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A proof of a conjecture of Shklyarov

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Abstract. We prove a conjecture of Shklyarov concerning the relationship between K. Saito's higher residue pairing and a certain pairing on the periodic cyclic homology of matrix factorization categories. Along the way, we give new proofs of a result of Shklyarov (Corollary 2 of [Adv. Math. 292 (2016), 181–209]) and Polishchuk–Vaintrob's Hirzebruch–Riemann–Roch formula for matrix factorizations (Theorem 4.1.4 (i) of [Duke Math. J. 161 (2012), 1863–1926]).

1. Introduction

Let $Q = \mathbb{C}[x_1, \dots, x_n]$, and let $\mathfrak{m}$ denote the maximal ideal $(x_1, \dots, x_n) \subseteq Q$. Fix $f \in \mathfrak{m}$, and assume the only singular point of the associated morphism $f : \mathrm{Spec}(Q) \to \mathbb{A}^1_{\mathbb{C}}$ is $\mathfrak{m}$. Let $mf(Q, f)$ denote the differential $\mathbb{Z}/2$-graded category of matrix factorizations of f; see Section 3.1 for the definition of $mf(Q, f)$. Shklyarov proves in [16, Theorem 1] that a certain pairing on the periodic cyclic homology of $mf(Q, f)$ coincides, up to a constant factor c_f (which possibly depends on f), with K. Saito's higher residue pairing, via the Hochschild–Kostant–Rosenberg (HKR) isomorphism. Shklyarov conjectures in [16, Conjecture 3] that $c_f = (-1)^{n(n+1)/2}$. The main goal of this paper is to prove this conjecture.

We begin by discussing Shklyarov's conjecture in more detail.

1.1. Background on Shklyarov's conjecture

Let $HN(mf(Q, f))$ denote the negative cyclic complex of $mf(Q, f)$, and let $HN_*(mf(Q, f))$ denote its homology. See, for instance, [2, Section 3] for the definition of the negative cyclic complex of a dg-category. The dg-category $mf(Q, f)$ is *proper*, i.e., each cohomology group of the ($\mathbb{Z}/2$-graded) morphism complex of any two objects is a finite dimensional $\mathbb{C}$-vector space. As with any such dg-category, there is a canonical pairing of $\mathbb{Z}/2$-graded $\mathbb{C}$-vector spaces,

$$K_{mf} : HN_*\big(mf(Q, f)\big) \times HN_*\big(mf(Q, f)\big) \to \mathbb{C}[[u]],$$

where u is an even degree variable. The pairing K_{mf} is defined exactly as in [16, p. 184], but with periodic cyclic homology HP_* replaced with HN_* and $\mathbb{C}((u))$ replaced with

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$\mathbb{C}[[u]]$. We note that K_{mf} is $\mathbb{C}[[u]]$-*sesquilinear*; i.e., for any $\alpha, \beta \in HN_*(mf(Q, f))$ and $g \in \mathbb{C}[[u]]$, we have

$$K_{mf}\big(g(u)\cdot\alpha, \beta\big) = g(u)K_{mf}(\alpha, \beta) = K_{mf}\big(\alpha, g(-u)\cdot\beta\big).$$

It follows from the work of Segal [14, Corollary 3.4] and Polishchuk–Positselski [9, Section 4.8] that there is a quasi-isomorphism

$$I_f : HN\big(mf(Q, f)\big) \xrightarrow{\simeq} \big(\Omega^\bullet_{Q/\mathbb{C}}[[u]], ud - df\big), \tag{1.1}$$

which generalizes the classical HKR theorem. The target of I_f is called the *twisted de Rham complex*, and it is a $\mathbb{Z}/2$-graded complex indexed by setting $\Omega^m_{Q/\mathbb{C}}$ to have (homological) degree m and u to have degree -2. (Since the twisted de Rham complex is $\mathbb{Z}/2$-graded, we could just as well say Ω^m has degree $-m$ and u has degree 2. Note that the map ud has degree -1 whereas df has degree 1, but since this is regarded as a $\mathbb{Z}/2$-graded complex, there is no problem.) In particular, we have an isomorphism

$$I_f : HN_n\big(mf(Q, f)\big) \xrightarrow{\cong} H^{(0)}_f,$$

where

$$H^{(0)}_f := H_n\big(\Omega^\bullet_{Q/\mathbb{C}}[[u]], ud - df\big) = \frac{\Omega^n_{Q/\mathbb{C}}[[u]]}{(ud - df)\cdot\Omega^{n-1}_{Q/\mathbb{C}}[[u]]}.$$

In [13], K. Saito equips the $\mathbb{C}[[u]]$-module $H^{(0)}_f$ with a pairing

$$K_f : H^{(0)}_f \times H^{(0)}_f \to \mathbb{C}[[u]]$$

known as the *higher residue pairing*. Shklyarov has proven the following result concerning the relationship between the canonical pairing and the higher residue pairing under the HKR isomorphism.

Theorem 1.2 ([16, Theorem 1]). *For each polynomial f as above, there is a constant $c_f \in \mathbb{C}$ (possibly depending on f) such that the diagram*

$$\begin{array}{ccc} HN_n\big(mf(Q, f)\big)^{\times 2} & \xrightarrow[\cong]{I_f \times I_f} & \big(H^{(0)}_f\big)^{\times 2} \\ & {\scriptstyle c_f\cdot u^n\cdot K_{mf}} \searrow \quad \swarrow {\scriptstyle K_f} & \\ & \mathbb{C}[[u]] & \end{array} \tag{1.3}$$

commutes.

Moreover, Shklyarov makes the following prediction.

Conjecture 1.4 ([16, Conjecture 3]). *For any f, $c_f = (-1)^{n(n+1)/2}$.*

1.2. Outline of the proof of Conjecture 1.4

The constant c_f can be determined from a related, but simpler, pairing on $HH_*(mf(Q,f))$, the Hochschild homology of $mf(Q,f)$. We recall that, for any dg-category $\mathcal{C}$, there is a short exact sequence

$$0 \to HN(\mathcal{C}) \xrightarrow{\cdot u} HN(\mathcal{C}) \to HH(\mathcal{C}) \to 0 \tag{1.5}$$

of complexes. It follows, for instance from (1.1), that $HN_*(mf(Q,f))$ and $HH_*(mf(Q,f))$ are concentrated in degree n (mod 2). The long exact sequence in homology induced by (1.5) therefore induces an isomorphism

$$HN_*\big(mf(Q,f)\big)/u \cdot HN_*\big(mf(Q,f)\big) \xrightarrow{\cong} HH_*\big(mf(Q,f)\big). \tag{1.6}$$

The pairing K_{mf} determines a well-defined pairing modulo u, which we write, via (1.6), as

$$\eta_{mf} : HH_*\big(mf(Q,f)\big) \times HH_*\big(mf(Q,f)\big) \to \mathbb{C}.$$

The isomorphism I_f is $\mathbb{C}[[u]]$-linear and, upon setting $u = 0$, it induces an isomorphism

$$I_f(0) : HH_n\big(mf(Q,f)\big) \xrightarrow{\cong} H_n\big(\Omega^\bullet_{Q/\mathbb{C}}, -df\big).$$

The higher residue pairing K_f has the form

$$K_f\left(\omega + \sum_{j\geq 1} \omega_j u^j, \omega' + \sum_{j \geq 1} \omega'_j u^j\right) = \langle \omega, \omega' \rangle_{\text{res}} u^n + \text{higher order terms},$$

where $\langle \omega, \omega' \rangle_{\text{res}}$ is the classical residue pairing determined by the partial derivatives of f. It is defined algebraically as

$$\langle g \cdot dx_1 \cdots dx_n, h \cdot dx_1 \cdots dx_n \rangle_{\text{res}} = \operatorname{res}\left[\begin{matrix} gh \cdot dx_1 \cdots dx_n \\ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \end{matrix}\right],$$

where the right-hand side is Grothendieck's residue symbol.

Thus, upon dividing the maps in diagram (1.3) by u^n and setting $u = 0$, we obtain the commutative triangle

$$\begin{array}{ccc} HH_n\big(mf(Q,f)\big) \times HH_n\big(mf(Q,f)\big) & \xrightarrow[\cong]{I_f(0) \times I_f(0)} & \dfrac{\Omega^n_{Q/\mathbb{C}}}{df \wedge \Omega^{n-1}_{Q/\mathbb{C}}} \times \dfrac{\Omega^n_{Q/\mathbb{C}}}{df \wedge \Omega^{n-1}_{Q/\mathbb{C}}} \\ & {\scriptstyle c_f \eta_{mf}} \searrow \quad \swarrow {\scriptstyle \langle -,- \rangle_{\text{res}}} & \\ & \mathbb{C}. & \end{array} \tag{1.7}$$

Since $I_f(0)$ is an isomorphism, and the residue pairing is nonzero, the value of c_f is uniquely determined by the commutativity of (1.7).

In this paper, we re-establish the commutativity of diagram (1.7) using techniques that differ from those used by Shklyarov. Our method results in an explicit calculation of c_f.

Theorem 1.8. *Shklyarov's conjecture holds: that is, for any f as above,*

$$c_f = (-1)^{n(n+1)/2}.$$

In fact, we prove the commutativity of diagram (1.7), and Theorem 1.8, in the case where Q is an essentially smooth algebra over a characteristic 0 field k, $\mathfrak{m}$ is a k-rational maximal ideal, and $f \in \mathfrak{m}$ is such that $\mathfrak{m}$ is the only singularity of the morphism $f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k$. The special case $k = \mathbb{C}$, $Q = \mathbb{C}[x_1, \dots, x_n]$, and $\mathfrak{m} = (x_1, \dots, x_n)$ yields Shklyarov's conjecture.

The general outline of our proof is summarized by the diagram

$$\begin{array}{ccc}
HH_n\big(mf(Q,f)\big) \times HH_n\big(mf(Q,f)\big) & \xrightarrow[\cong]{I_f(0)\times I_f(0)} & H_n\big(\Omega^\bullet_{Q/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q/k}, -df\big) \\
\Big\downarrow {\scriptstyle \mathrm{id}\times\Psi} & & \Big\downarrow {\scriptstyle \mathrm{id}\times(-1)^n} \\
HH_n\big(mf(Q,f)\big) \times HH_n\big(mf(Q,-f)\big) & \xrightarrow[\cong]{I_f(0)\times I_{-f}(0)} & H_n\big(\Omega^\bullet_{Q/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q/k}, df\big) \\
\Big\downarrow {\scriptstyle \star} & & \Big\downarrow {\scriptstyle \wedge} \\
HH_{2n}\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0)\big) & \xrightarrow{\varepsilon} & H_{2n}\mathbb{R}\Gamma_{\mathfrak{m}}\big(\Omega^\bullet_{Q_{\mathfrak{m}}/k}\big) \\
& {\scriptstyle (-1)^{n(n+1)/2}\,\mathrm{trace}} \searrow \quad k. \quad \swarrow {\scriptstyle \mathrm{res}} &
\end{array} \tag{1.9}$$

The map Ψ is induced by taking Q-linear duals; $\star$ is induced by a Künneth map followed by the tensor product of matrix factorizations; trace is defined in Section 4; res is Grothendieck's residue map; $\wedge$ is induced by exterior multiplication of differential forms, using that the complexes $(\Omega^\bullet_{Q/k}, \pm f)$ are supported on $\{\mathfrak{m}\}$; and the map ε is an HKR-type map. We prove that

(1) the diagram commutes (Lemma 3.11, Lemma 3.14, Corollary 3.26, and Theorem 4.36),

(2) the composition along the left side of this diagram is the canonical pairing η_{mf} (Lemma 4.23), and

(3) the composition along the right side of this diagram is the residue pairing $\langle -, - \rangle_{\mathrm{res}}$ (Proposition 4.34).

Finally, in Section 5, we use some of our results to give a new proof Polishchuk–Vaintrob's Hirzebruch–Riemann–Roch theorem for matrix factorizations [10, Theorem 4.1.4 (i)].

We note that a result closely related to the commutativity of (1.7) was also proven by Polishchuk–Vaintrob [10, Corollary 4.1.3]. More precisely, they prove that the residue pairing on $H_n(\Omega^\bullet_{Q/k}, -df) \times H_n(\Omega^\bullet_{Q/k}, df)$ and the canonical pairing on $HH_n(mf(Q, f)) \times HH_n(mf(Q, -f))$ coincide up to multiplication by $(-1)^{(n-1)n/2}$

via an isomorphism

$$\gamma : HH_n\big(mf(Q, f)\big) \xrightarrow{\cong} H_n\big(\Omega^\bullet_{Q/k}, df\big)$$

described in [10, (2.28)]. Combining this result of Polishchuk–Vaintrob with our Theorem 1.8 and the nondegeneracy of the residue pairing, we conclude that, if $\alpha, \alpha' \in HH_n(mf(Q, f))$,

$$\big\langle \gamma(\alpha), \gamma(\alpha')\big\rangle_{\text{res}} = \big\langle I_f(0)(\alpha), I_f(0)(\alpha')\big\rangle_{\text{res}}. \tag{1.10}$$

If one could prove (1.10) directly, one could simply combine [10, Corollary 4.1.3] with the commutativity of the top square of diagram (1.9) to quickly prove Shklyarov's conjecture. But we believe there is no way to prove (1.10) without going through Theorem 1.8.

2. Generalities on Hochschild homology for curved dg-categories

We review some background on Hochschild homology of curved dg-categories and establish some new results concerning pairings of such. Throughout this section, k is a field, and "graded" means Γ-graded for $\Gamma \in \{\mathbb{Z}, \mathbb{Z}/2\}$. We will eventually focus on the case $\Gamma = \mathbb{Z}/2$.

2.1. Hochschild homology of curved dg-categories

We refer the reader to [2, Section 2.1] for the definition of a curved differential Γ-graded category (henceforth referred to as a cdg-category). Recall that a cdg-category with just one object is a curved differential Γ-graded algebra (cdga).

For a cdg-category $\mathcal{C}$ whose objects form a set, define $HH(\mathcal{C})^\natural$ to be the Γ-graded k-vector space given by the direct sum totalization of the $\mathbb{Z} - \Gamma$-bicomplex which, in $\mathbb{Z}$-degree n, is the Γ-graded k-vector space

$$\bigoplus_{X_0,\dots,X_n \in \mathcal{C}} \begin{aligned} &\operatorname{Hom}(X_1, X_0) \otimes_k \Sigma \operatorname{Hom}(X_2, X_1) \otimes_k \cdots \\ &\otimes_k \Sigma \operatorname{Hom}(X_n, X_{n-1}) \otimes_k \Sigma \operatorname{Hom}(X_0, X_n). \end{aligned}$$

When $\mathcal{C}$ is *essentially small*, so that the isomorphism classes of objects in the Γ-graded category underlying $\mathcal{C}$ form a set (see [9, Section 2.6]), we define $HH(\mathcal{C})^\natural$ by first replacing $\mathcal{C}$ with a full subcategory consisting of a single object from each isomorphism class. From now on, we will tacitly assume all of our cdg-categories are essentially small. Given $\alpha_i \in \operatorname{Hom}(X_{i+1}, X_i)$ for $i = 0, \dots, n$ (with $X_{n+1} = X_0$), we write $\alpha_0[\alpha_1|\cdots|\alpha_n]$ for the element $\alpha_0 \otimes s\alpha_1 \otimes \cdots \otimes s\alpha_n$ of $HH(\mathcal{C})^\natural$.

The *Hochschild complex of* $\mathcal{C}$, denoted by $HH(\mathcal{C})$, is the above graded k-vector space equipped with the differential $b := b_2 + b_1 + b_0$, where b_2, b_1, b_0 are defined as in [2, Section 3.1]. Roughly, b_2 is the classical Hochschild differential induced by the composition law in $\mathcal{C}$, b_1 is induced by the differentials of $\mathcal{C}$, and b_0 is induced by the curvature elements of $\mathcal{C}$. When $\mathcal{C}$ has just one object with trivial curvature, then $\mathcal{C}$ is a dga, and the maps b_2 and b_1 are the classical ones (and $b_0 = 0$ in this case).

We will also need "Hochschild homology of the second kind," as introduced by Polishchuk–Positselski in [9] and by Căldăraru–Tu in [4]; the latter authors call this theory "Borel-Moore Hochschild homology". Define $HH^{II}(\mathcal{C})^\natural$ to be the Γ-graded k-vector space given as the direct *product* totalization of the above bicomplex. Equivalently, $HH^{II}(\mathcal{C})^\natural$ is the completion of $HH(\mathcal{C})^\natural$ under the topology determined by the evident filtration. Since b is continuous for this topology, it induces a differential on $HH^{II}(\mathcal{C})^\natural$, which we also write as b, and we write $HH^{II}(\mathcal{C})$ for the resulting chain complex.

2.2. The Künneth map for Hochschild homology of cdga's

For a cdga $\mathcal{A} = (A, d_A, h_A)$, we have

$$HH(A)^\natural = A \otimes_k T(\Sigma A),$$

where, for any graded k-vector space V, $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$. Recall that $T(V)$ is a commutative k-algebra under the *shuffle product*:

$$(v_1 \otimes \cdots \otimes v_p) \bullet (v_{p+1} \otimes \cdots \otimes v_{p+q}) = \sum_\sigma \pm v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p+q)},$$

where σ ranges over all (p, q)-shuffles. The sign is given by the usual rule for permuting homogeneous elements in a product.

Since A is also an algebra, $HH(A)^\natural$ has an algebra structure, whose multiplication rule will be written as

$$- \star - : HH(\mathcal{A})^\natural \otimes_k HH(\mathcal{A})^\natural \to HH(\mathcal{A})^\natural.$$

It is given explicitly as

$$x[a_1|\cdots|a_p] \star y[a_{p+1}|\cdots|a_{p+q}] = \sum_\sigma \pm xy[a_{\sigma(1)}|\cdots|a_{\sigma(p+q)}].$$

Note that the canonical inclusion $T(\Sigma A) \hookrightarrow HH(A)^\natural$ lands in the center of $HH(A)^\natural$ for the $\star$ multiplication.

If $\mathcal{B} = (B, d_B, h_B)$ is another cdga, the tensor product of $\mathcal{A}$ and $\mathcal{B}$ is defined to be

$$\mathcal{A} \otimes_k \mathcal{B} = (A \otimes_k B, d_A \otimes 1 + 1 \otimes d_B, h_A \otimes 1 + 1 \otimes h_B).$$

We define the *Künneth map*

$$-\tilde{\star}- : HH(\mathcal{A})^\natural \otimes_k HH(\mathcal{B})^\natural \to HH(\mathcal{A} \otimes_k \mathcal{B})^\natural$$

to be the composition of the tensor product of the maps induced by the canonical inclusions $HH(\mathcal{A})^\natural \hookrightarrow HH(\mathcal{A} \otimes \mathcal{B})^\natural$ and $HH(\mathcal{B})^\natural \hookrightarrow HH(\mathcal{A} \otimes \mathcal{B})^\natural$ with the $\star$ product for

$\mathcal{A} \otimes_k \mathcal{B}$. The $\star$ product on $HH(\mathcal{A})^\natural$ can be recovered from the Künneth map by setting $\mathcal{B} = \mathcal{A}$: the $\star$ product coincides with the composition

$$HH(\mathcal{A})^\natural \otimes_k HH(\mathcal{A})^\natural \xrightarrow{\tilde{\star}} HH(\mathcal{A} \otimes_k \mathcal{A})^\natural \xrightarrow{\mu_*} HH(\mathcal{A})^\natural,$$

where

$$\mu_* : (A \otimes_k A) \otimes_k T\big(\Sigma(A \otimes A)\big) \to A \otimes_k T(\Sigma A)$$

is induced by the multiplication map $\mu : A \otimes A \to A$.

