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Generators in $\mathcal{Z}$-stable C^*-algebras of real rank zero

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Abstract. We show that every separable C^*-algebra of real rank zero that tensorially absorbs the Jiang–Su algebra contains a dense set of generators. It follows that, in every classifiable, simple, nuclear C^*-algebra, a generic element is a generator.

1. Introduction

The generator problem for C^*-algebras asks to determine the minimal number of generators for a given C^*-algebra. One difficulty when studying this problem is that the minimal number of generators is a rather ill-behaved invariant; in particular, it may increase when passing to ideals or inductive limits. We showed in [13] that a much better behaved invariant is obtained by considering the minimal number n such that generating n-tuples are dense.

The most interesting aspect of the generator problem is to determine which C^*-algebras are generated by a single operator, and it is a major open question if every separable, simple C^*-algebra is singly generated. A natural variant of this problem is to describe the class of C^*-algebras that contain a dense set of generators. In previous work, we showed that this class includes all separable AF algebras [13, Theorem 7.3] and all separable, approximately subhomogeneous (ASH) algebras that are $\mathcal{Z}$-stable, that is, that tensorially absorb the Jiang–Su algebra $\mathcal{Z}$ [14, Theorem 5.10].

In this paper, we show that every separable, $\mathcal{Z}$-stable C^*-algebra of real rank zero contains a dense set of generators; see Theorem 5.3. This includes all separable, nuclear, purely infinite C^*-algebras of real rank zero (Corollary 5.5) and in particular every Kirchberg algebra (Corollary 5.6). Together with the previous result about $\mathcal{Z}$-stable ASH algebras, we deduce that every classifiable, simple, nuclear C^*-algebra contains a dense set of generators; see Corollary 5.7.

The main tool to prove these results is the generator rank, which was introduced in [13]. A unital, separable C^*-algebra A has generator rank at most n, denoted by $\mathrm{gr}(A) \leq n$, if the set of self-adjoint $(n+1)$-tuples that generate A as a C^*-algebra is dense; see Definition 2.1 for the general definition in the nonunital and nonseparable case. By [13, Proposition 5.6], A has generator rank zero if and only if A is commutative with totally

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disconnected spectrum. Since two self-adjoint elements a and b generate the same sub-C^*-algebra as the element $a+ib$, it follows that A has generator rank at most one if and only if A contains a dense set of generators.

Our main result (Theorem 5.3) shows that separable, $\mathcal{Z}$-stable C^*-algebras of real rank zero have generator rank one. To prove this, we heavily rely on the permanence properties of the generator rank that were established in [13]. In particular, the generator rank does not increase when passing to ideals, quotients, or inductive limits, and we can estimate the generator rank of extensions; see Theorem 2.4.

The general strategy is as follows: given a unital, separable, $\mathcal{Z}$-stable C^*-algebra A of real rank zero, we use that $\mathcal{Z}$ is the inductive limit of generalized dimension-drop algebras $Z_{2^\infty,3^\infty}$ (see the proof of Proposition 4.3 for the definition) to reduce the problem to computing the generator rank of $A\otimes Z_{2^\infty,3^\infty}$.

We obtain the estimate $\mathrm{gr}(A\otimes Z_{2^\infty,3^\infty})\leq 1$ by combining two results: first, we show that $A\otimes Z_{2^\infty,3^\infty}$ has finite generator rank (we verify that it is at most 8); see Proposition 4.3. Then, using a delicate construction of generators in $A\otimes Z_{2^\infty,3^\infty}$, we show that $\mathrm{gr}(A\otimes Z_{2^\infty,3^\infty})\leq n+1$ implies that $\mathrm{gr}(A\otimes Z_{2^\infty,3^\infty})\leq n$, for every $n\geq 1$; see Lemma 5.2.

To verify that $A\otimes Z_{2^\infty,3^\infty}$ has finite generator rank, we use the short exact sequence

$$0\to A\otimes C_0\big((0,1),M_{6^\infty}\big)\to A\otimes Z_{2^\infty,3^\infty}\to A\otimes(M_{2^\infty}\oplus M_{3^\infty})\to 0,$$

and it suffices to show that the ideal and the quotient have finite generator rank. To estimate the generator rank of $A\otimes C_0((0,1),M_{6^\infty})$, we use that it is an ideal in $A\otimes C_0([0,1],M_{6^\infty})$. In Section 3, we prove an upper bound for the generator rank of such algebras.

The algebra $A\otimes Z_{2^\infty,3^\infty}$ has a natural structure as a $C([0,1])$-algebra, with each fiber over $(0,1)$ isomorphic to $A\otimes M_{6^\infty}$, and with the fibers at 0 and 1 isomorphic to $A\otimes M_{3^\infty}$ and $A\otimes M_{2^\infty}$, respectively. We use a Stone–Weierstraß-type result that characterizes when a set of self-adjoint elements generates $A\otimes Z_{2^\infty,3^\infty}$: the elements have to generate each fiber, and they have to suitably separate the points in $[0,1]$. The assumption of real rank zero is crucial for both: it is used in the proof of Proposition 4.2 to show that the tensor product of A with a UHF algebra has generator rank one, and it is used in the proofs of Lemma 3.2 and Lemma 5.2 to construct self-adjoint elements that separate the points in $[0,1]$.

Notation

Given a C^*-algebra A, we use A_{sa} to denote the set of self-adjoint elements in A. We write M_d for the C^*-algebra of d-by-d complex matrices. Given a subset $F\subseteq A$ and $a\in A$, we write $a\in_\varepsilon F$ if there exists $b\in F$ such that $\|a-b\|<\varepsilon$. We set $\mathbb{N}:=\{0,1,2,\ldots\}$, the natural numbers including 0. The spectrum of an operator a is denoted by $\sigma(a)$.

2. The generator rank

In this section, we recall the definition and basic properties of the generator rank gr and its precursor gr_0 from [13].

Definition 2.1 ([13, Definitions 2.1 and 3.1]). Let A be a C^*-algebra. Then $\mathrm{gr}_0(A)$ is defined as the smallest integer $n \geq 0$ such that, for every $a_0, \ldots, a_n \in A_{\mathrm{sa}}$, $\varepsilon > 0$, and $c \in A$, there exist $b_0, \ldots, b_n \in A_{\mathrm{sa}}$ such that

$$\|b_j - a_j\| < \varepsilon \text{ for } j = 0, \ldots, n \quad \text{and} \quad c \in_\varepsilon C^*(b_0, \ldots, b_n).$$

If no such n exists, then $\mathrm{gr}_0(A) = \infty$. The *generator rank* of A is defined as $\mathrm{gr}(A) := \mathrm{gr}_0(\widetilde{A})$, where $\widetilde{A}$ denotes the minimal unitization of A.

For the definition and the basic properties of the real rank of C^*-algebras, we refer to [1, Section V.3.2, p. 452ff].

Remark 2.2. Let A be a C^*-algebra. If A is unital, then $\mathrm{gr}(A) = \mathrm{gr}_0(A)$ by definition. By [14, Theorem 5.5], we also have $\mathrm{gr}(A) = \mathrm{gr}_0(A)$ whenever A is subhomogeneous. By [13, Theorem 3.13], we have $\mathrm{gr}(A) = \max\{\mathrm{rr}(A), \mathrm{gr}_0(A)\}$. In particular, if A has real rank zero, then $\mathrm{gr}(A) = \mathrm{gr}_0(A)$.

In general, however, it is unclear whether $\mathrm{gr}(A) = \mathrm{gr}_0(A)$; see [13, Question 3.16].

For separable C^*-algebras, the generator rank and its precursor can be described by the denseness of generating tuples.

Theorem 2.3 ([13, Theorem 3.4]). *Let A be a* separable *C^*-algebra and $n \in \mathbb{N}$. Then $\mathrm{gr}_0(A) \leq n$ if and only if, for every $a_0, \ldots, a_n \in A_{\mathrm{sa}}$ and $\varepsilon > 0$, there exist $b_0, \ldots, b_n \in A_{\mathrm{sa}}$ such that*

$$\|b_j - a_j\| < \varepsilon \text{ for } j = 0, \ldots, n \quad \text{and} \quad A = C^*(b_0, \ldots, b_n).$$

Analogously, we obtain a characterization of $\mathrm{gr}(A) \leq n$ by denseness of generating tuples in $\widetilde{A}$.

We will repeatedly use the following permanence properties of the generator rank, which were shown in [13, Theorems 6.2 and 6.3].

Theorem 2.4. *Let $I \subseteq A$ be a closed, two-sided ideal in a C^*-algebra A. Then*

$$\max\left\{\mathrm{gr}(I), \mathrm{gr}(A/I)\right\} \leq \mathrm{gr}(A) \leq \mathrm{gr}(I) + \mathrm{gr}(A/I) + 1.$$

Moreover, if $A = \varinjlim_\lambda A_\lambda$ is an inductive limit, then

$$\mathrm{gr}(A) \leq \liminf_\lambda \mathrm{gr}(A_\lambda).$$

It is natural to expect that the generator rank of the direct sum of two C^*-algebras is the maximum of the generator ranks of the summands. We have verified this in the case that both summands have real rank zero (see Proposition 2.5 below) and whenever one of the summands is subhomogeneous [14, Proposition 5.8]. However, in general, this remains unclear; see [13, Question 6.4].

Proposition 2.5 ([13, Lemma 7.1]). *Let A, B be C^*-algebras of real rank zero. Then* $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$.

