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Leibniz bialgebras, relative Rota–Baxter operators, and the classical Leibniz Yang–Baxter equation

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Abstract. In this paper, first we introduce the notion of a Leibniz bialgebra and show that matched pairs of Leibniz algebras, Manin triples of Leibniz algebras, and Leibniz bialgebras are equivalent. Then we introduce the notion of a (relative) Rota–Baxter operator on a Leibniz algebra and construct the graded Lie algebra that characterizes relative Rota–Baxter operators as Maurer–Cartan elements. By these structures and the twisting theory of twilled Leibniz algebras, we further define the classical Leibniz Yang–Baxter equation, classical Leibniz r-matrices, and triangular Leibniz bialgebras. Finally, we construct solutions of the classical Leibniz Yang–Baxter equation using relative Rota–Baxter operators and Leibniz-dendriform algebras.

1. Introduction

The paper aims to establish the bialgebraic theory for Leibniz algebras. In particular, it answers the questions: What is a triangular Leibniz bialgebra?, What is a classical Leibniz Yang–Baxter equation?, and What is a classical Leibniz r-matrix?

1.1. Leibniz algebras and Leibniz bialgebras

Leibniz algebras were first discovered by Bloh who called them D-algebras [9]. Then Loday rediscovered this algebraic structure and called them Leibniz algebras [29, 31]. Recently Leibniz algebras was studied from different aspects due to applications in both mathematics and physics. In particular, integration of Leibniz algebras were studied in [12, 15] and deformation quantization of Leibniz algebras was studied in [16]. As the underlying structures of embedding tensors, Leibniz algebras also have applications in higher gauge theories [26, 36].

For a given algebraic structure, a bialgebra structure on this algebra is obtained by a comultiplication together with some compatibility conditions between the multiplication and the comultiplication. Among various bialgebras, Lie bialgebras have important applications in both mathematics and mathematical physics, e.g., a Lie bialgebra is the algebraic structure corresponding to a Poisson–Lie group and the classical structure of a quantized universal enveloping algebra [14, 17].

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The purpose of this paper is to study the bialgebra theory for Leibniz algebras with the motivation from the great importance of Lie bialgebras. We define a skew-symmetric quadratic Leibniz algebra using a skew-symmetric bilinear form. This is supported by the fact that the operad of Lie algebras is a cyclic operad, but the operad of Leibniz algebras is an anticyclic operad. Actually, it is observed by Chapoton in [13] using the operad theory that one should use the aforementioned skew-symmetric invariant bilinear form on a Leibniz algebra. We define Manin triples and dual representations of Leibniz algebras using the skew-symmetric quadratic Leibniz algebras. We introduce the notion of a Leibniz bialgebra and show that matched pairs of Leibniz algebras, Manin triples of Leibniz algebras, and Leibniz bialgebras are equivalent. Even though we obtain some nice results totally parallel to the context of Lie bialgebras, we need to emphasize that our bialgebra theory is not a generalization of Lie bialgebras, namely the restriction of our theory on Lie algebras is independent of Lie bialgebras.

1.2. Triangular Leibniz bialgebras: relative Rota–Baxter operator approach

Due to the importance of the classical Yang–Baxter equation and triangular Lie bialgebras, it is natural to define the Leibniz analogue of the classical Yang–Baxter equation and triangular Leibniz bialgebras. This is a very hard problem due to that the representation theory of Leibniz algebras is not good, e.g., there is no tensor product in the module category of Leibniz algebras. We solve this problem using relative Rota–Baxter operators and the twisting theory of twilled Leibniz algebras.

The notion of a relative Rota–Baxter operator (originally called an $\mathcal{O}$-operator) on a Lie algebra was introduced by Kupershmidt [27], which can be traced back to Bordemann [11]. A relative Rota–Baxter operator gives rise to a skew-symmetric r-matrix in a larger Lie algebra [4]. In the context of associative algebras, relative Rota–Baxter operators give rise to dendriform algebras [22, 30], play important role in the bialgebra theory [5], and lead to the splitting of operads [6].

The twisting theory was introduced by Drinfeld in [18] motivated by the study of quasi-Lie bialgebras and quasi-Hopf algebras. As a useful tool in the study of bialgebras, the twisting theory was further applied to associative algebras and Poisson geometry; see [23, 25, 32, 38] for more details.

In this paper, we introduce the notion of a relative Rota–Baxter operator on a Leibniz algebra. We construct the graded Lie algebra that characterizes relative Rota–Baxter operators as Maurer–Cartan elements. Using this graded Lie algebra, we give the definition of a classical Leibniz Yang–Baxter equation. Moreover, we give the twisting theory of twilled Leibniz algebras, by which we define triangular Leibniz bialgebras. We also use relative Rota–Baxter operators and Leibniz-dendriform algebras to give solutions of the classical Leibniz Yang–Baxter equations in some larger Leibniz algebras. Note that embedding tensors studied in [35] are special relative Rota–Baxter operators on Leibniz algebras, which play important roles in mathematical physics [10, 26].

1.3. Outline of the paper

In Section 2, we introduce the notions of a Manin triple of Leibniz algebras and a Leibniz bialgebra. We prove the equivalence between matched pairs of Leibniz algebras, Manin triples of Leibniz algebras, and Leibniz bialgebras. The main innovation is that we use a skew-symmetric invariant bilinear form instead of a symmetric invariant bilinear form in the definition of a skew-symmetric quadratic Leibniz algebra.

In Section 3, we make preparations for our later study of triangular Leibniz algebras. In Section 3.1, we give the graded Lie algebra that characterizes Leibniz algebras as Maurer–Cartan elements and some technical tools. In Section 3.2, we introduce the notion of a relative Rota–Baxter operator on a Leibniz algebra and construct the graded Lie algebra that characterizes it as a Maurer–Cartan element. In Section 3.3, we introduce the notion of a twilled Leibniz algebra. The twisting theory of twilled Leibniz algebras is studied in detail for the purpose to define triangular Leibniz bialgebras.

In Section 4, we study triangular Leibniz bialgebras. We define the classical Leibniz Yang–Baxter equation and a classical Leibniz r-matrix using the graded Lie algebra given in Section 3.2, and then define a triangular Leibniz bialgebra successfully using the twisting theory of a twilled Leibniz algebra given in Section 3.3.

In Section 5, we construct solutions of the classical Leibniz Yang–Baxter equation using relative Rota–Baxter operators and Leibniz-dendriform algebras.

Quantization of Lie bialgebras and deformation quantization of Leibniz algebras were studied in [16,19]. It is natural to study quasi-triangular Leibniz bialgebras and their quantization. On the other hand, classical r-matrices play an important role in the study of integrable systems. It is natural to investigate whether classical Leibniz r-matrices can be applied to some integrable systems. It is also natural to investigate the global objects corresponding to Leibniz bialgebras. We will study these questions in the future.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$ and finite-dimensional.

2. Skew-symmetric quadratic Leibniz algebras and Leibniz bialgebras

Definition 2.1. A Leibniz algebra is a vector space $\mathfrak{g}$ together with a bilinear operation $[\cdot,\cdot]_\mathfrak{g} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$\big[x, [y,z]_\mathfrak{g}\big]_\mathfrak{g} = \big[[x,y]_\mathfrak{g}, z\big]_\mathfrak{g} + \big[y, [x,z]_\mathfrak{g}\big]_\mathfrak{g}, \quad \forall x,y,z \in \mathfrak{g}.$$

This is in fact a left Leibniz algebra. In this paper, we only consider left Leibniz algebras.

A representation of a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ is a triple $(V; \rho^L, \rho^R)$, where V is a vector space and $\rho^L, \rho^R : \mathfrak{g} \to \mathfrak{gl}(V)$ are linear maps such that the following equalities hold for all $x, y \in \mathfrak{g}$:

$$\rho^L\big([x,y]_\mathfrak{g}\big) = \big[\rho^L(x), \rho^L(y)\big], \tag{2.1}$$

$$\rho^R\big([x,y]_{\mathfrak{g}}\big) = \big[\rho^L(x), \rho^R(y)\big], \tag{2.2}$$

$$\rho^R(y) \circ \rho^L(x) = -\rho^R(y) \circ \rho^R(x). \tag{2.3}$$

Define the left multiplication $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and the right multiplication $R : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $L_x y = [x, y]_{\mathfrak{g}}$ and $R_x y = [y, x]_{\mathfrak{g}}$, respectively, for all $x, y \in \mathfrak{g}$. Then $(\mathfrak{g}; L, R)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, which is called the regular representation. Define two linear maps $L^*, R^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ with $x \to L^*_x$ and $x \to R^*_x$ for all $x, y \in \mathfrak{g}, \xi \in \mathfrak{g}^*$, respectively, by

$$\langle L^*_x \xi, y\rangle = -\big\langle \xi, [x,y]_{\mathfrak{g}}\big\rangle, \quad \big\langle R^*_x \xi, y\big\rangle = -\big\langle \xi, [y,x]_{\mathfrak{g}}\big\rangle. \tag{2.4}$$

If there is a Leibniz algebra structure on the dual space $\mathfrak{g}^*$, we denote the left multiplication and the right multiplication by $\mathcal{L}$ and $\mathcal{R}$, respectively.

2.1. Skew-symmetric quadratic Leibniz algebras and the Leibniz analogue of the string Lie 2-algebra

It is observed by Chapoton in [13] using the operad theory that one needs to use skew-symmetric bilinear forms instead of symmetric bilinear forms on a Leibniz algebra. This is the key ingredient in our study of Leibniz bialgebras.

Definition 2.2 ([13]). A skew-symmetric quadratic Leibniz algebra is a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ equipped with a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2\mathfrak{g}^*$ such that the following invariant condition holds:

$$\omega\big(x, [y,z]_{\mathfrak{g}}\big) = \omega\big([x,z]_{\mathfrak{g}} + [z,x]_{\mathfrak{g}}, y\big), \quad \forall x,y,z \in \mathfrak{g}. \tag{2.5}$$

Remark 2.3. In the original definition of a nondegenerate skew-symmetric invariant bilinear form on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ given in [13], there is a superfluous condition

$$\omega\big(x, [y,z]_{\mathfrak{g}}\big) = -\omega\big([y,x]_{\mathfrak{g}}, z\big). \tag{2.6}$$

In fact, by (2.5), for all $x, y, z \in \mathfrak{g}$, we have

$$-\omega\big([y,x]_{\mathfrak{g}}, z\big) = \omega\big(z, [y,x]_{\mathfrak{g}}\big) = \omega\big([z,x]_{\mathfrak{g}} + [x,z]_{\mathfrak{g}}, y\big) = \omega\big(x, [y,z]_{\mathfrak{g}}\big).$$

Remark 2.4. Note that we use skew-symmetric bilinear forms instead of symmetric bilinear forms and use the invariant condition (2.5) instead of the invariant condition

$$B\big([x,y]_{\mathfrak{g}}, z\big) = B\big(x, [y,z]_{\mathfrak{g}}\big)$$

and this is the main ingredient in our study of Leibniz bialgebras. In [8], the authors use symmetric bilinear forms and the invariant condition $B([x,y]_{\mathfrak{g}}, z) = B(x, [y,z]_{\mathfrak{g}})$ to study Leibniz bialgebras so that one has to add some strong conditions. As we will see, everything in the following study is natural in the sense that we do not need to add any extra conditions on the Leibniz algebra.

Recall that a quadratic Lie algebra is a Lie algebra $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}})$ equipped with a nondegenerate symmetric bilinear form $B \in \mathrm{Sym}^2(\mathfrak{k}^*)$, which is invariant in the sense that

$$B\big([x,y]_{\mathfrak{g}}, z\big) = B\big(x, [y,z]_{\mathfrak{g}}\big), \quad \forall x,y,z \in \mathfrak{k}.$$

Associated to a quadratic Lie algebra $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}}, B)$, we have a closed 3-form $\bar{\Theta} \in \wedge^3\mathfrak{k}^*$ given by

$$\bar{\Theta}(x,y,z) = B\big(x, [y,z]_{\mathfrak{k}}\big),$$

which is known as the Cartan 3-form.

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$ be a skew-symmetric quadratic Leibniz algebra. Define $\Theta \in \otimes^3\mathfrak{g}^*$ by

$$\Theta(x,y,z) = \omega\big(x, [y,z]_{\mathfrak{g}}\big), \quad \forall x,y,z \in \mathfrak{g}. \tag{2.7}$$

This 3-tensor can be viewed as the Leibniz analogue of the Cartan 3-form on a quadratic Lie algebra as the following lemma shows.

Lemma 2.5. *With the above notations, Θ is a 3-cocycle on the Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with values in the trivial representation $(\mathbb{K}; 0, 0)$, i.e., $\partial\Theta = 0$.*

Proof. For all $x,y,z,w \in \mathfrak{g}$, by the fact that $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}}$ is a left center, we have

$$\begin{aligned}
(\partial\Theta)(x,y,z,w) &= -\Theta\big([x,y]_{\mathfrak{g}}, z, w\big) - \Theta\big(y, [x,z]_{\mathfrak{g}}, w\big) - \Theta\big(y, z, [x,w]_{\mathfrak{g}}\big)\\
&\quad + \Theta\big(x, [y,z]_{\mathfrak{g}}, w\big) + \Theta\big(x, z, [y,w]_{\mathfrak{g}}\big) - \Theta\big(x, y, [z,w]_{\mathfrak{g}}\big)\\
&= -\omega\big([x,y]_{\mathfrak{g}}, [z,w]_{\mathfrak{g}}\big) - \omega\big(y, \big[[x,z]_{\mathfrak{g}}, w\big]_{\mathfrak{g}}\big) - \omega\big(y, \big[z, [x,w]_{\mathfrak{g}}\big]_{\mathfrak{g}}\big)\\
&\quad + \omega\big(x, \big[[y,z]_{\mathfrak{g}}, w\big]_{\mathfrak{g}}\big) + \omega\big(x, \big[z, [y,w]_{\mathfrak{g}}\big]_{\mathfrak{g}}\big) - \omega\big(x, \big[y, [z,w]_{\mathfrak{g}}\big]_{\mathfrak{g}}\big)\\
&= \omega\big(y, \big[x, [z,w]_{\mathfrak{g}}\big]_{\mathfrak{g}}\big) - \omega\big(y, \big[[x,z]_{\mathfrak{g}}, w\big]_{\mathfrak{g}}\big) - \omega\big(y, \big[z, [x,w]_{\mathfrak{g}}\big]_{\mathfrak{g}}\big)\\
&\quad + \omega\big(x, \big[[y,z]_{\mathfrak{g}} + [z,y]_{\mathfrak{g}}, w\big]_{\mathfrak{g}}\big)\\
&= 0,
\end{aligned}$$

which finishes the proof. ∎

Consequently, given a skew-symmetric quadratic Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$, we can construct a Leibniz 2-algebra (2-term Lod_∞-algebra) [2, 28, 34], which can be viewed as the Leibniz analogue of the string Lie 2-algebra associated to a semisimple Lie algebra [3].

On the graded vector space $\mathbb{K} \oplus \mathfrak{g}$, define $l_1 : \mathbb{K} \to \mathfrak{g}$ to be the zero map and define l_2 and l_3 by

$$\begin{cases}
l_2(x,y) = [x,y]_{\mathfrak{g}}, & \forall x,y \in \mathfrak{g},\\
l_2(x,s) = l_2(s,x) = 0, & \forall x \in \mathfrak{g},\ s \in \mathbb{K},\\
l_3(x,y,z) = \Theta(x,y,z) = \omega\big(x, [y,z]_{\mathfrak{g}}\big), & \forall x,y,z \in \mathfrak{g}.
\end{cases}$$

Theorem 2.6. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$ be a skew-symmetric quadratic Leibniz algebra. Then $(\mathbb{K}, \mathfrak{g}, l_1 = 0, l_2, l_3)$ is a Leibniz 2-algebra.*

Proof. It follows from Lemma 2.5 and we omit the details. ∎

Remark 2.7. A semisimple Lie algebra with the Killing form is naturally a quadratic Lie algebra. How to construct a skew-symmetric bilinear form associated to a Leibniz algebra such that it is invariant in the sense of (2.5) is not known yet.

2.2. Matched pairs, Manin triples of Leibniz algebras, and Leibniz bialgebras

In this subsection, first we recall the notion of a matched pair of Leibniz algebras. Then we introduce the notions of a Manin triple of Leibniz algebras and a Leibniz bialgebra. Finally, we prove the equivalence between matched pairs of Leibniz algebras, Manin triples of Leibniz algebras, and Leibniz bialgebras.

