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Homotopy Rota–Baxter operators and post-Lie algebras

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Abstract. Rota–Baxter operators and the more general $\mathcal{O}$-operators, together with their interconnected pre-Lie and post-Lie algebras, are important algebraic structures, with Rota–Baxter operators and pre-Lie algebras instrumental in the Connes–Kreimer approach to renormalization of quantum field theory. This paper introduces the notions of a homotopy Rota–Baxter operator and a homotopy $\mathcal{O}$-operator on a symmetric graded Lie algebra. Their characterization by Maurer–Cartan elements of suitable differential graded Lie algebras is provided. Through the action of a homotopy $\mathcal{O}$-operator on a symmetric graded Lie algebra, we arrive at the notion of an operator homotopy post-Lie algebra, together with its characterization in terms of Maurer–Cartan elements. A cohomology theory of post-Lie algebras is established, with an application to 2-term skeletal operator homotopy post-Lie algebras.

1. Introduction

This paper carries out a homotopy study of Rota–Baxter operators, $\mathcal{O}$-operators (also called relative Rota–Baxter operators and generalized Rota–Baxter operators), and the related pre-Lie algebras and post-Lie algebras.

1.1. Background

Homotopy began with a fundamental notion in topology describing the continuous deformation of one function to another.

The first homotopy construction in algebra is the A_∞-algebra of Stasheff, arising from his work on homotopy characterization of connected based loop spaces [44]. Later related developments include the work of Boardman and Vogt [8] about E_∞-spaces on the infinite loop space, the work of Schlessinger and Stasheff [40] about L_∞-algebras on perturbations of rational homotopy types, and the work of Chapoton and Livernet [13] about pre-Lie$_\infty$-algebras. Homotopy of a large class of algebraic structures was obtained in the context of operads [31].

Homotopy structures, A_∞- and L_∞-algebras in particular, have many applications in mathematics and physics, in supergravity, string theory [6, 43], and especially in noncommutative geometry [13, 19, 23, 26, 48].

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Rota–Baxter associative algebras originated from the probability study of G. Baxter [7] and later found applications in the Connes–Kreimer algebraic approach to renormalization of quantum field theory [12, 14, 22, 32]. Indeed, in the fundamental algebraic Birkhoff factorization for regularization maps in the Connes–Kreimer approach, Rota–Baxter algebra is the target of the regularization. Further, the algebraic Birkhoff factorization can be interpreted as a factorization in a larger Rota–Baxter algebra [17, 18].

A Rota–Baxter operator on a Lie algebra is naturally the operator form of a classical r-matrix [41] under certain conditions. To better understand such connection with the classical Yang–Baxter equation and the related integrable systems, the notion of an $\mathcal{O}$-operator on a Lie algebra was introduced [9, 28].

An $\mathcal{O}$-operator naturally gives rise to a pre-Lie or more generally a post-Lie algebra, and also found broad applications in mathematics and mathematical physics [3, 5, 13, 46, 47]. Incidentally, there is a pre-Lie algebra structure in Feynman graphs [15], as the domain of the regularization. Pre-Lie$_\infty$-algebras have been related to renormalization [24].

1.2. Approach of the paper

Given the importance of Rota–Baxter algebras and $\mathcal{O}$-operators, and homotopy theories, it is useful to study the homotopy of these structures and establish the relationship to homotopy pre-Lie and post-Lie algebras. Such a study might further improve the understanding of the roles played by these structures in renormalization and other applications.

The operadic framework of homotopy is not yet general enough to include Rota–Baxter algebras. We thus take the more classical approach via differential graded Lie algebras (dgLa) and Maurer–Cartan elements whose idea can be traced back to the well-known work of Gerstenhaber [21] on deformations.

This approach is based on the principle that objects of a certain algebraic structure on a vector space are given by degree 1 solutions of the Maurer–Cartan equation in a suitable dgLa built from the vector space. When the vector space is replaced by a graded vector space, similar solutions give objects in the homotopy algebraic structure.

For instance, for a vector space V, the graded vector space $\oplus_{n=0}^{+\infty} \mathrm{Hom}(\wedge^n V, V)$ together with the Nijenhuis–Richardson bracket $[\cdot,\cdot]_{\mathsf{NR}}$ is a dgLa with the trivial derivation. A Lie algebra structure on V is precisely a degree 1 solution $\omega \in \mathrm{Hom}(\wedge^2 V, V)$ of the Maurer–Cartan equation. For a graded vector space $V = \oplus_{i\in\mathbb{Z}} V^i$, let $S(V)$ denote the symmetric algebra of V and let $\mathrm{Hom}^n(S(V), V)$ denote the space of degree n linear maps. Then with a similarly defined Nijenhuis–Richardson bracket $[\cdot,\cdot]_{\mathsf{NR}}$ on the graded vector space $\oplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(S(V), V)$ (see equation (2.2)), we again have a graded Lie algebra (gLa) and (curved) L_∞-algebras can be characterized as degree 1 solutions of the corresponding Maurer–Cartan equation.

In a recent study [30, 45], this approach was taken to give a Maurer–Cartan characterization of $\mathcal{O}$-operators and relative Rota–Baxter Lie algebras, and further to establish a deformation theory and its controlling cohomology for $\mathcal{O}$-operators and relative Rota–Baxter Lie algebras.

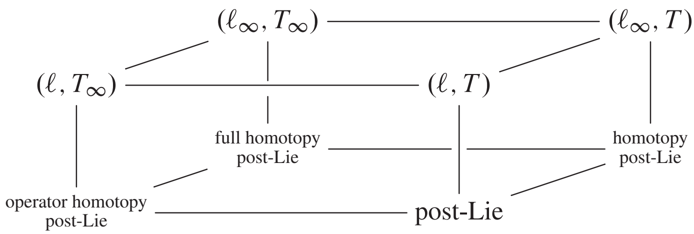
To illustrate our approach in a broader context, we focus on the Rota–Baxter Lie algebra for now and regard its algebraic structure as a pair (ℓ, T) consisting of a Lie bracket $\ell = [\cdot, \cdot]$ and a Rota–Baxter operator T. Then to obtain the homotopy of the Rota–Baxter Lie algebra, one can begin with taking homotopy of either the binary operation ℓ or the unary operation T. The homotopy of the Lie algebra $\ell_\infty = \{\ell_i\}_{i=1}^{+\infty}$ is well known as the L_∞-algebra. Together with the natural Rota–Baxter operator action as defined in [39], we have the Rota–Baxter homotopy Lie algebra (ℓ_∞, T) or Rota–Baxter L_∞-algebra. Similar to Rota–Baxter Lie algebra, a Rota–Baxter homotopy Lie algebra is expected to induce a homotopy post-Lie algebra, giving rise to the commutative diagram

$$\begin{array}{ccc} (\ell, T) & \longrightarrow & (\ell_\infty, T) \\ \Big\downarrow{\scriptstyle \text{RB action}} & & \Big\downarrow{\scriptstyle \text{RB action}} \\ \text{post-Lie} & \longrightarrow & \text{homotopy post-Lie} \end{array} \tag{1.1}$$

where the horizontal arrows are taking homotopy and the vertical arrows are taking actions of the Rota–Baxter operators.

