgebroids which are graded projective as A-corings induces a morphism of graded A-corings $\mathrm{gr}(\phi) : \mathrm{gr}(\mathcal{B}) \to \mathrm{gr}(\mathcal{B}')$.

Proof. It follows from Lemma 1.1 and Proposition 1.2, once proved that $\mathcal{B}$ with the filtration $\{F_n(\mathcal{B}) \mid n \in \mathbb{N}\}$ is a filtered A-coring. To this aim observe that, as a left A^e-submodule of $\mathcal{B}$, $F_n(\mathcal{B})$ is generated by $1_{\mathcal{B}}$ and by elements of the form $X_1 \cdots X_k$ for $1 \leq k \leq n$ and $X_i \in \mathrm{Prim}(\mathcal{B})$ for all i. By applying $\Delta_{\mathcal{B}}$ we find that

$$\Delta_{\mathcal{B}}(1_{\mathcal{B}}) = 1_{\mathcal{B}} \otimes_A 1_{\mathcal{B}} \in F_0(\mathcal{B}) \otimes_A F_0(\mathcal{B}),$$

and that

$$\Delta_{\mathcal{B}}(X_1 \cdots X_k) = \prod_{i=1}^{k}(X_i \otimes_A 1_{\mathcal{B}} + 1_{\mathcal{B}} \otimes_A X_i) = \sum_{t+s=k} X_{p_1} \cdots X_{p_t} \otimes_A X_{q_1} \cdots X_{q_s}$$

belongs to $\sum_{t+s=k} F_t(\mathcal{B}) \otimes_A F_s(\mathcal{B})$, where

$$\{p_i \mid 1 \leq i \leq t\} \cup \{q_j \mid 1 \leq j \leq s\} = \{1, \ldots, k\},$$

$p_1 < p_2 < \cdots < p_t$ and $q_1 < q_2 < \cdots < q_s$. By left A^e-linearity of $\Delta_{\mathcal{B}}$, we may conclude that it is filtered. On the other hand, $\varepsilon_{\mathcal{B}}$ is obviously filtered (by definition of the filtration on A). Therefore, $({}_s\mathcal{B}_{t^o}, \Delta_{\mathcal{B}}, \varepsilon_{\mathcal{B}})$ is in fact a filtered A-coring. ∎

In view of Lemma 4.11 and by mimicking [35, page 3140] and Section 1.2, we give the following definition.

Definition 4.12. A primitively generated A-bialgebroid $\mathcal{B}$ is called *graded projective* if each associated graded component $\mathrm{gr}_n(\mathcal{B})$ is a projective left A^e-module.

The following can be understood as an analogue of the celebrated Heyneman–Radford Theorem for coalgebras [20, Proposition 2.4.2] (extending the earlier Heyneman–Sweedler Theorem [21, Lemma 3.2.6]).

Theorem 4.13. *Let $\phi : \mathcal{B} \to \mathcal{B}'$ be a morphism of graded projective primitively generated A-bialgebroids. If $\mathrm{gr}(\mathcal{B})$ is strongly graded as an A-coring and ϕ is injective when restricted to the left A^e-submodule of $\mathcal{B}$ generated by $\mathrm{Prim}(\mathcal{B})$, then ϕ is injective.*

Proof. In order to prove that ϕ is injective, we are going to prove that $\mathrm{gr}(\phi)$ is injective. In view of [36, Chapter D, Corollary III.6], the latter implies that ϕ is injective as well.

To this aim, let us prove that $\mathrm{gr}_n(\phi)$ is injective for every $n \geq 0$. To begin with, let us prove that $\mathrm{gr}_0(\phi)$ is injective. Since $\mathrm{gr}_0(\mathcal{B}) = F_0(\mathcal{B})/F_{-1}(\mathcal{B}) = F_0(\mathcal{B}) = \Phi_{\mathcal{B}}(A^e)$, we may take $\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o)$ as generic element in $\ker(\mathrm{gr}_0(\phi))$. Then

$$0 = \mathrm{gr}_0(\phi)\Big(\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) \Big) = \sum_i s_{\mathcal{B}'}(a_i) t_{\mathcal{B}'}(b_i^o).$$

By applying $\varepsilon_{\mathcal{B}'}$ to both sides we find out that $\sum_i a_i b_i = 0$ in A, whence

$$\sum_i a_i \otimes b_i^o = \sum_i (a_i \otimes 1_A^o)(b_i \otimes 1_A^o - 1_A \otimes b_i^o)$$

in A^e and so

$$\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) = \sum_i s_{\mathcal{B}}(a_i)(s_{\mathcal{B}}(b_i) - t_{\mathcal{B}}(b_i^o)) \in A^e \cdot \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle.$$

Being ϕ injective on $A^e \cdot \mathsf{Prim}(\mathcal{B})$, we conclude that $\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) = 0$ and hence $\mathrm{gr}_0(\phi)$ is injective.

To prove that $\mathrm{gr}_1(\phi)$ is injective, notice that an element in $F_1(\mathcal{B})/F_0(\mathcal{B})$ is of the form

$$\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) + \sum_j s_{\mathcal{B}}(a_j') t_{\mathcal{B}}(b_j'^o) X_j s_{\mathcal{B}}(a_j'') t_{\mathcal{B}}(b_j''^o) + F_0(\mathcal{B})$$

$$\overset{(35)}{=} \sum_j s_{\mathcal{B}}(a_j' a_j'') t_{\mathcal{B}}((b_j'' b_j')^o) X_j + F_0(\mathcal{B})$$

for $X_j \in \mathsf{Prim}(\mathcal{B})$ and $a_i, b_i, a_j', b_j', a_j'', b_j'' \in A$ for all i, j. Therefore, we may assume that $\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) X_i + F_0(\mathcal{B})$ is a generic element belonging to $\ker(\mathrm{gr}_1(\phi))$. This implies that

$$0 = \mathrm{gr}_1(\phi)\Big(\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) X_i + F_0(\mathcal{B}) \Big) = \sum_i s_{\mathcal{B}'}(a_i) t_{\mathcal{B}'}(b_i^o) \phi(X_i) + F_0(\mathcal{B}')$$

and hence there exists $\sum_j s_{\mathcal{B}'}(a_j') t_{\mathcal{B}'}(b_j'^o) \in F_0(\mathcal{B}')$ such that

$$\sum_i s_{\mathcal{B}'}(a_i) t_{\mathcal{B}'}(b_i^o) \phi(X_i) + \sum_j s_{\mathcal{B}'}(a_j') t_{\mathcal{B}'}(b_j'^o) = 0$$

in $\mathcal{B}'$. By applying $\varepsilon_{\mathcal{B}'}$ again we find out that $\sum_j a_j' b_j' = 0$ and hence

$$\sum_j s_{\mathcal{B}'}(a_j') t_{\mathcal{B}'}(b_j'^o) = \sum_j s_{\mathcal{B}'}(a_j')(s_{\mathcal{B}'}(b_j') - t_{\mathcal{B}'}(b_j'^o)) \in A^e \cdot \langle s_{\mathcal{B}'} - t_{\mathcal{B}'} \rangle.$$