Definition 2.8 ([1]). Let $(\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1})$ and $(\mathfrak{g}_2, [\cdot,\cdot]_{\mathfrak{g}_2})$ be two Leibniz algebras. If there exists a representation (ρ_1^L, ρ_1^R) of $\mathfrak{g}_1$ on $\mathfrak{g}_2$ and a representation (ρ_2^L, ρ_2^R) of $\mathfrak{g}_2$ on $\mathfrak{g}_1$ such that the identities

$$\begin{aligned}&\rho_1^R(x)[u,v]_{\mathfrak{g}_2} - \big[u, \rho_1^R(x)v\big]_{\mathfrak{g}_2} + \big[v, \rho_1^R(x)u\big]_{\mathfrak{g}_2} - \rho_1^R\big(\rho_2^L(v)x\big)u \\ &\qquad + \rho_1^R\big(\rho_2^L(u)x\big)v = 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned}&\rho_1^L(x)[u,v]_{\mathfrak{g}_2} - \big[\rho_1^L(x)u, v\big]_{\mathfrak{g}_2} - \big[u, \rho_1^L(x)v\big]_{\mathfrak{g}_2} - \rho_1^L\big(\rho_2^R(u)x\big)v \\ &\qquad - \rho_1^R\big(\rho_2^R(v)x\big)u = 0, \end{aligned} \tag{2.9}$$

$$\big[\rho_1^L(x)u, v\big]_{\mathfrak{g}_2} + \rho_1^L\big(\rho_2^R(u)x\big)v + \big[\rho_1^R(x)u, v\big]_{\mathfrak{g}_2} + \rho_1^L\big(\rho_2^L(u)x\big)v = 0, \tag{2.10}$$

$$\begin{aligned}&\rho_2^R(u)[x,y]_{\mathfrak{g}_1} - \big[x, \rho_2^R(u)y\big]_{\mathfrak{g}_1} + \big[y, \rho_2^R(u)x\big]_{\mathfrak{g}_1} - \rho_2^R\big(\rho_1^L(y)u\big)x \\ &\qquad + \rho_2^R\big(\rho_1^L(x)u\big)y = 0, \end{aligned} \tag{2.11}$$

$$\begin{aligned}&\rho_2^L(u)[x,y]_{\mathfrak{g}_1} - \big[\rho_2^L(u)x, y\big]_{\mathfrak{g}_1} - \big[x, \rho_2^L(u)y\big]_{\mathfrak{g}_1} - \rho_2^L\big(\rho_1^R(x)u\big)y \\ &\qquad - \rho_2^R\big(\rho_1^R(y)u\big)x = 0, \end{aligned} \tag{2.12}$$

$$\big[\rho_2^L(u)x, y\big]_{\mathfrak{g}_1} + \rho_2^L\big(\rho_1^R(x)u\big)y + \big[\rho_2^R(u)x, y\big]_{\mathfrak{g}_1} + \rho_2^L\big(\rho_1^L(x)u\big)y = 0 \tag{2.13}$$

hold for all $x, y \in \mathfrak{g}_1$ and $u, v \in \mathfrak{g}_2$, then we call $(\mathfrak{g}_1, \mathfrak{g}_2; (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ a matched pair of Leibniz algebras.

Proposition 2.9 ([1]). *Let $(\mathfrak{g}_1, \mathfrak{g}_2; (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ be a matched pair of Leibniz algebras. Then there is a Leibniz algebra structure on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ defined by*

$$[x+u, y+v]_{\bowtie} = [x,y]_{\mathfrak{g}_1} + \rho_2^R(v)x + \rho_2^L(u)y + [u,v]_{\mathfrak{g}_2} + \rho_1^L(x)v + \rho_1^R(y)u. \tag{2.14}$$

In the Lie algebra context, to relate matched pairs of Lie algebras to Lie bialgebras and Manin triples for Lie algebras, we need the notion of the coadjoint representation, which is the dual representation of the adjoint representation. Now we investigate the dual representation in the Leibniz algebra context.

Lemma 2.10. *Let $(V; \rho^L, \rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then*

$$\big(V^*; (\rho^L)^*, -(\rho^L)^* - (\rho^R)^*\big)$$

is a representation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, which is called the dual representation of $(V; \rho^L, \rho^R)$.

Proof. It follows from (2.1)–(2.3) and we omit the details. ■

Definition 2.11. A Manin triple of Leibniz algebras is a triple $(\mathscr{G}, \mathfrak{g}_1, \mathfrak{g}_2)$, where

- $(\mathscr{G}, [\cdot,\cdot]_{\mathscr{G}}, \omega)$ is a skew-symmetric quadratic Leibniz algebra,
- both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isotropic subalgebras of $(\mathscr{G}, [\cdot,\cdot]_{\mathscr{G}})$,
- $\mathscr{G} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces.

Example 2.12. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Leibniz algebra. Then $(\mathfrak{g} \ltimes_{L^*, -L^*-R^*} \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple of Leibniz algebras, where the natural nondegenerate skew-symmetric bilinear form ω on $\mathfrak{g} \oplus \mathfrak{g}^*$ is given by

$$\omega(x+\xi, y+\eta) = \langle \xi, y\rangle - \langle \eta, x\rangle, \quad \forall x, y \in \mathfrak{g},\ \xi, \eta \in \mathfrak{g}^*. \tag{2.15}$$

For a Leibniz algebra $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*})$, let $\triangle : \mathfrak{g} \to \otimes^2\mathfrak{g}$ be the dual map of $[\cdot,\cdot]_{\mathfrak{g}^*} : \otimes^2\mathfrak{g}^* \to \mathfrak{g}^*$, i.e.,

$$\langle \triangle x, \xi \otimes \eta\rangle = \big\langle x, [\xi, \eta]_{\mathfrak{g}^*}\big\rangle.$$

Definition 2.13. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*})$ be Leibniz algebras. Then $(\mathfrak{g}, \mathfrak{g}^*)$ is called a Leibniz bialgebra if the following conditions hold:

(a) for all $x, y \in \mathfrak{g}$, we have

$$\tau\big((R_y \otimes \mathrm{Id})\triangle x\big) = (R_x \otimes \mathrm{Id})\triangle y,$$

where $\tau : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ is the exchange operator defined by $\tau(x\otimes y) = y \otimes x$;

(b) for all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}\triangle[x, y]_{\mathfrak{g}} = {}&\big((\mathrm{Id}\otimes R_y - L_y \otimes \mathrm{Id} - R_y \otimes \mathrm{Id}) \circ (\mathrm{Id} + \tau)\big)\triangle x\\ &+ (\mathrm{Id}\otimes L_x + L_x \otimes \mathrm{Id})\triangle y.\end{aligned}$$

Until now, we have recalled the notion of a matched pair and introduced the notions of a Manin triple of Leibniz algebras and a Leibniz bialgebra. Similar to the case of Lie algebras, these objects are equivalent when we consider the dual representation of the regular representation in a matched pair of Leibniz algebras. The following theorem is the main result in this section.

Theorem 2.14. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*})$ be two Leibniz algebras. Then the following conditions are equivalent:*

(i) $(\mathfrak{g}, \mathfrak{g}^*)$ *is a Leibniz bialgebra;*

(ii) $(\mathfrak{g}, \mathfrak{g}^*; (L^*, -L^* - R^*), (\mathscr{L}^*, -\mathscr{L}^* - \mathscr{R}^*))$ *is a matched pair of Leibniz algebras;*

(iii) $(\mathfrak{g}\oplus\mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ *is a Manin triple of Leibniz algebras, where the invariant skew-symmetric bilinear form on $\mathfrak{g}\oplus\mathfrak{g}^*$ is given by (2.15).*

Proof. First we prove that (ii) is equivalent to (iii).

Let $(\mathfrak{g},\mathfrak{g}^*;(L^*,-L^*-R^*),(\mathcal{L}^*,-\mathcal{L}^*-\mathcal{R}^*))$ be a matched pair of Leibniz algebras. Then $(\mathfrak{g}\oplus\mathfrak{g}^*,[\cdot,\cdot]_{\bowtie})$ is a Leibniz algebra, where $[\cdot,\cdot]_{\bowtie}$ is given by (2.14). We only need to prove that ω satisfies the invariant condition (2.5). For all $x,y,z\in\mathfrak{g}$ and $\xi,\eta,\alpha\in\mathfrak{g}^*$, we have

$$\begin{aligned}\omega\big(x+\xi,[y+\eta,z+\alpha]_{\bowtie}\big)&=\big\langle\xi,[y,z]_{\mathfrak{g}}\big\rangle-\langle\xi,\mathcal{L}^*_{\alpha}y\rangle-\langle\xi,\mathcal{R}^*_{\alpha}y\rangle+\langle\xi,\mathcal{L}^*_{\eta}z\rangle\\&\quad-\langle L^*_y\alpha,x\rangle+\langle L^*_z\eta,x\rangle+\langle R^*_z\eta,x\rangle-\big\langle[\eta,\alpha]_{\mathfrak{g}^*},x\big\rangle\\&=\big\langle\xi,[y,z]_{\mathfrak{g}}\big\rangle+\big\langle[\alpha,\xi]_{\mathfrak{g}^*},y\big\rangle+\big\langle[\xi,\alpha]_{\mathfrak{g}^*},y\big\rangle-\big\langle[\eta,\xi]_{\mathfrak{g}^*},z\big\rangle\\&\quad+\big\langle\alpha,[y,x]_{\mathfrak{g}}\big\rangle-\big\langle\eta,[z,x]_{\mathfrak{g}}\big\rangle-\big\langle\eta,[x,z]_{\mathfrak{g}}\big\rangle-\big\langle[\eta,\alpha]_{\mathfrak{g}^*},x\big\rangle.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\omega\big([x+\xi,z+\alpha]_{\bowtie}+[z+\alpha,x+\xi]_{\bowtie},y+\eta\big)\\&\quad=\omega\big([x,z]_{\mathfrak{g}}-\mathcal{R}^*_{\alpha}x-R^*_z\xi+[\xi,\alpha]_{\mathfrak{g}^*}+[z,x]_{\mathfrak{g}}-\mathcal{R}^*_{\xi}z-R^*_x\alpha+[\alpha,\xi]_{\mathfrak{g}^*},y+\eta\big)\\&\quad=-\langle R^*_z\xi,y\rangle+\big\langle[\xi,\alpha]_{\mathfrak{g}^*},y\big\rangle-\langle R^*_x\alpha,y\rangle+\big\langle[\alpha,\xi]_{\mathfrak{g}^*},y\big\rangle\\&\qquad-\big\langle\eta,[x,z]_{\mathfrak{g}}\big\rangle+\langle\eta,\mathcal{R}^*_{\alpha}x\rangle-\big\langle\eta,[z,x]_{\mathfrak{g}}\big\rangle+\langle\eta,\mathcal{R}^*_{\xi}z\rangle\\&\quad=\big\langle\xi,[y,z]_{\mathfrak{g}}\big\rangle+\big\langle[\xi,\alpha]_{\mathfrak{g}^*},y\big\rangle+\big\langle\alpha,[y,x]_{\mathfrak{g}}\big\rangle+\big\langle[\alpha,\xi]_{\mathfrak{g}^*},y\big\rangle\\&\qquad-\big\langle\eta,[x,z]_{\mathfrak{g}}\big\rangle-\big\langle[\eta,\alpha]_{\mathfrak{g}^*},x\big\rangle-\big\langle\eta,[z,x]_{\mathfrak{g}}\big\rangle-\big\langle[\eta,\xi]_{\mathfrak{g}^*},z\big\rangle.\end{aligned}$$

Thus, ω satisfies the invariant condition (2.5).

Conversely, let $(\mathcal{G},\mathfrak{g},\mathfrak{g}^*)$ be a Manin triple of Leibniz algebras with the invariant bilinear form given by (2.15). For all $x\in\mathfrak{g}$, $\xi\in\mathfrak{g}^*$, assume that

$$[x,\xi]_{\mathcal{G}}=\rho^L_1(x)\xi+\rho^R_2(\xi)x,\quad [\xi,x]_{\mathcal{G}}=\rho^R_1(x)\xi+\rho^L_2(\xi)x,$$

for some $\rho^L_1,\rho^R_1:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}^*)$ and $\rho^L_2,\rho^R_2:\mathfrak{g}^*\to\mathfrak{gl}(\mathfrak{g})$. By (2.5), we have

$$\begin{aligned}\big\langle\eta,\rho^R_2(\xi)x\big\rangle&=\omega\big(\eta,[x,\xi]_{\mathcal{G}}\big)=\omega\big([\eta,\xi]_{\mathfrak{g}^*}+[\xi,\eta]_{\mathfrak{g}^*},x\big)\\&=\langle\mathcal{R}_{\xi}\eta+\mathcal{L}_{\xi}\eta,x\rangle=-\langle\eta,\mathcal{R}^*_{\xi}x+\mathcal{L}^*_{\xi}x\rangle,\end{aligned}$$

which implies that $\rho^R_2=-\mathcal{L}^*-\mathcal{R}^*$, and

$$\begin{aligned}\big\langle\rho^L_1(x)\xi,y\big\rangle&=-\omega\big(y,[x,\xi]_{\mathcal{G}}\big)=-\omega\big([y,\xi]_{\mathcal{G}}+[\xi,y]_{\mathcal{G}},x\big)\\&=-\omega\big(\xi,[x,y]_{\mathfrak{g}}\big)=-\langle\xi,L_xy\rangle=\langle L^*_x\xi,y\rangle,\end{aligned}$$

which implies that $\rho^L_1=L^*$. Similarly, we have $\rho^R_1=-L^*-R^*$ and $\rho^L_2=\mathcal{L}^*$. Thus, $(\mathfrak{g},\mathfrak{g}^*;(L^*,-L^*-R^*),(\mathcal{L}^*,-\mathcal{L}^*-\mathcal{R}^*))$ is a matched pair.

Next we prove that (i) is equivalent to (ii).

Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}^*,[\cdot,\cdot]_{\mathfrak{g}^*})$ be Leibniz algebras. Consider their representations $(\mathfrak{g}^*;L^*,-L^*-R^*)$ and $(\mathfrak{g};\mathcal{L}^*,-\mathcal{L}^*-\mathcal{R}^*)$. For all $x,y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$, consider the left-hand side of (2.13); then we have

$$[\mathcal{L}^*_{\xi}x,y]_{\mathfrak{g}}+\mathcal{L}^*_{(-L^*_x-R^*_x)\xi}y+\big[(-\mathcal{L}^*_{\xi}-\mathcal{R}^*_{\xi})x,y\big]_{\mathfrak{g}}+\mathcal{L}^*_{L^*_x\xi}y=-\mathcal{L}^*_{R^*_x\xi}y-[\mathcal{R}^*_{\xi}x,y]_{\mathfrak{g}}.$$

Furthermore, by straightforward computations, we have

$$\begin{aligned}\big\langle -\mathcal{L}^*_{R^*_x\xi}y-[\mathcal{R}^*_\xi x,y]_{\mathfrak{g}},\eta\big\rangle &= \big\langle y,[R^*_x\xi,\eta]_{\mathfrak{g}^*}\big\rangle-\langle R_y\mathcal{R}^*_\xi x,\eta\rangle\\ &= \langle\triangle y,R^*_x\xi\otimes\eta\rangle-\langle\triangle x,R^*_y\eta\otimes\xi\rangle\\ &= -\big\langle(R_x\otimes\mathrm{Id})(\triangle y),\xi\otimes\eta\big\rangle+\big\langle(R_y\otimes\mathrm{Id})(\triangle x),\eta\otimes\xi\big\rangle\\ &= -\big\langle(R_x\otimes\mathrm{Id})(\triangle y),\xi\otimes\eta\big\rangle+\big\langle\tau\big((R_y\otimes\mathrm{Id})(\triangle x)\big),\xi\otimes\eta\big\rangle.\end{aligned}$$

Therefore, (2.13) is equivalent to

$$(R_x\otimes\mathrm{Id})(\triangle y)=\tau\big((R_y\otimes\mathrm{Id})(\triangle x)\big). \tag{2.16}$$

The left-hand side of (2.12) is equal to

$$\mathcal{L}^*_\xi[x,y]_{\mathfrak{g}}-[\mathcal{L}^*_\xi x,y]_{\mathfrak{g}}-[x,\mathcal{L}^*_\xi y]_{\mathfrak{g}}-\mathcal{L}^*_{(-L^*_x-R^*_x)\xi}y-(-\mathcal{L}^*_{(-L^*_y-R^*_y)\xi}-\mathcal{R}^*_{(-L^*_y-R^*_y)\xi})x.$$

