J. Noncommut. Geom. 16 (2022), 1445–1477
DOI 10.4171/JNCG/484

The inverse function theorem for curved L-infinity spaces

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Abstract. In this paper, we prove an inverse function theorem in derived differential geometry. More concretely, we show that a morphism of curved L_∞ spaces which is a quasi-isomorphism at a point has a local homotopy inverse. This theorem simultaneously generalizes the inverse function theorem for smooth manifolds and the Whitehead theorem for L_∞ algebras. The main ingredients are the obstruction theory for L_∞ homomorphisms (in the curved setting) and the homotopy transfer theorem for curved L_∞ algebras. Both techniques work in the A_∞ case as well.

1. Introduction

1.1. Curved L_∞ spaces

The notion of *curved L_∞ space* was introduced by Costello [8], as an alternative approach to derived differential geometry. In [8] an L_∞ space is defined as a pair $(M, \mathfrak{G})$ where M is a smooth manifold, and $\mathfrak{G}$ is a *curved L_∞ algebra* (see [17] for a definition) over the de Rham algebra Ω^*_M. In this paper, we shall work with a more down-to-earth notion of L_∞ space, following analogous constructions in the theory of dg-schemes by Behrend [2, 3], and Ciocan-Fontanine–Kapranov [6, 7]. This notion is however equivalent to the original one – see [22] for a proof of the equivalence. Interestingly, a related concept also appeared in the study of deformations of coisotropic submanifolds in symplectic geometry [14, 18]. Conceptually, the L_∞ approach to derived geometric structures is Koszul dual to the more classical approach using dg (or simplicial) commutative algebras as developed by Toën–Vezzosi [21]. Such structures naturally appear in various gauge theories, producing L_∞ enhancements of the associated Maurer–Cartan moduli spaces.

More precisely, throughout the paper, an L_∞ space $\mathbb{M} = (M, \mathfrak{g})$ is given by a pair of M a smooth manifold and $\mathfrak{g}$ a curved L_∞ algebra over the ring of smooth functions $C^\infty(M)$. We also require that $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \cdots \oplus \mathfrak{g}_d,$$

where each $\mathfrak{g}_i$ is a vector bundle in degree i. In particular, it has minimal degree 2 and maximal degree d for some $d \geq 2$. Conceptually, this grading condition reflects the fact that we are interested in derived schemes, not stacks.

2020 Mathematics Subject Classification. Primary 14A30; Secondary 18N99, 58H15.
Keywords. L_∞ space, inverse function theorem, derived differential geometry.

The main goal of this paper is to understand when maps of L_∞ spaces are "invertible". Let us introduce some terminology and notation in order to describe our results. Throughout the paper, we use the notation

$$\mu_k : \mathfrak{g}[1] \otimes \cdots \otimes \mathfrak{g}[1] \to \mathfrak{g}[1], \quad k \geq 0$$

to denote the (shifted) higher brackets of an L_∞ algebra. We shall use the same notation in the A_∞ case as well. Note that in the L_∞ case, the maps μ_k are all graded symmetric by definition. Given an L_∞ space $\mathbb{M} = (M, \mathfrak{g})$, the curvature term $\mu_0 \in \mathfrak{g}_2$ is a section of the bundle $\mathfrak{g}_2$. Let $p \in \mu_0^{-1}(0)$ be a point in its zero locus. The tangent complex at p is defined as the chain complex

$$T_p\mathbb{M} := T_pM \xrightarrow{\mathrm{d}\mu_0|_p} \mathfrak{g}_2|_p \xrightarrow{\mu_1|_p} \mathfrak{g}_3|_p \xrightarrow{\mu_1|_p} \cdots .$$

The fact that this is a chain complex follows from the L_∞ algebra equation together with $\mu_0(p) = 0$. A morphism between two curved L_∞ spaces $\mathbb{M} = (M, \mathfrak{g})$ and $\mathbb{N} = (N, \mathfrak{h})$ is given by a pair $\mathfrak{f} = (f, f^\sharp)$ where $f : M \to N$ is a smooth map and $f^\sharp = (f_1^\sharp, f_2^\sharp, \ldots) : \mathfrak{g} \to f^*\mathfrak{h}$ is a sequence of bundle maps which define an L_∞ homomorphism.

Let $\mathfrak{f} : \mathbb{M} \to \mathbb{N}$ be a morphism and $p \in M$ a point in the zero locus of the curvature of $\mathfrak{g}$. This morphism induces a map between the tangent complexes $d\mathfrak{f}_p : T_p\mathbb{M} \to T_{f(p)}\mathbb{N}$, explicitly given by

$$\begin{array}{ccccccc} T_pM & \longrightarrow & \mathfrak{g}_2|_p & \longrightarrow & \mathfrak{g}_3|_p & \longrightarrow & \cdots \\ {\scriptstyle df_p}\downarrow & & {\scriptstyle f_1^\sharp}\downarrow & & {\scriptstyle f_1^\sharp}\downarrow & & \\ T_{f(p)}N & \longrightarrow & \mathfrak{h}_2|_{f(p)} & \longrightarrow & \mathfrak{h}_3|_{f(p)} & \longrightarrow & \cdots \end{array}$$

We are now ready to state our first main result which states that if $d\mathfrak{f}_p$ is a quasi-isomorphism of chain complexes then $\mathfrak{f}$ is "locally invertible".

Theorem 1.1. *Let $(M, \mathfrak{g})$ and $(N, \mathfrak{h})$ be L_∞ spaces and $\mathfrak{f} = (f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ be an L_∞ morphism. Assume that the tangent map $d\mathfrak{f}_p$ is a quasi-isomorphism at $p \in (\mu_0^{\mathbb{M}})^{-1}(0)$. Then there exist open neighborhoods U of p and V of $f(p)$ such that the restriction*

$$\mathfrak{f}|_U : (U, \mathfrak{g}|_U) \to (V, \mathfrak{h}|_V)$$

is a homotopy equivalence[1] *of L_∞ spaces.*

In the case when both $\mathfrak{g}$ and $\mathfrak{h}$ are trivial, this is simply the inverse function theorem for smooth manifolds. When the L_∞ bundle $\mathfrak{g}$ is concentrated in degree 2, we obtain the notion of m-Kuranishi neighborhood in the work of Joyce [13], or Kuranishi chart (with trivial isotropy) introduced by Fukaya–Oh–Ohta–Ono [12]. In this special case,

[1]The notion of homotopy between L_∞ spaces is formulated in Definition 4.5.

our theorem essentially recovers [13, Theorem 4.16], but with a slightly different notion of homotopy. In the case when both M and N are a point, the above theorem recovers the well-known fact (see [19] and [15]) that quasi-isomorphisms between (uncurved) L_∞ algebras are homotopy equivalences – this statement is referred to as the Whitehead theorem for L_∞ algebras in [12].

In forthcoming work, we shall use the complex analytic version of the above theorem to prove local invariance of the Maurer–Cartan moduli space associated with *bounded* L_∞ algebras (see [22]). This result is essential to understand L_∞ enhancements of moduli spaces from gauge theory, such as moduli spaces of flat connections on a vector bundle.

We also expect a global version of Theorem 1.1 to hold. We formulate it in the following conjecture.

Conjecture 1.2. *Let $\mathfrak{f} = (f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ be an L_∞ morphism. Assume that the tangent map $d\mathfrak{f}_p$ is a quasi-isomorphism for all $p \in (\mu_0^{\mathbb{M}})^{-1}(0)$, and that the induced map $f : (\mu_0^{\mathbb{M}})^{-1}(0) \to (\mu_0^{\mathbb{N}})^{-1}(0)$ on the zero loci is a bijection. Then there exist open neighborhoods U of $(\mu_0^{\mathbb{M}})^{-1}(0)$ and V of $(\mu_0^{\mathbb{N}})^{-1}(0)$ such that the restriction*

$$\mathfrak{f}|_U : (U, \mathfrak{g}|_U) \to (V, \mathfrak{h}|_V)$$

is a homotopy equivalence[2] of L_∞ spaces.

At this moment we are able to prove this conjecture on the special case where both $\mathfrak{g}$ and $\mathfrak{h}$ are concentrated in degree 2. Our proof of this result uses a partition-of-unity argument similar to the one used by Joyce in [13] where an analogous result is proved for m-Kuranishi spaces.

The second main result of the paper is the construction of a *minimal chart* around a point $p \in \mu_0^{-1}(0) \subset M$. This construction generalizes the so-called minimal model construction for (uncurved) L_∞ algebras (see [9, 15, 23]). More precisely, we have the following:

Theorem 1.3. *Let $\mathbb{M} = (M, \mathfrak{g})$ be an L_∞ space and $p \in \mu_0^{-1}(0)$. Then there exists an L_∞ space $\mathbb{W} = (W, \mathfrak{h})$ with $W \subset M$ a submanifold containing p, such that*

- *the tangent complex $T_p\mathbb{W}$ has zero differential;*
- *there exists an open neighborhood $U \subset M$ of p which contains W and such that the inclusion map $i : W \to U$ extends to a homotopy equivalence*

$$(i, i^\sharp) : (W, \mathfrak{h}) \to (U, \mathfrak{g}|_U)$$

of L_∞ spaces. Here, the map $i^\sharp : \mathfrak{h} \to i^*\mathfrak{g}$ is an L_∞ homomorphism constructed explicitly using summation over trees. We refer to Section 5.2 for details.

[2] See Definition 4.5 for homotopy equivalence of L_∞ spaces. In this global setting it requires the existence of torsion-free, flat connections on M and N.

This theorem is one of the main ingredients in our proof of Theorem 1.1 but we expect it will have many other applications in derived differential geometry. For instance, the analogous statement in Derived Algebraic Geometry is an important step in the proof of the Darboux theorem for shifted symplectic derived schemes in [5].

1.2. About the proofs

The proof of Theorem 1.3 generalizes the homotopy transfer theorem (sometimes also called homological perturbation lemma) to the curved setting. We would like to point out that we do not impose any conditions on the curvature term, unlike the filtered case considered in [11, 12]. In that case, one assumes the algebra is equipped with a filtration and the curvature term lives in the positive part of the filtration. For these filtered algebras the transfer theorem was stated in [11] and proved in detail for A_∞ algebras in [12]. Our theorem is valid for general curved A_∞ or L_∞ algebras in the presence of a generalization of the usual homotopy retraction data (i, p, H). We refer to Section 3 for more details, but for example, the homotopy operator H must satisfy

$$\mu_1 H + H\mu_1 = ip - \mathsf{id} - H\mu_1^2 H,$$

which reduces to the usual homotopy identity in the uncurved case since $\mu_0 = 0$ implies that $\mu_1^2 = 0$. Remarkably, both the minimal L_∞ structure on $\mathfrak{h}$ and the L_∞ homomorphism $i^\sharp$ are given by summation over the same stable trees as in the uncurved setting, and the curvature term does not play a big role (see Section 3). In that sense, the construction is closer to the uncurved case than to the filtered one studied in [11, 12].

For the proof of Theorem 1.1 we first develop an obstruction theory for L_∞ (or A_∞) homomorphisms between curved L_∞ (respectively A_∞) algebras, again without making any additional assumptions on the ground ring or on the curvature term. Since we are in the general curved setting (and therefore have no good notion of convergence), the homomorphisms we consider are by definition strict (in the terminology of [12]), meaning the constant terms f_0 vanish. Therefore our theory is once again closer to the uncurved case than the filtered case for which an analogous theory was developed in [12].

Our obstruction theory is set up in a way which easily generalizes from L_∞ algebras to spaces. Combining this new obstruction theory with Theorem 1.3 we prove Theorem 1.1 using an argument analogous to that in [12].

We present our results in the smooth realm, meaning all the manifolds (and vector bundles, maps ...) are real C^∞ manifolds. But our proofs and constructions also work in the complex analytic setting, with only minor modifications, see Remark 4.6.

1.3. Other related works

A recent preprint [4] by Behrend–Liao–Xu obtained similar results, using the framework of categories of fibrant objects. While the definitions of L_∞ spaces[3] and the tangent

[3] In [4] the authors use derived manifolds.

complexes are clearly the same as ours, it is not clear at the moment of writing how Behrend–Liao–Xu's notion of homotopy between morphisms of L_∞ spaces is related to ours (see Definition 4.5). One could say our approach is more algebraic in the sense that both the obstruction theory and the homological perturbation technique are generalizations of the situation for uncurved algebras.

An interesting question is whether Theorem 1.1 and Theorem 1.3 admit generalizations to allow the tangent complex to have components in non-positive degrees. This is related to the notion of *shifted Lie algebroid structure* studied by Pym–Safronov [20].

1.4. Organization of the paper

In Section 2, we develop the obstruction theory for constructing A_∞/L_∞ homomorphisms between curved A_∞/L_∞ algebras. Section 3 generalizes the homotopy transfer theorem to the curved setting. In Section 4, we recall basic definitions of L_∞ spaces. In particular, we explicitly describe homotopies between L_∞ morphisms. In Section 5, we prove Theorem 1.1 and Theorem 1.3.

2. Obstruction theory

Let $(A, \mu_0^A, \mu_1^A, \ldots)$ and $(B, \mu_0^B, \mu_1^B, \ldots)$ be two curved A_∞/L_∞ algebras over a commutative ring R. In this section, we study the obstruction theory of A_∞/L_∞ homomorphisms from A to B. Classically, in the non-curved case, this is done by using a pro-nilpotent L_∞ algebra structure on the space $\mathsf{Hom}(T^c A[1], B)$ of cochains on A with values in B. In particular, the obstruction class to construct the $(n+1)$-th component of an A_∞/L_∞ homomorphism $(f_j)_{j=1}^n$ from A to B is a cohomology class

$$\mathfrak{o}\big((f_j)_{j=1}^n\big) \in H^1\big(\mathsf{Hom}(A[1]^{n+1}, B[1]), d = [\mu_1, -]\big),$$

determined by the first n components. However, adding the curvature term spoils the pro-nilpotent structure, and obviously the above obstruction space is not even defined as μ_1^A and μ_1^B might not square to zero. In this section, we define a variant of the above obstruction space which takes into account the appearance of curvatures, which allows us to extend the obstruction theory of A_∞/L_∞ homomorphisms to the curved setting.