Summing up,

$$
\begin{aligned}
0 &= \sum_i s_{\mathcal{B}'}(a_i) t_{\mathcal{B}'}(b_i^o) \phi(X_i) + \sum_j s_{\mathcal{B}'}(a_j')(s_{\mathcal{B}'}(b_j') - t_{\mathcal{B}'}(b_j'^o)) \\
&= \phi\Big(\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) X_i + \sum_j s_{\mathcal{B}}(a_j')(s_{\mathcal{B}}(b_j') - t_{\mathcal{B}}(b_j'^o)) \Big),
\end{aligned}
$$

but, being ϕ injective on $A^e \cdot \mathrm{Prim}(\mathcal{B})$, this yields that

$$
0 = \sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) X_i + \sum_j s_{\mathcal{B}}(a_j')(s_{\mathcal{B}}(b_j') - t_{\mathcal{B}}(b_j'^o))
$$

in $\mathcal{B}$ and hence

$$
\sum_i s_{\mathcal{B}}(a_i) t_{\mathcal{B}}(b_i^o) X_i + F_0(\mathcal{B}) = 0.
$$

Finally, let us prove that $\mathrm{gr}_n(\phi)$ is injective for all $n \geq 1$ by induction. We just showed the case $n = 1$. Assume that $\mathrm{gr}_1(\phi), \ldots, \mathrm{gr}_n(\phi)$ are all injective for a certain $n \geq 1$ and consider an element $z \in \ker(\mathrm{gr}_{n+1}(\phi))$. Consider also the canonical projections

$$
p_{h,k}^{\mathcal{B}} : \bigoplus_{i+j=n+1} \mathrm{gr}_i(\mathcal{B}) \otimes_A \mathrm{gr}_j(\mathcal{B}) \to \mathrm{gr}_h(\mathcal{B}) \otimes_A \mathrm{gr}_k(\mathcal{B})
$$

for $h + k = n + 1$, as in (9). For all p, q such that $p + q = n + 1$ we have that

$$
p_{h,k}^{\mathcal{B}'} \circ \Big(\bigoplus_{i+j=n+1} \mathrm{gr}_i(\phi) \otimes_A \mathrm{gr}_j(\phi) \Big) = (\mathrm{gr}_h(\phi) \otimes_A \mathrm{gr}_k(\phi)) \circ p_{h,k}^{\mathcal{B}}. \tag{54}
$$

Therefore, for all $1 \leq h \leq n$ we have that

$$
\begin{aligned}
0 &= (p_{h,k}^{\mathcal{B}'} \circ \Delta_{\mathrm{gr}(\mathcal{B}')}^{[n+1]} \circ (\mathrm{gr}_{n+1}(\phi)))(z) \\
&\overset{(10)}{=} \Big(p_{h,k}^{\mathcal{B}'} \circ \Big(\bigoplus_{i+j=n+1} \mathrm{gr}_i(\phi) \otimes_A \mathrm{gr}_j(\phi) \Big) \circ \Delta_{\mathrm{gr}(\mathcal{B})}^{[n+1]} \Big)(z) \\
&\overset{(54)}{=} \big((\mathrm{gr}_h(\phi) \otimes_A \mathrm{gr}_k(\phi)) \circ p_{h,k}^{\mathcal{B}} \circ \Delta_{\mathrm{gr}(\mathcal{B})}^{[n+1]} \big)(z).
\end{aligned}
$$

By the induction hypothesis and projectivity of $\mathrm{gr}_s(\mathcal{B})$ and $\mathrm{gr}_s(\mathcal{B}')$ as left A^e-modules for all $s \geq 0$, we know that $\mathrm{gr}_h(\phi) \otimes_A \mathrm{gr}_k(\phi)$ is injective and hence

$$
\big(p_{h,k}^{\mathcal{B}} \circ \Delta_{\mathrm{gr}(\mathcal{B})}^{[n+1]} \big)(z) = 0
$$

for all $h + k = n + 1$, $1 \leq h \leq n$. Since $\mathrm{gr}(\mathcal{B})$ is strongly graded by hypothesis, $p_{h,k}^{\mathcal{B}} \circ \Delta_{\mathrm{gr}(\mathcal{B})}^{[n+1]}$ is injective and hence $z = 0$.   ∎

Theorem 4.14. *Let $\mathcal{B}$ be an A-bialgebroid. Then we have an isomorphism*

$$
\mathcal{B} \cong A \odot U_{\Bbbk}(L) \odot A
$$

for an A-anchored Lie algebra (L, ω) if and only if

(CM1) L is a subalgebra of $\mathsf{Prim}(\mathcal{B})$ and $\mathsf{Prim}(\mathcal{B}) \cong L \ltimes_{\delta} \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle$,

(CM2) $\mathcal{B}$ is graded projective and primitively generated,

(CM3) the left A^e-submodule of $\mathcal{B}$ generated by L is 0 (in which case we require $\eta_{\mathcal{B}}$ to be injective) or it is free and generated by a $\mathbb{k}$-basis of L,

(CM4) $A^e \cdot \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle \cap A^e \cdot L = 0$ (in particular, $A^e \cdot \mathsf{Prim}(\mathcal{B}) = (A^e \cdot L) \oplus (A^e \cdot \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle))$.

Proof. We want to apply Theorem 4.13 to show that the conditions listed are sufficient. First of all, let us prove that for any A-anchored Lie algebra (L, ω) the Connes–Moscovici bialgebroid $\mathcal{B}_L$ is graded projective and that $\mathrm{gr}(\mathcal{B}_L)$ is strongly graded (we already know that $\mathcal{B}_L$ is primitively generated from Example 4.5 and Remark 4.7).

In view of (52), we know that $F_n(\mathcal{B}_L) = A \otimes F_n(U_{\mathbb{k}}(L)) \otimes A$. By exactness of the tensor product over a field, the short exact sequence of $\mathbb{k}$-vector spaces

$$0 \longrightarrow F_{n-1}(U_{\mathbb{k}}(L)) \longrightarrow F_n(U_{\mathbb{k}}(L)) \longrightarrow \mathrm{gr}_n(U_{\mathbb{k}}(L)) \longrightarrow 0$$

induces a short exact sequence of left A^e-modules

$$0 \longrightarrow A \otimes F_{n-1}(U_{\mathbb{k}}(L)) \otimes A \longrightarrow A \otimes F_n(U_{\mathbb{k}}(L)) \otimes A \longrightarrow A \otimes \mathrm{gr}_n(U_{\mathbb{k}}(L)) \otimes A \longrightarrow 0.$$

Therefore, we have $\mathrm{gr}_n(\mathcal{B}_L) \cong A \otimes \mathrm{gr}_n(U_{\mathbb{k}}(L)) \otimes A$ as left A^e-modules. In particular, $\mathrm{gr}_n(\mathcal{B}_L)$ is a free left A^e-module.