In this paper, we will pursue the other direction, by taking homotopy of the Rota–Baxter operator T and obtain $T_\infty := \{T_i\}_{i=0}^{+\infty}$, without taking homotopy of ℓ. We call the resulting structure (ℓ, T_∞) the *operator homotopy Rota–Baxter Lie algebra* to distinguish it from the above-mentioned Rota–Baxter homotopy Lie algebra. The action of T_∞ gives rise to a variation of the homotopy post-Lie algebra which we will call the *operator homotopy post-Lie algebra* (see Remark 3.5). This gives another commutative diagram shown as the front rectangle in equation (1.2) while the diagram in equation (1.1) is embedded as the right rectangle.

Eventually, the *full* homotopy of the Rota–Baxter Lie algebra should come from the combined homotopies of both the Lie algebra structure and the Rota–Baxter operator structure, tentatively denoted by (ℓ_∞, T_∞) and called the *full homotopy Rota–Baxter Lie algebra*. A suitable action of T_∞ on ℓ_∞ should give the *full homotopy post-Lie algebra* whose structure is still mysterious. These various homotopies of the Rota–Baxter Lie algebra, as well as their derived homotopies of the post-Lie algebra, could be put together and form the diagram



(1.2)

where going in the left and inside directions should be taking various homotopy, and going downward should be taking the actions of (homotopy) Rota–Baxter operators.

1.3. Outline of the paper

After some background on dgLas and Maurer–Cartan equations summarized in Section 2.1, we introduce in Section 2.2 the notion of a homotopy $\mathcal{O}$-operator with any weight (Definition 2.9). Using a derived bracket, we construct a dgLa which characterizes homotopy $\mathcal{O}$-operators of any weight as their Maurer–Cartan elements (Theorem 2.16).

In Section 3, by applying a homotopy $\mathcal{O}$-operator to a symmetric graded Lie algebra (sgLa), we obtain a variation of the homotopy post-Lie algebra, called the operator homotopy post-Lie algebra. From here we can specialize in several directions and obtain interesting applications. First, when the weight of the $\mathcal{O}$-operator is taken to be zero, we obtain homotopy $\mathcal{O}$-operators of weight zero. Since $\mathcal{O}$-operators of weight zero naturally derive pre-Lie algebras [39], it is expected that homotopy $\mathcal{O}$-operators of weight zero derive pre-Lie$_\infty$-algebras. We confirm this in Corollary 3.12, yielding the commutative diagram

$$\begin{array}{ccc} \mathcal{O}\text{-operators} & \xrightarrow{\text{homotopy}} & \text{homotopy } \mathcal{O}\text{-operators} \\ \downarrow & & \downarrow \\ \text{pre-Lie} & \xrightarrow{\text{homotopy}} & \text{pre-Lie}_\infty. \end{array} \tag{1.3}$$

In other words, the compositions of taking homotopy and taking operator action in either order give the pre-Lie$_\infty$-algebras.

Another application of our general construction is the characterization of post-Lie algebra structures on any given Lie algebra using Maurer–Cartan elements in a suitable dgLa (Corollary 3.7). There has been quite much interest on such constructions in the recent literature [10, 11, 20, 38].

In Section 4, we first consider the cohomology theory of post-Lie algebras. In the "abelian" case of pre-Lie algebras, the cohomology groups were first defined in [16] by derived functors and then in [34] by resolutions of algebras from the operadic Koszul duality theory. We explicitly establish a cohomology theory of post-Lie algebras which specializes to the above cohomology theory of pre-Lie algebras. The third cohomology group of a post-Lie algebra is applied in Section 4.3 to classify 2-term skeletal operator homotopy post-Lie algebras.

Notations. We assume that all the vector spaces are over a field of characteristic zero. For a homogeneous element x in a $\mathbb{Z}$-graded vector space, we also use x in the exponent, as in $(-1)^x$, to denote its degree in order to simplify the notation.

2. Homotopy $\mathcal{O}$-operators of weight λ

In this section, we introduce the notion of a homotopy $\mathcal{O}$-operator of weight λ, where λ is a constant. We construct a dgLa and show that homotopy $\mathcal{O}$-operators of weight λ can be characterized as its Maurer–Cartan elements to justify our definition.

2.1. Maurer–Cartan elements and Nijenhuis–Richardson brackets

We first recall some background needed in later sections.

Definition 2.1 ([31]). Let $(\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k, [\cdot,\cdot], d)$ be a dgLa. A degree 1 element $\theta \in \mathfrak{g}^1$ is called a *Maurer–Cartan element* of $\mathfrak{g}$ if it satisfies the *Maurer–Cartan equation*

$$d\theta + \frac{1}{2}[\theta,\theta] = 0. \tag{2.1}$$

A permutation $\sigma \in \mathbb{S}_n$ is called an $(i,n-i)$-shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. If $i = 0$ or n, we assume that $\sigma = \mathrm{Id}$. The set of all $(i,n-i)$-shuffles will be denoted by $\mathbb{S}_{(i,n-i)}$. The notion of an $(i_1,\ldots,i_k)$-shuffle and the set $\mathbb{S}_{(i_1,\ldots,i_k)}$ are defined analogously.

