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Coarse quotients of metric spaces and embeddings of uniform Roe algebras

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Abstract. We study embeddings of uniform Roe algebras which have "large range" in their codomain and the relation of those with coarse quotients between metric spaces. Among other results, we show that if Y has property A and there is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ with "large range" and so that $\Phi(\ell_\infty(X))$ is a Cartan subalgebra of $\mathrm{C}^*_u(Y)$, then there is a bijective coarse quotient $X \to Y$. This shows that the large-scale geometry of Y is, in some sense, controlled by the one of X. For instance, if X has finite asymptotic dimension, so does Y.

1. Introduction

Given a metric space X, one associates to it a C^*-algebra $\mathrm{C}^*_u(X)$, called the *uniform Roe algebra of* X, which captures some of the large-scale geometric properties of X (see Section 2 for precise definitions regarding uniform Roe algebras and coarse geometry). This algebra was introduced by J. Roe in the context of index theory of elliptical operators in noncompact manifolds [17, 18], but its uses have spread way beyond this field. As a sample application, their study entered the realm of mathematical physics in the context of topological materials and, in particular, of topological insulators. For instance, Y. Kubota has proposed the uniform Roe algebra of $\mathbb{Z}^n$ as a model for disordered topological materials while E. Ewert and R. Meyer have used its non-uniform version for the same purpose (see [10, 13]). In this context, in order to study various types of symmetries, it is important to better understand embeddings/isomorphisms of (uniform) Roe algebra.

This paper deals with preservation of the large-scale geometry of uniformly locally finite (u.l.f.) metric spaces under embeddings between their uniform Roe algebras—the study of such embeddings was initiated by I. Farah, A. Vignati, and the current author in [5]. (Recall that a metric space (X, d) is u.l.f. if, given $r > 0$, there is $N \in \mathbb{N}$ so that the balls of radius r in X contain at most N-many elements. Finitely generated groups and k-regular graphs are examples of such spaces.) Firstly, notice that an injective coarse map $X \to Y$ induces a canonical embedding $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$: given such $f : X \to Y$, the isometric embedding $u_f : \ell_2(X) \to \ell_2(Y)$ determined by $u_f \delta_x = \delta_{f(x)}$ gives the embedding $\mathrm{Ad}(u_f) : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ [5, Theorem 1.2]. If f is, furthermore, a coarse

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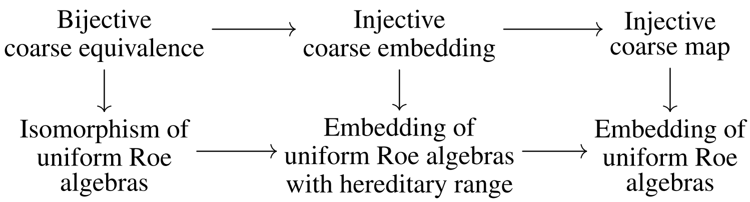

Figure 1. Relation between coarse properties of maps between metric spaces and embeddings of their uniform Roe algebras.

embedding, then the image of Φ is a hereditary subalgebra of $C^*_u(Y)$ and, if f is a bijective coarse equivalence, then Φ is an isomorphism (see Figure 1).

Rigidity questions for uniform Roe algebras deal with when the vertical arrows in Figure 1 can be reversed. The first two arrows are known to be reversable if the codomain space has G. Yu's property A (see [24, Theorem 6.13] and [5, Theorem 1.4]). As for the latter, the following holds: under some geometric conditions on Y, (1) a rank-preserving embedding $C^*_u(X) \to C^*_u(Y)$ gives a uniformly finite-to-one[1] coarse map $X \to Y$ [5, Theorem 5.4], and (2) a compact-preserving embedding $C^*_u(X) \to C^*_u(Y)$ gives a partition for X into finitely many pieces all of which can be mapped into Y by an injective coarse map [5, Theorem 1.2]. Although those results do not give injective coarse maps, their power comes from the fact that, for u.l.f. metric spaces, the existence of uniformly finite-to-one coarse maps $X \to Y$ often implies that X inherits large-scale geometric properties of Y—among them, we have property A, asymptotic dimension, and finite decomposition complexity [5, Corollary 1.3].

The current paper focuses on better understanding the form taken by embeddings $C^*_u(X) \to C^*_u(Y)$ and on obtaining results on geometry preservation which are "opposite" to the ones mentioned above. Precisely, we are interested in when the existence of an embedding $C^*_u(X) \to C^*_u(Y)$ can make Y inherit geometric properties of X (instead of X inheriting properties of Y). Obviously, we exclude the case in which the embedding $\Phi : C^*_u(X) \to C^*_u(Y)$ is an isomorphism from our range of interests. Indeed, if that is so, not only $\Phi^{-1} : C^*_u(Y) \to C^*_u(X)$ is an embedding and the previous results apply, but one can also obtain a coarse equivalence between X and Y under some geometric conditions (see [4, 21] for precise statements). We are then interested in non-isomorphic embeddings $C^*_u(X) \to C^*_u(Y)$ which have "large enough range"; forcing then the geometry of Y to be "controlled" by the one of X.

Before giving the definition of an embedding with "large enough range", we introduce the concept which motivated it. Recall that a surjective bounded linear map between Banach spaces is called a *quotient map*. By the open mapping theorem, the image of the unit ball under quotient maps contains a ball with positive radius. With that in mind, S. Zhang extended in [26] the concept of quotient maps to the (coarse) metric space category

[1]Recall that a map $f : X \to Y$ is *uniformly finite-to-one* if $\sup_{y\in Y} |f^{-1}(n)| < \infty$.

as follows: let X and Y be metric spaces and $f : X \to Y$. A coarse map f is a *coarse quotient map* if there is $K > 0$ such that for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$B\big(f(x), \varepsilon\big) \subset f\big(B(x, \delta)\big)^K$$

for all $x \in X$, where $f(B(x,\delta))^K$ is the K-neighborhood of $f(B(x,\delta))$.[2] Coarse quotients have been studied further in [2,27] and a weaker notion, called *weak coarse quotient map*, was studied in [12].

Notice that coarse quotients do not need to be coarse embeddings; not even if the map is a bijection. We refer to Section 2.4 for examples of coarse quotient maps. For the time being, a quick spoiler: if G is a finite group acting on a metric space X by coarse equivalences, then the canonical map from X to the orbit space X/G is a coarse quotient map.

As coarse quotients are objects in between coarse maps and coarse equivalences, there should be a diagram as the one in Figure 1 with coarse quotients appearing in the middle column. More precisely, as it is automatic from the definition that a coarse quotient $f : X \to Y$ is K-dense in Y (for K as above), there is not much loss in generality to look at *bijective* coarse quotients. The next definition introduces the terminology needed. Recall that the propagation of $a = [a_{xy}] \in \mathcal{B}(\ell_2(X))$ is defined as

$$\operatorname{prop}(a) = \sup\big\{d(x, y) \mid a_{xy} \neq 0\big\}.$$

Definition 1.1. Let X be a metric space and $A \subset \mathrm{C}^*_u(X)$ a C^*-subalgebra.

(1) Let $\varepsilon, k, \ell > 0$. An element $b \in \mathrm{C}^*_u(X)$ is *(ε, k, ℓ)-cobounded* if there are $a_1, \dots, a_\ell \in A$, with $\|a_i\| \leq \ell$, and $c_1, \dots, c_\ell \in \mathrm{C}^*_u(X)$, with $\operatorname{prop}(c_i) \leq k$ and $\|c_i\| \leq \ell$, so that

$$\Big\| b - \sum_{i=1}^{\ell} c_i a_i \Big\| \leq \varepsilon;$$

and b is *(ε, k)-cobounded* if it is (ε, k, ℓ)-cobounded for some ℓ.

(2) The subalgebra A is *cobounded in* $\mathrm{C}^*_u(X)$ if there is $k > 0$ so that every $b \in \mathrm{C}^*_u(X)$ is (ε, k)-cobounded for all $\varepsilon > 0$.

(3) The subalgebra A is *strongly-cobounded in* $\mathrm{C}^*_u(X)$ if there is $k > 0$ so that every contraction $b \in \mathrm{C}^*_u(X)$ is (ε, k, k)-cobounded for all $\varepsilon > 0$.

It is not hard to obtain nonisomorphic embeddings $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ whose images are strongly-cobounded in $\mathrm{C}^*_u(Y)$ and, in particular, cobounded (see Remark 4.3).

We now proceed to describe the main results of this paper. We start giving a characterization of the existence of bijective coarse quotients $X \to Y$ in terms of uniform Roe

[2]When extending quotient maps to metric spaces, the absence of linearity gives rise to two distinct approaches: extend the concept taking into account the (1) large-scale geometry of the metric spaces or (2) the small-scale geometry (i.e., its uniform structure). We deal with the former in this paper. For *uniform quotient maps*, see [1].

algebras. Notice that this characterization does not depend on extra geometric conditions on either X or Y.

Theorem 1.2. *Let X and Y be u.l.f. metric spaces. The following are equivalent:*

(1) *there is a bijective coarse quotient map $X \to Y$;*

(2) *there is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X)) = \ell_\infty(Y)$ and $\Phi(\mathrm{C}^*_u(X))$ is cobounded in $\mathrm{C}^*_u(Y)$.*

Theorem 1.2 should be compared with two known results: (1) X and Y are bijectively coarsely equivalent if and only if there is an isomorphism $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X)) = \ell_\infty(Y)$ [4, Theorem 8.1]; and (2) there is an injective coarse map $X \to Y$ if and only if there is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X)) \subset \ell_\infty(Y)$ (although not explicitly, this follows easily from [5, Theorem 4.3 and Lemma 5.3]).

While Theorem 1.2 gives a complete characterization of the existence of bijective coarse quotients, its hypothesis is too rigid. Indeed, the demand that $\Phi(\ell_\infty(X)) = \ell_\infty(Y)$ uses too much of the structure given by a choice of basis of $\ell_2(Y)$ (namely, its canonical orthonormal basis). Hence, it is interesting to obtain results under milder conditions on the embeddings.