To show that $\mathrm{gr}(\mathcal{B}_L)$ is strongly graded, consider an element

$$z := \sum_{\alpha, \beta} a_\alpha \otimes u_{\alpha, \beta} \otimes b_\beta \in A \otimes \mathrm{gr}_n(U) \otimes A$$

such that $(p_{h,k}^{\mathcal{B}_L} \circ \Delta_{\mathrm{gr}(\mathcal{B}_L)}^{[n]})(z) = 0$ for some h, k satisfying $h + k = n$, where the elements $\{a_\alpha\}_\alpha$ in A are linearly independent over $\mathbb{k}$ as well as the elements $\{b_\beta\}_\beta$. Now, consider the commutative diagram

$$
\begin{array}{ccc}
A \otimes \mathrm{gr}_n(U) \otimes A & \xrightarrow{\ a_\alpha^* \otimes \mathrm{gr}_n(U) \otimes b_\beta^*\ } & \mathrm{gr}_n(U) \\[4pt]
{\scriptstyle \Delta_{\mathrm{gr}(\mathcal{B}_L)}^{[n]}} \Big\downarrow & & \Big\downarrow {\scriptstyle \Delta_{\mathrm{gr}(U)}^{[n]}} \\[8pt]
\bigoplus_{i+j=n}(A \otimes \mathrm{gr}_i(U) \otimes A) \otimes_A (A \otimes \mathrm{gr}_j(U) \otimes A) & & \\[4pt]
{\scriptstyle p_{h,k}^{\mathcal{B}_L}} \Big\downarrow & & \bigoplus_{i+j=n} \mathrm{gr}_i(U) \otimes \mathrm{gr}_j(U) \\[8pt]
(A \otimes \mathrm{gr}_h(U) \otimes A) \otimes_A (A \otimes \mathrm{gr}_k(U) \otimes A) & & \Big\downarrow {\scriptstyle p_{h,k}^{\mathrm{gr}(U)}} \\[4pt]
{\scriptstyle \cong} \Big\downarrow & & \\[8pt]
A \otimes \mathrm{gr}_h(U) \otimes A \otimes \mathrm{gr}_k(U) \otimes A & \xrightarrow[\ a_\alpha^* \otimes \mathrm{gr}_h(U) \otimes 1_A^* \otimes \mathrm{gr}_k(U) \otimes b_\beta^*\ ]{} & \mathrm{gr}_h(U) \otimes \mathrm{gr}_k(U).
\end{array}
$$

If we plug z in it and we recall that $\mathrm{gr}(U)$ is strongly graded (that is, $p_{h,k}^{\mathrm{gr}(U)} \circ \Delta_{\mathrm{gr}(U)}^{[n]}$ is injective for all $n \geq 0$ and for all $h + k = n$) then we find $u_{\alpha,\beta} = 0$ for all α, β and hence $z = 0$.

Now, the inclusion of A-anchored Lie algebras $L \to \mathrm{Prim}(\mathcal{B})$ extends uniquely to a morphism of A-bialgebroids $\Psi : \mathcal{B}_L \to \mathcal{B}$, in view of the universal property of $\mathcal{B}_L$ (Corollary 3.5). Moreover, similarly to what we did in the proof of Corollary 4.9, one can show that Ψ is surjective (because $\mathcal{B}$ is primitively generated and $\mathbb{P}(\Psi)$ is surjective). Therefore, to conclude by applying Theorem 4.13 we are left to check that the candidate isomorphism Ψ is injective when restricted to $A^e \cdot \mathrm{Prim}(\mathcal{B}_L)$. Since

$$\mathrm{Prim}(\mathcal{B}_L) = (1_A \otimes L \otimes 1_A) \oplus \langle s_{\mathcal{B}_L} - t_{\mathcal{B}_L} \rangle,$$

a generic element in $A^e \cdot \mathrm{Prim}(\mathcal{B}_L)$ is of the form

$$\sum_{i,j} a_{i,j} \otimes x_i \otimes b_{i,j} + \sum_{h,k} a_{h,k}'' a_h' \otimes 1_U \otimes b_{h,k}'' - a_{h,k}'' \otimes 1_U \otimes a_h' b_{h,k}''$$

for $X_i \in L$ and $a_{i,j}, b_{i,j}, a_{h,k}'', b_{h,k}'', a_h' \in A$, where we may assume that the X_k's are elements of a $\mathbb{k}$-basis of L, without loss of generality. Thus,

$$0 = \Psi\Big(\sum_{i,j} a_{i,j} \otimes x_i \otimes b_{i,j} + \sum_{h,k} a_{h,k}'' a_h' \otimes 1_U \otimes b_{h,k}'' - a_{h,k}'' \otimes 1_U \otimes a_h' b_{h,k}''\Big)$$

$$= \sum_{i,j} s_{\mathcal{B}}(a_{i,j}) t_{\mathcal{B}}(b_{i,j}) X_i + \sum_{h,k} s_{\mathcal{B}}(a_{h,k}'' a_h') t_{\mathcal{B}}(b_{h,k}'') - \sum_{h,k} s_{\mathcal{B}}(a_{h,k}'') t_{\mathcal{B}}(a_h' b_{h,k}'').$$

By (CM4), this entails that

$$0 = \sum_{i,j} s_{\mathcal{B}}(a_{i,j}) t_{\mathcal{B}}(b_{i,j}) X_i \tag{55}$$

and

$$0 = \sum_{h,k} s_{\mathcal{B}}(a_{h,k}'' a_h') t_{\mathcal{B}}(b_{h,k}'') - s_{\mathcal{B}}(a_{h,k}'') t_{\mathcal{B}}(a_h' b_{h,k}''). \tag{56}$$

By (CM3), relation (55) yields that

$$\sum_j a_{i,j} \otimes b_{i,j} = 0$$

in A^e for all i. Relation (56), instead, implies that

$$0 = \eta_{\mathcal{B}}\Big(\sum_{h,k} a_{h,k}'' a_h' \otimes b_{h,k}'' - a_{h,k}'' \otimes a_h' b_{h,k}''\Big).$$

However, since $A^e \cdot L$ is a free left A^e-module with action given via $\eta_{\mathcal{B}}$ (or $\eta_{\mathcal{B}}$ is injective by hypothesis), $\eta_{\mathcal{B}}$ itself has to be injective and hence

$$0 = \sum_{h,k} a_{h,k}'' a_h' \otimes b_{h,k}'' - a_{h,k}'' \otimes a_h' b_{h,k}'',$$

which, in turn, yields

$$0 = \sum_{h,k} a''_{h,k} a'_h \otimes 1_U \otimes b''_{h,k} - a''_{h,k} \otimes 1_U \otimes a'_h b''_{h,k}.$$

Summing up, (CM1) ensures the existence of a morphism Ψ and (CM2) entails that Ψ is surjective. Conditions (CM3) and (CM4), instead, allow us to conclude that Ψ is injective on $A^e \cdot \mathsf{Prim}(\mathcal{B}_L)$ and hence, by Theorem 4.13, Ψ is injective on $\mathcal{B}_L$. Thus, Ψ is an isomorphism.