It is important to note that, for an algebra A, the $\star$ product does not, in general, make $HH(A)$ into a dga, since b_2 is not a derivation for the $\star$ multiplication unless A is commutative. But b_2 is a derivation for the Künneth map; see Lemma 2.6.

The $\star$ product does behave well with respect to b_1. In detail, recall that the tensor algebra functor $T(-)$ sends Γ-graded complexes of k-vector spaces to differential Γ-graded algebras under the shuffle product. Let d_T denote the differential on $T(\Sigma A)$ induced from the differential Σd on ΣA. Then $(T(\Sigma A), \bullet, d_T)$ is a dga, where $\bullet$ is the shuffle product. By examining the explicit formula for b_1, we see that

$$b_1 = d_A \otimes 1 + 1 \otimes d_T.$$

In other words, $(HH(\mathcal{A})^\natural, \star, b_1)$ is a dga, and it is given as a tensor product of dga's:

$$\big(HH(\mathcal{A})^\natural, \star, b_1\big) = (A, \cdot, d_A) \otimes \big(T(\Sigma A), \bullet, d_T\big),$$

where $\cdot$ is the multiplication rule for A.

If z is an element of A of even degree, then we have

$$1[z] \star a_0[a_1|\cdots|a_n] = \sum_i (-1)^{|a_0|+|a_1|+\cdots+|a_i|-i} a_0[a_1|\cdots|a_i|z|a_{i+1}|\cdots|a_n].$$

In particular, the component b_0 of the differential in $HH(\mathcal{A})$ is given by

$$b_0 = 1[h] \star -. \tag{2.1}$$

Since $1[h]$ is a central element of $(A \otimes T(\Sigma A), \star)$ of odd degree, it follows that

$$b_0(-) \star - = b_0(- \star -) = \pm - \star b_0(-). \tag{2.2}$$

The $\star$ product extends to HH^{II} since it is continuous for the topology on HH whose completion gives HH^{II}.

2.3. Functoriality of HH^{II} using the shuffle product

We recall that a morphism $\mathcal{A} = (A, d_A, h_A) \to \mathcal{B} = (B, d_B, h_B)$ of cdga's is given by a pair $\phi = (\rho, \beta)$, with $\rho : A \to B$ a morphism of Γ-graded k-algebras and $\beta \in B$ a degree 1 element, such that

- $\rho(d(a)) - d'(\rho(a)) = [\beta, \rho(a)]$ for all $a \in A$, and

- $\rho(h) = h' + d'(\beta) + \beta^2$.

Such a morphism is called *strict* if $\beta = 0$.

A strict morphism ϕ induces maps

$$\phi_* : HH(\mathcal{A}) \to HH(\mathcal{B}) \quad \text{and} \quad \phi_* : HH^{II}(\mathcal{A}) \to HH^{II}(\mathcal{B})$$

given by

$$\phi_*\big(a_0[a_1|\cdots|a_n]\big) = \rho(a_0)\big[\rho(a_1)|\cdots|\rho(a_n)\big].$$

A nonstrict morphism ϕ does not, in general, induce a map on Hochschild homology, but it does induce a map

$$\phi_* : HH^{II}(\mathcal{A}) \to HH^{II}(\mathcal{B})$$

given by sending $a_0[a_1|\cdots|a_n]$ to

$$\sum_{i_0,\dots,i_n \ge 0} (-1)^{i_0+\cdots+i_n} \rho(a_0)\big[\underbrace{\beta|\cdots|\beta}_{i_0 \text{ copies}}|\rho(a_1)|\underbrace{\beta|\cdots|\beta}_{i_1 \text{ copies}}|\rho(a_2)|\cdots|\rho(a_n)|\underbrace{\beta|\cdots|\beta}_{i_n \text{ copies}}\big]. \tag{2.3}$$

We next show how ϕ_* may also be defined using the $\star$ product. Suppose $b \in B$ is a degree 1 element, and let $\exp(1[b])$ denote the degree 0, central element of the algebra $(HH^{II}(B)^\natural, \star)$ given by evaluating the power series for the exponential function at $1[b]$:

$$\begin{aligned} \exp\big(1[b]\big) &= 1 + 1[b] + \frac{1}{2!}\big(1[b] \star 1[b]\big) + \frac{1}{3!}\big(1[b] \star 1[b] \star 1[b]\big) + \cdots \\ &= 1 + 1[b] + 1[b|b] + 1[b|b|b] + \cdots. \end{aligned}$$

The signs are correct, since $s(b) \in T(\Sigma B)$ has even degree. We have

$$\begin{aligned} \exp\big(1[b]\big) \star \big(b_0[b_1|\cdots|b_n]\big) &= \big(b_0[b_1|\cdots|b_n]\big) \star \exp\big(1[b]\big) \\ &= \sum_{i_0,\dots,i_n \ge 0} b_0\big[\underbrace{b|\cdots|b}_{i_0 \text{ copies}}|b_1|\underbrace{b|\cdots|b}_{i_1 \text{ copies}}|b_2|\cdots|b_n|\underbrace{b|\cdots|b}_{i_n \text{ copies}}\big]. \end{aligned}$$

By comparing formulas, we see that

$$\phi_* = \exp\big(1[-\beta]\big) \star \rho_*. \tag{2.4}$$

That is,

$$\begin{aligned} \phi_*\big(a_0[a_1|\cdots|a_n]\big) &= \exp\big(1[-\beta]\big) \star \rho(a_0)\big[\rho(a_1)|\cdots|\rho(a_n)\big] \\ &= \rho(a_0)\big[\rho(a_1)|\cdots|\rho(a_n)\big] \star \exp\big(1[-\beta]\big). \end{aligned}$$

2.4. The Künneth map for Hochschild homology of cdg-categories

For a pair of cdg-categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathcal{C} \otimes_k \mathcal{D}$ for the cdg-category whose objects are ordered pairs (C, D) with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ and such that

$$\operatorname{Hom}\big((C,D),(C',D')\big) = \operatorname{Hom}_{\mathcal{C}}(C,C') \otimes_k \operatorname{Hom}_{\mathcal{D}}(D,D'),$$

with differentials given in the standard way for a tensor product. The composition rules are the evident ones, and the curvature elements are defined by

$$h_{(C,D)} = h_C \otimes \mathrm{id}_D + \mathrm{id}_C \otimes h_D.$$

Note that, if $\mathcal{A} = (A, d_A, h_A)$ and $\mathcal{B} = (B, d_B, h_B)$ are cdga's, then this construction specializes to the construction given above as follows:

$$\mathcal{A} \otimes_k \mathcal{B} = \big(A \otimes_k B, d_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes d_B, h_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes h_B\big).$$

We define the *Künneth map* for the cdg-categories $\mathcal{C}$ and $\mathcal{D}$ to be the map

$$-\tilde{\star}- : HH(\mathcal{C})^\natural \otimes_k HH(\mathcal{D})^\natural \to HH(\mathcal{C} \otimes_k \mathcal{D})^\natural$$

given by

$$c_0[c_1|\cdots|c_m]\tilde{\star} d_0[d_1|\cdots|d_n] = \sum_\sigma \pm c_0 \otimes d_0[e_{\sigma(1)}|\cdots|e_{\sigma(m+n)}],$$

where σ ranges over all (m,n)-shuffles, and

$$e_i := \begin{cases} c_i \otimes \mathrm{id}, & \text{if } 1 \le i \le m, \text{ and} \\ \mathrm{id} \otimes d_{i-m}, & \text{if } m+1 \le i \le m+n. \end{cases}$$

This map extends to $HH^{II}(-)^\natural$:

$$-\tilde{\star}- : HH^{II}(\mathcal{C})^\natural \otimes_k HH^{II}(\mathcal{D})^\natural \to HH^{II}(\mathcal{C} \otimes \mathcal{D})^\natural.$$

Remark 2.5. There does not seem to be an analogue of the $\star$ product for a general cdg-category. The issue is that, in general, there is no "diagonal map"

$$\mathcal{C} \otimes_k \mathcal{C} \to \mathcal{C}.$$

Lemma 2.6. *For any two cdg-categories $\mathcal{C}$ and $\mathcal{D}$, the diagram*

$$\begin{array}{ccc} HH(\mathcal{C})^\natural \otimes_k HH(\mathcal{D})^\natural & \xrightarrow{-\tilde{\star}-} & HH(\mathcal{C} \otimes_k \mathcal{D})^\natural \\ \downarrow{\scriptstyle b_i \otimes \mathrm{id} + \mathrm{id} \otimes b_i} & & \downarrow{\scriptstyle b_i} \\ HH^{II}(\mathcal{C})^\natural \otimes_k HH(\mathcal{D})^\natural & \xrightarrow{-\tilde{\star}-} & HH(\mathcal{C} \otimes_k \mathcal{D})^\natural \end{array}$$

commutes for $i = 0, 1,$ and 2, and similarly for $HH^{II}(-)^\natural$. In particular,

$$-\tilde{\star}- : HH(\mathcal{C}) \otimes_k HH(\mathcal{D}) \to HH(\mathcal{C} \otimes_k \mathcal{D})$$

and

$$-\tilde{\star}- : HH^{II}(\mathcal{C}) \otimes_k HH^{II}(\mathcal{D}) \to HH^{II}(\mathcal{C} \otimes_k \mathcal{D})$$

are chain maps.

Proof. This follows from the definitions by a routine check. ■

2.5. Naturality of the Künneth map

We recall that a morphism $\mathcal{A} \to \mathcal{B}$ of cdg-categories is a pair $\phi = (F, \beta)$, where $F : \mathcal{A} \to \mathcal{B}$ is a morphism of categories enriched in Γ-graded k-vector spaces, and β is an assignment to each object X of $\mathcal{A}$ a degree 1 element $\beta_X \in \operatorname{End}_{\mathcal{B}}(F(X))$. The pair (F, β) is required to satisfy that

- for all $X, Y \in \operatorname{Ob}(\mathcal{A})$ and $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$,
$$F\big(\delta(f)\big) = \delta\big(F(f)\big) + \beta_Y \circ F(f) - (-1)^{|f|} F(f) \circ \beta_X,$$
where δ is the differential on $\operatorname{Hom}_{\mathcal{A}}(X, Y)$; and
- for all $X \in \operatorname{Ob}(\mathcal{A})$,
$$F(h_X) = h_{F(X)} + \delta(\beta_X) + \beta_X^2.$$

ϕ is called *strict* if $\beta_X = 0$ for all X.

Lemma 2.7. *Suppose $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ are curved differential Γ-graded categories, and $\phi = (F, \beta) : \mathcal{A} \to \mathcal{B}$, $\phi' = (F', \beta') : \mathcal{A}' \to \mathcal{B}'$ are morphisms of such categories. Then*

(1) *$\phi \otimes \phi' := (F \otimes F', \beta \otimes 1 + 1 \otimes \beta')$ is a morphism from $\mathcal{A} \otimes_k \mathcal{A}'$ to $\mathcal{B} \otimes_k \mathcal{B}'$, and, if ϕ and ϕ' are strict morphisms, then so is $\phi \otimes \phi'$;*

(2) *the diagram*

$$\begin{array}{ccc} HH^{II}(\mathcal{A}) \otimes_k HH^{II}(\mathcal{A}') & \xrightarrow{(\phi)_* \otimes (\phi')_*} & HH^{II}(\mathcal{B}) \otimes_k HH^{II}(\mathcal{B}') \\ \downarrow \tilde{\star} & & \downarrow \tilde{\star} \\ HH^{II}\big(\mathcal{A} \otimes_k \mathcal{A}'\big) & \xrightarrow{(\phi \otimes \phi')_*} & HH^{II}\big(\mathcal{B} \otimes_k \mathcal{B}'\big) \end{array}$$

commutes; and

(3) *if ϕ and ϕ' are strict morphisms, the corresponding diagram involving ordinary Hochschild homology commutes.*

Proof. The proof of (1) is a routine check, and (3) is an immediate consequence of (2). For (2), to simplify the notation, we assume the cdg-categories involved are cdg-algebras; the proof of the general claim is notationally more complicated but essentially the same. Write $\phi = (\rho, \beta)$, $\phi' = (\rho', \beta')$, so that, by (2.4),

$$\phi_* = \exp\big(1[-\beta]\big) \star \rho_* \quad \text{and} \quad \phi'_* = \exp\big(1[-\beta']\big) \star \rho'_*.$$

Let $\iota : HH^{II}(\mathcal{A}) \hookrightarrow HH^{II}(\mathcal{A} \otimes_k \mathcal{A}')$ and $\iota' : HH^{II}(\mathcal{A}') \hookrightarrow HH^{II}(\mathcal{A} \otimes_k \mathcal{A}')$ be the canonical inclusions. We have

$$\begin{aligned} \exp\big(1[-\beta]\big) \tilde{\star} \exp\big(1[-\beta']\big) &= \exp\big(\iota\big(1[-\beta]\big)\big) \star \exp\big(\iota'\big(1[-\beta']\big)\big) \\ &= \exp\big(1[-\beta \otimes 1 - 1 \otimes \beta']\big); \end{aligned}$$

the second equation holds since $\iota(1[-\beta])$ and $\iota'(1[-\beta'])$ commute. Therefore, for elements $\alpha \in HH^{II}(\mathcal{A})$ and $\alpha' \in HH^{II}(\mathcal{A}')$, using also the associativity of $\star$, we get

$$\begin{aligned}(\phi)_*(\alpha)\tilde{\star}(\phi')_*(\alpha') &= \big(\exp(1[-\beta]) \star \rho(\alpha)\big)\tilde{\star}\big(\exp\big(1[-\beta']\big) \star \rho'(\alpha')\big) \\ &= \big(\exp\big(1[-\beta]\big)\tilde{\star}\exp\big(1[-\beta']\big)\big) \star \big(\rho(\alpha)\tilde{\star}\rho'(\alpha')\big) \\ &= \exp\big(1[-\beta \otimes 1 - 1 \otimes \beta']\big) \star (\rho \otimes \rho')(\alpha\tilde{\star}\alpha') \\ &= (\phi \otimes \phi')_*(\alpha\tilde{\star}\alpha').\end{aligned}$$

■

3. Hochschild homology of matrix factorization categories

Let k be a field, and let Q be an essentially smooth k-algebra. Fix $f \in Q$.

3.1. Matrix factorizations

The dg-category $mf(Q, f)$ of *matrix factorizations of f over Q* is defined as follows.

- Objects are pairs (P, δ_P), where P is a finitely generated $\mathbb{Z}/2$-graded projective Q-module, and δ_P is an odd degree endomorphism of P such that $\delta_P^2 = f\,\mathrm{id}_P$.
- $\mathrm{Hom}_{mf(Q,f)}((P, \delta_P), (P', \delta_{P'}))$ is the $\mathbb{Z}/2$-graded complex $\mathrm{Hom}_Q(P, P')$ with differential ∂ given by
$$\partial(\alpha) = \delta_{P'}\alpha - (-1)^{|\alpha|}\alpha\delta_P$$
for α homogeneous. From now on, we will omit the subscript on $\mathrm{Hom}_{mf(Q,f)}(-,-)$.

We emphasize that f is allowed to be 0. The *homotopy category of* $mf(Q, f)$, denoted by $[mf(Q, f)]$, is the Q-linear category with the same objects as $mf(Q, f)$ and morphisms given by $\mathrm{Hom}_{[mf(Q,f)]}(-,-) := H^0\,\mathrm{Hom}(-,-)$.

Let $X, Y \in mf(Q, f)$, and let $\alpha_0, \alpha_1 \in \mathrm{Hom}(X, Y)$ be cocycles. We recall that α_0, α_1 are *homotopic* if there is an odd degree Q-linear map $h : X \to Y$ such that

$$hd_X + d_Y h = \alpha_0 - \alpha_1.$$

This is just the usual notion of a homotopy between morphisms of a $\mathbb{Z}/2$-graded complex, adapted verbatim to the setting of matrix factorizations. An object $X \in mf(Q, f)$ is *contractible* if id_X is null-homotopic. Morphisms in $mf(Q, f)$ that are cocycles are homotopic if and only if they are equal in $[mf(Q, f)]$.

Definition 3.1. Given $X \in mf(Q, f)$, the *support of* X is the set

$$\mathrm{supp}(X) = \big\{\mathfrak{p} \in \mathrm{Spec}(Q) \mid X_{\mathfrak{p}} \text{ is not a contractible object of } mf(Q_{\mathfrak{p}}, f)\big\}.$$

For a closed subset Z of $\mathrm{Spec}(Q)$, let $mf^Z(Q, f)$ denote the full dg-subcategory of $mf(Q, f)$ consisting of those X with $\mathrm{supp}(X) \subseteq Z$.

We record the following.

Proposition 3.2. *Let $X \in mf(Q, f)$.*

(1) *When $f = 0$,* $\operatorname{supp}(X)$ *is the set of points at which the $\mathbb{Z}/2$-complex X is not exact. Therefore, when $f = 0$, the notion of support defined above agrees with the usual notion of support for a $\mathbb{Z}/2$-graded complex.*

(2) *One has* $\operatorname{supp}(X) \subseteq \operatorname{Spec}(Q/f)$. *When f is a nonzero-divisor,* $\operatorname{supp}(X) \subseteq \operatorname{Sing}(Q/f)$.

Proof. (1) This is [1, Lemma 2.3]. (2) It is easy to check that any matrix factorization of a unit is contractible. Suppose f is a nonzero-divisor. By [8, Theorem 3.9], the homotopy category $[mf(Q, f)]$ is equivalent to the singularity category of Q/f, and the singularity category is trivial when Q/f is regular. ■

Remark 3.3. If f is a nonzero-divisor, so that the morphism of schemes $f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k$ is flat, then

$$\operatorname{Spec}(Q/f) \cap \operatorname{Sing}(f) = \operatorname{Sing}(Q/f),$$

where $\operatorname{Sing}(f)$ denotes the set of points of $\operatorname{Spec}(Q)$ at which the morphism $f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k$ is not smooth.