Let $V = \oplus_{k\in\mathbb{Z}}V^k$ be a $\mathbb{Z}$-graded vector space. We will denote by $S(V)$ the symmetric algebra of V. That is, $S(V) := T(V)/I$, where $T(V)$ is the tensor algebra and I is the 2-sided ideal of $T(V)$ generated by all homogeneous elements of the form

$$x \otimes y - (-1)^{xy} y \otimes x.$$

We will write

$$S(V) = \oplus_{i=0}^{+\infty} S^i(V).$$

Denote the product of homogeneous elements $v_1,\ldots,v_n \in V$ in $S^n(V)$ by $v_1 \odot \cdots \odot v_n$. The degree of $v_1 \odot \cdots \odot v_n$ is by definition the sum of the degree of v_i. For a permutation $\sigma \in \mathbb{S}_n$ and $v_1,\ldots,v_n \in V$, the Koszul sign $\varepsilon(\sigma; v_1,\ldots,v_n) \in \{-1,1\}$ is defined by

$$v_1 \odot \cdots \odot v_n = \varepsilon(\sigma; v_1,\ldots,v_n) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}$$

and the antisymmetric Koszul sign $\chi(\sigma; v_1,\ldots,v_n) \in \{-1,1\}$ is defined by

$$\chi(\sigma; v_1,\ldots,v_n) := (-1)^{\sigma}\varepsilon(\sigma; v_1,\ldots,v_n).$$

Denote by $\mathrm{Hom}^n(S(V),V)$ the space of degree n linear maps from the graded vector space $S(V)$ to the graded vector space V. Obviously, an element $f \in \mathrm{Hom}^n(S(V),V)$ is the sum of $f_i : S^i(V) \to V$. We will write $f = \sum_{i=0}^{+\infty} f_i$. Set $C^n(V,V) := \mathrm{Hom}^n(S(V),V)$ and $C^*(V,V) := \oplus_{n\in\mathbb{Z}}C^n(V,V)$. As the graded version of the classical Nijenhuis–Richardson bracket given in [37], the *Nijenhuis–Richardson bracket* $[\cdot,\cdot]_{\mathsf{NR}}$ on the graded vector space $C^*(V,V)$, for any $f = \sum_{i=0}^{+\infty} f_i \in C^m(V,V)$, $g = \sum_{j=0}^{+\infty} g_j \in C^n(V,V)$, is given by

$$[f,g]_{\mathsf{NR}} := f \circ g - (-1)^{mn} g \circ f, \tag{2.2}$$

where $f \circ g \in C^{m+n}(V,V)$ is defined by

$$f \circ g = \Big(\sum_{i=0}^{+\infty} f_i\Big) \circ \Big(\sum_{j=0}^{+\infty} g_j\Big) := \sum_{k=0}^{+\infty}\Big(\sum_{i+j=k+1} f_i \circ g_j\Big), \tag{2.3}$$

while $f_i \circ g_j \in \mathrm{Hom}(S^{i+j-1}(V), V)$ is defined by

$$\begin{aligned}(f_i \circ g_j)(v_1, \dots, v_{i+j-1}) \\ = \sum_{\sigma \in \mathbb{S}_{(j,i-1)}} \varepsilon(\sigma) f_i\big(g_j(v_{\sigma(1)}, \dots, v_{\sigma(j)}), v_{\sigma(j+1)}, \dots, v_{\sigma(i+j-1)}\big),\end{aligned}$$

with the convention that $f_0 \circ g_j := 0$[1] and $(f_j \circ g_0)(v_1, \dots, v_{j-1}) := f_j(g_0, v_1, \dots, v_{j-1})$.

See [25, 33] for the notion of a curved L_∞-algebra.

Theorem 2.2 ([1, 33]). *With notations as above, $(C^*(V,V), [\cdot,\cdot]_{\mathsf{NR}})$ is a graded Lie algebra* (gLa). *Its Maurer–Cartan elements $\sum_{i=0}^{+\infty} l_i$ are the curved L_∞-algebra structures on V.*

We denote a curved L_∞-algebra by $(V, \{l_k\}_{k=0}^{+\infty})$. A curved L_∞-algebra $(V, \{l_k\}_{k=0}^{+\infty})$ with $l_0 = 0$ is exactly an L_∞-algebra [29].

Definition 2.3 ([1]). A *symmetric graded Lie algebra (sgLa)* is a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ equipped with a bilinear bracket $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ of degree 1, satisfying

(1) (*graded symmetry*)

$$[x, y]_{\mathfrak{g}} = (-1)^{xy}[y, x]_{\mathfrak{g}},$$

(2) (*graded Leibniz rule*)

$$\big[x, [y, z]_{\mathfrak{g}}\big]_{\mathfrak{g}} = (-1)^{x+1}\big[[x, y]_{\mathfrak{g}}, z\big]_{\mathfrak{g}} + (-1)^{(x+1)(y+1)}\big[y, [x, z]_{\mathfrak{g}}\big]_{\mathfrak{g}}.$$

Here x, y, z are homogeneous elements in $\mathfrak{g}$, which also denote their degrees when in exponent.

An sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is just a curved L_∞-algebra $(\mathfrak{g}, \{l_k\}_{k=0}^{+\infty})$, in which $l_k = 0$ for all $k \geq 0, k \neq 2$.

For a graded vector space $V = \oplus_{i \in \mathbb{Z}} V^i$, the *suspension operator* $s : V \mapsto sV$ assigns V to the graded vector space $sV = \oplus_{i \in \mathbb{Z}} (sV)^i$ with $(sV)^i := V^{i-1}$. The natural degree 1 map $s : V \to sV$ is the identity map of the underlying vector space, sending $v \in V$ to its suspended copy $sv \in sV$. Likewise, the *desuspension operator* s^{-1} changes the grading of V according to the rule $(s^{-1}V)^i := V^{i+1}$. The degree -1 map $s^{-1} : V \to s^{-1}V$ is defined in the obvious way.

Example 2.4. Let V be a graded vector space. Then $s^{-1}\mathfrak{gl}(V)$ is an sgLa where the symmetric Lie bracket, for any $f \in \mathrm{Hom}^m(V,V)$, $g \in \mathrm{Hom}^n(V,V)$, is given by

$$[s^{-1}f, s^{-1}g] := (-1)^m s^{-1}[f, g]. \tag{2.4}$$

[1] The linear map f_0 is just a distinguished element $\Phi \in V^0$.

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'})$ be sgLas. A *homomorphism* from $\mathfrak{g}$ to $\mathfrak{g}'$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ of degree 0 such that

$$\phi\big([x_1, x_2]_{\mathfrak{g}}\big) = \big[\phi(x_1), \phi(x_2)\big]_{\mathfrak{g}'}, \quad \forall x_1, x_2 \in \mathfrak{g}.$$

Definition 2.5. A linear map of graded vector spaces $D : \mathfrak{g} \to \mathfrak{g}$ of degree n is called a *derivation of degree n* on an sgLa $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ if

$$D[x, y]_{\mathfrak{g}} = (-1)^n [Dx, y]_{\mathfrak{g}} + (-1)^{n(x+1)}[x, Dy]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}.$$

We denote the vector space of derivations of degree n by $\mathrm{Der}^n(\mathfrak{g})$. Denote $\mathrm{Der}(\mathfrak{g}) = \oplus_{n\in\mathbb{Z}} \mathrm{Der}^n(\mathfrak{g})$, which is a graded vector space.