Proposition 2.6 ([13, Proposition 5.6]). *A C^*-algebra A has generator rank zero if and only if A is commutative with totally disconnected spectrum.*

Remark 2.7. Let A be a separable C^*-algebra. Since two self-adjoint elements a and b generate the same sub-C^*-algebra as the element $a + ib$, and since the set of generators in A is always a G_δ-subset, we have $\mathrm{gr}_0(A) \leq 1$ if and only if the set of (non-self-adjoint) generators in A is a dense G_δ-subset; see [13, Remark 3.7].

If A has real rank zero, then $\mathrm{gr}_0(A) = \mathrm{gr}(A)$. Thus, a separable C^*-algebra A of real rank zero has generator rank at most one if and only if a generic element in A is a generator.

We end this section with a standard result that will be used in the proof of Lemma 5.2. We include a proof for completeness. For the definition and basic results of $C(X)$-algebras, we refer to [3, Section 2].

Given an element a in a $C(X)$-algebra, and $x \in X$, we let $a(x)$ denote the image of a in the fiber at x. We will repeatedly use that the map $X \to \mathbb{R}$, $x \mapsto \|a(x)\|$ is upper semicontinuous, and that $\|a\| = \sup_{x \in X} \|a(x)\|$; see [3, Lemma 2.1].

Lemma 2.8. *Let X be a compact Hausdorff space, let A be a unital $C(X)$-algebra, and let $B \subseteq A$ be a sub-C^*-algebra. Then the following are equivalent:*

(a) *we have $C(X) \subseteq B$;*

(b) *B separates the points in X in the sense that, for every distinct $x_0, x_1 \in X$, there exists $b \in B$ with $b(x_0) = 0$ and $b(x_1) = 1$.*

Proof. Assuming (b), we need to verify (a). We first show that $1 \in B$. For every $y \in X$, there exists $b_y \in B$ with $b_y(y) = 1$. Replacing b_y by $b_y b_y^*$, we may assume that b_y is positive. Set $U_y := \{z \in X : \|1 - b_y(z)\| < \frac{1}{2}\}$, which is open since $\|1 - b_y(\cdot)\|$ is upper semicontinuous. Hence, the compact set X is covered by $(U_y)_{y \in X}$, which allows us to choose $y_1, \ldots, y_N \in X$ such that X is covered by $\bigcup_{j=1}^N U_{y_j}$. Set $b := \sum_{j=1}^N b_{y_k} \in B$. Let $f : \mathbb{R} \to [0, 1]$ be a continuous function with $f(0) = 0$ and such that f takes value 1 on $[\frac{1}{2}, \infty)$. Set $c := f(b) \in B$. For each x, we have $b(x) \geq \frac{1}{2}$ and therefore $c(x) = 1$. It follows that $c = 1$, since $\|c - 1\| = \sup_{x \in X} \|c(x) - 1\| = 0$.

Claim 1. *Let $F \subseteq X$ be closed and $x \in X \setminus F$. Then there exists a positive $b \in B$ with $b|_F = 0$ and $b(x) = 1$.* To prove the claim, let $y \in F$. By assumption, there is $b_y \in B$ such that $b_y(y) = 0$ and $b_y(x) = 1$. As above, we may assume that b_y is positive, and we may then find $y_1, \ldots, y_N \in F$ such that F is covered by the sets $U_j := \{z \in X : \|b_{y_j}(z)\| < \frac{1}{2}\}$

for $j = 1, \dots, N$. Let $g\colon \mathbb{R} \to [0,1]$ be a continuous function with $g(1) = 1$ and such that g takes the value 0 on $(-\infty, \frac{1}{2}]$. For each j, set $c_j := g(b_{y_j}) \in B$. Then c_j vanishes on U_{y_j}, and $c_j(x) = 1$. Thus, $b := (c_1 c_2 \cdots c_N)(c_1 c_2 \cdots c_N)^*$ has the desired properties, which proves the claim.

Claim 2. *Let $F \subseteq G \subseteq X$ with F closed and G open. Then there exists $b \in B$ with $0 \le b \le 1$, $b|_{X \setminus G} = 0$, and $b|_F = 1$.* To prove the claim, set $C := X \setminus G$. Applying Claim 1 and using that $1 \in B$, for each $y \in C$ we obtain a positive $b_y \in B$ such that $b_y|_F = 1$ and $b_y(y) = 0$. As in Claim 1, we find $y_1, \dots, y_M$ such that C is covered by the sets $\{z \in X : \|b_{y_j}(z)\| < \frac{1}{2}\}$ for $j = 1, \dots, N$. With g as above, the element $d := g(b_{y_1}) \cdots g(b_{y_M})$ satisfies $d|_C = 0$ and $d|_F = 1$. Set $b := h(dd^*)$, where $h\colon \mathbb{R} \to \mathbb{R}$ is the function $h(t) = \min\{t, 1\}$. Then b has the claimed properties.

Claim 3. *Let $f\colon X \to [0,\infty)$ be continuous, and let $\varepsilon > 0$. Then there exists $b \in B$ with $\|b - f\| \le 2\varepsilon$.* Set $b_0 := 1$. For each $k \ge 1$, set

$$F_k := \{x \in X : f(x) \ge k\varepsilon\} \quad \text{and} \quad G_k := \{x \in X : f(x) > (k-1)\varepsilon\}.$$

Then G_k is open, F_k is closed, and $F_k \subseteq G_k$. Using Claim 2, choose $b_k \in B$ such that $0 \le b_k \le 1$, $b_k|_{X \setminus G_k} = 0$, and $b_k|_{F_k} = 1$.

Note that $b_k = 0$ for large enough k, which allows us to set $b := \varepsilon \sum_{k=0}^{\infty} b_k$. For each $x \in X$, we have $f(x) \le b(x) \le f(x) + 2\varepsilon$ by construction, and thus $\|b(x) - f(x)\| \le 2\varepsilon$. Hence, $\|b - f\| = \sup_{x \in X} \|b(x) - f(x)\| \le 2\varepsilon$.

It follows from Claim 3 that B contains every positive function in $C([0,1])$, which implies the result. ∎

3. Generator rank of $C([0,1], A)$

Throughout this section, we let A denote a unital, separable C^*-algebra of real rank zero and generator rank at most one, and we set $B := A \otimes C([0,1])$. We consider B with its natural $C([0,1])$-algebra structure, with each fiber isomorphic to A. The goal is to verify $\mathrm{gr}(B) \le 6$, which we accomplish in a series of lemmas.

Lemma 3.1. *Let $x_1, \dots, x_4 \in B_{\mathrm{sa}}$ and $\varepsilon > 0$. Then there exist $y_1, \dots, y_4 \in B_{\mathrm{sa}}$ such that*

$$\|y_k - x_k\| < \varepsilon \text{ for } k = 1, \dots, 4 \quad \text{and} \quad A = C^*\big(y_1(t), \dots, y_4(t)\big) \text{ for all } t \in [0,1].$$

Proof. Using that each $x_j\colon [0,1] \to A$ is uniformly continuous, choose $N \ge 1$ such that $\|x_j(t) - x_j(t')\| < \frac{\varepsilon}{3}$ for all $t, t' \in [0,1]$ with $|t - t'| \le \frac{1}{N}$.

Let $j \in \{0, \dots, N\}$. Then $x_1(\frac{2j}{2N})$ and $x_2(\frac{2j}{2N})$ belong to A_{sa}. Using that A is unital and separable with $\mathrm{gr}(A) \le 1$, we can apply Theorem 2.3 to obtain $y_1^{(2j)}, y_2^{(2j)} \in A_{\mathrm{sa}}$ such that

$$\left\| y_1^{(2j)} - x_1\left(\frac{2j}{2N}\right) \right\| < \frac{\varepsilon}{3}, \quad \left\| y_2^{(2j)} - x_2\left(\frac{2j}{2N}\right) \right\| < \frac{\varepsilon}{4}, \quad \text{and} \quad A = C^*(y_1^{(2j)}, y_2^{(2j)}).$$

Analogously, we obtain $y_3^{(2j)}, y_4^{(2j)} \in A_{\mathrm{sa}}$ such that

$$\left\| y_3^{(2j)} - x_3\left(\frac{2j}{2N}\right) \right\| < \frac{\varepsilon}{3}, \quad \left\| y_4^{(2j)} - x_4\left(\frac{2j}{2N}\right) \right\| < \frac{\varepsilon}{3}, \quad \text{and} \quad A = C^*(y_3^{(2j)}, y_4^{(2j)}).$$

For $j \in \{0, \ldots, N\}$, let $f_j, g_j \colon [0,1] \to [0,1]$ be continuous functions such that

(1) f_j takes the value 1 on $[\frac{2j}{2N}, \frac{2j+1}{2N}]$ for $j = 0, \ldots, N-1$;

(2) the collection $(f_j)_{j=0,\ldots,N}$ is a partition of unity subordinate to the family

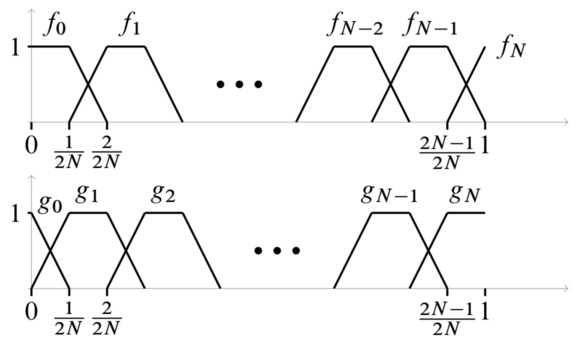
$$\left\{ \left[0, \frac{2}{2N}\right), \left(\frac{1}{2N}, \frac{4}{2N}\right), \ldots, \left(\frac{2N-3}{2N}, \frac{2N}{2N}\right), \left(\frac{2N-1}{2N}, \frac{2N}{2N}\right] \right\};$$