Let R be another essentially smooth k-algebra, and let $g \in R$. Given $X \in mf(Q, f)$ and $Y \in mf(R, g)$, we form the tensor product

$$X \otimes Y \in mf(Q \otimes_k R, f \otimes 1 + 1 \otimes g)$$

by adapting the notion of tensor product of $\mathbb{Z}/2$-graded complexes to matrix factorizations. The tensor product gives a dg-functor

$$mf(Q, f) \otimes_k mf(R, g) \to mf(Q \otimes_k R, f \otimes 1 + 1 \otimes g).$$

If Z and W are closed subsets of $\operatorname{Spec}(Q)$ and $\operatorname{Spec}(R)$, respectively, one has an induced functor

$$mf^Z(Q, f) \otimes_k mf^W(R, g) \to mf^{Z \times W}(Q \otimes_k R, f \otimes 1 + 1 \otimes g).$$

If $Q = R$, composing with multiplication in Q gives a functor

$$mf^Z(Q, f) \otimes_k mf^W(Q, g) \to mf^{Z \cap W}(Q, f + g).$$

We also have a duality functor D which determines an isomorphism of dg-categories

$$D : mf(Q, f)^{\mathrm{op}} \xrightarrow{\cong} mf(Q, -f).$$

The functor D sends an object $P = (P, \delta_P)$ of $mf(Q, f)$ to the object $P^* = (P^*, -\delta_P^*)$ of $mf(Q, -f)$, and it sends an element α of $\operatorname{Hom}(P_2, P_1)^{\mathrm{op}} = \operatorname{Hom}(P_1, P_2)$ to the element α^* of $\operatorname{Hom}(P_2^*, P_1^*)$. Note that $\alpha^*(\gamma) = (-1)^{|\alpha||\gamma|}\gamma \circ \alpha$. If $X \in mf^Z(X, f)^{\mathrm{op}}$ for some closed $Z \subseteq \mathrm{Spec}(Q)$, then $D(X) \in mf^Z(X, -f)$. If $X, Y \in mf(Q, f)$, there is a canonical isomorphism

$$\mathrm{Hom}(X, Y) \cong D(X) \otimes Y.$$

In particular, if $X \in mf^Z(Q, f)$ and $Y \in mf^W(Q, f)$, we have

$$\mathrm{Hom}(X, Y) \in mf^{Z \cap W}(Q, 0). \tag{3.4}$$

3.2. The HKR map

Assume for the rest of Section 3 that $\mathrm{char}(k) = 0$. Given a $\mathbb{Z}$-graded complex $(C^\bullet, d)$ of k-vector spaces, its $\mathbb{Z}/2$-*folding* is the $\mathbb{Z}/2$-graded complex whose even (resp., odd) component is $\bigoplus_{i \in \mathbb{Z}} C^{2i}$ (resp., $\bigoplus_{i \in \mathbb{Z}} C^{2i+1}$) and whose differential is given by d.

Let $\Omega^\bullet_{Q/k}$ denote the $\mathbb{Z}/2$-graded commutative Q-algebra given by the $\mathbb{Z}/2$-folding of the exterior algebra over $\Omega^1_{Q/k}$. That is,

$$\Omega^{\mathrm{even}}_{Q/k} = \bigoplus_j \Omega^{2j}_{Q/k} \quad \text{and} \quad \Omega^{\mathrm{odd}}_{Q/k} = \bigoplus_j \Omega^{2j+1}_{Q/k}.$$

We write $(\Omega^\bullet_{Q/k}, -df)$ for the $\mathbb{Z}/2$-graded complex of Q-modules with underlying graded Q-module $\Omega^\bullet_{Q/k}$ and with differential given by left multiplication by $-df \in \Omega^1_{Q/k}$.

Let Z be a closed subset of $\mathrm{Spec}(Q/f)$. The goal of the rest of this section is to study, for each triple (Q, f, Z), an HKR-type map,

$$\varepsilon_{Q,f,Z} : HH\big(mf^Z(Q, f)\big) \to \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big). \tag{3.5}$$

Here, $\mathbb{R}\Gamma_Z$ is the right adjoint of the inclusion functor $D^Z_{\mathbb{Z}/2}(Q) \subseteq D_{\mathbb{Z}/2}(Q)$, where $D_{\mathbb{Z}/2}(Q)$ denotes the derived category of $\mathbb{Z}/2$-graded Q-modules, and $D^Z_{\mathbb{Z}/2}(Q) \subseteq D_{\mathbb{Z}/2}(Q)$ the subcategory spanned by complexes with support contained in Z. It will be convenient for us to use the following Čech model for $\mathbb{R}\Gamma_Z$. Choose $g_1, \dots, g_m \in Q$ such that $Z = V(g_1, \dots, g_m)$, and let

$$\mathcal{C} = \mathcal{C}(g_1, \dots, g_m) = \bigotimes_j \big(Q \to Q[1/g_i]\big)$$

be the ($\mathbb{Z}/2$-folding of the) augmented Čech complex. It is well known that $\mathcal{C} \otimes_Q M$ models $\mathbb{R}\Gamma_Z(M)$ for any $M \in D_{\mathbb{Z}/2}(Q)$; i.e., the functor

$$\mathcal{C} \otimes_Q - : D_{\mathbb{Z}/2}(Q) \to D^Z_{\mathbb{Z}/2}(Q)$$

is right adjoint to the inclusion. From now on, given $g_1, \dots, g_m \in Q$ such that $V(g_1, \dots, g_m) = Z$, we will tacitly identify $\mathbb{R}\Gamma_Z(M)$ with $\mathcal{C} \otimes_Q M$. Note that, for any $\mathbb{Z}/2$-graded complex M of Q-modules that is supported in Z, the natural morphism of complexes

$$\mathcal{C} \otimes_Q M \to M \tag{3.6}$$

given by the tensor product of the augmentation map $\mathcal{C} \to Q$ with id_M is a quasi-isomorphism.

HKR maps for matrix factorization categories have been widely studied. Segal and Căldăraru–Tu give such an HKR map, involving Hochschild homology of the second kind and without a support condition, in [14, Corollary 3.4] and [4, Theorem 4.2], respectively; Efimov generalizes this result to the nonaffine setting in [6, Proposition 3.21]; and Preygel gives a map just as in (3.5) (but also in the not-necessarily-affine setting), and proves it is a quasi-isomorphism, in [11, Theorem 8.2.6 (iv)]. But [11] does not contain a concrete formula for where the HKR map (3.5) sends an element of the bar complex computing $HH(mf^Z(Q, f))$, and we will need such a formula later on. So, we develop our own version of (3.5).

3.2.1. Quasi-matrix factorizations. Define a curved dg-category $qmf(Q, f)$, the category of *quasi-matrix factorizations*, in the following way.

- Objects (P, δ_P) are defined in the same way as those of $mf(Q, f)$, except we remove the requirement that δ_P^2 is given by multiplication by f.
- Morphisms are defined in the same way as in $mf(Q, f)$.
- The curvature element of $\mathrm{End}_{qmf(Q,f)}(P, \delta_P)$ is $\delta_P^2 - f$.

$mf(Q, f)$ is precisely the full subcategory of $qmf(Q, f)$ spanned by objects with trivial curvature. Let $qmf(Q, f)^0$ denote the full subcategory of $qmf(Q, f)$ spanned by those objects (P, δ_P) such that $\delta_P = 0$. Note that the curvature element of an object in $qmf(Q, f)^0$ is $-f$. The pair $(Q, 0)$ determines an object of $qmf(Q, f)^0$, and its endomorphisms form the curved differential $\mathbb{Z}/2$-graded algebra $(Q, 0, -f)$. That is, we have inclusions

$$mf(Q, f) \hookrightarrow qmf(Q, f) \hookleftarrow qmf(Q, f)^0 \hookleftarrow (Q, 0, -f).$$

These functors are all *pseudo-equivalences*, in the language of [9, Section 1.5], and so, by [9, Lemma A, p. 5319], the induced maps

$$\begin{aligned} &HH^{II}\big(mf(Q, f)\big) \\ &\quad \to HH^{II}\big(qmf(Q, f)\big) \longleftarrow HH^{II}\big(qmf(Q, f)^0\big) \longleftarrow HH^{II}(Q, 0, -f) \end{aligned}$$

are all quasi-isomorphisms.

A key point is that there is a (nonstrict) cdg-functor

$$(F, \beta) : qmf(Q, f) \to qmf(Q, f)^0$$

given by $F(P, \delta_P) = (P, 0)$ and $\beta_{(P,\delta_P)} = \delta_P$. The induced map

$$(F, \beta)_* : HH^{II}\big(qmf(Q, f)\big) \to HH^{II}\big(qmf(Q, f)^0\big)$$

sends $\alpha_0[\alpha_1|\cdots|\alpha_n]$, where $\alpha_i \in \mathrm{Hom}((P_{i+1}, \delta_{i+1}), (P_i, \delta_i))$, to

$$\sum_{i_0,\ldots,i_n \geq 0} (-1)^{i_0+\cdots+i_n} \alpha_0\big[\overbrace{\delta_1|\cdots|\delta_1}^{i_0}|\alpha_1|\overbrace{\delta_2|\cdots|\delta_2}^{i_1}|\cdots|\alpha_n|\overbrace{\delta_0|\cdots|\delta_0}^{i_n}\big].$$

3.2.2. The supertrace. Given a $\mathbb{Z}/2$-graded finitely generated projective Q-module P, define the *supertrace* map

$$\mathrm{str} : \mathrm{End}_Q(P) \to Q$$

as the composition

$$\mathrm{End}_Q(P) \cong P^* \otimes_Q P \xrightarrow{\gamma \otimes p \mapsto \gamma(p)} Q$$

for homogeneous elements γ, p. Equivalently, for $\alpha \in \mathrm{End}_Q(P)$ we have

$$\mathrm{str}(\alpha) = \begin{cases} \mathrm{tr}(\alpha_0 : P_0 \to P_0) - \mathrm{tr}(\alpha_1 : P_1 \to P_1), & \text{if } \alpha \text{ has degree 0, and} \\ 0, & \text{if } \alpha \text{ has degree 1.} \end{cases}$$

Here, tr is the classical trace of an endomorphism of a projective module. We extend str to a map

$$\mathrm{End}_{\Omega^\bullet_{Q/k}}\left(P \otimes_Q \Omega^\bullet_{Q/k}\right) \cong \mathrm{End}_Q(P) \otimes_Q \Omega^\bullet_{Q/k} \xrightarrow{\mathrm{str} \otimes \mathrm{id}} \Omega^\bullet_{Q/k},$$

which we also write as str.

3.2.3. The HKR map without supports.

Definition 3.7. A *connection* on an object $(P, \delta_P) \in qmf(Q, f)$ is a k-linear map

$$\nabla : P \to \Omega^1_{Q/k} \otimes_Q P$$

of odd degree such that $\nabla(qp) = dq \otimes p + q\nabla(p)$, i.e., a *superconnection*, in the language of [12]. Notice that the definition does not involve δ_P.

Choose a connection ∇_P on each object $(P, 0) \in qmf(Q, f)^0$; we stipulate that the connection chosen for $Q \in qmf(Q, f)^0$ is the canonical one given by the de Rham differential, $d : Q \to \Omega^1_{Q/k}$. Define

$$\varepsilon^0 : HH^{II}\big(qmf(Q, f)^0\big)^\natural \to \Omega^\bullet_{Q/k}$$

by

$$\varepsilon^0\big(\alpha_0[\alpha_1|\cdots|\alpha_m]\big) = \frac{1}{m!}\,\mathrm{str}\left(\alpha_0\alpha_1'\cdots\alpha_m'\right),$$

where, for $\alpha : (P_1, 0) \to (P_2, 0)$, we set $\alpha' = \nabla_{P_2} \circ \alpha - (-1)^{|\alpha|}\alpha \circ \nabla_{P_1}$. By [2, Theorem 5.18], ε^0 gives a chain map

$$HH^{II}\big(qmf(Q, f)^0\big) \to \big(\Omega^\bullet_{Q/k}, -df\big).$$

Then the composition

$$\varepsilon^Q : HH^{II}(Q, 0, -f) \xrightarrow{\simeq} HH^{II}\big(qmf(Q, f)^0\big) \xrightarrow{\varepsilon^0} \big(\Omega^\bullet_{Q/k}, -df\big),$$

where the first map is induced by inclusion, is given by the classical HKR map

$$\varepsilon^Q\big(q_0[q_1|\cdots|q_n]\big) = \frac{q_0 dq_1 \cdots dq_n}{n!} \in \Omega^n_{Q/k}.$$

In particular, ε^0 is a quasi-isomorphism. $(F,\beta)_*$ is also a quasi-isomorphism, since

$$qmf(Q,f)^0 \xrightarrow{\simeq} qmf(Q,f) \xrightarrow{(F,\beta)} qmf(Q,f)^0$$

is the identity.

We define the HKR map

$$\varepsilon_{Q,f} : HH\big(mf(Q,f)\big) \to \big(\Omega^{\bullet}_{Q/k}, -df\big)$$

to be the composition

$$\begin{aligned} HH\big(mf(Q,f)\big) &\xrightarrow{\text{can}} HH^{II}\big(mf(Q,f)\big) \xrightarrow{\simeq} HH^{II}\big(qmf(Q,f)\big) \\ &\xrightarrow{(F,\beta)_*} HH^{II}\big(qmf(Q,f)^0\big) \xrightarrow{\varepsilon^0} \big(\Omega^{\bullet}_{Q/k}, -df\big), \end{aligned}$$

where "can" denotes the canonical map. A more explicit formula for $\varepsilon_{Q,f}$ is given as follows. Given objects $(P_0,\delta_0),\ldots,(P_n,\delta_n)$ of $mf(Q,f)$ and maps

$$P_0 \xleftarrow{\alpha_0} P_1 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_{n-1}} P_n \xleftarrow{\alpha_n} P_0,$$

set $\nabla_i = \nabla_{P_i}$. Then

$$\begin{aligned} &\varepsilon_{Q,f}\big(\alpha_0[\alpha_1|\cdots|\alpha_n]\big) \\ &\quad = \sum_{i_0,\ldots,i_n \geq 0} \frac{(-1)^{i_0+\cdots+i_n}}{(n+i_0+\cdots+i_n)!} \operatorname{str}\big(\alpha_0(\delta_1')^{i_0}\alpha_1' \cdots (\delta_n')^{i_{n-1}}\alpha_n'(\delta_0')^{i_n}\big), \end{aligned}$$

where, just as above,

$$\alpha_j' = \nabla_j \circ \alpha_j - (-1)^{|\alpha_j|}\alpha_j \circ \nabla_{j+1} \quad (\text{with } \nabla_{n+1} = \nabla_0),$$

and

$$\delta_j' = [\nabla_j, \delta_i] = \nabla_j \circ \delta_j + \delta_j \circ \nabla_j.$$

Note that the sum in this formula is finite, since $\Omega^j_{Q/k} = 0$ for $j > \dim(Q)$.

Summarizing, we have a commutative diagram

$$\begin{array}{ccccc} HH(mf(Q,f)) & \longrightarrow HH^{II}(mf(Q,f)) \xrightarrow{\simeq} & HH^{II}(qmf(Q,f)) & & \\ & & \simeq \downarrow (F,\beta)_* & & \\ & \searrow^{\varepsilon_{Q,f}} & HH^{II}(qmf(Q,f)^0) & \xleftarrow{\simeq} & HH^{II}(Q,0,-f) \\ & & \simeq \downarrow \varepsilon^0 & \swarrow_{\varepsilon^Q}^{\simeq} & \\ & & (\Omega_{Q/k}, -df). & & \end{array} \tag{3.8}$$

Notice that this implies that $\varepsilon_{Q,f}$ is independent, up to natural isomorphism in the derived category, of the choices of connections. In particular, the map on homology induced by $\varepsilon_{Q,f}$ is independent of such choices.

We include the following result, although it will not be needed in this paper.

Proposition 3.9. *If the only critical value of $f : \operatorname{Spec}(Q) \to \mathbb{A}^1$ is 0, $\varepsilon_{Q,f}$ is a quasi-isomorphism.*

Proof. By [9, Section 4.8, Corollary A], the canonical map

$$HH\big(mf(Q,f)\big) \to HH^{II}\big(mf(Q,f)\big)$$

is a quasi-isomorphism. The statement therefore follows from the commutativity of diagram (3.8). ∎

3.2.4. The HKR map with supports. We now define the HKR map for a general closed subset Z of $\operatorname{Spec}(Q)$. Composing $\varepsilon_{Q,f}$ with the natural map induced by the inclusion $mf^Z(Q,f) \subseteq mf(Q,f)$ gives a map

$$HH\big(mf^Z(Q,f)\big) \to \big(\Omega^\bullet_{Q/k}, -df\big). \tag{3.10}$$

By Proposition 3.2 (1) and (3.4), if $X, Y \in mf^Z(Q,f)$, $\operatorname{Hom}(X,Y)$ is a complex of Q-modules whose support is contained in Z. (When f is a nonzero-divisor, this complex is in fact supported on $Z \cap \operatorname{Sing}(Q/f)$.) It follows that each row of the bicomplex used to define $HH(mf^Z(Q,f))$ is supported on Z. Since $HH(mf^Z(Q,f))$ is the direct sum totalization of this bicomplex, we have that $HH(mf^Z(Q,f))$ is supported on Z. Adjointness thus gives a canonical isomorphism

$$\varepsilon_{Q,f,Z} : HH\big(mf^Z(Q,f)\big) \to \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big)$$

in $D(Q)$. In other words, $\varepsilon_{Q,f,Z}$ is represented in $D(Q)$ by the diagram

$$HH\big(mf^Z(Q,f)\big) \xleftarrow[\simeq]{(3.6)} \mathbb{R}\Gamma_Z HH\big(mf^Z(Q,f)\big) \xrightarrow{(3.10)} \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big).$$

We will sometimes refer to $\varepsilon_{Q,f,Z}$ as just ε, if no confusion can arise.

3.3. Relationship between the HKR map and the map $I_f(0)$

When $Q = \mathbb{C}[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$ is the only singular point of the map $f : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$, Shklyarov defines in [16, Section 4.1] an isomorphism

$$I_f(0) : HH_*\big(mf(Q,f)\big) \xrightarrow{\cong} H_*\big(\Omega^\bullet_{Q/k}, -df\big)$$

as follows. Let $\mathcal{A}_f$ be the endomorphism dga of the following matrix factorization (P, δ_P) which represents the residue field $Q/\mathfrak{m}$ in the singularity category of Q/f: choose polynomials $y_1, \dots, y_n \in Q$ so that $f = \sum_i x_i y_i$, let P be the $\mathbb{Z}/2$-graded exterior algebra over Q on generators $e_1, \dots, e_n$, and define a differential on P given by

$$\delta_P = \sum_i x_i e_i^* + y_i e_i.$$

Here, e_i^* is the Q-linear derivation of P determined by $e_i^*(e_j) = \delta_{ij}$. By a theorem of Dyckerhoff [5, Theorem 5.2(3)], the inclusion

$$\iota : \mathcal{A}_f \hookrightarrow mf(Q, f)$$

is a Morita equivalence. Since Hochschild homology is Morita invariant, the induced map

$$\iota_* : HH_*(\mathcal{A}_f) \xrightarrow{\cong} HH_*\big(mf(Q, f)\big)$$

is an isomorphism.