Furthermore, by straightforward computations, we have

$$\begin{aligned}&\big\langle\mathcal{L}^*_\xi[x,y]_{\mathfrak{g}}-[\mathcal{L}^*_\xi x,y]_{\mathfrak{g}}-[x,\mathcal{L}^*_\xi y]_{\mathfrak{g}}+\mathcal{L}^*_{L^*_x\xi}y+\mathcal{L}^*_{R^*_x\xi}y\\ &\quad-\mathcal{L}^*_{L^*_y\xi}x-\mathcal{L}^*_{R^*_y\xi}x-\mathcal{R}^*_{L^*_y\xi}x-\mathcal{R}^*_{R^*_y\xi}x,\eta\big\rangle\\ &=-\big\langle[x,y]_{\mathfrak{g}},[\xi,\eta]_{\mathfrak{g}^*}\big\rangle-\big\langle x,[\xi,R^*_y\eta]_{\mathfrak{g}^*}\big\rangle-\big\langle y,[\xi,L^*_x\eta]_{\mathfrak{g}^*}\big\rangle-\big\langle y,[L^*_x\xi,\eta]_{\mathfrak{g}^*}\big\rangle\\ &\quad-\big\langle y,[R^*_x\xi,\eta]_{\mathfrak{g}^*}\big\rangle+\big\langle x,[L^*_y\xi,\eta]_{\mathfrak{g}^*}\big\rangle+\big\langle x,[R^*_y\xi,\eta]_{\mathfrak{g}^*}\big\rangle+\big\langle x,[\eta,L^*_y\xi]_{\mathfrak{g}^*}\big\rangle\\ &\quad+\big\langle x,[\eta,R^*_y\xi]_{\mathfrak{g}^*}\big\rangle\\ &=-\big\langle\triangle[x,y]_{\mathfrak{g}},\xi\otimes\eta\big\rangle-\langle\triangle x,\xi\otimes R^*_y\eta\rangle-\langle\triangle y,\xi\otimes L^*_x\eta\rangle-\langle\triangle y,L^*_x\xi\otimes\eta\rangle\\ &\quad-\langle\triangle y,R^*_x\xi\otimes\eta\rangle+\langle\triangle x,L^*_y\xi\otimes\eta\rangle+\langle\triangle x,R^*_y\xi\otimes\eta\rangle+\langle\triangle x,\eta\otimes L^*_y\xi\rangle\\ &\quad+\langle\triangle x,\eta\otimes R^*_y\xi\rangle\\ &=-\big\langle\triangle[x,y]_{\mathfrak{g}},\xi\otimes\eta\big\rangle+\big\langle(\mathrm{Id}\otimes R_y)\triangle x,\xi\otimes\eta\big\rangle+\big\langle(\mathrm{Id}\otimes L_x)\triangle y,\xi\otimes\eta\big\rangle\\ &\quad+\big\langle(L_x\otimes\mathrm{Id})\triangle y,\xi\otimes\eta\big\rangle+\big\langle(R_x\otimes\mathrm{Id})\triangle y,\xi\otimes\eta\big\rangle-\big\langle(L_y\otimes\mathrm{Id})\triangle x,\xi\otimes\eta\big\rangle\\ &\quad-\big\langle(R_y\otimes\mathrm{Id})\triangle x,\xi\otimes\eta\big\rangle-\big\langle\tau\big((\mathrm{Id}\otimes L_y)+(\mathrm{Id}\otimes R_y)\big)\triangle x,\xi\otimes\eta\big\rangle.\end{aligned}$$

Therefore, (2.12) is equivalent to

$$\begin{aligned}\triangle[x,y]_{\mathfrak{g}}=&\big(\mathrm{Id}\otimes R_y-L_y\otimes\mathrm{Id}-R_y\otimes\mathrm{Id}-\tau\circ(\mathrm{Id}\otimes L_y)-\tau\circ(\mathrm{Id}\otimes R_y)\big)(\triangle x)\\ &+(\mathrm{Id}\otimes L_x+L_x\otimes\mathrm{Id}+R_x\otimes\mathrm{Id})(\triangle y).\end{aligned} \tag{2.17}$$

The left-hand side of (2.11) is equal to

$$\begin{aligned}&(-\mathcal{L}^*_\xi-\mathcal{R}^*_\xi)[x,y]_{\mathfrak{g}}+[x,\mathcal{L}^*_\xi y+\mathcal{R}^*_\xi y]_{\mathfrak{g}}-[y,\mathcal{L}^*_\xi x+\mathcal{R}^*_\xi x]_{\mathfrak{g}}\\ &\quad+\mathcal{L}^*_{L^*_y\xi}x+\mathcal{R}^*_{L^*_y\xi}x-\mathcal{L}^*_{L^*_x\xi}y-\mathcal{R}^*_{L^*_x\xi}y.\end{aligned}$$

Furthermore, by straightforward computations, we have

$$
\begin{aligned}
&\big\langle(-\mathcal{L}^*_\xi-\mathcal{R}^*_\xi)[x,y]_\mathfrak{g}+[x,\mathcal{L}^*_\xi y+\mathcal{R}^*_\xi y]_\mathfrak{g}-[y,\mathcal{L}^*_\xi x+\mathcal{R}^*_\xi x]_\mathfrak{g}\\
&\qquad+\mathcal{L}^*_{L^*_y\xi}x+\mathcal{R}^*_{L^*_y\xi}x-\mathcal{L}^*_{L^*_x\xi}y-\mathcal{R}^*_{L^*_x\xi}y,\eta\big\rangle\\
&\quad=\big\langle[x,y]_\mathfrak{g},[\xi,\eta]_{\mathfrak{g}^*}\big\rangle+\big\langle[x,y]_\mathfrak{g},[\eta,\xi]_{\mathfrak{g}^*}\big\rangle+\big\langle y,[\xi,L^*_x\eta]_{\mathfrak{g}^*}\big\rangle+\big\langle y,[L^*_x\eta,\xi]_{\mathfrak{g}^*}\big\rangle\\
&\qquad-\big\langle x,[\xi,L^*_y\eta]_{\mathfrak{g}^*}\big\rangle-\big\langle x,[L^*_y\eta,\xi]_{\mathfrak{g}^*}\big\rangle-\big\langle x,[L^*_y\xi,\eta]_{\mathfrak{g}^*}\big\rangle-\big\langle x,[\eta,L^*_y\xi]_{\mathfrak{g}^*}\big\rangle\\
&\qquad+\big\langle y,[L^*_x\xi,\eta]_{\mathfrak{g}^*}\big\rangle+\big\langle y,[\eta,L^*_x\xi]_{\mathfrak{g}^*}\big\rangle\\
&\quad=\big\langle\triangle[x,y]_\mathfrak{g},\xi\otimes\eta\big\rangle+\big\langle\tau\big(\triangle[x,y]_\mathfrak{g}\big),\xi\otimes\eta\big\rangle-\big\langle(\mathrm{Id}\otimes L_x)\triangle y,\xi\otimes\eta\big\rangle\\
&\qquad-\big\langle\tau\big((L_x\otimes\mathrm{Id})\triangle y\big),\xi\otimes\eta\big\rangle+\big\langle(\mathrm{Id}\otimes L_y)\triangle x,\xi\otimes\eta\big\rangle+\big\langle\tau\big((L_y\otimes\mathrm{Id})\triangle x\big),\xi\otimes\eta\big\rangle\\
&\qquad+\big\langle(L_y\otimes\mathrm{Id})\triangle x,\xi\otimes\eta\big\rangle+\big\langle\tau\big((\mathrm{Id}\otimes L_y)\triangle x\big),\xi\otimes\eta\big\rangle-\big\langle(L_x\otimes\mathrm{Id})\triangle y,\xi\otimes\eta\big\rangle\\
&\qquad-\big\langle\tau\big((\mathrm{Id}\otimes L_x)\triangle y\big),\xi\otimes\eta\big\rangle.
\end{aligned}
$$

Therefore, (2.11) is equivalent to

$$
\begin{aligned}
&\triangle[x,y]_\mathfrak{g}+\tau\big(\triangle[x,y]_\mathfrak{g}\big)\\
&\quad=\big(\mathrm{Id}\otimes L_x+\tau\circ(L_x\otimes\mathrm{Id})+L_x\otimes\mathrm{Id}+\tau\circ(\mathrm{Id}\otimes L_x)\big)\triangle y\\
&\qquad-\big(\mathrm{Id}\otimes L_y+\tau\circ(L_y\otimes\mathrm{Id})+L_y\otimes\mathrm{Id}+\tau\circ(\mathrm{Id}\otimes L_y)\big)\triangle x.
\end{aligned}
\tag{2.18}
$$

By (2.16) and (2.17), we deduce that

$$
\begin{aligned}
&\triangle[x,y]_\mathfrak{g}+\tau\big(\triangle[x,y]_\mathfrak{g}\big)\\
&\quad=\big(\underline{\mathrm{Id}\otimes R_y}-L_y\otimes\mathrm{Id}\underbrace{-R_y\otimes\mathrm{Id}}-\tau\circ(\mathrm{Id}\otimes L_y)\underline{-\tau\circ(\mathrm{Id}\otimes R_y)}\big)\triangle x\\
&\qquad+\big(\mathrm{Id}\otimes L_x+L_x\otimes\mathrm{Id}+\underbrace{R_x\otimes\mathrm{Id}}\big)\triangle y\\
&\qquad+\big(\underline{\tau\circ(\mathrm{Id}\otimes R_y)}-\tau\circ(L_y\otimes\mathrm{Id})\underbrace{-\tau\circ(R_y\otimes\mathrm{Id})}-\mathrm{Id}\otimes L_y\underline{-\mathrm{Id}\otimes R_y}\big)\triangle x\\
&\qquad+\big(\tau\circ(\mathrm{Id}\otimes L_x)+\tau\circ(L_x\otimes\mathrm{Id})+\underbrace{\tau\circ(R_x\otimes\mathrm{Id})}\big)\triangle y\\
&\quad=\text{the right-hand side of (2.18)}.
\end{aligned}
$$

Thus, by (2.12) and (2.13), we can deduce that (2.11) holds.

Consider the left-hand side of (2.10), it equals $-L^*_{\mathcal{R}^*_\xi x}\eta-[R^*_x\xi,\eta]_{\mathfrak{g}^*}$. For all $y\in\mathfrak{g}$, we have

$$
\big\langle-L^*_{\mathcal{R}^*_\xi x}\eta-[R^*_x\xi,\eta]_{\mathfrak{g}^*},y\big\rangle=\big\langle\eta,[\mathcal{R}^*_\xi x,y]_\mathfrak{g}\big\rangle+\langle\eta,\mathcal{L}^*_{R^*_x\xi}y\rangle=\big\langle\eta,[\mathcal{R}^*_\xi x,y]_\mathfrak{g}+\mathcal{L}^*_{R^*_x\xi}y\big\rangle,
$$

which implies that (2.10) is equivalent to (2.13).

Similarly, if (2.10) holds, we can deduce that (2.9) is equivalent to (2.12). Furthermore, by (2.9) and (2.10), we can deduce that (2.8) holds naturally. Therefore, $(\mathfrak{g},\mathfrak{g}^*;(L^*,-L^*-R^*),(\mathcal{L}^*,-\mathcal{L}^*-\mathcal{R}^*))$ is a matched pair of Leibniz algebras if and only if (2.16) and (2.17) hold. Note that (2.16) is exactly Condition (a) in Definition 2.13. Furthermore, if (2.16) holds, (2.17) is exactly Condition (b) in Definition 2.13. Thus,

$(\mathfrak{g}, \mathfrak{g}^*)$ is a Leibniz bialgebra if and only if $(\mathfrak{g}, \mathfrak{g}^*; (L^*, -L^* - R^*), (\mathcal{L}^*, -\mathcal{L}^* - \mathcal{R}^*))$ is a matched pair of Leibniz algebras. ■

Corollary 2.15. *Let* $(\mathfrak{g}, \mathfrak{g}^*)$ *be a Leibniz bialgebra. Then* $(\mathfrak{g}^*, \mathfrak{g})$ *is also a Leibniz bialgebra.*

3. (Relative) Rota–Baxter operators and the twisting theory

In this section, first we recall the graded Lie algebra whose Maurer–Cartan elements are Leibniz algebra structures, and define the bidegree of a multilinear map which is the technical tool in our later study. Then we introduce the notion of a relative Rota–Baxter operator on a Leibniz algebra, and construct the graded Lie algebra whose Maurer–Cartan elements are relative Rota–Baxter operators. Finally, we give the twisting theory of twilled Leibniz algebras. These structures and theories are the main ingredients in our later study of Leibniz bialgebras.

3.1. Lifts and bidegrees of multilinear maps

A permutation $\sigma \in \mathbb{S}_n$ is called an $(i, n-i)$-*shuffle* if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. If $i = 0$ or n, we assume that $\sigma = \mathrm{Id}$. The set of all $(i, n-i)$-shuffles will be denoted by $\mathbb{S}_{(i,n-i)}$. The notion of an $(i_1, \ldots, i_k)$-shuffle and the set $\mathbb{S}_{(i_1,\ldots,i_k)}$ are defined analogously.

Let $\mathfrak{g}$ be a vector space. We consider the graded vector space

$$C^*(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n\geq 1} C^n(\mathfrak{g}, \mathfrak{g}) = \bigoplus_{n\geq 1} \mathrm{Hom}(\otimes^n \mathfrak{g}, \mathfrak{g}).$$

The Balavoine bracket on the graded vector space $C^*(\mathfrak{g}, \mathfrak{g})$ is given by

$$[P, Q]_{\mathsf{B}} = P \bar{\circ} Q - (-1)^{pq} Q \bar{\circ} P, \tag{3.1}$$

for all $P \in C^{p+1}(\mathfrak{g}, \mathfrak{g})$, $Q \in C^{q+1}(\mathfrak{g}, \mathfrak{g})$, where $P \bar{\circ} Q \in C^{p+q+1}(\mathfrak{g}, \mathfrak{g})$ is defined by

$$P \bar{\circ} Q = \sum_{k=1}^{p+1} P \circ_k Q, \tag{3.2}$$

and $\circ_k$ is defined by

$$\begin{aligned}(P \circ_k Q)(x_1, \ldots, x_{p+q+1}) = &\sum_{\sigma \in \mathbb{S}_{(k-1,q)}} (-1)^{\sigma} (-1)^{(k-1)q} \\ &P\big(x_{\sigma(1)}, \ldots, x_{\sigma(k-1)}, Q(x_{\sigma(k)}, \ldots, x_{\sigma(k+q-1)}, x_{k+q}), x_{k+q+1}, \ldots, x_{p+q+1}\big).\end{aligned} \tag{3.3}$$

The following theorem is well known.

Theorem 3.1 ([7,21]). *With the above notations,* $(C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\mathsf{B}})$ *is a graded Lie algebra. Its Maurer–Cartan elements are precisely the Leibniz algebra structures on* $\mathfrak{g}$.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be vector spaces and elements in $\mathfrak{g}_1$ will be denoted by x, y, x_i and elements in $\mathfrak{g}_2$ will be denoted by u, v, v_i. Let $c : \mathfrak{g}_2^{\otimes n} \to \mathfrak{g}_1$ be a linear map. We can construct a linear map $\hat{c} \in C^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\hat{c}\big((x_1, v_1) \otimes \cdots \otimes (x_n, v_n)\big) := \big(c(v_1, \dots, v_n), 0\big).$$

In general, for a given linear map $f : \mathfrak{g}_{i(1)} \otimes \mathfrak{g}_{i(2)} \otimes \cdots \otimes \mathfrak{g}_{i(n)} \to \mathfrak{g}_j, i(1), \dots, i(n), j \in \{1, 2\}$, we define a linear map $\hat{f} \in C^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\hat{f} := \begin{cases} f & \text{on } \mathfrak{g}_{i(1)} \otimes \mathfrak{g}_{i(2)} \otimes \cdots \otimes \mathfrak{g}_{i(n)}, \\ 0 & \text{all other cases.} \end{cases}$$

We call the linear map $\hat{f}$ a horizontal lift of f, or simply a lift. Let $H : \mathfrak{g}_2 \to \mathfrak{g}_1$ be a linear map. Its lift is given by $\widehat{H}(x, v) = (H(v), 0)$. Obviously we have $\widehat{H} \circ \widehat{H} = 0$.