2.1. Definition of obstruction spaces

First, using the curvature term μ_0^A, we form the complex $C(A, B)$ as follows:

$$\begin{aligned} C(A,B) := \cdots \to \mathsf{Hom}(A[1]^{\otimes k}, B[1]) &\xrightarrow{\delta} \mathsf{Hom}(A[1]^{\otimes k-1}, B[1]) \to \cdots \\ &\to B[1] \to 0, \end{aligned}$$

$$\delta(\phi_k)(a_1, \ldots, a_{k-1}) := \sum_{j=0}^{k-1} (-1)^{|\phi_k|' + |a_1|' + \cdots + |a_j|'} \phi_k(a_1, \ldots, a_j, \mu_0^A, a_{j+1}, \ldots, a_{k-1}).$$

One verifies that $\delta^2 = 0$. We shall denote its cohomology by $D^k(A, B)$ where k is the tensor degree. Next, using the operators μ_1^A and μ_1^B, we define another operator d as

$$d : \mathsf{Hom}(A[1]^{\otimes k}, B[1]) \to \mathsf{Hom}(A[1]^{\otimes k}, B[1]),$$

$$d(\phi_k)(a_1, \dots, a_k) := \mu_1^B \phi_k(a_1, \dots, a_k) - \sum_{j=1}^{k} (-1)^{|\phi_k|' + \star} \phi_k(a_1, \dots, \mu_1^A(a_j), \dots, a_k)$$

where $\star = |a_1|' + \cdots + |a_{j-1}|'$. One can readily verify that $d\delta + \delta d = 0$. Thus, d induces a map on the δ-cohomology, i.e. we obtain maps

$$d : D^k(A, B) \to D^k(A, B), \quad \forall k \geq 0.$$

In complete generality we do not have $d^2 = 0$. However, we have the following:

Lemma 2.1. *If there exists an R-linear map $f : A \to B$ such that $f(\mu_0^A) = \mu_0^B$, then the composition $d^2 : D^k(A, B) \to D^k(A, B)$ equals zero.*

Proof. Choose any such f. Given $[\phi_k] \in D^k(A, B)$, define $\psi \in \mathsf{Hom}(A[1]^{k+1}, B[1])$ as

$$\psi := (-1)^{|\phi_k|'} \sum_{i=0}^{k-1} \phi_k(\mathsf{id}^i \otimes \mu_2^A \otimes \mathsf{id}^{k-1-i}) + \mu_2^B(\phi_k \otimes f) + \mu_2^B(f \otimes \phi_k).$$

One can verify that we have $d^2(\phi_k) = \delta\psi$. Hence $d^2 = 0$ in δ-cohomology. ■

Definition 2.2. Under the assumptions for A and B in Lemma 2.1, we set the k-th obstruction space $H^k(A, B)$ to be the degree one cohomology of the complex $(D^k(A, B), d)$, i.e.

$$H^k(A, B) := H^1\big(D^k(A, B), d\big).$$

2.2. Obstruction classes

Let $f_j : A[1]^{\otimes j} \to B[1]$ $(j = 1, \dots, n)$ be n multi-linear maps of cohomological degree zero, such that the following conditions hold:

(i) The A_∞ homomorphism axiom holds up to $(n-1)$ inputs, i.e. we have

$$\sum_{\substack{j \geq 0, i_1, \dots, i_j \geq 1 \\ i_1 + \cdots + i_j = k}} \mu_j^B(f_{i_1} \otimes \cdots \otimes f_{i_j}) = \sum_{\substack{r \geq 0, s \geq 0, t \geq 0 \\ r+s+t=k}} f_{r+t+1}(\mathsf{id}^{\otimes r} \otimes \mu_s^A \otimes \mathsf{id}^{\otimes t})$$

for all $0 \leq k \leq n-1$.

(ii) In the case with n inputs, we require that

$$\sum_{\substack{j \geq 0, i_1, \dots, i_j \geq 1 \\ i_1 + \cdots + i_j = n}} \mu_j^B(f_{i_1} \otimes \cdots \otimes f_{i_j}) - \sum_{\substack{r \geq 0, t \geq 0, s \geq 1 \\ r+s+t=n}} f_{r+t+1}(\mathsf{id}^{\otimes r} \otimes \mu_s^A \otimes \mathsf{id}^{\otimes t}) \quad (1)$$

is δ-exact, i.e. it lies in the image of $\delta : C^{n+1}(A, B) \to C^n(A, B)$.

Given such a collection of maps $f_j : A[1]^{\otimes j} \to B[1]$ $(j = 1, \ldots, n)$, we define its obstruction class as follows. Choose any f'_{n+1} of cohomological degree zero such that $\delta f'_{n+1}$ equals the expression in (1). Then we set

$$\mathsf{obs}_{n+1} := \sum_{\substack{j \geq 2, i_1, \ldots, i_j \geq 1 \\ i_1 + \cdots + i_j = n+1}} \mu^B_j (f_{i_1} \otimes \cdots \otimes f_{i_j}) - \sum_{\substack{r \geq 0, t \geq 0, s \geq 2 \\ r+s+t = n+1}} f_{r+t+1}(\mathsf{id}^{\otimes r} \otimes \mu^A_s \otimes \mathsf{id}^{\otimes t}) + df'_{n+1}.$$

Lemma 2.3. *The above expression* obs_{n+1} *is* δ*-closed. Moreover,* $d(\mathsf{obs}_{n+1})$ *is* δ*-exact. Thus, it represents a well-defined class which we denote by* $\mathfrak{o}((f_j)^n_{j=1}) \in H^{n+1}(A, B)$*. This class is independent of the choice of* f'_{n+1}*.*

Proof. Denote by F_n the extension of $\sum^n_{j=1} f_j$ as a coalgebra map $T(A[1]) \to T(B[1])$. Similarly, denote by $\widetilde{\mu}_k$ the extension of μ_k as coderivations on the tensor coalgebra. Condition (i) implies that we have

$$\widetilde{\mu}^B F_n = F_n \widetilde{\mu}^A : A[1]^N \to B[1]^M, \quad \forall -1 \leq N - M \leq n-2.$$

Observe also that $\delta(\phi) = (-1)^{|\phi|'} \phi \widetilde{\mu}_0$. Then we compute

$$\begin{aligned}\delta(\mathsf{obs}_{n+1}) &= -\widetilde{\mu}_{\neq 1} F_n \widetilde{\mu}_0 + F_n \widetilde{\mu}_{\geq 2} \widetilde{\mu}_0 - \widetilde{\mu}_1 f'_{n+1} \widetilde{\mu}_0 + f'_{n+1} \widetilde{\mu}_1 \widetilde{\mu}_0 \\ &= (-\widetilde{\mu}_{\neq 1} \widetilde{\mu} F_n + \widetilde{\mu}_{\neq 1} F_n \widetilde{\mu}_{\geq 1}) + F_n \widetilde{\mu}_{\geq 2} \widetilde{\mu}_0 - (\widetilde{\mu}_1 \widetilde{\mu} F_n + \widetilde{\mu}_1 F_n \widetilde{\mu}_{\geq 1}) \\ &\quad + (-\widetilde{\mu} F_n \widetilde{\mu}_1 + F_n \widetilde{\mu}_{\geq 1} \widetilde{\mu}_1).\end{aligned}$$

The two terms $-\widetilde{\mu}_{\neq 1} \widetilde{\mu} F_n$ and $-\widetilde{\mu}_1 \widetilde{\mu} F_n$ combine to give zero since $\widetilde{\mu}\widetilde{\mu} = 0$. The following three terms give

$$\widetilde{\mu}_{\neq 1} F_n \widetilde{\mu}_{\geq 1} + \widetilde{\mu}_1 F_n \widetilde{\mu}_{\geq 1} - \widetilde{\mu} F_n \widetilde{\mu}_1 = \widetilde{\mu} F_n \widetilde{\mu}_{\geq 2}.$$

Since there are n inputs, after applying $\widetilde{\mu}_{\geq 2}$, we are left with at most $n-1$ inputs to apply $\widetilde{\mu} F_n$. In this case, we may use the commutativity $\widetilde{\mu} F_n = F_n \widetilde{\mu}$, i.e. we have the above three terms sum up to

$$\widetilde{\mu}_{\neq 1} F_n \widetilde{\mu}_{\geq 1} + \widetilde{\mu}_1 F_n \widetilde{\mu}_{\geq 1} - \widetilde{\mu} F_n \widetilde{\mu}_1 = F_n \widetilde{\mu} \widetilde{\mu}_{\geq 2} = -F_n \widetilde{\mu} \widetilde{\mu}_0 - F_n \widetilde{\mu} \widetilde{\mu}_1.$$

The last equality follows from $\widetilde{\mu}\widetilde{\mu} = 0$. Now, putting these back into the calculation of $\delta(\mathsf{obs}_{n+1})$, we obtain

$$\begin{aligned}\delta(\mathsf{obs}_{n+1}) &= F_n \widetilde{\mu}_{\geq 2} \widetilde{\mu}_0 + F_n \widetilde{\mu}_{\geq 1} \widetilde{\mu}_1 - F_n \widetilde{\mu} \widetilde{\mu}_0 - F_n \widetilde{\mu} \widetilde{\mu}_1 \\ &= -F_n \widetilde{\mu}_1 \widetilde{\mu}_0 - F_n \widetilde{\mu}_0 \widetilde{\mu}_1 = 0.\end{aligned}$$

Here, we have used the A_∞ relation that $\widetilde{\mu}_0 \widetilde{\mu}_0 = 0$ and $\widetilde{\mu}_0 \widetilde{\mu}_1 + \widetilde{\mu}_1 \widetilde{\mu}_0 = 0$. Next, we prove that $d(\mathsf{obs}_{n+1})$ is δ-exact. We use the notation $F'_{n+1} : T(A[1]) \to T(B[1])$ to denote the extension of $f_1, \ldots, f_n, f'_{n+1}$ to the tensor coalgebra. By definition of f'_{n+1} we have that

$$\widetilde{\mu} F'_{n+1} = F'_{n+1} \widetilde{\mu} : A[1]^N \to B[1]^M, \quad \forall -1 \leq N - M \leq n-1.$$

Using the notation F'_{n+1}, we may write the obstruction as

$$\mathsf{obs}_{n+1} = \widetilde{\mu} F'_{n+1} - F'_{n+1}\widetilde{\mu}_{\geq 1} : A[1]^{n+1} \to B[1].$$

Applying the operator d to it yields

$$d(\mathsf{obs}_{n+1}) = \widetilde{\mu}_1\widetilde{\mu} F'_{n+1} - \widetilde{\mu}_1 F'_{n+1}\widetilde{\mu}_{\geq 1} + \widetilde{\mu} F'_{n+1}\widetilde{\mu}_1 - F'_{n+1}\widetilde{\mu}_{\geq 1}\widetilde{\mu}_1.$$

The first and the third terms give

$$\begin{aligned}
\widetilde{\mu}_1\widetilde{\mu} F'_{n+1} + \widetilde{\mu} F'_{n+1}\widetilde{\mu}_1 &= -\widetilde{\mu}_{\geq 2}\widetilde{\mu} F'_{n+1} + \widetilde{\mu} F'_{n+1}\widetilde{\mu}_1 \\
&= -\widetilde{\mu}_{\geq 2} F'_{n+1}\widetilde{\mu} + \widetilde{\mu} F'_{n+1}\widetilde{\mu}_1 \\
&= -\widetilde{\mu}_{\geq 2} F'_{n+1}\widetilde{\mu}_0 - \widetilde{\mu}_{\geq 2} F_n\widetilde{\mu}_{\geq 1} + \widetilde{\mu} F_n\widetilde{\mu}_1 + \widetilde{\mu}_1 f'_{n+1}\widetilde{\mu}_1 \\
&= -\widetilde{\mu}_{\geq 2} F'_{n+1}\widetilde{\mu}_0 - \widetilde{\mu}_{\geq 2} F_n\widetilde{\mu}_{\geq 2} + \widetilde{\mu}_1 f'_{n+1}\widetilde{\mu}_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-\widetilde{\mu}_1 F'_{n+1}\widetilde{\mu}_{\geq 1} - F'_{n+1}\widetilde{\mu}_{\geq 1}\widetilde{\mu}_1 &= -\widetilde{\mu}_1 F'_{n+1}\widetilde{\mu}_{\geq 1} - F'_{n+1}\widetilde{\mu}\widetilde{\mu}_1 \\
&= -\widetilde{\mu}_1 F'_{n+1}\widetilde{\mu}_{\geq 1} + F'_{n+1}\widetilde{\mu}\widetilde{\mu}_0 + F'_{n+1}\widetilde{\mu}\widetilde{\mu}_{\geq 2} \\
&= -\widetilde{\mu}_1 F'_{n+1}\widetilde{\mu}_{\geq 1} + F'_{n+1}\widetilde{\mu}\widetilde{\mu}_0 + \widetilde{\mu} F_n\widetilde{\mu}_{\geq 2} \\
&= F'_{n+1}\widetilde{\mu}\widetilde{\mu}_0 + \widetilde{\mu} F_n\widetilde{\mu}_{\geq 2} - \widetilde{\mu}_1 F_n\widetilde{\mu}_{\geq 2} - \widetilde{\mu}_1 f'_{n+1}\widetilde{\mu}_1 \\
&= F'_{n+1}\widetilde{\mu}\widetilde{\mu}_0 + \widetilde{\mu}_{\geq 2} F_n\widetilde{\mu}_{\geq 2} - \widetilde{\mu}_1 f'_{n+1}\widetilde{\mu}_1.
\end{aligned}$$

Adding the two equations together yields the desired formula

$$d(\mathsf{obs}_{n+1}) = -\widetilde{\mu}_{\geq 2} F'_{n+1}\widetilde{\mu}_0 + F'_{n+1}\widetilde{\mu}\widetilde{\mu}_0 = \delta\big(\widetilde{\mu}_{\geq 2} F'_{n+1} - F'_{n+1}\widetilde{\mu}\big).$$

Finally, to see that the class $\mathfrak{o}((f_j)_{j=1}^n) = [\mathsf{obs}_{n+1}]$ is independent of f'_{n+1}, let f''_{n+1} be another such map. Then we have the two obstructions differ by $d(f'_{n+1} - f''_{n+1})$ with $\delta(f'_{n+1} - f''_{n+1}) = 0$, this proves the two obstruction classes are equal. ∎

2.3. $A_{(n)}$ homomorphisms

In the following, we shall refer to a collection of maps $(f_j : A[1]^{\otimes j} \to B[1])_{j=1}^n$ satisfying conditions (i), (ii) in Section 2.2 as an $A_{(n)}$ homomorphism from A to B. Obviously, an $A_{(n)}$ homomorphism is also an $A_{(k)}$ homomorphism, for any $k \leq n$. Just as in the case of the usual A_∞ homomorphisms, one can compose $A_{(n)}$ homomorphisms, using the formula

$$(g \circ f)_j := \sum_{\substack{l, i_1, \ldots, i_l \geq 1 \\ i_1 + \cdots + i_l = j}} g_l(f_{i_1} \otimes \cdots \otimes f_{i_l}), \tag{2}$$

for $j = 1, \ldots, n$. If we denote by f'_{n+1} (and g'_{n+1}) a map such that $\delta f'_{n+1}$ equals the expression in (1), then we can define

$$(g \circ f)'_{n+1} := g'_{n+1}(f_1 \otimes \cdots \otimes f_1) + g_1(f'_{n+1}) + \sum_{\substack{l, i_1, \ldots, i_l \geq 1 \\ i_1 + \cdots + i_l = n+1}} g_l(f_{i_1} \otimes \cdots \otimes f_{i_l}).$$

Please note that this composition is strictly associative.

The simple but crucial observation is that an $A_{(n)}$ homomorphism lifts to an $A_{(n+1)}$ homomorphism if and only if its obstruction class $\mathfrak{o}((f_j)_{j=1}^n)$ vanishes.