(3) g_j takes the value 1 on $[\frac{2j-1}{2N}, \frac{2j}{2N}]$ for $j = 1, \ldots, N$;

(4) the collection $(g_j)_{j=0,\ldots,N}$ is a partition of unity subordinate to the family

$$\left\{ \left[0, \frac{1}{2N}\right), \left(\frac{0}{2N}, \frac{3}{2N}\right), \ldots, \left(\frac{2N-4}{2N}, \frac{2N-1}{2N}\right), \left(\frac{2N-2}{2N}, \frac{2N}{2N}\right] \right\}.$$

The functions are depicted in the following picture:



Set

$$y_1 := \sum_{j=0}^{N} y_1^{(2j)} f_j, \quad y_2 := \sum_{j=0}^{N} y_2^{(2j)} f_j, \quad y_3 := \sum_{j=0}^{N} y_3^{(2j)} g_j, \quad y_4 := \sum_{j=0}^{N} y_4^{(2j)} g_j.$$

To verify that $\|y_1 - x_1\| < \varepsilon$, we estimate $\|y_1(t) - x_1(t)\|$ for $t \in [0,1]$. For $t = 1$, we have $y_1(1) = y_1^{(2N)}$, and so

$$\|y_1(1) - x_1(1)\| = \|y_1(1) - y_1^{(2N)}\| < \frac{\varepsilon}{3} \leq \frac{2}{3}\varepsilon.$$

Given $t \in [0,1)$, let $j \in \{1, \ldots, N\}$ satisfy $t \in [\frac{2j-2}{2N}, \frac{2j}{2N})$. Since $|t - \frac{2j-2}{2N}|, |t - \frac{2j}{2N}| < \frac{2}{2N}$, we have

$$\|y_1^{(2j-2)} - x_1(t)\| \leq \left\| y_1^{(2j-2)} - x_1\left(\frac{2j-2}{2N}\right) \right\| + \left\| x_1\left(\frac{2j-2}{2N}\right) - x_1(t) \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \frac{2}{3}\varepsilon,$$

$$\|y_1^{(2j)} - x_1(t)\| \leq \left\| y_1^{(2j)} - x_1\left(\frac{2j}{2N}\right) \right\| + \left\| x_1\left(\frac{2j}{2N}\right) - x_1(t) \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \frac{2}{3}\varepsilon.$$

Using that $y_1(t) = y_1^{(2j-2)} f_{j-1}(t) + y_1^{(2j)} f_j(t)$ and that $f_{j-1}(t) + f_j(t) = 1$, we get

$$\|y_1(t) - x_1(t)\| \le \|y_1^{(2j-2)} - x_1(t)\| f_{j-1}(t) + \|y_1^{(2j)} - x_1(t)\| f_j(t) \le \frac{2}{3}\varepsilon.$$

Hence, $\|y_1 - x_1\| = \sup_{t\in[0,1]} \|y_1(t) - x_1(t)\| \le \frac{2}{3}\varepsilon < \varepsilon$. Analogously, one shows that $\|y_k - x_k\| < \varepsilon$ for $k = 2, 3, 4$.

It remains to verify that $\{y_1, \ldots, y_4\}$ generates A in each fiber. Given $t \in [0, 1]$, choose $l \in \{0, \ldots, 2N - 1\}$ such that $t \in [\frac{l}{2N}, \frac{l+1}{2N}]$. If l is even, set $j = \frac{l}{2}$. Then

$$y_1(t) = y_1^{(2j)} \quad \text{and} \quad y_2(t) = y_2^{(2j)}.$$

If l is odd, set $j = \frac{l+1}{2}$. Then

$$y_3(t) = y_3^{(2j)} \quad \text{and} \quad y_4(t) = y_4^{(2j)}.$$

Thus, either $\{y_1(t), y_2(t)\}$ or $\{y_3(t), y_4(t)\}$ generates A, and it follows that $A = C^*(y_1(t), \ldots, y_4(t))$ in either case. ∎

Lemma 3.2. *Let $x_1, x_2, x_3 \in B_{\mathrm{sa}}$ and $\varepsilon > 0$. Then there exist $y_1, y_2, y_3 \in B_{\mathrm{sa}}$ with $\|y_k - x_k\| < \varepsilon$ for $k = 1, 2, 3$, and such that $C^*(y_1, y_2, y_3)$ separates the points in* $[0, 1]$ *in the sense of Lemma 2.8* (b).

Proof. Using that each $x_j\colon [0, 1] \to A$ is uniformly continuous, choose $N \ge 1$ such that $\|x_j(t) - x_j(t')\| < \frac{\varepsilon}{4}$ for all $t, t' \in [0, 1]$ with $|t - t'| \le \frac{1}{N}$.

Let $j \in \{0, \ldots, N\}$. Using that A has real rank zero, we find invertible elements $y_1^{(3j)}, y_2^{(3j)}, y_3^{(3j)} \in A_{\mathrm{sa}}$ with finite spectra such that

$$\left\| y_1^{(3j)} - x_1\left(\frac{3j}{3N}\right) \right\| < \frac{\varepsilon}{4}, \quad \left\| y_1^{(3j)} - x_2\left(\frac{3j}{3N}\right) \right\| < \frac{\varepsilon}{4}, \quad \text{and} \quad \left\| y_1^{(3j)} - x_3\left(\frac{3j}{3N}\right) \right\| < \frac{\varepsilon}{4}.$$

By perturbing the elements, if necessary, we may assume that the spectra of $y_k^{(3j)}$ and $y_{k'}^{(3j')}$ are disjoint whenever $k \neq k'$ or $j \neq j'$. Choose $\mu > 0$ such that any two distinct points in $\{0\} \cup \bigcup_{k,j} \sigma(y_k^{(3j)})$ have distance at least 2μ. We may assume that $\mu < \frac{\varepsilon}{4}$.

For $j \in \{0, \ldots, N\}$, let $f_j, g_j, h_j\colon [0, 1] \to [0, 1]$ be continuous functions such that

(1) f_j takes the value 1 on $[\frac{3j}{3N}, \frac{3j+2}{3N}]$ for $j = 0, \ldots, N - 1$;

(2) the collection $(f_j)_{j=0,\ldots,N}$ is a partition of unity subordinate to the family

$$\left\{\left[0, \frac{3}{3N}\right), \left(\frac{2}{3N}, \frac{6}{2N}\right), \ldots, \left(\frac{3N-4}{3N}, \frac{3N}{3N}\right), \left(\frac{3N-1}{3N}, \frac{3N}{3N}\right]\right\};$$

(3) g_j takes the value 1 on $[\frac{3j-2}{3N}, \frac{3j}{2N}]$ for $j = 1, \ldots, N$;

(4) the collection $(g_j)_{j=0,\ldots,N}$ is a partition of unity subordinate to the family

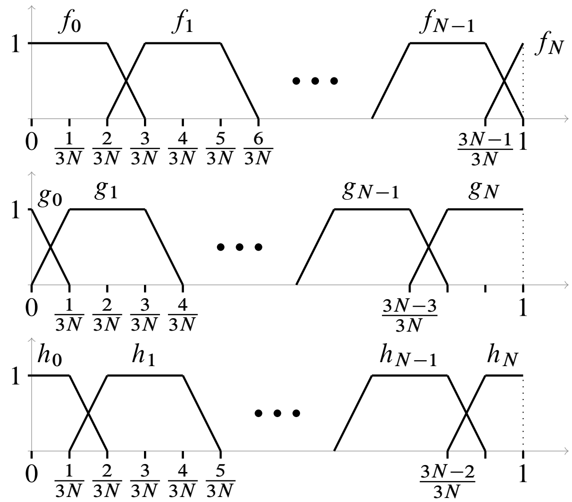
$$\left\{\left[0, \frac{1}{3N}\right), \left(\frac{0}{3N}, \frac{4}{3N}\right), \ldots, \left(\frac{3N-6}{3N}, \frac{3N-2}{3N}\right), \left(\frac{3N-3}{3N}, \frac{3N}{3N}\right]\right\};$$

(5) h_0 takes the value 1 on $[\frac{0}{3N}, \frac{1}{3N}]$; h_N takes the value 1 on $[\frac{3N-1}{3N}, \frac{3N}{3N}]$; and h_j takes the value 1 on $[\frac{3j-1}{3N}, \frac{3j+1}{3N}]$ for $j = 1, \dots, N-1$;

(6) the collection $(h_j)_{j=0,\dots,N}$ is a partition of unity subordinate to the family

$$\left\{\left[0, \frac{2}{3N}\right), \left(\frac{1}{3N}, \frac{5}{2N}\right), \dots, \left(\frac{3N-5}{3N}, \frac{3N-1}{3N}\right), \left(\frac{3N-2}{3N}, \frac{3N}{3N}\right]\right\}.$$

The functions are depicted in the following picture:



Define $y_1, y_2, y_3 \colon [0, 1] \to A$ by

$$y_1(t) := \sum_{j=0}^{N} (y_1^{(3j)} + t\mu) f_j(t), \quad y_2(t) := \sum_{j=0}^{N} (y_2^{(3j)} + t\mu) g_j(t),$$

$$y_3(t) := \sum_{j=0}^{N} (y_3^{(3j)} + t\mu) h_j(t),$$

for $t \in [0, 1]$.

