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Purely infinite corona algebras and extensions

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Abstract. We classify all essential extensions of the form

$$0 \to \mathcal{B} \to \mathcal{D} \to C(X) \to 0$$

where $\mathcal{B}$ is a nonunital, simple, separable, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra with continuous scale, and where X is a finite CW complex. In fact, we prove that there is a group isomorphism

$$\mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B}).$$

1. Introduction

Motivated by the problem of classifying essentially normal operators on a separable, infinite dimensional Hilbert space, Brown, Douglas and Fillmore (BDF) classified all C^*-algebra extensions of the form

$$0 \to \mathcal{K} \to \mathcal{D} \to C(X) \to 0$$

where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space, and X is a compact metric space. This was a starting point for much interesting phenomena in operator theory and has led to the rapid development of extension theory with many effective techniques (especially from KK theory) to compute the Ext-group $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ for many C^*-algebras $\mathcal{A}, \mathcal{B}$.

However, in general, $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ does not capture all unitary equivalence classes of extensions. Among other things, there can be many distinct unitary equivalence classes of trivial extensions, and also, an extension ϕ with $[\phi] = 0$ in $\mathrm{Ext}(C(X), \mathcal{B})$ need not be trivial. (For these and other shortcomings, see, for example, [36, 46], and [37].)

One of the implicit reasons for the success of the original BDF Theory is that $\mathbb{B}(l_2)$ and the Calkin algebra $\mathbb{B}(l_2)/\mathcal{K}$ have particularly nice structure. Among other things, $\mathbb{B}(l_2)$ has strict comparison and real rank zero (it is a von Neumann algebra), and $\mathbb{B}(l_2)/\mathcal{K}$ is

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simple purely infinite. (For example, the BDF–Voiculescu result that roughly speaking says that all essential extensions are absorbing would not be true without the simplicity of $\mathbb{B}(l_2)/\mathcal{K}$.[1])

It would be nice to find a class of corona algebras which generalize nice features from $\mathbb{B}(l_2)/\mathcal{K}$, with the goal of developing operator theory and extension theory in an agreeable context, among other things generalizing further the theories developed by BDF, Voiculescu and other workers. These ideas where clearly present[2] in the early literature.

Simple purely infinite corona algebras have been completely characterized. Recall that a simple C^*-algebra has *continuous scale* if, roughly speaking, it has a sequential approximate identity which is like a "Cauchy sequence". More precisely:

Definition 1.1. Let $\mathcal{B}$ be a nonunital, separable, simple C^*-algebra. Then $\mathcal{B}$ has *continuous scale* if $\mathcal{B}$ has an approximate identity $\{e_n\}_{n=1}^{\infty}$ such that $e_{n+1}e_n = e_n$ for all n, and for every $b \in \mathcal{B}_+ - \{0\}$, there exists an $N \geq 1$ such that for all $m > n \geq N$,

$$e_m - e_n \preceq b.$$

(See, for example, [45].)

In the above equation, $\preceq$ is a subequivalence relation for positive elements (generalizing Murray–von Neumann subequivalence for projections) given as follows: For a C^*-algebra $\mathcal{D}$, for $a, d \in \mathcal{D}_+$, $a \preceq d$ if there exists a sequence $\{x_n\}$ in $\mathcal{D}$ such that $x_n d x_n^* \to a$.

Often, we will write "continuous scale C^*-algebra" to mean a C^*-algebra with continuous scale. Many simple C^*-algebras have continuous scale. For example, every nonunital, σ-unital, simple purely infinite C^*-algebra has continuous scale. In fact, a σ-unital, stable, simple C^*-algebra has continuous scale if and only if it is simple purely infinite (see [45, Theorem 3.7]). Also, every nonelementary, separable, simple C^*-algebra has a hereditary C^*-subalgebra with continuous scale (e.g., see [41, Proposition 2.3]). On the other hand, the C^*-algebra $\mathcal{K}$ of compact operators on a separable, infinite dimensional Hilbert space does not have continuous scale.

Theorem 1.2. *Let $\mathcal{B}$ be a nonunital, separable, simple, nonelementary C^*-algebra. Then the following statements are equivalent:*

(1) *$\mathcal{B}$ has continuous scale.*

(2) *$\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple.*

(3) *$\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite.*

([41, 45]; see also [9, 55].)

[1] In fact, under a nuclearity hypothesis, Kasparov's KK^1 only classifies, up to unitary equivalence, the absorbing extensions.

[2] As a proper subset.

We note that purely infinite simple C^*-algebras have real rank zero ([57]). We further note that for a general nonunital, separable, simple C^*-algebra $\mathcal{D}$, an extension of $\mathcal{D}$ by $C(X)$ can often be decomposed in a way where one piece sits inside the minimal ideal of $\mathcal{M}(\mathcal{D})/\mathcal{D}$, and this piece is essentially an extension of a simple continuous scale C^*-algebra (e.g., [37]; see also [25]). Thus, simple purely infinite corona algebras are not just a very nice context, but are part of the general picture.[3]

Nonetheless, difficulties still arise that are not present in the case of $\mathbb{B}(l_2)/\mathcal{K}$. For example, for a simple continuous scale C^*-algebra $\mathcal{B}$, the K theory of $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}(\mathcal{B})/\mathcal{B}$ can be much more complicated than that of $\mathbb{B}(l_2)$ and $\mathbb{B}(l_2)/\mathcal{K}$. Moreover, in the case where $\mathcal{B}$ is nonstable, we do not have infinite repeats and the powerful tools of the classical theory of absorbing extensions (e.g., [2, 6, 13, 27, 29, 40, 53]) are no longer completely available.

In effect, one needs to develop a type of nonstable absorption theory, where one takes into account the fine structure of the K theory. Such a theory has previously been considered with definite results (e.g., [36, 37, 46]). The author of the aforementioned results studied the case where the ideal was a simple, nonunital, continuous scale algebra with real rank zero, stable rank one, strict comparison and unique tracial state. In the present paper, one of the results removes the unique tracial state condition, but with the addition of the highly restrictive condition of Jiang–Su-stability.

As part of the program, we also have results characterizing (not necessarily simple) purely infinite corona algebras. Under mild regularity conditions on a simple C^*-algebra $\mathcal{B}$, we have the following equivalences: $\mathcal{B}$ has quasicontinuous scale $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})$ has strict comparison $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})$ has finitely many ideals $\Leftrightarrow$ $\mathcal{I}_{\min} = \mathcal{I}_{\text{cont}}$ $\Leftrightarrow$ $V(\mathcal{M}(\mathcal{B}))$ has finitely many order ideals. We believe that this category is suitable to the development of a definitive and elegant extension theory, and should be the first case before the construction of an even more general theory. Furthermore, all such corona algebras have real rank zero,[4] and many other related fundamental results have been investigated (e.g., [24–26, 30, 44, 49]).

1.1. Notation

We end this section with some brief remarks on notation. In the last part, we also spell out some necessary prerequisites for reading this paper.

For a C^*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ denotes the multiplier algebra of $\mathcal{B}$. Thus, $\mathcal{C}(\mathcal{B}) =_{\text{df}} \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the corresponding corona algebra.

[3] And thus, also, the nonstable case is important even for understanding the stable case. We further note that nonstabilization has been a key part of some of the most interesting and difficult results in the field. See, for example, [4, 34, 38, 40].

[4] We note that real rank zero was a reoccurring, though implicit, theme in the proof of the original BDF index theorem. Moreover, the Kasparov technical lemma, which is a foundation for the construction of the Kasparov product and the important properties of KK, implies that the corona algebra of a σ-unital algebra is an SAW*-algebra, a property with formal similarities to real rank zero.

For each extension

$$0 \to \mathcal{B} \to \mathcal{D} \to \mathcal{C} \to 0$$

(of $\mathcal{B}$ by $\mathcal{C}$)[5], we will work with the corresponding *Busby invariant* which is a *-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$. We will always work with *essential extensions* which is equivalent to requiring that the corresponding Busby invariant be injective; hence, throughout the paper, when we write "extension", we mean essential extension. An extension is unital if the corresponding Busby invariant is a unital map.

Say that $\phi, \psi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two extensions. We say that ϕ and ψ are *unitarily equivalent* (and write $\phi \sim \psi$) if there exists a unitary $u \in \mathcal{M}(\mathcal{B})$ such that

$$\phi(c) = \pi(u)\psi(c)\pi(u)^*$$

for all $c \in \mathcal{C}$. Here, $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map.

$\mathbf{Ext}(\mathcal{C}, \mathcal{B})$ denotes the set of unitary equivalence classes of nonunital extensions of $\mathcal{B}$ by $\mathcal{C}$. If, in addition, $\mathcal{C}$ is unital, $\mathbf{Ext}_u(\mathcal{C}, \mathcal{B})$ is the set of unitary equivalence classes of unital extensions.

For a unital simple C^*-algebra $\mathcal{C}$, $T(\mathcal{C})$ denotes the tracial state space of $\mathcal{C}$. If $\mathcal{C}$ is a nonunital simple C^*-algebra, $T(\mathcal{C})$ will denote the class of (norm-)lower semicontinuous, densely defined traces which are normalized at a fixed element $e \in \mathcal{C}_+ - \{0\}$, where e is in the Pedersen ideal of $\mathcal{C}$ (of course, for statements in this paper involving $T(\mathcal{C})$, where $\mathcal{C}$ is nonunital, the choice of e will not be relevant). For $\tau \in T(\mathcal{C})$ (where $\mathcal{C}$ is unital or nonunital), for $c \in \mathcal{C}_+$, $d_\tau(c) =_{\text{df}} \lim_{n\to\infty} \tau(c^{1/n})$. (Good references are [24] and [25].)

For a C^*-algebra $\mathcal{D}$ and for $a, b \in \mathcal{D}_+$, $a \preceq b$ means that there exists a sequence $\{x_n\}$ in $\mathcal{D}$ such that $x_n b x_n^* \to a$. (This subequivalence generalizes Murray–von Neumann subequivalence for projections.) For $a \in \mathcal{D}_+$, we let $\text{her}_{\mathcal{D}}(a) =_{\text{df}} \overline{a\mathcal{D}a}$, the hereditary C^*-subalgebra of $\mathcal{D}$ generated by a. Sometimes, for simplicity, we write $\text{her}(a)$ in place of $\text{her}_{\mathcal{D}}(a)$. Similarly, for a C^*-subalgebra $\mathcal{C} \subseteq \mathcal{D}$, we let $\text{her}_{\mathcal{D}}(\mathcal{C})$ or $\text{her}(\mathcal{C})$ denote $\overline{\mathcal{C}\mathcal{D}\mathcal{C}}$, the hereditary C^*-subalgebra of $\mathcal{D}$ generated by $\mathcal{C}$. Finally, for a subset $S \subseteq \mathcal{D}$, we let $\text{Ideal}_{\mathcal{D}}(S)$ denote the ideal of $\mathcal{D}$ which is generated by S. Again, we often write $\text{Ideal}(S)$ in place of $\text{Ideal}_{\mathcal{D}}(S)$.

In this paper, any simple, separable, stably finite C^*-algebra is assumed to have the property that every quasitrace is a trace.

Throughout this paper, $\mathcal{Z}$ denotes the Jiang–Su algebra ([23]). A C^*-algebra $\mathcal{C}$ is said to be $\mathcal{Z}$*-stable* if $\mathcal{C} \otimes \mathcal{Z} \cong \mathcal{C}$.