Next, we recall the notion of homotopy between $A_{(n)}$ homomorphisms. Let $\Omega^*_{[0,1]}$ be piece-wise polynomial differential forms on the interval $[0,1]$ (as in [12, Definition 4.2.9]). This is a unital dg-algebra, therefore given an A_∞ algebra B we can easily define the tensor product $B \otimes \Omega^*_{[0,1]}$ (see [1] for details). Moreover, there are naive (also called linear) maps of A_∞ algebras $\mathsf{ev}_0, \mathsf{ev}_1 : B \otimes \Omega^*_{[0,1]} \to B$, given by "evaluating at $t = 0, 1$". Two $A_{(n)}$ homomorphisms $f = (f_1, \dots, f_n)$, $g = (g_1, \dots, g_n) : A \to B$ are called homotopic, denoted by $f \cong g$, if there exists an $A_{(n)}$ homomorphism

$$F = (F_1, \dots, F_n) : A \to B \otimes \Omega^*_{[0,1]}$$

such that $\mathsf{ev}_0 \circ F = f$ and $\mathsf{ev}_1 \circ F = g$. The following properties are standard:

- The homotopy relation "$\cong$" between $A_{(n)}$ morphisms is an equivalence relation.
- If $h : A' \to A$ is another $A_{(n)}$ morphism and $f \cong g$, then $f \circ h \cong g \circ h$.
- If $h : B \to B'$ is another $A_{(n)}$ morphism and $f \cong g$, then $h \circ f \cong h \circ g$.
- Two homotopic $A_{(n)}$ homomorphisms $(f_j)_{j=1}^n$ and $(g_j)_{j=1}^n$ from A to B have the same obstruction class, i.e. we have

  $$\mathfrak{o}\big((f_j)_{j=1}^n\big) = \mathfrak{o}\big((g_j)_{j=1}^n\big).$$

  Indeed, it is clear that we have

  $$\mathfrak{o}\big((f_j)_{j=1}^n\big) = (\mathsf{ev}_0)_* \mathfrak{o}\big((F_j)_{j=1}^n\big),$$
  $$\mathfrak{o}\big((g_j)_{j=1}^n\big) = (\mathsf{ev}_1)_* \mathfrak{o}\big((F_j)_{j=1}^n\big).$$

  Then observe that every cohomology class in $H^{n+1}(A, B \otimes \Omega^*_{[0,1]})$ can be represented by an element of the form $\phi \otimes \mathbb{1}$ with $\phi \in C^{n+1}(A, B)$ and $\mathbb{1}$ the constant function in $\Omega^*_{[0,1]}$, which clearly shows that applying the two evaluation maps ev_0 and ev_1 both yield $[\phi] \in H^{n+1}(A, B)$.

It is useful to spell out the definition of homotopy in the $n = 1$ case.

Lemma 2.4. *Two $A_{(1)}$ homomorphisms $f_1, g_1 : A \to B$ are $A_{(1)}$ homotopic if and only if there is a map $H : A \to B$ of degree -1, such that $H(\mu_0^A) = 0$ and $f_1 - g_1 - \mu_1^B H - H\mu_1^A$ is δ-exact.*

Proof. An $A_{(1)}$ homotopy gives a map $F_1 : A \to B \otimes \Omega^*_{[0,1]}$. If we write $F_1(a) = f_1^t(a) + (-1)^{|a|'} h_1^t(a) dt$, then the $A_{(1)}$ homomorphism equation for F is equivalent to: $f_1^1 = f_1$, $f_1^0 = g_1$, $h_1^t(\mu_0^A) = 0$ and $-\frac{df_1^t}{dt} + \mu_1^B h_1^t + h_1^t \mu_1^A$ is δ-exact. We obtain the desired equality by taking $H = \int_0^1 h_1^t dt$. ∎

Two A_∞ algebras A and B are called $A_{(n)}$ homotopic, if there exist $A_{(n)}$ homomorphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g$ and $g \circ f$ are both $A_{(n)}$ homotopic to the identity homomorphism.

2.4. Homotopy invariance of obstruction theory

The obstruction spaces are natural with respect to $A_{(1)}$ homomorphisms. Namely, fix two curved A_∞ algebras A and B. Assume that there exists an R-linear map $f : A \to B$ such that $f(\mu_0^A) = \mu_0^B$, so that the obstruction space $H^k(A, B)$ is defined. Let $h : B \to B'$ be an $A_{(1)}$ homomorphism. By definition, we have $hf(\mu_0^A) = h(\mu_0^B) = \mu_0^{B'}$, which shows that the obstruction space $H^k(A, B')$ is also defined. Furthermore, the morphism h induces a push-forward map

$$h_* : H^k(A, B) \to H^k(A, B'),$$

defined by $[\phi] \mapsto [h \circ \phi]$ where $\phi \in \mathsf{Hom}(A[1]^{\otimes k}, B[1])$ is a representative. Similarly, let $g : A'[1] \to A[1]$ be an $A_{(1)}$ homomorphism. We may define the pull-back map

$$g^* : H^k(A, B) \to H^k(A', B),$$

by $[\phi] \mapsto [\phi \circ (g \otimes \cdots \otimes g)]$ where we used k-copies of g in the tensor product.

Lemma 2.5. *Assume that $h : B \to B'$ and $g : A' \to A$ are both $A_{(1)}$ homotopy equivalences. Then both h_* and g^* are isomorphisms.*

Proof. It follows from Lemma 2.4 that if h_0 and h_1 are $A_{(1)}$ homotopic, then $(h_0)_* = (h_1)_*$ and $(h_0)^* = (h_1)^*$. This, together with the identities $(h_0 h_1)_* = (h_0)_*(h_1)_*$ and $(g_1 g_0)^* = (g_0)^*(g_1)^*$, immediately give the result. ∎

2.5. Whitehead theorem for curved A_∞ algebras

Proposition 2.6. *Let $f = (f_1, \ldots, f_n) : A \to B$ be an $A_{(n)}$ homomorphism. Assume that $\mathfrak{o}((f_j)_{j=1}^n) = 0$. Denote by $\mathcal{L}(f)$ the set of liftings of f to an $A_{(n+1)}$ homomorphism modulo the homotopy equivalence relation. Then $\mathcal{L}(f)$ carries a natural transitive action by the abelian group $H^0(D^{n+1}(A, B), d)$.*

Proof. Let f_{n+1} be a lift of f to an $A_{(n+1)}$ homomorphism. Denote by $[f_{n+1}]$ its equivalence class in $\mathcal{L}(f)$. Let $\beta : A[1]^{\otimes n+1} \to B[1]$ be a map representing an element $[\beta] \in H^0(D^{n+1}(A, B), d)$. We define the group action by the formula

$$[\beta].[f_{n+1}] := [f_{n+1} + \beta]. \tag{3}$$

To see that the action is independent of the choice of β, let β' be another representative of the class $[\beta]$. Thus, the difference $\beta' - \beta = d\alpha$ for some δ-closed morphism $\alpha : A[1]^{\otimes n+1} \to B[1]$. We may define a homotopy between the two extensions $(f_1, \ldots, f_{n+1} + \beta)$ and $(f_1, \ldots, f_{n+1} + \beta')$ by putting

$$\begin{aligned} F &: A \to B \otimes \Omega^*_{[0,1]}, \\ F_k &= f_k, \quad 1 \le k \le n, \\ F_{n+1} &= f_{n+1} + t \cdot \beta' + (1-t) \cdot \beta + \alpha \cdot dt. \end{aligned}$$

This shows that the action map (3) is independent of the choice of β.

Similarly, assume that f'_{n+1} is another representative of the lift class $[f_{n+1}]$, i.e. there exists a homotopy $H : A \to B \otimes \Omega^*_{[0,1]}$ between $(f_1, \dots, f_n, f_{n+1})$ and $(f_1, \dots, f_n, f'_{n+1})$. We simply change H_{n+1} to $H_{n+1} + \beta$, which gives a homotopy between the two lifts $(f_1, \dots, f_n, f_{n+1} + \beta)$ and $f_1, \dots, f_n, f'_{n+1} + \beta)$. This verifies that the action map is well defined.

Transitivity of the action map is clear: since any two lifts differ by some β that would represent a class in $H^0(D^{n+1}(A, B), d)$. ∎

Lemma 2.7. *Let $f : A \to B$ be an $A_{(n)}$ homomorphism. Assume that $h : B \to B'$ and $g : A' \to A$ are both $A_{(n+1)}$ homomorphisms. Then we have*

$$\mathfrak{o}\big(((hf)_j)_{j=1}^n\big) = (h_1)_* \mathfrak{o}\big((f_j)_{j=1}^n\big),$$
$$\mathfrak{o}\big(((fg)_j)_{j=1}^n\big) = (g_1)^* \mathfrak{o}\big((f_j)_{j=1}^n\big).$$

Moreover, the natural map $- \circ g : \mathcal{L}(f) \to \mathcal{L}(f \circ g)$ given by composition of $A_{(n+1)}$ homomorphisms is a homomorphism of $H^0(D^{n+1}(A, B), d)$-modules. Here, the $H^0(D^{n+1}(A, B), d)$-module structure of $\mathcal{L}(f \circ g)$ is via the group homomorphism

$$(g_1)^* : H^0\big(D^{n+1}(A, B), d\big) \to H^0\big(D^{n+1}(A', B), d\big).$$

Proof. The first statements can be proved as in the uncurved case, see [12, Theorem 4.5.1]. The second statement follows from the action map (3) and the formula for composition in (2). ∎

Theorem 2.8. *An A_∞ homomorphism $f = (f_1, f_2, \dots) : A \to B$ between curved A_∞ algebras is a homotopy equivalence if and only if the map f_1 is an $A_{(1)}$ homotopy equivalence.*

Proof. The only if part is trivial, so we prove the if part. Let $g_1 : B \to A$ be an $A_{(1)}$ homotopy inverse of $f_1 : A \to B$. We argue by induction on n that if we are given an $A_{(n)}$ homomorphism

$$g := (g_1, \dots, g_n) : B \to A$$

such that $g \circ f \cong \mathrm{id}$ as $A_{(n)}$ homomorphisms, then there exists $g_{n+1} : B[1]^{\otimes n+1} \to A[1]$ that extends g to an $A_{(n+1)}$ homomorphism $\widetilde{g} = (g_1, \dots, g_n, g_{n+1})$ such that $\widetilde{g} \circ f \cong \mathrm{id}$ as $A_{(n+1)}$ homomorphisms.

We first argue that $\mathfrak{o}(g) = 0$. Using the homotopy invariance of obstruction class, we have

$$f^* \mathfrak{o}(g) = \mathfrak{o}(g \circ f) = \mathfrak{o}(\mathrm{id}) = 0.$$

But f is an $A_{(1)}$ homotopy equivalence, thus f^* is an isomorphism, which shows that $\mathfrak{o}(g) = 0$. Similarly, one can argue that if $H : A \to A \otimes \Omega^*_{[0,1]}$ is an $A_{(n)}$ homotopy between id and $g \circ f$, then we also have $\mathfrak{o}(H) = 0$.

Now, consider the following diagram of maps, provided by Proposition 2.6 and Lemma 2.7:

$$\begin{array}{ccc} \mathcal{L}(H) & \xrightarrow{(\mathsf{ev}_0)_*} & \mathcal{L}(\mathsf{id}_A) \\ {\scriptstyle (\mathsf{ev}_1)_*}\downarrow & & \\ \mathcal{L}(g \circ f) & \xleftarrow{-\circ f} & \mathcal{L}(g). \end{array}$$

Observe that the upper-right corner admits a canonical lift by id_A. We claim that there exists a lift $\widetilde{H}$ of H such that

$$(\mathsf{ev}_0)_*(\widetilde{H}) = \mathsf{id}_A.$$

Indeed, let $\widetilde{H}'$ be any lift of H. By the transitivity of the action map, there exists an element $\beta \in H^0(D^{n+1}(A, A), d)$ such that

$$\beta.\big((\mathsf{ev}_0)_*(\widetilde{H}')\big) = \mathsf{id}_A.$$

Since ev_0 is a homotopy equivalence, there exists $\gamma \in H^0(D^{n+1}(A, A \otimes \Omega^*_{[0,1]}), d)$ such that $(\mathsf{ev}_0)_*\gamma = \beta$. Using Lemma 2.7 we obtain

$$(\mathsf{ev}_0)_*\big(\gamma.\widetilde{H}'\big) = \beta.\big((\mathsf{ev}_0)_*(\widetilde{H}')\big) = \mathsf{id}_A.$$

We set $\widetilde{H} := \gamma.\widetilde{H}'$. By the same argument, one can show that there exists a lift $\widetilde{g}$ of g such that

$$(\mathsf{ev}_1)_*(\widetilde{H}) = \widetilde{g} \circ f.$$

In conclusion, we obtained an $A_{(n+1)}$ homomorphism $\widetilde{g} : B \to A$ such that $\widetilde{g} \circ f \cong \mathsf{id}_A$.

Finally, we need to prove that $f \circ \widetilde{g} \cong \mathsf{id}_B$. Since $\widetilde{g}$ is also a weak equivalence, the conclusion above implies that there exists an $A_{(n+1)}$ homomorphism $f' : A \to B$ extending $(f_1, \dots, f_n)$ such that $f' \circ \widetilde{g} \cong \mathsf{id}_B$. Thus we have

$$f \circ \widetilde{g} \cong f' \circ \widetilde{g} \circ f \circ \widetilde{g} \cong f' \circ \widetilde{g} \cong \mathsf{id}_B,$$

which finishes the proof. ∎

Remark 2.9. Observe that in the uncurved case, according to Lemma 2.4 our notion of $A_{(1)}$ homotopy between morphisms of chain complexes agrees with the usual one. Furthermore, if we are over a field, quasi-isomorphic chain complexes are in fact homotopy equivalent. Thus, the above theorem easily implies the usual Whitehead theorem of uncurved A_∞ algebras over a field which states that a quasi-isomorphism between uncurved A_∞ algebras over a field is in fact a homotopy equivalence.

2.6. Curved L_∞ algebras

The previous discussion and results have direct analogues in the L_∞ setting. Let A and B be two curved L_∞ algebras. In this case, we set the δ-complex $C(A, B)$ as

$$\begin{aligned} C(A, B) :=\ & \cdots \to \mathsf{Hom}(\mathsf{sym}^k A[1], B[1]) \xrightarrow{\delta} \mathsf{Hom}(\mathsf{sym}^{k-1} A[1], B[1]) \to \cdots \\ & \to B[1] \to 0, \end{aligned}$$

$$\delta(\phi_k)(a_1 \cdots a_{k-1}) := (-1)^{|\phi_k|'} \cdot \phi_k(\mu_0^A \cdot a_1 \cdots a_{k-1}).$$

One verifies that $\delta^2 = 0$. As before, we denote its cohomology by $D^k(A, B)$ where k is the tensor degree. Using the operators μ_1^A and μ_1^B, we define another operator d by

$$d : \mathsf{Hom}(\mathsf{sym}^k A[1], B[1]) \to \mathsf{Hom}(\mathsf{sym}^k A[1], B[1]),$$

$$d(\phi_k)(a_1 \cdots a_k) := \mu_1^B \phi_k(a_1 \cdots a_k) - \sum_{j=1}^{k} (-1)^{\star} \phi_k(a_1 \cdots \mu_1^A(a_j) \cdots a_k)$$

with $\star = |\phi_k|' + |a_1|' + \cdots + |a_{j-1}|'$. Here sym stands for the graded symmetric algebra.

If there exists an R-linear map $f : A \to B$ such that $f(\mu_0^A) = \mu_0^B$, we set the k-th obstruction space $H^k(A, B)$ to be $H^k(A, B) := H^1(D^k(A, B), d)$.

Let $f_j : \mathsf{sym}^j A[1] \to B[1]$ $(j = 1, \ldots, n)$ be n multi-linear maps of cohomological degree zero. We call the sequence $(f_1, \ldots, f_n)$ an $L_{(n)}$ morphism if the following conditions hold:

(i) The L_∞ homomorphism axiom holds up to $(n-1)$ inputs, i.e. for all $0 \le m \le n-1$ we have

$$\sum_k \frac{1}{k!} \sum_\sigma \varepsilon_\sigma \cdot \mu_k^B \big(f_{i_1}(a_{\sigma(1)} \cdots) \cdots f_{i_k}(\cdots a_{\sigma(m)})\big)$$
$$= \sum_{r \ge 0} \sum_\tau \varepsilon_\tau \cdot f_{m-r+1}\big(\mu_r^A(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(m)}\big)$$

where σ is a $(i_1, \ldots, i_k)$ type shuffle, and τ is a $(r, n-r)$ type shuffle, and ε_σ and ε_τ are Koszul signs associated with these permutations.