To verify that $\|y_1 - x_1\| < \varepsilon$, we estimate $\|y_1(t) - x_1(t)\|$ for $t \in [0, 1]$, similarly as in the proof of Lemma 3.1. For $t = 1$, we obtain

$$\left\|y_1(1) - x_1(1)\right\| = \left\|y_1^{(3j)} + \mu - x_1(1)\right\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \frac{3}{4}\varepsilon.$$

Given $t \in [0, 1)$, let $j \in \{1, \dots, N\}$ satisfy $t \in [\frac{3j-3}{3N}, \frac{3j}{3N})$. Using at the second step that $|t - \frac{3j}{3N}| < \frac{3}{3N}$, we have

$$\left\|y_1^{(3j)} + t\mu - x_1(t)\right\| \leq \mu + \left\|y_1^{(3j)} - x_1\left(\frac{3j}{3N}\right)\right\| + \left\|x_1\left(\frac{3j}{3N}\right) - x_1(t)\right\| \leq \frac{3}{4}\varepsilon.$$

Analogously, we obtain $\|y_1^{(3j-3)} + t\mu - x_1(t)\| \leq \frac{3}{4}\varepsilon$. We deduce that

$$\|y_1(t) - x_1(t)\| = \big\|\big((y_1^{(3j-3)} + t\mu) f_{j-1}(t) + (y_1^{(3j)} + t\mu) f_j(t)\big) - x_1(t)\big\| \leq \frac{3}{4}\varepsilon.$$

Hence, $\|y_1 - x_1\| = \sup_{t\in[0,1]} \|y_1(t) - x_1(t)\| \leq \frac{3}{4}\varepsilon < \varepsilon$. Analogously, one shows that $\|y_k - x_k\| < \varepsilon$ for $k = 2, 3$.

It remains to verify that $C^*(y_1, y_2, y_3)$ separates the points in $[0, 1]$ in the sense of Lemma 2.8 (b). Let $s, t \in [0, 1]$ be distinct. Choose $k \in \{1, 2, 3\}$ such that both s and t are contained in the set

$$\bigcup_{j\in\mathbb{Z}} \Big[\frac{3j + k - 1}{3N}, \frac{3j + k + 1}{3N}\Big].$$

Let us consider the case $k = 1$. Choose $j, j' \in \{0, \dots, N\}$ such that $s \in [\frac{3j}{3N}, \frac{3j+2}{3N}]$ and $t \in [\frac{3j'}{3N}, \frac{3j'+2}{3N}]$. Then

$$y_1(s) = y_1^{(3j)} + s\mu \quad \text{and} \quad y_1(t) = y_1^{(3j')} + t\mu.$$

Since $s \neq t$, and by choice of μ, it follows that $y_1(s)$ and $y_1(t)$ have finite spectra that are disjoint and do not contain 0. Hence, there exists a continuous function $f\colon \mathbb{R} \to [0, 1]$ such that $f(y_1(s)) = 0$ and $f(y_1(t)) = 1$. Then the element $b := f(y_1) \in B$ satisfies $b(s) = 0$ and $b(t) = 1$.

The cases $k = 2, 3$ are analogous, using y_2 and y_3. ∎

Proposition 3.3. *Let A be a unital, separable C^*-algebra of real rank zero and generator rank at most one. Set $B := A \otimes C([0, 1])$. Then we have* $\mathrm{gr}(B) \leq 6$.

Proof. We show that every 7-tuple in B_{sa} can be approximated by a tuple that generates B. Since B is separable and unital, this verifies $\mathrm{gr}(B) \leq 6$; see Theorem 2.3.

Let $x_1, \dots, x_7 \in B_{\mathrm{sa}}$ and $\varepsilon > 0$. Applying Lemma 3.1 for $x_1, \dots, x_4$, we obtain $y_1, \dots, y_4 \in B_{\mathrm{sa}}$ such that

$$\|y_j - x_j\| < \varepsilon \text{ for } j = 1, \dots, 4 \quad \text{and} \quad A = C^*\big(y_1(t), \dots, y_4(t)\big) \text{ for all } t \in [0, 1].$$

Applying Lemma 3.2 for x_5, x_6, x_7, we obtain $y_5, y_6, y_7 \in B_{\mathrm{sa}}$ such that $\|y_k - x_k\| < \varepsilon$ for $k = 5, 6, 7$, and such that $C^*(y_5, y_6, y_7)$ separates the points in $[0, 1]$ in the sense of Lemma 2.8 (b).

Set $D := C^*(y_1, \dots, y_7) \subseteq B$. Then D exhausts each fiber of B and, moreover, separates the points in $[0, 1]$. Hence, $D = B$ by [15, Lemma 3.2], which shows that $\{y_1, \dots, y_7\}$ generates B, as desired. ∎

Remark 3.4. The proof of Proposition 3.3 can be generalized to show the following: if A is a unital, separable C^*-algebra of real rank zero, then $B := A \otimes C([0, 1])$ satisfies $\mathrm{gr}(B) \leq 2\,\mathrm{gr}(A) + 4$.

4. Establishing finite generator rank

In this section, we prove that unital, separable, $\mathcal{Z}$-stable C^*-algebras of real rank zero have finite generator rank; see Proposition 4.3. In the next section, we will successively reduce the upper bound for the generator rank of such algebras down to one.

We start with a lemma that simplifies the computation of the generator rank for C^*-algebras that absorb a strongly self-absorbing C^*-algebra. For the definition and basic results of strongly self-absorbing C^*-algebras, we refer to [17]. Given a strongly self-absorbing C^*-algebra D, a C^*-algebra A is said to be D*-stable* if $A \cong A \otimes D$. Since every strongly self-absorbing C^*-algebra is nuclear, we do not need to specify the tensor product. Typical examples of strongly self-absorbing C^*-algebras are UHF algebras of infinite type, the Jiang–Su algebra $\mathcal{Z}$, and the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$.

Lemma 4.1. *Let D be a strongly self-absorbing C^*-algebra, let A be a separable, D-stable C^*-algebra, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *we have $\mathrm{gr}_0(A) \leq n$;*

(2) *for every $x_0, \dots, x_n \in A_{\mathrm{sa}}$ and $\varepsilon > 0$, there exist $y_0, \dots, y_n \in (A \otimes D)_{\mathrm{sa}}$ such that*

$$\left\| y_j - (x_j \otimes 1) \right\| < \varepsilon \text{ for } j = 0, \dots, n \quad \text{and} \quad A \otimes D = C^*(y_0, \dots, y_n);$$

(3) *for every $x_0, \dots, x_n \in A_{\mathrm{sa}}$, $\varepsilon > 0$, and $z \in A$, there exist $y_0, \dots, y_n \in (A \otimes D)_{\mathrm{sa}}$ such that*

$$\left\| y_j - (x_j \otimes 1) \right\| < \varepsilon \text{ for } j = 0, \dots, n \quad \text{and} \quad z \otimes 1 \in_\varepsilon C^*(y_0, \dots, y_k).$$

Proof. Since $A \cong A \otimes D$, we have $\mathrm{gr}_0(A) = \mathrm{gr}_0(A \otimes D)$. Using that A is separable, it follows from Theorem 2.3 that (1) implies (2). It is clear that (2) implies (3). Assuming (3), let us verify (1). To show $\mathrm{gr}_0(A \otimes D) \leq n$, let $a_0, \dots, a_n \in (A \otimes D)_{\mathrm{sa}}$, $\varepsilon > 0$, and $c \in A \otimes D$. We need to find $b_0, \dots, b_n \in (A \otimes D)_{\mathrm{sa}}$ such that

$$\|b_j - a_j\| < \varepsilon \text{ for } j = 0, \dots, n \quad \text{and} \quad c \in_\varepsilon C^*(b_0, \dots, b_n).$$

Since D is strongly self-absorbing and A is separable and D-stable, there exists a *-isomorphism $\Phi\colon A \to A \otimes D$ that is approximately unitarily equivalent to the inclusion $\iota\colon A \to A \otimes D$ given by $\iota(a) = a \otimes 1$; that is, there exists a sequence $(u_m)_m$ of unitaries in $A \otimes D$ such that $\lim_{m\to\infty} u_m \Phi(a) u_m^* = \iota(a)$ for all $a \in A$; see [17, Theorem 2.2]. Set

$$x_j := \Phi^{-1}(a_j) \text{ for } j = 0, \dots, n \quad \text{and} \quad z := \Phi^{-1}(c).$$

Using that $u_m^*(x_j \otimes 1)u_m \to a_j$ for $j = 0, \dots, n$ and $u_m^*(z \otimes 1)u_m \to c$, we can choose m such that

$$\left\| u_m^*(x_j \otimes 1)u_m - a_j \right\| < \frac{\varepsilon}{2} \text{ for } j = 0, \dots, n \quad \text{and} \quad \left\| u_m^*(z \otimes 1)u_m - c \right\| < \frac{\varepsilon}{2}.$$

By assumption (3), we obtain $y_0, \dots, y_n \in (A \otimes D)_{\mathrm{sa}}$ such that

$$\|y_j - (x_j \otimes 1)\| < \frac{\varepsilon}{2} \text{ for } j = 0, \dots, n \quad \text{and} \quad z \otimes 1 \in_{\varepsilon/2} C^*(y_0, \dots, y_n).$$

For $j = 0, \dots, n$, we have

$$\|u_m^* y_j u_m - a_j\| \le \|u_m^* y_j u_m - u_m^*(x_j \otimes 1)u_m\| + \|u_m^*(x_j \otimes 1)u_m - a_j\| < \varepsilon.$$

Moreover, from $z \otimes 1 \in_{\varepsilon/2} C^*(y_0, \dots, y_n)$, it follows that

$$u_m^*(z \otimes 1)u_m \in_{\varepsilon/2} C^*(u_m^* y_0 u_m, \dots, u_m^* y_n u_m),$$

and thus

$$c \in_\varepsilon C^*(u_m^* y_0 u_m, \dots, u_m^* y_n u_m),$$

which shows that $u_m^* y_0 u_m, \dots, u_m^* y_n u_m$ have the desired properties. ∎

Proposition 4.2. *Let A be a separable, unital C^*-algebra of real rank zero that tensorially absorbs a UHF algebra of infinite type. Then* $\mathrm{gr}(A) \le 1$.