Let $\mathcal{A}, \mathcal{C}$ be C^*-algebras. Throughout this paper, we will write that a map $\phi : \mathcal{A} \to \mathcal{C}$ is *c.p.c.* if it is linear and completely positive contractive. Let $\mathcal{F} \subset \mathcal{A}$ be a finite subset and let $\delta > 0$. A c.p.c. map $\psi : \mathcal{A} \to \mathcal{C}$ is said to be $\mathcal{F}$-δ*-multiplicative* if $\|\psi(fg) - \psi(f)\psi(g)\| < \delta$ for all $f, g \in \mathcal{F}$.

We will be using, without definition or explanation, many notations and results from KK theory and other theories. The reader will be required to be familiar with the references listed below.

[5]In the literature, the terminology is sometimes reversed and this is sometimes called an "extension of $\mathcal{C}$ by $\mathcal{B}$". Following Arveson, BDF, Voiculescu and others, we prefer "$\mathcal{B}$ by $\mathcal{C}$".

Good references for basic multiplier algebra theory, extension theory, K theory, and KK theory are [5, 28, 39, 54]. See also [24–26] for much of the advanced multiplier algebra machinery. We emphasize that we will be extensively using, without definition or explanation, notation and results from [5, 28, 39].

For the notation and basic KK-theoretic tools (which, again, we will freely use without definition or explanation), we refer the reader to [8, 17, 19, 22, 36, 38–40, 43, 46–48], and the references therein. We emphasize, once more, the nonstable aspects of the theory which can be found in, say, [19] as well as other references mentioned above.

References for simple continuous scale algebras are [45] and [41]. Section 1 of [42] contains computations of the K theory for the multiplier and corona algebras of simple, separable, continuous scale C^*-algebras with real rank zero, stable rank one and strict comparison (see also [44, Propositions 4.2, 4.4 and Corollary 4.6]; and also [12]). Other good sources are [24] and [25]. We note that simple continuous scale algebras play a key role in recent outstanding breakthroughs (see, for example, [11]). The reader should also be familiar with [1–4, 6, 7, 20, 21, 33, 34, 53, 56, 57].

The first version of this paper, with essentially the same proof, was typed up in 2016, and starting 2016, many talks on the main result of the present paper were given in multiple conferences in Canada, China, and the USA.

We thank the referee for many helpful comments that improved the presentation of the paper. Among other things, this includes the referee's pointing out the arguments of [16, Paragraphs 5.11 and 5.12] to us, which greatly simplified some of the key proofs in the paper.

2. Some results in nonstable absorption

This section is a brief exposition of some results from [48]. Precursors to the results in this section are [2, 6, 13, 27, 31, 37, 46, 53]. This section has the flavor of operator theory, especially Halmos' proof of the Weyl–von Neumann–Berg Theorem. (See also the historical remark before Definition 2.2.) We note that this is true also for later parts of the paper (e.g., see Proposition 4.6). Recall, from the end of the first section, that all our extensions are assumed to be essential.

The following definition/lemma is [48, Remark 1 (after Proposition 2.5) and Subsection 3.1].

Definition 2.1 (And also Lemma). Let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra, and let X be a compact metric space. Then there is an addition on the class of nonunital extensions of $\mathcal{B}$ by $C(X)$. More precisely, say that $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two nonunital *-monomorphisms. Then the *BDF sum* of ϕ and ψ is given by

$$S\phi(\cdot)S^* + T\psi(\cdot)T^*$$

where $S, T \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ are isometries such that $SS^* + TT^* \leq 1$. We denote the above sum by $\phi \oplus \psi$.

The above sum is well defined up to unitary equivalence. Thus, the above sum induces an addition and hence a semigroup structure on $\mathbf{Ext}(C(X), \mathcal{B})$.

Suppose, in addition, that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$. Then the same holds for $\mathbf{Ext}_u(C(X), \mathcal{B})$, but where, in the above, we require that $SS^* + TT^* = 1$.

The concepts of *null* and *totally trivial* extensions (see Definitions 2.2 and 2.5) are due to Lin (e.g., see [46] and [37]), though we have modified the definitions. Early versions of these concepts were already present in [6].

Recall that in the original BDF case, where X is a compact subset of the plane, uniqueness of the trivial element of $\text{Ext}(C(X), \mathcal{K})$ essentially follows from the Weyl–von Neumann–Berg Theorem. Recall also that for a simple, separable, real rank zero C^*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ has the classical Weyl–von Neumann Theorem if and only if $\mathcal{M}(\mathcal{B})$ has real rank zero (e.g., [56,57]; see also [15,33]). This is perhaps one clue for the reasons for the assumption that $\mathcal{M}(\mathcal{B})$ has real rank zero in some early papers (see, for example, [37,46]). All this also indicates the operator-theoretic nature of the present study.

Recall, from the end of the first section, that for extensions ϕ and ψ, $\phi \sim \psi$ means that ϕ and ψ are unitarily equivalent. The next definition is for both the unital and nonunital cases.

Definition 2.2. Let $\mathcal{B}$ be a simple, nonunital, separable, continuous scale C^*-algebra. Let X be a compact metric space and let $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be an essential extension.

(1) ϕ is said to be *null* if there exists a commutative AF-subalgebra $\mathcal{C} \subset \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that $\text{Ran}(\phi) \subseteq \mathcal{C}$ and $[p] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$ for every projection $p \in \mathcal{C}$.

(2) ϕ is said to be *self-absorbing* if $\phi \oplus \phi \sim \phi$.

Proposition 2.3. *Let $\mathcal{B}$ be a nonunital, simple, separable C^*-algebra with continuous scale and let X be a compact metric space. Then we have the following:*

(1) *There exists a null extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$. Moreover, we can require ϕ to be nonunital or unital (if, additionally, $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$).*

(2) *Every null extension $C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is self-absorbing.*

(3) *Any two unital self-absorbing extensions $C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are unitarily equivalent. The same holds for any two nonunital self-absorbing extensions.*

(4) *Every self-absorbing extension must be null.*

Proof. This is [48, Theorem 3.4]. ■

Theorem 2.4. *Let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra. Let X be a compact metric space.*

Then $\mathbf{Ext}(C(X), \mathcal{B})$ is a group where the zero element is the class of a null extension. If, in addition, $[1_{\mathcal{C}(\mathcal{B})}]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$, then the same holds for $\mathbf{Ext}_u(C(X), \mathcal{B})$.

Proof. This is [48, Theorem 3.5]. Another proof that $\mathbf{Ext}(C(X), \mathcal{B})$ is a group can be found in [50, Theorem 2.10]. ■

Definition 2.5. Let $\mathcal{B}$ be a nonunital, separable C^*-algebra, and let X be a compact metric space. An extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is *totally trivial* if there exist a strictly converging properly increasing sequence $\{e_n\}_{n=1}^{\infty}$ of projections in $\mathcal{B}$, and a dense sequence $\{x_n\}_{n=1}^{\infty}$ in X, with each term repeating infinitely many times, such that

$$\phi = \pi \circ \psi$$

where $\psi : C(X) \to \mathcal{M}(\mathcal{B})$ is the *-homomorphism given by

$$\psi(f) =_{\text{df}} \sum_{n=1}^{\infty} f(x_n)(e_n - e_{n-1}),$$

and where $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map. (Here, $e_0 =_{\text{df}} 0$.)

Sometimes, to save writing, we call a *-homomorphism $\psi : C(X) \to \mathcal{M}(\mathcal{B})$ a *totally trivial extension* if it has the form as in Definition 2.5 above.

Theorem 2.6. *Let X be a finite CW complex and let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra with real rank zero, stable rank one and weak unperforation. Then an extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is null if and only if ϕ is totally trivial and $K_0(\phi) = 0$.*

Proof. This is a part of [48, Theorem 4.10]. ∎

3. Aspects of operator theory: *KK* theory

In this section, we gather some relevant results which have their origins in BDF theory and closely related phenomena like Lin's important work on almost commuting self-adjoint matrices. This interesting phenomena have had manifold implications including the important uniqueness and stable uniqueness theorems (e.g., [17–19, 38, 40, 43]). We also briefly discuss results concerning the complementary problem of stable existence.

As noted at the end of the first section, we will be freely using, without definition or explanation, notation and basic results from standard references on KK theory, especially with regard to parts of the theory concerning existence, uniqueness, absorbing extensions and nonstable aspects of the theory. A key reference is [39]. Other references are [5, 8, 18, 19, 28, 38, 40, 43], and other references listed at the end of the first section. The reader is assumed to be familiar with the notation and contents of these references.

Firstly, recall that one of our standing hypotheses is that for the unital, separable, simple, stably finite C^*-algebras discussed in this paper, we always assume that every quasitrace is a trace.

Let X be a compact metric space and let $\mathcal{A}$ be a C^*-algebra. Recall that a *-homomorphism $\phi : C(X) \to \mathcal{A}$ is said to be *finite dimensional* if there exist $x_1, x_2, \ldots, x_n \in X$ and pairwise orthogonal projections $p_1, p_2, \ldots, p_n \in \mathcal{A}$ such that $\phi(f) = \sum_{j=1}^{n} f(x_j)p_j$ for all $f \in C(X)$. In this case, the *spectrum* $\text{sp}(\phi)$ of ϕ, is defined to be $\text{sp}(\phi) =_{\text{df}} \{x_1, x_2, \ldots, x_n\}$.

For all $m \geq 1$, let Y_m be the 2-dimensional CW complex obtained by attaching a 2-cell to S^1 via the degree m map from S^1 to S^1. Let $C_0(Y_m)$ be the C^*-algebra of continuous functions on Y_m which vanish at a fixed point $\infty \in Y_m$. Recall that $K_0(C(Y_m)) = \mathbb{Z} \oplus \mathbb{Z}/m$, $K_0(C_0(Y_m)) = \mathbb{Z}/m$, and $K_1(C(Y_m)) = K_1(C_0(Y_m)) = 0$. We note that Y_m is actually the space $T_{II,m}$ in [10, Paragraph 4.2(b)].

Recall that for any unital C^*-algebra $\mathcal{C}$, $\mathbb{P}(\mathcal{C})$ is the notation for the collection of projections in $\bigcup_{n=1}^{\infty} \mathbb{M}_n(C(S^1) \otimes C(Y_m) \otimes \mathcal{C})$. Recall that $\underline{K}(\mathcal{C})^+$, the image of $\mathbb{P}(\mathcal{C})$ in $\underline{K}(\mathcal{C})$, is a positive cone for $\underline{K}(\mathcal{C})$. (See, for example, [8, Definition 4.4], [36, Section 2.1] or [39].) To simplify notation, for a subset $\mathcal{P} \subseteq \mathbb{P}(\mathcal{C})$, we also use $\mathcal{P}$ to denote its image in $\underline{K}(\mathcal{C})^+$.

Definition 3.1. Let X be a compact metric space, and let $\mathcal{A}$ be a C^*-algebra. Let $\mathcal{P} \subseteq \mathbb{P}(C(X))$. Then $\mathfrak{N}_{\mathcal{P}}$ denotes the set of all maps $\alpha : \mathcal{P} \to \underline{K}(\mathcal{A})$ such that there exists a finite dimensional *-homomorphism $\phi : C(X) \to \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi]|_{\mathcal{P}} = \alpha$.

Let $\mathfrak{N}$ denote the set of $\alpha \in KL(\mathcal{C}(X), \mathcal{A})$ such that there exists a finite dimensional *-homomorphism $\phi : C(X) \to \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi] = \alpha$.