From now on, we identify P with $Q \otimes_{\mathbb{C}} \Lambda$, where $\Lambda = \Lambda_{\mathbb{C}}(e_1, \dots, e_n)$, and $\mathcal{A}_f$ with $Q \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\Lambda)$. Shklyarov defines a quasi-isomorphism

$$\alpha : HH(\mathcal{A}_f) \xrightarrow{\simeq} (\Omega^\bullet_{Q/k}, -df)$$

as the composition

$$HH(\mathcal{A}_f) \xrightarrow{\exp(-1[\delta_P])} HH^{II}(\mathcal{A}_f) \xrightarrow{\varepsilon'} (\Omega^\bullet_{Q/k}, -df),$$

where

$$\varepsilon'\big(q_0 \otimes \alpha_0[q_1 \otimes \alpha_1|\cdots|q_n \otimes \alpha_n]\big) = \frac{(-1)^{\sum_{i \text{ odd}} |\alpha_i|}}{n!} \mathrm{str}(\alpha_0 \cdots \alpha_n) q_0 dq_1 \cdots dq_n.$$

Finally, $I_f(0)$ is the composition

$$HH_*\big(mf(Q, f)\big) \xrightarrow{\iota_*^{-1}} HH_*(\mathcal{A}_f) \xrightarrow{\alpha} H_*\big(\Omega^\bullet_{Q/k}, -df\big).$$

Lemma 3.11. *The map ε' coincides with the map $\varepsilon_{Q,f}$ restricted to $HH(\mathrm{End}(P))$ for the choice of connection ∇_P defined as $\nabla_P(q \otimes \alpha) = dq \otimes \alpha$. Thus, $I_f(0) = \varepsilon_{Q,f}$.*

Proof. We have

$$\begin{aligned} &\varepsilon_{Q,f}\big(q_0 \otimes \alpha_0[q_1 \otimes \alpha_1|\cdots|q_n \otimes \alpha_n]\big) \\ &\quad = \frac{1}{n!} \mathrm{str}\big((q_0 \otimes \alpha_0)(dq_1 \otimes \alpha_1)\cdots(dq_n \otimes \alpha_n)\big) \\ &\quad = \frac{(-1)^{\sum_i i|\alpha_i|}}{n!} \mathrm{str}(\alpha_0 \cdots \alpha_n) q_0 dq_1 \cdots dq_n \\ &\quad = \frac{(-1)^{\sum_{i \text{ odd}} |\alpha_i|}}{n!} \mathrm{str}(\alpha_0 \cdots \alpha_n) q_0 dq_1 \cdots dq_n. \end{aligned}$$

■

3.4. Compatibility of the HKR map with taking duals

Shklyarov proves in [15, Proposition 3.2] that, for any differential $\mathbb{Z}/2$-graded algebra $\mathcal{A}$, there is a canonical isomorphism of complexes

$$\Phi : HH(\mathcal{A}) \xrightarrow{\cong} HH(\mathcal{A}^{\mathrm{op}}) \tag{3.12}$$

given by

$$a_0[a_1|\cdots|a_n] \mapsto (-1)^{n+\sum_{1\le i<j\le n}(|a_i|-1)(|a_j|-1)} a_0^{\mathrm{op}}[a_n^{\mathrm{op}}|\cdots|a_1^{\mathrm{op}}],$$

where, for $a \in \mathcal{A}$, a^{op} denotes a regarded as an element of $\mathcal{A}^{\mathrm{op}}$. The same formula gives an isomorphism

$$HH(\mathcal{C}) \xrightarrow{\cong} HH(\mathcal{C}^{\mathrm{op}})$$

for any curved differential Γ-graded category $\mathcal{C}$, where $\Gamma \in \{\mathbb{Z}, \mathbb{Z}/2\}$.

Composing Φ and D, where D is the dualization functor defined in Section 3.1, we obtain the isomorphism of complexes

$$\Psi : HH\big(mf^Z(Q,f)\big) \xrightarrow{\cong} HH\big(mf^Z(Q,-f)\big) \tag{3.13}$$

given explicitly by

$$\Psi\big(a_0[a_1|\cdots|a_n]\big) = (-1)^{n+\sum_{1\le i<j\le n}(|a_i|-1)(|a_j|-1)} a_0^*[a_n^*|\cdots|a_1^*].$$

Lemma 3.14. *The diagram*

$$\begin{array}{ccc} HH\big(mf^Z(Q,f)\big) & \xrightarrow{\varepsilon_{Q,f,Z}} & \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\gamma} \\ HH\big(mf^Z(Q,-f)\big) & \xrightarrow{\varepsilon_{Q,-f,Z}} & \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, df\big) \end{array}$$

commutes in $D(Q)$, where γ is $\mathbb{R}\Gamma_Z$ applied to the map whose restriction to $\Omega^j_{Q/k}$ is multiplication by $(-1)^j$ for all j.

Proof. The map $\varepsilon_{Q,f,Z}$ factors as

$$HH\big(mf^Z(Q,f)\big) \to \mathbb{R}\Gamma_Z HH\big(mf(Q,f)\big) \xrightarrow{\varepsilon_{Q,f}} \big(\Omega^\bullet_{Q/k}, -df\big),$$

where the first map is the canonical one. $\varepsilon_{Q,-f,Z}$ factors similarly. Since the diagram

$$\begin{array}{ccc} HH\big(mf^Z(Q,f)\big) & \longrightarrow & \mathbb{R}\Gamma_Z HH\big(mf(Q,f)\big) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\mathbb{R}\Gamma_Z(\Psi)} \\ HH\big(mf^Z(Q,-f)\big) & \longrightarrow & \mathbb{R}\Gamma_Z HH\big(mf(Q,-f)\big) \end{array}$$

evidently commutes, we may assume $Z = \mathrm{Spec}(Q)$.

Recall from (3.8) that $\varepsilon_{Q,f}$ fits into a commutative diagram

$$\begin{array}{ccccc} HH\big(mf(Q,f)\big) & \xrightarrow{\theta} & HH^{II}\big(qmf(Q,f)^0\big) & \xleftarrow[\simeq]{\mathrm{can}} & HH^{II}(Q,0,-f) \\ & {\scriptstyle\varepsilon_{Q,f}}\searrow & \simeq\downarrow{\scriptstyle\varepsilon^0} & \swarrow{\scriptstyle\varepsilon^Q}\,\simeq & \\ & & \big(\Omega^\bullet_{Q/k}, -df\big), & & \end{array} \tag{3.15}$$

where

$$\theta\big(\alpha_0[\alpha_1|\cdots|\alpha_n]\big) = \sum_{i_0,\dots,i_n\geq 0} (-1)^{i_0+\cdots+i_n}\alpha_0\big[\delta_1^{i_0}|\alpha_1|\delta_2^{i_1}|\cdots|\alpha_n|\delta_0^{i_n}\big].$$

Here, δ^i stands for $\overbrace{\delta|\cdots|\delta}^{i}$.

The map Ψ extends to a map

$$\Psi : HH^{II}\big(qmf(Q,f)^0\big) \to HH^{II}\big(qmf(Q,-f)^0\big)$$

using the same formula, and this map in turn restricts to a map

$$\Psi : HH^{II}(Q,0,-f) \to HH^{II}(Q,0,f)$$

given by

$$\Psi\big(q_0[q_1|\cdots|q_n]\big) = (-1)^{n+\binom{n}{2}} q_0[q_n|\cdots|q_1].$$

We claim that the diagram

$$\begin{array}{ccccc} HH\big(mf(Q,f)\big) & \xrightarrow{\theta} & HH^{II}\big(qmf(Q,f)^0\big) & \xleftarrow[\simeq]{\text{can}} & HH^{II}(Q,0,-f) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} \\ HH\big(mf(Q,-f)\big) & \xrightarrow{\theta} & HH^{II}\big(qmf(Q,-f)^0\big) & \xleftarrow[\simeq]{\text{can}} & HH^{II}(Q,0,f) \end{array} \tag{3.16}$$

commutes. This is evident for the right square. As for the left, the element $\alpha_0[\alpha_1|\cdots|\alpha_n]$ is mapped via $\Psi\circ\theta$ to

$$\sum_{i_0,\dots,i_n\geq 0} (-1)^I(-1)^{n+I+\sum_{1\leq i<j\leq n}(|\alpha_i|-1)(|\alpha_j|-1)}\alpha_0^*\big[\big(\delta_0^*\big)^{i_n}|\alpha_n^*|\cdots|\big(\delta_2^*\big)^{i_1}|\alpha_1^*|\big(\delta_1^*\big)^{i_0}\big],$$

where $I = i_0+\cdots+i_n$. The sign is correct since $|\delta_i|-1$ is even for all i. The map $\theta\circ\Psi$ sends $\alpha_0[\alpha_1|\cdots|\alpha_n]$ to

$$\sum_{j_0,\dots,j_n\geq 0} (-1)^J(-1)^{n+\sum_{1\leq i<j\leq n}(|\alpha_i|-1)(|\alpha_j|-1)}\alpha_0^*\big[\big(-\delta_0^*\big)^{j_0}|\alpha_n^*|\cdots|\big(-\delta_2^*\big)^{j_{n-1}}|\alpha_1^*|\big(-\delta_1^{j_n}\big)^*\big],$$

where $J = j_0+\cdots+j_n$. The reason for the minus sign in $(-\delta_j^*)^i$ is that the differential of $(P,d)^*$ is $-d^*$. Since these two expressions are equal, the left square commutes.

Using the commutativity of the diagrams (3.15) and (3.16), it suffices to prove that the square

$$\begin{array}{ccc} HH^{II}(Q,0,-f) & \xrightarrow{\Psi} & HH^{II}(Q,0,f) \\ \downarrow{\scriptstyle\varepsilon^Q} & & \downarrow{\scriptstyle\varepsilon^Q} \\ (\Omega^\bullet_{Q/k},-df) & \xrightarrow{\gamma} & (\Omega^\bullet_{Q/k},df) \end{array}$$

commutes. This holds since $\Omega^\bullet_{Q/k}$ is graded commutative, so that

$$\begin{aligned}\gamma\varepsilon^Q\big(q_0[q_1|\cdots|q_n]\big) &= \frac{(-1)^n}{n!} q_0 dq_1 \cdots dq_n \\ &= \frac{(-1)^{n+\binom{n}{2}}}{n!} q_0 dq_n \cdots dq_1 \\ &= \varepsilon^Q \Psi\big(q_0[q_1|\cdots|q_n]\big). \end{aligned}$$ ■

3.5. Multiplicativity of the HKR map

Let (Q, f, Z) and (R, g, W) be triples consisting of an essentially smooth k-algebra, an element of the algebra, and a closed subset of the spectrum of the algebra. The tensor product of matrix factorizations (Section 3.1), along with the Künneth map for Hochschild homology of dg-categories (Section 2.4), gives a pairing

$$\begin{aligned} - \tilde{\star} - : HH\big(mf^Z(Q, f)\big) \otimes_k HH\big(mf^W(R, g)\big) \\ \to HH\big(mf^{Z\times W}(Q \otimes_k R, f \otimes 1 + 1 \otimes g)\big). \end{aligned} \tag{3.17}$$

Write $f + g$ for the element $f \otimes 1 + 1 \otimes g \in Q \otimes_k R$. Multiplication in $\Omega^\bullet_{Q\otimes_k R/k}$ defines a pairing of complexes of $Q \otimes_k R$-modules

$$- \wedge - : \big(\Omega^\bullet_{Q/k}, -df\big) \otimes_k \big(\Omega^\bullet_{R/k}, -dg\big) \to \big(\Omega^\bullet_{Q\otimes_k R/k}, -df - dg\big).$$

We compose this with the canonical maps $\mathbb{R}\Gamma_Z(\Omega^\bullet_{Q/k}, -df) \to (\Omega^\bullet_{Q/k}, -df)$ and $\mathbb{R}\Gamma_W(\Omega^\bullet_{R/k}, -dg) \to (\Omega^\bullet_{R/k}, -dg)$ to obtain the map

$$\mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big) \otimes_k \mathbb{R}\Gamma_W\big(\Omega^\bullet_{R/k}, -dg\big) \to \big(\Omega^\bullet_{Q\otimes_k R/k}, -df - dg\big).$$

The source of this map is supported on the closed subset $Z \times W$ of $\mathrm{Spec}(Q \otimes_k R) = \mathrm{Spec}(Q) \times_k \mathrm{Spec}(R)$. Thus, by adjointness, we obtain a pairing

$$\begin{aligned} - \wedge - : \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big) \otimes_Q \mathbb{R}\Gamma_W\big(\Omega^\bullet_{R/k}, -dg\big) \\ \to \mathbb{R}\Gamma_{Z\times W}\big(\Omega^\bullet_{Q\otimes_k R/k}, -df - dg\big). \end{aligned} \tag{3.18}$$

A key fact is that the pairings (3.17) and (3.18) are compatible via the HKR maps.

Proposition 3.19. *The diagram*

$$\begin{array}{ccc} HH\big(mf^Z(Q, f)\big) \otimes_k HH\big(mf^W(R, g)\big) & \xrightarrow{\varepsilon_{Q,f,Z} \otimes \varepsilon_{R,f,W}} & \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big) \otimes_k \mathbb{R}\Gamma_W\big(\Omega^\bullet_{R/k}, -dg\big) \\ \downarrow {\scriptstyle -\tilde{\star}-} & & \downarrow {\scriptstyle \wedge} \\ HH\big(mf^{Z\times W}(Q \otimes_k R, f + g)\big) & \xrightarrow{\varepsilon_{Q\otimes_k R, f+g, Z\times W}} & \mathbb{R}\Gamma_{Z\times W}\big(\Omega^\bullet_{Q\otimes_k R/k}, -df - dg\big) \end{array}$$

in $D(Q \otimes_k R)$ commutes.

Proof. It is enough to show the diagrams

$$\begin{CD}
HH\big(mf^{Z}(Q,f)\big)\otimes_k HH\big(mf^{W}(R,g)\big) @>>> \mathbb{R}\Gamma_Z HH\big(mf(Q,f)\big)\otimes_k \mathbb{R}\Gamma_W HH\big(mf(R,g)\big)\\
@V{-\tilde{\star}-}VV @VV{-\tilde{\star}-}V\\
HH\big(mf^{Z\times W}(Q\otimes_k R, f+g)\big) @>>> \mathbb{R}\Gamma_{Z\times W} HH\big(mf(Q\otimes_k R, f+g)\big)
\end{CD} \tag{3.20}$$

and

$$\begin{CD}
\mathbb{R}\Gamma_Z HH\big(mf(Q,f)\big)\otimes_k \mathbb{R}\Gamma_W HH\big(mf(R,g)\big) @>>> \mathbb{R}\Gamma_Z\big(\Omega^\bullet_{Q/k}, -df\big)\otimes_k \mathbb{R}\Gamma_W\big(\Omega^\bullet_{R/k}, -dg\big)\\
@V{-\tilde{\star}-}VV @VV{\wedge}V\\
\mathbb{R}\Gamma_{Z\times W} HH\big(mf(Q\otimes_k R, f+g)\big) @>>> \mathbb{R}\Gamma_{Z\times W}\big(\Omega^\bullet_{Q\otimes_k R/k}, -df-dg\big)
\end{CD} \tag{3.21}$$

commute. Here, the right-most vertical map in (3.20) (which coincides with the left-most vertical map in (3.21)) is defined in a manner similar to the map (3.18), and the horizontal maps in (3.20) are the canonical ones. The commutativity of (3.20) is clear. As for (3.21), it suffices to show the diagram

$$\begin{CD}
HH\big(mf(Q,f)\big)\otimes_k HH\big(mf(R,g)\big) @>{\varepsilon_{Q,f}\otimes\varepsilon_{R,g}}>> \big(\Omega^\bullet_{Q/k}, -df\big)\otimes_k \big(\Omega^\bullet_{R/k}, -dg\big)\\
@V{-\tilde{\star}-}VV @VV{\wedge}V\\
HH\big(mf(Q\otimes_k R, f+g)\big) @>{\varepsilon_{Q\otimes_k R, f+g}}>> \big(\Omega^\bullet_{Q\otimes_k R/k}, -df-dg\big)
\end{CD}$$

in $D(Q\otimes_k R)$ commutes. Factoring the HKR maps as in diagram (3.8), it suffices to show the squares

$$\begin{CD}
HH\big(mf(Q,f)\big)\otimes_k HH\big(mf(R,g)\big) @>>> HH^{II}\big(qmf^0(Q,f)\big)\otimes_k HH^{II}\big(qmf^0(R,g)\big)\\
@V{-\tilde{\star}-}VV @VV{-\tilde{\star}-}V\\
HH\big(mf(Q\otimes_k R, f+g)\big) @>>> HH^{II}\big(qmf^0(Q\otimes_k R, f+g)\big)
\end{CD} \tag{3.22}$$

and

$$\begin{CD}
HH^{II}\big(qmf(Q,f)\big)\otimes_k HH^{II}\big(qmf(R,g)\big) @>{\varepsilon^0\otimes\varepsilon^0}>> \big(\Omega^\bullet_{Q/k}, -df\big)\otimes_k \big(\Omega^\bullet_{R/k}, -dg\big)\\
@V{-\tilde{\star}-}VV @VV{\wedge}V\\
HH^{II}\big(qmf(Q\otimes_k R, f+g)\big) @>{\varepsilon^0}>> \big(\Omega^\bullet_{Q\otimes_k R/k}, -df-dg\big)
\end{CD} \tag{3.23}$$

commute. It follows immediately from Lemma 2.7 that (3.22) commutes. The square

$$\begin{CD}
HH^{II}(Q,-f)\otimes_k HH^{II}(R,-g) @>{\simeq}>> HH^{II}\big(qmf(Q,f)\big)\otimes_k HH^{II}\big(qmf(R,g)\big)\\
@V{-\tilde{\star}-}VV @VV{-\tilde{\star}-}V\\
HH^{II}(Q\otimes_k R, -f-g) @>{\simeq}>> HH^{II}\big(qmf(Q\otimes_k R, f+g)\big)
\end{CD} \tag{3.24}$$

evidently commutes, and concatenating this diagram with (3.23) gives a commutative diagram. It follows that (3.23) commutes. ■

For an essentially smooth k-algebra Q, any element $f \in Q$, and any pair of closed subsets Z and W of $\operatorname{Spec}(Q)$, there is a pairing

$$HH\big(mf^{Z}(Q,f)\big) \times HH\big(mf^{W}(Q,-f)\big) \xrightarrow{\star} HH\big(mf^{Z\cap W}(Q,0)\big) \tag{3.25}$$

defined by composing the Künneth map

$$\begin{aligned} &HH\big(mf^{Z}(Q,f)\big) \times HH\big(mf^{W}(Q,-f)\big) \\ &\xrightarrow{\tilde{\star}} HH\big(mf^{Z\times W}(Q\otimes_k Q, f\otimes 1 - 1\otimes f)\big) \end{aligned}$$

with the map

$$HH\big(mf^{Z\times W}(Q\otimes_k Q, f\otimes 1 - 1\otimes f)\big) \to HH\big(mf^{Z\cap W}(Q,0)\big)$$

induced by the multiplication map $Q\otimes Q \to Q$. The previous result, along with the functoriality of the HKR map, yields the following corollary.

Corollary 3.26. *The diagram*

$$\begin{array}{ccc} HH\big(mf^{Z}(Q,f)\big)\otimes_k HH\big(mf^{W}(Q,-f)\big) & \xrightarrow{\varepsilon_{Q,f,Z}\otimes\varepsilon_{Q,-f,Z}} & \mathbb{R}\Gamma_Z(\Omega^\bullet_{Q/k},-df)\otimes_k \mathbb{R}\Gamma_W(\Omega^\bullet_{Q/k},df) \\ \downarrow & & \downarrow \wedge \\ HH\big(mf^{Z\cap W}(Q,0)\big) & \xrightarrow{\varepsilon_{Q,0,Z\cap W}} & \mathbb{R}\Gamma_{Z\cap W}\Omega^\bullet_{Q/k} \end{array}$$

in $D(Q\otimes_k Q)$ commutes.