We have the following in the presence of G. Yu's property A:

Theorem 1.3. *Let X and Y be u.l.f. metric spaces, and assume that Y has property A. If there is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X))$ is a Cartan subalgebra of $\mathrm{C}^*_u(Y)$ and $\Phi(\mathrm{C}^*_u(X))$ is strongly-cobounded in $\mathrm{C}^*_u(Y)$, then there is a bijective coarse quotient $X \to Y$.*

Roughly speaking, property A is needed in Theorem 1.3 for three reasons: (1) selecting a map $X \to Y$, (2) assuring the map can be taken to be a bijection, and (3) to guarantee that $(\mathrm{C}^*_u(Y), \Phi(\ell_\infty(X)))$ is a *Roe Cartan pair*. The former can actually be obtained under milder geometric assumptions on Y (see Theorem 1.4). As for the latter, recall that Roe Cartan pairs were introduced by S. White and R. Willett in [24] as follows: a subalgebra $B \subset \mathrm{C}^*_u(Y)$ is called *coseparable* if there is a countable $S \subset \mathrm{C}^*_u(Y)$ so that $\mathrm{C}^*_u(Y) = \mathrm{C}^*(B, S)$. Then $(\mathrm{C}^*_u(Y), B)$ is a *Roe Cartan pair* if B is a coseparable Cartan subalgebra of $\mathrm{C}^*_u(Y)$ which is isomorphic to $\ell_\infty(\mathbb{N})$ (see Section 2.2). It is open if a Cartan subalgebra of $\mathrm{C}^*_u(Y)$ isomorphic to $\ell_\infty(\mathbb{N})$ is automatically coseparable. However, it has been recently shown that this is indeed the case if Y has property A [6, Corollary 6.3].

We now describe a version of Theorem 1.3 which holds under weaker geometric conditions. A metric space (X, d) is *sparse* if there is a partition $X = \bigsqcup_n X_n$ into finite subsets so that $d(X_n, X_m) \to \infty$ as $n + m \to \infty$. Also, we say that X *yields only compact ghost projections* if all ghost projections in $\mathrm{C}^*_u(X)$ are compact.[3] The weaker geometric property which we look at is the one of *all sparse subspaces of X yielding only compact ghost*

[3] Recall, an operator $a = [a_{xy}] \in \mathcal{B}(\ell_2(X))$ is called a *ghost* if for all $\varepsilon > 0$ there is a finite $A \subset X$ so that $|a_{xy}| < \varepsilon$ for all $x, y \notin A$ (see Section 2.1 for details).

projections. This is a fairly broad property: indeed, it is not only implied by property A, but also by coarse embeddability into ℓ_2 and, more generally, by the validity of the coarse Baum–Connes conjecture with coefficients (see [4, Lemma 7.3] and [3, Theorem 5.3]).

The following is a version of Theorem 1.3 outside the scope of property A. Since property A is not assumed, we loose bijectivity and coseparability of the range is needed in the hypothesis.

Theorem 1.4. *Let X and Y be u.l.f. metric spaces, and assume that all sparse subspaces of Y yield only compact ghost projections. If there is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X))$ is a coseparable Cartan subalgebra of $\mathrm{C}^*_u(Y)$ and $\Phi(\mathrm{C}^*_u(X))$ is strongly-cobounded in $\mathrm{C}^*_u(Y)$, then there is a uniformly finite-to-one coarse quotient $X \to Y$.*

Theorem 1.4 can be applied to obtain restrictions to the geometry of Y. Without getting into details, we mention that the next result is a corollary of Theorem 1.4 and results on the "coarse version" of a function $X \to Y$ being n-to-1. Precisely, we show that a uniformly finite-to-one coarse quotient $X \to Y$ between u.l.f. metric spaces is automatically *coarsely n-to-1* for some $n \in \mathbb{N}$ (see Definition 3.2 and Proposition 3.3).

Corollary 1.5. *Let X and Y be u.l.f. metric spaces, and assume that all sparse subspaces of Y yield only compact ghost projections. Suppose that there is an embedding*

$$\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$$

so that $\Phi(\ell_\infty(X))$ is a coseparable Cartan subalgebra of $\mathrm{C}^*_u(Y)$ and $\Phi(\mathrm{C}^*_u(X))$ is strongly-cobounded in $\mathrm{C}^*_u(Y)$. Then, the following hold.

(1) *If X has finite asymptotic dimension, so does Y.*
(2) *If X has property A, so does Y.*
(3) *If X has asymptotic property C, so does Y.*
(4) *If X has straight finite decomposition complexity, so does Y.*

In Section 5, we provide formulas for strongly continuous embeddings between uniform Roe algebras (see Theorems 5.3 and 5.5). Precisely, all isomorphisms $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ between uniform Roe algebras are spacially implemented, i.e., there is a unitary $u : \ell_2(X) \to \ell_2(Y)$ so that $\Phi = \mathrm{Ad}(u)$ [21, Lemma 3.1]. This was later generalized for embeddings $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ onto a hereditary subalgebra of $\mathrm{C}^*_u(Y)$ [5, Lemma 6.1]. Notice that spacially implemented embeddings are automatically strongly continuous and rank-preserving. We generalize the two results above by showing that an embedding $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ is spacially implemented if and only if it is strongly continuous and rank-preserving; see Theorem 5.3 (two other characterizations of such embeddings in terms of compact operators are also given). Moreover, we prove an equivalent result for strongly continuous embeddings $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ which send rank 1 operators to rank n operators, where $n \in \mathbb{N} \cup \{\infty\}$; see Theorem 5.5.

We finish the paper stating some natural problems which are left open; see Section 6.

2. Preliminaries

2.1. Uniform Roe algebras

Given a Hilbert space H, $\mathcal{B}(H)$ denotes the C*-algebra of bounded operators on H and $\mathcal{K}(H)$ denotes its ideal of compact operators. Given a set X, $(\delta_x)_{x\in X}$ denotes the standard unit basis of the Hilbert space $\ell_2(X)$ and 1_X denotes the identity on $\ell_2(X)$. Given $x, y \in X$, e_{xy} denotes the partial isometry in $\mathcal{B}(\ell_2(X))$ so that

$$e_{xy}\delta_x = \delta_y \quad \text{and} \quad e_{xy}\delta_z = 0 \text{ for all } z \neq x.$$

Given $A \subset X$, we write $\chi_A = \text{SOT-}\sum_{x\in X} e_{xx}$; so $1_X = \chi_X$.

If (X, d) is a metric space, $a \in \mathcal{B}(\ell_2(X))$ and $r > 0$, we say that the *propagation of a is at most r*, and write $\text{prop}(a) \leq r$, if

$$d(x, y) > r \text{ implies } \langle a\delta_x, \delta_y\rangle = 0, \quad \text{for all } x, y \in X.$$

The *support of a* is defined by

$$\text{supp}(a) = \big\{(x, y) \in X \times X \mid \langle a\delta_x, \delta_y\rangle \neq 0\big\}.$$

Given a metric space (X, d), the subset of all operators in $\mathcal{B}(\ell_2(X))$ with finite propagation is a $*$-algebra called the *algebraic uniform Roe algebra of X*, denoted by $\mathrm{C}^*_u[X]$. The closure of $\mathrm{C}^*_u[X]$ is a C*-algebra called the *uniform Roe algebra of X*, denoted by $\mathrm{C}^*_u(X)$.

Instead of presenting the original definition of property A, we give a definition in terms of ghost operators which is better suited for our goals:

Definition 2.1. Let X be a uniformly locally finite metric space.

(1) An operator $a \in \mathcal{B}(\ell_2(X))$ is called a *ghost* if for all $\varepsilon > 0$ there is a finite $A \subset X$ so that

$$\big|\langle a\delta_x, \delta_y\rangle\big| \leq \varepsilon \quad \text{for all } x, y \in X \setminus A.$$

(2) The metric space X has *property A* if all ghost operators in $\mathrm{C}^*_u(X)$ are compact.[4]

We recall the notions of coarse-likeness introduced in [4].

Definition 2.2. Let X and Y be a metric space.

(1) Given $\varepsilon > 0$ and $r > 0$, an operator $a \in \mathcal{B}(\ell_2(X))$ can be *ε-r-approximated* (equivalently, a is *ε-r-approximable*) if there is $b \in \mathrm{C}^*_u(X)$ with $\text{prop}(b) \leq r$ so that $\|a - b\| \leq \varepsilon$.

(2) A map $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ is *coarse-like* if for all $\varepsilon, s > 0$ there is $r > 0$ so that $\Phi(a)$ can be ε-r-approximated for all contractions $a \in \mathrm{C}^*_u(X)$ with $\text{prop}(a) \leq s$.

[4] This is not the original definition of property A, but it is equivalent to it by [20, Theorem 1.3]. For its original definition, we refer the reader to [25, Definition 2.1].

The following is a simple consequence of [4, Lemma 4.9] (see [5, Proposition 3.3] for a precise proof; cf. [4, Theorem 4.4]).[5]

Theorem 2.3. *Let X and Y be uniformly locally finite metrics spaces and $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ a strongly continuous linear operator. Then Φ is coarse-like.*

2.2. Roe Cartan pairs

Recall, given a C*-algebra A, that a C*-subalgebra $B \subset A$ is called a *Cartan subalgebra of A* if

(1) B is a maximal abelian self-adjoint subalgebra of A,

(2) B contains an approximate unit for A,

(3) the normalizer of B in A, i.e., $\{a \in A \mid aBa^*, \subset a^*Ba \subset B\}$, generates A as a C*-algebra, and

(4) there is a faithful condition expectation $A \to B$.

If X is a u.l.f. metric space, then $\ell_\infty(X)$ is a Cartan subalgebra of $\mathrm{C}^*_u(X)$ [24, Proposition 4.10].

Let A be a unital C*-algebra and $B \subset A$ a Cartan subalgebra of A. We say that (A, B) is *Roe Cartan pair* if

(1) A contains the algebra of compact operators on a infinite dimensional Hilbert space as an essential ideal,

(2) B is isomorphic to the C*-algebra $\ell_\infty(\mathbb{N})$, and

(3) B is *co-separable* in A, i.e., there is a countable $S \subset A$ so that $A = \mathrm{C}^*(B, S)$.