Proposition 3.2. *Let X be a finite CW complex, and let $m =_{\text{df}} 2\dim(X) + 1$. Let $\varepsilon > 0$, a finite subset $\mathcal{F} \subset C(X)$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ be given. Then there exist a $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that the following statement is true:*

For every unital C^*-algebra $\mathcal{A}$ and for every unital c.p.c. $\mathcal{G}$-δ-multiplicative map $\phi : C(X) \to \mathcal{A}$, there exists a unital c.p.c. $\mathcal{F}$-ε-multiplicative map $\psi : C(X) \to \mathbb{M}_m(\mathcal{A})$ such that

$$[\phi \oplus \psi]|_{\mathcal{P}} \in \mathfrak{N}_{\mathcal{P}}.$$

Proof. This follows from [35, Corollary 1.24]. ■

Theorem 3.3. *Let X be a finite CW complex, $\varepsilon > 0$ and $\mathcal{F} \subset C(X)$ a finite subset. Then there exists a nonempty finite subset $\mathcal{E} \subset C(X)_+ - \{0\}$ such that for all $\lambda > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following holds:*

For all unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebras $\mathcal{A}$, for every unital $\mathcal{G}$-δ-multiplicative c.p.c. map $\phi : C(X) \to \mathcal{A}$, if $[\phi]|_{\mathcal{P}} : \mathcal{P} \to \underline{K}(\mathcal{A})$ lies in $\mathfrak{N}_{\mathcal{P}}$, and

$$d_\tau(\phi(g)) > \lambda$$

*for all $\tau \in T(\mathcal{A})$ and all $g \in \mathcal{E}$, then there exists a unital *-homomorphism $\psi : C(X) \to \mathcal{A}$ with finite dimensional range such that*

$$\|\phi(f) - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. This is [47, Theorem 2.12]. ■

Let $\{\mathcal{A}_N\}_{N=1}^{\infty}$ be a sequence of unital C^*-algebras. Define $\prod_b K_0(\mathcal{A}_N)$ to consist of all those $\{x_N\}_{N=1}^{\infty} \in \prod K_0(\mathcal{A}_N)$ such that there exists an $L \geq 1$ (dependent on $\{x_N\}$) where for all $N \geq 1$, x_N can be represented by $[p_N] - [q_N]$ where p_N, q_N are projections in $\mathbb{M}_L(\mathcal{A}_N)$. (Of course, here $\prod K_0(\mathcal{A}_N)$ is our notation for $\prod_{N=1}^{\infty} K_0(\mathcal{A}_N)$.)

We thank the referee for pointing out to us the arguments of the next two results, which are essentially from [16, Paragraphs 5.11 and 5.12] (see also [18]).

Lemma 3.4 (See [16, Paragraph 5.12]). *Let $\{\mathcal{A}_N\}_{N=1}^{\infty}$ be a sequence of simple, unital, separable, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebras.*

Then for all $m \geq 2$,

$$\text{Kernel}\Big(\prod_b K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod_b K_0(\mathcal{A}_N)\Big) = \text{Kernel}\Big(\prod K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod K_0(\mathcal{A}_N)\Big)$$

and

$$\text{Cokernel}\Big(\prod_b K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod_b K_0(\mathcal{A}_N)\Big) = \text{Cokernel}\Big(\prod K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod K_0(\mathcal{A}_N)\Big).$$

In the above, $\times m$ is the group homomorphism between abelian groups, given by $x \mapsto mx$. (It takes x to x added to itself m times.)

Proof. For all $N \geq 1$, since $\mathcal{A}_N$ is simple, finite, $\mathcal{Z}$-stable and has real rank zero, it has strict comparison of positive elements by traces and cancellation of projections, and the image of $K_0(\mathcal{A}_N)_+$ is (uniform norm) dense in $\text{Aff}(T(\mathcal{A}_N))_+$ (see, for example, [51]). Hence, every torsion element of $K_0(\mathcal{A}_N)$ can be represented by the difference of classes of two projections in $\mathcal{A}_N$. From this, we get the first equality

$$\text{Kernel}\Big(\prod_b K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod_b K_0(\mathcal{A}_N)\Big) = \text{Kernel}\Big(\prod K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod K_0(\mathcal{A}_N)\Big).$$

Next, for all $N \geq 1$, since $\mathcal{A}_N$ is nonelementary, simple and has real rank zero, $K_0(\mathcal{A}_N)$ is weakly divisible, i.e., for every $x \in K_0(\mathcal{A}_N)_+$, for every $n \geq 2$, there exist $x_1, x_2 \in K_0(\mathcal{A}_N)_+$ for which $x = nx_1 + (n+1)x_2$ (e.g., see [51]). Hence, since $\mathcal{A}_N$ has strict comparison of positive elements by traces and cancellation of projections, for every $x \in K_0(\mathcal{A}_N)_+$, there exist $x', x'' \in K_0(\mathcal{A}_N)_+$ for which $x = x' + mx''$ and x' is the class of a projection in $\mathcal{A}_N$. From this we get the second equality

$$\text{Cokernel}\Big(\prod_b K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod_b K_0(\mathcal{A}_N)\Big) = \text{Cokernel}\Big(\prod K_0(\mathcal{A}_N) \stackrel{\times m}{\to} \prod K_0(\mathcal{A}_N)\Big).$$

■

Recall that for a C^*-algebra $\mathcal{C}$, for $j = 0, 1$ and for all $m \geq 2$, $K_j(\mathcal{C}; \mathbb{Z}/m) = K_j(C_0(Y_m) \otimes \mathcal{C})$ and $K_j(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m) = K_j(C(Y_m) \otimes \mathcal{C}) = K_j(\mathcal{C}) \oplus K_j(\mathcal{C}; \mathbb{Z}/m)$. (See, for example, [39].)

Lemma 3.5 (See [16, Paragraphs 5.11, 5.12] and [18]). *Let $\{\mathcal{A}_N\}_{N=1}^{\infty}$ be a sequence of simple, unital, separable, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebras.*

Then

$$K_0\Big(\prod \mathcal{A}_N\Big) = \prod_b K_0(\mathcal{A}_N) \qquad \text{and} \qquad K_1\Big(\prod \mathcal{A}_N\Big) = \prod K_1(\mathcal{A}_N),$$

and for $j = 0, 1$ and all $m \geq 2$,

$$K_j\Big(\prod \mathcal{A}_N; \mathbb{Z}/m\Big) = \prod K_j(\mathcal{A}_N; \mathbb{Z}/m).$$

Proof. Since $\mathcal{A}_N$ is simple $\mathcal{Z}$-stable finite and has real rank zero, it has weak unperforation, stable rank one and weakly divisible K_0 group (see, for example, [51]). Hence, by [18, Corollary 2.1],

$$K_0\Big(\prod \mathcal{A}_N\Big) = \prod_b K_0(\mathcal{A}_N) \qquad \text{and} \qquad K_1\Big(\prod \mathcal{A}_N\Big) = \prod K_1(\mathcal{A}_N). \tag{3.6}$$

The standard mod p K theory exact sequences induce the following commuting diagram where both rows are exact (e.g., see [52, Proposition 1.6], [16, Paragraph 5.11], [39, Section 5.8]):

$$\begin{array}{ccccccccc}
\prod K_0(\mathcal{A}_N) & \xrightarrow{\times m} & \prod K_0(\mathcal{A}_N) & \longrightarrow & \prod K_0(\mathcal{A}_N; \mathbb{Z}/m) & \longrightarrow & \prod K_1(\mathcal{A}_N) & \xrightarrow{\times m} & \prod K_1(\mathcal{A}_N) \\
\uparrow (1) & & \uparrow (1) & & \uparrow (2) & & \uparrow (3) & & \uparrow (3) \\
K_0(\prod \mathcal{A}_N) & \xrightarrow{\times m} & K_0(\prod \mathcal{A}_N) & \longrightarrow & K_0(\prod \mathcal{A}_N; \mathbb{Z}/m) & \longrightarrow & K_1(\prod \mathcal{A}_N) & \xrightarrow{\times m} & K_1(\prod \mathcal{A}_N)
\end{array} \tag{3.7}$$

Firstly, by [18, Corollary 2.1], the vertical maps (1) and (2) are injective. By (3.6), the vertical map (3) is an isomorphism.

Let G, G', H, H' be the abelian groups that are given as follows:

$$\begin{aligned}
G &=_{\text{df}} \text{Cokernel}\Big(\prod K_0(\mathcal{A}_N) \overset{\times m}{\to} \prod K_0(\mathcal{A}_N)\Big), \\
G' &=_{\text{df}} \text{Cokernel}\Big(K_0\Big(\prod \mathcal{A}_N\Big) \overset{\times m}{\to} K_0\Big(\prod \mathcal{A}_N\Big)\Big), \\
H &=_{\text{df}} \text{Kernel}\Big(\prod K_1(\mathcal{A}_N) \overset{\times m}{\to} \prod K_1(\mathcal{A}_N)\Big),
\end{aligned}$$

and

$$H' =_{\text{df}} \text{Kernel}\Big(K_1\Big(\prod \mathcal{A}_N\Big) \overset{\times m}{\to} K_1\Big(\prod \mathcal{A}_N\Big)\Big).$$

Then the commuting diagram (3.7) induces the following commuting diagram where the rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & \prod K_0(\mathcal{A}_N; \mathbb{Z}/m) & \longrightarrow & H & \longrightarrow & 0 \\
& & \uparrow (1') & & \uparrow (2) & & \uparrow (3') & & \\
0 & \longrightarrow & G' & \longrightarrow & K_0(\prod \mathcal{A}_N; \mathbb{Z}/m) & \longrightarrow & H' & \longrightarrow & 0
\end{array}$$

where the vertical maps ($1'$) and (2) are injective and the vertical map ($3'$) is an isomorphism. But by the second equality in Lemma 3.4 and by (3.6), the map ($1'$) is an isomorphism. Thus, the map (2) is an isomorphism.