Remark 2.6. A derivation of degree n on an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is just a homotopy derivation $\{\theta_k\}_{k=1}^{+\infty}$ of degree n on $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, in which $\theta_k = 0$ for all $k \geq 2$. See [25] for more details.

By a straightforward check, we obtain the following proposition.

Proposition 2.7. *With the above notations,* $(s^{-1}\mathrm{Der}(\mathfrak{g}), [\cdot,\cdot])$ *is a symmetric graded Lie subalgebra of* $(s^{-1}\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot])$*, where the bracket* $[\cdot,\cdot]$ *is defined by* (2.4).

Definition 2.8. An *action* of an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on an sgLa$(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ is a homomorphism of graded vector spaces $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ of degree 1 such that $s^{-1} \circ \rho : \mathfrak{g} \to s^{-1}\mathrm{Der}(\mathfrak{h})$ is an sgLa homomorphism.

In particular, if $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ is abelian, we obtain an action of an sgLa on a graded vector space. It is obvious that $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is an action of the sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on itself, which is called the *adjoint action*.

Let ρ be an action of an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on a graded vector space V. For $x \in \mathfrak{g}^i$, we have $\rho(x) \in \mathrm{Hom}^{i+1}(V, V)$. Moreover, there is an sgLa structure on the direct sum $\mathfrak{g} \oplus V$, for any $x_1, x_2 \in \mathfrak{g}, v_1, v_2 \in V$, given by

$$[x_1 + v_1, x_2 + v_2]_{\rho} := [x_1, x_2]_{\mathfrak{g}} + \rho(x_1)v_2 + (-1)^{x_1x_2}\rho(x_2)v_1.$$

This sgLa is called the *semidirect product* of the sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and $(V;\rho)$, and denoted by $\mathfrak{g} \ltimes_{\rho} V$.

2.2. Homotopy $\mathcal{O}$-operators of weight λ

Now we give the main notion of this paper.

Definition 2.9. Let ρ be an action of an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on an sgLa$(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$. A degree 0 element

$$T = \sum_{i=0}^{+\infty} T_i \in \mathrm{Hom}(S(\mathfrak{h}), \mathfrak{g}) \quad \text{with } T_i \in \mathrm{Hom}(S^i(\mathfrak{h}), \mathfrak{g})$$

is called a *homotopy $\mathcal{O}$-operator of weight* λ on an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with respect to the action ρ if the following equalities hold for all $p \geq 0$ and all homogeneous elements $v_1, \dots, v_p \in \mathfrak{h}$:

$$\begin{aligned}
&\sum_{1\leq i<j\leq p} (-1)^{\alpha} T_{p-1}\big(\lambda[v_i, v_j]_{\mathfrak{h}}, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_p\big)\\
&+ \sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(l,1,p-l-1)}}} \varepsilon(\sigma) T_{k-1}\big(\rho\big(T_l(v_{\sigma(1)}, \dots, v_{\sigma(l)})\big)v_{\sigma(l+1)}, v_{\sigma(l+2)}, \dots, v_{\sigma(p)}\big)\\
&= \frac{1}{2}\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,l)}}} \varepsilon(\sigma)\big[T_{k-1}(v_{\sigma(1)}, \dots, v_{\sigma(k-1)}), T_l(v_{\sigma(k)}, \dots, v_{\sigma(p)})\big]_{\mathfrak{g}},
\end{aligned} \tag{2.5}$$

where

$$\alpha = v_i(v_1 + \cdots + v_{i-1}) + v_j(v_1 + \cdots + v_{j-1}) + v_i v_j.$$

Remark 2.10. The linear map T_0 is just an element $\Omega \in \mathfrak{g}^0$. Below are the generalized Rota–Baxter identities for $p = 0, 1, 2$:

$$\begin{aligned}
[\Omega, \Omega]_{\mathfrak{g}} &= 0,\\
T_1\big(\rho(\Omega)v_1\big) &= \big[\Omega, T_1(v_1)\big]_{\mathfrak{g}},\\
\big[T_1(v_1), T_1(v_2)\big]_{\mathfrak{g}} &= T_1\big(\rho\big(T_1(v_1)\big)v_2 + (-1)^{v_1v_2}\rho\big(T_1(v_2)\big)v_1 + \lambda[v_1, v_2]_{\mathfrak{h}}\big)\\
&\quad + T_2\big(\rho(\Omega)v_1, v_2\big) + (-1)^{v_1v_2} T_2\big(\rho(\Omega)v_2, v_1\big)\\
&\quad - \big[\Omega, T_2(v_1, v_2)\big]_{\mathfrak{g}}.
\end{aligned}$$

Remark 2.11. If the sgLa reduces to a Lie algebra and the action reduces to an action of a Lie algebra on another Lie algebra, the above definition reduces to the definition of an *$\mathcal{O}$-operator of weight* λ [9, 28] on a Lie algebra[2]. More precisely, the linear map $T : \mathfrak{h} \to \mathfrak{g}$ satisfies

$$[Tu, Tv]_{\mathfrak{g}} = T\big(\rho(Tu)(v) - \rho(Tv)(u) + \lambda[u, v]_{\mathfrak{h}}\big), \quad \forall u, v \in \mathfrak{h}.$$

Definition 2.12. A degree 0 element

$$R = \sum_{i=0}^{+\infty} R_i \in \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}) \quad \text{with } R_i \in \mathrm{Hom}(S^i(\mathfrak{g}), \mathfrak{g})$$

is called a *homotopy Rota–Baxter operator of weight* λ on an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ if the follow-

[2] An $\mathcal{O}$-operator was called a generalized Rota–Baxter operator in [46] and a relative Rota–Baxter operator in [39].

ing equalities hold for all $p \geq 0$ and all homogeneous elements $x_1, \dots, x_p \in \mathfrak{g}$,

$$\begin{aligned}
&\sum_{1 \leq i < j \leq p} (-1)^{\alpha} R_{p-1}\big(\lambda[x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_p\big) \\
&+ \sum_{\substack{k+l=p+1 \\ \sigma \in \mathbb{S}_{(l,1,p-l-1)}}} \varepsilon(\sigma) R_{k-1}\big(\big[R_l(x_{\sigma(1)}, \dots, x_{\sigma(l)}), x_{\sigma(l+1)}\big]_{\mathfrak{g}}, x_{\sigma(l+2)}, \dots, x_{\sigma(p)}\big) \\
&= \frac{1}{2} \sum_{\substack{k+l=p+1 \\ \sigma \in \mathbb{S}_{(k-1,l)}}} \varepsilon(\sigma) \big[R_{k-1}(x_{\sigma(1)}, \dots, x_{\sigma(k-1)}), R_l(x_{\sigma(k)}, \dots, x_{\sigma(p)})\big]_{\mathfrak{g}},
\end{aligned}$$

where $\alpha = x_i(x_1 + \cdots + x_{i-1}) + x_j(x_1 + \cdots + x_{j-1}) + x_i x_j$.