Roe Cartan pairs were recently introduced to the literature in [24]. Notice that, as $\ell_\infty(X)$ is a Cartan subalgebra of $\mathrm{C}^*_u(X)$, it is clear that $(\mathrm{C}^*_u(X), \ell_\infty(X))$ is a Roe Cartan pair for any u.l.f. metric space X. Moreover, we notice that it is not known whether co-separability is a necessary property. Precisely, if $(\mathrm{C}^*_u(X), B)$ satisfies (2), it is not known whether B is automatically co-separable. By [6, Corollary 6.3], this is indeed the case if X has property A.

The importance of Roe Cartan pairs to our goals lies in the following theorem:

Theorem 2.4 ([24, Theorem B]). *Let (A, B) be a Roe Cartan pair. Then there is a u.l.f. metric space X and an isomorphism $\Phi : A \to \mathrm{C}^*_u(X)$ so that $\Phi(B) = \ell_\infty(X)$.*

2.3. Coarse geometry

Given a metric space (X, d), $x \in X$, and $\varepsilon > 0$, we denote by $B(x, \varepsilon)$ the closed unit ball centered at x of radius ε. Given a subset $A \subset X$ and $K > 0$, we write

$$A^K = \{x \in X \mid d(x, A) \le K\}.$$

[5]The hypothesis of [5, Proposition 3.33] actually demands Φ to be compact-preserving. However, this is so as its proof used an earlier version of [4, Lemma 4.9] which required compactness. The (newer) published version of [4, Lemma 4.9] does not do so, hence the proof of [5, Proposition 3.33] holds for noncompact preserving Φ's.

A metric space (X, d) is u.l.f. if $\sup_{x\in X} |B(x,r)| < \infty$ for all $r > 0$, where $|B(x,r)|$ denotes the cardinality of $B(x,r)$.

Let (X, d) and (Y, ∂) be metric spaces and $f : X \to Y$ a map. The *modulus of uniform continuity of f* is defined by

$$\omega_f(t) = \{\partial\big(f(x), f(y)\big) \mid d(x,y) \le t\}$$

for $t \ge 0$. Then f is called *coarse* if $\omega_f(t) < \infty$ for all $t \ge 0$. Given another map $g : X \to Y$, we say that f is *close* to g, and write $f \sim g$, if

$$\sup_{x\in X} \partial\big(f(x), g(x)\big) < \infty.$$

The map f is a *coarse equivalence* if f is coarse and there is a coarse map $h : Y \to X$ so that $f \circ h \sim \mathrm{Id}_Y$ and $h \circ f \sim \mathrm{Id}_X$.

Given $K > 0$, we say that f is *K-co-coarse* if for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$B\big(f(x), \varepsilon\big) \subset f\big(B(x, \delta)\big)^K$$

for all $x \in X$. The map f is *co-coarse* if it is *K-co-coarse* for some $K > 0$. If f is both coarse and co-coarse, f is a *coarse quotient*.

Proposition 2.5. *Let (X, d) and (Y, ∂) be metric spaces and $f, g : X \to Y$ close maps. If f is a coarse quotient, so is g.*

Proof. Coarseness is well known to be preserved under closeness. Moreover, if $m = \sup_{x\in X} \partial(f(x), g(x))$ and $K > 0$ is so that f is K-co-coarse, then it is straightforward to check that g is $(K + m)$-co-coarse. ∎

From now on, we assume throughout the paper that d and ∂ are the metrics on X and Y, respectively.

2.4. Examples of coarse quotients

Firstly, notice that every coarse equivalence is a coarse quotient map [26, Proposition 2.5]. But coarse equivalences are far from the only examples of coarse quotient maps. For instance, given $n, m \in \mathbb{N}$ with $n < m$, the standard projection $q : \mathbb{Z}^m \to \mathbb{Z}^n$ is clearly a coarse quotient map (and clearly not a coarse equivalence). The map $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2n$ for $n \in \mathbb{N}$ and $f(n) = -2n + 1$ for all $n \in \mathbb{Z} \setminus \mathbb{N}$ is also a coarse quotient. In this case, f is a bijective coarse quotient which is not a coarse embedding/equivalence. Similar constructions give us bijective coarse quotients $\mathbb{Z}^n \to \mathbb{N}^n$ for all $n \in \mathbb{N}$.

More generally, group actions give us natural examples of coarse quotient maps. Precisely, let (X, d) be a discrete metric space and G a group acting on X. We say that G acts on X *uniformly by coarse equivalences* if each $g \in G$ acts on X by a coarse equivalence and if there is a map $\omega : [0, \infty) \to [0, \infty)$ so that

$$d(g \cdot x, g \cdot y) \le \omega\big(d(x, y)\big)$$

for all $x, y \in X$ and all $g \in G$. If G is a finite group which acts on X by coarse equivalences, then it is automatic that G acts on X uniformly by coarse equivalences.

Given $G \curvearrowright X$, denote the orbit space of this action by X/G, i.e., define an equivalence $\sim_G$ on X by letting $x \sim_G y$ if there is $g \in G$ with $g \cdot y = x$ and X/G is the set of $\sim_G$-equivalence classes. The coarse structure of X induces a coarse structure on X/G. Precisely, we can endow X/G with the metric

$$\partial\big([x],[y]\big) = \min\Big\{ \sup_{x' \in [x]} \inf_{y' \in [y]} d(x', y'), \sup_{y' \in [y]} \inf_{x' \in [x]} d(x', y') \Big\}$$

for all $[x], [y] \in X/G$. As X is discrete, ∂ is clearly a metric on X/G. Let $q : X \to X/G$ be the quotient map. If G acts on X uniformly by coarse equivalences, then it is straightforward to check that q is a coarse quotient map. The coarse geometry of those spaces was studied in [12].

3. Coarse quotients and geometry preservation

Coarse properties are those mathematical properties of metric spaces which are preserved under coarse equivalence. Some large-scale properties are also stable under coarse embeddings, i.e., if X coarsely embeds into Y and Y has such property, then so does X. Moreover, for u.l.f. metric spaces, it is also known that some large-scale properties are stable under the existence of uniformly finite-to-one coarse maps—for instance, this holds for property A, asymptotic dimension, and finite dimension decomposition; see [5, Proof of Corollary 1.3, Proposition 2.5, and Corollary 5.8].

In this section, we show that the "opposite" phenomena can happen for uniformly finite-to-one coarse *quotient* maps $X \to Y$. Precisely, we show that the existence of such maps is often enough so that large-scale geometric properties of X pass to Y. This is the case for finite asymptotic dimension, straight finite decomposition complexity, asymptotic property C, and property A (Corollary 3.7).

We start noticing that compositions of coarse quotients are coarse quotients. This is essentially done in [26, Proposition 2.5], but we include a short proof for the reader's convenience.

Proposition 3.1. *Let X, Y, and Z be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be coarse quotient maps. Then $g \circ f$ is a coarse quotient. In particular, $g \circ f$ is a coarse quotient map given that f is a coarse quotient map and g is a coarse equivalence.*

Proof. If both f and g are coarse quotient maps, then both are coarse and so is their composition. We are left to show that $g \circ f$ is co-coarse. For that, fix $K > 0$ so that f and g are K-co-coarse. Given $\varepsilon > 0$, let $\delta > 0$ be so that $B(g(y), \varepsilon) \subset g(B(y, \delta))^K$ for all $y \in Y$, and let $\gamma > 0$ be so that $B(f(x), \delta) \subset f(B(x, \gamma))^K$ for all $x \in X$. Then, for $L = K + \omega_g(K)$, we have

$$B\big(g \circ f(x), \varepsilon\big) \subset g\big(B\big(f(x), \delta\big)\big)^K \subset g\big(f\big(B(y, \gamma)\big)^K\big)^K \subset g\big(f\big(B(y, \gamma)\big)\big)^L$$

for all $x \in X$. Hence the assignment $\varepsilon \mapsto \gamma$ witness that $g \circ f$ is L-co-coarse. ∎

Our main tool in order to obtain coarse geometry preservation is based on the next concept. This was introduced in [14, Section 3.2] as property "B_n" and it is the "coarse version" of a function being n-to-1.

Definition 3.2. Let X and Y be metric spaces. Given $f : X \to Y$ and $n \in \mathbb{N}$, we say that f is *coarsely n-to-1* if for each $s > 0$ there is $r > 0$ so that for all $B \subset Y$, with $\mathrm{diam}(B) \leq s$, there are $A_1, \ldots, A_n \subset X$, with $\mathrm{diam}(A_i) \leq r$ for all $i \in \{1, \ldots, n\}$, so that $f^{-1}(B) \subset \bigcup_{i=1}^{n} A_i$. The map $f : X \to Y$ is called *uniformly coarsely finite-to-one* if f is coarsely n-to-1 for some $n \in \mathbb{N}$.

The next proposition is the main result of this section.

Proposition 3.3. *Let X be a metric space and Y a u.l.f. metric space. Any uniformly finite-to-one coarse quotient map $X \to Y$ is uniformly coarsely finite-to-one.*

We need a couple of preliminary results before proving Proposition 3.3. We start by noticing that "coarsely n-to-1" is a coarse property for maps between metric spaces.