Proof. Let D be a UHF algebra of infinite type such that A is D-stable. Since A is unital, we have $\mathrm{gr}(A) = \mathrm{gr}_0(A)$. To verify condition (3) of Lemma 4.1, let $a, b \in A_{\mathrm{sa}}$, $\varepsilon > 0$, and $c \in A$. We need to find $x, y \in (A \otimes D)_{\mathrm{sa}}$ such that

$$\|x - (a \otimes 1)\| < \varepsilon, \quad \|y - (b \otimes 1)\| < \varepsilon, \quad \text{and} \quad c \otimes 1 \in_\varepsilon C^*(x, y).$$

Since A has real rank zero, we may assume that a is invertible and that its spectrum $\sigma(a)$ is finite, so that there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R} \setminus \{0\}$ and pairwise orthogonal projections $p_1, \dots, p_n \in A$ that sum to 1_A such that $a = \sum_{j=1}^n \lambda_j p_j$. Choose $\mu > 0$ such that μ is strictly smaller than the distance between any two values in $\sigma(a) \cup \{0\}$. We may assume that $\mu < \varepsilon$.

Choose $d \ge 5$ such that D admits a unital embedding $M_d \subseteq D$. Using that c can be written as a linear combination of four positive, invertible elements, we can choose positive, invertible elements $c_2, c_3 \dots, c_d \in A$ such that

$$\|c_j\| < \frac{\varepsilon}{d}, \text{ for } j = 2, \dots, d \quad \text{and} \quad c \in C^*(c_2, \dots, c_d).$$

Let $(e_{j,k})_{j,k=1,\dots,d}$ be matrix units for M_d. We define $x, y \in A \otimes M_d$ as

$$x := a \otimes 1 + \sum_{j=1}^d \frac{j}{d}\mu e_{jj} \quad \text{and} \quad y := b \otimes 1 + \sum_{j=2}^d c_j \otimes (e_{1j} + e_{j1}).$$

As matrices, these elements look as follows:

$$x = \begin{pmatrix} a + \frac{1}{d}\mu & 0 & \cdots & 0 \\ 0 & a + \frac{2}{d}\mu & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a + \mu \end{pmatrix}, \quad y = \begin{pmatrix} b & c_2 & c_3 & \cdots & c_d \\ c_2 & b & 0 & \cdots & 0 \\ c_3 & 0 & b & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ c_d & 0 & \cdots & 0 & b \end{pmatrix}.$$

Then

$$\|x-(a\otimes 1)\| = \left\| \sum_{j=1}^{d} \frac{j}{d}\mu e_{jj} \right\| = \mu < \varepsilon$$

and

$$\|y-(b\otimes 1)\| = \left\| \sum_{j=2}^{d} c_j \otimes (e_{1j}+e_{j1}) \right\| \leq \sum_{j=2}^{d} \|c_j\| < \varepsilon.$$

Set $B := C^*(x,y) \subseteq A\otimes M_d \subseteq A\otimes D$. We will verify that $z\otimes 1\in B$. The j-th element on the diagonal of x is $a+\frac{j}{d}\mu$, whose spectrum is

$$\sigma\Big(a+\frac{j}{d}\mu\Big) = \Big\{\lambda_1+\frac{j}{d}\mu, \dots, \lambda_n+\frac{j}{d}\mu\Big\}.$$

Given distinct $j,k\in\{1,\dots,d\}$, it follows from the choice of μ that the spectra of $a+\frac{j}{d}\mu$ and $a+\frac{k}{d}\mu$ are finite disjoint sets not containing 0. For $j\in\{1,\dots,d\}$, let $f_j\colon\mathbb{R}\to[0,1]$ be a continuous function that takes the value 1 on $\sigma(a+\frac{j}{d}\mu)$ and the value 0 on $\{0\}\cup\bigcup_{k\neq j}\sigma(a+\frac{k}{d}\mu)$. Then

$$1\otimes e_{jj} = f_j(x) \in B \subseteq A\otimes M_d.$$

Thus, B contains the diagonal matrix units of M_d.

To show that B also contains the other matrix units, we follow ideas of Olsen and Zame from [9]. Given $k\in\{2,\dots,d\}$, we have

$$c_k\otimes e_{1k} = (1\otimes e_{11})y(1\otimes e_{kk}) \in B.$$

Then

$$c_k^2\otimes e_{11} = (c_j\otimes e_{1k})(c_j\otimes e_{1k})^* \in B.$$

Since c_k is positive and invertible, we have $c_k^{-1}\in C^*(c_k^2)\subseteq A$, and it follows that $c_k^{-1}\otimes e_{11}\in B$. Hence,

$$1\otimes e_{1k} = (c_k^{-1}\otimes e_{11})(c_k\otimes e_{1k}) \in B.$$

It follows that $1\otimes M_d\subseteq B$. For each $k\in\{2,\dots,d\}$, we deduce that

$$c_k\otimes 1 = \sum_{j=1}^{d}(1\otimes e_{j1})(c_k\otimes e_{1k})(1\otimes e_{kj}) \in B.$$

Since $c\in C^*(c_2,\dots,c_d)$, we get $c\otimes 1\in B\subseteq A\otimes D$, as desired. ∎

We use M_{2^∞} to denote the UHF algebra of type 2^∞, and similarly for M_{3^∞}.

Proposition 4.3. *Let A be a unital, separable C^*-algebra of real rank zero. Then* $\mathrm{gr}(A\otimes\mathcal{Z})\leq 8$.

Proof. Let $Z_{2^\infty,3^\infty}$ be the generalized dimension-drop algebra given by

$$Z_{2^\infty,3^\infty} = \left\{ f \in C\big([0,1], M_{2^\infty} \otimes M_{3^\infty}\big) : f(0) \in M_{2^\infty} \otimes 1,\ f(1) \in 1 \otimes M_{3^\infty} \right\}.$$

By [12, Theorem 3.4], $\mathcal{Z}$ is an inductive limit of a sequence of C^*-algebras each isomorphic to $Z_{2^\infty,3^\infty}$. Hence, $A \otimes \mathcal{Z}$ is isomorphic to an inductive limit of C^*-algebras isomorphic to $A \otimes Z_{2^\infty,3^\infty}$. Using that the generator rank behaves well with respect to inductive limits (Theorem 2.4), we get

$$\mathrm{gr}(A \otimes \mathcal{Z}) \leq \liminf_{n\to\infty} \mathrm{gr}(A \otimes Z_{2^\infty,3^\infty}) = \mathrm{gr}(A \otimes Z_{2^\infty,3^\infty}).$$

It thus suffices to verify $\mathrm{gr}(A \otimes Z_{2^\infty,3^\infty}) \leq 8$. Set

$$I := \left\{ f \in Z_{2^\infty,3^\infty} : f(0) = f(1) = 0 \right\}.$$

Then I is a closed, two-sided ideal in $Z_{2^\infty,3^\infty}$ and $Z_{2^\infty,3^\infty}/I \cong M_{2^\infty} \oplus M_{3^\infty}$. Since $Z_{2^\infty,3^\infty}$ is nuclear, we obtain a short exact sequence

$$0 \to A \otimes I \to A \otimes Z_{2^\infty,3^\infty} \to A \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0.$$

Note that $A \otimes I$ is isomorphic to a closed, two-sided ideal in $A \otimes M_{6^\infty} \otimes C([0,1])$.

Since real rank zero is preserved by passing to matrix algebras and inductive limits, we get $\mathrm{rr}(A \otimes M_{k^\infty}) = 0$ for $k = 2, 3, 6$. By Proposition 4.2, we obtain $\mathrm{gr}(A \otimes M_{k^\infty}) \leq 1$ for $k = 2, 3, 6$. Using that the generator rank does not increase when passing to closed, two-sided ideals (Theorem 2.4) at the first step and Proposition 3.3 for $A \otimes M_{6^\infty}$ at the second step, we get

$$\mathrm{gr}(A \otimes I) \leq \mathrm{gr}\left(A \otimes M_{6^\infty} \otimes C\big([0,1]\big)\right) \leq 6.$$

By Proposition 2.5, we have

$$\mathrm{gr}\left(A \otimes (M_{2^\infty} \oplus M_{3^\infty})\right) = \max\left\{ \mathrm{gr}(A \otimes M_{2^\infty}), \mathrm{gr}(A \otimes M_{3^\infty}) \right\} \leq 1.$$

Applying the estimate for the generator rank of an extension (Theorem 2.4), we get

$$\mathrm{gr}(A \otimes Z_{2^\infty,3^\infty}) \leq \mathrm{gr}(A \otimes I) + \mathrm{gr}\left(A \otimes (M_{2^\infty} \oplus M_{3^\infty})\right) + 1 \leq 8,$$

as desired. ∎

5. Establishing generator rank one

In this section, we prove our main result: separable, $\mathcal{Z}$-stable C^*-algebras of real rank zero have generator rank one; see Theorem 5.3. We deduce some interesting corollaries, most importantly that every classifiable, simple, nuclear C^*-algebra has generator rank one; see Corollary 5.7.