We will be especially interested in the case where $Z\cap W = \{\mathfrak{m}\}$.

4. Proof of Shklyarov's conjecture

Throughout this section, we assume

- k is a field,
- Q is a regular k-algebra, and
- $\mathfrak{m}$ is a k-rational maximal ideal of Q; i.e. the canonical map $k \to Q/\mathfrak{m}$ is an isomorphism.

Let us review our progress on the proof of Conjecture 1.4. Recall from the introduction that, to prove the conjecture, it suffices to show that diagram (1.9) commutes, the composition along the left side of this diagram computes the pairing η_{mf}, and the composition along the right side computes the residue pairing. So far, we have shown the two interior squares of (1.9) commute: this follows from Lemma 3.11, Lemma 3.14, and Corollary 3.26. In this section, we show the left side of the diagram gives the canonical pairing η_{mf} (Lemma 4.23), the right side of the diagram gives the residue pairing (Proposition 4.34), and the bottom triangle commutes (Theorem 4.36).

4.1. Computing $HH(mf^{\mathfrak{m}}(Q,0))$

We carry out a calculation of the Hochschild homology of the dg-category $mf^{\mathfrak{m}}(Q,0)$ that we will use repeatedly throughout the rest of the paper. Let n denote the Krull dimension of $Q_{\mathfrak{m}}$. We recall that a sequence $x_1,\dots,x_n \in \mathfrak{m}$ is called a *system of parameters* if $x_1,\dots,x_n$ generate an $\mathfrak{m}$-primary ideal, and a system of parameters is called *regular* if the elements generate $\mathfrak{m}$.

Fix a regular system of parameters $x_1,\dots,x_n$ for $Q_{\mathfrak{m}}$, and set $K = \mathrm{Kos}_{Q_{\mathfrak{m}}}(x_1,\dots,x_n) \in mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)$, the $\mathbb{Z}/2$-folded Koszul complex on the x_i's. Explicitly, K is the differential $\mathbb{Z}/2$-graded algebra whose underlying algebra is the exterior algebra over $Q_{\mathfrak{m}}$ generated by $e_1,\dots,e_n$ with $d^K(e_i) = x_i$. The differential $\mathbb{Z}/2$-graded $Q_{\mathfrak{m}}$-algebra $\mathcal{E} := \mathrm{End}_{mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)}(K)$ is generated by odd degree elements $e_1,\dots,e_n,e_1^*,\dots,e_n^*$ satisfying $e_i^2 = 0 = (e_i^*)^2$, $[e_i,e_j] = 0 = [e_i^*,e_j^*]$, and $[e_i,e_j^*] = \delta_{ij}$; and the differential $d^{\mathcal{E}}$ is determined by the equations $d^{\mathcal{E}}(e_i) = x_i$ and $d^{\mathcal{E}}(e_i^*) = 0$. Let Λ be the dg-k-subalgebra of $\mathcal{E}$ generated by the e_i^*. So, Λ is an exterior algebra over k on n generators, with trivial differential. The inclusion $\Lambda \subseteq \mathcal{E}$ is a quasi-isomorphism of differential $\mathbb{Z}/2$-graded k-algebras. Since Λ is graded commutative, $HH_*(\Lambda)$ is a k-algebra under the shuffle product, and, by a standard calculation, there is an isomorphism

$$\Lambda \otimes_k k[y_1,\dots,y_n] \xrightarrow{\cong} HH_*(\Lambda), \tag{4.1}$$

of k-algebras, where $e_i^* \otimes 1 \mapsto e_i^*[\,]$, and $1 \otimes y_i \mapsto 1[e_i^*]$. Here, and throughout the paper, we use the notation $\alpha_0[\,]$ to denote an element of a Hochschild complex of the form $\alpha_0[\alpha_1|\cdots|\alpha_n]$ with $n = 0$.

Lemma 4.2. *The canonical morphisms*

$$\mathcal{E} \hookrightarrow mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0) \tag{4.3}$$

and

$$mf^{\mathfrak{m}}(Q,0) \to mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0) \tag{4.4}$$

of dg-categories are Morita equivalences. In particular, one has canonical quasi-isomorphisms

$$HH(\Lambda) \xrightarrow{\simeq} HH\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)\big) \xleftarrow{\simeq} HH\big(mf^{\mathfrak{m}}(Q,0)\big). \tag{4.5}$$

Proof. To prove (4.3) is a Morita equivalence, we prove the thick closure of K in the homotopy category $[mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)]$ is all of $[mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)]$. Let $\mathcal{D}$ denote the derived category of all $\mathbb{Z}/2$-complexes of finitely generated $Q_{\mathfrak{m}}$-modules whose homology groups are finite dimensional over k. Since $Q_{\mathfrak{m}}$ is regular, it follows from [1, Proposition 3.4] that the canonical functor

$$\big[mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)\big] \to \mathcal{D}$$

is an equivalence. It therefore suffices to show $\mathrm{Thick}(K) = \mathcal{D}$; in fact, we need only show every object in $\mathcal{D}$ with free components is in $\mathrm{Thick}(K)$.

Let X be an object of $\mathcal{D}$ with free components. We may assume that X is *minimal*, i.e., that $k \otimes_{Q_\mathfrak{m}} X$ is a direct sum of copies of k and Σk. The isomorphism $K \xrightarrow{\cong} k$ in $\mathcal{D}$ induces an isomorphism

$$K \otimes_{Q_\mathfrak{m}} X \xrightarrow{\cong} k \otimes_{Q_\mathfrak{m}} X,$$

and therefore $K \otimes_{Q_\mathfrak{m}} X \in \operatorname{Thick}(k)$. It thus suffices to prove $X \in \operatorname{Thick}(K \otimes_{Q_\mathfrak{m}} X)$. Since

$$K \otimes_{Q_\mathfrak{m}} X \cong \operatorname{Kos}_{Q_\mathfrak{m}}(x_1) \otimes_{Q_\mathfrak{m}} \cdots \otimes_{Q_\mathfrak{m}} \operatorname{Kos}_{Q_\mathfrak{m}}(x_n) \otimes_{Q_\mathfrak{m}} X,$$

it suffices to show that, for every $Y \in \mathcal{D}$ whose components are free $Q_\mathfrak{m}$-modules, and every $x \in \mathfrak{m} \setminus \{0\}$, $Y \in \operatorname{Thick}(Y/xY)$. Using induction and the exact sequence

$$0 \to Y/x^{n-1}Y \xrightarrow{x} Y/x^nY \to Y/xY \to 0,$$

we get $Y/x^nY \in \operatorname{Thick}(Y/xY)$ for all n. Observing that $\operatorname{End}_{\mathcal{D}}(Y)[1/x] = 0$, choose $n \gg 0$ such that multiplication by x^n on Y determines the zero map in $\mathcal{D}$. The distinguished triangle

$$Y \xrightarrow{x^n} Y \to Y/x^n \to \Sigma Y$$

in $\mathcal{D}$ therefore splits, implying that Y is a summand of Y/x^n. Thus, $Y \in \operatorname{Thick}(Y/x^n) \subseteq \operatorname{Thick}(Y/xY)$.

As for (4.4), the functor $[mf^\mathfrak{m}(Q,0)] \to [mf^\mathfrak{m}(Q_\mathfrak{m},0)]$ is fully faithful, since $\operatorname{Hom}_{[mf^\mathfrak{m}(Q,0)]}(X,Y)$ is supported in $\{\mathfrak{m}\}$ for any X, Y. It follows that the induced map

$$\big[mf^\mathfrak{m}(Q,0)\big]^{\text{idem}} \to \big[mf^\mathfrak{m}(Q_\mathfrak{m},0)\big]^{\text{idem}} \tag{4.6}$$

on idempotent completions is fully faithful, so we need only to show that (4.6) is essentially surjective. By the above argument, it suffices to show that K is in the essential image of (4.6). Choose a Q-free resolution F of k; $F_\mathfrak{m}$ is homotopy equivalent to the Koszul complex on the x_i's, and so the $\mathbb{Z}/2$-folding of $F_\mathfrak{m}$ is isomorphic to K in $[mf^\mathfrak{m}(Q_\mathfrak{m},0)]$. ∎

Remark 4.7. Let $\widehat{Q}$ denote the $\mathfrak{m}$-adic completion of Q. Letting $\widehat{Q}$ play the role of Q in Lemma 4.2 implies that the inclusion

$$\operatorname{End}_{mf^\mathfrak{m}(\widehat{Q},0)}\big(K \otimes_{Q_\mathfrak{m}} \widehat{Q}\big) \hookrightarrow mf^\mathfrak{m}(\widehat{Q},0)$$

is a Morita equivalence. The same proof that shows the map (4.4) in Lemma 4.2 is a Morita equivalence shows the canonical map

$$mf^\mathfrak{m}(Q,0) \to mf^\mathfrak{m}(\widehat{Q},0)$$

is a Morita equivalence.

4.2. The trace map

We define an even degree map

$$\text{trace} : HH_*\big(mf^{\mathfrak{m}}(Q,0)\big) \to k$$

of $\mathbb{Z}/2$-graded k-vector spaces, with k concentrated in even degree, as follows. Let $\operatorname{Perf}_{\mathbb{Z}/2}(k)$ denote the dg-category of $\mathbb{Z}/2$-graded complexes of (not necessarily finitely dimensional) k-vector spaces having finite dimensional homology. There is a dg-functor $mf^{\mathfrak{m}}(Q,0) \to \operatorname{Perf}_{\mathbb{Z}/2}(k)$ induced by restriction of scalars along the structural map $k \to Q$ that induces a map

$$u : HH_*\big(mf^{\mathfrak{m}}(Q,0)\big) \to HH_*\big(\operatorname{Perf}_{\mathbb{Z}/2}(k)\big),$$

and there is a canonical isomorphism

$$v : k \xrightarrow{\cong} HH_*\big(\operatorname{Perf}_{\mathbb{Z}/2}(k)\big)$$

given by $a \mapsto a[\,]$. Here, k is considered as a $\mathbb{Z}/2$-graded complex concentrated in even degree, and, on the right, a is regarded as an endomorphism of this complex. We define

$$\text{trace} := v^{-1}u.$$

In the rest of this subsection, we establish several technical properties of the trace map that we will need later on.

Given an object $(P,\delta_P)\in mf^{\mathfrak{m}}(Q,0)$, there is a canonical map of complexes $\operatorname{End}(P)\to HH(mf^{\mathfrak{m}}(Q,0))$ given by $\alpha \mapsto \alpha[\,]$ and hence an induced map

$$H_*\big(\operatorname{End}(P)\big) \to HH_*\big(mf^{\mathfrak{m}}(Q,0)\big). \tag{4.8}$$

Proposition 4.9. *If $(P,\delta_P) \in mf^{\mathfrak{m}}(Q,0)$, and α is an even degree endomorphism of P, the composition*

$$H_0\big(\operatorname{End}(P)\big) \xrightarrow{(4.8)} HH_0\big(mf^{\mathfrak{m}}(Q,0)\big) \xrightarrow{\text{trace}} k$$

sends α to the supertrace of the endomorphism of $H_*(P)$ induced by α:

$$\begin{aligned}\text{trace}\big(\alpha[\,]\big) &= \operatorname{str}\big(H_*(\alpha) : H_*(P) \to H_*(P)\big)\\ &= \operatorname{tr}\big(H_0(\alpha) : H_0(P) \to H_0(P)\big) - \operatorname{tr}\big(H_1(\alpha) : H_1(P) \to H_1(P)\big).\end{aligned}$$

In particular,

$$\text{trace}\big(\mathrm{id}_P[\,]\big) = \dim_k H_0(P) - \dim_k H_1(P).$$

Proof. Let $\operatorname{Vect}_{\mathbb{Z}/2}(k)$ denote the subcategory of $\operatorname{Perf}_{\mathbb{Z}/2}(k)$ spanned by finite-dimensional $\mathbb{Z}/2$-graded vector spaces with trivial differential. It is well known that the inclusion

$\mathrm{Vect}_{\mathbb{Z}/2}(k) \hookrightarrow \mathrm{Perf}_{\mathbb{Z}/2}(k)$ induces a quasi-isomorphism on Hochschild homology. Composing the map $\mathrm{End}(H_*(P)) \to HH_*(\mathrm{Vect}_{\mathbb{Z}/2}(k))$ given by $\alpha \mapsto \alpha[]$ with the canonical map $H_*(\mathrm{End}(P)) \to \mathrm{End}(H_*(P))$ gives a map

$$H_*\big(\mathrm{End}(P)\big) \to HH_*\big(\mathrm{Vect}_{\mathbb{Z}/2}(k)\big). \tag{4.10}$$

We first show that the square

$$\begin{array}{ccc} H_*\big(\mathrm{End}(P)\big) & \xrightarrow{(4.8)} & HH_*\big(mf^{\mathfrak{m}}(Q,0)\big) \\ \Big\downarrow{\scriptstyle (4.10)} & & \Big\downarrow{\scriptstyle u} \\ HH_*\big(\mathrm{Vect}_{\mathbb{Z}/2}(k)\big) & \xrightarrow{\cong} & HH_*\big(\mathrm{Perf}_{\mathbb{Z}/2}(k)\big) \end{array} \tag{4.11}$$

commutes. Let β be an even degree cycle in $\mathrm{End}(P)$, and let $H_*(\beta)$ denote the induced endomorphism of $H_*(P)$. We must show the cycles $\beta[]$ and $H_*(\beta)[]$ coincide in $HH_*(\mathrm{Perf}_{\mathbb{Z}/2}(k))$. To see this, choose even degree k-linear chain maps

$$\iota : H_*(P) \to P, \quad \pi : P \to H_*(P)$$

such that

- $\pi \circ \iota = \mathrm{id}_{H_*(P)}$, and
- $\iota \circ \pi$ is homotopic to id_P via a ($\mathbb{Z}/2$-graded) homotopy h, i.e.,

$$\iota \circ \pi - \mathrm{id}_P = \delta_P \circ h + h \circ \delta_P.$$

Applying the Hochschild differential b to

$$\pi[\beta \circ \iota] \in \mathrm{Hom}\big(P, H_*(P)\big) \otimes \mathrm{Hom}\big(H_*(P), P\big) \subseteq HH\big(\mathrm{Perf}_{\mathbb{Z}/2}(k)\big),$$

we get

$$\begin{aligned} b\big(\pi[\beta \circ \iota]\big) = (b_2 + b_1)\big(\pi[\beta \circ \iota]\big) = b_2\big(\pi[\beta \circ \iota]\big) &= (\pi \circ \beta \circ \iota)[] - (\iota \circ \pi \circ \beta)[] \\ &= H_*(\beta)[] - (\iota \circ \pi \circ \beta)[]. \end{aligned}$$

Next, observe that

$$(b_2 + b_1)\big((h \circ \beta)[]\big) = b_1\big((h \circ \beta)[]\big) = (\iota \circ \pi \circ \beta - \beta)[].$$

It follows that diagram (4.11) commutes.

The isomorphism

$$v : k \xrightarrow{\cong} HH_*\big(\mathrm{Perf}_{\mathbb{Z}/2}(k)\big)$$

factors as

$$k \xrightarrow{\cong} HH_*(k) \xrightarrow{\cong} HH_*\big(\mathrm{Vect}_{\mathbb{Z}/2}(k)\big) \xrightarrow{\cong} HH_*\big(\mathrm{Perf}_{\mathbb{Z}/2}(k)\big),$$

where each map is the evident canonical one. There is a chain map $HH(\mathrm{Vect}_{\mathbb{Z}/2}(k)) \to HH(k)$ given by the generalized trace map described in [14, Section 2.3.1] and an evident isomorphism $HH_*(k) \xrightarrow{\cong} k$. It follows from [14, Lemma 2.12] that composing these maps gives the inverse of

$$k \xrightarrow{\cong} HH_*(k) \xrightarrow{\cong} HH_*\big(\mathrm{Vect}_{\mathbb{Z}/2}(k)\big).$$

As discussed in [14, P. 872], the generalized trace sends a class of the form $\alpha_0[\,]$ to $\mathrm{str}(\alpha_0)[\,]$. The statement now follows from the commutativity of (4.11). ∎

Remark 4.12. If Z and W are closed subsets of $\mathrm{Sing}(Q/f)$ that satisfy $Z \cap W = \{\mathfrak{m}\}$, then, from (3.25), we obtain the pairing

$$HH_*\big(mf^Z(Q,f)\big) \times HH_*\big(mf^W(Q,-f)\big) \xrightarrow{\star} HH_*\big(mf^{\mathfrak{m}}(Q,0)\big).$$

By Proposition 4.9, given $X \in mf^Z(Q,f)$ and $Y \in mf^W(Q,-f)$, the composition

$$\begin{aligned} H_*\big(\mathrm{End}(X)\big) \times H_*\big(\mathrm{End}(Y)\big) &\to HH_*\big(mf^Z(Q,f)\big) \times HH_*\big(mf^W(Q,-f)\big) \\ &\xrightarrow{\star} HH_*\big(mf^{\mathfrak{m}}(Q,0)\big) \xrightarrow{\text{trace}} k \end{aligned}$$

sends a pair of endomorphisms (α,β) to $\mathrm{tr}(H_0(\alpha\otimes\beta)) - \mathrm{tr}(H_1(\alpha\otimes\beta))$. In particular, it sends $(\mathrm{id}_X, \mathrm{id}_Y)$ to

$$\theta(X,Y) := \dim_k H_0(X\otimes Y) - \dim_k H_1(X \otimes Y).$$

Recall from Subsection 4.1 the folded Koszul complex K and the exterior algebra $\Lambda \subseteq \mathrm{End}_{mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)}(K)$. Denote by $\eta : \Lambda \to k$ the augmentation map that sends e_i^* to 0.

Proposition 4.13. *The composition*

$$HH_*(\Lambda) \xrightarrow{(4.5)} HH_*\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)\big) \xrightarrow{\text{trace}} k \tag{4.14}$$

coincides with

$$HH_*(\Lambda) \xrightarrow{HH_*(\eta)} HH_*(k) \xrightarrow{\cong} k, \tag{4.15}$$

where the second map in (4.15) is the canonical isomorphism. In particular, if $\alpha_0[\alpha_1|\cdots|\alpha_n]$ is a cycle in $HH(\Lambda)$, where $n>0$, the map (4.14) sends $\alpha_0[\alpha_1|\cdots|\alpha_n]$ to 0.