The fact that the conditions (CM1)–(CM4) are necessary is clear. ∎

Remark 4.15. In the context of the proof above, observe that if $L = 0$, then $\mathsf{Prim}(\mathcal{B}) = \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle$. If moreover $\mathcal{B}$ is primitively generated, then $\Psi : A \odot \Bbbk \odot A \to \mathcal{B}$ is surjective and it coincides with $\eta_{\mathcal{B}} : A^e \to \mathcal{B}$ up to the isomorphism $A \odot \Bbbk \odot A \cong A^e$. This is the point where injectivity of $\eta_{\mathcal{B}}$ enters the picture.

Example 4.16. Let $A = \mathsf{Mat}_n(\Bbbk)$ for $n \geq 2$ and let $\mathcal{B} = \mathsf{End}_{\Bbbk}(A)$ as in Example 1.7(c). It follows from the Skolem–Noether theorem that every derivation of A is inner. In particular, $\mathsf{Prim}(\mathcal{B}) = \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle$ and conditions (CM1) and (CM4) are satisfied. Moreover, as we have seen in Example 4.10, $\mathcal{B}$ is also primitively generated and, in fact, $\mathsf{gr}_n(\mathcal{B}) = 0$ for all $n \geq 1$.

In order to apply Theorem 4.14, we are left to show that $\eta_{\mathcal{B}}$ is injective (notice that we already know it is surjective), but a straightforward computation reveals that $\eta_{\mathcal{B}}$ coincides with the composition of isomorphisms

$$\mathsf{Mat}_n(\Bbbk) \otimes \mathsf{Mat}_n(\Bbbk)^o \xrightarrow{\mathsf{Mat}_n(\Bbbk) \otimes (-)^{\mathsf{T}}} \mathsf{Mat}_n(\Bbbk) \otimes \mathsf{Mat}_n(\Bbbk) \cong \mathsf{Mat}_{n^2}(\Bbbk) \cong \mathsf{End}_{\Bbbk}(\mathsf{Mat}_n(\Bbbk)).$$

Therefore, (CM2) and (CM3) are satisfied as well and, by Theorem 4.14,

$$\mathcal{B} \cong A \odot U_{\Bbbk}(0) \odot A.$$

4.3. Bialgebroids over commutative algebras

A slightly more favorable situation is provided by the case of bialgebroids over a commutative base.

Let us assume henceforth that A is a commutative $\Bbbk$-algebra. This implies that now we can consider the target $t_{\mathcal{B}}$ of an A-bialgebroid $\mathcal{B}$ as an algebra map $t_{\mathcal{B}} : A \to \mathcal{B}$ and hence we will omit the $(-)^o$. By Lemma 4.1, we may consider the quotient Lie algebra

$$\overline{\mathsf{Prim}(\mathcal{B})} := \mathsf{Prim}(\mathcal{B})/\langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle,$$

which is A-anchored with anchor $\overline{\omega}_{\mathcal{B}}$ induced by $\omega_{\mathcal{B}}$, because now $\langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle$ is abelian with zero anchor. This induces a well-defined functor

$$\mathbb{P}' : \mathsf{Bialgd}_A \to \mathsf{AnchLie}_A, \qquad \mathcal{B} \mapsto \overline{\mathsf{Prim}(\mathcal{B})}.$$

As we have seen in Proposition 4.3 and in Corollary 4.4, $\mathcal{B}_L$ satisfies the additional property that $\mathrm{Prim}(\mathcal{B}_L) \cong \overline{\mathrm{Prim}(\mathcal{B}_L)} \ltimes_\delta \langle s_{\mathcal{B}_L} - t_{\mathcal{B}_L} \rangle$ as A-anchored Lie algebras. If we restrict our attention to the full subcategory $\overline{\mathrm{Bialgd}}_A$ of Bialgd_A composed by all those A-bialgebroids $\mathcal{B}$ such that $\mathrm{Prim}(\mathcal{B}) \cong \overline{\mathrm{Prim}(\mathcal{B})} \ltimes_\delta \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle$ as A-anchored Lie algebras, then the functors $\mathbb{B}$ and $\mathbb{P}'$ induce functors

$$\overline{\mathbb{P}} : \overline{\mathrm{Bialgd}}_A \to \mathrm{AnchLie}_A, \qquad \mathcal{B} \mapsto \overline{\mathrm{Prim}(\mathcal{B})}$$

and

$$\overline{\overline{\mathbb{B}}} : \mathrm{AnchLie}_A \to \overline{\mathrm{Bialgd}}_A, \qquad (L, \omega) \mapsto A \odot U_{\Bbbk}(L) \odot A.$$

It can be shown that, in this case, we always have a natural isomorphism

$$\overline{\gamma}_L := \left(L \xrightarrow{\gamma_L} \mathrm{Prim}(\mathcal{B}_L) \twoheadrightarrow \overline{\mathrm{Prim}(\mathcal{B}_L)} \right)$$

inducing a surjective map

$$\mathrm{Bialgd}_A(\mathcal{B}_L, \mathcal{B}) \to \mathrm{AnchLie}_A((L, \omega), (\overline{\mathrm{Prim}(\mathcal{B})}, \overline{\omega}_{\mathcal{B}})),$$

which, however, is not injective in general (that is, $\overline{\overline{\mathbb{B}}}$ and $\overline{\mathbb{P}}$ are not adjoint functors). Nevertheless, we may always consider a "preferred" morphism of A-bialgebroids

$$\overline{\epsilon}_{\mathcal{B}} := \left(A \odot U_{\Bbbk}\big(\overline{\mathrm{Prim}(\mathcal{B})}\big) \odot A \xrightarrow{A \odot U_{\Bbbk}(\iota_{\mathcal{B}}) \odot A} A \odot U_{\Bbbk}\big(\mathrm{Prim}(\mathcal{B})\big) \odot A \xrightarrow{\epsilon_{\mathcal{B}}} \mathcal{B} \right)$$

induced by a chosen injection $\iota_{\mathcal{B}} : \overline{\mathrm{Prim}(\mathcal{B})} \to \mathrm{Prim}(\mathcal{B})$ and by the counit of the adjunction in Theorem 3.6. Since the hypothesis on $\mathrm{Prim}(\mathcal{B})$ ensures that

$$\mathbb{P}(\overline{\epsilon}_{\mathcal{B}}) : \mathrm{Prim}(A \odot U_{\Bbbk}\big(\overline{\mathrm{Prim}(\mathcal{B})}\big) \odot A) \to \mathrm{Prim}(\mathcal{B})$$

is an epimorphism, we may conclude that $\mathcal{B}$ is primitively generated if and only if $\overline{\epsilon}_{\mathcal{B}}$ is surjective, as in Corollary 4.9, and we may restate Theorem 4.14 in the present framework.