Recall that the dimension-drop algebra $Z_{2,3}$ is defined as

$$Z_{2,3} := \big\{ f \in C\big([0,1], M_2 \otimes M_3\big) : f(0) \in M_2 \otimes 1,\ f(1) \in 1 \otimes M_3 \big\}.$$

Below, we always view $Z_{2,3}$ as the subalgebra of $C([0,1), M_6)$ given by the next result.

Lemma 5.1. *The dimension-drop algebra $Z_{2,3}$ is isomorphic to the subalgebra of $C([0,1], M_6)$ consisting of the continuous functions $f\colon [0,1] \to M_6$ that satisfy*

$$f(0) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & & & & \\ \alpha_{21} & \alpha_{22} & & & & \\ & & \alpha_{11} & \alpha_{12} & & \\ & & \alpha_{21} & \alpha_{22} & & \\ & & & & \alpha_{11} & \alpha_{12} \\ & & & & \alpha_{21} & \alpha_{22} \end{pmatrix},$$

$$f(1) = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & & & \\ \beta_{21} & \beta_{22} & \beta_{23} & & & \\ \beta_{31} & \beta_{32} & \beta_{33} & & & \\ & & & \beta_{33} & \beta_{31} & \beta_{32} \\ & & & \beta_{13} & \beta_{11} & \beta_{12} \\ & & & \beta_{23} & \beta_{21} & \beta_{22} \end{pmatrix},$$

for some $\alpha_{jk}, \beta_{jk} \in \mathbb{C}$.

Proof. Using an identification of $M_2 \otimes M_3$ with M_6, we naturally view $Z_{2,3}$ as a subalgebra of $C([0,1], M_6)$. Let $u \in M_6$ be the permutation matrix given as

$$u := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $t \mapsto v_t$ be a continuous path of unitaries in M_6 with $v_0 = 1$ and $v_1 = u$. Then v is a unitary in $C([0,1], M_6)$ that conjugates $Z_{2,3} \subseteq C([0,1], M_6)$ onto the subalgebra of functions described in the statement. ∎

Lemma 5.2. *Let A be a unital, separable C^*-algebra of real rank zero, and let $n \in \mathbb{N}$ such that* $\mathrm{gr}(A) \leq n+2$. *Let $x_0, \dots, x_{n+1} \in A_{\mathrm{sa}}$, $\varepsilon > 0$, and $z \in A$. Then there exist $y_0, \dots, y_{n+1} \in (A \otimes Z_{2,3})_{\mathrm{sa}}$ such that*

$$\big\| y_j - (x_j \otimes 1) \big\| < \varepsilon \text{ for } j = 0, \dots, n+1 \quad \text{and} \quad z \otimes 1 \in_\varepsilon C^*(y_0, \dots, y_{n+1}).$$

Proof. Consider the $n+3$ elements $\frac{\varepsilon}{2}, x_0, \dots, x_{n+1} \in A_{\mathrm{sa}}$. Using that A is unital, separable with $\mathrm{gr}(A) \leq n+2$, apply Theorem 2.3 to obtain $a, x_0', \dots, x_{n+1}' \in A_{\mathrm{sa}}$ such that

$$\left\| a - \frac{\varepsilon}{2} \right\| < \frac{\varepsilon}{2}, \quad \|x_j' - x_j\| < \varepsilon \text{ for } j = 0, \dots, n+1, \quad \text{and} \quad A = C^*(a, x_0', \dots, x_{n+1}').$$

Note that a is positive and invertible. To simplify notation, set $b := x_0'$ and $c := x_1'$. Choose a polynomial p in noncommuting variables such that

$$\left\| z - p(a, b, c, x_2', \dots, x_{n+1}') \right\| < \varepsilon.$$

Since p is continuous as a map $A_{\mathrm{sa}}^{n+3} \to A$, we can choose $\delta > 0$ such that every $\bar{c} \in A_{\mathrm{sa}}$ with $\|\bar{c} - c\| < \delta$ satisfies

$$\left\| z - p(a, b, \bar{c}, x_2', \dots, x_{n+1}') \right\| < \varepsilon.$$

Since A has real rank zero, we can choose an invertible element $c' \in A_{\mathrm{sa}}$ with finite spectrum $\sigma(c')$ and such that $\|c' - c\| < \frac{\delta}{2}$. Then $c' = \sum_{j \in J} \lambda_j q_j$ for some finite set J, some pairwise disjoint real numbers λ_j (the eigenvalues of c'), and some pairwise orthogonal projections q_j that sum to 1_A.

Choose $\mu > 0$ such that any two points in $\{0\} \cup \{\lambda_j : j \in J\}$ have distance strictly larger than 4μ. We may assume that $2\mu < \varepsilon$ and $2\mu < \frac{\delta}{2}$.

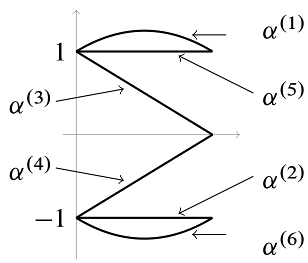
Next, we define auxiliary functions $\alpha^{(k)} \colon [0,1] \to \mathbb{R}$ for $k = 1, \dots, 6$ by

$$\alpha^{(1)}(t) := 1 + t(1-t), \quad \alpha^{(5)}(t) := 1, \quad \alpha^{(3)}(t) := 1 - t$$

and

$$\alpha^{(6)}(t) := -\alpha^{(1)}(t), \quad \alpha^{(2)}(t) := -\alpha^{(5)}(t), \quad \alpha^{(4)}(t) := -\alpha^{(3)}(t),$$

for $t \in [0,1]$. The functions are shown in the following picture:



We note the following properties:

(a) $\alpha^{(k)}$ is continuous with $\|\alpha^{(k)}\|_\infty < 2$, for $k = 1, \dots, 6$;

(b) $\alpha^{(1)}(0) = \alpha^{(5)}(0) = \alpha^{(3)}(0)$ and $\alpha^{(4)}(0) = \alpha^{(2)}(0) = \alpha^{(6)}(0)$;

(c) $\alpha^{(1)}(1) = \alpha^{(5)}(1)$, $\alpha^{(3)}(1) = \alpha^{(4)}(1)$, and $\alpha^{(2)}(1) = \alpha^{(6)}(1)$;

(d) $\alpha^{(6)}(t) < \alpha^{(2)}(t) < \alpha^{(4)}(t) < \alpha^{(3)}(t) < \alpha^{(5)}(t) < \alpha^{(1)}(t)$ for each $t \in (0,1)$.

For each $k = 1, \dots, 6$, we define $f_{kk}\colon [0,1] \to A$ by

$$f_{kk}(t) := \sum_{j \in J} \big(\lambda_j + \mu\alpha^{(k)}(t)\big) q_j$$

for $t \in [0,1]$. We let $e_{kl} \in M_6$ denote the matrix units, for $k, l = 1, \dots, 6$. Then define $f, g\colon [0,1] \to A \otimes M_6$ by

$$\begin{aligned} f(t) &:= \sum_{k=1}^{6} f_{kk}(t) \otimes e_{kk}, \\ g(t) &:= b \otimes 1 + a \otimes (e_{12} + e_{21} + e_{56} + e_{65}) + \mu t(e_{23} + e_{32} + e_{46} + e_{64}) \\ &\quad + (1-t)a(e_{34} + e_{43}) \end{aligned}$$

for $t \in [0,1]$.

This means that $f(t)$ and $g(t)$ have the following matrix form:

$$f(t) := \begin{pmatrix} f_{11} & & \\ & \ddots & \\ & & f_{66} \end{pmatrix}, \quad g(t) := \left(\begin{array}{cc|cc|cc} b & a & & & & \\ a & b & \mu t & & & \\ \hline & \mu t & b & (1-t)a & & \\ & & (1-t)a & b & & \mu t \\ \hline & & & & b & a \\ & & & \mu t & a & b \end{array}\right).$$

We view elements in $A \otimes Z_{2,3}$ as continuous functions $[0,1] \to A \otimes M_6$. By (a), f is continuous. Using (b) and (c), we deduce that f belongs to $A \otimes Z_{2,3}$. We also have $g \in A \otimes Z_{2,3}$. Set

$$B := C^*(f, g, x_2' \otimes 1, \dots, x_{n+1}' \otimes 1) \subseteq A \otimes Z_{2,3}.$$

For each $t \in [0,1]$, we let $B(t) \subseteq A \otimes M_6$ be the image of B under the evaluation map $A \otimes Z_{2,3} \to A \otimes M_6$, $h \mapsto h(t)$. We use e_{kl} to denote the matrix units in M_6.