Remark 2.13. A homotopy Rota–Baxter operator

$$R = \sum_{i=0}^{+\infty} R_i \in \operatorname{Hom}\big(S(\mathfrak{g}), \mathfrak{g}\big)$$

of weight λ on an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a homotopy $\mathcal{O}$-operator of weight λ with respect to the adjoint action ad. If, moreover, the sgLa reduces to a Lie algebra, then the resulting linear operator $R : \mathfrak{g} \to \mathfrak{g}$ is a *Rota–Baxter operator of weight* λ [7, 22] in the sense that

$$\big[R(x), R(y)\big]_{\mathfrak{g}} = R\big(\big[R(x), y\big]_{\mathfrak{g}} + \big[x, R(y)\big]_{\mathfrak{g}} + \lambda[x, y]_{\mathfrak{g}}\big), \quad \forall x, y \in \mathfrak{g}.$$

In the sequel, we construct a dgLa and show that homotopy $\mathcal{O}$-operators of weight λ can be characterized as its Maurer–Cartan elements to justify our definition of homotopy $\mathcal{O}$-operators of weight λ. For this purpose, we recall the derived bracket construction of graded Lie algebras. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, d)$ be a dgLa. We define a new bracket on $s\mathfrak{g}$ by

$$[sx, sy]_d := (-1)^x s[dx, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \tag{2.6}$$

The new bracket is called the *derived bracket* [27]. It is well known that the derived bracket is a graded Leibniz bracket on the shifted graded space $s\mathfrak{g}$. Note that the derived bracket is not graded skew-symmetric in general. We recall a basic result.

Proposition 2.14 ([27]). *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, d)$ *be a* dgLa*, and let* $\mathfrak{h} \subset \mathfrak{g}$ *be a subalgebra which is abelian, i.e.,* $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{g}} = 0$*. If the derived bracket is closed on* $s\mathfrak{h}$*, then* $(s\mathfrak{h}, [\cdot,\cdot]_d)$ *is a* gLa.

Let ρ be an action of an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on an sgLa$(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$. Consider the graded vector space $C^*(\mathfrak{h}, \mathfrak{g}) := \oplus_{n \in \mathbb{Z}} \operatorname{Hom}^n(S(\mathfrak{h}), \mathfrak{g})$. Define a linear map

$$\mathrm{d} : \operatorname{Hom}^n\big(S(\mathfrak{h}), \mathfrak{g}\big) \to \operatorname{Hom}^{n+1}\big(S(\mathfrak{h}), \mathfrak{g}\big)$$

by

$$(\mathrm{d}g)_p(v_1, \dots, v_p) = \sum_{1 \leq i < j \leq p} (-1)^{\alpha} g_{p-1}\big(\lambda[v_i, v_j]_{\mathfrak{h}}, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_p\big), \tag{2.7}$$

where

$$\alpha = n + 1 + v_i(v_1 + \cdots + v_{i-1}) + v_j(v_1 + \cdots + v_{j-1}) + v_i v_j.$$

Also define a graded bracket operation

$$\begin{aligned}
&[\![\cdot,\cdot]\!] : \mathrm{Hom}^m\big(S(\mathfrak{h}),\mathfrak{g}\big) \times \mathrm{Hom}^n\big(S(\mathfrak{h}),\mathfrak{g}\big) \to \mathrm{Hom}^{m+n+1}\big(S(\mathfrak{h}),\mathfrak{g}\big),\\
&[\![f,g]\!]_p(v_1,\ldots,v_p)\\
&= -\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(l,1,p-l-1)}}} \varepsilon(\sigma) f_{k-1}\Big(\rho\big(g_l(v_{\sigma(1)},\ldots,v_{\sigma(l)})\big)v_{\sigma(l+1)}, v_{\sigma(l+2)},\ldots,v_{\sigma(p)}\Big)\\
&+ (-1)^{\alpha} \sum_{\substack{k+l=s+1\\ \sigma\in\mathbb{S}_{(k-1,1,p-k)}}} \varepsilon(\sigma) g_l\Big(\rho\big(f_{k-1}(v_{\sigma(1)},\ldots,v_{\sigma(k-1)})\big)v_{\sigma(k)}, v_{\sigma(k+1)},\ldots,v_{\sigma(p)}\Big)\\
&- \sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,l)}}} (-1)^{\beta}\varepsilon(\sigma)\big[f_{k-1}(v_{\sigma(1)},\ldots,v_{\sigma(k-1)}), g_l(v_{\sigma(k)},\ldots,v_{\sigma(p)})\big]_{\mathfrak{g}},
\end{aligned}\tag{2.8}$$

where $\alpha = (m+1)(n+1)$, $\beta = n(v_{\sigma(1)} + \cdots + v_{\sigma(k-1)}) + m + 1$ and

$$f = \sum_i f_i \in \mathrm{Hom}^m(S(\mathfrak{h}),\mathfrak{g}), \quad g = \sum g_i \in \mathrm{Hom}^n(S(\mathfrak{h}),\mathfrak{g})$$

with $f_i, g_i \in \mathrm{Hom}(S^i(\mathfrak{h}),\mathfrak{g})$ and $v_1,\ldots,v_p \in \mathfrak{h}$. Here we write $\mathrm{d}g = \sum_i (\mathrm{d}g)_i$ with $(\mathrm{d}g)_i \in \mathrm{Hom}(S^i(\mathfrak{h}),\mathfrak{g})$, and $[\![f,g]\!] = \sum_i [\![f,g]\!]_i$ with $[\![f,g]\!]_i \in \mathrm{Hom}(S^i(\mathfrak{h}),\mathfrak{g})$.

Theorem 2.15. *For an action ρ of an* sgLa$(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ *on an* sgLa$(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$*, with the notations above, the triple* $(sC^*(\mathfrak{h},\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d})$ *is a* dgLa.