(ii) In the case with n inputs, we require that

$$\sum_{k,\sigma} \frac{1}{k!} \mu_k^B (f_{i_1} \otimes \cdots \otimes f_{i_k}) \mathsf{Sh}_\sigma - \sum_{r \ge 1, \tau} f_{n-r+1}(\mu_r^A \otimes \mathsf{id}^{\otimes n-r}) \mathsf{Sh}_\tau \tag{4}$$

is δ-exact, i.e. it lies in the image of $\delta : C^{n+1}(A, B) \to C^n(A, B)$. Here, σ, τ are as above and Sh_σ is the map that permutes the inputs according to the shuffle σ.

Given an $L_{(n)}$ homomorphism $(f_1, \ldots, f_n) : A \to B$, we define its obstruction class as follows. Choose any f'_{n+1} of cohomological degree zero such that $\delta f'_{n+1}$ equals the expression in (4). Then we set

$$\begin{aligned}\mathsf{obs}_{n+1} := & \sum_{\substack{k \ge 2, \sigma \\ i_1 + \cdots + i_k = n+1}} \frac{1}{k!} \mu_k^B (f_{i_1} \otimes \cdots \otimes f_{i_k}) \mathsf{Sh}_\sigma \\ & - \sum_{r \ge 2, \tau} f_{n-r+2}(\mu_r^A \otimes \mathsf{id}^{\otimes n+1-r}) \mathsf{Sh}_\tau + d f'_{n+1},\end{aligned}$$

and define the obstruction class by

$$\mathfrak{o}\big((f_j)_{j=1}^n\big) = [\mathsf{obs}_{n+1}] \in H^{n+1}(A, B).$$

Again, one can verify (similar to Lemma 2.3) that this class is well defined and independent of the choice of f'_{n+1}.

The formal properties of the obstruction theory still holds in the L_∞ case, with which we deduce the following result for curved L_∞ algebras. Again, in the uncurved case and over a field, this result immediately implies the classical Whitehead theorem of L_∞ algebras that quasi-isomorphisms are also homotopy equivalences.

Theorem 2.10. *An L_∞ homomorphism $f = (f_1, f_2, \ldots) : A \to B$ between curved L_∞ algebras is a homotopy equivalence if and only if f_1 is an $L_{(1)}$ homotopy equivalence.*

3. Homotopy transfer of curved algebras

In this section, we prove a curved version of the homological perturbation lemma. This works for both A_∞ and L_∞ algebras in the presence of a new version of homotopy retraction data in the curved case. This is used to construct minimal charts of L_∞ spaces in Section 5.2.

3.1. Curved homotopy retraction data

Let us consider the following situation: we are given a curved A_∞ algebra (A, m_k), a graded vector space V, R-linear degree zero maps $i : V \to A$ and $p : A \to V$ and an R-linear map $H : A \to A$ of degree -1. Assume there is $C \in V$ of degree 2 satisfying $i(C) = m_0$ and moreover:

$$m_1 H + H m_1 = ip - \mathrm{id}_A - H m_1^2 H, \tag{5}$$

$$p m_1 H = 0, \quad H m_1 i = 0. \tag{6}$$

We have the following:

Theorem 3.1. *In the situation described, there is a curved A_∞ algebra structure on V with $\mu_0 = C$ and $\mu_1 = p m_1 i$. Moreover, there is an A_∞ homomorphism $\varphi : (V, \mu_k) \to (A, m_k)$ with $\varphi_1 = i$.*

A common application of this theorem is to construct "minimal" algebras. In that case, we have the side conditions $Hi = pH = 0$ and $pi = \mathrm{id}$. In the presence of these extra conditions, (6) follows from (5).

Before we go into the proof we describe the maps μ_k, for $k \geq 2$,

$$\mu_k = \sum_{T \in \Gamma_k} \mu_T,$$

where Γ_k is the set of rooted stable planar trees with k-leaves.

We use T as a flow chart to define a map $\mu_T : V^{\otimes k} \to V$. We assign to each $v \in V(T)$ the map $m_{\mathrm{val}(v)}$; to the internal edges we assign H; and finally we assign p to the root and

i to the leaves. For example, the tree in Figure 1 gives the map

$$\mu_T(u_1,u_2,u_3,u_4,u_5,u_6) \\ = p\circ m_3(H\circ m_3(i(u_1),i(u_2),i(u_3)),i(u_4),H\circ m_2(i(u_5),i(u_6))).$$

We would like to point out that these are exactly the same formulas as in the uncurved case. In particular, the m_0 term plays no role in the formulas for μ_k, $k\geq 2$.

To prove these maps define an A_∞ algebra we will need the following auxiliary maps. Let $T\in\Gamma_k$, denote by $E(T)$ the set of edges of T and by $e(T)$ the set of internal edges of T. For each $T\in\Gamma_k$, we define $\overline{T}$ as the tree T with one additional vertex in each internal edge of T. Given $e\in E(\overline{T})$ we define $\hat{\mu}_{\overline{T},e}$ in the same way as μ_T with the extra assignment of m_1 to the edge e. Given $e\in e(T)$ we define $\mu^{\Pi}_{T,e}$, $\mu^{\mathrm{id}}_{T,e}$ and $\mu^{\gamma}_{T,e}$ in the same way as μ_T, but with $\Pi = ip$ (respectively id and $\gamma := Hm_1^2H$) assigned to e instead of H.

Proof of Theorem 3.1. One can easily check the first two A_∞ equations for μ_k using equations (5), (6). For two or more inputs we define

$$\hat{\mu}_{k,\beta}(u_1,\dots,u_k) = \sum_{\substack{T\in\Gamma_k,\\ e\in E(T)}} (-1)^{|T_e|}\hat{\mu}_{\overline{T},e}(u_1,\dots,u_k)$$

where $|T_e| = \sum_{i=1}^{m_e}|u_i|'$ with m_e defined as the smallest $1\leq j\leq k$ such that the path from the i-th leaf to the root does not include e, for all $i<j$. Then, given $e\in e(T)$, denote by E_- and E_+ the edges of $\overline{T}$ contained in e. Equation (5) implies

$$\hat{\mu}_{\overline{T},E_-} + \hat{\mu}_{\overline{T},E_+} = \mu^{\Pi}_{T,e} - \mu^{\mathrm{id}}_{T,e} - \mu^{\gamma}_{T,e}.$$

Therefore,

$$\hat{\mu}_k = \sum_{\substack{T\in\Gamma_k,\\ e\in E(T)\setminus e(T)}} (-1)^{|T_e|}\hat{\mu}_{\overline{T},e} + \sum_{\substack{T\in\Gamma_k,\\ e\in e(T)}} (-1)^{|T_e|}(\mu^{\Pi}_{T,e} - \mu^{\mathrm{id}}_{T,e} - \mu^{\gamma}_{T,e}). \tag{7}$$

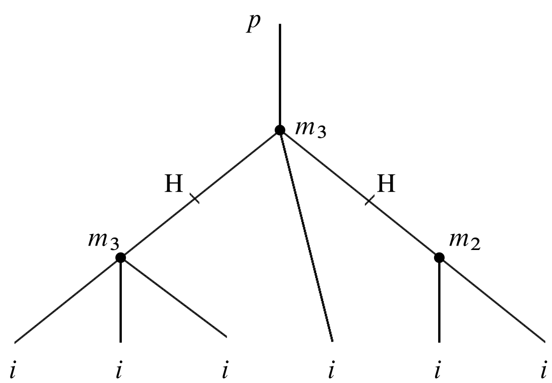

Figure 1. Example of an element of Γ_6.

On the other hand,

$$\widehat{\mu}_k = \sum_{T\in\Gamma_k} \sum_{v\in V(T)} \sum_{\substack{e\in E(\overline{T})\\ v\in\partial e}} (-1)^{|T_e|}\widehat{\mu}_{\overline{T},e},$$

and the A_∞ equation implies that

$$\widehat{\mu}_k = -\sum_{T\in\Gamma_k} \sum_{v\in V(T)} \sum_{\substack{S\in\Gamma_k,\\ e\in e(S),\\ S/e=T}} (-1)^{|S_e|}\mu^{\mathrm{id}}_{S,e} - \sum_{\substack{S\in\Gamma_{k+1},\\ S/i=T}} (-1)^{|S_e|}\mu_S(\ldots,u_{i-1},\mu_0,u_i,\ldots). \quad (8)$$

Here S/e is the tree obtained from S by collapsing the edge e and S/i is the tree obtained by deleting the i-th leaf of S. Putting (7) and (8) together we conclude

$$\sum_{T\in\Gamma_k}\Big(\sum_{e\in e(T)} (-1)^{|T_e|}(\mu^{\Pi}_{T,e} - \mu^{\gamma}_{T,e}) + \widehat{\mu}_{T,r} + \sum_{i=1}^{k}(-1)^{|T_e|}\widehat{\mu}_{T,l_i}\Big) + \sum_{T\in\Gamma_k} \sum_{\substack{S\in\Gamma_{k+1},\\ S/i=T}} (-1)^{|S_e|}\mu_S(\ldots,u_{i-1},\mu_0,u_i,\ldots) = 0, \quad (9)$$

where r is the edge adjacent to the root and the l_i are the edges adjacent to the leaves of T. It follows from the definition of μ_T that

$$\sum_{T\in\Gamma_k} \sum_{e\in e(T)} (-1)^{|T_e|}\mu^{\Pi}_{T,e} = \sum_{\substack{k_1\neq 0,1,\\ k_2\neq 1}} (-1)^*\mu_{k_2}(u_1,\ldots,\mu_{k_1}(u_{i+1},\ldots,u_{i+k_1}),\ldots,u_k).$$

Equations (5) and (6) imply $pm_1 = \mu_1 p - pm_1^2 H$. This combined with the A_∞ equation gives

$$\widehat{\mu}_{T,r} = \mu_1\mu_T + \mu_{C_2\circ_2 T}(\mu_0,\ldots) + (-1)^*\mu_{C_2\circ_1 T}(\ldots,\mu_0),$$

where C_2 is the unique tree with two leaves and $C_2 \circ_i T$ is the tree obtained by grafting the root of T to the i-th leaf of C_2.

Analogously, the identity $m_1 i = i\mu_1 - Hm_1^2 i$ implies

$$\sum_{i=1}^{k}(-1)^{|T_e|}\widehat{\mu}_{T,l_i} = \sum_i \mu_T(\ldots,\mu_1(u_i),\ldots) + \sum_i (-1)^*\mu_{T\circ_i C_2}(\ldots,\mu_0,u_i,\ldots) + (-1)^*\mu_{T\circ_i C_2}(\ldots,u_i,\mu_0,\ldots).$$

Finally, using the fact that $-\gamma(u) = Hm_2(m_0, H(u)) + (-1)^{|u|}m_2(H(u), m_0)$ we have

$$(-1)^{|T_e|+1}\mu^{\gamma}_{T,e} = (-1)^*\mu_{T_1\circ_i C_2\circ_2 T_2}(\ldots,\mu_0,u_i,\ldots) + (-1)^*\mu_{T_1\circ_i C_2\circ_1 T_2}(\ldots,\mu_0,u_{i+j+1},\ldots),$$

where T_1 and T_2 are the trees obtained from cutting T along the edge e and j is the number of leaves in T_2. These last four identities prove that equation (9) is equivalent to the A_∞ algebra equation for the μ_k.

The construction of map $\varphi : V \to A$ is similar. We put $\varphi_1 = i$ and $\varphi_k = \sum_{T \in \Gamma_k} \varphi_T$ where the map φ_T is defined in the same way as μ_T, the only difference is that we assign H to the root vertex (instead of p as in the case of μ_T). Similarly we define the auxiliary maps

$$\widehat{\varphi}_k = \sum_{\substack{T \in \Gamma_k,\\ e \in e(\overline{T})}} (-1)^{|T_e|} \widehat{\varphi}_{\overline{T},e},$$

and for each $e \in e(T)$ we define $\varphi^{\Pi}_{T,e}$, $\varphi^{\mathrm{id}}_{T,e}$ and $\varphi^{\gamma}_{T,e}$.

The same argument we used above applies to show

$$\sum_{T \in \Gamma_k} \Big(\sum_{e \in e(T)} (-1)^{|T_e|} (\varphi^{\Pi}_{T,e} - \varphi^{\gamma}_{T,e}) + \widehat{\varphi}_{T,r} + \sum_{i=1}^{k} (-1)^{|T_e|} \widehat{\varphi}_{T,l_i} \Big) + \sum_{T \in \Gamma_k} \sum_{\substack{S \in \Gamma_{k+1},\\ S/i = T}} (-1)^{|S_e|} \varphi_S(\ldots, u_{i-1}, \mu_0, u_i, \ldots) = 0. \tag{10}$$

Now, using (5) again, we see that

$$\widehat{\varphi}_{T,r} = -m_1 \circ \varphi_T + i \circ \mu_T - \varphi^{\mathrm{id}}_{T,r} - \varphi^{\gamma}_{T,r},$$

and

$$\varphi^{\mathrm{id}}_{T,r} = m_j(\varphi_{T_1}, \ldots, \varphi_{T_j}),$$

where j is the valency of the vertex of T closest to the root and T_i are the trees obtained from cutting T at the incoming edges at that vertex. One can now see by the same argument that equation (10) is equivalent to the A_∞ homomorphism equation for φ,

$$\sum_{j,\, i_1 + \ldots + i_j = k} m_j\big(\varphi_{i_1}(u_1, \ldots, u_{i_1}), \ldots, \varphi_{i_j}(\ldots, u_k)\big) - \sum_{\substack{0 \le j \le k,\\ 0 \le i \le k-j}} (-1)^* \varphi_{k-j+1}\big(u_1, \ldots, \mu_j(u_{i+1}, \ldots, u_{i+j}), \ldots, u_k\big) = 0. \qquad \blacksquare$$

Remark 3.2. In the case of uncurved A_∞ algebras, there are also explicit formulas for a homomorphism $\psi : (A, m_k) \to (V', \mu_k)$ with $\psi_1 = p$ and a homotopy $\mathscr{H} : (A, m_k) \to (A, m_k)$ between $\varphi \circ \psi$ and id_A. See [16] for this construction.

3.2. The L_∞ case

The discussion in the L_∞ case is very much the same as the A_∞ case, except that instead of using planar stable rooted trees in the formulas, one uses isomorphism classes of stable rooted trees.

The only difference is how to define the map μ_T for each tree T (as opposed to a planar tree): we pick $\widetilde{T}$ a planar embedding of T and define $\mu_{\widetilde{T}}$ as before. Then we take

$$\mu_T = \frac{1}{|\mathrm{Aut}(T)|} \mu_{\widetilde{T}} \circ \mathrm{Sh}$$

where Sh is the symmetrization map and $|\mathsf{Aut}(T)|$ is the order of the automorphism group of T. We refer to [10, Section 4] for a detailed treatment of this construction.

The rest of the proof is exactly the same.

4. The category of curved L_∞ spaces

In this section, we recall basic definitions of curved L_∞ spaces, morphisms between these spaces, and describe the notion of homotopy between morphisms.

4.1. Curved L_∞ spaces

A curved L_∞ space (sometimes shortened to L_∞ space) is a pair $(M, \mathfrak{g})$ where M is a smooth manifold, and $\mathfrak{g}$ is a $\mathbb{Z}$-graded vector bundle over M of the form

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \cdots \oplus \mathfrak{g}_d$$

for some $d \geq 2$, together with bundle maps $\mu_k : \mathsf{sym}^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ of degree one such that the L_∞ equation holds,

$$\sum_{k=0}^{n} \sum_{\sigma \in \mathsf{Sh}(k,n-k)} \varepsilon_\sigma \cdot \mu_{n-k+1}\big(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}\big) = 0,$$

where $\mathsf{Sh}(k, n-k)$ consists of $(k, n-k)$ type shuffles, and ε_σ is the Koszul sign associated with the permutation $a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma(1)}, \ldots, a_{\sigma(n)}$ with the a's considered as elements of $\mathfrak{g}[1]$.