Proposition 3.4. *Let X and Y be metric spaces and let $f, g : X \to Y$ be close maps. Given $n \in \mathbb{N}$, if f is coarsely n-to-1, so is g.*

Proof. As f and g are close, let $k = \sup_{x \in X} \partial(f(x), g(x)) < \infty$. Given $s > 0$, let $r > 0$ be as in Definition 3.2 for $s + 2k$, f, and n. Fix $B \subset Y$ with $\mathrm{diam}(B) \leq s$. As $\mathrm{diam}(B^k) \leq s + 2k$, our choice of r gives $A_1, \ldots, A_n \subset X$, with $\mathrm{diam}(A_i) \leq r$ for all $i \in \{1, \ldots, n\}$, so that $f^{-1}(B^k) \subset \bigcup_{i=1}^{n} A_i$. Then $g^{-1}(B) \subset \bigcup_{i=1}^{n} A_i$. ■

Lemma 3.5. *Let X be a metric space and Y a u.l.f. metric space. Any injective coarse quotient $X \to Y$ is uniformly coarsely finite-to-one.*

Proof. Let $f : X \to Y$ be an injective coarse quotient map. Without loss of generality, assume that f is surjective. Fix $K > 0$ so that f is K-co-coarse and let $m = \sup_{y \in Y} |B(y, K)|$. Let us show that f is coarsely m-to-1. Fix $\varepsilon > 0$. By our choice of K, there is $\delta = \delta(\varepsilon) > 0$ so that

$$B\big(f(x), 2\varepsilon\big) \subset f\big(B(x, \delta)\big)^K$$

for all $x \in X$.

Fix $y \in Y$. We construct a finite sequence $(x_i)_i$ of elements of X by induction as follows. Pick $x_1 \in X$ so that $f(x_1) \in B(y, \varepsilon)$. Let $\ell \in \mathbb{N}$ and assume that $x_1, \ldots, x_\ell \in X$ have been chosen. If

$$B(y, \varepsilon) \subset \bigcup_{i=1}^{\ell} f\big(B(x_i, 3\delta)\big),$$

we stop the procedure and $(x_i)_{i=1}^{\ell}$ is the outcome of it. If not, then pick $x_{\ell+1} \in X$ so that

$$f(x_{\ell+1}) \in B(y, \varepsilon) \setminus \bigcup_{i=1}^{\ell} f\big(B(x_i, 3\delta)\big).$$

This completes the induction. As $B(y,\varepsilon)$ contains finitely many elements, this procedure is finite.

Let $(x_i)_{i=1}^{\ell}$ be the finite sequence obtained by the procedure above. Let us show that $\ell \leq m$. Assume by contradiction that $\ell > m$ and let $z = f(x_{m+1})$. As each $f(x_i)$ belongs to $B(y,\varepsilon)$, we have that $\partial(z, f(x_i)) \leq 2\varepsilon$ for all $i \in \{1,\ldots,\ell\}$. Hence, by our choice of δ, there is a finite sequence $(w_i)_{i=1}^{m}$ in X so that $d(x_i, w_i) \leq \delta$ and $\partial(z, f(w_i)) \leq K$ for all $i \in \{1,\ldots,m\}$. By the construction of $(x_i)_{i=1}^{\ell}$, $f(x_i) \notin f(B(x_j, 3\delta))$ for $j < i$. Hence, $d(x_i, x_j) > 3\delta$ for all $i \neq j$, which implies that $(w_i)_{i=1}^{m}$ is a distinct sequence. As f is injective, $(f(w_i))_{i=1}^{m}$ is a distinct sequence. As

$$z \notin \bigcup_{i=1}^{m} f\big(B(x_i, 3\delta)\big),$$

$\{z, f(w_1), \ldots, f(w_m)\}$ is a subset of $B(z,K)$ with $m+1$ elements. This contradicts our choice of m.

As $\ell = m$, $B(y,\varepsilon) \subset \bigcup_{i=1}^{m} f(B(x_i, 3\delta))$. As f is injective, this implies that

$$f^{-1}\big(B(y,\varepsilon)\big) \subset \bigcup_{i=1}^{m} B\big(x_i, 3\delta\big).$$

As y was arbitrary, the assignment $\varepsilon \mapsto 3\delta(\varepsilon)$ witnesses that f is a coarsely m-to-1 map. ∎

Lemma 3.6. *Let X and Y be metric spaces, and let $f : X \to Y$ be a uniformly finite-to-one map. If Y is u.l.f., then there is a u.l.f. metric space Z, with $Y \subset Z$, and an injective map $g : X \to Z$ which is close to f. Moreover, the inclusion $Y \hookrightarrow Z$ is a coarse equivalence.*

Proof. Let $n = \sup_{y\in Y} |f^{-1}(y)|$. Let $Z = Y \times \{1,\ldots,n\}$ and define a metric ∂_Z on Z by letting

$$\partial_Z\big((y,i),(z,j)\big) = \partial(y,z) + 1$$

for all distinct $(y,i),(z,j) \in Z$. As (Y,∂) is u.l.f., so is (Z,∂_Z). For each $y \in Y$, enumerate $f^{-1}(y)$, say $f^{-1}(y) = \{x_1^y, \ldots, x_{i(y)}^y\}$. As $X = \bigsqcup_{y\in Y}\{x_1^y,\ldots,x_{i(y)}^y\}$, we can define a map $g : X \to Z$ by letting $g(x_j^y) = (y,j)$ for all $y \in Y$ and all $j \in \{1,\ldots,i(y)\}$. It is clear that f is close to g and that $Y \times \{1\} \hookrightarrow Z$ is a coarse equivalence. By identifying Y with $Y \times \{1\}$, we can assume that $Y \subset Z$. ∎

Proof of Proposition 3.3. Let $f : X \to Y$ be a uniformly finite-to-one coarse quotient map. Let Z and g given Lemma 3.6 be applied to $f : X \to Y$. As the inclusion $i : Y \hookrightarrow Z$ is a coarse equivalence, $f = i \circ f : X \to Z$ is a coarse quotient map (Proposition 3.1). Hence, as f is close to g, g is a coarse quotient map (Proposition 2.5). As g is injective, Lemma 3.5 gives us that g is uniformly coarsely finite-to-one. Using that f and g are close to each other once again, this shows that f is uniformly coarsely finite-to-one (Proposition 3.4). ∎

We can now use Proposition 3.3 in order to obtain that some coarse properties of X pass to Y in the presence of a uniformly finite-to-one coarse quotient map $X \to Y$. For brevity, we do not introduce the geometric properties mentioned in the corollary below. Instead, we refer the reader to [11, Section 1.E] for the definition of asymptotic dimension, [16, Definitions 2.7.7] for the definition of asymptotic property C, and [7, Definition 2.2] for the definition of straight finite decomposition (property A has been defined in Definition 2.1).

Corollary 3.7. *Let X and Y be metric spaces and assume that Y is u.l.f. and that there is a uniformly finite-to-one coarse quotient map $X \to Y$. The following holds.*

1. *If X has finite asymptotic dimension, so does Y.*
2. *If X has property A, so does Y.*
3. *If X has asymptotic property C, so does Y.*
4. *If X has straight finite decomposition complexity, so does Y.*

Proof. (1) This follows from Proposition 3.3 and [14, Theorem 1.4].

(2) This follows from Proposition 3.3 and [9, Corollary 7.5].

(3) This follows from Proposition 3.3 and [9, Theorem 6.2].

(4) This follows from Proposition 3.3 and [9, Theorems 8.4 and 8.7]. ■

4. Cobounded embeddings between uniform Roe algebras

In this section, we study embeddings into uniform Roe algebras whose images are "large". We obtain that such embeddings can often be enough to guarantee the existence of coarse quotient maps between the spaces. Theorems 1.2, 1.3, 1.4, and Corollary 1.5 are proven in this section.

The next proposition is well known and its proof can be found for instance in [4, Proposition 2.4]. Recall, given $K > 0$, that a subset A of a metric space X is called K-*separated* if $d(x, y) \geq K$ for all distinct $x, y \in A$.

Proposition 4.1. *Let X be a u.l.f. metric space. Given any $K > 0$, there is $n \in \mathbb{N}$ and a partition*

$$X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$$

so that each X_i is K-separated.

Proposition 4.2. *Let X and Y be u.l.f. metric spaces, and let $f : X \to Y$ be an injective map so that $f : X \to f(X)$ is a coarse quotient map. Then there is a spacially implemented embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X)) = \ell_\infty(f[X])$ and $\Phi(\mathrm{C}^*_u(X))$ is cobounded in $\mathrm{C}^*_u(f[X])$.*

Proof. Let $u_f : \ell_2(X) \to \ell_2(Y)$ be the isometric embedding given by $u_f \delta_x = \delta_{f(x)}$ for all $x \in X$. Then it is easy to see that $\Phi = \mathrm{Ad}(u_f) : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ is an embedding (see [5, Theorem 1.2] for a detailed proof). Since it is clear that $\Phi(\ell_\infty(X)) = \ell_\infty(f[X])$, we only need to notice that $\Phi(\mathrm{C}^*_u(X))$ is cobounded in $\mathrm{C}^*_u(f[X])$.

Fix $K > 0$ so that f is K-co-bounded and let $Z = f[X]$. Fix $\varepsilon > 0$ and let $a \in \mathrm{C}^*_u[Z]$ with $\mathrm{prop}(a) \le \varepsilon$. Without loss of generality, assume that $\varepsilon > K$. By our choice of K, there is $\delta > 0$ so that

$$B\big(f(x), \varepsilon\big) \subset f\big(B(x, \delta)\big)^K$$

for all $x \in X$. As Y is u.l.f., there is $n = n(\varepsilon, Y) \in \mathbb{N}$ and a partition $Z = \bigsqcup_{i=1}^n Z_i$ so that each Z_i is 3ε-separated (Proposition 4.1). For each $(i, j) \in \{1, \dots, n\}^2$, let $a(i, j) = \chi_{Z_i} a \chi_{Z_j}$.

Fix $i, j \in \{1, \dots, n\}$. For simplicity, let $b = a(i, j)$. Clearly, $\mathrm{prop}(b) \le \mathrm{prop}(a) \le \varepsilon$. Hence, as Z_i and Z_j are 3ε-separated, there are 3ε-separated sequences $(y_n)_n$ and $(y'_n)_n$ in Z so that

$$b = \sum_{n\in\mathbb{N}} b_n e_{y_n y'_n}, \quad \text{where } b_n = \langle b\delta_{y_n}, \delta_{y'_n}\rangle \text{ for all } n \in \mathbb{N}.$$

In particular, $\partial(y_n, y'_n) \le \varepsilon$ for all $n \in \mathbb{N}$. As $f : X \to Z$ is bijective, fix a sequence $(x_n)_n$ of distinct elements in X so that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. By our choice of δ, for each $n \in \mathbb{N}$, there are $z_n \in X$ so that $d(x_n, z_n) \le \delta$ and $\partial(f(z_n), y'_n) \le K$. As $(y'_n)_n$ is 3ε-separated and $K < \varepsilon$, it follows that $(f(z_n))_n$ is a sequence of distinct elements and, as f is injective, so is $(z_n)_n$. In particular,

$$c = \mathrm{SOT}\text{-}\sum_{n\in\mathbb{N}} b_n e_{x_n z_n}$$

is well defined and, as $\mathrm{prop}(c) \le \delta$, $c \in \mathrm{C}^*_u(X)$. Similarly, $d = \mathrm{SOT}\text{-}\sum_{n\in\mathbb{N}} e_{f(z_n) y'_n}$ is also well defined and it has propagation at most K.