So we have a group isomorphism

$$K_0\Big(\prod \mathcal{A}_N; \mathbb{Z}/m\Big) \cong \prod K_0(\mathcal{A}_N; \mathbb{Z}/m).$$

By a similar argument, we have a group isomorphism

$$K_1\Big(\prod \mathcal{A}_N; \mathbb{Z}/m\Big) \cong \prod K_1(\mathcal{A}_N; \mathbb{Z}/m). \qquad \blacksquare$$

Recall that for all $m \geq 1$, $\mathcal{I}_m$ is the notation for the (nonunitized) dimension drop algebra $\mathcal{I}_m = \{f \in C[0,1] \otimes \mathbb{M}_m : f(0) = 0 \text{ and } f(1) \in \mathbb{C}\}$, and $\widetilde{\mathcal{I}_m}$ is the unitization of $\mathcal{I}_m$ (i.e., the unitized dimension drop algebra). Recall that for a C^*-algebra $\mathcal{C}$ and for $m \geq 2$, $K_*(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m) = KK(\widetilde{\mathcal{I}_m}, C(S^1) \otimes \mathcal{C})$, $K_0(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m) = KK(\widetilde{\mathcal{I}_m}, \mathcal{C})$, $K_*(\mathcal{C}; \mathbb{Z}/m) = KK(\mathcal{I}_m, C(S^1) \otimes \mathcal{C})$, and $K_0(\mathcal{C}; \mathbb{Z}/m) = KK(\mathcal{I}_m, \mathcal{C})$. Under the above identifications, recall that

$$K_0(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m)^{++} =_{\mathrm{df}} \{[\phi] : \phi : \widetilde{\mathcal{I}_m} \to \mathcal{C} \otimes \mathcal{K} \text{ is a *-homomorphism}\},$$
$$K_*(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m)^{++} =_{\mathrm{df}} \{[\phi] : \phi : \widetilde{\mathcal{I}_m} \to C(S^1) \otimes \mathcal{C} \otimes \mathcal{K} \text{ is a *-homomorphism}\},$$
$$K_0(\mathcal{C}; \mathbb{Z}/m) = \{[\psi] : \psi : \mathcal{I}_m \to \mathcal{C} \otimes \mathcal{K} \text{ is a *-homomorphism}\},$$

and

$$K_*(\mathcal{C}; \mathbb{Z}/m) = \{[\psi] : \psi : \mathcal{I}_m \to C(S^1) \otimes \mathcal{C} \otimes \mathcal{K} \text{ is a *-homomorphism}\}.$$

$\underline{K}(\mathcal{C})^{++}$ is the subsemigroup of $\underline{K}(\mathcal{C})$ generated by $K_*(\mathcal{C})^{++}$ and $K_*(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m)^{++}$ (for all $m \geq 2$). Recall that $\underline{K}(\mathcal{C})^{++}$ is a cone for $\underline{K}(\mathcal{C})$ (see, for example, [8, Definitions 4.4 and 4.5, Proposition 4.9, Theorem 5.2], etc.).

For each $x \in \underline{K}(\mathcal{C})$, we denote $x = \{x(j,m)\}_{0 \leq j \leq 1, 0 \leq m < \infty}$, where $x(j,0) \in K_j(\mathcal{C})$, $x(j,1) = 0$ and for all $m \geq 2$, $x(j,m) \in K_j(\mathcal{C}; \mathbb{Z}/m)$.

Recall that if $\mathcal{C}$ is a unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra, and if X is a compact metric space, then $C(X) \otimes \mathcal{C} \otimes \mathcal{K}$ has strict comparison of projections by traces in $T(C(X) \otimes \mathcal{C})$ ([51, Corollary 4.10]; the hypothesis of exactness is replaced by our standing assumption that all quasitraces are traces).

In general, for a C^*-algebra $\mathcal{C}$, $\underline{K}(\mathcal{C})^+$ and $\underline{K}(\mathcal{C})^{++}$ need not to coincide. However, this is so under additional hypotheses. The next lemma is well known and straightforward. We do the easy formal computation for the convenience of the reader.

Lemma 3.8. *Suppose that $\mathcal{C}$ is a unital, separable, simple, finite, real rank zero and $\mathcal{Z}$-stable C^*-algebra. Then*

$$\underline{K}(\mathcal{C})^+ = \underline{K}(\mathcal{C})^{++} = \{0\} \cup \{x \in \underline{K}(\mathcal{C}) : x(0,0) > 0\}.$$

We note that in [8, Definition 4.6], for a simple C^*-algebra $\mathcal{C}$, the set $\{0\} \cup \{x \in \underline{K}(\mathcal{C}) : x(0,0) > 0\}$ is denoted by $\underline{K}(\mathcal{C})_+$.

Proof. That

$$\underline{K}(\mathcal{C})^+ = \{0\} \cup \{x \in \underline{K}(\mathcal{C}) : x(0,0) > 0\}$$

follows from the fact that for any compact metric space X, $C(X) \otimes \mathcal{C} \otimes \mathcal{K}$ has strict comparison for projections. (See two paragraphs before this lemma.)

It is clear, from the definition of $\underline{K}(\mathcal{C})^{++}$, that

$$\underline{K}(\mathcal{C})^{++} \subseteq \{0\} \cup \{x \in \underline{K}(\mathcal{C}) : x(0,0) > 0\}.$$

So it suffices to prove the reverse inclusion.

Say that $x \in \underline{K}(\mathcal{C})$ such that $x(0,0) > 0$. Let $y \in \underline{K}(\mathcal{C})$ so that

$$x = x(0,0) + y.$$

By the hypotheses on $\mathcal{C}$, it is well known that we can find $l \geq 1$, $m_j \geq 1$, $p_j \in \mathrm{Proj}(1_{C(S^1)} \otimes \mathcal{C} \otimes \mathcal{K})$, and *-homomorphisms $\psi_j : \mathcal{I}_{m_j} \to p_j(C(S^1) \otimes \mathcal{C} \otimes \mathcal{K})p_j$ for $1 \leq j \leq l$, such that

i. $p_j \perp p_k$ for all $j \neq k$,

ii. $x(0,0) = \sum_{j=1}^{l} [p_j]$, and

iii. $y = \sum_{j=1}^{l} [\psi_j]$.

For $1 \leq j \leq l$, let $\phi_j : \widetilde{\mathcal{I}_{m_j}} \to p_j(C(S^1) \otimes \mathcal{C} \otimes \mathcal{K})p_j$ be given by

$$\phi_j|_{\mathcal{I}_{m_j}} = \psi_j \qquad \text{and} \qquad \phi_j(1) = p_j.$$

Then

$$\sum_{j=1}^{l} [\phi_j] \in \underline{K}(\mathcal{C})^{++}$$

and

$$\sum_{j=1}^{l} [\phi_j] = x(0,0) + y = x.$$ ■

For $j = 0, 1$ and $n \geq 0$, we will use the standard notation for the Bockstein operations

$$\begin{aligned} \rho_n^j &: K_j(\mathcal{A}) \to K_j(\mathcal{A}; \mathbb{Z}/n), \\ \beta_n^j &: K_j(\mathcal{A}; \mathbb{Z}/n) \to K_{j+1}(\mathcal{A}), \\ \kappa_{n,mn}^j &: K_j(\mathcal{A}; \mathbb{Z}/mn) \to K_j(\mathcal{A}; \mathbb{Z}/n), \\ \kappa_{mn,n}^j &: K_j(\mathcal{A}; \mathbb{Z}/n) \to K_j(\mathcal{A}; \mathbb{Z}/mn). \end{aligned}$$

And we are using their standard definitions as Kasparov products.

The next computation is formal and trivial.

Lemma 3.9. *Let* $\{\mathcal{A}_N\}_{N=1}^{\infty}$ *be a sequence of unital, separable, simple, finite, real rank zero,* $\mathcal{Z}$*-stable* C^**-algebras.*

Under the identifications

$$K_0\Big(\prod \mathcal{A}_N\Big) = \prod_b K_0(\mathcal{A}_N)$$

and

$$K_0\Big(\prod \mathcal{A}_N;\mathbb{Z}/n\Big) = \prod K_0(\mathcal{A}_N;\mathbb{Z}/n)$$

(see Lemma 3.5), the Bockstein operation

$$\rho_n^0 : K_0\Big(\prod \mathcal{A}_N\Big) \to K_0\Big(\prod \mathcal{A}_N;\mathbb{Z}/n\Big)$$

is given by

$$\rho_n^0 =_{\mathrm{df}} \prod \rho_{n,N}^0$$

where for all N*,*

$$\rho_{n,N}^0 : K_0(\mathcal{A}_N) \to K_0(\mathcal{A}_N;\mathbb{Z}/n)$$

is the corresponding Bockstein operation for the N*th component algebra.*

Similar statements hold for ρ_n^1, β_n^j, $\kappa_{n,mn}^j$, $\kappa_{mn,n}^j$.

Proof. We prove the statement for ρ_n^0. The proofs for the other Bockstein operations are similar.

Recall that for a unital C^*-algebra $\mathcal{C}$, the Bockstein operation

$$\rho_n^0 : K_0(\mathcal{C}) \to K_0(\mathcal{C};\mathbb{Z}/n)$$

is given by taking the Kasparov product with the element $[\delta_1] \in KK(\mathcal{I}_n, \mathbb{C})$, where $\delta_1 : \mathcal{I}_n \to \mathbb{C}$ is the *-homomorphism $\delta_1(f) =_{\mathrm{df}} f(1)$. (Recall that, for this Bockstein operation, we are using the identifications $K_0(\mathcal{C}) = KK(\mathbb{C}, \mathcal{C})$ and $K_0(\mathcal{C};\mathbb{Z}/n) = KK(\mathcal{I}_n, \mathcal{C})$.)

Let $x \in K_0(\prod \mathcal{A}_N) = \prod_b K_0(\mathcal{A}_N)$. We want to prove that $\rho_n^0(x) = \prod \rho_{n,N}^0(x_N)$. Since each $\mathcal{A}_N$ is nonelementary, simple, unital, real rank zero, stable rank one and weakly unperforated, the elements of $K_0(\prod \mathcal{A}_N)$ are generated by the classes of projections in $\prod \mathcal{A}_N$. Hence, we may assume that

$$x = \prod [p_N]$$

where $p_N \in \mathcal{A}_N$ is a projection, for all N.

Let $\psi : \mathbb{C} \to \prod \mathcal{A}_N$ be the *-homomorphism given by

$$\psi(1) =_{\mathrm{df}} \prod p_N.$$

For all N, let $\psi_N : \mathbb{C} \to \mathcal{A}_N$ be the *-homomorphism given by

$$\psi_N(1) =_{\mathrm{df}} p_N.$$

Under the identification $K_0(\prod \mathcal{A}_N) = KK(\mathbb{C}, \prod \mathcal{A}_N)$,

$$x = [\psi].$$

Hence,

$$\begin{aligned}\rho_n^0(x) &= [\psi \circ \delta_1] = \Big[\prod(\psi_N \circ \delta_1)\Big] \\ &= \prod[\psi_N \circ \delta_1] = \prod \rho_{n,N}^0(x_N).\end{aligned}$$

■

Let X be a finite CW complex. Recall that there exists a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that for any C^*-algebra $\mathcal{D}$, if $\alpha, \beta \in KL(C(X), \mathcal{D})$ satisfy that $\alpha|_{\mathcal{P}} = \beta|_{\mathcal{P}}$ then $\alpha = \beta$ in $KL(C(X), \mathcal{D})$.

Lemma 3.10. *Let X be a finite CW complex, $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given. Then there exist an $N \geq 1$ and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following statement is true:*

Suppose that $\mathcal{A}$ is a unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra, and say that $\alpha \in KK(C(X), \mathcal{A})$ satisfies that

$$\alpha([1_{C(X)}]) = [1_{\mathcal{A}}]$$

in $K_0(\mathcal{A})$ and

$$\alpha([p]) \geq 0$$

for all $p \in \mathcal{P}$.

*Then there exist a unital c.p.c. $\mathcal{F}$-ε-multiplicative map $\phi : C(X) \to \mathbb{M}_{N+1}(\mathcal{A})$ and a unital finite dimensional *-homomorphism $\psi : C(X) \to \mathbb{M}_N(\mathcal{A})$ such that*

$$[\phi] = \alpha + [\psi]$$

in $KK(C(X), \mathcal{A})$.

Proof. Let X be a finite CW complex. We may assume that X is connected. Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given.