Proof. If C is a $\mathbb{Z}$-graded complex, denote its $\mathbb{Z}/2$-folding by $\mathrm{Fold}(C)$. Similarly, given a differential $\mathbb{Z}$-graded category $\mathcal{C}$, define a differential $\mathbb{Z}/2$-graded category $\mathrm{Fold}(\mathcal{C})$ with the same objects as $\mathcal{C}$ and morphism complexes given by taking the $\mathbb{Z}/2$-foldings of the morphism complexes of $\mathcal{C}$. In this proof, we use the notation $HH^{\mathbb{Z}}(-)$ (resp., $HH^{\mathbb{Z}/2}(-)$) to denote the Hochschild complex of a differential $\mathbb{Z}$-graded (resp., $\mathbb{Z}/2$-graded) category. We observe that, if $\mathcal{C}$ is a differential $\mathbb{Z}$-graded category,

$$\mathrm{Fold}\big(HH_*^{\mathbb{Z}}(\mathcal{C})\big) = HH_*^{\mathbb{Z}/2}\big(\mathrm{Fold}(\mathcal{C})\big). \tag{4.16}$$

Let $\mathrm{Perf}^{\mathfrak{m}}(Q)$ denote the dg-category of perfect complexes of Q-modules with support in $\{\mathfrak{m}\}$, and let $\mathrm{Perf}_{\mathbb{Z}}(k)$ denote the differential $\mathbb{Z}$-graded category of complexes of (not necessarily finite dimensional) k-vector spaces with finite dimensional total homology. As in the $\mathbb{Z}/2$-graded case, there is an isomorphism

$$\widetilde{v} : k \xrightarrow{\cong} HH_*^{\mathbb{Z}}(\mathrm{Perf}_{\mathbb{Z}}(k)),$$

where k is concentrated in degree 0, given by $a \mapsto a[\,]$.

Let $\widetilde{K}$ denote the $\mathbb{Z}$-graded Koszul complex on the regular system of parameters $x_1, \ldots, x_n$ for $Q_{\mathfrak{m}}$ chosen in Subsection 4.1, so that the $\mathbb{Z}/2$-folding of $\widetilde{K}$ is K. Similarly, denote by $\widetilde{\Lambda}$ the subalgebra (with trivial differential) of $\mathrm{End}(\widetilde{K})$, defined in the same way as Λ, so that the $\mathbb{Z}/2$-folding of $\widetilde{\Lambda}$ is Λ. Notice that every α_i appearing in our cycle $\alpha_0[\alpha_1|\cdots|\alpha_n]$ can be considered as an element of $\widetilde{\Lambda}$.

We consider the composition

$$\begin{aligned} HH_*^{\mathbb{Z}}(\widetilde{\Lambda}) \to HH_*^{\mathbb{Z}}(\mathrm{End}(\widetilde{K})) \to HH_*^{\mathbb{Z}}(\mathrm{Perf}^{\mathfrak{m}}(Q)) \\ \to HH_*^{\mathbb{Z}}(\mathrm{Perf}_{\mathbb{Z}}(k)) \xrightarrow{(\widetilde{v})^{-1}} k \end{aligned} \tag{4.17}$$

of maps of $\mathbb{Z}$-graded k-vector spaces. We claim (4.17) coincides with the composition

$$HH_*^{\mathbb{Z}}(\widetilde{\Lambda}) \to HH_*^{\mathbb{Z}}(k) \xrightarrow{\cong} k, \tag{4.18}$$

where the first map is induced by the augmentation map $\widetilde{\Lambda} \to k$. We need only check this in degree 0. $HH_0^{\mathbb{Z}}(\widetilde{\Lambda})$ is a 1-dimensional k-vector space generated by $\mathrm{id}_K[\,]$. The map (4.18) sends $\mathrm{id}_K[\,]$ to 1, and, by (the $\mathbb{Z}$-graded version of) Lemma 4.9, the map (4.17) does as well.

Applying Fold(−) to (4.17), and using (4.16), we arrive at a composition

$$HH_*^{\mathbb{Z}/2}(\Lambda) \to HH_*^{\mathbb{Z}/2}(\mathrm{Fold}(\mathrm{Perf}^{\mathfrak{m}}(Q))) \to k$$

of maps of $\mathbb{Z}/2$-graded complexes of k-vector spaces, which may be augmented to a commutative diagram

$$\begin{array}{ccccc} HH_*^{\mathbb{Z}/2}(\Lambda) & \longrightarrow & HH_*^{\mathbb{Z}/2}(\mathrm{Fold}(\mathrm{Perf}^{\mathfrak{m}}(Q))) & \longrightarrow & k \\ & \searrow^{(4.5)} & \downarrow & \nearrow_{\mathrm{trace}} & \\ & & HH_*^{\mathbb{Z}/2}(mf^{\mathfrak{m}}(Q,0)). & & \end{array} \tag{4.19}$$

On the other hand, applying Fold(−) to (4.18), and once again applying (4.16), we get the map (4.15). ■

Lemma 4.20. *Suppose Q and Q' are regular k-algebras, and $\mathfrak{m} \subseteq Q$, $\mathfrak{m}' \subseteq Q'$ are k-rational maximal ideals. Let $g : Q \to Q'$ be a k-algebra map such that $g^{-1}(\mathfrak{m}') = \mathfrak{m}$,* *the induced map $Q_{\mathfrak{m}} \to Q'_{\mathfrak{m}'}$ is flat, and $g(\mathfrak{m})Q'_{\mathfrak{m}'} = \mathfrak{m}'Q'_{\mathfrak{m}'}$. Then g induces a quasi-isomorphism*

$$g_* : HH\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0)\big) \xrightarrow{\simeq} HH\big(mf^{\mathfrak{m}'}(Q'_{\mathfrak{m}'}, 0)\big),$$

and

$$\operatorname{trace}_{Q'_{\mathfrak{m}'}} \circ g_* = \operatorname{trace}_{Q_{\mathfrak{m}}}.$$

Proof. Let $\widehat{Q}$ (resp., $\widehat{Q'}$) denote the $\mathfrak{m}$-adic (resp., $\mathfrak{m}'$-adic) completion of Q (resp., Q'). The assumptions on g imply that it induces an isomorphism $\widehat{Q} \xrightarrow{\cong} \widehat{Q'}$. The first assertion follows since the canonical maps

$$HH_*\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0)\big) \to HH_*\big(mf^{\mathfrak{m}}(\widehat{Q}, 0)\big)$$

and

$$HH_*\big(mf^{\mathfrak{m}'}(Q'_{\mathfrak{m}'}, 0)\big) \to HH_*\big(mf^{\mathfrak{m}}(\widehat{Q'}, 0)\big)$$

are isomorphisms by Remark 4.7.

As for the second assertion, let $n = \dim(Q_{\mathfrak{m}})$, choose a regular system of parameters $x_1, \dots, x_n$ of $Q_{\mathfrak{m}}$, and construct the exterior algebra Λ using this system of parameters, as in Subsection 4.1. The hypotheses ensure that $g(x_1), \dots, g(x_n)$ form a regular system of parameters for $Q'_{\mathfrak{m}'}$, and we let Λ' be the associated exterior algebra. We have a commutative diagram

$$\begin{array}{ccc} HH_*(\Lambda) & \xrightarrow{\cong} & HH_*(\Lambda') \\ \downarrow{\cong} & & \downarrow{\cong} \\ HH_*\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0)\big) & \longrightarrow & HH_*\big(mf^{\mathfrak{m}'}(Q'_{\mathfrak{m}'}, 0)\big), \end{array}$$

where the vertical isomorphisms are as in Lemma 4.2. By Proposition 4.13, it now suffices to observe that the composition

$$HH_*(\Lambda) \xrightarrow{\cong} HH_*(\Lambda') \to k,$$

where the second map is induced by the augmentation $\Lambda' \to k$, coincides with the map induced by the augmentation $\Lambda \to k$. ■

Lemma 4.21. *Suppose Q, Q' are essentially smooth k-algebras and $\mathfrak{m}' \subseteq Q'$, $\mathfrak{m}'' \subseteq Q''$ are k-rational maximal ideals. Set $Q = Q' \otimes_k Q''$ and $\mathfrak{m} = \mathfrak{m}' \otimes_k Q'' + Q' \otimes_k \mathfrak{m}''$. Then Q is an essentially smooth k-algebra, $\mathfrak{m}$ is a k-rational maximal ideal of Q, and the diagram*

$$\begin{array}{ccc} HH_*\big(mf^{\mathfrak{m}'}(Q'_{\mathfrak{m}'}, 0)\big) \otimes_k HH_*\big(mf^{\mathfrak{m}''}(Q''_{\mathfrak{m}''}, 0)\big) & \xrightarrow{\tilde{\star}} & HH_*\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0)\big) \\ \downarrow{\scriptstyle \operatorname{trace} \otimes \operatorname{trace}} & & \downarrow{\scriptstyle \operatorname{trace}} \\ k \otimes_k k & \xrightarrow{\cong} & k \end{array}$$

commutes.

Proof. The first two assertions are standard facts. As for the final one, let n' and n'' denote the dimensions of $Q'_{\mathfrak{m}'}$ and $Q''_{\mathfrak{m}''}$, resp. Choose regular systems of parameters $x_1, \dots, x_{n'}$ and $y_1, \dots, y_{n''}$ of $Q'_{\mathfrak{m}'}$ and $Q''_{\mathfrak{m}''}$, resp., so that $x_1, \dots, x_{n'}, y_1, \dots, y_{n''}$ form a regular system of parameters of $Q_{\mathfrak{m}}$. As in the proof of Lemma 4.20, let Λ, Λ', and Λ'' be exterior algebras associated to these systems of parameters, as constructed in Subsection 4.1. By Lemma 2.7, we have a commutative square

$$\begin{array}{ccc} HH_*(\Lambda') \otimes_k HH_*(\Lambda'') & \xrightarrow{\cong} & HH_*\big(mf^{\mathfrak{m}'}(Q'_{\mathfrak{m}'}, 0)\big) \otimes_k HH_*\big(mf^{\mathfrak{m}''}(Q''_{\mathfrak{m}''}, 0)\big) \\ \Big\downarrow \tilde{\star} & & \Big\downarrow \tilde{\star} \\ HH_*(\Lambda) & \xrightarrow{\cong} & HH_*\big(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0)\big), \end{array}$$

where the horizontal isomorphisms are as in Lemma 4.2. By Proposition 4.13, it now suffices to observe that the composition

$$\Lambda \xrightarrow{\cong} \Lambda' \otimes_k \Lambda'' \to k,$$

where the second map is the tensor product of the augmentations, coincides with the augmentation $\Lambda \to k$. ∎

4.3. The canonical pairing on Hochschild homology

A k-linear differential $\mathbb{Z}/2$-graded category $\mathcal{C}$ is called *proper* if, for all pairs of objects (X, Y), $\dim_k H_i \operatorname{Hom}_{\mathcal{C}}(X, Y) < \infty$ for $i = 0, 1$.

Definition 4.22. For a proper differential $\mathbb{Z}/2$-graded category $\mathcal{C}$, the *canonical pairing for Hochschild homology* is the map

$$\eta_{\mathcal{C}}(-,-) : HH_*(\mathcal{C}) \otimes_k HH_*(\mathcal{C}) \to k$$

given by the composition

$$\begin{aligned} HH_*(\mathcal{C}) \otimes_k HH_*(\mathcal{C}) &\xrightarrow{\mathrm{id} \otimes \Phi} HH_*(\mathcal{C}) \otimes_k HH_*(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\tilde{\star}} HH_*(\mathcal{C} \otimes_k \mathcal{C}^{\mathrm{op}}) \\ &\xrightarrow{HH((X,Y) \mapsto \operatorname{Hom}_{\mathcal{C}}(Y,X))} HH_*\big(\operatorname{Perf}_{\mathbb{Z}/2}(k)\big) \xleftarrow{\cong} k, \end{aligned}$$

where Φ is the map defined in (3.12).

When $\operatorname{Sing}(Q/f) = \{\mathfrak{m}\}$, $mf(Q, f)$ is proper, so we have the canonical pairing

$$\eta_{mf} : HH_*\big(mf(Q, f)\big) \otimes_k HH_*\big(mf(Q, f)\big) \to k.$$

Lemma 4.23. *When* $\operatorname{Sing}(Q/f) = \{\mathfrak{m}\}$*,* η_{mf} *coincides with the pairing given by the composition*

$$\begin{aligned} &HH_*\big(mf(Q, f)\big) \otimes_k HH_*\big(mf(Q, f)\big) \\ &\xrightarrow{\mathrm{id} \otimes \Psi} HH_*\big(mf(Q, f)\big) \otimes_k HH_*\big(mf(Q, -f)\big) \xrightarrow{\star} HH_*\big(mf^{\mathfrak{m}}(Q, 0)\big) \xrightarrow{\mathrm{trace}} k, \end{aligned}$$

where Ψ *is defined in* (3.13).

Proof. By Lemma 2.7, there is a commutative square

$$\begin{array}{ccc} HH_*(mf(Q,f)) \otimes_k HH_*(mf(Q,f)^{\mathrm{op}}) & \xrightarrow{HH(\mathrm{id})\otimes HH(D)} & HH_*(mf(Q,f)) \otimes_k HH_*(mf(Q,-f)) \\ \downarrow{\scriptstyle -\tilde{\star}-} & & \downarrow{\scriptstyle -\tilde{\star}-} \\ HH_*(mf(Q,f) \otimes_k mf(Q,f)^{\mathrm{op}}) & \xrightarrow{HH(\mathrm{id}\otimes D)} & HH_*(mf(Q,f) \otimes_k mf(Q,-f)), \end{array}$$

where D is the dg-functor defined in Subsection 3.4. Therefore, it suffices to show the composition

$$\begin{aligned} HH_*(mf(Q,f) \otimes_k mf(Q,f)^{\mathrm{op}}) &\xrightarrow{1\otimes D} HH_*(mf(Q,f) \otimes_k mf(Q,-f)) \\ &\xrightarrow{\mathrm{can}} HH_*(mf^{\mathfrak{m}}(Q,0)) \\ &\xrightarrow{\mathrm{Forget}} HH_*(\mathrm{Perf}_{\mathbb{Z}/2}(k)) \end{aligned}$$

coincides with the map induced by the dg-functor

$$mf(Q,f) \otimes_k mf(Q,f)^{\mathrm{op}} \to \mathrm{Perf}_{\mathbb{Z}/2}(k)$$

given by $(X,Y) \mapsto \mathrm{Hom}_{mf}(Y,X)$, and this is clear. ∎

4.4. The residue map

Assume that Q is an essentially smooth k-algebra and $\mathfrak{m}$ is a k-rational maximal ideal of Q. Let n be the Krull dimension of $Q_{\mathfrak{m}}$. In this subsection, we recall the definition of Grothendieck's residue map

$$\mathrm{res}^G : H^n_{\mathfrak{m}}(\Omega^n_{Q_{\mathfrak{m}}/k}) \to k$$

and some of its properties. Recall from Subsection 3.2 that for any system of parameters $x_1, \dots, x_n$ of $Q_{\mathfrak{m}}$, we have a canonical isomorphism

$$H^n_{\mathfrak{m}}(\Omega^n_{Q_{\mathfrak{m}}/k}) \cong H^n(\mathscr{C}(x_1,\dots,x_n) \otimes_{Q_{\mathfrak{m}}} \Omega^n_{Q_{\mathfrak{m}}/k}). \tag{4.24}$$

We will temporarily use $\mathbb{Z}$-gradings and index things cohomologically, using superscripts. In particular, $\Omega^{\bullet}_{Q_{\mathfrak{m}}/k}$ is a graded $Q_{\mathfrak{m}}$-module with $\Omega^j_{Q_{\mathfrak{m}}/k}$ declared to have cohomological degree j.

We introduce some notation that will be convenient when computing with the augmented Čech complex. First form the exterior algebra over $Q_{\mathfrak{m}}[1/x_1,\dots,1/x_n]$ on (cohomological) degree 1 generators $\alpha_1,\dots,\alpha_n$, and make it a complex with differential given as left multiplication by the degree 1 element $\sum_i \alpha_i$. We identify $\mathscr{C}(x_1,\dots,x_n)$ as the subcomplex whose degree j component is

$$\bigoplus_{i_1<\cdots<i_j} Q_{\mathfrak{m}}\left[\frac{1}{x_{i_1}\cdots x_{i_j}}\right]\alpha_{i_1}\cdots\alpha_{i_j}.$$

Define

$$E(x_1,\dots,x_n) := \frac{Q_\mathfrak{m}[1/x_1,\dots,1/x_n]}{\sum_j Q_\mathfrak{m}[1/x_1,\dots,\widehat{1/x_j},\dots,1/x_n]}.$$

Since $x_1,\dots,x_n$ is a regular sequence, there is an isomorphism

$$E(x_1,\dots,x_n) \xrightarrow{\cong} H^n\big(\mathcal{C}(x_1,\dots,x_n)\big)$$

sending $\overline{g}$ to $\overline{g\alpha_1\cdots\alpha_n}$ for $g \in Q_\mathfrak{m}[1/x_1,\dots,1/x_n]$. Using that $\Omega^n_{Q_\mathfrak{m}/k}$ is a flat $Q_\mathfrak{m}$-module, we obtain the isomorphism

$$H^n\big(\mathcal{C}(x_1,\dots,x_n) \otimes_{Q_\mathfrak{m}} \Omega^n_{Q_\mathfrak{m}/k}\big) \cong E(x_1,\dots,x_n) \otimes_{Q_\mathfrak{m}} \Omega^n_{Q_\mathfrak{m}/k}. \tag{4.25}$$

Every element of $E(x_1,\dots,x_n) \otimes_{Q_\mathfrak{m}} \Omega^n_{Q_\mathfrak{m}/k}$ is a sum of terms of the form

$$\frac{1}{x_1^{a_1}\cdots x_n^{a_n}} \otimes \omega$$

with $a_i \geq 1$ and $\omega \in \Omega^n_{Q_\mathfrak{m}/k}$, and this element corresponds to

$$\frac{\alpha_1\cdots\alpha_n}{x_1^{a_1}\cdots x_n^{a_n}} \otimes \omega \in H^n\big(\mathcal{C}(x_1,\dots,x_n) \otimes_{Q_\mathfrak{m}} \Omega^n_{Q_\mathfrak{m}/k}\big) \tag{4.26}$$

under the isomorphism (4.25).

Definition 4.27. Given a system of parameters $x_1,\dots,x_n$ for $Q_\mathfrak{m}$, integers $a_i \geq 1$ for each $1 \leq i \leq n$, and an n-form $\omega \in \Omega^n_{Q_\mathfrak{m}/k}$, the *generalized fraction*

$$\left[\frac{\omega}{x_1^{a_1},\dots,x_n^{a_n}}\right] \in H^n_\mathfrak{m}\big(\Omega^n_{Q_\mathfrak{m}/k}\big)$$

is the class corresponding to the element in (4.26) under the canonical isomorphism (4.24).

To define Grothendieck's residue map, we now assume that $x_1,\dots,x_n$ is a *regular* system of parameters. Since $\mathfrak{m}$ is k-rational, the $\mathfrak{m}$-adic completion $\widehat{Q}$ of Q is isomorphic to the ring of formal power series $k[[x_1,\dots,x_n]]$, and a basis for $E(x_1,\dots,x_n)$ as a k-vector space is given by the set $\left\{\frac{1}{x_1^{a_1}\cdots x_n^{a_n}} \mid a_i \geq 1\right\}$. We also have that $\Omega^n_{Q_\mathfrak{m}/k}$ is a free $Q_\mathfrak{m}$-module of rank one spanned by $dx_1\cdots dx_n$. It follows that the set

$$\left\{\left[\frac{dx_1\cdots dx_n}{x_1^{a_1},\dots,x_n^{a_n}}\right] \mid a_i \geq 1\right\}$$

is a k-basis of $H^n_\mathfrak{m}(\Omega^n_{Q_\mathfrak{m}/k})$.