Claim 1a. *Let* $t \in (0,1)$. *Then* $1 \otimes M_6 \subseteq B(t)$. To prove the claim, note that the spectrum of $f_{kk}(t)$ is

$$\sigma\big(f_{kk}(t)\big) = \big\{\lambda_j + \alpha_j^{(k)}(t) : j \in J\big\},$$

for $k = 1, \dots, 6$. Using (d), we obtain that $\sigma(f_{kk}(t))$ and $\sigma(f_{ll}(t))$ are disjoint whenever $k \neq l$.

Given $k \in \{1, \dots, 6\}$, let $h_k\colon \mathbb{R} \to [0,1]$ be a continuous function that takes the value 1 on $\sigma(f_{kk}(t))$ and that takes the value 0 on $\{0\} \cup \bigcup_{l \neq k} \sigma(f_{ll}(t))$. Then $1 \otimes e_{kk} = h_k(f(t)) \in B(t)$. Thus, $B(t)$ contains the diagonal matrix units of $1 \otimes M_6$.

It follows that $B(t)$ contains

$$1 \otimes e_{23} = \frac{1}{\mu t}(1 \otimes e_{22}) g(t) (1 \otimes e_{33}) \quad \text{and} \quad 1 \otimes e_{46} = \frac{1}{\mu t}(1 \otimes e_{44}) g(t) (1 \otimes e_{66}).$$

We have $a \otimes e_{12} = (1 \otimes e_{11})g(t)(1 \otimes e_{22}) \in B(t)$, and thus

$$a^2 \otimes e_{22} = (a \otimes e_{12})^*(a \otimes e_{12}) \in B(t).$$

As in the proof of Proposition 4.2, we use that a is positive and invertible, to deduce $a^{-1} \otimes e_{22} \in B(t)$, and thus

$$1 \otimes e_{12} = (a \otimes e_{12})(a^{-1} \otimes e_{22}) \in B(t).$$

Analogously, we get that $B(t)$ contains $1 \otimes e_{34}$ and $1 \otimes e_{56}$. It follows that $B(t)$ contains $1 \otimes e_{kl}$ for every $k, l \in \{1, \dots, 6\}$, and so $1 \otimes M_6 \subseteq B(t)$. Similarly, one proves:

Claim 1b. *We have*

$$1 \otimes (e_{kl} + e_{k+2,l+2} + e_{k+4,l+4}) \in B(0), \quad \text{for } k, l \in \{1, 2\}.$$

Claim 1c. *We have* $1 \otimes (e_{33} + e_{44}) \in B(1)$ *and*

$$1 \otimes (e_{kl} + e_{k+4,l+4}) \in B(1), \quad \textit{for } k, l \in \{1, 2\},$$

and

$$1 \otimes (e_{l3} + e_{l+4,4}),\ 1 \otimes (e_{3l} + e_{4,l+4}) \in B(1), \quad \textit{for } l \in \{1, 2\}.$$

Claim 2. *Let* $t \in [0, 1]$. *Then* $z \otimes 1_{M_6} \in_\varepsilon B(t)$. First, we assume that $t \in (0, 1)$. By Claim 1a, we have $1 \otimes e_{11}, 1 \otimes e_{21} \in B(t)$, and thus

$$a \otimes e_{11} = (1 \otimes e_{11})g(t)(1 \otimes e_{21}) \in B(t).$$

Analogously, we obtain

$$b \otimes e_{11},\ f_{11}(t) \otimes e_{11},\ x'_2 \otimes e_{11}, \dots, x'_{n+1} \otimes e_{11} \in B(t).$$

We have

$$\left\| f_{11}(t) - c \right\| \le \left\| f_{11}(t) - c' \right\| + \|c' - c\| < 2\mu + \frac{\delta}{2} \le \delta.$$

By choice of δ, we get

$$\left\| z - p\big(a, b, f_{11}(t), x'_2, \dots, x'_{n+1}\big) \right\| < \varepsilon,$$

and thus

$$z \otimes e_{11} \in_\varepsilon B(t).$$

It follows that $z \otimes e_{kk} \in_\varepsilon B(t)$ for each k, and consequently $z \otimes 1 \in_\varepsilon B(t)$.

Next, we consider the case $t = 0$. Set $\tilde{e}_{kl} := e_{kl} + e_{k+2,l+2} + e_{k+4,l+4} \in M_6$ for $k, l \in \{1, 2\}$. By Claim 1b, we have $1 \otimes \tilde{e}_{kl} \in B(0)$ for each k, l, and thus

$$a \otimes \tilde{e}_{11} = (1 \otimes \tilde{e}_{11})g(0)(1 \otimes \tilde{e}_{21}) \in B(0).$$

Analogously, we obtain

$$b \otimes \tilde{e}_{11},\ f_{11}(0) \otimes \tilde{e}_{11},\ x'_2 \otimes \tilde{e}_{11}, \dots, x'_{n+1} \otimes \tilde{e}_{11} \in B(0).$$

Arguing as in the proof of Claim 1a, we get $z \otimes \tilde{e}_{11} \in_\varepsilon B(0)$. It follows that $z \otimes \tilde{e}_{22} \in_\varepsilon B(0)$, and consequently $z \otimes 1 \in_\varepsilon B(0)$.

Similarly, one proves $z \otimes 1 \in_\varepsilon B(1)$.

Claim 3. *Let $s, t \in [0,1]$ with $s \neq t$. Then there exists $d \in B$ such that $d(s) = 0$ and $d(t) = 1$.*
We first consider the case $s < t$. By choice of μ, the intervals $[\lambda_j - 2\mu, \lambda_j + 2\mu]$ are pairwise disjoint for $j \in J$. We may, therefore, choose a continuous function $h\colon \mathbb{R} \to [0,1]$ that takes the value 0 on

$$S := \bigcup_{j \in J} \left([\lambda_j - 2\mu, \lambda_j - (1-s)\mu] \cup [\lambda_j + (1-s)\mu, \lambda_j + 2\mu] \right)$$

and the value 1 on

$$T := \bigcup_{j \in J} [\lambda_j - (1-t)\mu, \lambda_j + (1-t)\mu].$$

Note that T consists of the real numbers that have distance at most $(1-t)\mu$ to some λ_j. For the purposes below, one could consider S as the real numbers that have distance at least $(1-s)\mu$ to each λ_j.

Set $c := h(f) \in B$, the element obtained by applying functional calculus for h to f. Since $f = \mathrm{diag}(f_{11}, \ldots, f_{66})$, we have $c = \mathrm{diag}(h(f_{11}), \ldots, h(f_{66}))$.

Let $r \in [0,1]$. We have

$$\sigma\big(f_{kk}(r)\big) = \big\{\lambda_j + \mu\alpha^{(k)}(r) : j \in J\big\}.$$

If $k \in \{1,5,2,6\}$, then $|\alpha^{(k)}(r)| \geq 1$, and thus

$$\big|\lambda_j - \big(\lambda_j + \mu\alpha^{(k)}(r)\big)\big| \geq \mu \geq (1-s)\mu,$$

whence

$$\sigma\big(f_{kk}(r)\big) \subseteq S.$$

It follows that

$$h(f_{11}) = h(f_{55}) = h(f_{22}) = h(f_{66}) = 0.$$

For $k \in \{3,4\}$, we have

$$\big|\lambda_j - \big(\lambda_j + \mu\alpha^{(k)}(s)\big)\big| = \mu|1-s| \quad \text{and} \quad \big|\lambda_j - \big(\lambda_j + \mu\alpha^{(k)}(t)\big)\big| = \mu|1-t|,$$

whence

$$\sigma\big(f_{kk}(s)\big) \subseteq S \quad \text{and} \quad \sigma\big(f_{kk}(t)\big) \subseteq T.$$

It follows that

$$h(f_{33})(s) = h(f_{44})(s) = 0 \quad \text{and} \quad h(f_{33})(t) = h(f_{44})(t) = 1.$$

In conclusion, we have

$$c(s) = 0 \quad \text{and} \quad c(t) = 1 \otimes (e_{33} + e_{44}).$$

If $t < 1$, then, by Claim 1a, there exist $b_{kl} \in B$ such that $b_{kl}(t) = 1 \otimes e_{kl}$ for $k, l \in \{1, \dots, 6\}$. Then the following element has the desired properties:

$$d := c + b_{13}cb_{31} + b_{23}cb_{32} + b_{53}cb_{35} + b_{63}cb_{36}.$$

If $t = 1$, then, by Claim 1c, there exist $b_{k3}, b_{3l} \in B$ such that $b_{k3}(1) = 1 \otimes (e_{k3} + e_{k+4,4})$ and $b_{3l}(1) = 1 \otimes (e_{3l} + e_{4,l+4})$ for $k, l \in \{1, 2\}$. Then the following element has the desired properties:

$$d := c + b_{13}cb_{31} + b_{23}cb_{32}.$$

Next, let us indicate how to proceed in the case $s > t$. Let $h\colon \mathbb{R} \to [0,1]$ be a continuous function that takes the value 0 on

$$S := \bigcup_{j \in J} \left([\lambda_j - 2\mu, \lambda_j - \mu] \cup \left[\lambda_j - (1-s)\mu, \lambda_j + (1-s)\mu\right] \cup [\lambda_j + \mu, \lambda_j + 2\mu] \right)$$

and the value 1 on

$$T := \bigcup_{j \in J} \left\{ \lambda_j - (1-t)\mu, \lambda_j + (1-t)\mu \right\}.$$

The element $c := h(f) \in B$ satisfies $c(s) = 0$ and $c(t) \neq 0$. Similar as in the case $s < t$, one then constructs $d \in B$ such that $d(s) = 0$ and $d(t) = 1$, which proves the claim.