Theorem 4.17. *Let $\mathcal{B}$ be an A-bialgebroid over a commutative algebra A. Then*

$$\mathcal{B} \cong A \odot U_{\Bbbk}\big(\overline{\mathrm{Prim}(\mathcal{B})}\big) \odot A$$

if and only if

- (CM1) $\mathrm{Prim}(\mathcal{B}) \cong \overline{\mathrm{Prim}(\mathcal{B})} \ltimes_\delta \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle$,
- (CM2) $\mathcal{B}$ *is graded projective and primitively generated,*
- (CM3) *the left A^e-submodule of $\mathcal{B}$ generated by $\overline{\mathrm{Prim}(\mathcal{B})}$ is 0 (in which case we require $\eta_{\mathcal{B}}$ to be injective) or it is free and generated by a $\Bbbk$-basis of $\overline{\mathrm{Prim}(\mathcal{B})}$,*
- (CM4) $A^e \cdot \langle s_{\mathcal{B}} - t_{\mathcal{B}} \rangle \cap A^e \cdot \overline{\mathrm{Prim}(\mathcal{B})} = 0$.

Example 4.18. Let $A := \mathbb{C}[X]$ and consider the A-bialgebroid

$$\mathcal{H} := \mathbb{C}\left[x, y, t, z, \frac{1}{t} \right]$$

studied in [15, §5.6] and inspired by the coordinate ring of the Malgrange's groupoid. Its bialgebroid structure is uniquely determined by

$$
\begin{aligned}
&s_{\mathcal{H}}(X) = x, &&t_{\mathcal{H}}(X) = y, &&\varepsilon_{\mathcal{H}}(x) = X = \varepsilon_{\mathcal{H}}(y),\\
&\varepsilon_{\mathcal{H}}(t) = 1, &&\varepsilon_{\mathcal{H}}(z) = 0, &&\Delta_{\mathcal{H}}(x) = x \otimes_A 1,\\
&\Delta_{\mathcal{H}}(y) = 1 \otimes_A y, &&\Delta_{\mathcal{H}}(t) = t \otimes_A t, &&\Delta_{\mathcal{H}}(z) = z \otimes_A t + t^2 \otimes_A z
\end{aligned}
$$

and ordinary multiplication and unit. The ideal $\mathcal{I} = \langle t - 1 \rangle$ generated by $t - 1$ in $\mathcal{H}$ is a bi-ideal (it is an ideal by construction and a coideal by [8, §2.4]) and hence the quotient $\mathcal{H}/\mathcal{I}$ is an A-bialgebroid. It can be identified with $\mathcal{B} := \mathbb{C}[u, v, w]$ with

$$
\begin{aligned}
&s(X) = u, &&t(X) = w, &&\varepsilon(u) = X = \varepsilon(w), &&\varepsilon(v) = 0,\\
&\Delta(u) = u \otimes_A 1, &&\Delta(w) = 1 \otimes_A w, &&\Delta(v) = v \otimes_A 1 + 1 \otimes_A v.
\end{aligned}
$$

The space of primitive elements of $\mathcal{B}$ is

$$
\mathsf{Prim}(\mathcal{B}) = \langle u^i - w^i \mid i \geq 0 \rangle \oplus \mathbb{C}v
$$

with $\overline{\mathsf{Prim}(\mathcal{B})} \cong \mathbb{C}v$, whence (CM1), (CM3) and (CM4) are satisfied (in this case, $\delta \equiv 0$ by Lemma 2.6, as everything is commutative). Concerning (CM2), we observe that $\mathcal{B}$ with the foregoing structures is the free left A^e-module generated by $\{v^k \mid k \geq 0\}$, whence it is primitively generated and graded projective (even free). Thus, Theorem 4.17 ensures that

$$
\mathcal{B} \cong A \odot U_{\mathbb{C}}(\mathbb{C}v) \odot A.
$$

The same conclusion could have been drawn by observing that $\mathcal{B} \cong \mathbb{C}[X] \otimes \mathbb{C}[Y] \otimes \mathbb{C}[X]$ is the scalar extension commutative (Hopf) $\mathbb{C}[X]$-bialgebroid obtained from the Hopf algebra $\mathbb{C}[Y]$ and that, as Hopf algebras, $\mathbb{C}[Y] \cong U_{\mathbb{C}}(\mathbb{C}Y)$.

4.4. Final Remarks

An additional step which deserve to be taken is to restrict the attention further to those A-bialgebroids $\mathcal{B}$ over a commutative algebra A such that $s_{\mathcal{B}} = t_{\mathcal{B}}$ (for example, cocommutative A-bialgebroids). However, in this case the Connes–Moscovici construction is not the correct construction to look at. One may prove that the assignment

$$
A \odot U_{\mathbb{k}}(L) \odot A \to A \# U_{\mathbb{k}}(L), \qquad a \otimes u \otimes b \mapsto ab \otimes u
$$

provides a surjective homomorphism of A-bialgebroids with kernel the ideal generated by $\langle s_{\mathcal{B}_L} - t_{\mathcal{B}_L} \rangle$ in $\mathcal{B}_L$. The A-bialgebroid structure on $A \# U_{\mathbb{k}}(L)$ is that of extension of scalars with trivial coaction on A, that is to say,

$$
\begin{aligned}
s = t &: a \mapsto a \otimes 1_{U_{\mathbb{k}}(L)},\\
\varepsilon &: a \otimes u \mapsto a\varepsilon(u),\\
\Delta &: a \otimes u \mapsto \sum (a \otimes u_{(1)}) \otimes_A (1_A \otimes u_{(2)})
\end{aligned}
$$

and semi-direct product algebra structure, that is,

$$(a \otimes u)(b \otimes v) = \sum a(u_{(1)} \cdot b) \otimes u_{(2)}v \qquad \text{and} \qquad 1_{A \# U_\Bbbk(L)} = 1_A \otimes 1_{U_\Bbbk(L)}.$$

In view of the results from Section 2.2 and Section 3.2, the foregoing observations suggest that $A \# U_\Bbbk(L)$ would be the right A-bialgebroid construction to consider, in order to recover the universal property of Theorem 2.9 and an adjunction as in Theorem 3.6. Nevertheless, we keep this question for a future investigation.

5. The finite-dimensional case and flat bimodule connections

In this last section, we show how, for a finite-dimensional A-anchored Lie algebra (L, ω), representations of (A, L, ω) in the sense of Corollary 2.10 naturally give rise to a family of A-bimodules with flat bimodule connection for the first-order differential calculus $\mathsf{Hom}_\Bbbk(L, A)$. As a consequence, we see how the latter forms a closed monoidal category with closed monoidal underlying functor to A-bimodules and with the associated Connes–Moscovici bialgebroid $A \odot U_\Bbbk(L) \odot A$ as the bialgebroid representing it in the spirit of [18]. For the general theory of differential calculi and bimodule connections, we refer the reader to [4, 12].