Suppose, to contrary, that the conclusion of Lemma 3.10 is false. Let $\{\mathcal{P}_N\}_{N=1}^\infty$ be an increasing sequence of finite subsets of $\mathbb{P}(C(X))$, $\{\mathcal{A}_N\}_{N=1}^\infty$ a sequence of unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebras, and $\alpha_N \in KK(C(X), \mathcal{A}_N)$ for all $N \geq 1$ such that

$$\alpha_N([1_{C(X)}]) = [1_{\mathcal{A}_N}]$$

in $K_0(\mathcal{A}_N)$,

$$\alpha_N([p]) \geq 0$$

for all $p \in \mathcal{P}_N$,

$$\bigcup_{N=1}^\infty [\mathcal{P}_N] = \underline{K}(C(X))^+,$$

and there are no unital c.p.c. $\mathcal{F}$-ε-multiplicative map $\phi' : C(X) \to \mathbb{M}_{N+1}(\mathcal{A}_N)$ and no unital finite dimensional *-homomorphism $\psi' : (C) \to \mathbb{M}_N(\mathcal{A}_N)$ for which

$$[\phi'] = \alpha_N + [\psi']$$

in $KK(C(X), \mathcal{A}_N)$, for all $N \geq 1$.

We denote the above statement by "(*)".

Let $\beta : \underline{K}(C(X)) \to \prod_{N=1}^{\infty} \underline{K}(\mathcal{A}_N)$ be the group homomorphism given by

$$\beta =_{\mathrm{df}} \prod_{N=1}^{\infty} \alpha_N.$$

Now, let $P \in \mathbb{P}(C(X))$ be arbitrary. For simplicity, let us assume that $m \geq 2$ is such that $[P] \in K_0(C(X); \mathbb{Z} \oplus \mathbb{Z}/m) = K_0(C(Y_m) \otimes C(X))$. Choose $M \geq 1$ so that $[P] \leq M[1_{C(Y_m) \otimes C(X)}] = M[1_{C(X)}]$ in $K_0(C(Y_m) \otimes C(X)) = K_0(C(X)) \oplus K_0(C(X); \mathbb{Z}/m)$.

Since $\{\alpha_N\}$ is asymptotically positive for sufficiently large N, we must have that $\alpha_N([P]) \geq 0$ and $\alpha_N([P]) \leq M\alpha_N([1_{C(X)}]) = M[1_{\mathcal{A}_N}]$. Hence, since $C(Y_m) \otimes \mathcal{A}_N$ has strict comparison for projections for all N, we must have that

$$\beta([P]) \in \prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m).$$

Since P was arbitrary, we must have that

$$\mathrm{Ran}(\beta|_{K_0(C(X);\mathbb{Z}\oplus\mathbb{Z}/m)}) \subseteq \prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m).$$

Hence, by Lemma 3.5, we have that

$$\mathrm{Ran}(\beta) \subseteq \underline{K}\Big(\prod \mathcal{A}_N\Big).$$

Since every α_N respects the Bockstein operations, it follows, by Lemma 3.9, that β respects the Bockstein operations. Hence,

$$\beta \in \mathrm{Hom}_{\Lambda}\left(\underline{K}(C(X)), \underline{K}\Big(\prod \mathcal{A}_N\Big)\right).$$

Hence, by [39, Theorem 6.1.11] (see also [38, Theorem 5.9])[6], there is an integer $L \geq 1$, a unital c.p.c. $\mathcal{F}$-$\varepsilon/2$-multiplicative map $\Phi : C(X) \to \mathbb{M}_{L+1} \otimes \prod_{N=1}^{\infty} \mathcal{A}_N$, and a finite dimensional unital *-homomorphism $\Psi : C(X) \to \mathbb{M}_L \otimes \prod_{N=1}^{\infty} \mathcal{A}_N$ such that

$$[\Phi] = \beta + [\Psi]$$

in $KK(C(X), \prod_{N=1}^{\infty} \mathcal{A}_N)$. (Recall that X is a finite CW complex.)

[6]The hypothesis of separability of the codomain algebra in Lin's existence result, can be easily removed.

We have decompositions

$$\Phi = \prod_{N=1}^{\infty} \phi_N \qquad \text{and} \qquad \Psi = \prod_{N=1}^{\infty} \psi_N$$

where for all N, $\phi_N : C(X) \to \mathbb{M}_{L+1} \otimes \mathcal{A}_N$ is a unital c.p.c. $\mathcal{F}$-$\varepsilon/2$-multiplicative map, and $\psi_N : C(X) \to \mathbb{M}_L \otimes \mathcal{A}_N$ is a unital finite dimensional *-homomorphism such that

$$[\phi_N] = \alpha_N + [\psi_N]$$

in $KK(C(X), \mathcal{A}_N)$. This contradicts (*). ∎

The next result is straightforward, but we nonetheless sketch a proof.

Lemma 3.11. *Let X be a connected finite CW complex. Then there exists a finite subset $\mathcal{P}_X \subset \mathrm{Proj}(C(X) \otimes \mathcal{K})$ for which the following is true:*

For every finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$, there exists an integer $N \geq 1$ such that for every unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra $\mathcal{A}$, for every $\alpha \in KL(C(X), \mathcal{A})$ with $\alpha([1_{C(X)}]) = [1_{\mathcal{A}}]$ in $K_0(\mathcal{A})$ and

$$\alpha([p]) \geq 0$$

*for all $p \in \mathcal{P}_X$, and for every unital finite dimensional *-homomorphism $\psi : C(X) \to \mathbb{M}_N \otimes \mathcal{A}$, we have that*

$$(\alpha + [\psi])|_{\mathcal{P}} \geq 0.$$

Sketch of proof. Since X is a finite CW complex, let $\mathcal{F} \subset \mathrm{Proj}(C(X) \otimes \mathcal{K})$ be a finite set whose image, in $K_0(C(X))^+$, generates $K_0(C(X))$. Say that $\mathcal{F} = \{p_1, \ldots, p_K\}$. For all $1 \leq j \leq K$, let $M_j \geq 1$ be such that $p_j \preceq \bigoplus^{M_j} 1_{C(X)}$.

Define $\mathcal{P}_X =_{\text{df}} \{1_{C(X)}, p_j, r_j : 1 \leq j \leq K\}$, where for all j, $r_j \in C(X) \otimes \mathcal{K}$ is a projection such that $[r_j] = (M_j + 1)[1_{C(X)}] - [p_j]$.

Say $\mathcal{P} = \{q_1, q_2, \ldots, q_L\} \subseteq \mathbb{P}(C(X))$. For each $1 \leq j \leq L$, let $q_j(0,0)$ be a projection of q_j into $\mathrm{Proj}(C(X) \otimes \mathcal{K})$. (Recall that for all j, q_j is a projection in $C(Y_m) \otimes C(S^1) \otimes C(X) \otimes \mathcal{K}$ for some m dependent on j. By taking a point evaluation, with point in $Y_m \times S^1$, we get a projection in $C(X) \otimes \mathcal{K}$.) Then $[q_j(0,0)]$ is the K_0 piece of $[q_j]$ in $K_*(C(X); \mathbb{Z} \oplus \mathbb{Z}/m) = K_0(C(X)) \oplus K_1(C(X)) \oplus K_*(C(X); \mathbb{Z}/m)$.

For all l, let $m_{l,j}$ be integers such that

$$[q_l(0,0)] = \sum_{j=1}^{K} m_{l,j}[p_j].$$

Let $N = \sum_{l=1}^{L} \sum_{j=1}^{K} (M_j + 10)|m_{l,j}| + 1$. We would be done since, by Lemma 3.8, an element of $\underline{K}(\mathcal{A})$ is positive if and only if it is either zero or its K_0 component is strictly positive. ∎

Corollary 3.12. *Let X be a finite CW complex. Then there is a finite subset*

$$\mathcal{P}_X \subset \text{Proj}(C(X) \otimes \mathcal{K})$$

for which the following is true:

For every $\varepsilon > 0$ and finite subset $\mathcal{F} \subset C(X)$, there exists $N \geq 1$ such that for every unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra $\mathcal{A}$, and for every $\alpha \in KK(C(X), \mathcal{A})$ such that

$$\alpha([1_{C(X)}]) = [1_{\mathcal{A}}]$$

in $K_0(\mathcal{A})$ and

$$\alpha([p]) \geq 0$$

*for all $p \in \mathcal{P}_X$, there exists a unital c.p.c. $\mathcal{F}$-ε-multiplicative map $\phi : C(X) \to \mathbb{M}_{N+1}(\mathcal{A})$ and a unital finite dimensional *-homomorphism $\psi : C(X) \to \mathbb{M}_N(\mathcal{A})$ such that*

$$[\phi] = \alpha + [\psi]$$

in $KK(C(X), \mathcal{A})$.

Recall that an extension of C^*-algebras

$$0 \to \mathcal{B} \to \mathcal{D} \to \mathcal{A} \to 0$$

is said to be *quasidiagonal* if there exists an approximate unit $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{B}$, consisting of an increasing sequence of projections, such that for all $x \in \mathcal{D}$,

$$\|xe_n - e_n x\| \to 0$$

as $n \to \infty$.

Proposition 3.13. *Let X be a compact metric space, and let $\mathcal{B}$ be a nonunital, σ-unital, simple C^*-algebra with real rank zero and continuous scale.*

Suppose that $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is an essential extension such that

$$[\phi] = 0$$

in $KL(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$. Then ϕ is quasidiagonal.

Proof. This follows from [36, Theorem 1.5] (see also [42, Theorems 7.10 and 7.11]). ∎

4. A nonstable Brown–Douglas–Fillmore Theorem

We move towards the technical, operator-theoretic argument of Proposition 4.6. This result and its proof has many precursors, including the Weyl–von Neumann Theorem and its many generalizations over the years. (The reader is expected to be comfortable with the references in the last paragraph of Section 1.) We have tried to make the proof easy to read. Nonetheless, we expect the proof of Proposition 4.6 to not be too easy, even for a very well-prepared reader.

Recall, from the previous sections, that we will be using much notation and results from KK theory without definition and explanation. (See the references from previous sections, especially those from the end of Section 1.) Recall, also, our standing assumption that for separable, simple, unital, stably finite C^*-algebras, we always assume that every quasitrace is a trace.

We would additionally like to remind the reader of the references (which the reader should be familiar with) [24, 25, 41, 45] and [44]. Recall that for a compact convex set K, $\mathrm{Aff}(K)$ is the collection of all real-valued affine continuous functions on K. Recall that with the uniform norm and the natural strict order (i.e., the order where f is below g if $f(s) < g(s)$ for all $s \in K$), $\mathrm{Aff}(K)$ is an ordered Banach space. We let $\mathrm{LAff}(K)$ denote the class of affine lower semicontinuous functions from K to $(-\infty, \infty]$.

Recall also that $\mathrm{Aff}(K)_{++}$ ($\mathrm{LAff}(K)_{++}$) denotes the functions in $\mathrm{Aff}(K)$ (respectively $\mathrm{LAff}(K)$) which are strictly positive at every point in K.

Let $\mathcal{B}$ be a nonunital, separable, stably finite, simple C^*-algebra. Here, we follow the previously mentioned and universally well-established convention by fixing a nonzero positive element of the Pedersen ideal of $\mathcal{B}$ (if $\mathcal{B}$ has real rank zero, a nonzero projection would do) and defining $T(\mathcal{B})$ to be the set of all densely defined, (norm-)lower semicontinuous traces on $\mathcal{B}$ that are normalized at that fixed positive element. In what follows, when $\mathcal{B}$ is nonunital, the choice of that nonzero positive Pedersen ideal element will not be relevant. (When $\mathcal{B}$ is unital, we always take the Pedersen ideal element to be the unit.) It is well known that $T(\mathcal{B})$, with the topology of pointwise convergence on $\mathrm{Ped}(\mathcal{B})$, is a compact convex set (see [14]).