Claim 3 verifies condition (b) in Lemma 2.8, whence B contains $C([0,1])$. Thus, B is a $C([0,1])$-subalgebra of $A \otimes Z_{2,3}$. If π_t denotes the quotient map from $A \otimes Z_{2,3}$ to the fiber at t, then Claim 2 shows that $\pi_t(z \otimes 1) \in_\varepsilon \pi_t(B) = B(t)$ for every $t \in [0,1]$. By [3, Lemma 2.1], we get $z \otimes 1 \in_\varepsilon B$, which proves the result. ■

Theorem 5.3. *Let A be a separable, $\mathcal{Z}$-stable C^*-algebra of real rank zero. Then A has generator rank one. In particular, a generic element of A is a generator.*

Proof. We first prove the theorem under the additional assumption that A is unital. In this case, we have $\mathrm{gr}(A) \leq 8$ by Proposition 4.3. Next, we successively reduce the upper bound for $\mathrm{gr}(A)$.

Claim. *Let $n \in \mathbb{N}$ such that* $\mathrm{gr}(A) \leq n + 2$. *Then* $\mathrm{gr}(A) \leq n + 1$. To prove the claim, we verify condition (3) of Lemma 4.1. Let $x_0, \dots, x_{n+1} \in A_{\mathrm{sa}}$, $\varepsilon > 0$, and $z \in A$. We need to find $y_0, \dots, y_{n+1} \in (A \otimes \mathcal{Z})_{\mathrm{sa}}$ such that

$$\left\| y_j - (x_j \otimes 1) \right\| < \varepsilon \text{ for } j = 0, \dots, n+1 \quad \text{and} \quad z \otimes 1 \in_\varepsilon C^*(y_0, \dots, y_{n+1}).$$

By identifying $Z_{2,3}$ with a unital sub-C^*-algebra of $\mathcal{Z}$, elements $y_0, \dots, y_{n+1}$ with the desired properties are provided by Lemma 5.2.

Applying the claim seven times, we obtain that $\mathrm{gr}(A) \leq 1$.

If A is nonunital, we use that A is separable and has real rank zero to choose an increasing approximate unit $(p_n)_n$ of projections in A; see [2, Proposition 2.9]. For each n, we consider the unital corner $A_n := p_n A p_n$. By [2, Corollary 2.8], real rank zero passes to hereditary sub-C^*-algebras. By [17, Corollary 3.1], $\mathcal{Z}$-stability passes to hereditary sub-C^*-algebras. Thus, A_n is a unital, separable, $\mathcal{Z}$-stable C^*-algebra of real rank zero, and thus $\mathrm{gr}(A_n) \leq 1$. By Theorem 2.4, we get

$$\mathrm{gr}(A) \leq \liminf_n \mathrm{gr}(A_n) = 1.$$

Since A is noncommutative (if nonzero), we have $\mathrm{gr}(A) \neq 0$ by Proposition 2.6, and so $\mathrm{gr}(A) = 1$. Since A is separable and has real rank zero, $\mathrm{gr}(A) = 1$ means that generators in A are a dense G_δ-subset; see Remark 2.7 ■

Remark 5.4. Using the methods developed in [13, Section 4], one can remove the assumption of separability in Theorem 5.3: every $\mathcal{Z}$-stable C^*-algebra of real rank zero has generator rank one. They key point is that, for every $\mathcal{Z}$-stable C^*-algebra A and every separable sub-C^*-algebra $B_0 \subseteq A$, there exists a separable, $\mathcal{Z}$-stable sub-C^*-algebra $B \subseteq A$ with $B_0 \subseteq B$. Similar methods are used in the proof of Corollary 5.5 below.

For the definition and the basic properties of pure infiniteness for nonsimple C^*-algebras, we refer to [7].

Corollary 5.5. *Every nuclear, purely infinite C^*-algebra of real rank zero has generator rank one.*

Proof. Let A be a nuclear, purely infinite C^*-algebra of real rank zero. As in [13, Paragraph 4.1], we let $\mathrm{Sub}_{\mathrm{sep}}(A)$ denote the collection of separable sub-C^*-algebras of A; a subset $\mathcal{S} \subseteq \mathrm{Sub}_{\mathrm{sep}}(A)$ is *σ-complete* if $\overline{\bigcup \mathcal{T}} \in \mathcal{S}$ for every countable, directed subset $\mathcal{T} \subseteq \mathcal{S}$; a subset $\mathcal{S} \subseteq \mathrm{Sub}_{\mathrm{sep}}(A)$ is *cofinal* if, for every $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$, there is $B \in \mathcal{S}$ with $B_0 \subseteq B$.

We let $\mathcal{S}_{\mathrm{nuc}}$, $\mathcal{S}_{\mathrm{pi}}$, and $\mathcal{S}_{\mathrm{rr0}}$ denote the sets of separable sub-C^*-algebras of A that are nuclear, purely infinite, and have real rank zero, respectively. It follows from [1, Paragraph II.9.6.5 and Proposition IV.3.1.9] that $\mathcal{S}_{\mathrm{nuc}}$ is σ-complete and cofinal. Using [7, Proposition 4.18 and Corollary 4.22], it follows that $\mathcal{S}_{\mathrm{pi}}$ is σ-complete and cofinal. Lastly, it was noted at the end of [13, Paragraph 4.1] that real rank zero satisfies the "Löwenheim–Skolem condition," which means that $\mathcal{S}_{\mathrm{rr0}}$ is σ-complete and cofinal. Set

$$\mathcal{S} := \mathcal{S}_{\mathrm{nuc}} \cap \mathcal{S}_{\mathrm{pi}} \cap \mathcal{S}_{\mathrm{rr0}}.$$

It is well known that the intersection of countably many σ-complete, cofinal subsets is again σ-complete and cofinal, whence $\mathcal{S}$ is σ-complete and cofinal.

Let $B \in \mathcal{S}$. Then B is a separable, nuclear, purely infinite C^*-algebra of real rank zero. It follows from [8, Theorem 9.1] that B is $\mathcal{O}_\infty$-stable, and thus $\mathcal{Z}$-stable. By Theorem 5.3, we have $\mathrm{gr}(B) = 1$. Since A is the inductive limit of the system $\mathcal{S}$ (indexed over itself), we obtain $\mathrm{gr}(A) \leq 1$ by Theorem 2.4. Since purely infinite C^*-algebras are by definition noncommutative, we have $\mathrm{gr}(A) \neq 0$ by Proposition 2.6, and so $\mathrm{gr}(A) = 1$. ■

Recall that a *Kirchberg algebra* is a separable, simple, nuclear, purely infinite C^*-algebra. By Zhang's theorem [1, Proposition V.3.2.12, p. 454], every simple, purely infinite C^*-algebra has real rank zero. Thus, Kirchberg algebras have real rank zero. Applying Corollary 5.5, we obtain the following corollary.

Corollary 5.6. *Every Kirchberg algebra has generator rank one.*

Let us say that a simple, nuclear C^*-algebra is *classifiable* if it is unital, separable, $\mathcal{Z}$-stable and satisfies the universal coefficient theorem (UCT). By the recent breakthrough in the Elliott classification program [4–6, 16], two simple, nuclear, classifiable C^*-algebras are isomorphic if and only if their Elliott invariants (K-theoretic and tracial data) are isomorphic.

Corollary 5.7. *Let A be a unital, separable, simple, nuclear, $\mathcal{Z}$-stable C^*-algebra satisfying the UCT. Then A has generator rank one. In particular, a generic element in A is a generator.*

Proof. By [10, Theorem 4.1.10], A is either stably finite or purely infinite. In the second case, A is a Kirchberg algebra and we obtain $\mathrm{gr}(A) = 1$ by Corollary 5.6. (The purely infinite case does not require the UCT.)

In the first case, it follows from [16, Theorem 6.2 (iii)] that A is an ASH algebra. By [14, Theorem 5.10], every $\mathcal{Z}$-stable ASH algebra has generator rank one. Thus, we have $\mathrm{gr}(A) = 1$ in either case.

Since A is unital and separable, $\mathrm{gr}(A) = 1$ means that generators in A are a dense G_δ-subset; see Remark 2.7. ∎

Remarks 5.8. (1) It seems likely that the proof of the main theorem can be generalized to show the following: if A is a unital, separable, $\mathcal{Z}$-stable C^*-algebra such that $A \otimes M_{2^\infty}$ and $A \otimes M_{3^\infty}$ have real rank zero, then A has generator rank one.

(2) Let A be a unital, separable, $\mathcal{Z}$-stable C^*-algebra. By [15, Theorem 3.8], A is singly generated. Our results show that, under additional assumptions, A even contains a dense set of generators. It is reasonable to expect that every $\mathcal{Z}$-stable C^*-algebra has generator rank one. However, by [13, Proposition 3.10], the real rank is a lower bound for the generator rank, and it is not known that every $\mathcal{Z}$-stable C^*-algebra has real rank at most one.

Note that every unital, separable, *simple*, $\mathcal{Z}$-stable C^*-algebra has real rank at most one: it is either purely infinite and then has real rank zero or it is stably finite and thus has stable rank one by [11, Theorem 6.7], which entails real rank at most one. Therefore, the following question has no obvious obstruction:

Question 5.9. Does every unital, separable, *simple*, $\mathcal{Z}$-stable C^*-algebra have generator rank one?

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