We denote by $\mathfrak{g}^{l,k}$ the direct sum of all $(l + k)$-tensor powers of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, where l (resp. k) is the number of $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$). By the properties of the Hom-functor, we have

$$C^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong \sum_{l+k=n} \mathrm{Hom}(\mathfrak{g}^{l,k}, \mathfrak{g}_1) \oplus \sum_{l+k=n} \mathrm{Hom}(\mathfrak{g}^{l,k}, \mathfrak{g}_2), \tag{3.4}$$

where the isomorphism is the horizontal lift.

Definition 3.2. A linear map $f \in \mathrm{Hom}(\otimes^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2), \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ has a bidegree $l|k$, which is denoted by $\|f\| = l|k$ if f satisfies the following conditions

- (i) $l + k + 1 = n$;
- (ii) if X is an element in $\mathfrak{g}^{l+1,k}$, then $f(X) \in \mathfrak{g}_1$;
- (iii) if X is an element in $\mathfrak{g}^{l,k+1}$, then $f(X) \in \mathfrak{g}_2$;
- (iv) all the other case, $f(X) = 0$.

A linear map f is said to be homogeneous if f has a bidegree. We have $l + k \geq 0$, $k, l \geq -1$ because $n \geq 1$ and $l + 1, k + 1 \geq 0$. For instance, the lift $\widehat{H} \in C^1(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ of $H : \mathfrak{g}_2 \to \mathfrak{g}_1$ has the bidegree $-1|1$.

It is obvious that we have the following lemmas.

Lemma 3.3. *Let $f_1, \dots, f_k \in C^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ be homogeneous linear maps and the bidegrees of f_i are different. Then $f_1 + \cdots + f_k = 0$ if and only if $f_1 = \cdots = f_k = 0$.*

Lemma 3.4. *If $\|f\| = -1|l$ (resp. $l| - 1$) and $\|g\| = -1|k$ (resp. $k| - 1$), then $[f, g]_{\mathrm{B}} = 0$.*

Proof. Assume that $\|f\| = -1|l$ and $\|g\| = -1|k$. Then f and g are both horizontal lifts of linear maps in $C^*(\mathfrak{g}_2, \mathfrak{g}_1)$. By the definition of the lift, we have $f \circ_i g = g \circ_j f = 0$ for any i, j. Thus, we have $[f, g]_{\mathrm{B}} = 0$. ■

Lemma 3.5. *Let $f \in C^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ and $g \in C^m(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ be homogeneous linear maps with bidegrees $l_f|k_f$ and $l_g|k_g$, respectively. Then the composition $f \circ_i g \in C^{n+m-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ is a homogeneous linear map of the bidegree $l_f + l_g|k_f + k_g$.*

Lemma 3.6. *If* $\|f\| = l_f|k_f$ *and* $\|g\| = l_g|k_g$, *then* $[f,g]_{\mathsf{B}}$ *has the bidegree* $l_f + l_g|k_f + k_g$.

Proof. By Lemma 3.5 and (3.1), we have $\|[f,g]_{\mathsf{B}}\| = l_f + l_g|k_f + k_g$. ■

3.2. (Relative) Rota–Baxter operators

First we introduce the notion of a (relative) Rota–Baxter operator and give some examples.

Definition 3.7. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Leibniz algebra.

(i) A linear operator $R : \mathfrak{g} \to \mathfrak{g}$ is called a Rota–Baxter operator if

$$\big[R(x), R(y)\big]_{\mathfrak{g}} = R\big(\big[R(x), y\big]_{\mathfrak{g}} + \big[x, R(y)\big]_{\mathfrak{g}}\big), \quad \forall x, y \in \mathfrak{g}. \tag{3.5}$$

(ii) Let $(V; \rho^L, \rho^R)$ be a representation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. A relative Rota–Baxter operator on $\mathfrak{g}$ with respect to the representation $(V; \rho^L, \rho^R)$ is a linear map $K : V \to \mathfrak{g}$ such that

$$[Kv_1, Kv_2]_{\mathfrak{g}} = K\big(\rho^L(Kv_1)v_2 + \rho^R(Kv_2)v_1\big), \quad \forall v_1, v_2 \in V. \tag{3.6}$$

(iii) Let $K : V \to \mathfrak{g}$ (resp. $K' : V' \to \mathfrak{g}'$) be a relative Rota–Baxter operator on $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ (resp. $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'})$) with respect to the representation $(V; \rho^L, \rho^R)$ (resp. $(V; \rho^{L'}, \rho^{R'})$). A homomorphism from K to K' is a pair (ϕ, φ), where $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Leibniz algebra homomorphism, $\varphi : V \to V'$ is a linear map such that for all $x \in \mathfrak{g}, u \in V$,

$$K' \circ \varphi = \phi \circ K, \tag{3.7}$$

$$\varphi\rho^L(x)(u) = \rho^{L'}\big(\phi(x)\big)\big(\varphi(u)\big), \tag{3.8}$$

$$\varphi\rho^R(x)(u) = \rho^{R'}\big(\phi(x)\big)\big(\varphi(u)\big). \tag{3.9}$$

In particular, if ϕ and φ are invertible, then (ϕ, φ) is called an isomorphism from K to K'.

We denote by RBOLeibniz the category of relative Rota–Baxter operators on Leibniz algebras.

Remark 3.8. When $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a Lie algebra and $\rho^R = -\rho^L$, we obtain the notion of a relative Rota–Baxter operator (an $\mathcal{O}$-operator) on a Lie algebra with respect to a representation.

Example 3.9. Consider the 2-dimensional Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot])$ given with respect to a basis $\{e_1, e_2\}$ by

$$[e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1.$$

Let $\{e_1^*, e_2^*\}$ be the dual basis. Then $K = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a relative Rota–Baxter operator on $(\mathfrak{g}, [\cdot,\cdot])$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ if and only if

$$[Ke_i^*, Ke_j^*] = K(L^*_{Ke_i^*}e_j^* - L^*_{Ke_j^*}e_i^* - R^*_{Ke_j^*}e_i^*), \quad \forall i, j = 1, 2.$$

Thus, we obtain

$$\begin{aligned}
a_{21}(a_{11}+a_{21}) &= a_{12}(a_{11}+a_{21}),\\
a_{22}(a_{11}+a_{21}) &= 0,\\
a_{21}(a_{12}+a_{22}) &= a_{22}a_{11}+(a_{12}+2a_{22})a_{12},\\
a_{22}(a_{21}+a_{12}+2a_{22}) &= 0,\\
a_{22}(a_{11}+a_{21}) &= -a_{22}(a_{11}+a_{12}),\\
-a_{22}(a_{21}+a_{22}) &= 0,\\
a_{22}(a_{12}+a_{22}) &= 0.
\end{aligned}$$

Summarizing the above discussion, we have the following.

(i) If $a_{22}=0$, then $K=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & 0\end{pmatrix}$ is a relative Rota–Baxter operator on $(\mathfrak{g},[\cdot,\cdot])$ with respect to the representation $(\mathfrak{g}^*;L^*,-L^*-R^*)$ if and only if

$$(a_{12}-a_{21})a_{12}=(a_{12}-a_{21})(a_{11}+a_{21})=0.$$

In particular, any $K=\begin{pmatrix} a & b\\ b & 0\end{pmatrix}$ or $K=\begin{pmatrix} a & 0\\ -a & 0\end{pmatrix}$ is a relative Rota–Baxter operator.

(ii) If $a_{22}\neq 0$, then $K=\begin{pmatrix} cca_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ is a relative Rota–Baxter operator on $(\mathfrak{g},[\cdot,\cdot])$ with respect to the representation $(\mathfrak{g}^*;L^*,-L^*-R^*)$ if and only if

$$a_{11}=-a_{12}=-a_{21}=a_{22}.$$

In the sequel, we construct the graded Lie algebra that characterizes relative Rota–Baxter operators as Maurer–Cartan elements.

Let $(V;\rho^L,\rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$. Then there is a Leibniz algebra structure on $\mathfrak{g}\oplus V$ given by

$$[x+u,y+v]_\ltimes=[x,y]_\mathfrak{g}+\rho^L(x)v+\rho^R(y)u, \tag{3.10}$$

for all $x,y\in\mathfrak{g}$, $u,v\in V$. This Leibniz algebra is called the semidirect product of $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ and $(V;\rho^L,\rho^R)$, and denoted by $\mathfrak{g}\ltimes_{\rho^L,\rho^R}V$. We denote the above semidirect product Leibniz multiplication by $\hat{\mu}_1$.

Consider the graded vector space

$$C^*(V,\mathfrak{g}):=\bigoplus_{n\geq 1}C^n(V,\mathfrak{g})=\bigoplus_{n\geq 1}\mathrm{Hom}(\otimes^n V,\mathfrak{g}).$$

Theorem 3.10. *With the above notations, $(C^*(V,\mathfrak{g}),\{\cdot,\cdot\})$ is a graded Lie algebra, where the graded Lie bracket $\{\cdot,\cdot\}:C^m(V,\mathfrak{g})\times C^n(V,\mathfrak{g})\to C^{m+n}(V,\mathfrak{g})$ is defined by*

$$\{g_1,g_2\}=(-1)^{|g_1|}\big[[\hat{\mu}_1,\hat{g}_1]_\mathsf{B},\hat{g}_2\big]_\mathsf{B}, \tag{3.11}$$

for all $g_1\in C^m(V,\mathfrak{g})$, $g_2\in C^n(V,\mathfrak{g})$. Moreover, its Maurer–Cartan elements are relative Rota–Baxter operators on the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ with respect to the representation $(V;\rho^L,\rho^R)$.

Proof. The graded Lie algebra $(C^*(V,\mathfrak{g}),\{\cdot,\cdot\})$ is obtained via the derived bracket [24, 39]. In fact, the Balavoine bracket $[\cdot,\cdot]_{\mathsf{B}}$ associated to the direct sum vector space $\mathfrak{g}\oplus V$ gives rise to a graded Lie algebra $(C^*(\mathfrak{g}\oplus V,\mathfrak{g}\oplus V),[\cdot,\cdot]_{\mathsf{B}})$. Since $\widehat{\mu}_1$ is the semidirect product Leibniz algebra structure on the vector space $\mathfrak{g}\oplus V$, by Theorem 3.1, we deduce that $(C^*(\mathfrak{g}\oplus V,\mathfrak{g}\oplus V),[\cdot,\cdot]_{\mathsf{B}},d=[\widehat{\mu}_1,\cdot]_{\mathsf{B}})$ is a differential graded Lie algebra. Obviously $C^*(V,\mathfrak{g})$ is an abelian subalgebra. Further, we define the derived bracket on the graded vector space $C^*(V,\mathfrak{g})$ by

$$\{g_1,g_2\}:=(-1)^{|g_1|}[d(\hat{g}_1),\hat{g}_2]_{\mathsf{B}}=(-1)^{|g_1|}\big[[\widehat{\mu}_1,\hat{g}_1]_{\mathsf{B}},\hat{g}_2\big]_{\mathsf{B}},$$

for all $g_1\in C^m(V,\mathfrak{g})$, $g_2\in C^n(V,\mathfrak{g})$. By Lemma 3.6, the derived bracket $\{\cdot,\cdot\}$ is closed on $C^*(V,\mathfrak{g})$, which implies that $(C^*(V,\mathfrak{g}),\{\cdot,\cdot\})$ is a graded Lie algebra. Moreover, it is straightforward to obtain the concrete graded Lie bracket $\{\cdot,\cdot\}$ on the graded vector space $C^*(V,\mathfrak{g})=\bigoplus_{k=1}^{+\infty}C^k(V,\mathfrak{g})$.

For all $K\in C^1(V,\mathfrak{g})$ and $v_1,v_2\in V$, we have

$$\{K,K\}(v_1,v_2)=2\big([Kv_1,Kv_2]_{\mathfrak{g}}-K\big(\rho^L(Kv_1)v_2+\rho^R(Kv_2)v_1\big)\big).$$

Thus, Maurer–Cartan elements are precisely relative Rota–Baxter operators on $\mathfrak{g}$ with respect to the representation $(V;\rho^L,\rho^R)$. ■

This is the main ingredient in our later study of the classical Leibniz Yang–Baxter equation and the classical Leibniz r-matrix. Recently, using this graded Lie algebra, the cohomology theory of relative Rota–Baxter operators on Leibniz algebras was established in [37].

3.3. Twilled Leibniz algebras and the twisting theory

Let $(\mathcal{G},[\cdot,\cdot]_{\mathcal{G}})$ be a Leibniz algebra with a decomposition into two subspaces[1], $\mathcal{G}=\mathfrak{g}_1\oplus\mathfrak{g}_2$. For later convenience, we use Ω to denote the multiplication $[\cdot,\cdot]_{\mathcal{G}}$, i.e.,

$$\Omega\big((x,u),(y,v)\big):=\big[(x,u),(y,v)\big]_{\mathcal{G}}.$$

Lemma 3.11. *Any $\Omega\in C^2(\mathcal{G},\mathcal{G})$ is uniquely decomposed into four homogeneous linear maps of bidegrees* $2|-1$, $1|0$, $0|1$, *and* $-1|2$,

$$\Omega=\widehat{\phi}_1+\widehat{\mu}_1+\widehat{\mu}_2+\widehat{\phi}_2.$$

Proof. By (3.4), $C^2(\mathcal{G},\mathcal{G})$ is decomposed into

$$C^2(\mathcal{G},\mathcal{G})=(2|-1)+(1|0)+(0|1)+(-1|2),$$

where $(i|j)$ is the space of linear maps of the bidegree $i|j$. By Lemma 3.3, Ω is uniquely decomposed into homogeneous linear maps of bidegrees $2|-1$, $1|0$, $0|1$, and $-1|2$. ■

[1] Here $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not necessarily subalgebras.

Denote $P_{\mathfrak{g}_i}[X,Y]_{\mathcal{G}}$ by $[X,Y]_i$, for $X,Y \in \mathcal{G}$, $i=1,2$, where $P_{\mathfrak{g}_1}$ and $P_{\mathfrak{g}_2}$ are the natural projections from $\mathcal{G}$ to $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. The multiplication $[(x,u),(y,v)]_{\mathcal{G}}$ is uniquely decomposed by the canonical projections $P_{\mathfrak{g}_1}$ and $P_{\mathfrak{g}_2}$ into eight multiplications:

$$[x,y]_{\mathcal{G}} = \big([x,y]_1,[x,y]_2\big), \quad [x,v]_{\mathcal{G}} = \big([x,v]_1,[x,v]_2\big),$$
$$[u,y]_{\mathcal{G}} = \big([u,y]_1,[u,y]_2\big), \quad [u,v]_{\mathcal{G}} = \big([u,v]_1,[u,v]_2\big).$$

Write $\Omega = \widehat{\phi}_1 + \widehat{\mu}_1 + \widehat{\mu}_2 + \widehat{\phi}_2$ as in Lemma 3.11. Then we obtain

$$\widehat{\phi}_1\big((x,u),(y,v)\big) = \big(0,[x,y]_2\big), \tag{3.12}$$
$$\widehat{\mu}_1\big((x,u),(y,v)\big) = \big([x,y]_1,[x,v]_2+[u,y]_2\big), \tag{3.13}$$
$$\widehat{\mu}_2\big((x,u),(y,v)\big) = \big([x,v]_1+[u,y]_1,[u,v]_2\big), \tag{3.14}$$
$$\widehat{\phi}_2\big((x,u),(y,v)\big) = \big([u,v]_1,0\big). \tag{3.15}$$

Observe that $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are lifted linear maps of $\phi_1(x,y) := [x,y]_2$ and $\phi_2(u,v) := [u,v]_1$.

Definition 3.12. The triple $(\mathcal{G},\mathfrak{g}_1,\mathfrak{g}_2)$ is called a twilled Leibniz algebra if $\phi_1=\phi_2=0$, or, equivalently, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras of $\mathcal{G}$.

Lemma 3.13. *The triple $(\mathcal{G},\mathfrak{g}_1,\mathfrak{g}_2)$ is a twilled Leibniz algebra if and only if the following three conditions hold:*

$$\frac{1}{2}[\widehat{\mu}_1,\widehat{\mu}_1]_{\mathsf{B}} = 0, \tag{3.16}$$
$$[\widehat{\mu}_1,\widehat{\mu}_2]_{\mathsf{B}} = 0, \tag{3.17}$$
$$\frac{1}{2}[\widehat{\mu}_2,\widehat{\mu}_2]_{\mathsf{B}} = 0. \tag{3.18}$$

Proof. By Lemma 3.6 and Lemma 3.3, the proof is straightforward. ∎

Proposition 3.14. *There is a one-to-one correspondence between matched pairs of Leibniz algebras and twilled Leibniz algebras.*

Proof. Let $(\mathfrak{g}_1,\mathfrak{g}_2;(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ be a matched pair of Leibniz algebras. By Proposition 2.9, we obtain that $(\mathfrak{g}_1\oplus\mathfrak{g}_2,[\cdot,\cdot]_{\bowtie})$ is a Leibniz algebra. We denote this Leibniz algebra simply by $\mathfrak{g}_1\bowtie\mathfrak{g}_2$. Then $(\mathfrak{g}_1\bowtie\mathfrak{g}_2,\mathfrak{g}_1,\mathfrak{g}_2)$ is a twilled Leibniz algebra.