In order to formulate a good notion of homotopy between morphisms of L_∞ spaces we will need "special" connections on T_M and $\mathfrak{g}$. Therefore we make the following assumption:

T_M has a torsion-free, flat connection and $\mathfrak{g}$ has a flat connection.

In fact, it would be enough for most purposes to require the existence of these connections on an open neighborhood of $\mu_0^{-1}(0)$. But for simplicity we stick to the whole M.

The main results in this paper are local, meaning M is an open ball in $\mathbb{R}^n$, therefore this assumption is trivially satisfied.

A morphism from $(M, \mathfrak{g})$ to $(N, \mathfrak{h})$ is a pair $\mathfrak{f} = (f, f^\sharp)$ where $f : M \to N$ is a smooth map, and $f^\sharp : \mathfrak{g} \to f^*\mathfrak{h}$ is a homomorphism of L_∞ algebras. This means a sequence of (degree zero) bundle maps $f_k^\sharp : \mathsf{sym}^k(\mathfrak{g}[1]) \to f^*\mathfrak{h}[1]$ satisfying

$$\begin{aligned} &\sum_k \frac{1}{k!} \sum_\sigma \varepsilon_\sigma \cdot \mu_k\big(f_{i_1}^\sharp(a_{\sigma(1)} \cdots) \cdots f_{i_k}^\sharp(\cdots a_{\sigma(n)})\big) \\ &\quad = \sum_r \sum_\tau \varepsilon_\tau \cdot f_{n-r+1}^\sharp\big(\mu_r(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}\big), \end{aligned}$$

where σ is a $(i_1, \ldots, i_k)$ type shuffle, and τ is a $(r, n-r)$ type shuffle. On the left-hand side there is an abuse of notation: μ_k stands for $f^*\mu_k$.

Morphisms of L_∞ spaces can be composed similarly to the algebra case. Given L_∞ morphisms $\mathfrak{e} : (M', \mathfrak{g}') \to (M, \mathfrak{g})$ and $\mathfrak{f} : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ we define $\mathfrak{f} \circ \mathfrak{e} := (f \circ e, f^\sharp \circ e^\sharp)$ where

$$(f^\sharp \circ e^\sharp)_n(a_1 \cdots a_k) = \frac{1}{k!} \sum_\sigma \varepsilon_\sigma \cdot e^*(f_k^\sharp)\big(e_{i_1}^\sharp(a_{\sigma(1)} \cdots) \cdots e_{i_k}^\sharp(\cdots a_{\sigma(n)})\big). \tag{11}$$

As in the algebra case, we also define $L_{(n)}$ morphisms between curved L_∞ spaces.

4.2. Extensions of L_∞ structures

Let $(M, \mathfrak{g})$ be an L_∞ space. By our assumptions, we can choose a torsion-free, flat connection on T_M and also a flat connection on the bundle $\mathfrak{g}$. We set

$$\tilde{\mathfrak{g}} := T_M \oplus \mathfrak{g},$$

with T_M at cohomological degree one. The L_∞ structure on $\mathfrak{g}$ naturally extends to $\tilde{\mathfrak{g}}$ by inductively applying the formula

$$\mu_{k+1}(X \cdot \alpha_1 \cdots \alpha_k) := \nabla_X \mu_k(\alpha_1 \cdots \alpha_k) - \sum_{j=1}^{k} \mu_k(\alpha_1 \cdots \nabla_X \alpha_j \cdots \alpha_k). \tag{12}$$

Using the torsion-freeness and the flatness, one can verify that when pulling out tangent vectors using the above formula, the choice of order does not matter, i.e. we have that

$$\mu_{k+2}(X \cdot Y \cdot \alpha_1 \cdots \alpha_k) = \mu_{k+2}(Y \cdot X \cdot \alpha_1 \cdots \alpha_k)$$

for any two tangent vectors $X, Y \in T_M$.

Lemma 4.1. *Equation* (12) *defines an L_∞ algebra structure on $\tilde{\mathfrak{g}}$.*

Proof. We prove the L_∞ identity by induction on the total number of tangent vectors. Indeed, when there is no tangent vector, the L_∞ identity holds since $\mathfrak{g}$ forms an L_∞ algebra to begin with. We want to verify the L_∞ identity

$$\sum_{k=1}^{n} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \varepsilon_\sigma \cdot \mu_{n-k+1}\big(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}\big) = 0.$$

It is enough to consider the case when all the inputs a's are flat with respect to the chosen connection ∇. Now, we pick a tangent vector, say a_1, among the inputs and apply equation (12) to pull it out of the inputs. If a_1 falls into $a_{\sigma(1)}, \ldots, a_{\sigma(k)}$, we obtain terms of the form

$$\sum \sum \varepsilon_\sigma \mu_{n-k+1}(\nabla_{a_1} \mu_{k-1}(\cdots), \ldots)$$

When a_1 falls into $a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}$, we get

$$\nabla_{a_1}\Big(\sum \sum \varepsilon_\sigma \mu_{n-k}(\mu_k(\cdots), \ldots)\Big) - \sum \sum \varepsilon_\sigma \mu_{n-k+1}\big(\nabla_{a_1} \mu_{k-1}(\cdots), \ldots\big).$$

Thus, their sum yields $\nabla_{a_1}(\sum \sum \varepsilon_\sigma \mu_{n-k}(\mu_k(\cdots), \ldots))$ which vanishes by induction. ∎

L_∞ morphisms between L_∞ spaces can also be extended to the tangent bundles. More precisely, let $(f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ be a morphism of L_∞ spaces and, as above, choose torsion-free and flat connections on both spaces and consider the extended L_∞ algebras $\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}}$. We extend the homomorphism $f^\sharp$ to a homomorphism

$$f^\sharp : \widetilde{\mathfrak{g}} \to f^*\widetilde{\mathfrak{h}},$$

which we still denote by $f^\sharp$. The formula of extension is the same as in equation (12), i.e. we inductively define

$$f^\sharp_{k+1}(X \cdot \alpha_1 \cdots \alpha_k) := \nabla_X f^\sharp_k(\alpha_1 \cdots \alpha_k) - \sum_{j=1}^{k} f^\sharp_k(\alpha_1 \cdots \nabla_X \alpha_j \cdots \alpha_k). \tag{13}$$

The difference is that here we need $k \geq 1$. When $k = 0$ we define the map $f^\sharp_1 : T_M \to f^*T_N$ to be the tangent map df.

Lemma 4.2. *The maps defined in equation* (13) *form an* L_∞ *morphism* $f^\sharp : \widetilde{\mathfrak{g}} \to f^*\widetilde{\mathfrak{h}}$.

Proof. We need to verify that

$$\sum_k \frac{1}{k!} \sum_\sigma \varepsilon_\sigma \cdot \mu_k\big(f^\sharp_{i_1}(a_{\sigma(1)} \cdots) \cdots f^\sharp_{i_k}(\cdots a_{\sigma(n)})\big)$$
$$= \sum_r \sum_\tau \varepsilon_\tau \cdot f^\sharp_{n-r+1}\big(\mu_r(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}\big).$$

Let us pick up a tangent vector, say a_1, among the inputs. Also we assume that all the input vectors are flat. If a_1 is inside f_{i_j}, and $i_j = 1$, we get

$$\sum_k \frac{1}{(k-1)!} \sum_\sigma \varepsilon_\sigma \cdot \nabla_{a_1} \mu_{k-1}\big(f^\sharp_{i_1}(a_{\sigma(1)} \cdots) \cdots \widehat{f_1(a_1)} \cdots f^\sharp_{i_k}(\cdots a_{\sigma(n)})\big)$$
$$- \sum_k \frac{1}{(k-1)!} \sum_\sigma \varepsilon_\sigma \cdot \mu_{k-1}\big(f^\sharp_{i_1}(a_{\sigma(1)} \cdots) \cdots \widehat{f_1(a_1)} \cdots \nabla_{a_1} f^\sharp_{i_l}(\cdots) \cdots f^\sharp_{i_k}(\cdots a_{\sigma(n)})\big).$$

The coefficient becomes $\frac{1}{(k-1)!}$ since there are k possible choices of j. The second term cancels precisely the terms with a_1 inside f_{i_j} with $i_j \geq 2$. Thus, the left-hand side is equal to (by induction on the total number of tangent vectors)

$$\nabla_{a_1}\Big(\sum_k \frac{1}{(k-1)!} \sum_\sigma \varepsilon_\sigma \cdot \mu_{k-1}\big(f^\sharp_{i_1}(a_{\sigma(1)} \cdots) \cdots \widehat{f_1(a_1)} \cdots f^\sharp_{i_k}(\cdots a_{\sigma(n)})\big)\Big)$$
$$= \nabla_{a_1}\Big(\sum_r \sum_\tau \varepsilon_\tau \cdot f^\sharp_{n-r}\big(\mu_r(\cdots) \cdots\big)\Big)$$
$$= \sum_r \sum_\tau \varepsilon_\tau \cdot f^\sharp_{n-r+1}\big(\mu_r(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}\big)$$

which is exactly the right-hand side. ∎

These extensions of L_∞ spaces induced by the choice of connections are in fact independent of these choices up to isomorphism.

Lemma 4.3. *Let ∇ and ∇' be torsion-free and flat connections. Let $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ be the associated extended L_∞ algebras. Then there is an isomorphism*

$$\Phi^{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}'$$

defined by $\Phi_1^{\mathfrak{g}} = \mathsf{id}$, $\Phi_2^{\mathfrak{g}}(X,\alpha) = (\nabla'_X - \nabla_X)(\alpha)$, and for $k \geq 3$ by the recursive formula

$$\Phi_k^{\mathfrak{g}}(X \cdot \alpha_1 \cdots \alpha_{k-1}) := \nabla'_X \Phi_{k-1}^{\mathfrak{g}}(\alpha_1 \cdots \alpha_{k-1}) - \sum_{j=1}^{k-1} \Phi_{k-1}^{\mathfrak{g}}(\alpha_1 \cdots \nabla_X \alpha_j \cdots \alpha_k).$$

Moreover, this isomorphism is natural: given an L_∞ morphism $(f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ and different choices of connections, the induced extended L_∞ morphisms $f^\sharp : \tilde{\mathfrak{g}} \to f^*\tilde{\mathfrak{h}}$ and $(f^\sharp)' : \tilde{\mathfrak{g}}' \to f^*\tilde{\mathfrak{h}}'$ satisfy $\Phi^{\mathfrak{h}} \circ f^\sharp = (f^\sharp)' \circ \Phi^{\mathfrak{g}}$.

Proof. As before, we prove the L_∞ homomorphism equation by induction on the number of inputs that are tangent vectors. When there is no tangent vector, the operations μ_k and μ'_k agree and Φ is just the identity. Let us now pick up a tangent vector, say a_1, among the inputs. For simplicity we assume that all the inputs are flat with respect to ∇. Let us consider the left-hand side of the L_∞ equation, when a_1 is inside Φ_{i_j} with $i_j = 1$, we get

$$\begin{aligned}\sum_k \frac{1}{(k-1)!} \sum_\sigma \varepsilon_\sigma \cdot \nabla'_{a_1} \mu'_{k-1}\big(\Phi_{i_1}(a_{\sigma(1)} \cdots) \cdots \widehat{\Phi_1(a_1)} \cdots \Phi_{i_k}(\cdots a_{\sigma(n)})\big) \\ - \sum_k \frac{1}{(k-1)!} \sum_\sigma \varepsilon_\sigma \cdot \mu'_{k-1}\big(\Phi_{i_1}(a_{\sigma(1)} \cdots) \cdots \widehat{\Phi_1(a_1)} \cdots \\ \cdot \nabla'_{a_1} \Phi_{i_l}(\cdots) \cdots \Phi_{i_k}(\cdots a_{\sigma(n)})\big),\end{aligned}$$

using the definition of μ'_k and the fact $\Phi_1 = \mathsf{id}$. The second sum above exactly cancels with the other terms in the left-hand side of the L_∞ homomorphism equation with a_1 inside f_{i_j} with $i_j \geq 2$. This is because the a_i are ∇ flat and $\Phi_2(a_1, a_j) = \nabla'_{a_1} a_j$. Therefore, by induction hypothesis, the left-hand side equals

$$\begin{aligned}\nabla'_{a_1}\Big(\sum_r \sum_\tau \varepsilon_\tau \cdot \Phi_{n-r}\big(\mu_r(a_{\tau(2)} \cdots) \cdots a_{\tau(n)}\big)\Big) \\ = \nabla'_{a_1}\big(\mu_{n-1}(a_2 \cdots a_n)\big) + \sum_{r \leq n-2} \sum_\tau \varepsilon_\tau \cdot \Phi_{n+1-r}\big(a_1 \cdot \mu_r(a_{\tau(2)} \cdots) \cdots a_{\tau(n)}\big) \\ + \sum_{r \leq n-2} \sum_\tau \varepsilon_\tau \cdot \Phi_{n-r}\big(\nabla_{a_1}\big(\mu_r(a_{\tau(2)} \cdots)\big) \cdots a_{\tau(n)}\big).\end{aligned} \tag{14}$$

Here, the first term equals $\Phi_2(a_1, \mu_{n-1}(a_2 \cdots a_n))$ and in the third term we have

$$\nabla_{a_1}\big(\mu_r(a_{\tau(2)} \cdots)\big) = \mu_{r+1}(a_1 \cdot a_{\tau(2)} \cdots).$$

Hence (14) equals

$$\sum_r \sum_\tau \varepsilon_\tau \cdot \Phi_{n-r+1}\big(\mu_r(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}\big).$$

More precisely, the first two terms in (14) correspond in the above to the terms where a_1 is outside the μ_r.

The proof of the naturality statement is entirely analogous and we omit it. ■

4.3. Homotopy of L_∞ morphisms

In order to define the notion of homotopy we need to consider the space version of tensoring with $\Omega^*_{[0,1]}$ as in Section 2.3. Let $(M,\mathfrak{g})$ and $(N,\mathfrak{h})$ be two L_∞ spaces and let $F : M \times [0,1] \to N$ be a smooth map. Consider the graded bundle $F^*\widetilde{\mathfrak{h}}_{[0,1]} := F^*\widetilde{\mathfrak{h}} \otimes \pi_2^*\Omega^*_{[0,1]}$, where $\pi_2 : M \times [0,1] \to [0,1]$ is the projection and denote $\mu^t_k = (f^t)^*\mu_k$, $f^t = F(-,t)$. On $F^*\widetilde{\mathfrak{h}}_{[0,1]}$ we define the operations

$$\begin{aligned} \mu_0^{\otimes} &:= \mu_0^t - (dF/dt)dt, \\ \mu_1^{\otimes}(x(t) + y(t)dt) &:= \mu_1^t(x(t)) + \mu_1^t(y(t))dt \\ &\quad + (-1)^{|x(t)|}\nabla_{d/dt}(x(t)))dt, \\ \mu_k^{\otimes}(\ldots, x_i(t) + y_i(t)dt, \ldots) &:= \mu_k^t(\ldots, x_i(t), \ldots) \\ &\quad + \sum_i (-1)^{\dagger}\mu_k^t(x_1(t), \ldots, y_i(t), \ldots, x_k(t))dt, \end{aligned} \tag{15}$$

for $k \geq 2$, where $\dagger = \sum_{a=i+1}^k |x_a|'$.