Notice that $b = d\Phi(c)$. As $\varepsilon > 0$ and $i, j \in \{1, \dots, n\}$ were arbitrary, we are done. ■

Remark 4.3. Notice that if $f : \mathbb{Z} \to \mathbb{N}$ is the bijective coarse quotient given by $f(n) = 2n$ for $n \in \mathbb{N}$ and $f(n) = -2n + 1$ for $n \in \mathbb{Z} \setminus \mathbb{N}$, then $\Phi = \mathrm{Ad}(u_f)$ is actually strongly-cobounded. Indeed, let I and P denote the odd and even natural numbers, respectively. Then any $a \in \mathrm{C}^*_u(\mathbb{N})$ can be written as

$$a = \chi_I a \chi_I + \chi_P a \chi_P + \chi_P a \chi_I + \chi_I a \chi_P.$$

Clearly, $\chi_I a \chi_I, \chi_P a \chi_P \in \Phi(\mathrm{C}^*_u(\mathbb{Z}))$. Let $g : P \to I$ be the bijection given by $g(n) = n - 1$ for all $n \in P$, and let $c = \mathrm{SOT}\text{-}\sum_{n\in P} e_{ng(n)}$; so $\mathrm{prop}(c) = 1$. Moreover, it is clear that $c\chi_P a \chi_I, c^*\chi_I a \chi_P \in \Phi(\mathrm{C}^*_u(\mathbb{Z}))$. As $\chi_P a \chi_I = c^* c \chi_P a \chi_I$ and $\chi_I a \chi_P = cc^* \chi_P a \chi_I$, it easily follows that $\Phi(\mathrm{C}^*_u(\mathbb{Z}))$ is strongly-cobounded in $\mathrm{C}^*_u(\mathbb{N})$.

We do not know if an arbitrary bijective coarse quotient $X \to Y$ is enough to produce an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ with strongly-cobounded range (see Problem 6.3).

For the next technical lemma, we introduce a weakening of Definition 1.1.

Definition 4.4. Let X be a metric space and let $A_1 \subset A_2$ be C*-subalgebras of $\mathrm{C}^*_u(X)$. Given $\varepsilon > 0$, the algebra A_1 is called *ε-almost cobounded in A_2* if there is $k > 0$ so that for all $b \in A_2$ there are $a_1, \dots, a_\ell \in A_1$ and $c_1, \dots, c_\ell \in A_2$, with $\mathrm{prop}(c_i) \le k$, so that

$$\Big\| b - \sum_{i=1}^{\ell} c_i a_i \Big\| \le \varepsilon.$$

Lemma 4.5. *Let X and Y be u.l.f. metric spaces, and let $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ be an embedding so that $\Phi(\ell_\infty(X)) \subset \ell_\infty(Y)$ and $\Phi(e_{xx})$ has rank 1 for all $x \in X$. Let $Z \subset Y$ be a subset so that*

$$\chi_Z = \mathrm{SOT}\text{-}\sum_{x \in X} \Phi(e_{xx}).$$

Assume that $\Phi(\mathrm{C}^*_u(X))$ is ε-almost cobounded in $\chi_Z \mathrm{C}^*_u(Y) \chi_Z$ for some $\varepsilon \in (0, 1)$. Then there exists a bijective coarse quotient map $X \to Z$.

Proof. As Φ is a $*$-homomorphism, the hypothesis implies that each $\Phi(e_{xx})$ is a projection of rank 1 in $\ell_\infty(Z)$. Hence, for each $x \in X$ there is $y_x \in Z$ so that $\Phi(e_{xx}) = e_{y_x y_x}$. Define a map $f : X \to Y$ by letting $f(x) = y_x$ for each $x \in X$. Notice that $Z = f(X)$. Also, the map f is clearly injective, and, by [5, Lemma 5.3], it is also coarse.[6]

We now show that $f : X \to Z$ is co-coarse. For that, fix $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}$ which witness that $\Phi(\mathrm{C}^*_u(X))$ is ε-almost cobounded in $\chi_Z \mathrm{C}^*_u(Y) \chi_Z$. Assume for a contradiction that f is not k-co-coarse. Hence, there exists $\gamma > 0$ and sequences $(x_n)_n$ and $(y_n)_n$ in X so that

(1) $\partial(f(x_n), f(y_n)) \le \gamma$ for all $n \in \mathbb{N}$, and

(2) for all $z \in X$, $\partial(f(y_n), f(z)) \le k$ implies that $d(x_n, z) \ge n$.

Without loss of generality, by going to a subsequence, we can assume that $f(x_n) \neq f(x_m)$ and $f(y_n) \neq f(y_m)$ for all $n \neq m$. Indeed, if this is not the case, then there is an infinite $N \subset \mathbb{N}$ so that either $f(x_n) = f(x_m)$, for all $n, m \in N$, or $f(y_n) = f(y_m)$, for all $n, m \in N$. Then (1) and u.l.f.-ness of Y imply that, going to a further infinite subset of N if necessary, we can assume that $f(x_n) = f(x_m)$ and $f(y_n) = f(y_m)$, for all $n, m \in N$. As f is injective, there is $x, y \in X$ so that $x = x_n$ and $y = y_n$ for all $n \in N$. However, (2) implies that $d(x, y) = d(x_n, y_n) \ge n$ for all $n \in N$; contradiction.

As $f(x_n) \neq f(x_m)$ and $f(y_n) \neq f(y_m)$ for all $n \neq m$ in N, (1) above implies that $\sum_{n \in N} e_{f(x_n) f(y_n)}$ converges in the strong operator topology and it belongs to $\mathrm{C}^*_u(Y)$. By our choice of k, there are $a_1, \dots, a_\ell \in \mathrm{C}^*_u(X)$ and $c_1, \dots, c_\ell \in \chi_Z \mathrm{C}^*_u(Y) \chi_Z$, with $\mathrm{prop}(c_i) \le k$, so that

$$\Big\| \chi_Z \Big(\sum_{n \in N} e_{f(x_n) f(y_n)} \Big) \chi_Z - \sum_{i=1}^{\ell} c_i \Phi(a_i) \Big\| \le \varepsilon.$$

[6]Notice that, although not explicit in the statement, the only assumption needed in order for [5, Lemma 5.3] to hold for Φ is that $\Phi(e_{xx})$ has rank 1 for all $x \in X$.

Hence, we have that, for all $n \in N$,

$$\begin{aligned}
&\Big\| e_{f(y_n)f(y_n)} \Big(\sum_{i=1}^{\ell} c_i \Phi(a_i) \Big) e_{f(x_n)f(x_n)} \Big\| \\
&\quad \geq \Big\| e_{f(y_n)f(y_n)} \chi Z \Big(\sum_{i \in \mathbb{N}} e_{f(x_i)f(y_i)} \Big) \chi Z e_{f(x_n)f(x_n)} \Big\| - \varepsilon \\
&\quad = \| e_{f(x_n)f(y_n)} \| - \varepsilon \\
&\quad = 1 - \varepsilon.
\end{aligned}$$

For each $n \in N$, let $B_n = B(f(y_n), k) \cap Z$ and let $A_n = f^{-1}(B_n)$. As each c_i has propagation at most k and $\mathrm{supp}(c_i) \subset Z \times Z$, we have that $e_{f(y_n)f(y_n)} c_i = e_{f(y_n)f(y_n)} c_i \chi_{B_n}$ for all $n \in \mathbb{N}$ and all $i \in \{1, \ldots, \ell\}$. Therefore, as $\Phi(\chi_{A_n}) = \chi_{B_n}$, we have that

$$\begin{aligned}
&\Big\| e_{f(y_n)f(y_n)} \Big(\sum_{i=1}^{\ell} c_i \Phi(a_i) \Big) e_{f(x_n)f(x_n)} \Big\| \\
&\quad = \Big\| \sum_{i=1}^{\ell} e_{f(y_n)f(y_n)} c_i \chi_{B_n} \Phi(a_i) e_{f(x_n)f(x_n)} \Big\| \\
&\quad = \Big\| \sum_{i=1}^{\ell} e_{f(y_n)f(y_n)} c_i \Phi(\chi_{A_n}) \Phi(a_i) \Phi(e_{x_n x_n}) \Big\| \\
&\quad \leq \max_i \|c_i\| \sum_{i=1}^{\ell} \|\chi_{A_n} a_i e_{x_n x_n}\|
\end{aligned}$$

for all $n \in N$. By going to a subsequence, a simple pigeonhole argument allows us to assume that, for some $i \in \{1, \ldots, \ell\}$, we have

$$\inf_{n \in \mathbb{N}} \|\chi_{A_n} a_i e_{x_n x_n}\| > 0.$$

By the definition of $(A_n)_n$ and (2) above, we have that $\lim_n d(x_n, A_n) = \infty$. This, together with the previous inequality, contradicts the fact that $a_i \in \mathrm{C}^*_u(X)$. ■

Theorem 4.6. *Let X and Y be uniformly locally finite metric spaces. The following are equivalent.*

1. *There is a bijective coarse quotient map $X \to Y$.*
2. *There is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X)) = \ell_\infty(Y)$ and $\Phi(\mathrm{C}^*_u(X))$ is cobounded in $\mathrm{C}^*_u(Y)$.*
3. *There is $\varepsilon \in (0, 1)$ and an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ so that $\Phi(\ell_\infty(X)) = \ell_\infty(Y)$ and $\Phi(\mathrm{C}^*_u(X))$ is ε-almost cobounded in $\mathrm{C}^*_u(Y)$.*

Proof. (1)⇒(2) This follows from Proposition 4.2.