Conversely, if $(\mathcal{G},\mathfrak{g}_1,\mathfrak{g}_2)$ is a twilled Leibniz algebra, then (ρ_1^L,ρ_1^R) is a representation of $\mathfrak{g}_1$ on $\mathfrak{g}_2$ and (ρ_2^L,ρ_2^R) is a representation of $\mathfrak{g}_2$ on $\mathfrak{g}_1$, where $\rho_1^L,\rho_1^R,\rho_2^L,\rho_2^R$ are defined by

$$\rho_1^L(x)u=[x,u]_2, \quad \rho_1^R(x)u=[u,x]_2, \quad \rho_2^L(u)x=[u,x]_1, \quad \rho_2^R(u)x=[x,u]_1.$$

By Lemma 3.13, $[\widehat{\mu}_1,\widehat{\mu}_2]_{\mathsf{B}}=0$, which is equivalent to (2.8)–(2.13). Thus, $(\mathfrak{g}_1,\mathfrak{g}_2;(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ is a matched pair of Leibniz algebras. ∎

Let $(\mathcal{G},[\cdot,\cdot]_{\mathcal{G}})$ be a Leibniz algebra with a decomposition into two subspaces, $\mathcal{G}=\mathfrak{g}_1\oplus\mathfrak{g}_2$, and $\Omega=\widehat{\phi}_1+\widehat{\mu}_1+\widehat{\mu}_2+\widehat{\phi}_2$ the Leibniz multiplication. Let $\widehat{H}$ be the lift of a linear map $H : \mathfrak{g}_2 \to \mathfrak{g}_1$. Then $e^{[\cdot,\hat{H}]_{\mathsf{B}}}$ is an automorphism of the graded Lie algebra $(C^*(\mathcal{G},\mathcal{G}),[\cdot,\cdot]_{\mathsf{B}})$.

Definition 3.15. The transformation $\Omega^H := e^{[\cdot,\hat{H}]_{\mathsf{B}}}\Omega$ is called a twisting of Ω by H.

Lemma 3.16. $\Omega^H = e^{-\hat{H}} \circ \Omega \circ (e^{\hat{H}} \otimes e^{\hat{H}})$.

Proof. For all $(x_1,v_1),(x_2,v_2) \in \mathcal{G}$, we have

$$\begin{aligned}[\Omega,\hat{H}]_{\mathsf{B}}\big((x_1,v_1),(x_2,v_2)\big) &= (\Omega\bar{\circ}\hat{H} - \hat{H}\bar{\circ}\Omega)\big((x_1,v_1),(x_2,v_2)\big)\\ &= \Omega\big(\big(H(v_1),0\big),(x_2,v_2)\big) + \Omega\big((x_1,v_1),\big(H(v_2),0\big)\big)\\ &\quad - \hat{H}\big(\Omega\big((x_1,v_1),(x_2,v_2)\big)\big).\end{aligned}$$

By $\hat{H} \circ \hat{H} = 0$, we have

$$\begin{aligned}&\big[[\Omega,\hat{H}]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}}\big((x_1,v_1),(x_2,v_2)\big)\\ &\quad= [\Omega,\hat{H}]_{\mathsf{B}}\big(\big(H(v_1),0\big),(x_2,v_2)\big) + [\Omega,\hat{H}]_{\mathsf{B}}\big((x_1,v_1),\big(H(v_2),0\big)\big)\\ &\qquad - \hat{H}\big([\Omega,\hat{H}]_{\mathsf{B}}\big((x_1,v_1),(x_2,v_2)\big)\big)\\ &\quad= 2\Omega\big(\big(H(v_1),0\big),\big(H(v_2),0\big)\big) - 2\hat{H}\Omega\big(\big(H(v_1),0\big),(x_2,v_2)\big)\\ &\qquad - 2\hat{H}\Omega\big((x_1,v_1),\big(H(v_2),0\big)\big).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\big[\big[[\Omega,\hat{H}]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}}\big((x_1,v_1),(x_2,v_2)\big) &= -6\hat{H}\Omega\big(\big(H(v_1),0\big),\big(H(v_2),0\big)\big),\\ \underbrace{\big[\cdots\big[}_{i}[\Omega,\hat{H}]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}},\ldots,\hat{H}\big]_{\mathsf{B}}\big((x_1,v_1),(x_2,v_2)\big) &= 0, \quad \forall i \geq 4,\end{aligned}$$

and

$$e^{[\cdot,\hat{H}]_{\mathsf{B}}}\Omega = \Omega + [\Omega,\hat{H}]_{\mathsf{B}} + \frac{1}{2}\big[[\Omega,\hat{H}]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}} + \frac{1}{6}\big[\big[[\Omega,\hat{H}]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}},\hat{H}\big]_{\mathsf{B}}. \tag{3.19}$$

Thus, we have

$$\begin{aligned}\Omega^H &= \Omega - \hat{H}\circ\Omega + \Omega\circ(\hat{H}\otimes \mathrm{Id}) + \Omega\circ(\mathrm{Id}\otimes\hat{H})\\ &\quad - \hat{H}\circ\Omega\circ(\mathrm{Id}\otimes\hat{H}) - \hat{H}\circ\Omega\circ(\hat{H}\otimes\mathrm{Id})\\ &\quad + \Omega\circ(\hat{H}\otimes\hat{H}) - \hat{H}\circ\Omega\circ(\hat{H}\otimes\hat{H}).\end{aligned}$$

By $\hat{H} \circ \hat{H} = 0$, we have

$$\begin{aligned}&e^{-\hat{H}}\circ\Omega\circ(e^{\hat{H}}\otimes e^{\hat{H}})\\ &\quad= (\mathrm{Id}-\hat{H})\circ\Omega\circ\big((\mathrm{Id}+\hat{H})\otimes(\mathrm{Id}+\hat{H})\big)\\ &\quad= \Omega + \Omega\circ(\mathrm{Id}\otimes\hat{H}) + \Omega\circ(\hat{H}\otimes\mathrm{Id}) + \Omega\circ(\hat{H}\otimes\hat{H}) - \hat{H}\circ\Omega\\ &\qquad - \hat{H}\circ\Omega\circ(\mathrm{Id}\otimes\hat{H}) - \hat{H}\circ\Omega\circ(\hat{H}\otimes\mathrm{Id}) - \hat{H}\circ\Omega\circ(\hat{H}\otimes\hat{H}).\end{aligned}$$

Thus, we obtain that $\Omega^H = e^{-\hat{H}} \circ \Omega \circ (e^{\hat{H}} \otimes e^{\hat{H}})$. ∎

Proposition 3.17. *The twisting Ω^H is a Leibniz algebra structure on $\mathcal{G}$.*

Proof. By $\Omega^H = e^{-\hat{H}} \circ \Omega \circ (e^{\hat{H}} \otimes e^{\hat{H}})$, we have

$$[\Omega^H, \Omega^H]_{\mathsf{B}} = 2\Omega^H \bar{\circ} \Omega^H = 2e^{-\hat{H}} \circ (\Omega \bar{\circ} \Omega) \circ (e^{\hat{H}} \otimes e^{\hat{H}} \otimes e^{\hat{H}})$$
$$= e^{-\hat{H}} \circ [\Omega, \Omega]_{\mathsf{B}} \circ (e^{\hat{H}} \otimes e^{\hat{H}} \otimes e^{\hat{H}}) = 0,$$

which implies that Ω^H is a Leibniz algebra structure on $\mathcal{G}$ by Theorem 3.1. ∎

Corollary 3.18. $e^{\hat{H}} : (\mathcal{G}, \Omega^H) \to (\mathcal{G}, \Omega)$ *is an isomorphism between Leibniz algebras.*

The twisting operations are completely determined by the following result.

Proposition 3.19. *Write $\Omega := \hat{\phi}_1 + \hat{\mu}_1 + \hat{\mu}_2 + \hat{\phi}_2$ and $\Omega^H := \hat{\phi}_1^H + \hat{\mu}_1^H + \hat{\mu}_2^H + \hat{\phi}_2^H$. Then one has*

$$\begin{aligned}
\hat{\phi}_1^H &= \hat{\phi}_1, \\
\hat{\mu}_1^H &= \hat{\mu}_1 + [\hat{\phi}_1, \hat{H}]_{\mathsf{B}}, \\
\hat{\mu}_2^H &= \hat{\mu}_2 + [\hat{\mu}_1, \hat{H}]_{\mathsf{B}} + \frac{1}{2}\big[[\hat{\phi}_1, \hat{H}]_{\mathsf{B}}, \hat{H}\big]_{\mathsf{B}}, \\
\hat{\phi}_2^H &= \hat{\phi}_2 + [\hat{\mu}_2, \hat{H}]_{\mathsf{B}} + \frac{1}{2}\big[[\hat{\mu}_1, \hat{H}]_{\mathsf{B}}, \hat{H}\big]_{\mathsf{B}} + \frac{1}{6}\big[\big[[\hat{\phi}_1, \hat{H}]_{\mathsf{B}}, \hat{H}\big]_{\mathsf{B}}, \hat{H}\big]_{\mathsf{B}}.
\end{aligned}$$

Proof. It follows from a direct but tedious computation. We omit the details. ∎

In the sequel, we consider a special case of the above twisting theory. Let $(V; \rho^L, \rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Consider the twilled Leibniz algebra $(\mathfrak{g} \ltimes_{\rho^L,\rho^R} V, \mathfrak{g}, V)$. Denote the Leibniz multiplication $[\cdot,\cdot]_{\ltimes}$ by Ω. Write $\Omega = \hat{\mu}_1 + \hat{\mu}_2$. Then $\hat{\mu}_2 = 0$.

Theorem 3.20. *With the above notations, let $H : V \to \mathfrak{g}$ be a linear map. The twisting $((\mathfrak{g} \oplus V, \Omega^H), \mathfrak{g}, V)$ is a twilled Leibniz algebra if and only if H is a relative Rota–Baxter operator on the Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to the representation $(V; \rho^L, \rho^R)$. Moreover, the Leibniz algebra structure on V is given by*

$$[u, v]_H := \rho^L\big(H(u)\big)v + \rho^R\big(H(v)\big)u, \quad \forall u, v \in V. \tag{3.20}$$

Proof. By Proposition 3.19, the twisting has the form

$$\hat{\mu}_1^H = \hat{\mu}_1, \tag{3.21}$$
$$\hat{\mu}_2^H = [\hat{\mu}_1, \hat{H}]_{\mathsf{B}}, \tag{3.22}$$
$$\hat{\phi}_2^H = \frac{1}{2}\big[[\hat{\mu}_1, \hat{H}]_{\mathsf{B}}, \hat{H}\big]_{\mathsf{B}}. \tag{3.23}$$

Thus, the twisting $((\mathfrak{g} \oplus V, \Omega^H), \mathfrak{g}, V)$ is a twilled Leibniz algebra if and only if $\hat{\phi}_2^H = 0$, which implies that H is a relative Rota–Baxter operator by Theorem 3.10.

By Lemma 3.13, we deduce that $\hat{\mu}_2^H$ is a Leibniz algebra multiplication on V. It is straightforward to deduce that the multiplication on V is given by (3.20). ∎

4. The classical Leibniz Yang–Baxter equation and triangular Leibniz bialgebras

In this section, first we construct a Leibniz bialgebra using a symmetric relative Rota–Baxter operator. Then we define the classical Leibniz Yang–Baxter equation using the graded Lie algebra obtained in Theorem 3.10. Its solutions are called classical Leibniz r-matrices. Using the twisting theory given in Section 3, we define a triangular Leibniz bialgebra successfully. Finally, we generalize a result by Semenov-Tian-Shansky [33] about the relation between the operator form and the tensor form of a classical r-matrix to the context of Leibniz algebras.

Let $K : \mathfrak{g}^* \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$. Let Ω be the Leibniz multiplication of the semidirect product Leibniz algebra $\mathfrak{g} \ltimes_{L^*, -L^*-R^*} \mathfrak{g}^*$. By Theorem 3.20, $((\mathfrak{g}\oplus\mathfrak{g}^*, \Omega^K), \mathfrak{g}, \mathfrak{g}^*)$ is a twilled Leibniz algebra. Moreover, by Corollary 3.18, $e^{\hat{K}} : (\mathfrak{g} \oplus \mathfrak{g}^*, \Omega^K) \to (\mathfrak{g} \oplus \mathfrak{g}^*, \Omega)$ is an isomorphism between Leibniz algebras.

First by Theorem 3.20, we have the following corollary.

Corollary 4.1. *Let $K : \mathfrak{g}^* \to \mathfrak{g}$ be a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$. Then $\mathfrak{g}^*_K := (\mathfrak{g}^*, [\cdot,\cdot]_K)$ is a Leibniz algebra, where $[\cdot,\cdot]_K$ is given by*

$$[\xi, \eta]_K = L^*_{K\xi}\eta - L^*_{K\eta}\xi - R^*_{K\eta}\xi, \quad \forall \xi, \eta \in \mathfrak{g}^*.$$

Proposition 4.2. *Let $K : \mathfrak{g}^* \to \mathfrak{g}$ be a relative Rota–Baxter operator on $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$. Then $e^{\hat{K}}$ preserves the bilinear form ω given by (2.15) if and only if $K^* = K$. Here K^* is the dual map of K, i.e., $\langle K\xi, \eta\rangle = \langle \xi, K^*\eta\rangle$, for all $\xi, \eta \in \mathfrak{g}^*$.*

Proof. By $\hat{K} \circ \hat{K} = 0$, we have $e^{\hat{K}} = \mathrm{Id} + \hat{K}$. For all $x, y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$, we have

$$\begin{aligned}
&\omega\big(e^{\hat{K}}(x+\xi), e^{\hat{K}}(y+\eta)\big)\\
&\quad= \omega(x+\xi, y+\eta) + \omega\big(x+\xi, K(\eta)\big) + \omega\big(K(\xi), y+\eta\big) + \omega\big(K(\xi), K(\eta)\big)\\
&\quad= \omega(x+\xi, y+\eta) + \big\langle \xi, K(\eta)\big\rangle - \big\langle \eta, K(\xi)\big\rangle\\
&\quad= \omega(x+\xi, y+\eta) + \big\langle (K^* - K)\xi, \eta\big\rangle.
\end{aligned}$$

Thus, $\omega(e^{\hat{K}}(x+\xi), e^{\hat{K}}(y+\eta)) = \omega(x+\xi, y+\eta)$ if and only if $K^* = K$. ■

Proposition 4.3. *Let $K : \mathfrak{g}^* \to \mathfrak{g}$ be a relative Rota–Baxter operator on $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ and $K^* = K$. Then $(\mathfrak{g} \oplus \mathfrak{g}^*, \Omega^K)$ is a skew-symmetric quadratic Leibniz algebra with the invariant bilinear form ω given by (2.15) and $e^{\hat{K}}$ is an isomorphism from the skew-symmetric quadratic Leibniz algebra $(\mathfrak{g} \oplus \mathfrak{g}^*, \Omega^K)$ to $(\mathfrak{g} \oplus \mathfrak{g}^*, \Omega)$.*

Proof. Since $e^{\widehat{K}}$ is a Leibniz algebra isomorphism and preserves the bilinear form ω, for all $X, Y, Z \in \mathfrak{g} \oplus \mathfrak{g}^*$, we have

$$\begin{aligned}\omega\big(X, \Omega^K(Y, Z)\big) &= \omega\big(X, e^{-\widehat{K}}\Omega(e^{\widehat{K}}Y, e^{\widehat{K}}Z)\big) = \omega\big(e^{\widehat{K}}X, \Omega(e^{\widehat{K}}Y, e^{\widehat{K}}Z)\big)\\ &= \omega\big(\Omega(e^{\widehat{K}}X, e^{\widehat{K}}Z) + \Omega(e^{\widehat{K}}Z, e^{\widehat{K}}X), e^{\widehat{K}}Y\big)\\ &= \omega\big(\Omega^K(X, Z) + \Omega^K(Z, X), Y\big),\end{aligned}$$

which implies that $(\mathfrak{g} \oplus \mathfrak{g}^*, \Omega^K)$ is a skew-symmetric quadratic Leibniz algebra. It is obvious that $e^{\widehat{K}}$ is an isomorphism from the quadratic Leibniz algebra $(\mathfrak{g} \oplus \mathfrak{g}^*, \Omega^K)$ to $(\mathfrak{g} \oplus \mathfrak{g}^*, \Omega)$. ∎

By Corollary 4.1, Proposition 4.3, and Theorem 2.14, we obtain the following.