Lemma 4.4. *The operations $\mu_k^{\otimes}$ define a curved L_∞ algebra structure on $F^*\widetilde{\mathfrak{h}}_{[0,1]}$.*

Proof. The proof is standard, it follows from the fact that the tensor product of an L_∞ algebra and a commutative dg-algebra is again an L_∞ algebra together with the relation

$$\nabla_{\frac{d}{dt}}\mu_k^t(a_1, \ldots, a_k) = \mu_{k+1}^{\otimes}\Big(\frac{\partial F}{\partial t}dt, a_1, \ldots, a_k\Big),$$

for flat a_i. ■

We are now ready to define homotopy.

Definition 4.5. Two L_∞ morphisms $(f^0, f^{0,\sharp})$ and $(f^1, f^{1,\sharp})$ from $(M,\mathfrak{g})$ to $(N,\mathfrak{h})$ are homotopic if there exists a map $F : M \times [0,1] \to N$, together with an L_∞ homomorphism

$$F^\sharp : \pi_1^*\widetilde{\mathfrak{g}} \to F^*\widetilde{\mathfrak{h}}_{[0,1]},$$

where $\pi_1 : M \times [0,1] \to M$ is the projection map, satisfying the following conditions:

- It is compatible with the connection, i.e.

$$F^\sharp_{k+1}(X \cdot \alpha_1 \cdots \alpha_k) := \nabla_X F^\sharp_k(\alpha_1 \cdots \alpha_k) - \sum_{j=1}^k F^\sharp_k(\alpha_1 \cdots \nabla_X \alpha_j \cdots \alpha_k)$$

  and $F^\sharp_1(X) = dF(X)$, $\forall X \in T_M$.

- The following boundary conditions hold:

$$(F, F^\sharp)|_{t=0} = (f^0, f^{0,\sharp}),$$
$$(F, F^\sharp)|_{t=1} = (f^1, f^{1,\sharp}).$$

Note that by Lemma 4.3, this definition is independent of the choice of ∇.

Like usual we say an L_∞ morphism $\mathfrak{f} : \mathbb{M} \to \mathbb{N}$ is a homotopy equivalence if there is another L_∞ morphism $\mathfrak{e} : \mathbb{N} \to \mathbb{M}$ such that both $\mathfrak{f} \circ \mathfrak{e}$ and $\mathfrak{e} \circ \mathfrak{f}$ are homotopic to the identity L_∞ morphism.

Remark 4.6. The above definition can be easily adapted to the complex analytic setting when both $(M, \mathfrak{g})$ and $(N, \mathfrak{h})$ are holomorphic L_∞ spaces such that the underlying complex manifolds M and N admits holomorphic torsion-free and flat connections. More precisely, we simply require that F and $F^\sharp$ be fiberwise holomorphic, i.e. they are smooth in the t direction, and holomorphic whenever we fix a value $t \in [0, 1]$. This is more transparent with explicit formulas of $F^\sharp$ in the following paragraph.

It will be helpful to unwind this definition. The compatibility condition implies the morphism $F^\sharp$ is determined by its values on the elements of $\pi_1^* \mathfrak{g}$. We write

$$F_k^\sharp(a_1, \ldots, a_k) = f_k^t(a_1, \ldots, a_k) + (-1)^{\sum_i |a_i|'} h_k^t(a_1, \ldots, a_k)dt.$$

Then the L_∞ morphism equation for $F^\sharp$ is equivalent to

(1) the maps $(f_k^t)_{k\geq 1}$ define an L_∞ homomorphism $\mathfrak{g} \to (f^t)^*\mathfrak{h}$;

(2) the maps h_k^t satisfy the equations $h_1^t(\mu_0) = \frac{\partial F}{\partial t}$ and for $n \geq 1$

$$\begin{aligned}\sum_k \frac{1}{(k-1)!} \sum_\sigma \varepsilon_\sigma \mu_k^t \big(h_{i_1}^t(a_{\sigma(1)} \cdots) f_{i_2}^t \cdots f_{i_k}^t(\cdots a_{\sigma(n)})\big) \\ + \sum_{j\geq 0} \sum_\tau \varepsilon_\tau h_{n-j+1}^t \big(\mu_j(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}\big) \\ = \nabla_{\frac{d}{dt}} f_n^t(a_1, \ldots, a_n) \qquad (16)\end{aligned}$$

where σ is a $(i_1, \ldots, i_k)$ type shuffle, τ is a $(r, n-r)$ type shuffle and the a_i are flat.

Proposition 4.7. (**a**) *Homotopy of L_∞ morphisms is an equivalence relation.*

(**b**) *Let $(f^0, f^{0,\sharp})$ and $(f^1, f^{1,\sharp})$ be homotopic L_∞ morphisms. Then $(f^0, f^{0,\sharp}) \circ (d, d^\sharp)$ and $(f^1, f^{1,\sharp}) \circ (d, d^\sharp)$ are homotopic, and $(e, e^\sharp) \circ (f^0, f^{0,\sharp})$ and $(e, e^\sharp) \circ (f^1, f^{1,\sharp})$ are homotopic, for any composable L_∞ morphisms $(d, d^\sharp)$, $(e, e^\sharp)$.*

Proof. For **(a)** first note that a diffeomorphism $\rho : [0, 1] \to [0, 1]$ induces, by pull-back, an L_∞ homomorphism $\rho^* : F^*\widetilde{\mathfrak{h}}_{[0,1]} \to (F_\rho)^*\widetilde{\mathfrak{h}}_{[0,1]}$, where $F_\rho := F \circ (\mathrm{id} \times \rho)$. Now, given a homotopy $(F, F^\sharp)$ from $(f^0, f^{0,\sharp})$ to $(f^1, f^{1,\sharp})$, take $\rho(t) = 1 - t$ and consider the pair $(F_\rho, F_\rho^\sharp := \rho^* \circ F^\sharp)$. This defines a homotopy from $(f^1, f^{1,\sharp})$ to $(f^0, f^{0,\sharp})$, which shows symmetry of the homotopy relation. For transitivity let ρ be a non-decreasing diffeomorphism which is constant in neighborhoods of 0 and 1 in [0, 1]. For this choice of ρ, $\mathfrak{F}_\rho := (F_\rho, F_\rho^\sharp)$ is a new homotopy from $(f^0, f^{0,\sharp})$ to $(f^1, f^{1,\sharp})$. Given a homotopy $(G, G^\sharp)$ from $(f^1, f^{1,\sharp})$ to $(f^2, f^{2,\sharp})$ we consider $\mathfrak{G}_\rho$, as before and define the concatenation $\mathfrak{F}_\rho \bullet \mathfrak{G}_\rho$ by

$$F_\rho \bullet G_\rho(x,t) = \begin{cases} F_\rho(x, 2t), & t \leq 1/2, \\ G_\rho(x, 2t-1), & t \geq 1/2, \end{cases}$$

and analogously $F_\rho^\sharp \bullet G_\rho^\sharp$. By our choice of ρ these are smooth maps and can be easily seen to determine a homotopy from $(f^0, f^{0,\sharp})$ to $(f^2, f^{2,\sharp})$.

For **(b)** we prove only the second statement as they are analogous. Let $(F, F^\sharp)$ be a homotopy from $(f^0, f^{0,\sharp})$ to $(f^1, f^{1,\sharp})$ and $e^\sharp : \mathfrak{h} \to e^*\mathfrak{h}'$ be an L_∞ homomorphism. It is easy to check there is an induced L_∞ homomorphism $\widetilde{e}^\sharp : F^*\widetilde{\mathfrak{h}}_{[0,1]} \to (e \circ F)^*\widetilde{\mathfrak{h}'}_{[0,1]}$. Now the pair $(e \circ F, \widetilde{e}^\sharp \circ F^\sharp)$ defines the required homotopy from $(e, e^\sharp) \circ (f^0, f^{0,\sharp})$ to $(e, e^\sharp) \circ (f^1, f^{1,\sharp})$. ■

5. The inverse function theorem for L_∞ spaces

In this section, we first adapt the obstruction theory of Section 2 to the case of L_∞ spaces. Then we prove Theorem 1.1 and Theorem 1.3.

5.1. Obstruction theory for morphisms between L_∞ spaces

Much of the discussion on the obstruction theory for A_∞ and L_∞ homomorphisms in Section 2 translates without significant changes to the L_∞ space setting. When we are given two L_∞ spaces $(M, \mathfrak{g})$, $(N, \mathfrak{h})$ and a smooth map $f : M \to N$, we can define a differential δ on $\bigoplus_k \mathsf{Hom}(\mathsf{sym}^k(\mathfrak{g}[1]), f^*\mathfrak{h}[1])$, as in Section 2.1,

$$\delta(\phi_k)(a_1, \ldots, a_{k-1}) := (-1)^{|\phi_k|'}\phi_k(\mu_0, a_1, \ldots, a_{k-1}).$$

We denote by $D^k(\mathfrak{g}, f^*\mathfrak{h})$ the δ cohomology and, assuming there is a map $f_1^\sharp$ satisfying $f_1^\sharp(\mu_0) = f^*\mu_0$, we define the differential

$$d\phi(a_1, \ldots, a_k) := \mu_1\phi(a_1, \ldots, a_k) - (-1)^{|\phi|' + |a_i|'\sum_{l=1}^{i-1}|a_l|'}\phi(\mu_1(a_i), a_1 \ldots, \widehat{a_i}, \ldots, a_k),$$

as before μ_1 in the first term really stands for $f^*\mu_1$. As in Definition 2.2, we define the obstruction space

$$H^k(\mathfrak{g}, f^*\mathfrak{h}) := H^1\big(D^k(\mathfrak{g}, f^*\mathfrak{h}), d\big).$$

We also define a sequence of maps $(f_1^\sharp, \ldots, f_n^\sharp) : \mathfrak{g} \to f^*\mathfrak{h}$ (together with f) to be an $L_{(n)}$ morphism, if it satisfies the L_∞ homomorphism equation for $0 \leq k \leq n-1$

inputs and for n inputs up to a δ-exact term. We define, in the same way as in Section 2, an obstruction class

$$\mathfrak{o}\big((f_j^{\sharp})_{j=1}^n\big) \in H^{n+1}(\mathfrak{g}, f^*\mathfrak{h}).$$

This obstruction class vanishes if and only if the map can be lifted to an $L_{(n+1)}$ morphism.

Lemma 5.1. *The obstruction class* $\mathfrak{o}((f_j^{\sharp})_{j=1}^n) \in H^{n+1}(\mathfrak{g}, f^*\mathfrak{h})$ *vanishes if and only if the corresponding class (in the extended* L_∞ *algebras)* $\mathfrak{o}((f_j^{\sharp})_{j=1}^n) \in H^{n+1}(\tilde{\mathfrak{g}}, f^*\tilde{\mathfrak{h}})$ *vanishes.*

Proof. If there exists $f_{n+1}^{\sharp} : \mathsf{sym}^{n+1}(\mathfrak{g}[1]) \to f^*\mathfrak{h}[1]$, we may extend it using equation (13) to obtain an $L_{(n+1)}$ homomorphism on the extended L_∞ algebras, which implies that $\mathfrak{o}((f_j^{\sharp})_{j=1}^n) \in H^{n+1}(\tilde{\mathfrak{g}}, f^*\tilde{\mathfrak{h}})$ vanish. Conversely, if the latter obstruction class vanishes, we simply restrict the map $f_{n+1}^{\sharp}$ to $\mathsf{sym}^{n+1}(\mathfrak{g}[1]) \to f^*\mathfrak{h}[1]$. ■

Extra work is needed to formulate the homotopy invariance of obstruction spaces and classes. Let $(F, F^{\sharp})$ be a homotopy between two $L_{(1)}$ morphisms $(f^0, f^{0,\sharp})$ and $(f^1, f^{1,\sharp})$. Denote by $\iota_a : M \to M \times [0,1]$ the inclusion map $\iota_a(x) = (x, a)$. For an element $\varphi \in \mathsf{Hom}(\mathsf{sym}^{n+1}\pi_1^*\tilde{\mathfrak{g}}[1], F^*\tilde{\mathfrak{h}}_{[0,1]}[1])$, we have $\iota_a^*(\varphi) \in \mathsf{Hom}(\mathsf{sym}^{n+1}\tilde{\mathfrak{g}}[1], (f^a)^*\tilde{\mathfrak{h}})$, for $a = 0, 1$. It is easy to see this assignment induces a map on obstruction spaces $\mathsf{ev}_a : H^{n+1}(\pi_1^*\tilde{\mathfrak{g}}, F^*\tilde{\mathfrak{h}}_{[0,1]}) \to H^{n+1}(\tilde{\mathfrak{g}}, (f^a)^*\tilde{\mathfrak{h}})$, which we call the evaluation map.

Proposition 5.2. *The evaluation map*

$$\mathsf{ev}_a : H^{n+1}(\pi_1^*\tilde{\mathfrak{g}}, F^*\tilde{\mathfrak{h}}_{[0,1]}) \to H^{n+1}(\tilde{\mathfrak{g}}, (f^a)^*\tilde{\mathfrak{h}})$$

is an isomorphism.

Proof. Both cases are identical, we will prove the statement for $a = 0$, by constructing i, a homotopy inverse to ev_0. Given $\varphi \in \mathsf{Hom}(\mathsf{sym}^{n+1}\tilde{\mathfrak{g}}[1], (f^0)^*\tilde{\mathfrak{h}})$, we define $\mathsf{i}\varphi(a_1, \dots, a_k)$ by taking the (unique) flat extension of $\varphi(a_1, \dots, a_k)$ in the t-direction and so obtain an element of $F^*\tilde{\mathfrak{h}}$. It is clear, i commutes with the δ differential and hence it induces a map from $D^k(\tilde{\mathfrak{g}}, (f^0)^*\tilde{\mathfrak{h}})$ to $D^k(\pi_1^*\tilde{\mathfrak{g}}, F^*\tilde{\mathfrak{h}}_{[0,1]})$. We pick a flat frame of the bundle $F^*\tilde{\mathfrak{h}}$ and compute, for a δ-closed φ,

$$\begin{aligned} d(\mathsf{i}(\varphi)) - \mathsf{i}(d(\varphi)) = (\mu_1^t - \mu_1^0) \circ \mathsf{i}\varphi &= \int_0^t \nabla_{d/ds}\mu_1^s \circ \mathsf{i}\varphi \, ds \\ &= \int_0^t \mu_2^s\Big(\frac{\partial F}{\partial s}, \mathsf{i}\varphi\Big)\, ds = \int_0^t \mu_2^s(h_1^s(\mu_0), \mathsf{i}\varphi)\, ds \\ &= \delta\bigg(\int_0^t \mu_2^s(h_1^s \cdot \mathsf{i}\varphi)\, ds\bigg). \end{aligned}$$