Suppose that $\mathcal{B}$, as in the previous paragraph (and nonunital), also has real rank zero. Fix an approximate unit $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{B}$, consisting of an increasing sequence of projections. Recall that every nonzero $A \in \mathcal{M}(\mathcal{B})_+$ induces an element $\widehat{A} \in \mathrm{LAff}(T(\mathcal{B}))_{++}$ which is defined by

$$\widehat{A}(\tau) =_{\mathrm{df}} \lim_{n\to\infty} \tau(e_n A e_n).$$

(Note that when $\mathcal{B}$ has continuous scale, $\widehat{A} \in \mathrm{Aff}(T(\mathcal{B}))_{++}$, i.e., $\widehat{A}$ is continuous.)

The above then extends naturally to a map

$$\widehat{\cdot} : (\mathbb{M}_n \otimes \mathcal{M}(\mathcal{B}))_+ \to \mathrm{LAff}(T(\mathcal{B}))_{++} \cup \{0\}$$

for all n. Recall that there is an ordered group homomorphism

$$\chi : K_0(\mathcal{B}) \to \mathrm{Aff}(T(\mathcal{B}))$$

which is given by

$$\chi([p]) =_{\mathrm{df}} \widehat{[p]} =_{\mathrm{df}} \widehat{p},$$

for all $[p] \in K_0(\mathcal{B})_+$.

Next, recall that if $\mathcal{C}$ is a C^*-algebra and $a \in \mathcal{C}_+$ with $\|a\| \leq 1$ and $a^2 \approx a$ close enough, then there is a projection p such that $p \approx a$ very close. We introduce the notation

$$\lceil a \rceil =_{\mathrm{df}} p$$

which is well defined up to Murray–von Neumann equivalence.

Finally, for all C^*-algebras $\mathcal{C}, \mathcal{D}$, for any linear map $\sigma : \mathcal{C} \to \mathcal{D}$, we denote again by σ the natural induced linear map $\mathbb{M}_n \otimes \mathcal{C} \to \mathbb{M}_n \otimes \mathcal{D}$, for all n. And for a nonunital C^*-algebra $\mathcal{B}$, recall that $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map.

We remind the reader of the following result, which was essentially proven by Lin in 1991:

Theorem 4.1. *Let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra with real rank zero, stable rank one, and weakly unperforated K_0 group. Then we have the following:*

(1) $(K_0(\mathcal{M}(\mathcal{B})), K_0(\mathcal{M}(\mathcal{B}))_+) = (\mathrm{Aff}(T(\mathcal{B})), \mathrm{Aff}(T(B))_{++} \cup \{0\})$.

(2) *For any two projections $P, Q \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$, $P \sim Q$ if and only if $\tau(P) = \tau(Q)$ for all $\tau \in T(\mathcal{B})$.*

(3) *For any $f \in \mathrm{Aff}(T(\mathcal{B}))_{++}$, there exist $k \geq 1$ and a projection $P \in \mathbb{M}_k \otimes \mathcal{M}(\mathcal{B}) - \mathbb{M}_k \otimes \mathcal{B}$ such that $\widehat{P} = f$. Moreover, if $f(\tau) < \tau(1_{\mathcal{M}(\mathcal{B})})$ for all $\tau \in T(\mathcal{B})$, then we can choose $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$.*

(4) *The six-term exact sequence (for the ideal $\mathcal{B} \subset \mathcal{M}(\mathcal{B})$) induces a short exact sequence*

$$0 \to \mathrm{Aff}(T(\mathcal{B}))/\chi(K_0(\mathcal{B})) \to K_0(\mathcal{M}(\mathcal{B})/\mathcal{B}) \to K_1(\mathcal{B}) \to 0.$$

Proof. The first three statements were proven in [32]. A more widely available version is [42, Theorem 1.4] (see also [12] and [44]). The last statement can be found in [42, Corollary 1.5]. ■

Lemma 4.2. *Let X be a compact metric space, and let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra. Let $\phi, \psi : C(X) \to \mathcal{C}(\mathcal{B})$ be nonunital essential extensions.*

Then there exists a nonunital essential extension $\phi' : C(X) \to \mathcal{C}(\mathcal{B})$ such that

$$\phi \sim \phi' \oplus \psi.$$

(Recall that $\oplus$ means BDF sum and $\sim$ means unitary equivalence with unitary coming from $\mathcal{M}(\mathcal{B})$.)

Proof. This follows immediately from Theorem 2.4. ■

Lemma 4.3. *Let X be a finite CW complex, and let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra with real rank zero, stable rank one and weak unperforation. Let $\phi : C(X) \to \mathcal{C}(\mathcal{B})$ be a nonunital essential extension such that*

$$KL(\phi) = 0.$$

Let ψ be any nonunital essential null extension (which exists by Proposition 2.3).

Then there exists a nonunital essential quasidiagonal extension $\phi' : C(X) \to \mathcal{C}(\mathcal{B})$ for which

$$\phi \sim \phi' \oplus \psi$$

and

$$KL(\phi') = 0.$$

Proof. By Lemma 4.2, let $\phi' : C(X) \to \mathcal{C}(\mathcal{B})$ be a nonunital essential extension such that

$$\phi \sim \phi' \oplus \psi.$$

Now, since

$$KL(\phi) = KL(\psi) = 0,$$
$$KL(\phi') = 0.$$

Hence, by Proposition 3.13, ϕ' is quasidiagonal.

∎

Lemma 4.4. *Let X be a finite CW complex, and let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra. Let*

$$K_0(C(X)) = F \oplus T$$

where F is a finitely generated free abelian group and T is a finite torsion group, and for all $1 \leq j \leq n$, let $p_j, q_j \in \mathrm{Proj}(C(X) \otimes \mathcal{K})$ be such that if

$$e_j =_{\mathrm{df}} [p_j] - [q_j],$$

then $\{e_1, e_2, \ldots, e_n\}$ is a basis for F.

Let $\Phi : C(X) \to \mathcal{M}(\mathcal{B})$ be a c.p.c. almost multiplicative map which is sufficiently multiplicative so that $\lceil \Phi(p_j) \rceil$ and $\lceil \Phi(q_j) \rceil$ are well defined as elements of $\mathrm{Proj}(\mathcal{M}(\mathcal{B}))$, for all j. Suppose that

$$\widehat{\lceil \Phi(p_j) \rceil}, \widehat{\lceil \Phi(q_j) \rceil} \in \chi(K_0(\mathcal{B}))$$

for all j. Then there exists an $\alpha \in KL(C(X), \mathcal{B})$ such that

$$\widehat{\alpha(e_j)} = \widehat{\lceil \Phi(p_j) \rceil} - \widehat{\lceil \Phi(q_j) \rceil}$$

for all j, and

$$\alpha|_T = 0.$$

Proof. For all $1 \leq j \leq n$, let $e'_j \in K_0(\mathcal{B})$ be such that

$$\widehat{e'_j} = \widehat{\lceil \Phi(p_j) \rceil} - \widehat{\lceil \Phi(q_j) \rceil}.$$

Let $\beta : (K_0(C(X)), K_1(C(X))) \to (K_0(\mathcal{B}), K_1(\mathcal{B}))$ be given by

$$\beta(e_j) = e'_j$$

for all j,

$$\beta|_T = 0, \qquad \text{and} \qquad \beta|_{K_1(C(X))} = 0.$$

By the Universal Coefficient Theorem, we can lift β to $\alpha \in KL(C(X), \mathcal{B})$. ∎

Lemma 4.5. *Let X be a finite CW complex and let $\mathcal{B}$ be a nonunital, separable, simple, continuous scale C^*-algebra. Then there exist $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ such that the following statements hold:*

Let $\{r_n\}$ be a sequence of pairwise orthogonal projections in $\mathcal{B}$ for which $\sum r_n$ converges strictly in $\mathcal{M}(\mathcal{B})$, and let $\phi_n : C(X) \to r_n \mathcal{B} r_n$ be a c.p.c. $\mathcal{F}$-ε-multiplicative map for all n, such that for all $f, g \in C(X)$,

$$\|\phi_n(fg) - \phi_n(f)\phi_n(g)\| \to 0$$

as $n \to \infty$. Let

$$\Phi =_{\mathrm{df}} \sum \phi_n$$

where the sum converges in the pointwise strict topology. Suppose that Φ satisfies the hypotheses of Lemma 4.4 and induces an essential extension. Apply Lemma 4.4 to Φ to get $\alpha \in KL(C(X), \mathcal{B})$.

Then for every $p \in \mathrm{Proj}(C(X) \otimes \mathcal{K})$,

$$\widehat{\alpha([p])} - \sum_{n=1}^{N} \widehat{[\phi_n]([p])} = \sum_{n=N+1}^{\infty} \widehat{[\phi_n]([p])}.$$

Hence, for the given p, there exists an $M \geq 1$ such that for all $N \geq M$,

$$\widehat{\alpha([p])} - \sum_{n=1}^{N} \widehat{[\phi_n]([p])} > 0.$$

In particular, suppose, in addition, that $\mathcal{B}$ has real rank zero, stable rank one and weak unperforation. Then for all $N \geq M$,

$$\alpha([p]) - \sum_{n=1}^{N} [\phi_n]([p]) \geq 0$$

in $K_0(\mathcal{B})$.

Proof. As in the statement and proof of Lemma 4.4, let

$$K_0(C(X)) = F \oplus T$$

where F is a finitely generated free abelian group and T is a finite torsion group.

Let $p_j, q_j \in \mathrm{Proj}(C(X) \otimes \mathcal{K})$ be such that if

$$e_j =_{\mathrm{df}} [p_j] - [q_j]$$

for $1 \leq j \leq m$ then $\{e_1, e_2, \ldots, e_m\}$ is a basis for F.