Theorem 4.4. *Let $K : \mathfrak{g}^* \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ and $K^* = K$. Then $(\mathfrak{g}, \mathfrak{g}^*_K)$ is a Leibniz bialgebra, where the Leibniz algebra $\mathfrak{g}^*_K$ is given in Corollary 4.1.*

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Leibniz algebra. We consider the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. By Theorem 3.10, we have the following corollary.

Corollary 4.5. *Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Leibniz algebra. Then $(C^*(\mathfrak{g}^*, \mathfrak{g}), \{\cdot,\cdot\})$ is a graded Lie algebra, where $\{\cdot,\cdot\}$ is given by (3.11).*

In the sequel, to define the classical Leibniz Yang–Baxter equation, we transfer the above graded Lie algebra structure to the tensor space.

For all $k \geq 1$ and $P \in \otimes^{k+1}\mathfrak{g}$, $\xi_1, \dots, \xi_{k+1} \in \mathfrak{g}^*$, we define

$$\begin{aligned}&\Psi : \otimes^{k+1}\mathfrak{g} \to \mathrm{Hom}(\otimes^k \mathfrak{g}^*, \mathfrak{g})\\ &\big\langle \Psi(P)(\xi_1, \dots, \xi_k), \xi_{k+1}\big\rangle = \langle P,\ \xi_1 \otimes \cdots \otimes \xi_k \otimes \xi_{k+1}\rangle\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\Upsilon : \mathrm{Hom}(\otimes^k \mathfrak{g}^*, \mathfrak{g}) \to \otimes^{k+1}\mathfrak{g}\\ &\big\langle \Upsilon(f), \xi_1 \otimes \cdots \otimes \xi_k \otimes \xi_{k+1}\big\rangle = \big\langle f(\xi_1, \dots, \xi_k), \xi_{k+1}\big\rangle.\end{aligned} \tag{4.2}$$

Obviously, we have $\Psi \circ \Upsilon = \mathrm{Id}$, $\Upsilon \circ \Psi = \mathrm{Id}$.

Theorem 4.6. *Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Leibniz algebra. Then there is a graded Lie bracket $[[\cdot,\cdot]]$ on the graded space $\bigoplus_{k\geq 2}(\otimes^k \mathfrak{g})$ given by*

$$[[P, Q]] := \Upsilon\big\{\Psi(P), \Psi(Q)\big\}, \quad \forall P \in \otimes^{m+1}\mathfrak{g},\ Q \in \otimes^{n+1}\mathfrak{g}.$$

Proof. By $\Psi \circ \Upsilon = \mathrm{Id}$, $\Upsilon \circ \Psi = \mathrm{Id}$, we transfer the graded Lie algebra structure on $C^*(\mathfrak{g}^*, \mathfrak{g})$ to that on the graded space $\bigoplus_{k\geq 2}(\otimes^k \mathfrak{g})$. ∎

The general formula of $[[P, Q]]$ is very sophisticated. But for $P = x \otimes y$ and $Q = z \otimes w$, there is an explicit expression, which is enough for our application.

Lemma 4.7. *For $x \otimes y, z \otimes w \in \mathfrak{g} \otimes \mathfrak{g}$, one has*

$$\begin{aligned}[][x \otimes y, z \otimes w]] = {} & z \otimes [w, x]_{\mathfrak{g}} \otimes y - [w, x]_{\mathfrak{g}} \otimes z \otimes y - [x, w]_{\mathfrak{g}} \otimes z \otimes y \\ & + z \otimes x \otimes [w, y]_{\mathfrak{g}} + x \otimes z \otimes [y, w]_{\mathfrak{g}} + x \otimes [y, z]_{\mathfrak{g}} \otimes w \\ & - [y, z]_{\mathfrak{g}} \otimes x \otimes w - [z, y]_{\mathfrak{g}} \otimes x \otimes w. \end{aligned} \tag{4.3}$$

Proof. For all $\xi \in \mathfrak{g}^*$, we have $\Psi(x \otimes y)(\xi) = \langle x, \xi \rangle y$. By Corollary 4.5, for all $\xi_1, \xi_2 \in \mathfrak{g}^*$, we have

$$\begin{aligned} & \{\Psi(x \otimes y), \Psi(z \otimes w)\}(\xi_1, \xi_2) \\ & = -\Psi(x \otimes y)(L^*_{\Psi(z\otimes w)\xi_1}\xi_2) + \Psi(x \otimes y)(L^*_{\Psi(z\otimes w)\xi_2}\xi_1) \\ & \quad + \Psi(x \otimes y)(R^*_{\Psi(z\otimes w)\xi_2}\xi_1) + [\Psi(z \otimes w)\xi_1, \Psi(x \otimes y)\xi_2]_{\mathfrak{g}} \\ & \quad + [\Psi(x \otimes y)\xi_1, \Psi(z \otimes w)\xi_2]_{\mathfrak{g}} - \Psi(z \otimes w)(L^*_{\Psi(x\otimes y)\xi_1}\xi_2) \\ & \quad + \Psi(z \otimes w)(L^*_{\Psi(x\otimes y)\xi_2}\xi_1) + \Psi(z \otimes w)(R^*_{\Psi(x\otimes y)\xi_2}\xi_1). \end{aligned}$$

Thus, for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}^*$, we have

$$\begin{aligned} & \langle [[x \otimes y, z \otimes w]], \xi_1 \otimes \xi_2 \otimes \xi_3 \rangle = \langle \{\Psi(x \otimes y), \Psi(z \otimes w)\}(\xi_1, \xi_2), \xi_3 \rangle \\ & = -\langle \Psi(x \otimes y)(L^*_{\Psi(z\otimes w)\xi_1}\xi_2), \xi_3 \rangle + \langle \Psi(x \otimes y)(L^*_{\Psi(z\otimes w)\xi_2}\xi_1), \xi_3 \rangle \\ & \quad + \langle \Psi(x \otimes y)(R^*_{\Psi(z\otimes w)\xi_2}\xi_1), \xi_3 \rangle + \langle [\Psi(z \otimes w)\xi_1, \Psi(x \otimes y)\xi_2]_{\mathfrak{g}}, \xi_3 \rangle \\ & \quad + \langle [\Psi(x \otimes y)\xi_1, \Psi(z \otimes w)\xi_2]_{\mathfrak{g}}, \xi_3 \rangle - \langle \Psi(z \otimes w)(L^*_{\Psi(x\otimes y)\xi_1}\xi_2), \xi_3 \rangle \\ & \quad + \langle \Psi(z \otimes w)(L^*_{\Psi(x\otimes y)\xi_2}\xi_1), \xi_3 \rangle + \langle \Psi(z \otimes w)(R^*_{\Psi(x\otimes y)\xi_2}\xi_1), \xi_3 \rangle \\ & = -\langle z, \xi_1 \rangle \langle x, L^*_w \xi_2 \rangle \langle y, \xi_3 \rangle + \langle z, \xi_2 \rangle \langle x, L^*_w \xi_1 \rangle \langle y, \xi_3 \rangle \\ & \quad + \langle z, \xi_2 \rangle \langle x, R^*_w \xi_1 \rangle \langle y, \xi_3 \rangle + \langle z, \xi_1 \rangle \langle x, \xi_2 \rangle \langle [w, y]_{\mathfrak{g}}, \xi_3 \rangle \\ & \quad + \langle x, \xi_1 \rangle \langle z, \xi_2 \rangle \langle [y, w]_{\mathfrak{g}}, \xi_3 \rangle - \langle x, \xi_1 \rangle \langle z, L^*_y \xi_2 \rangle \langle w, \xi_3 \rangle \\ & \quad + \langle x, \xi_2 \rangle \langle z, L^*_y \xi_1 \rangle \langle w, \xi_3 \rangle + \langle x, \xi_2 \rangle \langle z, R^*_y \xi_1 \rangle \langle w, \xi_3 \rangle \\ & = \langle z, \xi_1 \rangle \langle [w, x]_{\mathfrak{g}}, \xi_2 \rangle \langle y, \xi_3 \rangle - \langle [w, x]_{\mathfrak{g}}, \xi_1 \rangle \langle z, \xi_2 \rangle \langle y, \xi_3 \rangle \\ & \quad - \langle [x, w]_{\mathfrak{g}}, \xi_1 \rangle \langle z, \xi_2 \rangle \langle y, \xi_3 \rangle + \langle z, \xi_1 \rangle \langle x, \xi_2 \rangle \langle [w, y]_{\mathfrak{g}}, \xi_3 \rangle \\ & \quad + \langle x, \xi_1 \rangle \langle z, \xi_2 \rangle \langle [y, w]_{\mathfrak{g}}, \xi_3 \rangle + \langle x, \xi_1 \rangle \langle [y, z]_{\mathfrak{g}}, \xi_2 \rangle \langle w, \xi_3 \rangle \\ & \quad - \langle [y, z]_{\mathfrak{g}}, \xi_1 \rangle \langle x, \xi_2 \rangle \langle w, \xi_3 \rangle - \langle [z, y]_{\mathfrak{g}}, \xi_1 \rangle \langle x, \xi_2 \rangle \langle w, \xi_3 \rangle, \end{aligned}$$

which implies that (4.3) holds. ∎

Moreover, we can obtain the tensor form of a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$.

Proposition 4.8. *Let $K : \mathfrak{g}^* \to \mathfrak{g}$ be a linear map.*

(i) *K is a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ if and only if the tensor form $\Upsilon(K) \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies*

$$[[\Upsilon(K), \Upsilon(K)]] = 0.$$

(ii) $K = K^*$ *if and only if* $\Upsilon(K) = \tau(\Upsilon(K))$, *i.e.*, $\Upsilon(K) \in \mathrm{Sym}^2(\mathfrak{g})$.

Proof. Since Ψ is a graded Lie algebra isomorphism from the graded Lie algebra $(\bigoplus_{k\geq 2}(\otimes^k \mathfrak{g}), [[\cdot,\cdot]])$ to $(C^*(\mathfrak{g}^*, \mathfrak{g}), \{\cdot,\cdot\})$, we deduce that $\{K, K\} = 0$ if and only if $[[\Upsilon(K), \Upsilon(K)]] = 0$. The other conclusion is obvious. ■

Definition 4.9. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Leibniz algebra and $r \in \mathrm{Sym}^2(\mathfrak{g})$. Then equation

$$[[r, r]] = 0 \tag{4.4}$$

is called the classical Leibniz Yang–Baxter equation in $\mathfrak{g}$ and r is called a classical Leibniz r-matrix.

Example 4.10. Consider the 2-dimensional Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot])$ defined with respect to a basis $\{e_1, e_2\}$ by

$$[e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1.$$

For all $r = ae_1 \otimes e_1 + be_1 \otimes e_2 + be_2 \otimes e_1 + ce_2 \otimes e_2 \in \mathrm{Sym}^2(\mathfrak{g})$, by Lemma 4.7, we have

$$\begin{aligned}[[r, r]] = {} & 4c(a+b)e_2 \otimes e_1 \otimes e_1 - 2c(a+b)e_1 \otimes e_2 \otimes e_1 \\ & - 2c(a+b)e_1 \otimes e_1 \otimes e_2 + 2c(b+c)e_2 \otimes e_1 \otimes e_2 \\ & - 4c(b+c)e_1 \otimes e_2 \otimes e_2 + 2c(b+c)e_2 \otimes e_2 \otimes e_1.\end{aligned}$$

Summarizing the above computation, we have

(i) if $c = 0$, then any $r = ae_1 \otimes e_1 + b(e_1 \otimes e_2 + e_2 \otimes e_1)$ is a classical Leibniz r-matrix;

(ii) if $c \neq 0$, then $[[r, r]] = 0$ if and only if $a = c = -b$. Thus, any

$$r = c(e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_2)$$

is a classical Leibniz r-matrix.

Remark 4.11. For $r = ae_1 \otimes e_1 + be_1 \otimes e_2 + be_2 \otimes e_1 + ce_2 \otimes e_2$, we have $r^\sharp(e_1^*, e_2^*) = (e_1, e_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where $r^\sharp = \Psi(r) : \mathfrak{g}^* \to \mathfrak{g}$ is defined by $\langle r^\sharp(\xi), \eta\rangle = \langle r, \xi \otimes \eta\rangle$ for all $\xi, \eta \in \mathfrak{g}^*$. The above classical Leibniz r-matrices actually correspond to symmetric relative Rota–Baxter operators given in Example 3.9.

Corollary 4.12. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Leibniz algebra and* $r \in \mathrm{Sym}^2(\mathfrak{g})$ *a solution of the classical Leibniz Yang–Baxter equation in* $\mathfrak{g}$. *Then* $(\mathfrak{g}, \mathfrak{g}^*_{r^\sharp})$ *is a Leibniz bialgebra.*

Proof. By $r \in \mathrm{Sym}^2(\mathfrak{g})$ and $[[r, r]] = 0$, we deduce that $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ and $(r^\sharp)^* = r^\sharp$. By Theorem 4.4, we obtain that $(\mathfrak{g}, \mathfrak{g}^*_{r^\sharp})$ is a Leibniz bialgebra. ■

Definition 4.13. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Leibniz algebra and $r \in \mathrm{Sym}^2(\mathfrak{g})$ a solution of the classical Leibniz Yang–Baxter equation in $\mathfrak{g}$. We call the Leibniz bialgebra $(\mathfrak{g}, \mathfrak{g}^*_{r^\sharp})$ the triangular Leibniz bialgebra associated to the classical Leibniz r-matrix r.

Remark 4.14. In Section 2, we define a Leibniz bialgebra, which is equivalent to a Manin triple of Leibniz algebras. Note that there is no cohomology theory that can be used in the theory of Leibniz bialgebras. Thus, there is not an obvious way to define a "coboundary Leibniz bialgebra". Nevertheless, using the twisting method in the theory of twilled Leibniz algebras, we define triangular Leibniz bialgebras successfully.

In the Lie algebra context, we know that the dual description of a classical r-matrix is a symplectic structure on a Lie algebra. Now we investigate the dual description of a classical Leibniz r-matrix. A symmetric 2-form $\mathcal{B} \in \mathrm{Sym}^2(\mathfrak{g}^*)$ on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ induces a linear map $\mathcal{B}^\natural : \mathfrak{g} \to \mathfrak{g}^*$ by

$$\langle \mathcal{B}^\natural(x), y \rangle := \mathcal{B}(x, y), \quad \forall x, y \in \mathfrak{g}.$$

$\mathcal{B}$ is said to be nondegenerate if $\mathcal{B}^\natural : \mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism. Similarly, $r \in \mathrm{Sym}^2(\mathfrak{g})$ is said to be nondegenerate if $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is an isomorphism.

Proposition 4.15. *$r \in \mathrm{Sym}^2(\mathfrak{g})$ is a nondegenerate solution of the classical Leibniz Yang–Baxter equation in a Leibniz algebra $\mathfrak{g}$ if and only if the symmetric nondegenerate bilinear form $\mathcal{B}$ on $\mathfrak{g}$ defined by*

$$\mathcal{B}(x, y) := \langle (r^\sharp)^{-1}(x), y \rangle, \quad \forall x, y \in \mathfrak{g}, \tag{4.5}$$

satisfies the "closed" condition

$$\mathcal{B}(z, [x, y]_{\mathfrak{g}}) = -\mathcal{B}(y, [x, z]_{\mathfrak{g}}) + \mathcal{B}(x, [y, z]_{\mathfrak{g}}) + \mathcal{B}(x, [z, y]_{\mathfrak{g}}). \tag{4.6}$$

Proof. Let $r \in \mathrm{Sym}^2(\mathfrak{g})$ be nondegenerate. It is obvious that $\mathcal{B}$ is symmetric and nondegenerate.