(2)⇒(3) This is straightforward.

(3)⇒(1) As Φ is a $*$-homomorphism, each $\Phi(e_{xx})$ is a projection. Hence, as the restriction $\Phi \restriction \ell_\infty(X) : \ell_\infty(X) \to \ell_\infty(Y)$ is an isomorphism, $\Phi(e_{xx})$ has rank 1 for each $x \in X$. Moreover, this isomorphism also implies that $1_Y = \text{SOT-}\sum_{x\in X} \Phi(e_{xx})$. So, the result now follows from Lemma 4.5. ∎

Proof of Theorem 1.2. This follows from Theorem 4.6. ∎

We are ready to prove Theorems 1.3 and 1.4. Since our proofs give us slightly stronger results than the ones stated in Section 1, we need the following more general version of Definition 1.1 (3).

Definition 4.7. Let X be a metric space and let $A_1 \subset A_2$ be a C^*-subalgebra of $\mathrm{C}^*_u(X)$. We say that A_1 is *strongly-cobounded in* A_2 if there is $k \in \mathbb{N}$ so that for all $\varepsilon > 0$ and all contractions $b \in A_2$, there are $a_1, \ldots, a_k \in A_1$, with $\|a_i\| \le k$, and $c_1, \ldots, c_k \in A_2$, with $\mathrm{prop}(c_i) \le k$ and $\|c_i\| \le k$, so that

$$\left\| b - \sum_{i=1}^{k} c_i a_i \right\| \le \varepsilon.$$

We now prove the main theorem of this section. Theorems 1.3 and 1.4 will follow from it.

Theorem 4.8. *Let X and Y be u.l.f. metric spaces, and assume that Y has property A. Suppose that there is an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ and a hereditary C^*-subalgebra A of $\mathrm{C}^*_u(Y)$ so that $(A, \Phi(\ell_\infty(X)))$ is a Roe Cartan pair and $\Phi(\mathrm{C}^*_u(X))$ is strongly-cobounded in A. Then there is $Z \subset Y$ and a bijective coarse quotient $X \to Z$.*

Proof. Fix $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ and $A \subset \mathrm{C}^*_u(Y)$ as above. By hypothesis, $(A, \Phi(\ell_\infty(X)))$ is a Roe Cartan pair, so Theorem 2.4 implies that there is a u.l.f. metric space Z and an isomorphism $\Psi : \mathrm{C}^*_u(Z) \to A$ so that $\Psi(\ell_\infty(Z)) = \Phi(\ell_\infty(X))$. Let $\Theta = \Psi^{-1} \circ \Phi$, so $\Theta : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Z)$ is an embedding so that $\Theta(\ell_\infty(X)) = \ell_\infty(Z)$.

Claim 4.9. Given $\varepsilon > 0$, $\Theta(\mathrm{C}^*_u(X))$ is ε-almost cobounded in $\mathrm{C}^*_u(Z)$.

Proof. Let $\varepsilon > 0$. Fix $k \in \mathbb{N}$ which witness that $\Phi(\mathrm{C}^*_u(X))$ is strongly-cobounded in A and let $\delta = \varepsilon/(k^3 + 1)$. As A is a hereditary subalgebra of $\mathrm{C}^*_u(Y)$, [5, Lemma 6.1] gives an isometric embedding $u : \ell_2(Z) \to \ell_2(Y)$ so that $\Psi = \mathrm{Ad}(u)$. As $\mathrm{Ad}(u^*) : \mathrm{C}^*_u(Y) \to \mathrm{C}^*_u(Z)$ is compact-preserving and strongly continuous, Theorem 2.3 implies that $\mathrm{Ad}(u^*)$ is coarse-like. Notice that $\Psi^{-1} = \mathrm{Ad}(u^*) \restriction A$. So, coarse-likeness gives $R > 0$ so that $\Psi^{-1}(c)$ is δ-R-approximable for all contractions $c \in A$ with $\mathrm{prop}(c) \le k$.

Let $b \in \mathrm{C}^*_u(Z)$. By our choice of k, there are $a_1, \ldots, a_k \in \mathrm{C}^*_u(X)$, with $\|a_i\| \le k$, and $c_1, \ldots, c_k \in A$, with $\mathrm{prop}(c_i) \le k$ and $\|c_i\| \le k$, and so that

$$\left\| \Psi(b) - \sum_{i=1}^{k} c_i \Phi(a_i) \right\| \le \delta.$$

For each $i \in \{1, \dots, k\}$, let $d_i \in \mathrm{C}^*_u(Z)$ be so that $\mathrm{prop}(d_i) \le R$ and

$$\big\|\Psi^{-1}(c_i) - \|c_i\| d_i\big\| \le \delta \|c_i\|.$$

Then

$$\begin{aligned}
&\Big\|b - \sum_{i=1}^{k} \|c_i\| d_i \Theta(a_i)\Big\| \\
&\quad \le \Big\|b - \sum_{i=1}^{k} \Psi^{-1}(c_i \Phi(a_i))\Big\| + \Big\|\sum_{i=1}^{k} \Psi^{-1}(c_i)\Theta(a_i) - \sum_{i=1}^{k} \|c_i\| d_i \Theta(a_i)\Big\| \\
&\quad \le \Big\|\Psi(b) - \sum_{i=1}^{k} c_i \Phi(a_i)\Big\| + \sum_{i=1}^{k} \big\|\Psi^{-1}(c_i) - \|c_i\| d_i\big\| \cdot \|a_i\| \\
&\quad \le \delta + \delta k^3.
\end{aligned}$$

By our choice of δ, we are done. ■

As $\Theta(\ell_\infty(X)) = \ell_\infty(Z)$, the previous claim and Theorem 4.6 imply that there is a bijective coarse quotient map $f : X \to Z$. As Y has property A and A is a hereditary subalgebra of $\mathrm{C}^*_u(Y)$, [5, Theorem 1.4] gives us an injective coarse embedding $g : Z \to Y$. By Proposition 3.1, $g \circ f : X \to g(Z)$ is a coarse quotient. So we are done. ■

Proof of Theorem 1.3. If Y has property A, then $\mathrm{C}^*_u(Z)$ is isomorphic to $\mathrm{C}^*_u(Y)$ if and only if Z and Y are bijectively coarsely equivalent [24, Corollary 6.13]. Therefore, the result follows by running the proof of Theorem 4.8 for $A = \mathrm{C}^*_u(Y)$, and using [24, Corollary 6.13] instead of [5, Theorem 1.4] in the last paragraph of the proof. Indeed, the map $g : Z \to Y$ obtained at the end of the proof of Theorem 4.8 becomes a bijection, and so does the coarse quotient $g \circ f : X \to Y$. ■

Proof of Theorem 1.4. If the sparse subspaces of Y yield only compact ghost projections, then an isomorphism between $\mathrm{C}^*_u(Z)$ and $\mathrm{C}^*_u(Y)$ implies that Y and Z are coarsely equivalent (this follows form the proof of [5, Theorem 1.4 and Corollary 1.5]; equivalently, this is given by [3, Theorem 1.3]). The result then follows by running the proof of Theorem 4.8 for $A = \mathrm{C}^*_u(Y)$ and the result above instead of [5, Theorem 1.4] in the last paragraph of the proof. ■

Proof of Corollary 1.5. This follows straightforwardly from Theorem 1.4 and Corollary 3.7. ■

5. Characterization of spacially implemented embeddings

The study of embeddings between uniform Roe algebras is highly dependent on the embeddings being spacially implemented or strongly continuous and rank-preserving. In this section, we prove Theorem 5.3 and show that those two kinds of embedding coincide. We also give two other characteristics of such embeddings and prove a version of it for non-rank-preserving strongly continuous embeddings (see Theorem 5.5).

For that, we need a result of [5]. For that, recall that given a metric space X, an operator $a \in \mathcal{B}(\ell_2(X))$ is called *quasi-local* if for all $\varepsilon > 0$ there is $S > 0$ so that $d(A, B) > S$ implies that $\|\chi_A a \chi_B\| \leq \varepsilon$ for all $A, B \subset X$. The norm closure of all quasi-local operators forms a C^*-algebra called the *quasi-local algebra of* X, and this algebra is denoted by $\mathrm{C}^*_{ql}(X)$. Clearly, $\mathrm{C}^*_u(X) \subset \mathrm{C}^*_{ql}(X)$ and it remains open whether this inclusion is an equality; when this is so, the metric space X is called *quasi-local*. We know however that this is so if X has property A [23, Theorem 3.3].

Before stating [5, Theorem 4.3], notice that, if $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ is an embedding, then $(\Phi(\chi_F))_{F \subset X, |F| < \infty}$ is an increasing net of projections. Therefore,

$$p = \mathrm{SOT}\text{-}\sum_{x \in X} \Phi(e_{xx})$$

is well defined.

Theorem 5.1 ([5, Theorem 4.3]). *Let X and Y be u.l.f. metric spaces and $\Phi \colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ an embedding. Then, the projection $p = \mathrm{SOT}\text{-}\sum_{x \in X} \Phi(e_{xx})$ belong to $\mathrm{C}^*_{ql}(Y)$ and the map*

$$a \in \mathrm{C}^*_u(X) \mapsto p\Phi(a)p \in \mathrm{C}^*_{ql}(Y)$$

is a strongly continuous embedding. Moreover, if Φ is compact-preserving, then $p \in \mathrm{C}^*_u(Y)$; so the map above is a strongly continuous embedding into $\mathrm{C}^*_u(Y)$.

Theorem 5.1 allows us to obtain the following characterization of strong continuity of embeddings between uniform Roe algebras.