5.1. Anchored Lie algebras, representations and bimodule connections

Recall (e.g. from [4, Definition 1.1] or [12, §6]) that a *(generalised) first-order differential calculus* over a $\Bbbk$-algebra A is an A-bimodule Ω^1 with a derivation $d \in \mathsf{Der}_\Bbbk(A, \Omega^1)$ (the *exterior derivative*), that is, a $\Bbbk$-linear map $d : A \to \Omega^1$ satisfying

$$d(ab) = d(a) \cdot b + a \cdot d(b)$$

for all $a, b \in A$. A first-order differential calculus (Ω^1, d) is *parallelised with cotangent codimension n* if Ω^1 is free as a left and right A-module, with basis of cardinality n ([4, Definition 1.2]).

Proposition 5.1. *There is a bijective correspondence between $\Bbbk$-linear morphisms $\omega : L \to \mathsf{Der}_\Bbbk(A)$ and derivations $d : A \to \mathsf{Hom}_\Bbbk(L, A)$ making of $\Omega^1 = \mathsf{Hom}_\Bbbk(L, A)$ a first-order differential calculus, where the A-bimodule structure on Ω^1 is provided by the regular A-bimodule structure on A. In particular, the first-order differential calculus $(\Omega^1 = \mathsf{Hom}_\Bbbk(L, A), d)$ associated with (A, L, ω) is parallelised with cotangent dimension n if and only if L is a finite-dimensional $\Bbbk$-vector space of dimension n.*

Proof. The first claim follows from the bijective correspondence

$$\Psi_A : \mathsf{Hom}_\Bbbk(L, \mathsf{Hom}_\Bbbk(A, N)) \longleftrightarrow \mathsf{Hom}_\Bbbk(A, \mathsf{Hom}_\Bbbk(L, N))$$
$$f \longmapsto [a \mapsto [X \mapsto f(X)(a)]] \qquad (57)$$
$$[X \mapsto [a \mapsto g(a)(X)]] \longmapsfrom g$$

induced by the Hom-tensor adjunction, where N is any A-bimodule. The second one follows from the fact that $\mathsf{Hom}_{\Bbbk}(L, A) \cong A^{\dim_{\Bbbk}(L)}$ as left and right A-module. ∎

Let $\omega : L \to \mathsf{Der}_{\Bbbk}(A)$ be a $\Bbbk$-linear map. By following [22, §1], we set $\Omega^0 := A$ and

$$\Omega^n := \mathsf{Hom}_{\Bbbk}(\Lambda^n L, A),$$

for all $n \geq 1$, that is, Ω^n is the space of n-linear alternating functions from $L \times \cdots \times L$ to A. For the sake of the interested reader, these are the n-dimensional A-cochains of [9, §23]. For every $f \in \Omega^n$ and for all $X, X_1, \ldots, X_n \in L$ we set

$$(X \cdot f)(X_1 \otimes \cdots \otimes X_n) := \omega(X)\big(f(X_1 \otimes \cdots \otimes X_n)\big)$$

$$- \sum_{i=1}^{n} f(X_1 \otimes \cdots \otimes [X, X_i] \otimes \cdots \otimes X_n),$$

$$f_X(X_1 \otimes \cdots \otimes X_{n-1}) := f(X \otimes X_1 \otimes \cdots \otimes X_{n-1}),$$

and we define the differentials $d : \Omega^n \to \Omega^{n+1}$ inductively by setting $d(a)(X) = \omega(X)(a)$ and

$$d(f)_X = X \cdot f - d(f_X) \tag{58}$$

for all $a \in A$, $X \in L$, $f \in \Omega^n$ with $n \geq 1$. In the notation of [12, §3], (58) reads

$$(i_X \circ d)(f) = L_X(f) - (d \circ i_X)(f).$$

Lemma 5.2. *The complex* $(\Omega^\bullet, d)$ *is a cochain complex if and only if* $\omega : L \to \mathsf{Der}_{\Bbbk}(A)$ *is a morphism of Lie algebras.*

Proof. If ω is a morphism of Lie algebras then A is an L-module and the claim follows from [22, §1]. Conversely, if $(\Omega^\bullet, d)$ is a cochain complex then, for all $a \in A$, $X, Y \in L$,

$$0 = d^2(a)(X \otimes Y) = \omega(X)\omega(Y)(a) - \omega(Y)\omega(X)(a) - \omega\big([X, Y]\big)(a). \qquad ∎$$

Therefore, if (L, ω) is an A-anchored Lie algebra, then $(\Omega^\bullet, d)$ provides a DGA with

$$(f \wedge g)(X_1 \otimes \cdots \otimes X_{p+q}) := \sum_{S} (-1)^{\nu(S)} f(X_{s_1} \otimes \cdots \otimes X_{s_p}) g(X_{t_1} \otimes \cdots \otimes X_{t_q}) \tag{59}$$

as multiplication for all $f \in \Omega^p$, $g \in \Omega^q$, where the sum runs over all ordered subsets $S = (s_1, \ldots, s_p)$ of $\{1, 2, \ldots, p + q\}$, $(t_1, \ldots, t_q)$ is an ordered complement of S and $\nu(S) = \sum_{j=1}^{q} S(j)$ with $S(j)$ equal to the number of indices i for which s_i is greater than t_j.

Let (Ω^1, d) be a differential calculus on the algebra A. Recall from [12, §10] that a *left bimodule connection* (also called *left bimodule covariant derivative*) on an A-bimodule M

is a $\Bbbk$-linear map $\nabla : M \to \Omega^1 \otimes_A M$ (the *left connection*) together with an A-bimodule map $\sigma : M \otimes_A \Omega^1 \to \Omega^1 \otimes_A M$ (the *intertwining map*) such that

$$\nabla(a \cdot m) = a \cdot \nabla(m) + d(a) \otimes_A m \qquad \text{and} \qquad \nabla(m \cdot a) = \nabla(m) \cdot a + \sigma(m \otimes_A d(a)) \tag{60}$$

for all $a \in A$, $m \in M$. The *curvature* $R : M \to \Omega^2 \otimes_A M$ of ∇ is defined by

$$R(m) := \big((d \otimes_A M) - (\wedge \otimes_A M) \circ (\Omega^1 \otimes_A \nabla)\big)\big(\nabla(m)\big)$$

(i.e., $R = \nabla^2$) and ∇ is said to be *flat* if $R = 0$. Furthermore, a bimodule with left connection (M, ∇, σ) is called *extendable* ([4, Definition 4.10]) if for all $n \geq 1$, σ extends to $\sigma_n : M \otimes_A \Omega^n \to \Omega^n \otimes_A M$ such that

$$(\wedge \otimes_A M) \circ (\Omega^n \otimes_A \sigma_m) \circ (\sigma_n \otimes_A \Omega^m) = \sigma_{m+n} \circ (M \otimes_A \wedge)$$

as maps from $M \otimes_A \Omega^n \otimes_A \Omega^m$ to $\Omega^{m+n} \otimes_A M$.