Let α be given as in Lemma 4.4 with the above p_j, q_j, e_j. Let $\varepsilon > 0$ be small enough and a finite subset $\mathcal{F} \subset C(X)$ be big enough so that for any unital C^*-algebra $\mathcal{C}$, for any c.p.c. map $\sigma : C(X) \to \mathcal{C}$, if σ is $\mathcal{F}$-ε-multiplicative, then σ induces a well-defined element $[\sigma] \in KL(C(X), \mathcal{C})$, $\lceil \sigma(p_j) \rceil$, $\lceil \sigma(q_j) \rceil$ are well-defined elements of $\mathrm{Proj}(C(X) \otimes \mathcal{K})$, and

$$[\sigma]([p_j]) = [\lceil \sigma(p_j) \rceil] \qquad \text{and} \qquad [\sigma]([q_j]) = [\lceil \sigma(q_j) \rceil]$$

for all j. Let $p \in \mathrm{Proj}(C(X) \otimes \mathcal{K})$ be given. Let

$$[p] = x_F + x_T$$

where $x_F \in F$ and $x_T \in T$. Let

$$x_F = \sum_{j=1}^{m} n_j e_j.$$

Hence,

$$\begin{aligned} \widehat{\alpha([p])} &= \widehat{\alpha(x_F)} \qquad \text{(since } \alpha(x_T) \text{ is a torsion element)} \\ &= \sum_{j=1}^{m} n_j \widehat{\alpha(e_j)} \\ &= \sum_{j=1}^{m} n_j (\widehat{\lceil \Phi(p_j) \rceil} - \widehat{\lceil \Phi(q_j) \rceil}) \qquad \text{(by definition of } \alpha) \\ &= \sum_{j=1}^{m} n_j \left(\widehat{\left\lceil \sum_{k=1}^{\infty} \phi_k(p_j) \right\rceil} - \widehat{\left\lceil \sum_{k=1}^{\infty} \phi_k(q_j) \right\rceil} \right). \end{aligned}$$

Also,

$$\begin{aligned} \sum_{k=1}^{N} \widehat{[\phi_k]([p])} &= \sum_{k=1}^{N} \widehat{[\phi_k](x_F)} \qquad \text{(since } [\phi_k](x_T) \text{ is a torsion element)} \\ &= \sum_{k=1}^{N} \sum_{j=1}^{m} n_j \widehat{[\phi_k](e_j)} \\ &= \sum_{k=1}^{N} \sum_{j=1}^{m} n_j (\widehat{\phi_k(\lceil p_j \rceil)} - \widehat{\phi_k(\lceil q_j \rceil)}) \qquad \text{(since } \phi_k \text{ is } \mathcal{F}\text{-}\varepsilon\text{-multiplicative)} \\ &= \sum_{j=1}^{m} n_j \left(\widehat{\sum_{k=1}^{N} \phi_k(\lceil p_j \rceil)} - \widehat{\sum_{k=1}^{N} \phi_k(\lceil q_j \rceil)} \right). \end{aligned}$$

Then

$$\begin{aligned}
&\widehat{\alpha([p])} - \widehat{\sum_{k=1}^{N}[\phi_k]([p])} \\
&\quad = \sum_{j=1}^{m} n_j \left(\sum_{k=N+1}^{\infty} \widehat{\lceil \phi_k(p_j) \rceil} - \sum_{k=N+1}^{\infty} \widehat{\lceil \phi_k(q_j) \rceil} \right) \\
&\quad = \sum_{k=N+1}^{\infty} \sum_{j=1}^{m} n_j \left(\widehat{\lceil \phi_k(p_j) \rceil} - \widehat{\lceil \phi_k(q_j) \rceil} \right) \\
&\quad = \sum_{k=N+1}^{\infty} \widehat{[\phi_k](x_F)} \qquad \text{(since each } \phi_k \text{ is } \mathcal{F}\text{-}\varepsilon\text{-multiplicative)} \\
&\quad = \sum_{k=N+1}^{\infty} \widehat{[\phi_k]([p])} \qquad \text{(since each } [\phi_k](x_T) \text{ is a torsion element).}
\end{aligned}$$

But for all sufficiently large N, for all $k \geq N$,

$$[\phi_k]([p]) \geq 0$$

in $K_0(\mathcal{B})$, and we are done. ∎

Proposition 4.6. *Let $\mathcal{B}$ be a nonunital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra with continuous scale. Let X be a finite CW complex. Suppose that*

$$\phi : C(X) \to \mathcal{C}(\mathcal{B})$$

*is a nonunital *-monomorphism such that $KL(\phi) = 0$.*

Then ϕ is a null extension.

Proof. For simplicity, let us first assume that X is a connected finite CW complex. Let $\mathcal{B}$ be as in the hypotheses, and suppose that $\phi : C(X) \to \mathcal{C}(\mathcal{B})$ is a nonunital essential extension such that

$$KL(\phi) = 0.$$

By Theorem 2.6, it suffices to prove that ϕ is totally trivial.

By Lemma 4.3,

$$\phi \sim \widetilde{\phi} \oplus \psi^{(1)} \oplus \psi^{(2)} \tag{4.7}$$

where $\psi^{(d)}$ is a nonunital essential null extension ($d = 1, 2$),

$$KL(\widetilde{\phi}) = 0$$

and $\widetilde{\phi}$ is a nonunital essential quasidiagonal extension.

Moreover, let $\{\phi_n\}$, $\{\psi_n^{(d)}\}$ ($d = 1, 2, 3$) be sequences of c.p.c. maps from $C(X)$ to $\mathcal{B}$ such that the following statements are true:

(a) $\{\phi_n(1)\}$ is a sequence of pairwise orthogonal projections for which $\sum \phi_n(1)$ converges strictly.

(b) For all $d = 1, 2, 3$, $\{\psi_n^{(d)}(1)\}$ is a sequence of pairwise orthogonal projections for which $\sum \psi_n^{(d)}(1)$ converges strictly.

(c) For all m, n, k,

$$\phi_m(1) \perp \psi_n^{(1)}(1) \perp \psi_k^{(2)}(1) \perp \phi_m(1).$$

(d) For all $d = 1, 2, 3$, $\psi_n^{(d)}$ is a finite dimensional *-homomorphism.

(e)

$$\widetilde{\phi} = \pi\Big(\sum \phi_n\Big).$$

(f) For $d = 1, 2$,

$$\psi^{(d)} = \pi\Big(\sum \psi_n^{(d)}\Big).$$

(g) The map $C(X) \to \mathcal{C}(\mathcal{B})$ given by

$$f \mapsto \pi\Big(\sum \psi_n^{(3)}(f)\Big)$$

is an essential extension.

Let $\{\varepsilon_n\}$ be a strictly decreasing sequence in $(0, 1)$ such that

$$\varepsilon_n \to 0$$

as $n \to \infty$. Let $B_{C(X)}$ denote the closed unit ball of $C(X)_+$. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of $B_{C(X)}$ such that

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{F}_n} = B_{C(X)}.$$

Apply Theorem 3.3 to each $(\varepsilon_n, \mathcal{F}_n)$ to get a finite subset $\mathcal{E}_n \subset C(X)_+ - \{0\}$. We may assume that

$$1 \in \mathcal{E}_n \subset \mathcal{E}_{n+1}$$

for all n. For all n, let $p_n^{(2)} \in \mathcal{B}$ be a nonzero projection such that

$$p_n^{(2)} \preceq p$$

for every nonzero projection $p \in \text{Ran}(\psi_n^{(2)})$.

For $d = 1, 2$, throwing away finitely many initial terms from $\{\psi_n^{(d)}\}$ and replacing each $\psi_n^{(d)}$ with an appropriate block of the form $\sum_{k=l}^{m} \psi_k^{(d)}$ if necessary, we may assume that the spectrum of $\psi_n^{(d)}$ is ε_n-dense in X.

Doing the same again for $d = 1, 3$, we may assume that for $d = 1, 3$, for all n, for all $f \in \mathcal{E}_n$,

$$\psi_n^{(d)}(f) > 0$$

and

$$10\tau(\psi_n^{(d)}(1)) < \tau(p_n^{(2)}) \tag{4.8}$$

for all $\tau \in T(\mathcal{B})$. Hence, for $d = 1, 3$, there exists a nonzero projection $p_n^{(d)} \in \mathcal{B}$ such that

$$p_n^{(d)} \preceq p \tag{4.9}$$

for every nonzero projection $p \in \mathrm{Ran}(\psi_n^{(d)})$. Note that this implies that for all $f \in \mathcal{E}_n$, there is a projection $q \in \mathcal{B}$ such that $q \sim p_n^{(d)}$ and

$$q\psi_n^{(d)}(f) = \psi_n^{(d)}(f)q \in (0,\infty)q. \tag{4.10}$$

We may assume that

$$p_n^{(d)} \preceq p_{n+1}^{(d)}$$

for all n. For all n and for $d = 1, 3$, let

$$\lambda_n^{(d)} =_{\mathrm{df}} \min_{\tau \in T(\mathcal{B})} \frac{\tau(p_n^{(d)})}{2\tau(\psi_n^{(d)}(1))} > 0. \tag{4.11}$$

For all n, apply Theorem 3.3 to ε_n, $\mathcal{F}_n$, $\mathcal{E}_n$, $\lambda_n^{(d)}$ $(d = 1, 3)$. We then get ε'_n, $\mathcal{F}'_n$, $\mathcal{P}'_n$. We may assume that

$$\varepsilon'_{n+1} < \varepsilon'_n < \varepsilon_n,$$
$$\mathcal{F}_n \subseteq \mathcal{F}'_n \subseteq \mathcal{F}'_{n+1} \subseteq B_{C(X)}$$

and

$$\mathcal{P}'_n \subseteq \mathcal{P}'_{n+1}$$

for all n, such that

$$\varepsilon'_n \to 0$$

as $n \to \infty$, and

$$\bigcup_{n=1}^{\infty} [\mathcal{P}'_n] = \underline{K}(C(X))^+.$$

For all n, apply Proposition 3.2 to X, $m = 2\dim(X) + 1$, ε'_n, $\mathcal{F}'_n$, $\mathcal{P}'_n$ to get ε''_n, $\mathcal{F}''_n$. We may assume that

$$\varepsilon''_{n+1} < \varepsilon''_n < \varepsilon'_n$$

and

$$\mathcal{F}'_n \subseteq \mathcal{F}''_n \subseteq \mathcal{F}''_{n+1} \subseteq B_{C(X)}.$$

As a consequence,

$$\varepsilon''_n \to 0$$

as $n \to \infty$, and

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{F}''_n} = B_{C(X)}.$$

For all n, apply Proposition 3.2 to X, $m = 2\dim(X) + 1$, ε''_n, $\mathcal{F}''_n$, $\mathcal{P}'_n$ to get ε'''_n, $\mathcal{F}'''_n$. We may assume that for all n,

$$\varepsilon'''_{n+1} < \varepsilon'''_n < \varepsilon''_n$$

and

$$\mathcal{F}''_n \subseteq \mathcal{F}'''_n \subseteq \mathcal{F}'''_{n+1} \subseteq B_{C(X)}.$$

As a consequence,

$$\varepsilon'''_n \to 0$$

as $n \to \infty$, and

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{F}'''_n} = B_{C(X)}.$$

Now, for all n, apply Corollary 3.12 to X, ε'''_n, $\mathcal{F}'''_n$ to get $N_n \geq 1$. Let

$$\Phi =_{\mathrm{df}} \sum_{n=1}^{\infty} \phi_n.$$

Since $KL(\widetilde{\phi}) = 0$, throwing away finitely many initial terms ϕ_j if necessary, we may assume that $\{\phi_n\}$ and Φ satisfies the hypotheses of Lemma 4.5. Apply Lemma 4.5 to $\{\phi_n\}$ and Φ to get $\alpha \in KL(C(X), \mathcal{B})$.