Corollary 5.2. *Let X and Y be u.l.f. metric spaces and $\Phi \colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ an embedding. Let $p = \mathrm{SOT}\text{-}\sum_{x \in X} \Phi(e_{xx})$. The following are equivalent:*

(1) *Φ is strongly continuous, and*

(2) *$p = \Phi(1_X)$.*

Moreover, if either Y is quasi-local or Φ is compact-preserving, then the items above are also equivalent to

(3) *let A be the hereditary subalgebra of $\mathrm{C}^*_u(Y)$ generated by $\Phi(\mathrm{C}^*_u(X))$. Then,*

$$a\Phi\big(\mathcal{K}\big(\ell_2(X)\big)\big) = 0$$

implies that $a = 0$ for all $a \in A$.

Proof. (1)⇒(2) If Φ is strongly continuous, $p = \mathrm{SOT}\text{-}\sum_{x \in X} \Phi(e_{xx}) = \Phi(1_X)$.

(2)⇒(1) This follows from Theorem 5.1.

(2)⇒(3) Let a belong to A. If $a\Phi(\mathcal{K}(\ell_2(X))) = 0$, then $a\Phi(\sum_{x \in F} e_{xx}) = 0$ for all finite $F \subset X$. So $ap = a\Phi(1_X) = 0$. As $\Phi(1_X)$ is the unit of A, it follows that $a = 0$.

(3)⇒(2) By Theorem 5.1, if either Y is quasi-local or Φ is compact-preserving, then $p \in \mathrm{C}^*_u(Y)$. Let $q = \Phi(1_X) - p$, so $q \in \mathrm{C}^*_u(Y)$ and $q \leq \Phi(1_X)$. Hence, by the definition of A, we have that $q \in A$. If $a \in \mathcal{K}(\ell_2(X))$, then $a = \lim_n \chi_{X_n} a \chi_{X_n}$, where $(X_n)_n$ is an increasing sequence of finite subsets of X so that $X = \bigcup_n X_n$. Then

$$q\Phi(a) = \lim_n \big(\big(\Phi(1_X) - p\big)\Phi(\chi_{X_n})\Phi(a\chi_{X_n})\big) = 0.$$

Therefore, by the arbitrariness of a and the hypothesis, this implies that $q = 0$, i.e., $p = \Phi(1_X)$. ∎

We point out that (3) of Corollary 5.2 cannot be weakened to “$a\Phi(\mathcal{K}(\ell_2(X))) = 0$ implies $a = 0$ for all $a \in \Phi(\mathrm{C}^*_u(X))$”. We refer to Remark 5.4 for a further discussion on that.

The next result shows that spacially implemented embeddings and embeddings which are both strongly continuous and rank-preserving coincide.

Theorem 5.3. *Let X and Y be u.l.f. metric spaces and $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ an embedding. The following are equivalent:*

(1) *Φ is spacially implemented by an isometric embedding $\ell_2(X) \to \ell_2(Y)$,*

(2) *Φ is rank-preserving and strongly continuous,*

(3) *$\Phi(1_X)\mathcal{K}(\ell_2(Y))\Phi(1_X)$ is contained in $\Phi(\mathrm{C}^*_u(X))$, and*

(4) *there is a subspace $H' \subset H$ so that $\mathcal{K}(H') \subset \Phi(\mathrm{C}^*_u(X))$ and $\mathcal{K}(H')$ is an essential ideal of the hereditary subalgebra generated by $\Phi(\mathrm{C}^*_u(X))$.*

Proof. The implication (1)⇒(2) is straightforward. Moreover, if $\Phi = \mathrm{Ad}(u)$ for an isometric embedding $u : \ell_2(X) \to \ell_2(Y)$, then

$$u\mathcal{K}\big(\ell_2(X)\big)u^* = \mathcal{K}\big(\mathrm{Im}(u)\big) = uu^*\mathcal{K}\big(\ell_2(Y)\big)uu^* = \Phi(1_X)\mathcal{K}\big(\ell_2(Y)\big)\Phi(1_X).$$

So the implication (1)⇒(3) follows. As $\Phi(\mathrm{C}^*_u(X)) \subset \mathcal{B}(\mathrm{Im}(u))$, $\mathcal{K}(\mathrm{Im}(u))$ is an essential ideal of $\Phi(\mathrm{C}^*_u(X))$. So the implication (1)⇒(4) follows.

(2)⇒(1) As Φ is rank-preserving, $\Phi(e_{xx})$ is a rank 1 projection for each $x \in X$. So, for each $x \in X$, pick a normalized $\xi_x \in \ell_2(X)$ so that $\Phi(e_{xx})$ is the projection onto $\mathrm{span}\{\xi_x\}$. Strong continuity gives that $\Phi(1_X) = \text{SOT-}\sum_{x\in X}\Phi(e_{xx})$. Hence, $(\Phi(e_{xx}))_n$ is a maximal set of rank 1 projections for the Hilbert space $H' = \mathrm{Im}(\Phi(1_X))$, so $(\xi_n)_n$ is an orthonormal basis for H'. As $e_{xy} \in \mathrm{C}^*_u(X)$, it follows that $p_{xy} = \langle\cdot,\xi_x\rangle\xi_y \in \Phi(\mathrm{C}^*_u(X))$ for all $x, y \in X$. Hence, $\Phi(\mathrm{C}^*_u(X))$ contains $\mathcal{K}(H')$.

This implies that $\Phi(\mathcal{K}(\ell_2(X)))$ is a nontrivial ideal of $\mathcal{K}(H')$. So, the equality

$$\Phi\big(\mathcal{K}\big(\ell_2(X)\big)\big) = \mathcal{K}(H')$$

must hold; hence there is a unitary $u : \ell_2(X) \to H'$ so that $\Phi(a) = uau^*$ for all $a \in \mathcal{K}(\ell_2(X))$ (see [15, Theorem 2.4.8]). On can easily check that $\Phi = \mathrm{Ad}(u)$ (cf. [22, Lemma 3.1]).

(3)⇒(2) Let us notice that $\Phi(e_{xx})$ has rank 1 for all $x \in X$. Indeed, if this is not the case for some $x \in X$, there is a rank 1 projection $q < \Phi(e_{xx})$. As $q = \Phi(1_X)q\Phi(1_X) \in \Phi(\mathrm{C}^*_u(X))$, there is a projection $q' \in \mathrm{C}^*_u(X)$ with $\Phi(q') = q$. Then $0 < q' < e_{xx}$; contradiction.

As $\Phi(e_{xx})$ has rank 1 for all $x \in X$, we obtain that $\Phi(e_{xy})$ has rank 1 for all $x, y \in X$; $\Phi(e_{xy})$ is actually a partial isometry taking $\mathrm{Im}(\Phi(e_{xy}))$ onto $\mathrm{Im}(\Phi(e_{yy}))$. Therefore, $\Phi \restriction \mathcal{B}(\ell_2(F))$ is rank-preserving for all finite $F \subset X$. As $\bigcup_{F \subset X, |F| < \infty} \mathcal{B}(\ell_2(F))$ is dense in the finite rank operators, it easily follows that Φ is rank-preserving. In particular, Φ is compact-preserving.

As $\Phi(e_{xx}) \leq \Phi(1_X)$ for all $x \in X$, we have that $p = \text{SOT-}\sum_{x \in X} \Phi(e_{xx}) \leq \Phi(1_X)$. If this is a strict inequality, pick a rank 1 projection $p_1 \leq \Phi(1_X) - p$. Then

$$p_1 = \Phi(1_X) p_1 \Phi(1_X) \in \Phi\big(\mathrm{C}^*_u(X)\big).$$

Pick a nonzero projection $p_2 \in \mathrm{C}^*_u(X)$ with $\Phi(p_2) = p_1$. Then

$$\|e_{xx} p_2\| = \big\|\Phi(e_{xx})\Phi(p_2)\big\| \leq \|p p_1\| = 0$$

for all $x \in X$. So $p_2 = 0$; contradiction. Then $p = \Phi(1_X)$ and we have $\Phi = \mathrm{Ad}(p) \circ \Phi$. Hence, as Φ is compact-preserving, Theorem 5.1 implies that Φ is strongly continuous.

(4)⇒(3) Fix $H' \subset H$ as in the hypothesis and denote by A the hereditary C^*-algebra generated by $\Phi(\mathrm{C}^*_u(X))$. We claim that $\Phi(1_X) = 1_{H'}$. Indeed, as $\mathcal{K}(H') \subset \Phi(\mathrm{C}^*_u(X))$, all finite rank projections $q \in \mathcal{K}(H')$ are below $\Phi(1_X)$. Hence, $1_{H'} \leq \Phi(1_X)$. Let $q = \Phi(1_X) - 1_{H'}$, so $q \in A$. As $\mathcal{K}(H')q = 0$ and $\mathcal{K}(H')$ is an essential ideal of A, $q = 0$.

This shows that

$$\Phi(1_X)\mathcal{K}\big(\ell_2(Y)\big)\Phi(1_X) = 1_{H'}\mathcal{K}\big(\ell_2(Y)\big)1_{H'} = \mathcal{K}(H') \subset \Phi\big(\mathrm{C}^*_u(X)\big),$$

so we are done. ■

Remark 5.4. As noticed in [5, Proposition 4.1], there are embeddings between uniform Roe algebras which are not strongly continuous. Moreover, the embedding can also be taken to be rank-preserving. We recall the example: let $X = \{n^2 \mid n \in \mathbb{N}\}$ and let $\mathcal{U}$ be a nonprincipal ultrafilter on X. Let $\Phi_1 : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(X)$ be the identity and let

$$\Phi_2(\chi_A) = \begin{cases} \chi_A, & \text{if } A \in \mathcal{U}, \\ 0, & \text{if } A \notin \mathcal{U}. \end{cases}$$

The map Φ_2 extends to a $*$-homomorphism $\Phi_2 : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(X)$ which sends $\mathcal{K}(\ell_2(X))$ to zero. The map

$$\Phi = \Phi_1 \oplus \Phi_2 : a \in \mathrm{C}^*_u(X) \mapsto \Phi_1(a) \oplus \Phi_2(a) \in \mathrm{C}^*_u(X) \oplus \mathrm{C}^*_u(X)$$

is a rank-preserving embedding which is not strongly continuous. Moreover, as $\mathrm{C}^*_u(X) \oplus \mathrm{C}^*_u(X) \subset \mathrm{C}^*_u(X \sqcup X)$ (where $X \sqcup X$ is any metric space whose metric restricted to both copies of X coincide with X's original metric), this map can be considered as a map $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(X \sqcup X)$.