Proof. By Theorem 2.2, the graded Nijenhuis–Richardson bracket $[\cdot,\cdot]_{\mathsf{NR}}$ associated to the direct sum vector space $\mathfrak{g}\oplus\mathfrak{h}$ gives rise to a gLa$(C^*(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\mathsf{NR}})$. Obviously,

$$C^*(\mathfrak{h},\mathfrak{g}) := \oplus_{n\in\mathbb{Z}} \mathrm{Hom}^n\big(S(\mathfrak{h}),\mathfrak{g}\big)$$

is an abelian subalgebra. We denote the symmetric graded Lie brackets $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{\mathfrak{h}}$ by $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{h}}$, respectively. Since ρ is an action of the sgLa$(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$, $\mu_{\mathfrak{g}}+\rho$ is a semidirect product sgLa structure on $\mathfrak{g}\oplus\mathfrak{h}$. Theorem 2.2 implies that $\mu_{\mathfrak{g}}+\rho$ and $\lambda\mu_{\mathfrak{h}}$ are Maurer–Cartan elements of the gLa$(C^*(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\mathsf{NR}})$. Define a differential $d_{\mu_{\mathfrak{g}}+\rho}$ on $(C^*(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\mathsf{NR}})$ via

$$d_{\mu_{\mathfrak{g}}+\rho} := [\mu_{\mathfrak{g}}+\rho,\cdot]_{\mathsf{NR}}.$$

Further, we define the derived bracket on the graded vector space $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}^n(S(\mathfrak{h}),\mathfrak{g})$ by

$$[\![f,g]\!] := (-1)^m [d_{\mu_{\mathfrak{g}}+\rho} f, g]_{\mathsf{NR}} = (-1)^m \big[[\mu_{\mathfrak{g}}+\rho, f]_{\mathsf{NR}}, g\big]_{\mathsf{NR}},\tag{2.9}$$

for all $f = \sum f_i \in \mathrm{Hom}^m(S(\mathfrak{h}), \mathfrak{g})$, $g = \sum_i g_i \in \mathrm{Hom}^n(S(\mathfrak{h}), \mathfrak{g})$. Write

$$[\mu_{\mathfrak{g}} + \rho, f]_{\mathsf{NR}} = \sum_{i=0}^{+\infty} [\mu_{\mathfrak{g}} + \rho, f]^i_{\mathsf{NR}},$$

where

$$[\mu_{\mathfrak{g}} + \rho, f]^i_{\mathsf{NR}} \in \mathrm{Hom}\big(S^i(\mathfrak{h}), \mathfrak{g}\big).$$

By (2.3), for all $k \geq 2$, $x_1, \ldots, x_k \in \mathfrak{g}$ and $v_1 \ldots, v_k \in \mathfrak{h}$, we have

$$\begin{aligned}
&[\mu_{\mathfrak{g}} + \rho, f]^k_{\mathsf{NR}}\big((x_1, v_1), \ldots, (x_k, v_k)\big)\\
&= \big((\mu_{\mathfrak{g}} + \rho) \circ f_{k-1} - (-1)^m f_{k-1} \circ (\mu_{\mathfrak{g}} + \rho)\big)\big((x_1, v_1), \ldots, (x_k, v_k)\big)\\
&= \sum_{i=1}^{k} (-1)^{\alpha} (\mu_{\mathfrak{g}} + \rho)\big(f_{k-1}\big((x_1, v_1), \ldots, \widehat{(x_i, v_i)}, \ldots, (x_k, v_k)\big), (x_i, v_i)\big)\\
&\quad - (-1)^m \sum_{1\leq i<j\leq k} (-1)^{\beta} f_{k-1}\big((\mu_{\mathfrak{g}} + \rho)\big((x_i, v_i), (x_j, v_j)\big), (x_1, v_1), \ldots, \widehat{(x_i, v_i)},\\
&(x_{i+1}, v_{i+1}), \ldots, \widehat{(x_j, v_j)}, \ldots, (x_k, v_k)\big)\\
&= \sum_{i=1}^{k} (-1)^{\alpha} (\mu_{\mathfrak{g}} + \rho)\big(\big(f_{k-1}(v_1, \ldots, \hat{v}_i, \ldots, v_k), 0\big), (x_i, v_i)\big)\\
&\quad - (-1)^m \sum_{1\leq i<j\leq k} (-1)^{\beta} f_{k-1}\big(\big([x_i, x_j]_{\mathfrak{g}}, \rho(x_i)v_j + (-1)^{v_i v_j} \rho(x_j)v_i\big),\\
&(x_1, v_1), \ldots, \widehat{(x_i, v_i)}, \ldots, \widehat{(x_j, v_j)}, \ldots, (x_k, v_k)\big)\\
&= \sum_{i=1}^{k} (-1)^{\alpha} \big(\big[f_{k-1}(v_1, \ldots, \hat{v}_i, \ldots, v_k), x_i\big]_{\mathfrak{g}}, \rho\big(f_{k-1}(v_1, \ldots, \hat{v}_i, \ldots, v_k)\big)v_i\big)\\
&\quad - (-1)^m \sum_{1\leq i<j\leq k} (-1)^{\beta} \big(f_{k-1}\big(\rho(x_i)v_j, v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k\big), 0\big)\\
&\quad - (-1)^m \sum_{1\leq i<j\leq k} (-1)^{\beta} \big(f_{k-1}\big((-1)^{v_i v_j} \rho(x_j)v_i, v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k\big), 0\big).
\end{aligned}$$

Here

$$\begin{aligned}
\alpha &= v_i(v_{i+1} + \cdots + v_k),\\
\beta &= v_i(v_1 + \cdots + v_{i-1}) + v_j(v_1 + \cdots + v_{j-1}) + v_i v_j.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[\mu_{\mathfrak{g}} + \rho, f]^0_{\mathsf{NR}} &= 0,\\
[\mu_{\mathfrak{g}} + \rho, f]^1_{\mathsf{NR}}(x_1, v_1) &= \big([f_0, x_1]_{\mathfrak{g}}, \rho(f_0)v_1\big).
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
&\big[[\mu_{\mathfrak g}+\rho, f]_{\mathsf{NR}}, g\big]^p_{\mathsf{NR}}\big((x_1,v_1),\dots,(x_p,v_p)\big)\\
&\quad=\Big(\sum_{k+l=p+1}[\mu_{\mathfrak g}+\rho, f]^k_{\mathsf{NR}}\circ g_l\Big)\big((x_1,v_1),\dots,(x_p,v_p)\big)\\
&\qquad-(-1)^{(m+1)n}\Big(\sum_{k+l=p+1} g_l\circ[\mu_{\mathfrak g}+\rho, f]^k_{\mathsf{NR}}\Big)\big((x_1,v_1),\dots,(x_p,v_p)\big).
\end{aligned}$$