Assume henceforth that (L, ω) is an A-anchored Lie algebra such that L is finite-dimensional as $\Bbbk$-vector space with dual basis $\{e_i, e_i^* \mid i = 1, \ldots, n\}$. For every A-bimodule M we have an isomorphism of A-bimodules

$$\begin{aligned} \Phi_M : \Omega^1 \otimes_A M &\longleftrightarrow \operatorname{Hom}_{\Bbbk}(L, M) \\ f \otimes_A m &\longmapsto [X \mapsto f(X) \cdot m] \\ \textstyle\sum_i 1_A e_i^* \otimes_A \varphi(e_i) &\longleftarrow\!\shortmid \varphi. \end{aligned} \tag{61}$$

Inspired by Corollary 2.10, we may take as definition of representation of (A, L, ω) an A-bimodule M together with a Lie algebra map $\rho : L \to \operatorname{End}_{\Bbbk}(M)$ such that

$$\rho(X)(a \cdot m \cdot b) = a \cdot \rho(X)(m) \cdot b + \omega(X)(a) \cdot m \cdot b + a \cdot m \cdot \omega(X)(b) \tag{62}$$

for all $a, b \in A$, $X \in L$, $m \in M$.

Proposition 5.3. *There is a bijective correspondence between A-bimodules M together with a $\Bbbk$-linear morphism $\rho : L \to \operatorname{End}_{\Bbbk}(M)$ and A-bimodules M together with a $\Bbbk$-linear morphism $\nabla : M \to \Omega^1 \otimes_A M$. Furthermore, for every A-bimodule M we have an A-bilinear isomorphism $\sigma : M \otimes_A \Omega^1 \to \Omega^1 \otimes_A M$ uniquely determined by*

$$\Phi_M(\sigma(m \otimes_A f))(X) = m \cdot f(X) \tag{63}$$

for all $m \in M$, $f \in \Omega^1$ and $X \in L$, where Φ_M is the isomorphism (61).

Proof. Since we have the isomorphism of A-bimodules Φ_M of (61) and an analogue Ψ_M of the bijective correspondence (57), we have a bijective correspondence between maps $\rho : L \to \operatorname{End}_{\Bbbk}(M)$ and maps $\nabla : M \to \Omega^1 \otimes_A M$ given by the relation

$$\Phi_M \circ \nabla = \Psi_M(\rho). \tag{64}$$

Furthermore, we may consider the chain of A-bilinear isomorphisms

$$M \otimes_A \Omega^1 \cong M \otimes_A A \otimes L^* \cong M \otimes L^* \cong \mathsf{Hom}_\Bbbk(L, M) \cong L^* \otimes M$$
$$\cong L^* \otimes A \otimes_A M \cong \Omega^1 \otimes_A M$$

which descends from the finite-dimensionality of L. It is straightforward to check that the composition σ of the latter ones satisfies (63). ∎

Lemma 5.4. *Under the correspondence established in Proposition* 5.3, (M, ∇, σ) *is an A-bimodule with left bimodule connection and invertible intertwining satisfying* (63) *if and only if* (62) *holds.*

Proof. For all $a \in A$, $m \in M$, $X \in L$, we compute, up to the isomorphism (61),

$$\Phi_M\big(\nabla(a \cdot m)\big)(X) = \rho(X)(a \cdot m),$$
$$\Phi_M\big(a \cdot \nabla(m) + d(a) \otimes_A m\big)(X) = a \cdot \rho(X)(m) + \omega(X)(a) \cdot m,$$
$$\Phi_M\big(\nabla(m \cdot a)\big)(X) = \rho(X)(m \cdot a)$$

and, in view of (63),

$$\Phi_M\big(\nabla(m) \cdot a + \sigma(m \otimes_A d(a))\big)(X) = \rho(X)(m) \cdot a + m \cdot \omega(X)(a). \qquad \blacksquare$$

Lemma 5.5. *Under the correspondence established in Lemma* 5.4, (M, ∇, σ) *is, in addition, an A-bimodule with flat left bimodule connection and invertible intertwining satisfying* (63) *if and only if* $\rho : L \to \mathsf{End}_\Bbbk(M)$ *is, in addition, a Lie algebra map.*

Proof. Observe that the A-bilinear isomorphism $\Omega^1 \otimes_A M \cong \mathsf{Hom}_\Bbbk(L, M)$ of (61) can be extended to an A-bilinear isomorphism $\Omega^n \otimes_A M \cong \mathsf{Hom}_\Bbbk(\Lambda^n L, M)$. In particular, for every $m \in M$ we can see $R(m)$ as an element in $\mathsf{Hom}_\Bbbk(\Lambda^2 L, M)$, up to the latter isomorphism. For all $X, Y \in L$ and for every $m \in M$ we have that $R(m)$ satisfies

$$R(m)(X \otimes Y) = -\rho([X, Y])(m) - \rho(Y)(\rho(X)(m)) + \rho(X)(\rho(Y)(m)). \qquad \blacksquare$$

Summing up, we proved the following theorem.

Theorem 5.6. *We have an isomorphism of categories between the category of representations* (M, ρ) *of* (A, L, ω) *in the sense of Corollary* 2.10 *and the full subcategory of the category of A-bimodules with flat left bimodule connection and invertible intertwining* (M, ∇, σ) *such that σ satisfies* (63).

The interested reader may check that an A-bimodule with left bimodule connection (M, ∇, σ) as in Theorem 5.6 is extendable and satisfies the condition of [4, Lemma 4.74].

Theorem 5.7. *Let* (L, ω) *be an A-anchored Lie algebra with L of finite dimension n. For* $(\Omega^\bullet, d)$ *induced by* (A, L, ω) *as in Proposition* 5.1, *the category of A-bimodules with flat left bimodule connection and invertible intertwining* (M, ∇, σ), *where σ satisfies* (63), *is a closed monoidal category with closed monoidal underlying functor to A-bimodules.*

Proof. By Theorem 5.6, the category of A-bimodules with flat left bimodule connection and invertible intertwining (M, ∇, σ), where σ satisfies (63), is isomorphic to the category of representations of (A, L, ω) which, in turn, is isomorphic to the category of $A \odot U_{\Bbbk}(L) \odot A$-modules (by Corollary 2.10). Since $U_{\Bbbk}(L)$ is an involutive Hopf algebra, $A \odot U_{\Bbbk}(L) \odot A$ is a Hopf algebroid with antipode in the sense of Böhm and Szlachányi [6, Definition 4.1] (see [27, Theorem 3.1], [39, Remark 2.6]). In particular, it is a $\times_A$-Hopf algebra in the sense of Schauenburg [46, Theorem and Definition 3.5], whence the category of $A \odot U_{\Bbbk}(L) \odot A$-modules is a closed monoidal category with closed monoidal forgetful functor to A-bimodules. ∎

Let us conclude this subsection with an elementary example.