Notice that $\mathcal{K}(\ell_2(X)) \oplus \{0\}$ is an essential ideal of $\Phi(\mathrm{C}^*_u(X))$. Therefore, this example shows that (4) cannot be weakened to "there is a subspace $H' \subset H$ so that $\mathcal{K}(H')$ is an essential ideal of $\Phi(\mathrm{C}^*_u(X))$".

Given $n \in \mathbb{N}$, an operator between operator algebras is called *1-to-n rank-preserving* if it sends operators of rank 1 to operators of rank n. We now look at strongly continuous 1-to-n rank-preserving maps between uniform Roe algebras.

Given a metric space X and $n \in \mathbb{N} \cup \{\infty\}$, we have a canonical embedding $I_n = I_{X,n}$: $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(X) \otimes \mathcal{B}(\mathbb{C}^n)$. Precisely,

$$I_n : a = [a_{xy}] \mapsto \text{SOT-}\lim_{F \subset X, |F| < \infty} \sum_{x,y \in F} a_{xy} e_{xy} \otimes \mathrm{Id}_n \in \mathrm{C}^*_u(X) \otimes \mathcal{B}(\mathbb{C}^n)$$

(if $n = \infty$, $\mathbb{C}^n$ is assumed to be ℓ_2). In other words, identifying $\mathrm{C}^*_u(X) \otimes \mathcal{B}(\mathbb{C}^n)$ with a subalgebra of $\mathcal{B}(\ell_2(X, \mathbb{C}^n))$ in the standard way, and thinking about operators on $\mathcal{B}(\ell_2(X, \mathbb{C}^n))$ as X-by-X matrices with entries in $\mathcal{B}(\mathbb{C}^n)$, we have that $I_n([a_{xy}]) = [a_{xy} \mathrm{Id}_n]$ for all $a = [a_{xy}] \in \mathrm{C}^*_u(X)$. The map I_n is clearly strongly continuous.

We now prove the n-version of Theorem 5.3.

Theorem 5.5. *Let X and Y be u.l.f. metric spaces and $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ an embedding. Given $n \in \mathbb{N} \cup \{\infty\}$, the following are equivalent.*

(1) *There is an isometric embedding $u : \ell_2(X, \mathbb{C}^n) \to \ell_2(Y)$ so that $\Phi = \mathrm{Ad}(u) \circ I_n$.*

(2) *Φ is 1-to-n rank-preserving and strongly continuous.*

Proof. As I_n is 1-to-n rank-preserving and $\mathrm{Ad}(u) : \mathcal{B}(\ell_2(X, \mathbb{C}^n)) \to \mathcal{B}(\ell_2(Y))$ is rank-preserving, (1)$\Rightarrow$(2) follows.

(2)$\Rightarrow$(1) Fix $x_0 \in X$. As $\mathrm{Im}(\Phi(e_{x_0 x_0}))$ has dimension n, fix a surjetive isometry v : $\mathbb{C}^n \to \mathrm{Im}(\Phi(e_{x_0 x_0}))$. Notice that $\Phi(e_{x_0 x})$ is a partial isometry which takes $\mathrm{Im}(\Phi(e_{x_0 x_0}))$ isometrically onto $\mathrm{Im}(\Phi(e_{xx}))$, for all $x \in X$. Define an isometric embedding

$$u : \ell_2(X, \mathbb{C}^n) \to \ell_2(Y)$$

by letting

$$u\delta_x \otimes \xi = \Phi(e_{x_0 x}) v \xi$$

for all $x \in X$ and all $\xi \in \mathbb{C}^n$. So, $u \restriction \ell_2(\{x\}, \mathbb{C}^n)$ is a surjective isometry between $\ell_2(\{x\}, \mathbb{C}^n)$ and $\mathrm{Im}(\Phi(e_{xx}))$ for all $x \in X$. Moreover, as Φ is strongly continuous, u is an isometry onto

$$H = \overline{\mathrm{span}}\big\{\mathrm{Im}\big(\Phi(e_{xx})\big) \mid x \in X\big\} = \mathrm{Im}\big(\Phi(1_X)\big).$$

In particular, $u^*[H^\perp] = 0$.

Let us notice that $\Phi = \mathrm{Ad}(u) \circ I_n$. As Φ is strongly continuous, it is enough to show that $\Phi(e_{xy}) = \mathrm{Ad}(u) \circ I_n(e_{xy})$ for all $x, y \in X$. Fix $x, y \in X$. As $u^*[H^\perp] = 0$ and $\Phi(e_{xy})\xi = 0$ for all $\xi \in H^\perp$, it is enough to show that $\Phi(e_{xy})\xi = \mathrm{Ad}(u) \circ I_n(e_{xy})\xi$ for all $\xi \in H$.

Fix $\xi \in H$. As

$$H = \Big(\bigoplus_{x\in X} \mathrm{Im}\,(\Phi(e_{xx}))\Big)_{\ell_2},$$

write $\xi = (\xi_x)_{x\in X}$, where $\xi_x \in \mathrm{Im}(\Phi(e_{xx}))$ for each $x \in X$. If $z \neq x$, $\Phi(e_{xx})\xi_z = 0$ as $\Phi(e_{xx})$ and $\Phi(e_{zz})$ are orthogonal projections. As $u^*(\xi_z) \in \ell_2(\{z\}, \mathbb{C}^n)$, we also have that

$$\mathrm{Ad}(u) \circ I_n(e_{xy})\xi_z = ue_{xy} \otimes \mathrm{Id}_n u^*\xi_z = 0.$$

Hence, we only need to notice that $\Phi(e_{xy})\xi_x = \mathrm{Ad}(u) \circ I_n(e_{xy})\xi_x$. This follows since the formula of u gives that

$$ue_{xy} \otimes \mathrm{Id}_n u^*\xi_z = \Phi(e_{x_0y})vv^*\Phi(e_{xx_0})\xi_x = \Phi(e_{x_0y})\Phi(e_{xx_0})\xi_x = \Phi(e_{xy})\xi_x,$$

so we are done. ∎

6. Questions

We finish the paper with a selection of natural questions left unsolved. Firstly, when working with quotient maps, it is natural to look at sections of such maps. Recall, given a surjection $f : X \to Y$, that we say that a map $g : Y \to X$ is a *section of f* if $f \circ g = \mathrm{Id}_Y$. In the coarse category, the notion of coarse-section is more natural: precisely, $g : Y \to X$ is a *coarse-section of f* if it is coarse and $f \circ g \sim \mathrm{Id}_Y$. The following is straightforward.

Proposition 6.1. *Let $f : X \to Y$ be a coarse quotient between metric spaces. If $g : Y \to X$ is a section of f which is coarse, then $g : Y \to g(Y)$ is a bijective coarse equivalence. If $g : Y \to X$ is a coarse-section of f, then $g : Y \to g(Y)$ is a coarse equivalence.* ∎

Finding conditions for the existence of a coarse-section of a given coarse quotient would be interesting. More generally, the following problem holds.

Problem 6.2. Let X and Y be u.l.f. metric spaces and assume that there is a uniformly finite-to-one coarse quotient $X \to Y$. Does Y coarsely embed into X?

While Theorem 1.2 gives us an embedding with cobounded range, Theorem 1.3 assumes strong-coboundedness on the embedding. Without assuming some further structure on the metric space X, we do not know if a bijective coarse quotient is enough to obtain an embedding with strongly-cobounded range (see Remark 4.3).

Problem 6.3. Let X and Y be u.l.f. metric spaces and $f : X \to Y$ a bijective coarse quotient map. Is there an embedding $\Phi : \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ with strongly-cobounded range so that $\Phi(\ell_\infty(X))$ is a Cartan subalgebra of $\mathrm{C}^*_u(Y)$?

Although Corollary 3.7 shows that the existence of a uniformly finite-to-one coarse quotient $X \to Y$ gives us that Y has finite asymptotic dimension provided that the same holds for X, we do not have a precise bounded for $\mathrm{asydim}(Y)$.

Problem 6.4. Say X and Y are u.l.f. metric spaces, and assume that there is a uniformly finite-to-one coarse map $X \to Y$. Does it follow that $\mathrm{asydim}(Y) \leq \mathrm{asydim}(X)$?

It is known that, for proper metric spaces X with $\mathrm{asydim}(X) < \infty$, we have that $\mathrm{asydim}(X) = \dim(\nu X)$, where νX is the Higson corona of X (see [8, Theorem 6.2] for this result and [19, Section 2.3] for details on the Higson corona). Hence, in order to give a positive answer to Problem 6.4, it would be enough to give a positive answer to the following problem:

Problem 6.5. Let X and Y be u.l.f. metric spaces and assume that there is a bijective coarse quotient $X \to Y$. Does it follow that $\dim(\nu X) = \dim(\nu Y)$?

A positive answer to the next problem would be very useful in order to extend the current rigidity results in the literature to non-rank-preserving embeddings $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$.

Problem 6.6. Can $v : \mathbb{C}^n \to \mathrm{Im}(\Phi(e_{x_0 x_0}))$ in the proof of Theorem 5.5 be chosen so that $\mathrm{Ad}(u)(\mathrm{C}^*_u(X) \otimes \mathrm{M}_n(\mathbb{C})) \subset \mathrm{C}^*_u(Y)$?

On a different direction, but still trying to better understand non-rank-preserving embeddings $\mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$, a version of Theorem 5.1 would be very interesting.

Problem 6.7. Let X and Y be u.l.f. metric spaces and $\Phi: \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(Y)$ an embedding. Is there a projection $p \in \mathrm{C}^*_{ql}(Y)$ so that the map

$$a \in \mathrm{C}^*_u(X) \mapsto p\Phi(a)p \in \mathrm{C}^*_{ql}(Y)$$

is a rank-preserving embedding?

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