Now, let $\{M_n\}$ be a strictly increasing sequence of positive integers for which the following statements hold:

i. We have

$$\left(\alpha - \sum_{j=1}^{M_n} [\phi_j]\right)([p]) \geq 0$$

for all $p \in \mathcal{P}_X$. (Recall that, by Lemma 4.5, $\alpha - \sum_{j=1}^{M} [\phi_j]$ is asymptotically positive on $K_0(C(X))$. Also, by Lemma 3.8, for all $x \in \underline{K}(\mathcal{B}) - \{0\}$, $x \geq 0$ if and only if the K_0 part of x is strictly positive.)

ii. For all n, for all $j \geq M_n$, ϕ_j is $\mathcal{F}'''_n$-ε'''_n-multiplicative.

iii. For all n, let $r_n \in \mathcal{B}$ be a projection such that

$$[r_n]_{K_0(\mathcal{B})} = \alpha([1]) - \sum_{j=1}^{M_n} [\phi_j]([1]).$$

Then for all $\tau \in T(\mathcal{B})$,

$$(1 + 5m + m^2)(1 + N_n)\tau(r_n) < \min\{\tau(p_n^{(1)}), \tau(p_n^{(3)})\}. \tag{4.12}$$

(Recall that $m = 2\dim(X) + 1$.)

iv. Increasing the M_n if necessary, we may assume that for all n and $\tau \in T(\mathcal{B})$,

$$N_{n+1}\tau(r_{n+1}) < N_n \tau(r_n). \tag{4.13}$$

To simplify notation, for all l, let

$$\beta_l =_{\text{df}} \alpha - \sum_{j=1}^{l} [\phi_j].$$

We have that for all $n \geq 2$ and all $p \in \mathcal{P}_X$,

$$\beta_{M_n}([p]) \geq 0.$$

So by Corollary 3.12, there exists a unital c.p.c. $\mathcal{F}_n'''$-ε_n'''-multiplicative map $\eta_n : C(X) \to r_n \mathcal{B} r_n \otimes \mathbb{M}_{N_n+1}$ and a unital finite dimensional *-homomorphism

$$fd_n' : C(X) \to r_n \mathcal{B} r_n \otimes \mathbb{M}_{N_n}$$

such that

$$[\eta_n] = \beta_{M_n} + [fd_n']. \tag{4.14}$$

By Proposition 3.2, let $\xi_n : C(X) \to (r_n \mathcal{B} r_n) \otimes \mathbb{M}_{N_n+1} \otimes \mathbb{M}_m$ be a unital c.p.c. $\mathcal{F}_n''$-ε_n''-multiplicative map and let $fd_n'' : C(X) \to (r_n \mathcal{B} r_n) \otimes \mathbb{M}_{N_n+1} \otimes \mathbb{M}_{m+1}$ be a unital finite dimensional *-homomorphism such that

$$[\eta_n \oplus \xi_n]|_{\mathcal{P}_n'} = [fd_n'']|_{\mathcal{P}_n'}. \tag{4.15}$$

By Proposition 3.2 again, let $\zeta_n : C(X) \to (r_n \mathcal{B} r_n) \otimes \mathbb{M}_{N_n+1} \otimes \mathbb{M}_m \otimes \mathbb{M}_m$ be a c.p.c. $\mathcal{F}_n'$-ε_n'-multiplicative map and let $fd_n''' : C(X) \to (r_n \mathcal{B} r_n) \otimes \mathbb{M}_{N_n+1} \otimes \mathbb{M}_m \otimes \mathbb{M}_{m+1}$ be a finite dimensional *-homomorphism such that

$$[\xi_n \oplus \zeta_n]|_{\mathcal{P}_n'} = [fd_n''']|_{\mathcal{P}_n'}. \tag{4.16}$$

For all n, from (4.15), we have that

$$[\eta_n] - [\eta_{n+1}] + [\xi_n] - [\xi_{n+1}]|_{\mathcal{P}_n'} = [fd_n''] - [fd_{n+1}'']|_{\mathcal{P}_n'}.$$

(Recall that $\mathcal{P}_n' \subseteq \mathcal{P}_{n+1}'$.) But from (4.14) and the definition of β_l, we have that

$$[\eta_n] - [\eta_{n+1}] = \sum_{j=M_n+1}^{M_{n+1}} [\phi_j] + [fd_n'] - [fd_{n+1}'].$$

Hence,

$$\sum_{j=M_n+1}^{M_{n+1}} [\phi_j] + [\xi_n] - [\xi_{n+1}]|_{\mathcal{P}_n'} = [fd_n''] - [fd_{n+1}''] + [fd_{n+1}'] - [fd_n']|_{\mathcal{P}_n'}.$$

From this and (4.16), we have that

$$\sum_{j=M_n+1}^{M_{n+1}} [\phi_j] + [\xi_n] + [\zeta_{n+1}]|_{\mathcal{P}_n'} = [fd_n''] - [fd_{n+1}''] + [fd_{n+1}'] - [fd_n'] + [fd_{n+1}''']|_{\mathcal{P}_n'}.$$

Since X is connected, by the definitions of the fd'_l, fd''_l and fd'''_l and by (4.13) (from the definition of r_n), we can find a nonzero projection $p_n \in \mathcal{B} \otimes \mathcal{K}$ with

$$\tau(p_n) = (N_n m + m + 1)\tau(r_n) - (N_{n+1} m + m + 1)\tau(r_{n+1}) \\ + (N_{n+1} + 1)m(m+1)\tau(r_{n+1})$$

for all $\tau \in T(\mathcal{B})$, and we can find a unital finite dimensional *-homomorphism $fd_n^{(4)} : C(X) \to p_n(\mathcal{B} \otimes \mathcal{K})p_n$ such that

$$[fd_n^{(4)}] = [fd''_n] - [fd''_{n+1}] + [fd'_{n+1}] - [fd'_n] + [fd'''_{n+1}].$$

Hence,

$$\sum_{j=M_n+1}^{M_{n+1}} [\phi_j] + [\xi_n] + [\zeta_{n+1}]|_{\mathcal{P}'_n} = [fd_n^{(4)}]|_{\mathcal{P}'_n} \tag{4.17}$$

for all n.

Note that

$$\tau(\xi_n(1) + \zeta_n(1)) = (N_n + 1)m(m+1)\tau(r_n)$$

for all $\tau \in T(\mathcal{B})$. From this, Theorem 3.3, (4.15), the definition of $\lambda_n^{(3)}$ in (4.11), and (4.12) in the definition of the M_n, we have that there exists a unital finite dimensional *-homomorphism

$$FD'_{0,n} : C(X) \to (fd'''_n(1) \oplus \psi_n^{(3)}(1))(\mathcal{B} \otimes \mathcal{K})(fd'''_n(1) \oplus \psi_n^{(3)}(1))$$

such that some unitary conjugate of $\xi_n \oplus \zeta_n \oplus \psi_n^{(3)}$ is within ε_n of $FD'_{0,n}$ on $\mathcal{F}_n$.

From the definition of $FD'_{0,n}$, by (4.8), and since the spectrum of $\psi_n^{(2)}$ is ε_n-dense in X, by conjugating $FD'_{0,n}$ by a unitary if necessary, we can find finite dimensional *-homomorphisms FD'_n, FD''_n, with pairwise orthogonal ranges such that

$$FD'_n + FD''_n = \psi_n^{(2)}$$

and $FD'_{0,n}$ is within ε_n of FD'_n on the closed unit ball of $C(X)$.

Hence, replacing $\xi_n, \zeta_n, \psi_n^{(3)}$ by appropriate (simultaneous) unitary conjugates if necessary, we may assume that

$$\xi_n(1) + \zeta_n(1) + \psi_n^{(3)}(1) \leq \psi_n^{(2)}(1)$$

and $\xi_n + \zeta_n + \psi_n^{(3)}$ is within $2\varepsilon_n$ of FD'_n on $\mathcal{F}_n$. We denote this assumption by (A).

Also, note that for all n, for all $\tau \in T(\mathcal{B})$,

$$\tau\left(\sum_{j=M_n+1}^{M_{n+1}} \phi_j(1) + \xi_n(1) + \zeta_{n+1}(1)\right) < (N_n + 1)(1 + m + (m+1)(m+2))\tau(r_n).$$

(We are also using (4.13), among other things.)

Hence, by Theorem 3.3, (4.17), and (4.12), there exists a unital finite dimensional *-homomorphism

$$FD_n''' : C(X) \to \mathrm{her}_{\mathcal{B}}\left(\xi_n(1) + \zeta_{n+1}(1) + \sum_{j=M_n+1}^{M_{n+1}} \phi_j(1) + \psi_n^{(1)}(1)\right)$$

such that on $\mathcal{F}_n$, FD_n''' is within ε_n of $\xi_n + \zeta_{n+1} + \sum_{j=M_n+1}^{M_{n+1}} \phi_j + \psi_n^{(1)}$. Denote this (B).

To keep the rest of the argument simple, we introduce a notation. For all c.p.c. maps $\sigma, \mu : C(X) \to \mathcal{M}(\mathcal{B})$, we let

$$\sigma \sim_{\mathcal{B}} \mu$$

mean that there exists a unitary $u \in \mathcal{M}(\mathcal{B})$ such that for all $f \in C(X)$,

$$u\sigma(f)u^* - \mu(f) \in \mathcal{B}.$$

Let $\overline{\phi} : C(X) \to \mathcal{M}(\mathcal{B})$ be a c.p.c. map that lifts ϕ, i.e., $\pi \circ \overline{\phi} = \phi$. Then,

$$\begin{aligned}
\overline{\phi} &\sim_{\mathcal{B}} \sum \phi_j \oplus \sum \psi_n^{(1)} \oplus \sum \psi_n^{(2)} && \text{(by (4.7))}\\
&\sim_{\mathcal{B}} \sum_{n=1}^{\infty} \sum_{j=M_n+1}^{M_{n+1}} \phi_j \oplus \sum \psi_n^{(1)} \oplus \sum (FD_n' \oplus FD_n'') \\
&\sim_{\mathcal{B}} \sum_{n=1}^{\infty} \left(\sum_{j=M_n+1}^{M_{n+1}} \phi_j \oplus \xi_n \oplus \zeta_{n+1} \oplus \psi_n^{(3)} \oplus \psi_n^{(1)} \right) \oplus \sum FD_n'' && \text{(by (A))}\\
&\sim_{\mathcal{B}} \sum FD_n''' \oplus \psi_n^{(3)} \oplus \sum FD_n'' && \text{(by (B))}.
\end{aligned}$$

Hence, ϕ is totally trivial. Since $KL(\phi) = 0$, ϕ is null by Theorem 2.6. ∎

Let X be a compact metric space and $\mathcal{C}$ be a unital C^*-algebra. Let

$$\gamma : KK(C(X), \mathcal{C}) \to \mathrm{Hom}(K_*(C(X)), K_*(\mathcal{C}))$$

be the surjective map from the Universal coefficient theorem. We let $KK_u(C(X), \mathcal{C})$ be the set of elements $x \in KK(C(X), \mathcal{C})$ such that $\gamma(x)$ maps $[1_{C(X)}]$ (in $K_0(C(X))$) to $[1_{\mathcal{C}}]$ (in $K_0(\mathcal{C})$).

Finally, denote by $\mathfrak{J}$ the natural group homomorphism

$$\mathfrak{J} : \mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B}) : [\phi]_{\mathbf{Ext}} \mapsto [\phi]_{KK}.$$

Similarly, we have the analogous map

$$\mathfrak{J} : \mathbf{Ext}_u(C(X), \mathcal{B}) \to KK_u(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

which is a group homomorphism when $[1_{\mathcal{C}(\mathcal{B})}]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$.

Theorem 4.18. *Let X be a finite CW complex, and let $\mathcal{B}$ be a nonunital, simple, separable, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra with continuous scale.*

Then the map

$$\mathfrak{J} : \mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

is a group isomorphism.

Proof. The argument is a minor variation on the argument of [36, Theorem 2.10]. We provide the proof for the convenience of the reader.

Firstly, surjectivity of the map $\mathfrak{J}$ follows from [35, Theorem 1.17]. Next, injectivity of $\mathfrak{J}$ follows from Proposition 4.6. ∎

A trivial argument, using Theorem 4.18, gives us the unital case:

Theorem 4.19. *Let X be a finite CW complex and let $\mathcal{B}$ be a nonunital, simple, separable, finite, real rank zero, $\mathcal{Z}$-stable C^*-algebra with continuous scale such that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$. Then the map*

$$\mathfrak{J} : \mathbf{Ext}_u(C(X), \mathcal{B}) \to KK_u(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

is a group isomorphism.

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