Definition 4.28. Grothendieck's residue map $\operatorname{res}^G : H^n_\mathfrak{m}(\Omega^n_{Q/k}) \to k$ is the unique k-linear map such that, if $x_1,\dots,x_n$ is a regular system of parameters of $Q_\mathfrak{m}$, then

$$\operatorname{res}^G\left[\frac{dx_1\cdots dx_n}{x_1^{a_1},\dots,x_n^{a_n}}\right] = \begin{cases} 1 & \text{if } a_i = 1 \text{ for all } i\text{, and} \\ 0 & \text{otherwise.}\end{cases} \tag{4.29}$$

See [7, Theorem 5.2] for a proof that this definition is independent of the choice of $x_1, \dots, x_n$.

We now revert to the $\mathbb{Z}/2$-grading used throughout most of this paper. In particular, we regard $\Omega^\bullet_{Q_\mathfrak{m}/k}$ as a $\mathbb{Z}/2$-graded $Q_\mathfrak{m}$-module with $\Omega^j_{Q_\mathfrak{m}/k}$ located in degree $j \pmod 2$, and we use subscripts to indicate degrees.

Definition 4.30. The *residue map* for the $\mathbb{Z}/2$-graded $Q_\mathfrak{m}$-module $\Omega^\bullet_{Q_\mathfrak{m}/k}$ is the map

$$\operatorname{res} = \operatorname{res}_{Q,\mathfrak{m}} : H_{2n}\mathbb{R}\Gamma_\mathfrak{m}(\Omega^\bullet_{Q_\mathfrak{m}/k}) \to k,$$

defined as the composition

$$H_{2n}\mathbb{R}\Gamma_\mathfrak{m}(\Omega^\bullet_{Q_\mathfrak{m}/k}) \twoheadrightarrow H_{2n}\mathbb{R}\Gamma_\mathfrak{m}(\Sigma^{-n}\Omega^n_{Q_\mathfrak{m}/k}) \cong H^n_\mathfrak{m}(\Omega^n_{Q_\mathfrak{m}/k}) \xrightarrow{\operatorname{res}^G} k,$$

where the first map is induced by the canonical projection $\Omega^\bullet_{Q_\mathfrak{m}/k} \twoheadrightarrow \Sigma^{-n}\Omega^n_{Q_\mathfrak{m}/k}$.

We will need the following two properties of the residue map.

Lemma 4.31. *Suppose Q and Q' are essentially smooth k-algebras and $\mathfrak{m} \subseteq Q$, $\mathfrak{m}' \subseteq Q'$ are k-rational maximal ideals. Let $g : Q \to Q'$ be a k-algebra map such that $g^{-1}(\mathfrak{m}') = \mathfrak{m}$, the induced map $Q_\mathfrak{m} \to Q'_{\mathfrak{m}'}$ is flat, and $g(\mathfrak{m})Q'_{\mathfrak{m}'} = \mathfrak{m}'Q'_{\mathfrak{m}'}$. Then $Q_\mathfrak{m}$ and $Q'_{\mathfrak{m}'}$ have the same Krull dimension, say n; g induces an isomorphism*

$$g_* : H_{2n}\mathbb{R}\Gamma_\mathfrak{m}(\Omega^\bullet_{Q_\mathfrak{m}/k}) \xrightarrow{\cong} H_{2n}\mathbb{R}\Gamma_{\mathfrak{m}'}(\Omega^\bullet_{Q'_{\mathfrak{m}'}/k})$$

of k-vector spaces; and one has

$$\operatorname{res}_{Q',\mathfrak{m}'} \circ g_* = \operatorname{res}_{Q,\mathfrak{m}}.$$

Proof. Let $x_1, \dots, x_n$ be a regular system of parameters for $Q_\mathfrak{m}$, and set $x'_i = g(x_i)$. The assumptions on g give that $x'_1, \dots, x'_n$ form a regular system of parameters for $Q'_{\mathfrak{m}'}$, and hence the induced map on completions is an isomorphism. The first two assertions follow.

The map $E(x_1, \dots, x_n) \otimes_{Q_\mathfrak{m}} \Omega^n_{Q_\mathfrak{m}/k} \to E(x'_1, \dots, x'_n) \otimes_{Q'_{\mathfrak{m}'}} \Omega^n_{Q'_{\mathfrak{m}'}/k}$ induced by g sends $\frac{\alpha_1 \cdots \alpha_n}{x_1^{a_1} \cdots x_n^{a_n}} \otimes dx_1 \cdots dx_n$ to the expression obtained by substituting x'_i for x_i, and thus

$$g_*\left[\frac{dx_1 \cdots dx_n}{x_1^{a_1}, \dots, x_n^{a_n}}\right] = \left[\frac{dx'_1 \cdots dx'_n}{(x'_1)^{a_1}, \dots, (x'_n)^{a_n}}\right].$$

The equation $\operatorname{res}_{Q',\mathfrak{m}'} \circ g_* = \operatorname{res}_{Q,\mathfrak{m}}$ follows from (4.29). ∎

Lemma 4.32. *Let $(Q', \mathfrak{m}')$, $(Q'', \mathfrak{m}'')$, and $(Q, \mathfrak{m}) = (Q' \otimes_k Q'', \mathfrak{m}' \otimes_k Q'' + Q' \otimes_k \mathfrak{m}'')$ be as in Lemma 4.21. Set $m = \dim(Q')$ and $n = \dim(Q'')$. The diagram*

$$\begin{array}{ccc} H_{2m}\mathbb{R}\Gamma_{\mathfrak{m}}(\Omega^\bullet_{Q'_{\mathfrak{m}'}/k}) \otimes_k H_{2n}\mathbb{R}\Gamma_{\mathfrak{m}''}(\Omega^\bullet_{Q''_{\mathfrak{m}''}/k}) & \xrightarrow{\wedge} & H_{2m+2n}\mathbb{R}\Gamma_\mathfrak{m}(\Omega^\bullet_{Q_\mathfrak{m}/k}) \\ \Big\downarrow{\scriptstyle \operatorname{res}_{Q',\mathfrak{m}'} \otimes \operatorname{res}_{Q'',\mathfrak{m}''}} & & \Big\downarrow{\scriptstyle \operatorname{res}_{Q,\mathfrak{m}}} \\ k \otimes_k k & \xrightarrow{\cong} & k \end{array}$$

commutes up to the sign $(-1)^{mn}$.

Proof. It suffices to prove that the analogous diagram given by replacing $\Omega^{\bullet}_{Q'_{\mathfrak{m}'}/k}$ and $\Omega^{\bullet}_{Q''_{\mathfrak{m}''}/k}$ with $\Omega^{m}_{Q'_{\mathfrak{m}'}/k}$ and $\Omega^{n}_{Q''_{\mathfrak{m}''}/k}$ commutes. Let $x'_1, \dots, x'_m$ and $x''_1, \dots, x''_n$ be regular systems of parameters for $Q'_{\mathfrak{m}'}$ and $Q''_{\mathfrak{m}''}$. Then, upon identifying x'_i and x''_j with the elements $x'_i \otimes 1$ and $1 \otimes x''_i$ of $Q_{\mathfrak{m}}$, the sequence $x'_1, \dots, x'_m, x''_1, \dots, x''_n$ forms a regular system of parameters for $Q_{\mathfrak{m}}$. We use these three regular systems of parameters to identify $H_{2m}\mathbb{R}\Gamma_{\mathfrak{m}'}(\Omega^{m}_{Q'_{\mathfrak{m}'}/k})$ with $H_{2m}(\mathcal{C}(x'_1, \dots, x'_m) \otimes_{Q'_{\mathfrak{m}'}} \Omega^{m}_{Q'_{\mathfrak{m}'}/k})$ and similarly for Q'' and Q. Under these identifications, the map labeled $\wedge$ in the diagram sends

$$\frac{\alpha'_1 \cdots \alpha'_m}{x'_1 \cdots x'_m} \otimes dx'_1 \cdots dx'_m \otimes \frac{\alpha''_1 \cdots \alpha''_n}{x''_1 \cdots x''_n} \otimes dx''_1 \cdots dx''_n$$

to

$$(-1)^{mn} \frac{\alpha'_1 \cdots \alpha'_m \alpha''_1 \cdots \alpha''_n}{x'_1 \cdots x'_m x''_1 \cdots x''_m} \otimes dx'_1 \cdots dx'_m dx''_1 \cdots dx''_n,$$

with the sign arising since the dx'_i's and α''_j's have odd degree. The result now follows from Definition 4.27 and (4.29). ∎

4.5. The residue pairing

We assume Q, k, and $\mathfrak{m}$ are as in Subsection 4.4. All gradings in this section are $\mathbb{Z}/2$-gradings. Fix $f \in Q$, and assume $\operatorname{Sing}(f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k) = \{\mathfrak{m}\}$. Then the canonical map

$$(\Omega^{\bullet}_{Q/k}, -df) \to (\Omega^{\bullet}_{Q_{\mathfrak{m}}/k}, -df)$$

is a quasi-isomorphism, and the only nonzero homology module is

$$\frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \cong \frac{\Omega^n_{Q_{\mathfrak{m}}/k}}{df \wedge \Omega^{n-1}_{Q_{\mathfrak{m}}/k}},$$

located in degree $n := \dim(Q_{\mathfrak{m}})$. Choose a regular system of parameters

$$x_1, \dots, x_n \in \mathfrak{m} Q_{\mathfrak{m}}.$$

Then $dx_1, \dots, dx_n$ forms a $Q_{\mathfrak{m}}$-basis for $\Omega^1_{Q_{\mathfrak{m}}/k}$, and we write

$$\partial_1, \dots, \partial_n \in \operatorname{Der}_k(Q_{\mathfrak{m}}) = \operatorname{Hom}_{Q_{\mathfrak{m}}}(\Omega^1_{Q_{\mathfrak{m}}/k}, Q_{\mathfrak{m}})$$

for the associated dual basis. Set $f_i = \partial_i(f)$. The sequence $f_1, \dots, f_n$ forms a system of parameters for $Q_{\mathfrak{m}}$. For example, when $Q_{\mathfrak{m}} = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$, we have $\partial_i = \partial/\partial x_i$, so that $f_i = \partial f/\partial x_i$.

Definition 4.33. With the notation of the previous paragraph, the *residue pairing* is the map

$$\langle -, - \rangle_{\mathrm{res}} : \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \times \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \to k$$

that sends a pair $(g\,dx_1 \cdots dx_n, h\,dx_1 \cdots dx_n)$ to $\operatorname{res}^G \left[\frac{gh\,dx_1 \cdots dx_n}{f_1, \dots, f_n}\right]$.

Proposition 4.34. *The residue pairing coincides with the composition*

$$
\begin{aligned}
\frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \times \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} &= H_n\big(\Omega^\bullet_{Q/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q/k}, -df\big) \\
&\xrightarrow{\cong} H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \\
&\xrightarrow{\mathrm{id}\times(-1)^n} H^n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \times H^n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, df\big) \\
&\xleftarrow{\cong} H_n \mathbb{R}\Gamma_\mathfrak{m}\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, df\big) \\
&\xrightarrow{\text{Künneth}} H_{2n}\big(\mathbb{R}\Gamma_\mathfrak{m}\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \otimes_{Q_\mathfrak{m}} \big(\Omega^\bullet_{Q_\mathfrak{m}/k}, df\big)\big) \\
&\xrightarrow{\wedge} H_{2n} \mathbb{R}\Gamma_\mathfrak{m}\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, 0\big) \\
&\xrightarrow{\mathrm{res}} k.
\end{aligned}
$$

In particular, it is well defined and independent of the choice of regular system of parameters.

Proof. We need a formula for the inverse of the canonical isomorphism

$$
H_n \mathbb{R}\Gamma_\mathfrak{m}\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \xrightarrow{\cong} H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big). \tag{4.35}
$$

Since the isomorphism is $Q_\mathfrak{m}$-linear, we just need to know where the inverse sends $dx_1 \wedge \cdots \wedge dx_n$. Note that $\mathcal{C}(x_1, \dots, x_n) \otimes_{Q_\mathfrak{m}} \Omega^\bullet_{Q_\mathfrak{m}/k}$ is a graded-commutative $Q_\mathfrak{m}$-algebra (but not a dga), and the differential is left multiplication by $\sum_i \alpha_i - f_i dx_i$. Observe that the element

$$
\begin{aligned}
\omega &:= \Big(-\frac{1}{f_1}\alpha_1 + dx_1\Big)\Big(-\frac{1}{f_2}\alpha_2 + dx_2\Big)\cdots\Big(-\frac{1}{f_n}\alpha_n + dx_n\Big) \\
&= (-1)^n \frac{1}{f_1 \cdots f_n}(\alpha_1 - f_1 dx_1)(\alpha_2 - f_2 dx_2)\cdots(\alpha_n - f_n dx_n) \\
&\in \mathcal{C} \otimes_{Q_\mathfrak{m}} \big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big)
\end{aligned}
$$

is a cocycle, and it maps to $dx_1 \wedge \cdots \wedge dx_n \in H_n(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df)$ via (4.35). Therefore, the composition

$$
\begin{aligned}
\frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \times \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} &\xrightarrow{\cong} H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \\
&\xrightarrow{\mathrm{id}\times(-1)^n} H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \times H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, df\big) \\
&\xleftarrow{\cong} H_n\big(\mathcal{C} \otimes_{Q_\mathfrak{m}} \big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big)\big) \times H_n\big(\Omega^\bullet_{Q_\mathfrak{m}/k}, df\big) \\
&\xrightarrow{\text{Künneth}} H_{2n}\big(\mathcal{C} \otimes_{Q_\mathfrak{m}} \big(\Omega^\bullet_{Q_\mathfrak{m}/k}, -df\big) \otimes_{Q_\mathfrak{m}} \big(\Omega^\bullet_{Q_\mathfrak{m}/k}, df\big)\big)
\end{aligned}
$$

sends $(g dx_1 \cdots dx_n, h dx_1 \cdots dx_n)$ to

$$g \prod_i \left(-\frac{1}{f_i} \alpha_i + dx_i \right) \otimes (-1)^n h dx_1 \wedge \cdots \wedge dx_n.$$

Under the composition

$$\begin{aligned} &H_{2n}(\mathscr{C} \otimes_{Q_\mathfrak{m}} (\Omega^\bullet_{Q_\mathfrak{m}/k}, -df) \otimes_{Q_\mathfrak{m}} (\Omega^\bullet_{Q_\mathfrak{m}/k}, df)) \\ &\xrightarrow{\wedge} H_{2n}(\mathscr{C} \otimes_{Q_\mathfrak{m}} (\Omega^\bullet_{Q_\mathfrak{m}/k}, 0)) \xrightarrow{\cong} E \otimes_{Q_\mathfrak{m}} \Omega^n_{Q_\mathfrak{m}/k}, \end{aligned}$$

this element maps to

$$\frac{gh}{f_1 \cdots f_n} \otimes dx_1 \wedge \cdots \wedge dx_n,$$

which is sent to $\operatorname{res}^G \left[\frac{gh dx_1 \cdots dx_n}{f_1, \ldots, f_n} \right] \in k$ by the residue map. ■

4.6. Relating the trace and residue maps

Our goal in this subsection is to prove the following theorem.

Theorem 4.36. *Let k be a field of characteristic 0, Q an essentially smooth k-algebra, and $\mathfrak{m}$ a k-rational maximal ideal of Q. Then the diagram*

$$\begin{array}{ccc} HH_0(mf^{\mathfrak{m}}(Q_\mathfrak{m}, 0)) & \xrightarrow{\varepsilon} & H_{2n}\mathbb{R}\Gamma_\mathfrak{m}(\Omega^\bullet_{Q_\mathfrak{m}/k}) \\ & {\scriptstyle (-1)^{n(n+1)/2}\,\mathrm{trace}} \searrow \quad \swarrow {\scriptstyle \mathrm{res}} & \\ & k & \end{array}$$

commutes, where $n = \dim(Q_\mathfrak{m})$.

Our strategy for proving this theorem is to reduce it to the very special case when $Q = k[x]$ and $\mathfrak{m} = (x)$ and then to prove it in that case via an explicit calculation.

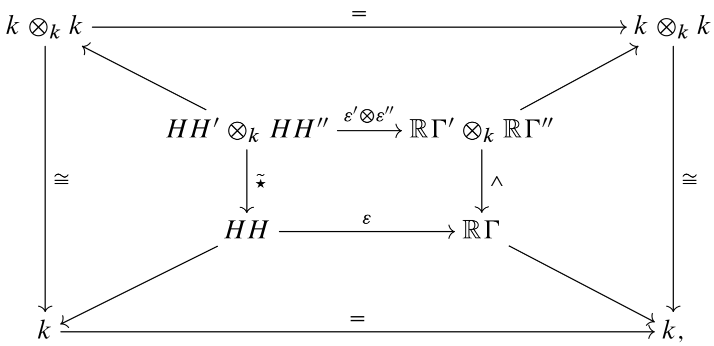
Lemma 4.37. *Given a pair $(Q, \mathfrak{m})$ and $(Q', \mathfrak{m}')$ satisfying the hypotheses of Theorem 4.36, suppose there is a k-algebra map $g : Q \to Q'$ such that $g^{-1}(\mathfrak{m}') = \mathfrak{m}$, the induced map $Q_\mathfrak{m} \to Q'_{\mathfrak{m}'}$ is flat, and $\mathfrak{m} Q'_{\mathfrak{m}'} = \mathfrak{m}' Q'_{\mathfrak{m}'}$. Then*

1. *Theorem 4.36 holds for $(Q, \mathfrak{m})$ if and only if it holds for $(Q', \mathfrak{m}')$;*
2. *Theorem 4.36 holds provided it holds in the special case where $Q = k[t_1, \ldots, t_n]$ and $\mathfrak{m} = (t_1, \ldots, t_n)$.*

Proof. (1) follows from Lemmas 4.20 and 4.31 and the naturality of the HKR map ε. As for (2), for $(Q, \mathfrak{m})$ as in Theorem 4.36, applying (1) to the map $g : Q \to Q_\mathfrak{m}$ allows us to reduce to the case when Q is local. In this case, let $x_1, \ldots, x_n$ be a regular system of parameters for Q, define $g : k[t_1, \ldots, t_n] \to Q$ to be the k-algebra map sending t_i to x_i, and apply (1) to g. ■

Lemma 4.38. *Suppose Q', Q'' are essentially smooth k-algebras, and $\mathfrak{m}' \subseteq Q'$, $\mathfrak{m}'' \subseteq Q''$ are k-rational maximal ideals. Let $Q = Q' \otimes_k Q''$ and $\mathfrak{m} = \mathfrak{m}' \otimes_k Q'' + Q' \otimes_k \mathfrak{m}''$. If Theorem 4.36 holds for each of $(Q', \mathfrak{m}')$ and $(Q'', \mathfrak{m}'')$, then it also holds for $(Q, \mathfrak{m})$. In particular, the theorem holds in general if it holds for the special case $Q = k[x]$, $\mathfrak{m} = (x)$.*