Example 5.8. Let A be any algebra and $\partial \in \mathsf{Der}_{\Bbbk}(A)$ a derivation on A. The *(generalised) derivation calculus* over A (see [4, Example 1.7]) is given by the regular A-bimodule $\Omega^1 = A$ with exterior derivative $d = \partial$. If we let $L := \Bbbk Y$ and $\omega : L \to \mathsf{Der}_{\Bbbk}(A), Y \mapsto \partial$, then $\Omega^1 \cong \mathsf{Hom}_{\Bbbk}(L, A)$ as A-bimodules. The datum of a left connection ∇ on an A-bimodule M is equivalent to the datum of a $\Bbbk$-linear endomorphism $h : M \to M$ such that the left-hand side of (60) is satisfied,

$$h(a \cdot m) = a \cdot h(m) + \partial(a) \cdot m$$

for all $a \in A, m \in M$. That is to say, a left module structure over the skew polynomial ring $A[Y; \partial]$ (also called Ore extension; see [33, §1.2.3] and [38, §I.1]). The left connection ∇ is a bimodule connection if and only if there exists an A-bilinear endomorphism $\sigma : M \to M$ such that

$$h(m \cdot a) = h(m) \cdot a + \sigma(m) \cdot \partial(a)$$

for all $a \in A, m \in M$. Since the additional condition (63) implies that $\sigma = \mathsf{Id}_M$, we have that an A-bimodule with left bimodule connection as in Theorem 5.7 is an A-bimodule together with a $\Bbbk$-linear endomorphism h such that

$$h(a \cdot m) = a \cdot h(m) + \partial(a) \cdot m \qquad \text{and} \qquad h(m \cdot a) = h(m) \cdot a + m \cdot \partial(a)$$

for all $a \in A, m \in M$. These are bimodules over $A[Y; \partial]$ whose left and right action by Y coincide. In the particular case of $A = \mathbb{C}[X]$ and $\partial : \mathbb{C}[X] \to \mathbb{C}[X]$ uniquely determined by $\partial(X) = 1$, an A-bimodule is a $\mathbb{C}$-vector space M together with two commuting endomorphisms f and g (representing the left and the right action by X) and the datum of a left bimodule connection ∇ as in Theorem 5.7 on M is equivalent to the datum of a third endomorphism $h : M \to M$ such that

$$h \circ f = f \circ h + \mathsf{Id}_M \qquad \text{and} \qquad h \circ g = g \circ h + \mathsf{Id}_M.$$

These are exactly the modules over the bialgebroid $\mathcal{B} = A \odot U_{\mathbb{C}}(\mathbb{C}v) \odot A$ of Example 4.18.

5.2. Final remarks and further directions

As correctly noticed by the referee, there is a relationship between the theory of connections on A-(bi)modules with respect to the so called *derivation-based differential calculi* studied by M. Dubois-Violette and collaborators (see [12] for a summary of the topic) and the connections arising as discussed above from a slightly more general notion than anchored Lie algebras. We sketch it briefly here, for the sake of the interested reader, but it will be the subject of a future work.

Anchored Lie algebras and Lie–Rinehart algebras are the extreme cases of a more general, unifying notion. A *central anchored Lie algebra* over A is a Lie algebra L endowed with a (left) $Z(A)$-module structure $Z(A) \otimes L \to L, z \otimes X \mapsto z \cdot X$, and with a Lie algebra morphism $\omega : L \to \mathrm{Der}_{\Bbbk}(A)$ such that

$$\omega(z \cdot X) = z \cdot \omega(X) \qquad \text{and} \qquad [X, z \cdot Y] = z \cdot [X, Y] + \omega(X)(z) \cdot Y$$

for all $z \in Z(A)$ and $X, Y \in L$, where $Z(A)$ is the centre of A. Anchored Lie algebras correspond to the case $Z(A) = \Bbbk$, while Lie–Rinehart algebras correspond to the case $Z(A) = A$. The study of central anchored Lie algebras and, in particular, of the flat bimodule connections arising from them as in Section 5.1 is related to the flat left bimodule $\underline{\Omega}^1_{\mathrm{Der}}(A)$-connections on central A-bimodules with respect to the derivation-based differential calculus $\underline{\Omega}^1_{\mathrm{Der}}(A)$ introduced by Dubois-Violette and Michor in [14, §3.4].

To see how, recall that $\underline{\Omega}^1_{\mathrm{Der}}(A)$ is, by definition, ${}_{Z(A)}\mathrm{Hom}(\mathrm{Der}_{\Bbbk}(A), A)$ and that a central A-bimodule is an A-bimodule whose left and right $Z(A)$-module structures coincide (i.e., $z \cdot m = m \cdot z$ for all $z \in Z(A)$ and $m \in M$). Now, given a central anchored Lie algebra (A, L, ω), Proposition 5.1 can be adapted to prove that

$$ {}_{Z(A)}\mathrm{Hom}(L, \mathrm{Der}_{\Bbbk}(A)) \cong \mathrm{Der}_{\Bbbk}(A, {}_{Z(A)}\mathrm{Hom}(L, A)), $$

whence $\Omega^1 := {}_{Z(A)}\mathrm{Hom}(L, A)$ becomes a first-order differential calculus on A via the derivation induced by ω. If we define a representation of (A, L, ω) to be a central A-bimodule M together with a Lie algebra map $\rho : L \to \mathrm{End}_{\Bbbk}(M)$ such that

$$\rho(z \cdot X)(m) = z \cdot \rho(X)(m),$$
$$\rho(X)(a \cdot m \cdot b) = a \cdot \rho(X)(m) \cdot b + \omega(X)(a) \cdot m \cdot b + a \cdot m \cdot \omega(X)(b)$$

for all $a, b \in A, X \in L, m \in M$, and if we assume that L is finitely generated and projective as a (left) $Z(A)$-module, then one can prove, along the lines of Section 5.1, that we have an isomorphism of categories between the category of representations (M, ρ) of (A, L, ω) and the full subcategory of the category of central A-bimodules with flat left bimodule connection and invertible intertwining (M, ∇, σ) such that σ satisfies the analogue of (63).

In the particular case of $L = \mathrm{Der}_{\Bbbk}(A)$ with ω equal to the identity, $\Omega^1 = \underline{\Omega}^1_{\mathrm{Der}}(A)$ by definition and we find that representations of $(A, \mathrm{Der}_{\Bbbk}(A), \mathrm{Id})$ in a central A-bimodule M are derivation-based flat connections on M in the sense of [13, §2] (see also [12, page 296]).

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