By a straightforward computation, we have

$$\begin{aligned}
&\big([\mu_{\mathfrak g}+\rho, f]^k_{\mathsf{NR}}\circ g_l\big)\big((x_1,v_1),\dots,(x_p,v_p)\big)\\
&=\sum_{\sigma\in\mathbb S_{(l,p-l)}}\varepsilon(\sigma)[\mu_{\mathfrak g}+\rho, f]^k_{\mathsf{NR}}\big(g_l\big((x_{\sigma(1)},v_{\sigma(1)}),\dots,(x_{\sigma(l)},v_{\sigma(l)})\big),\\
&(x_{\sigma(l+1)},v_{\sigma(l+1)}),\dots,(x_{\sigma(p)},v_{\sigma(p)})\big)\\
&=\sum_{\sigma\in\mathbb S_{(l,p-l)}}\varepsilon(\sigma)[\mu_{\mathfrak g}+\rho, f]^k_{\mathsf{NR}}\big((g_l(v_{\sigma(1)},\dots,v_{\sigma(l)}),0),\\
&(x_{\sigma(l+1)},v_{\sigma(l+1)}),\dots,(x_{\sigma(p)},v_{\sigma(p)})\big)\\
&=\sum_{\sigma\in\mathbb S_{(l,p-l)}}\varepsilon(\sigma)(-1)^{\overline{\alpha}}\big(\big[f_{k-1}(v_{\sigma(l+1)},\dots,v_{\sigma(p)}),g_l(v_{\sigma(1)},\dots,v_{\sigma(l)})\big]_{\mathfrak g},0\big)\\
&\quad-(-1)^m\sum_{\sigma\in\mathbb S_{(l,p-l)}}\varepsilon(\sigma)\\
&\quad\times\sum_{j=l+1}^{p}(-1)^{\overline{\beta}}\big(f_{k-1}\big(\rho\big(g_l(v_{\sigma(1)},\dots,v_{\sigma(l)})\big)v_{\sigma(j)},v_{\sigma(l+1)},\dots,\hat v_{\sigma(j)},\dots,v_{\sigma(p)}\big),0\big),
\end{aligned}$$

where $\overline{\alpha}=(v_{\sigma(1)}+\cdots+v_{\sigma(l)}+n)(v_{\sigma(l+1)}+\cdots+v_{\sigma(p)})$ and $\overline{\beta}=v_{\sigma(j)}(v_{\sigma(l+1)}+\cdots+v_{\sigma(j-1)})$. For any $\sigma\in\mathbb S_{(l,p-l)}$, we define $\tau=\tau_\sigma\in\mathbb S_{(p-l,l)}$ by

$$\tau(i)=\begin{cases}\sigma(i+l), & 1\le i\le p-l;\\ \sigma(i-p+l), & p-l+1\le i\le p.\end{cases}$$

Thus $\varepsilon(\tau;v_1,\dots,v_p)=\varepsilon(\sigma;v_1,\dots,v_p)(-1)^{(v_{\sigma(1)}+\cdots+v_{\sigma(l)})(v_{\sigma(l+1)}+\cdots+v_{\sigma(p)})}$. In fact, the elements of $\mathbb S_{(l,p-l)}$ are in bijection with the elements of $\mathbb S_{(p-l,l)}$. Moreover, by $k+l=p+1$, we have

$$\begin{aligned}
&\sum_{\sigma\in\mathbb S_{(l,p-l)}}\varepsilon(\sigma)(-1)^{\overline{\alpha}}\big(\big[f_{k-1}(v_{\sigma(l+1)},\dots,v_{\sigma(p)}),g_l(v_{\sigma(1)},\dots,v_{\sigma(l)})\big]_{\mathfrak g},0\big)\\
&\quad=\sum_{\tau\in\mathbb S_{(k-1,l)}}\varepsilon(\tau)(-1)^{\xi}\big(\big[f_{k-1}(v_{\tau(1)},\dots,v_{\tau(k-1)}),g_l(v_{\tau(k)},\dots,v_{\tau(p)})\big]_{\mathfrak g},0\big),
\end{aligned}$$

where $\xi = n(v_{\tau(1)} + \cdots + v_{\tau(k-1)})$. For any $\sigma \in \mathbb{S}_{(l,p-l)}$ and $l+1 \le j \le p$, we define $\tau = \tau_{\sigma,j} \in \mathbb{S}_{(l,1,p-l-1)}$ by

$$\tau(i) = \begin{cases} \sigma(i), & 1 \le i \le l; \\ \sigma(j), & i = l+1; \\ \sigma(i-1), & l+2 \le i \le j; \\ \sigma(i), & j+1 \le i \le p. \end{cases}$$

Thus we have

$$\varepsilon(\tau; v_1, \dots, v_p) = \varepsilon(\sigma; v_1, \dots, v_p)(-1)^{v_{\sigma(j)}(v_{\sigma(l+1)} + \cdots + v_{\sigma(j-1)})}.$$