Proof. For brevity, let $HH' = HH_0(mf^{\mathfrak{m}'}(Q'_{\mathfrak{m}'}, 0))$, $HH'' = HH_0(mf^{\mathfrak{m}''}(Q''_{\mathfrak{m}''}, 0))$, and $HH_0 = HH_0(mf^{\mathfrak{m}}(Q_{\mathfrak{m}}, 0))$, and similarly $\mathbb{R}\Gamma' = H_{\dim(Q'_{\mathfrak{m}'})}\mathbb{R}\Gamma_{\mathfrak{m}'}(\Omega^{\bullet}_{Q'_{\mathfrak{m}'}/k})$, etc. We consider the diagram



where the diagonal maps are the appropriate trace or residue maps. The left and right trapezoids commute by Lemmas 4.21 and 4.32, the middle square commutes by Proposition 3.19, the top trapezoid commutes by assumption, and the outer square obviously commutes. It follows from (4.1) and Lemma 4.2 that $HH' \otimes_k HH'' \xrightarrow{\tilde{\star}} HH$ is an isomorphism. A diagram chase now shows that the bottom trapezoid commutes, which gives the first assertion. The second assertion is an immediate consequence of the first assertion and Lemma 4.37. ■

Proof of Theorem 4.36*.* By Lemma 4.38, we need only to show that

$$\operatorname{res} \circ \varepsilon = -\operatorname{trace}$$

in the case where $Q = k[x]$ and $\mathfrak{m} = (x)$. Let K be the Koszul complex on x, considered as a differential $\mathbb{Z}/2$-graded algebra, as in Section 4.1, and let $\mathcal{E} = \operatorname{End}_{mf^{(x)}(k[x]_{(x)}, 0)}(K_{(x)})$. Recall from Section 4.1 that $\mathcal{E}$ is the differential $\mathbb{Z}/2$-graded Q-algebra generated by odd degree elements e, e^* satisfying the relations $e^2 = 0 = (e^*)^2$ and $[e, e^*] = 1$, and the differential $d^{\mathcal{E}}$ is given by $d^{\mathcal{E}}(e) = x$ and $d^{\mathcal{E}}(e^*) = 0$. By Lemma 4.2, we have an isomorphism

$$k[y] \xrightarrow{\cong} HH_0\big(mf^{(x)}\big(k[x]_{(x)}, 0\big)\big),$$

where

$$y \mapsto \operatorname{id}_K[e^*] \in HH(\mathcal{E}) \subseteq HH\big(mf^{(x)}\big(k[x]_{(x)}, 0\big)\big),$$

and, more generally,

$$y^j \mapsto j!\operatorname{id}_K\big[\overbrace{e^*|\cdots|e^*}^{j}\big], \quad \text{for } j \geq 0.$$

As usual, we identify $H_2\mathbb{R}\Gamma_{(x)}(\Omega^\bullet_{k[x]_{(x)}/k})$ with $\frac{k[x]_{(x)}[x^{-1}]}{k[x]_{(x)}} \cdot \alpha \otimes_{k[x]_{(x)}} \Omega^1_{k[x]_{(x)}/k}$, where $|\alpha| = 1$. Theorem 4.36 follows from the calculations

(1) $\operatorname{res}(\frac{\alpha}{x} \otimes dx) = 1$,

(2) $\operatorname{res}(\frac{\alpha}{x^i} \otimes dx) = 0$ for all $i > 1$,

(3) $\operatorname{trace}(y^0) = 1$,

(4) $\operatorname{trace}(y^j) = 0$ for all $j \geq 1$, and

(5) $\varepsilon(y^j) = -j!(\frac{\alpha}{x^{j+1}} \otimes dx)$ for all $j \geq 0$.

In fact, (1) and (2) follow from the definition of the residue map, and (3) and (4) follow from Propositions 4.9 and 4.13, so it remains only to establish (5).

Recall that the map ε is induced by the diagram

$$k[y] \xrightarrow{\cong} HH_0(\mathcal{E}) \xleftarrow{\cong} H_2\mathbb{R}\Gamma_{(x)} HH(\mathcal{E}) \xrightarrow{\mathbb{R}\Gamma_{(x)}\varepsilon'} H_2\mathbb{R}\Gamma_{(x)}\big(\Omega^\bullet_{k[x]_{(x)}/k}\big), \tag{4.39}$$

where ε' denotes the composition

$$HH(\mathcal{E}) \xrightarrow{(\mathrm{id}, d_K)_*} HH^{II}(\mathcal{E}^0) \xrightarrow{\varepsilon^0} \Omega^\bullet_{k[x]_{(x)}/k}.$$

Here, $\mathcal{E}^0$ is the same as $\mathcal{E}$, but with trivial differential, (id, d_K) is a morphism $\mathcal{E} \to \mathcal{E}^0$ of curved dga's (with trivial curvature), and ε^0 is as defined in 3.2.3.

We need to calculate the inverse of the isomorphism $H_2\mathbb{R}\Gamma_{(x)} HH(\mathcal{E}) \xrightarrow{\cong} HH_0(\mathcal{E})$ occurring in (4.39). As usual, we make the identification

$$\mathbb{R}\Gamma_{(x)} HH(\mathcal{E}) = HH(\mathcal{E}) \oplus HH(\mathcal{E})[1/x] \cdot \alpha.$$

The differential on the right is $\partial := b + \alpha$, where α denotes left multiplication by α; note that $\alpha^2 = 0$. So, for a class $\gamma + \gamma'\alpha$, we have

$$\partial(\gamma + \alpha\gamma') = b(\gamma) - b(\gamma')\alpha + \gamma\alpha.$$

With this notation, the quasi-isomorphism $\mathbb{R}\Gamma_{(x)} HH(\mathcal{E}) \xrightarrow{\simeq} HH(\mathcal{E})$ is given by setting $\alpha = 0$.

For $j \geq 0$, we define

$$y^{(j)} = \frac{1}{j!} y^j = \mathrm{id}_K\big[\overbrace{e^*|e^*|\cdots|e^*}^{j \text{ terms}}\big]$$

and

$$\omega_j = e\big[\overbrace{e^*|e^*|\cdots|e^*}^{j \text{ terms}}\big] \in HH(\mathcal{E})[1/x].$$

Then, for $j \geq 0$, we have

$$b(\omega_j) = xy^{(j)} - y^{(j-1)},$$

where $y^{(-1)} := 0$, from which we get

$$b\left(\frac{1}{x}\omega_j + \frac{1}{x^2}\omega_{j-1} + \cdots + \frac{1}{x^{j+1}}\omega_0\right) = y^{(j)}.$$

It follows that, for each $j \geq 0$, the class

$$y^{(j)} + \alpha\left(\frac{1}{x}\omega_j + \frac{1}{x^2}\omega_{j-1} + \cdots \frac{1}{x^{j+1}}\omega_0\right)$$

is a cycle in $\mathbb{R}\Gamma_{(x)}HH(\mathcal{E})$ that maps to $y^{(j)} \in HH(\mathcal{E})$ under the canonical map $\mathbb{R}\Gamma_{(x)}(HH(\mathcal{E})) \to HH(\mathcal{E})$. We conclude that the inverse of

$$H_2\mathbb{R}\Gamma_{(x)}HH(\mathcal{E}) \xrightarrow{\cong} HH_0(\mathcal{E}) = k[y]$$

maps y^j to the class of

$$\eta_j := y^j + j!\alpha\left(\frac{1}{x}\omega_j + \frac{1}{x^2}\omega_{j-1} + \cdots + \frac{1}{x^{j+1}}\omega_0\right)$$

for each $j \geq 0$, and hence

$$\varepsilon(y^j) = \mathbb{R}\Gamma_{(x)}\varepsilon'(\eta_j).$$

Recall that ε' sends $\theta_0[\theta_1|\cdots|\theta_n] \in HH(\mathcal{E})$ to

$$\sum(-1)^{j_0+\cdots+j_n}\frac{1}{(n+J)!}\operatorname{str}\left(\theta_0(d_K')^{j_0}\theta_1'\cdots\theta_n'(d_K')^{j_n}\right),$$

where the derivatives are computed relative to any specified flat connection on K. Using the Levi-Civita connection associated to the basis $\{1, e\}$ of K, we get $e' = 0$, $(e^*)' = 0$, and hence $d_K' = -e^*dx$. It follows that

$$\begin{aligned}
\varepsilon'(\omega_j) &= 0 \quad \text{for } j \geq 1,\\
\varepsilon'(\omega_0) &= \operatorname{str}(e) + \operatorname{str}(ee^*dx),\\
\varepsilon'(y^{(j)}) &= 0 \quad \text{for } j \geq 1, \text{ and}\\
\varepsilon'(y^{(0)}) &= \operatorname{str}(\mathrm{id}_K) + \operatorname{str}(e^*dx).
\end{aligned}$$

It is easy to see that $\operatorname{str}(ee^*) = -1$, $\operatorname{str}(e^*) = 0$, $\operatorname{str}(e) = 0$, and $\operatorname{str}(\mathrm{id}_K) = 0$, so that $\varepsilon'(\omega_0) = -dx$, $\varepsilon'(\omega_j) = 0$ for all $j \geq 0$, and $\varepsilon'(y^j) = 0$ for all j. We obtain

$$\varepsilon(y^j) = \mathbb{R}\Gamma_{(x)}\varepsilon'(\eta_j) = -j!\left(\frac{\alpha}{x^{j+1}} \otimes dx\right)$$

for all $j \geq 0$, as needed. ∎

4.7. Proof of the conjecture

Let $Q = \mathbb{C}[x_1, \dots, x_n]$ and $f \in \mathfrak{m} = (x_1, \dots, x_n) \subseteq Q$, and assume $\mathfrak{m}$ is the only singular point of the morphism $f : \operatorname{Spec}(Q) \to \mathbb{A}^1$. As discussed in the introduction, a result of Shklyarov [16, Corollary 2] states that there is a commutative diagram

$$\begin{array}{ccc} HH_n\big(mf(Q,f)\big)^{\times 2} & \xrightarrow[\cong]{I_f(0)\times I_f(0)} & \left(\dfrac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}}\right)^{\times 2} \\ & {\scriptstyle c_f\eta_{mf}}\searrow \qquad \swarrow{\scriptstyle \langle -,-\rangle_{\mathrm{res}}} & \\ & \mathbb{C}, & \end{array} \tag{4.40}$$

for some constant c_f which possibly depends on f.

Theorem 4.41. *Let k be a field of characteristic 0, Q an essentially smooth k-algebra, $\mathfrak{m}$ a k-rational maximal ideal, and f an element of $\mathfrak{m}$ such that $\mathfrak{m}$ is the only singularity of the morphism $f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k$. Then the diagram*

$$\begin{array}{ccc} HH_n\big(mf(Q,f)\big)^{\times 2} & \xrightarrow[\cong]{\varepsilon\times\varepsilon} & \left(\dfrac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}}\right)^{\times 2} \\ & {\scriptstyle (-1)^{n(n+1)/2}\eta_{mf}}\searrow \qquad \swarrow{\scriptstyle \langle -,-\rangle_{\mathrm{res}}} & \\ & k & \end{array}$$

commutes.

Proof. Consider the diagram

$$\begin{array}{ccc} HH_n(mf(Q,f))\times HH_n(mf(Q,f)) & \xrightarrow{\varepsilon\times\varepsilon} & H_n(\Omega^\bullet_Q, -df)\times H_n(\Omega^\bullet_Q, -df) \\ \downarrow{\scriptstyle \mathrm{id}\times\Psi} & & \downarrow{\scriptstyle \mathrm{id}\times(-1)^n} \\ HH_n(mf(Q,f))\times HH_n(mf(Q,-f)) & \xrightarrow{\varepsilon\times\varepsilon} & H_n(\Omega^\bullet_Q, -df)\times H_n(\Omega^\bullet_Q, df) \\ \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \wedge} \\ HH_{2n}(mf^{\mathfrak{m}}(Q_{\mathfrak{m}},0)) & \xrightarrow{\varepsilon} & H_{2n}\mathbb{R}\Gamma_{\mathfrak{m}}(\Omega^\bullet_{Q_{\mathfrak{m}}}) \\ & {\scriptstyle (-1)^{n(n+1)/2}\,\mathrm{trace}}\searrow \qquad \swarrow{\scriptstyle \mathrm{res}} & \\ & k. & \end{array} \tag{4.42}$$

The top square commutes by Lemma 3.14, the square in the middle commutes by Corollary 3.26, and the triangle at the bottom commutes by Theorem 4.36. By Lemma 4.23, the map

$$HH_n\big(mf(Q,f)\big) \times HH_n\big(mf(Q,f)\big) \to k$$

obtained by composing the maps along the left edge of (4.42) is $(-1)^{n(n+1)/2}\eta_{mf}$. By Proposition 4.34, the map

$$\left(\frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}}\right)^{\times 2} = H_n(\Omega^\bullet_Q, -df)^{\times 2} \to k$$

obtained by composing the maps along the right edge of (4.42) is $\langle -, - \rangle_{\mathrm{res}}$. ∎

Corollary 4.43. *Conjecture 1.4 holds. That is, for $f \in \mathfrak{m} = (x_1, \dots, x_n) \subseteq Q = \mathbb{C}[x_1, \dots, x_n]$ such that $\mathfrak{m}$ is the only singularity of the morphism $f : \operatorname{Spec}(Q) \to \mathbb{A}^1_k$, the unique constant c_f that makes diagram (1.3) commute is $(-1)^{n(n+1)/2}$, as predicted by Shklyarov.*

Proof. Under these assumptions, $\varepsilon = I_f(0)$ by Lemma 3.11. Theorem 4.41 thus implies that the value $c_f = (-1)^{n(n+1)/2}$ causes the diagram (4.40) to commute. As discussed in the introduction, this uniquely determines the value of c_f, and the unique constant c_f which makes diagram (4.40) commute is the same as that which makes diagram (1.3) commute. ∎

5. Recovering Polishchuk–Vaintrob's Hirzebruch–Riemann–Roch formula for matrix factorizations

Assume k, Q, $\mathfrak{m}$, and f are as in the statement of Theorem 4.41. We recall that, given objects $X, Y \in mf(Q, f)$, the *Euler pairing* applied to the pair (X, Y) is given by

$$\chi(X, Y) = \dim_k H_0 \operatorname{Hom}(X, Y) - \dim_k H_1 \operatorname{Hom}(X, Y).$$

In this final section, we give a new proof of a theorem due to Polishchuk–Vaintrob that relates the Euler pairing to the residue pairing via the Chern character map.

The following is an immediate consequence of the commutativity of diagram (4.42) in the proof of Theorem 4.41.

Corollary 5.1. *Let k, Q, $\mathfrak{m}$, and f be as in the statement of Theorem 4.41, and assume $n = \dim(Q_{\mathfrak{m}})$ is even. Then the triangle*

$$\begin{array}{ccc} HH_0(mf(Q,f)) \otimes_k HH_0(mf(Q,-f)) & \xrightarrow{\varepsilon \otimes_k \varepsilon} & \dfrac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \otimes_k \dfrac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}} \\ & \searrow \quad k \quad \swarrow_{\langle -,- \rangle_{\mathrm{res}}} & \end{array}$$

commutes, where the left diagonal map is $(-1)^{n(n+1)/2}\operatorname{trace} \circ (- \star -)$, and ε denotes the composition of the HKR map and the isomorphism $H_n(\Omega^\bullet_{Q/k}, \pm df) \xrightarrow{\cong} \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}}$.

Let $X \in mf(Q, f)$. We recall that the *Chern character of* X

$$\mathrm{ch}(X) \in HH_0\big(mf(Q, f)\big)$$

is the class represented by

$$\mathrm{id}_X[\,] \in \mathrm{End}(X) \subseteq HH\big(mf(Q, f)\big).$$

Assume now that n is even. The isomorphism

$$\varepsilon : HH_0\big(mf(Q, f)\big) \xrightarrow{\cong} \frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}}$$

sends $\mathrm{ch}(X)$ to the class

$$\frac{1}{n!}\,\mathrm{str}\,\big((\delta'_X)^n\big),$$

where $\delta'_X = [\nabla, \delta_X]$ for any choice of connection ∇ on X. Abusing notation, we also denote this element of $\frac{\Omega^n_{Q/k}}{df \wedge \Omega^{n-1}_{Q/k}}$ as $\mathrm{ch}(X)$.

For example, if the components of X are free, then, upon choosing bases, we may represent δ_X as a pair of square matrices (A, B) satisfying $AB = f I_r = BA$. Using the Levi-Civita connection associated to this choice of basis, we have

$$\mathrm{ch}(X) = \frac{2}{n!}\,\mathrm{tr}\,\big(\overbrace{dAdB \cdots dAdB}^{n \text{ factors}}\big). \tag{5.2}$$

Recall from Remark 4.12 that, for $X \in mf(Q, f)$ and $Y \in mf(Q, -f)$, $\theta(X, Y)$ is given by

$$\dim_k H_0(X \otimes Y) - \dim_k H_1(X \otimes Y),$$

and we have

$$\theta(X, Y) = \mathrm{trace}\,\big(\mathrm{ch}(X) \star \mathrm{ch}(Y)\big). \tag{5.3}$$

Corollary 5.4. *Under the assumptions of Corollary 5.1,*

(1) *if* $X \in mf(Q, f)$ *and* $Y \in mf(Q, -f)$, *then*

$$\theta(X, Y) = (-1)^{\binom{n}{2}}\big\langle \mathrm{ch}(X), \mathrm{ch}(Y)\big\rangle_{\mathrm{res}};$$

(2) *if* $X, Y \in mf(Q, f)$, *then*

$$\chi(X, Y) = (-1)^{\binom{n}{2}}\big\langle \mathrm{ch}(X), \mathrm{ch}(Y)\big\rangle_{\mathrm{res}}.$$

Remark 5.5. Corollary 5.4 (2) is Polishchuk–Vaintrob's Hirzebruch–Riemann–Roch formula for matrix factorizations [10, Theorem 4.1.4 (i)].

Proof. (1) is immediate from Corollary 5.1 and (5.3). We now prove (2). Without loss of generality, we may assume Q is local, so that the underlying $\mathbb{Z}/2$-graded Q-modules of X and Y are free. Given a matrix factorization $(P, \delta_P) \in mf(Q, f)$ written in terms of its $\mathbb{Z}/2$-graded components as

$$(\delta_1 : P_1 \to P_0, \delta_0 : P_0 \to P_1),$$

we define a matrix factorization $N(P, \delta_P) \in mf(Q, -f)$ with components

$$(\delta_1 : P_1 \to P_0, -\delta_0 : P_0 \to P_1).$$

We have

$$\big\langle \mathrm{ch}(X), \mathrm{ch}\,\big(N(Y)\big)\big\rangle_{\mathrm{res}} = (-1)^{\binom{n}{2}} \theta\big(X, N(Y)\big) = \chi(X, Y).$$

The first equality follows from (1), and the second equality follows from [3, Corollary 8.5] and [1, Proposition 3.18]; note that $(-1)^{\binom{n}{2}} = (-1)^{n/2}$, since n is even, and also that the notation χ in [1, Proposition 3.18] has a different meaning than it does here. It suffices to show $\mathrm{ch}(N(Y)) = (-1)^{n/2}\,\mathrm{ch}(Y)$, and this is clear by (5.2). ■

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