Since $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is an invertible linear map, for all $x, y, z \in \mathfrak{g}$, there are $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}^*$ such that $r^\sharp(\xi_1) = x$, $r^\sharp(\xi_2) = y$, and $r^\sharp(\xi_3) = z$. Since $r^\sharp$ is a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to the representation $(\mathfrak{g}^*; L^*, -L^* - R^*)$ and $(r^\sharp)^* = r^\sharp$, we have

$$\begin{aligned}
\mathcal{B}(z, [x, y]_{\mathfrak{g}}) &= \langle (r^\sharp)^{-1}(z), [x, y]_{\mathfrak{g}} \rangle = \langle \xi_3, [r^\sharp(\xi_1), r^\sharp(\xi_2)]_{\mathfrak{g}} \rangle \\
&= \langle \xi_3, r^\sharp(L^*_{r^\sharp(\xi_1)}\xi_2) \rangle - \langle \xi_3, r^\sharp(L^*_{r^\sharp(\xi_2)}\xi_1) \rangle - \langle \xi_3, r^\sharp(R^*_{r^\sharp(\xi_2)}\xi_1) \rangle \\
&= \langle r^\sharp(\xi_3), L^*_{r^\sharp(\xi_1)}\xi_2 \rangle - \langle r^\sharp(\xi_3), L^*_{r^\sharp(\xi_2)}\xi_1 \rangle - \langle r^\sharp(\xi_3), R^*_{r^\sharp(\xi_2)}\xi_1 \rangle \\
&= -\langle [r^\sharp(\xi_1), r^\sharp(\xi_3)]_{\mathfrak{g}}, \xi_2 \rangle + \langle [r^\sharp(\xi_2), r^\sharp(\xi_3)]_{\mathfrak{g}}, \xi_1 \rangle + \langle [r^\sharp(\xi_3), r^\sharp(\xi_2)]_{\mathfrak{g}}, \xi_1 \rangle \\
&= -\mathcal{B}([x, z]_{\mathfrak{g}}, y) + \mathcal{B}([z, y]_{\mathfrak{g}}, x) + \mathcal{B}([y, z]_{\mathfrak{g}}, x).
\end{aligned}$$

Thus, $\mathcal{B}$ satisfies (4.6). ∎

At the end of this section, we generalize a Semenov-Tian-Shansky's result in [33] to the context of Leibniz algebras.

Lemma 4.16. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$ *be a skew-symmetric quadratic Leibniz algebra. Then* $\omega^{\natural} : \mathfrak{g} \to \mathfrak{g}^*$ *is an isomorphism from the regular representation* $(\mathfrak{g}; L, R)$ *to its dual representation* $(\mathfrak{g}^*; L^*, -L^* - R^*)$.

Proof. For all $x, y, z \in \mathfrak{g}$, by (2.6) we have

$$\begin{aligned}\langle \omega^{\natural}(L_x y) - L_x^* \omega^{\natural}(y), z\rangle &= \omega\big([x,y]_{\mathfrak{g}}, z\big) + \langle \omega^{\natural}(y), L_x z\rangle\\ &= \omega\big([x,y]_{\mathfrak{g}}, z\big) + \omega\big(y, [x,z]_{\mathfrak{g}}\big) = 0.\end{aligned}$$

Thus, we have $\omega^{\natural} \circ L_x = L_x^* \circ \omega^{\natural}$. By (2.5), we have

$$\begin{aligned}\langle \omega^{\natural}(R_x y) - (-L_x^* - R_x^*)\omega^{\natural}(y), z\rangle &= \omega\big([y,x]_{\mathfrak{g}}, z\big) - \langle \omega^{\natural}(y), (L_x + R_x)z\rangle\\ &= \omega\big([y,x]_{\mathfrak{g}}, z\big) - \omega\big(y, [x,z]_{\mathfrak{g}} + [z,x]_{\mathfrak{g}}\big) = 0.\end{aligned}$$

Thus, we have $\omega^{\natural} \circ R_x = (-L_x^* - R_x^*) \circ \omega^{\natural}$. ∎

Theorem 4.17. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$ *be a skew-symmetric quadratic Leibniz algebra and* $K : \mathfrak{g}^* \to \mathfrak{g}$ *a linear map. Then* K *is a relative Rota–Baxter operator on* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *with respect to the representation* $(\mathfrak{g}^*; L^*, -L^* - R^*)$ *if and only if* $K \circ \omega^{\natural}$ *is a Rota–Baxter operator on* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

Proof. For all $x, y \in \mathfrak{g}$, by Lemma 4.16, we have

$$\begin{aligned}&(K \circ \omega^{\natural})\big(\big[K\omega^{\natural}(x), y\big]_{\mathfrak{g}} + \big[x, K\omega^{\natural}(y)\big]_{\mathfrak{g}}\big)\\ &\quad = K\big(\omega^{\natural}(L_{K\omega^{\natural}(x)} y) + \omega^{\natural} R_{K\omega^{\natural}(y)} x\big)\\ &\quad = K\big(L^*_{K\omega^{\natural}(x)}\omega^{\natural}(y) - L^*_{K\omega^{\natural}(y)}\omega^{\natural}(x) - R^*_{K\omega^{\natural}(y)}\omega^{\natural}(x)\big),\end{aligned}$$

which implies the conclusion. ∎

Corollary 4.18. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$ *be a skew-symmetric quadratic Leibniz algebra. Then* $r \in \mathrm{Sym}^2(\mathfrak{g})$ *is a solution of the classical Leibniz Yang–Baxter equation in* $\mathfrak{g}$ *if and only if* $r^{\sharp} \circ \omega^{\natural}$ *is a Rota–Baxter operator on* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$*, i.e.,*

$$\big[r^{\sharp} \circ \omega^{\natural}(x), r^{\sharp} \circ \omega^{\natural}(y)\big]_{\mathfrak{g}} = (r^{\sharp} \circ \omega^{\natural})\big(\big[r^{\sharp} \circ \omega^{\natural}(x), y\big]_{\mathfrak{g}} + \big[x, r^{\sharp} \circ \omega^{\natural}(y)\big]_{\mathfrak{g}}\big).$$

Remark 4.19. In [20], the authors defined $R_{\pm}$-matrix for Leibniz algebras as a direct generalization of Semenov-Tian-Shansky's approach in [33], without any bialgebra theory for Leibniz algebras. It is straightforward to see that their R_+-matrices in a Leibniz algebra are simply Rota–Baxter operators on the Leibniz algebra. By the above corollary, if r is a classical Leibniz r-matrix in a skew-symmetric quadratic Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \omega)$, then $r^{\sharp} \circ \omega^{\natural}$ is an R_+-matrix.

Our bialgebra theory for Leibniz algebras enjoys many good properties parallelling to that for Lie algebras. This justifies its correctness.

5. Solutions of the classical Leibniz Yang–Baxter equations

In this section, first we show that a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation $(V; \rho^L, \rho^R)$ gives rise to a solution of the classical Leibniz Yang–Baxter equation in a larger Leibniz algebra. Then we introduce the notion of a Leibniz-dendriform algebra, which is the underlying algebraic structure of a relative Rota–Baxter operator on a Leibniz algebra. Leibniz-dendriform algebras play important roles in our study of the classical Leibniz Yang–Baxter equation. There is a natural solution of the classical Leibniz Yang–Baxter equation in the semidirect product Leibniz algebra $A \ltimes_{L^*_{\lhd}, -L^*_{\lhd}-R^*_{\rhd}} A^*$ associated to a Leibniz-dendriform algebra $(A, \rhd, \lhd)$.

Lemma 5.1. *Let $(V; \rho^L, \rho^R)$ be a representation of a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then, the dual representation $(\mathfrak{g}^* \oplus V; L^*_{\ltimes}, -L^*_{\ltimes} - R^*_{\ltimes})$ of the regular representation $(\mathfrak{g} \oplus V^*; L_{\ltimes}, R_{\ltimes})$ of the semidirect product Leibniz algebra $\mathfrak{g} \ltimes_{(\rho^L)^*, -(\rho^L)^* - (\rho^R)^*} V^*$ has the following properties:*

$$\begin{aligned} L^*_{\ltimes}(x)\xi &= L^*_x\xi \in \mathfrak{g}^*, & L^*_{\ltimes}(x)v &= \rho^L(x)v \in V,\\ L^*_{\ltimes}(\chi)\xi &= 0, & L^*_{\ltimes}(\chi)v &\in \mathfrak{g}^*,\\ (-L^*_{\ltimes} - R^*_{\ltimes})(x)\xi &= (-L^*_x - R^*_x)\xi, & (-L^*_{\ltimes} - R^*_{\ltimes})(x)v &= \rho^R(x)v \in V,\\ (-L^*_{\ltimes} - R^*_{\ltimes})(\chi)\xi &= 0, & (-L^*_{\ltimes} - R^*_{\ltimes})(\chi)v &\in \mathfrak{g}^*, \end{aligned}$$

for all $x \in \mathfrak{g}$, $v \in V$, $\xi \in \mathfrak{g}^*$, $\chi \in V^*$.

Theorem 5.2. *A linear map $K : V \to \mathfrak{g}$ is a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation $(V; \rho^L, \rho^R)$ if and only if $K + K^*$ is a relative Rota–Baxter operator on the Leibniz algebra $\mathfrak{g} \ltimes_{(\rho^L)^*, -(\rho^L)^* - (\rho^R)^*} V^*$ with respect to the dual representation $(\mathfrak{g}^* \oplus V; L^*_{\ltimes}, -L^*_{\ltimes} - R^*_{\ltimes})$ of the regular representation $(\mathfrak{g} \oplus V^*; L_{\ltimes}, R_{\ltimes})$, i.e., the tensor form $\Upsilon(K + K^*)$ is a solution of the classical Leibniz Yang–Baxter equation in the Leibniz algebra $\mathfrak{g} \ltimes_{(\rho^L)^*, -(\rho^L)^* - (\rho^R)^*} V^*$.*

Proof. Let $K : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to a representation $(V; \rho^L, \rho^R)$. By Lemma 5.1, for all $u, v \in V$, we have

$$\begin{aligned} &(K+K^*)\big(L^*_{\ltimes}((K+K^*)u)v - (L^*_{\ltimes}+R^*_{\ltimes})((K+K^*)v)u\big) - \big[(K+K^*)u, (K+K^*)v\big]_{\ltimes}\\ &\quad= (K+K^*)\big(\rho^L(Ku)v + \rho^R(Kv)u\big) - [Ku, Kv]_{\ltimes}\\ &\quad= K\big(\rho^L(Ku)v + \rho^R(Kv)u\big) - [Ku, Kv]_{\mathfrak{g}}\\ &\quad= 0. \end{aligned} \tag{5.1}$$

For all $u, v \in V$, $\xi \in \mathfrak{g}^*$, we have

$$\begin{aligned} &\big\langle (K+K^*)\big(L^*_{\ltimes}((K+K^*)u)\xi + (-L^*_{\ltimes} - R^*_{\ltimes})((K+K^*)\xi)u\big)\\ &\qquad - \big[(K+K^*)u, (K+K^*)\xi\big]_{\ltimes}, v\big\rangle\\ &\quad= \big\langle K^*\big(L^*_{Ku}\xi + (-L^*_{\ltimes} - R^*_{\ltimes})(K^*\xi)u\big) - (\rho^L)^*(Ku)K^*\xi, v\big\rangle\\ &\quad= \big\langle L^*_{Ku}\xi + (-L^*_{\ltimes} - R^*_{\ltimes})(K^*\xi)u, Kv\big\rangle + \big\langle K^*\xi, \rho^L(Ku)v\big\rangle \end{aligned}$$

$$
\begin{aligned}
&= -\langle \xi, [Ku, Kv]_{\mathfrak{g}} \rangle - \langle u, (\rho^L)^*(Kv)K^*\xi + (\rho^R)^*(Kv)K^*\xi \rangle \\
&\quad + \langle u, (\rho^L)^*(Kv)K^*\xi \rangle + \langle \xi, K(\rho^L(Ku)v) \rangle \\
&= -\langle \xi, [Ku, Kv]_{\mathfrak{g}} \rangle - \langle u, (\rho^R)^*(Kv)K^*\xi \rangle + \langle \xi, K\big(\rho^L(Ku)v\big) \rangle \\
&= -\langle \xi, [Ku, Kv]_{\mathfrak{g}} \rangle + \langle K(\rho^R(Kv)u), \xi \rangle + \langle \xi, K\big(\rho^L(Ku)v\big) \rangle \\
&= 0.
\end{aligned}
$$

Similarly, we can show that for all $X, Y \in \mathfrak{g}^* \oplus V$, we have

$$
\begin{aligned}
&(K + K^*)\big(L^*_{\ltimes}\big((K + K^*)X\big)Y + (-L^*_{\ltimes} - R^*_{\ltimes})\big((K + K^*)Y\big)X\big) \\
&\quad - \big[(K + K^*)X, (K + K^*)Y\big]_{\ltimes} = 0,
\end{aligned}
$$

which implies that $K + K^*$ is a relative Rota–Baxter operator on the Leibniz algebra $\mathfrak{g} \ltimes_{(\rho^L)^*, -(\rho^L)^* - (\rho^R)^*} V^*$ with respect to the representation $(\mathfrak{g}^* \oplus V; L^*_{\ltimes}, -L^*_{\ltimes} - R^*_{\ltimes})$.

Conversely, let $K + K^*$ be a relative Rota–Baxter operator on the Leibniz algebra $\mathfrak{g} \ltimes_{(\rho^L)^*, -(\rho^L)^* - (\rho^R)^*} V^*$ with respect to the representation $(\mathfrak{g}^* \oplus V; L^*_{\ltimes}, -L^*_{\ltimes} - R^*_{\ltimes})$. By (5.1), we deduce that $K : V \to \mathfrak{g}$ is a relative Rota–Baxter operator on $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ with respect to the representation $(V; \rho^L, \rho^R)$. ■

In the sequel, we introduce the notion of a Leibniz-dendriform algebra as the underlying algebraic structure of a relative Rota–Baxter operator on a Leibniz algebra.

Definition 5.3. A Leibniz-dendriform algebra is a vector space A equipped with two binary operations $\rhd$ and $\lhd$: $A \otimes A \to A$ such that for all $x, y, z \in A$, we have

$$(x \lhd y) \lhd z = x \lhd (y \lhd z) - y \lhd (x \lhd z) - (x \rhd y) \lhd z, \tag{5.2}$$

$$x \lhd (y \rhd z) = (x \lhd y) \rhd z + y \rhd (x \lhd z) + y \rhd (x \rhd z), \tag{5.3}$$

$$x \rhd (y \rhd z) = (x \rhd y) \rhd z + y \lhd (x \rhd z) - x \rhd (y \lhd z). \tag{5.4}$$

Let $(A, \rhd, \lhd)$ and $(A', \rhd', \lhd')$ be Leibniz-dendriform algebras. A linear map $f : A \to A'$ is called a homomorphism if for all $x, y \in A$,

$$f(x \rhd y) = f(x) \rhd' f(y), \quad f(x \lhd y) = f(x) \lhd' f(y).$$

Let Leib-Dend be the category whose objects are Leibniz-dendriform algebras and morphisms are the homomorphisms of Leibniz-dendriform algebras. Leibniz-dendriform algebras are generalizations of Leibniz algebras and left-symmetric algebras as the following two examples show.

Example 5.4. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Leibniz algebra. We define two binary operations $\rhd$ and $\lhd$: $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ by

$$x \lhd y = [x, y]_{\mathfrak{g}}, \quad x \rhd y = 0, \quad \forall x, y \in \mathfrak{g}.$$

Then $(\mathfrak{g}, \rhd, \lhd)$ is a Leibniz-dendriform algebra.

Conversely, a Leibniz-dendriform algebra in which the operation $\rhd = 0$ is exactly a Leibniz algebra.

Example 5.5. Let $(V, \cdot_V)$ be a left-symmetric algebra, i.e., the following equality holds:

$$(x \cdot_V y) \cdot_V z - x \cdot_V (y \cdot_V z) = (y \cdot_V x) \cdot_V z - y \cdot_V (x \cdot_V z).$$

We define two binary operations $\rhd$ and $\lhd$: $V \otimes V \to V$ by

$$x \lhd y = x \cdot_V y, \quad x \rhd y = -y \cdot_V x, \quad \forall x, y \in V.$$

Then $(V, \rhd, \lhd)$ is a Leibniz-dendriform algebra.

Conversely, a Leibniz-dendriform algebra in which $x \rhd y = -y \rhd x$ is exactly a left-symmetric algebra.