Here, h_1^t is the map coming from the definition of homotopy in (16). Hence, i induces a map between the corresponding obstruction spaces. Clearly, we have $\mathsf{ev}_0 \circ \mathsf{i} = \mathsf{id}$. Let

$$K : D^k(\pi_1^*\tilde{\mathfrak{g}}, F^*\tilde{\mathfrak{h}}_{[0,1]}) \to D^k(\pi_1^*\tilde{\mathfrak{g}}, F^*\tilde{\mathfrak{h}}_{[0,1]})$$

be the map induced by the integration map

$$K(\varphi^t + \psi^t dt)(a_1, \ldots, a_k) = (-1)^{|\psi|'+\star} \int_0^t \psi^s(a_1, \ldots, a_k)\, ds,$$

where $\star := |a_1|' + \cdots + |a_k|'$. We claim that $\mathsf{i} \circ \mathsf{ev}_0 - \mathsf{id} = dK + Kd$. For $\phi := \varphi^t + \psi^t dt$ we compute (omitting the inputs)

$$\begin{aligned}(dK + Kd)(\phi) &= (-1)^{|\psi|'+\star}\mu_1^t \int_0^t \psi^s\, ds - \nabla_{\frac{d}{dt}} \int_0^t \psi^s\, ds\, dt - (-1)^{\star+1} \int_0^t \psi^s \widetilde{\mu_1}\, ds \\ &\quad - \int_0^t \nabla_{\frac{d}{ds}} \varphi^s\, ds + (-1)^{|\psi|'+\star+1} \int_0^t \mu_1^s \psi^s\, ds - (-1)^{\star} \int_0^t \psi^s \widetilde{\mu_1}\, ds \\ &= (-1)^{|\psi|'+\star}\Big(\int_0^t \nabla_{\frac{d}{ds}} \mu_1^s K(\psi^s ds)\, dt - \int_0^t \mu_1^s\big(\nabla_{\frac{d}{ds}} K(\psi^s ds)\big)\, dt \Big) \\ &\quad - \psi^t dt - \varphi^t + \varphi^0 \\ &= \mathsf{iev}_0(\phi) - \phi + (-1)^{|\psi|'+\star} \int_0^t \mu_2^s\Big(\frac{\partial F}{\partial s} \cdot K(\psi^s ds)\Big)\, dt \\ &= \mathsf{iev}_0(\phi) - \phi - (-1)^{|\psi|'+\star}\delta\Big(\int_0^t \mu_2^s\Big(h_1^s, \int_0^s \psi^u\, du\Big)\, dt \Big).\end{aligned}$$

In the last equality we have used the fact that $h_1^s(\mu_0) = \frac{\partial F}{\partial s}$. Thus, we conclude that $\mathsf{i} \circ \mathsf{ev}_0 = \mathsf{id}_{H^{n+1}(\pi_1^*\widetilde{\mathfrak{g}}, F^*\widetilde{\mathfrak{h}}_{[0,1]})}$ and therefore ev_0 is an isomorphism. ∎

Corollary 5.3. *Let $(f^0, f^{0,\sharp})$ and $(f^1, f^{1,\sharp})$ be two homotopic $L_{(n)}$ morphisms. Then $(f^0, f^{0,\sharp})$ lifts to an $L_{(n+1)}$ morphism if and only if $(f^1, f^{1,\sharp})$ does.*

Proof. The morphism $(f^a, f^{a,\sharp})$ lifts to an $L_{(n+1)}$ morphism if and only if $\mathfrak{o}((f_j^{a,\sharp})_{j=1}^n)$ vanishes. Let $(F, F^\sharp)$ be the homotopy between the two $L_{(n)}$ morphisms. We can easily see

$$\mathfrak{o}\big((f_j^{a,\sharp})_{j=1}^n\big) = \mathsf{ev}_a\big(\mathfrak{o}\big((F_j^\sharp)_{j=1}^n\big)\big).$$

By the previous proposition, ev_a is an isomorphism and therefore $\mathfrak{o}((f_j^{a,\sharp})_{j=1}^n)$ vanishes if and only if $\mathfrak{o}((F_j^\sharp)_{j=1}^n)$ does. ∎

The push-forward $(\mathfrak{e})_*$ and pull-back $(\mathfrak{d})^*$ maps on the obstruction space, under an $L_{(1)}$ morphism, are defined in the same manner as in the algebra case. We have the analogue to Lemma 2.5.

Lemma 5.4. *Let $\mathfrak{e} = (e, e_1^\sharp) : (N, \mathfrak{h}) \to (N', \mathfrak{h}')$ and $\mathfrak{d} = (d, d_1^\sharp) : (M', \mathfrak{g}') \to (M, \mathfrak{g})$ be $L_{(1)}$ homotopy equivalences. Then both maps*

$$\begin{aligned}(\mathfrak{e})_* &: H^n(\mathfrak{g}, f^*\mathfrak{h}) \to H^n(\mathfrak{g}, (e \circ f)^*\mathfrak{h}'),\\ (\mathfrak{d})^* &: H^n(\mathfrak{g}, f^*\mathfrak{h}) \to H^n(\mathfrak{g}', (f \circ d)^*\mathfrak{h})\end{aligned}$$

are isomorphisms.

Proof. Let $\mathfrak{E}$ be an $L_{(1)}$ homotopy between two $L_{(1)}$ morphisms $\mathfrak{e}^0$ and $\mathfrak{e}^1$. Observe that $\mathrm{ev}_a \circ (\mathfrak{E})_* = (\mathfrak{e}^a)_*$ for $a = 0, 1$. Since ev_a is an isomorphism, by Proposition 5.2, we conclude $(\mathfrak{e}^a)_*$ is an isomorphism if and only if $(\mathfrak{E})_*$ is an isomorphism. Now, let $\bar{\mathfrak{e}}$ be an $L_{(1)}$ homotopy inverse for $\mathfrak{e}$, then by the previous argument $(\mathfrak{e} \circ \bar{\mathfrak{e}})_*$ (and $(\bar{\mathfrak{e}} \circ \mathfrak{e})_*$) is an isomorphism. Hence we conclude $(\mathfrak{e})_*$ is an isomorphism from the equality $(\mathfrak{e} \circ \bar{\mathfrak{e}})_* = (\mathfrak{e})_* \circ (\bar{\mathfrak{e}})_*$.

The same argument proves the statement for $(\mathfrak{d})^*$. ■

With these preparations, we may deduce the following result which is the analogue of Theorem 2.8 in the case of L_∞ spaces. Its proof is essentially the same: using the previous results we prove analogues of Proposition 2.6 and Lemma 2.7 which lead to the following theorem. Since this involves only minor modifications we omit its proof.

Theorem 5.5. *An L_∞ space homomorphism $\mathfrak{f} = (f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ is a homotopy equivalence if and only if $(f, f_1^\sharp)$ is an $L_{(1)}$ homotopy equivalence.*

Remark 5.6. Unlike in the algebraic case (see Remark 2.9) the Whitehead theorem of L_∞ spaces (Theorem 1.1) does not immediately follow from the above result. The proof that quasi-isomorphisms are $L_{(1)}$ homotopies in the curved situation is considerably harder. In the remaining part of the section, we shall first prove Theorem 1.3 on the existence of minimal charts. Then we make use of the minimal charts to prove the desired Whitehead theorem.

5.2. Minimal charts

Let $\mathbb{M} = (M, \mathfrak{g})$ be an L_∞ space and let $p \in M$ be a point in the zero-set of μ_0. As in the introduction we define the tangent complex of $(M, \mathfrak{g})$ at p to be

$$T_p\mathbb{M} := T_pM \xrightarrow{\nabla\mu_0|_p} \mathfrak{g}_2|_p \xrightarrow{\mu_1|_p} \mathfrak{g}_3|_p \xrightarrow{\mu_1|_p} \mathfrak{g}_4|_p \xrightarrow{\mu_1|_p} \cdots \xrightarrow{\mu_1|_p} \mathfrak{g}_N|_p. \tag{17}$$

The fact that this is indeed a complex follows from the L_∞ algebra equation together with the condition $\mu_0|_p = 0$. Also note that the first map is independent of the connection.

Let $\mathfrak{f} = (f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ be an L_∞ morphism and $p \in \mu_0^{-1}(0)$. It easily follows from the definition of morphism that df and $f_1^\sharp$ induce a chain map $T_p\mathfrak{f} : T_p\mathbb{M} \to T_{f(p)}\mathbb{N}$.

Definition 5.7. Let $(M, \mathfrak{g})$ be an L_∞ space and let $p \in \mu_0^{-1}(0)$. We say $(M, \mathfrak{g})$ is *minimal* at p if all the maps in the complex $T_p\mathbb{M}$ are zero.

A morphism $\mathfrak{f} : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ is called a *quasi-isomorphism* at p if the chain map $T_p\mathfrak{f}$ induces an isomorphism in cohomology.

We have the following easy lemma.

Lemma 5.8. *An L_∞ morphism which is an $L_{(1)}$ homotopy equivalence is a quasi-isomorphism at any point.*

For any open set $W \subseteq M$ we can restrict the L_∞ structure to W and so obtain a new L_∞ space $(W, \mathfrak{g}|_W)$. We define a chart at p to be an L_∞ space $(N, \mathfrak{h})$, with $n_p \in \mu_0^{-1}(0)$, together with an L_∞ homotopy equivalence $\mathrm{i} = (i, i^\sharp) : (N, \mathfrak{h}) \to (W, \mathfrak{g}|_W)$ for some neighborhood W of p in M, such that $i(n_p) = p$. We say the chart is *minimal* if $(N, \mathfrak{h})$ is minimal at n_p.

The main step in the proof of the inverse function theorem for L_∞ spaces is the construction of minimal charts. We will do it in two steps.

Proposition 5.9. *Let $(M, \mathfrak{g})$ be an L_∞ space and $q \in \mu_0^{-1}(0)$. There is a chart at q, $(N, \mathfrak{h})$ with the property $\nabla \mu_0^N|_{n_q} = 0$.*

Proof. In a neighborhood U of q with coordinates $(x_1, \ldots, x_n)$, trivialize the bundle $\mathfrak{g}_2$ and write $\mu_0 = s = (s_1, \ldots, s_m) : U \to \mathbb{R}^m$. If $\nabla \mu_0|_q \neq 0$, there is i, j such that $\frac{\partial s_i}{\partial x_j}(q) \neq 0$. Hence, $N = {s_i}^{-1}(0) \cap W$, for some small open set $W \subset U$, is a smooth submanifold. It follows from the inverse function theorem (for smooth manifolds) that we can find coordinates on W, $(x_1, \ldots, x_{n-1}, y)$ such that $N = \{(x_1, \ldots, x_{n-1}, 0)\}$. Moreover, we can decompose the bundle $\mathfrak{g}_2|_N = E_2 \oplus C_2$ such that $s(x, y) = (\nu(x, y), y)$. We define $E_k = \mathfrak{g}_k|_N$ for $k \geq 3$. We define the map $\iota : N \to M$ as $\iota(x) = (x, 0)$. Additionally we denote by i the inclusion $E \to \mathfrak{g}$ and by p the projection $\mathfrak{g} \to E$. We claim the operations $\lambda_0 := \nu|_N$ and $\lambda_k = \mu_k|_E$, $k \geq 1$ define an L_∞ space. Indeed this is a degenerate case of Theorem 3.1 where we take $H = 0$. Please note that even though equation (21) does not hold on $\mathfrak{g}_2$, the theorem still holds since it is enough to have equation (21) hold on $\mathfrak{g}_{\geq 3}$, since this is the only situation where it is applied.

Therefore we have an L_∞ space

$$\Big(N, \mathfrak{E} := \bigoplus_{k \geq 2} E_k, \lambda_k\Big).$$

Moreover, there is an L_∞ homomorphism $\iota^\sharp : \mathfrak{E} \to \iota^* \mathfrak{g}$, with $\iota_1^\sharp = i|_N$. We now construct an $L_{(1)}$ homotopy inverse to $(\iota, \iota^\sharp)$. For this purpose, we define the maps $\Pi : M \times [0, 1] \to M$, $\Pi(x, y, t) = (x, ty)$ and $\pi_1^{t,\sharp} : \mathfrak{g} \to (\Pi^t)^* \mathfrak{g}$ by the formula

$$\begin{bmatrix} \mathsf{id} & -\int_t^1 \frac{\partial \nu}{\partial y}(x, sy)\, ds \\ 0 & t \cdot \mathsf{id} \end{bmatrix} : \mathfrak{g}_2 \to (\Pi^t)^* \mathfrak{g}_2,$$

and $\pi_1^{t,\sharp} = \mathsf{id}$ on $\mathfrak{g}_{k \geq 3}$. We claim the pair $(P, P_1^\sharp)$, where $P(x, y) = x$ and $P_1^\sharp = p\pi_1^{0,\sharp}$, is an $L_{(1)}$ morphism from $(M, \mathfrak{g})$ to $(N, \mathfrak{E})$, and moreover it is an $L_{(1)}$ homotopy inverse to $(\iota, \iota^\sharp)$. We first show that $\pi_1^{t,\sharp}$ is an $L_{(1)}$ homomorphism. An easy computation gives

$$\pi_1^{t,\sharp}(\mu_0) = (\nu(x, ty), ty) = (\Pi^t)^*(\mu_0).$$

In the decomposition $\mathfrak{g}_2|_N = E_2 \oplus C_2$, we write $\mu_1 = (\varphi, \alpha)$ and compute

$$(\pi_1^{t,\sharp}\mu_1 - \mu_1^t \pi_1^{t,\sharp})|_{\mathfrak{g}_2} = \Big(\varphi - \varphi^t, \alpha - t\alpha^t + \varphi^t \cdot \int_t^1 \frac{\partial \nu}{\partial y}(sy)\, ds\Big).$$

We claim this is δ-exact. We define

$$P_2 = \int_t^1 \mu_2^s(K \otimes \mathrm{id}) - \mu_2^s(\mathrm{id} \otimes K)\, ds,$$

where $K : \mathfrak{g}_2 \to T_M$ is the map defined as $K(e,c) = c\frac{\partial}{\partial y}$, and compute

$$\begin{aligned}\delta(P_2)(e,c) &= \int_t^1 \mu_2^s\Big(y\frac{\partial}{\partial y},(e,c)\Big) - \mu_2^s\Big(\mu_0, c\frac{\partial}{\partial y}\Big)\, ds \\ &= \int_t^1 \nabla_{\frac{d}{ds}}\mu_1^s((e,c)) - c\frac{\partial \mu_1}{\partial y}(x,sy)(\mu_0)\, ds \\ &= \Big(\varphi - \varphi^t,\ \alpha - \alpha^t - \int_t^1 \frac{\partial \varphi}{\partial y}(sy)\cdot \nu(x,y)\, ds - \int_t^1 \frac{\partial \alpha}{\partial y}(sy) y\, ds\Big)(e,c).\end{aligned} \tag{18}$$

Therefore $\delta(P_2) = (\pi_1^{t,\sharp}\mu_1 - \mu_1^t\pi_1^{t,\sharp})|_{\mathfrak{g}_2}$ is equivalent to the following identity:

$$\int_t^1 \varphi(x,ty)\cdot\frac{\partial \nu}{\partial y}(sy)\, ds + \int_t^1 \frac{\partial \varphi}{\partial y}(sy)\cdot \nu(x,y)\, ds = t\alpha^t - \alpha,$$

which in turn follows from the fact that the left-hand side equals

$$\int_t^1 \frac{\partial(\varphi\cdot\nu)}{\partial y}(x,sy)\, ds. \tag{19}$$

The L_∞ relation $\mu_1(\mu_0) = 0$ implies $\varphi\cdot\nu = -\alpha\cdot y$, hence

$$\int_t^1 \frac{\partial(\varphi\cdot\nu)}{\partial y}(x,sy)\, ds = -\int_t^1 \frac{\partial\alpha}{\partial y}(x,sy)sy + \alpha(x,sy)\, ds = t\alpha(x,ty) - \alpha(x,y).$$

Here, the last equality is given by integration by parts. Similarly one shows that

$$(\pi_1^{t,\sharp}\mu_1 - \mu_1^t\pi_1^{t,\sharp})|_{\mathfrak{g}_{\geq 3}} = \delta\Big(\int_t^1 \mu_2^s(K\otimes \mathrm{id})\Big).$$

Hence we conclude that $\pi_1^{t,\sharp}$ is an $L_{(1)}$ homomorphism.