Proposition 5.6. *Let $(A, \rhd, \lhd)$ be a Leibniz-dendriform algebra. Then the binary operation $[\cdot,\cdot]_{\rhd,\lhd} : A \otimes A \to A$ given by*

$$[x, y]_{\rhd,\lhd} = x \lhd y + x \rhd y, \quad \forall x, y \in A, \tag{5.5}$$

defines a Leibniz algebra, which is called the sub-adjacent Leibniz algebra of $(A, \rhd, \lhd)$ and $(A, \rhd, \lhd)$ is called a compatible Leibniz-dendriform algebra structure on $(A, [\cdot,\cdot]_{\rhd,\lhd})$.

Proof. For all $x, y, z \in A$, we have

$$\begin{aligned}\big[x, [y, z]_{\rhd,\lhd}\big]_{\rhd,\lhd} &= [x, y \lhd z + y \rhd z]_{\rhd,\lhd}\\ &= x \lhd (y \lhd z) + x \rhd (y \lhd z) + x \lhd (y \rhd z) + x \rhd (y \rhd z).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\big[[x, y]_{\rhd,\lhd}, z\big]_{\rhd,\lhd} + \big[y, [x, z]_{\rhd,\lhd}\big]_{\rhd,\lhd}\\ &= [x \lhd y + x \rhd y, z]_{\rhd,\lhd} + [y, x \lhd z + x \rhd z]_{\rhd,\lhd}\\ &= (x \lhd y) \lhd z + (x \lhd y) \rhd z + (x \rhd y) \lhd z + (x \rhd y) \rhd z\\ &\quad + y \lhd (x \lhd z) + y \rhd (x \lhd z) + y \lhd (x \rhd z) + y \rhd (x \rhd z).\end{aligned}$$

Thus, $(A, [\cdot,\cdot]_{\rhd,\lhd})$ is a Leibniz algebra. ∎

Example 5.7. Let V be a vector space. On the vector space $\mathfrak{gl}(V) \oplus V$, for all $A, B \in \mathfrak{gl}(V)$, $u, v \in V$, define two binary operations $\rhd$ and $\lhd$: $(\mathfrak{gl}(V) \oplus V) \otimes (\mathfrak{gl}(V) \oplus V) \to \mathfrak{gl}(V) \oplus V$ by

$$(A + u) \lhd (B + v) = AB + Av, \quad (A + u) \rhd (B + v) = -BA.$$

Then $(\mathfrak{gl}(V) \oplus V, \rhd, \lhd)$ is a Leibniz-dendriform algebra. Its sub-adjacent Leibniz algebra is exactly the one underlying an omni-Lie algebra introduced by Weinstein in [40].

Let $(A, \rhd, \lhd)$ be a Leibniz-dendriform algebra. Define two linear maps $L_\lhd : A \to \mathfrak{gl}(A)$ and $R_\rhd : A \to \mathfrak{gl}(A)$ by

$$L_\lhd(x)y = x \lhd y, \quad R_\rhd(x)y = y \rhd x, \quad \forall x, y \in A. \tag{5.6}$$

Proposition 5.8. *Let $(A, \rhd, \lhd)$ be a Leibniz-dendriform algebra. Then $(A; L_\lhd, R_\rhd)$ is a representation of the sub-adjacent Leibniz algebra $(A, [\cdot,\cdot]_{\rhd,\lhd})$. Moreover, the identity map $\mathrm{Id}_A : A \to A$ is a relative Rota–Baxter operator on the Leibniz algebra $(A, [\cdot,\cdot]_{\rhd,\lhd})$ with respect to the representation $(A; L_\lhd, R_\rhd)$.*

Proof. By (5.2), for all $x, y, z \in A$, we have

$$\begin{aligned}&\big(L_\lhd([x,y]_{\rhd,\lhd}) - [L_\lhd(x), L_\lhd(y)]\big)z\\&\quad= [x,y]_{\rhd,\lhd} \lhd z - x \lhd (y \lhd z) + y \lhd (x \lhd z) = 0.\end{aligned}$$

Thus, we have $L_\lhd([x,y]_{\rhd,\lhd}) = [L_\lhd(x), L_\lhd(y)]$. By (5.3), we have

$$\big(R_\rhd([x,y]_{\rhd,\lhd}) - [L_\lhd(x), R_\rhd(y)]\big)z = z \rhd [x,y] - x \lhd (z \rhd y) + (x \lhd z) \rhd y = 0,$$

which implies that $R_\rhd([x,y]_{\rhd,\lhd}) = [L_\lhd(x), R_\rhd(y)]$. By (5.3) and (5.4), we have

$$\begin{aligned}\big(R_\rhd(y)L_\lhd(x) + R_\rhd(y)R_\rhd(x)\big)z &= (x \lhd z) \rhd y + (z \rhd x) \rhd y\\&= x \lhd (z \rhd y) - z \rhd (x \lhd y) - z \rhd (x \rhd y)\\&\quad + z \rhd (x \rhd y) - x \lhd (z \rhd y) + z \rhd (x \lhd y) = 0.\end{aligned}$$

Thus, we have $R_\rhd(y)L_\lhd(x) = -R_\rhd(y)R_\rhd(x)$. Therefore, $(A; L_\lhd, R_\rhd)$ is a representation of the sub-adjacent Leibniz algebra $(A, [\cdot,\cdot]_{\rhd,\lhd})$. Moreover, we have

$$\mathrm{Id}_A\big(L_\lhd(\mathrm{Id}_A(x))y + R_\rhd(\mathrm{Id}_A(y))x\big) = x \lhd y + x \rhd y = \big[\mathrm{Id}_A(x), \mathrm{Id}_A(y)\big]_{\rhd,\lhd}.$$

Thus, we obtain that $\mathrm{Id}_A : A \to A$ is a relative Rota–Baxter operator on the Leibniz algebra $(A, [\cdot,\cdot]_{\rhd,\lhd})$ with respect to the representation $(A; L_\lhd, R_\rhd)$. ■

Proposition 5.9. *Let $(A, \rhd, \lhd)$ and $(A', \rhd', \lhd')$ be Leibniz-dendriform algebras and f a homomorphism from A to A'. Then (f, f) is a relative Rota–Baxter operator homomorphism from Id_A to $\mathrm{Id}_{A'}$.*

Proof. It follows a direct computation. We omit the details. ■

Proposition 5.10. *Let $K : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ with respect to a representation $(V; \rho^L, \rho^R)$. Then there is a Leibniz-dendriform algebra structure on V given by*

$$u \rhd_K v := \rho^R(Kv)u, \quad u \lhd_K v := \rho^L(Ku)v, \quad \forall u, v \in V. \tag{5.7}$$

Proof. By (2.1) and (3.6), we have

$$\begin{aligned}&u \lhd_K (v \lhd_K w) - v \lhd_K (u \lhd_K w) - (u \rhd_K v) \lhd_K w - (u \lhd_K v) \lhd_K w\\&= \rho^L(Ku)\rho^L(Kv)w - \rho^L(Kv)\rho^L(Ku)w - \rho^L\big(K(\rho^R(Kv)u)\big)w - \rho^L\big(K(\rho^L(Ku)v)\big)w\\&= \rho^L([Ku, Kv]_\mathfrak{g})w - \rho^L\big(K(\rho^R(Kv)u)\big)w - \rho^L\big(K(\rho^L(Ku)v)\big)w\\&= 0,\end{aligned}$$

which implies that (5.2) in Definition 5.3 holds.

Similarly, we can show that (5.3) and (5.4) also hold. ■

Proposition 5.11. *Let $K : V \to \mathfrak{g}$ and $K' : V' \to \mathfrak{g}'$ be two relative Rota–Baxter operators and (ϕ, φ) a homomorphism (resp. an isomorphism) from K to K'. Then φ is a homomorphism (resp. an isomorphism) of Leibniz-dendriform algebras from $(V, \rhd_K, \lhd_K)$ to $(V', \rhd_{K'}, \lhd_{K'})$.*

Proof. For all $u, v \in V$, we have

$$\begin{aligned}\varphi(u \rhd_K v) &= \varphi\big(\rho^R(Kv)u\big) = \rho^{R'}\big(\phi(Kv)\big)\varphi(u)\\ &= \rho^{R'}\big(K'\varphi(v)\big)\varphi(u) = \varphi(u) \rhd_{K'} \varphi(v).\end{aligned}$$

Similarly, we obtain $\varphi(u \lhd_K v) = \varphi(u) \lhd_{K'} \varphi(v)$. ■

Theorem 5.12. *Propositions 5.8 and 5.9 give us a functor* $\mathsf{F} : \mathsf{Leib\text{-}Dend} \to \mathsf{RBOLeibniz}$. *Conversely, Propositions 5.10 and 5.11 give us a functor* $\mathsf{G} : \mathsf{RBOLeibniz} \to \mathsf{Leib\text{-}Dend}$. *Moreover,* F *is a left adjoint for* G.

Proof. Let $(A, \rhd, \lhd)$ be a Leibniz-dendriform algebra. Then $\mathrm{Id}_A : A \to A$ is a relative Rota–Baxter operator on the Leibniz algebra $(A, [\cdot,\cdot]_{\rhd,\lhd})$ with respect to the representation $(A; L_{\lhd}, R_{\rhd})$. Furthermore, for all $x, y \in A$, we have

$$\begin{aligned}x \rhd_{\mathrm{Id}_A} y &= R_{\rhd}(y)x = x \rhd y,\\ x \lhd_{\mathrm{Id}_A} y &= L_{\lhd}(x)y = x \lhd y.\end{aligned}$$

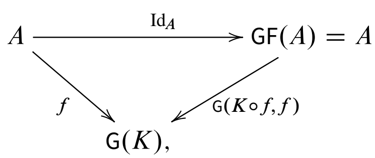
Thus, we have $(\mathsf{GF})(A, \rhd, \lhd) = (A, \rhd, \lhd)$. Let f be a Leibniz-dendriform algebra homomorphism from $(A, \rhd, \lhd)$ to $(A', \rhd', \lhd')$. By Proposition 5.9 and Proposition 5.11, we have $(\mathsf{GF})(f) = f$. We deduce that $\mathsf{GF} = \mathrm{Id}_{\mathsf{Leib}}\text{-}\mathsf{Dend}$. Thus, we have the identity natural transformation η (the unit of the adjunction)

$$\eta = \mathrm{Id}_{\mathrm{Id}_{\mathsf{Leib}}\text{-}\mathsf{Dend}} : \mathrm{Id}_{\mathsf{Leib}}\text{-}\mathsf{Dend} \to \mathsf{GF} = \mathrm{Id}_{\mathsf{Leib}}\text{-}\mathsf{Dend}.$$

Moreover, for any Leibniz-dendriform algebra $(A, \rhd, \lhd)$, relative Rota–Baxter operator $K : V \to \mathfrak{g}$ and Leibniz-dendriform algebra homomorphism $f : A \to \mathsf{G}(K)$, we have

$$\begin{aligned}f(x \rhd y) &= f(x) \rhd_K f(y) = \rho^R\big(Kf(y)\big)f(x),\\ f(x \lhd y) &= f(x) \lhd_K f(y) = \rho^L\big(Kf(x)\big)f(y).\end{aligned}$$

Thus, there is exactly one relative Rota–Baxter operator homomorphism $(K \circ f, f)$ from $\mathrm{Id}_A : A \to A$ to $K : V \to \mathfrak{g}$ such that the following diagram commutes:



which implies that F is a left adjoint for G. ■

By Example 5.4 and Proposition 5.10, we get the following conclusion.

Corollary 5.13. *Let $K : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to an antisymmetric representation $(V; \rho^L, \rho^R = 0)$. Then there is a Leibniz algebra structure on V given by*

$$[u, v]_K := \rho^L(Ku)v, \quad \forall u, v \in V. \tag{5.8}$$

By Example 5.5 and Proposition 5.10, we get the following conclusion.

Corollary 5.14. *Let $K : V \to \mathfrak{g}$ be a relative Rota–Baxter operator on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to a symmetric representation $(V; \rho^L, \rho^R = -\rho^L)$. Then there is a left-symmetric algebra structure on V given by*

$$u \cdot_K v := \rho^L(Ku)v, \quad \forall u, v \in V. \tag{5.9}$$

We give a sufficient and necessary condition for the existence of a compatible Leibniz-dendriform algebra structure on a Leibniz algebra.

Proposition 5.15. *There is a compatible Leibniz-dendriform algebra structure on a Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ if and only if there exists an invertible relative Rota–Baxter operator $K : V \to \mathfrak{g}$ on $\mathfrak{g}$ with respect to a representation $(V; \rho^L, \rho^R)$. Furthermore, the compatible Leibniz-dendriform algebra structure on $\mathfrak{g}$ is given by*

$$x \rhd y := K\big(\rho^R(y)K^{-1}x\big), \quad x \lhd y := K\big(\rho^L(x)K^{-1}y\big), \quad \forall x, y \in \mathfrak{g}. \tag{5.10}$$

Proof. Let $K : V \to \mathfrak{g}$ be an invertible relative Rota–Baxter operator on $\mathfrak{g}$ with respect to a representation $(V; \rho^L, \rho^R)$. By Proposition 5.10, there is a Leibniz-dendriform algebra on V given by

$$u \rhd_K v := \rho^R(Kv)u, \quad u \lhd_K v := \rho^L(Ku)v, \quad \forall u, v \in V.$$

Since K is an invertible relative Rota–Baxter operator, we obtain that

$$\begin{aligned} x \rhd y &:= K(K^{-1}x \rhd K^{-1}x) = K\big(\rho^R(y)K^{-1}x\big), \\ x \lhd y &:= K(K^{-1}x \lhd K^{-1}y) = K\big(\rho^L(x)K^{-1}y\big) \end{aligned}$$

is a Leibniz-dendriform algebra on $\mathfrak{g}$. By (3.6), we have

$$\begin{aligned} x \rhd y + x \lhd y &= K\big(\rho^R(y)K^{-1}x\big) + K\big(\rho^L(x)K^{-1}y\big) \\ &= K\big(\rho^R\big(K(K^{-1}y)\big)K^{-1}x\big) + K\big(\rho^L\big(K(K^{-1}x)\big)K^{-1}y\big) \\ &= [x, y]_{\mathfrak{g}}. \end{aligned}$$

On the other hand, let $(\mathfrak{g}, \rhd, \lhd)$ be a compatible Leibniz-dendriform algebra of the Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. By Proposition 5.8, $(\mathfrak{g}; L_{\lhd}, R_{\rhd})$ is a representation of the Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Moreover, $\mathrm{Id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ is a relative Rota–Baxter operator on the Leibniz algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to the representation $(\mathfrak{g}; L_{\lhd}, R_{\rhd})$. ■

Theorem 5.16. *Let* $(A, \triangleright, \triangleleft)$ *be a Leibniz-dendriform algebra. Then*

$$r := \sum_{i=1}^{n} (e_i^* \otimes e_i + e_i \otimes e_i^*) \tag{5.11}$$

is a symmetric solution of the classical Leibniz Yang–Baxter equation in the Leibniz algebra $A \ltimes_{L^*_{\triangleleft}, -L^*_{\triangleleft} - R^*_{\triangleright}} A^*$, *where* $\{e_1, \dots, e_n\}$ *is a basis of* A *and* $\{e_1^*, \dots, e_n^*\}$ *is its dual basis. Moreover, r is nondegenerate and the induced bilinear form* $\mathcal{B}$ *on* $A \ltimes_{L^*_{\triangleleft}, -L^*_{\triangleleft} - R^*_{\triangleright}} A^*$ *is given by*

$$\mathcal{B}(x + \xi, y + \eta) = \langle \xi, y \rangle + \langle \eta, x \rangle. \tag{5.12}$$

Proof. Since $(A, \triangleright, \triangleleft)$ is a Leibniz-dendriform algebra, the identity map $\mathrm{Id}_A : A \to A$ is a relative Rota–Baxter operator on the sub-adjacent Leibniz algebra $(A, [\cdot, \cdot]_{\triangleright, \triangleleft})$ with respect to the representation $(A; L_{\triangleleft}, R_{\triangleright})$. By Theorem 5.2, $r = \sum_{i=1}^{n} (e_i^* \otimes e_i + e_i \otimes e_i^*)$ is a symmetric solution of the classical Leibniz Yang–Baxter equation in $A \ltimes_{L^*_{\triangleleft}, -L^*_{\triangleleft} - R^*_{\triangleright}} A^*$. It is obvious that the corresponding bilinear form $\mathcal{B} \in \mathrm{Sym}^2(A \oplus A^*)$ is given by (5.12). The proof is finished. ∎

The above results can be viewed as the Leibniz analogue of the results given in [4].

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