We observe that $P^\sharp \circ \iota^\sharp = \mathrm{id}$, $\pi_1^{0,\sharp} = \iota^\sharp \circ P^\sharp$ and $\pi_1^{0,\sharp} = \mathrm{id}$. Therefore, in order to conclude that $(P, P\sharp)$ is an $L_{(1)}$ homotopy inverse to $(\iota,\iota^\sharp)$ it is enough, by (16), to define

$$h_1^t = \begin{cases} K, & \mathfrak{g}_2, \\ 0, & \mathfrak{g}_{\geq 3}, \end{cases}$$

and check the following identities:

$$h_1^t(\mu_0) = \frac{\partial \Pi}{\partial t}, \qquad \frac{\partial \pi_1^{t,\sharp}}{\partial t} = \mu_1^t h_1^t + h_1^t\mu.$$

We have thus proved that the L_∞ morphism $(\iota, \iota^\sharp)$ is an $L_{(1)}$ homotopy equivalence. It now follows from Theorem 5.5 that $(\iota, \iota^\sharp)$ is an L_∞ homotopy equivalence and therefore $(N, \mathfrak{E})$ is a chart at q. By construction, the rank of $\nabla \mu_0|_q$ in n is strictly smaller than in the original space M, hence, by applying the previous construction finitely many times, we can find a chart at q such that $\nabla \mu_0|_q$ has rank zero as claimed in the statement. ∎

We now state and prove Theorem 1.3 in the Introduction.

Theorem 5.10. *Let $(M, \mathfrak{g})$ be an L_∞ space and $q \in \mu_0^{-1}(0)$. There is a minimal chart at q.*

Proof. Proposition 5.9 implies that we can assume $\nabla \mu_0|_q = 0$. We pick a sub-bundle $E_3 \subseteq \mathfrak{g}_3$ such that $E_3|_q \oplus \mathsf{Im}(\mu_1|_q) = \mathfrak{g}_3|_q$. Then $\widetilde{\mu_1} : \mathfrak{g}_2 \to \mathfrak{g}_3/E_3$ is surjective at q, since $\mathsf{Im}\mu_1|_q = \mathsf{Im}\widetilde{\mu_1}|_q$. Hence $\widetilde{\mu_1}$ is surjective on some neighborhood of q, which we denote by U. This implies that $A_2 := \ker \widetilde{\mu_1}|_U$ is a sub-bundle of $\mathfrak{g}_2$. We pick a complement $A_2 \oplus B_2 = \mathfrak{g}_2$ and define $C_3 = \mu_1(B_2) \subseteq \mathfrak{g}_3|_U$. By construction, $\mu_1|_{B_2}$ is injective and therefore C_3 is a bundle. Moreover, $\mathfrak{g}|_3 = E_3 \oplus C_3$.

Next, we pick $E_4 \subset \mathfrak{g}_4$ such that $E_4|_q \oplus \mathsf{Im}(\mu_1|_q) = \mathfrak{g}_4|_q$ and define $A_3 := \ker \widetilde{\mu_1} : E_3 \to \mathfrak{g}_4/E_4$. We pick a complement $A_3 \oplus B_3 = E_3$ and define $C_4 := \mu_1(B_3)$. We repeat this argument, for all k and obtain a decomposition $\mathfrak{g}_k = A_k \oplus B_k \oplus C_k$ in a neighborhood of q, here $C_2 = 0$. In this decomposition, the map μ_1 takes the form

$$\mu_1 = \begin{bmatrix} \varphi & 0 & \alpha \\ \psi & 0 & \beta \\ 0 & \varepsilon & \gamma \end{bmatrix}.$$

Note that ε is an isomorphism, so we can define the degree -1 map

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varepsilon^{-1} \\ 0 & 0 & 0 \end{bmatrix} : \mathfrak{g} \to \mathfrak{g}.$$

On $\mathfrak{g}_2$, H is defined to be zero. We denote by i the inclusion $A \to \mathfrak{g}$ and by p the projection onto A. We have the following identities on $\mathfrak{g}$, which are easy to check:

$$H \circ i = 0, \quad p \circ H = 0, \tag{20}$$

$$H\mu_1 + \mu_1 H = i \circ p - \mathsf{id}_{\mathfrak{g}} - H\mu_1^2 H. \tag{21}$$

The first L_∞ equation $\mu_1(\mu_0) = 0$ implies that $\mu_0 = (\nu, 0)$, since ε is an isomorphism. Hence, $i(\nu)|_N = \iota^* \mu_0$ and we have all the data and conditions as in Theorem 3.1. Therefore, Theorem 3.1 constructs an L_∞ space

$$\left(M, \mathfrak{A} := \bigoplus_{k \geq 2} A_k, \lambda_k\right),$$

with $\lambda_0 = \nu$ and $\lambda_1 = p\mu_1 i$. Moreover, there is an L_∞ morphism $(\iota, \iota^\sharp) : (U, \mathfrak{A}) \to (U, \mathfrak{g}|_U)$, with $\iota = \mathrm{id}$ and $\iota_1^\sharp = i$. Also observe that, by definition of A_k, $\mu_1|_q(A_k) = 0$ and thus $(U, \mathfrak{A})$ is minimal at q.

The last step is to construct an $L_{(1)}$ homotopy inverse to $(\iota, \iota^\sharp)$ and then appeal to Theorem 5.5 to conclude that $(\iota, \iota^\sharp)$ is an L_∞ homotopy equivalence. For this purpose, we define the maps $\pi_1^{t,\sharp} : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\begin{bmatrix} \mathrm{id} & 0 & 0 \\ 0 & t \cdot \mathrm{id} & 0 \\ 0 & 0 & t \cdot \mathrm{id} \end{bmatrix} : \mathfrak{g} \to \mathfrak{g}.$$

We first show that $\pi_1^{t,\sharp}$ is an $L_{(1)}$ homomorphism. Since $\mu_0 = (\nu, 0)$ we have $\pi_1^t(\mu_0) = \mu_0$. Next, we define the map

$$P_2 = (1-t)\big(H\mu_2(p \otimes p) + p\mu_2(H \otimes \mathrm{id}) - p\mu_2(\mathrm{id} \otimes H)\big).$$

A simple computation using (21) gives

$$\mu_1 \pi_1^{t,\sharp} - \pi_1^{t,\sharp}\mu_1 = \delta(P_2).$$

In particular, we have shown that $P_1^\sharp := \Pi\pi^{0,\sharp}$ (where $\Pi : \mathfrak{g} \to A$ is the projection) is an $L_{(1)}$ morphism from $\mathfrak{g}$ to $\mathfrak{A}$.

It is obvious that $P_1^\sharp \circ \iota^\sharp = \mathrm{id}_{\mathfrak{A}}$. Finally, we need to show that $\iota^\sharp \circ P_1^\sharp$ is $L_{(1)}$ homotopic to the identity. First note $\pi^{1,\sharp} = \mathrm{id}_{\mathfrak{g}}$ and $\pi^{0,\sharp} = \iota^\sharp \circ P_1^\sharp$. We define $h_1^t = -H$ and easily check

$$h_1^t(\mu_0) = 0, \qquad \frac{\partial \pi^{t,\sharp}}{\partial t} = h_1^t \mu_1 + \mu_1 h_1^t - \delta(Q_2),$$

where $Q_2 = H\mu_2(H \otimes \mathrm{id}) + H\mu_2(\mathrm{id} \otimes H)$. This completes the proof that $(\iota, \iota^\sharp)$ is an $L_{(1)}$ and therefore an L_∞ homotopy equivalence. ■

We are now ready to prove Theorem 1.1 in the introduction.

Theorem 5.11. *Let $(M, \mathfrak{g})$ and $(N, \mathfrak{h})$ be L_∞ spaces and $\mathfrak{f} = (f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ be an L_∞ morphism. Assume that $\mathfrak{f}$ is a quasi-isomorphism at $q \in \mu_0^{-1}(0)$. Then there are neighborhoods U of q and V of $f(q)$ such that $f(U) \subseteq V$ and*

$$\mathfrak{f}|_U : (U, \mathfrak{g}|_U) \to (V, \mathfrak{h}|_V)$$

is an L_∞ homotopy equivalence.

Proof. Theorem 5.10 provides minimal charts at p and $f(p)$ and hence we have the following diagram:

$$\begin{array}{ccc} (L, \mathfrak{a}) & \underset{\mathfrak{p}}{\overset{\mathfrak{i}}{\rightleftarrows}} & (U, \mathfrak{g}|_U) \\ {\scriptstyle \mathfrak{F}^{-1}}\uparrow\downarrow{\scriptstyle \mathfrak{F}} & & \downarrow{\scriptstyle \mathfrak{f}_U} \\ (\widetilde{L}, \widetilde{\mathfrak{a}}) & \underset{\widetilde{\mathfrak{p}}}{\overset{\widetilde{\mathfrak{i}}}{\rightleftarrows}} & (V, \mathfrak{h}|_V). \end{array}$$

Here the pairs $\mathfrak{i}$, $\mathfrak{p}$ and $\tilde{\mathfrak{i}}, \tilde{\mathfrak{p}}$ are homotopy inverses and $\mathfrak{J} := \tilde{\mathfrak{p}} \circ \mathfrak{f}_U \circ \mathfrak{i}$. It follows from Lemma 5.8 that $\mathfrak{J}$ is a quasi-isomorphism at $n_q =: p(q)$, but since $(L, \mathfrak{a})$ and $(\tilde{L}, \tilde{\mathfrak{a}})$ are minimal at n_q and $J(n_q)$, we conclude that J is a local diffeomorphism and $J_1^\sharp|_{n_q}$ is an isomorphism. After restricting to small neighborhoods U' and V' of n_q and $J(n_q)$ we have that J is a diffeomorphism and $J_1^\sharp$ is an isomorphism of bundles. Therefore we can solve equation (11) inductively on n to find a strict L_∞ inverse to $J^\sharp$, which we denote by $\mathfrak{J}^{-1}$.

We make the neighborhoods U and V smaller, if necessary, to ensure the restrictions of $\mathfrak{i}$, $\mathfrak{p}$ (and $\tilde{\mathfrak{i}}, \tilde{\mathfrak{p}}$) are homotopy inverses on $(U', \mathfrak{a}|_{U'})$ (and $(V', \tilde{\mathfrak{a}}|_{V'})$). We define $\mathfrak{K} := \mathfrak{i} \circ \mathfrak{J}^{-1} \circ \tilde{\mathfrak{p}}$. By construction $\mathfrak{J}^{-1} \circ \tilde{\mathfrak{p}} \circ \mathfrak{f}_U \cong \mathfrak{p}$, therefore

$$\mathfrak{K} \circ \mathfrak{f}_U \cong \mathfrak{i} \circ \mathfrak{J}^{-1} \circ \tilde{\mathfrak{p}} \circ \mathfrak{f}_U \cong \mathfrak{i} \circ \mathfrak{p} \cong \mathrm{id}.$$

Similarly, $\mathfrak{f}_U \circ \mathfrak{i} \circ \mathfrak{J}^{-1} \cong \tilde{\mathfrak{i}}$, hence $\mathfrak{f}_U \circ \mathfrak{K} \cong \tilde{\mathfrak{i}} \circ \tilde{\mathfrak{p}} \cong \mathrm{id}$. Thus, $\mathfrak{K}$ is a homotopy inverse to $\mathfrak{f}_U$. ∎

Acknowledgments. We are grateful to Jim Stasheff for sending us his comments and suggestions of an earlier version of the paper.

References

[1] L. Amorim, Tensor product of filtered A_∞-algebras. *J. Pure Appl. Algebra* **220** (2016), no. 12, 3984–4016 Zbl 1344.18012 MR 3517566

[2] K. Behrend, Differential graded schemes I: Perfect resolving algebras. 2002, arXiv:math/0212225

[3] K. Behrend, Differential graded schemes II: The 2-category of differential graded schemes. 2002, arXiv:math/0212226

[4] K. Behrend, H. Liao, and P. Xu, Derived differential geometry. 2020, arXiv:2006.01376

[5] C. Brav, V. Bussi, and D. Joyce, A Darboux theorem for derived schemes with shifted symplectic structure. *J. Amer. Math. Soc.* **32** (2019), no. 2, 399–443 Zbl 1423.14009 MR 3904157

[6] I. Ciocan-Fontanine and M. Kapranov, Derived Quot schemes. *Ann. Sci. École Norm. Sup. (4)* **34** (2001), no. 3, 403–440 Zbl 1050.14042 MR 1839580

[7] I. Ciocan-Fontanine and M. Kapranov, Virtual fundamental classes via dg-manifolds. *Geom. Topol.* **13** (2009), no. 3, 1779–1804 Zbl 1159.14002 MR 2496057

[8] K. Costello, A geometric construction of the Witten genus, II. 2011, arXiv:1112.0816

[9] V. A. Dolgushev, A. E. Hoffnung, and C. L. Rogers, What do homotopy algebras form? *Adv. Math.* **274** (2015), 562–605 Zbl 1375.18053 MR 3318161

[10] D. Fiorenza and M. Manetti, L_∞ structures on mapping cones. *Algebra Number Theory* **1** (2007), no. 3, 301–330 Zbl 1166.17010 MR 2361936

[11] K. Fukaya, Deformation theory, homological algebra and mirror symmetry. In *Geometry and physics of branes (Como, 2001)*, pp. 121–209, Ser. High Energy Phys. Cosmol. Gravit., IOP, Bristol, 2003 Zbl 1058.81002 MR 1950958

[12] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*. AMS/IP Stud. Adv. Math. 46, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009 Zbl 1181.53002 MR 2553465

[13] D. Joyce, A new definition of Kuranishi space. 2014, arXiv:1409.6908

[14] H. V. Lê and Y.-G. Oh, Deformations of coisotropic submanifolds in locally conformal symplectic manifolds. *Asian J. Math.* **20** (2016), no. 3, 553–596 Zbl 1352.53072 MR 3528832

[15] J.-L. Loday and B. Vallette, *Algebraic operads*. Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012 Zbl 1260.18001 MR 2954392

[16] M. Markl, Transferring A_∞ (strongly homotopy associative) structures. *Rend. Circ. Mat. Palermo (2) Suppl.* (2006), no. 79, 139–151 Zbl 1112.18007 MR 2287133

[17] M. Markl, *Deformation theory of algebras and their diagrams*. CBMS Reg. Conf. Ser. Math. 116, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012 Zbl 1267.16029 MR 2931635

[18] Y.-G. Oh and J.-S. Park, Deformations of coisotropic submanifolds and strong homotopy Lie algebroids. *Invent. Math.* **161** (2005), no. 2, 287–360 Zbl 1081.53066 MR 2180451

[19] A. Proute, *Algèbres différentielles fortement homotopiquement associatives*. Thèse d'Etat, Université Paris VII, 1984

[20] B. Pym and P. Safronov, Shifted symplectic Lie algebroids. *Int. Math. Res. Not. IMRN* (2020), no. 21, 7489–7557 Zbl 1475.53090 MR 4176831

[21] B. Toën and G. Vezzosi, Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.* **193** (2008), no. 902 Zbl 1145.14003 MR 2394633

[22] J. Tu, Homotopy L-infinity spaces. 2014, arXiv:1411.5115

[23] B. Vallette, Homotopy theory of homotopy algebras. *Ann. Inst. Fourier (Grenoble)* **70** (2020), no. 2, 683–738 Zbl 07210769 MR 4105949

Received 15 February 2021; revised 10 June 2021.

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