Then

$$\begin{aligned} &\sum_{\sigma \in \mathbb{S}_{(l,p-l)}} \varepsilon(\sigma) \sum_{j=l+1}^{p} (-1)^{\bar{\beta}} \big(f_{k-1}\big(\rho\big(g_l(v_{\sigma(1)}, \dots, v_{\sigma(l)})\big)v_{\sigma(j)}, \\ &\qquad\qquad v_{\sigma(l+1)}, \dots, \hat{v}_{\sigma(j)}, \dots, v_{\sigma(p)}\big), 0\big) \\ &= \sum_{\tau \in \mathbb{S}_{(l,1,p-l-1)}} \varepsilon(\tau)\big(f_{k-1}\big(\rho\big(g_l(v_{\tau(1)}, \dots, v_{\tau(l)})\big)v_{\tau(l+1)}, v_{\tau(l+2)}, \dots, v_{\tau(p)}\big), 0\big). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\big([\mu_{\mathfrak{g}} + \rho, f]^k_{\mathsf{NR}} \circ g_l\big)\big((x_1, v_1), \dots, (x_p, v_p)\big) \\ &= \sum_{\sigma \in \mathbb{S}_{(k-1,l)}} \varepsilon(\sigma)(-1)^{n(v_{\sigma(1)} + \cdots + v_{\sigma(k-1)})} \\ &\qquad \times \big(\big[f_{k-1}(v_{\sigma(1)}, \dots, v_{\sigma(k-1)}), g_l(v_{\sigma(k)}, \dots, v_{\sigma(p)})\big]_{\mathfrak{g}}, 0\big) \\ &\quad - (-1)^m \sum_{\sigma \in \mathbb{S}_{(l,1,p-l-1)}} \varepsilon(\sigma)\big(f_{k-1}\big(\rho\big(g_l(v_{\sigma(1)}, \dots, v_{\sigma(l)})\big)v_{\sigma(l+1)}, v_{\sigma(l+2)}, \dots, v_{\sigma(p)}\big), 0\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\big(g_l \circ [\mu_{\mathfrak{g}} + \rho, f]^k_{\mathsf{NR}}\big)\big((x_1, v_1), \dots, (x_p, v_p)\big) \\ &\quad = \sum_{\sigma \in \mathbb{S}_{(k,n-k)}} \varepsilon(\sigma) g_l\big([\mu_{\mathfrak{g}} + \rho, f]^k_{\mathsf{NR}}\big((x_{\sigma(1)}, v_{\sigma(1)}), \dots, (x_{\sigma(k)}, v_{\sigma(k)})\big), \\ &\quad (x_{\sigma(k+1)}, v_{\sigma(k+1)}), \dots, (x_{\sigma(p)}, v_{\sigma(p)})\big) \\ &\quad = \sum_{\sigma \in \mathbb{S}_{(k,p-k)}} \varepsilon(\sigma) \sum_{i=1}^{k} (-1)^{\alpha'} \big(g_l\big(\rho\big(f_{k-1}(v_{\sigma(1)}, \dots, \hat{v}_{\sigma(i)}, \dots, v_{\sigma(k)})\big)v_{\sigma(i)}, \\ &\qquad\qquad v_{\sigma(k+1)} \dots, v_{\sigma(p)}\big), 0\big), \end{aligned}$$

where $\alpha' = v_{\sigma(i)}(v_{\sigma(i+1)} + \cdots + v_{\sigma(k)})$. For $\sigma \in \mathbb{S}_{(k,p-k)}$ and $1 \leq i \leq k$, we define $\tau = \tau_{\sigma,i} \in \mathbb{S}_{(k-1,1,p-k)}$ by

$$\tau(j) = \begin{cases} \sigma(j), & 1 \leq j \leq i-1; \\ \sigma(j+1), & i \leq j \leq k-1; \\ \sigma(i), & j = k; \\ \sigma(j), & k+1 \leq j \leq p. \end{cases}$$

Thus we have $\varepsilon(\tau; v_1, \ldots, v_p) = \varepsilon(\sigma; v_1, \ldots, v_p)(-1)^{v_{\sigma(i)}(v_{\sigma(i+1)}+\cdots+v_{\sigma(k)})}$. Then we have

$$\begin{aligned}&\big(g_l \circ [\mu_{\mathfrak{g}} + \rho, f]^k_{\mathsf{NR}}\big)\big((x_1, v_1), \ldots, (x_p, v_p)\big)\\ &= \sum_{\sigma \in \mathbb{S}_{(k-1,1,p-k)}} \varepsilon(\sigma)\Big(g_l\big(\rho\big(f_{k-1}(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)})\big)v_{\sigma(k)}, v_{\sigma(k+1)} \ldots, v_{\sigma(p)}\big), 0\Big).\end{aligned}$$

By (2.9), we obtain that the derived bracket $[\![\cdot,\cdot]\!]$ is closed on $sC^*(\mathfrak{h},\mathfrak{g})$, and it is given by (2.8). Therefore, $(sC^*(\mathfrak{h},\mathfrak{g}), [\![\cdot,\cdot]\!])$ is a gLa.

Moreover, we define a linear map $\mathrm{d} = [\lambda\mu_{\mathfrak{h}}, \cdot]_{\mathsf{NR}}$ on the graded space $C^*(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{g}\oplus\mathfrak{h})$. By simple computations, we obtain that d is closed on the subspace $sC^*(\mathfrak{h},\mathfrak{g})$, and is given by (2.7). By $[\lambda\mu_{\mathfrak{h}}, \lambda\mu_{\mathfrak{h}}]_{\mathsf{NR}} = 0$, we obtain that $\mathrm{d}^2 = 0$. Moreover, by $\operatorname{Im}\rho \subset \operatorname{Der}(\mathfrak{h})$, we have $[\mu_{\mathfrak{g}} + \rho, \lambda\mu_{\mathfrak{h}}]_{\mathsf{NR}} = 0$. Thus, we deduce that d is a derivation of the gLa$(sC^*(\mathfrak{h},\mathfrak{g}), [\![\cdot,\cdot]\!])$. Therefore, $(sC^*(\mathfrak{h},\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d})$ is a dgLa. The proof is finished. ■

Homotopy $\mathcal{O}$-operators of weight λ can be characterized as Maurer–Cartan elements of the above dgLa. Note that an element

$$T = \sum_{i=0}^{+\infty} T_i \in \operatorname{Hom}(S(\mathfrak{h}), \mathfrak{g})$$

is of degree 0 if and only if the corresponding element $T \in s\operatorname{Hom}(S(\mathfrak{h}), \mathfrak{g})$ is of degree 1.

Theorem 2.16. *Let ρ be an action of an* sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *on an* sgLa$(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$. *A degree* 0 *element*

$$T = \sum_{i=0}^{+\infty} T_i \in \operatorname{Hom}(S(\mathfrak{h}), \mathfrak{g})$$

is a homotopy $\mathcal{O}$-operator of weight λ on $\mathfrak{g}$ with respect to the action ρ if and only if $T = \sum_{i=0}^{+\infty} T_i$ is a Maurer–Cartan element of the dgLa$(sC^*(\mathfrak{h},\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d})$, *i.e.,*

$$\mathrm{d}T + \frac{1}{2}[\![T, T]\!] = 0.$$

Proof. For a degree 0 element $T = \sum_{i=0}^{+\infty} T_i$ of the graded vector space $C^*(\mathfrak{h},\mathfrak{g})$, we write

$$\mathrm{d}T + \frac{1}{2}[\![T, T]\!] = \sum_i \Big(\mathrm{d}T + \frac{1}{2}[\![T, T]\!]\Big)_i,$$

where $(\mathrm{d}T + \frac{1}{2}[\![T, T]\!])_i \in \mathrm{Hom}(S^i(\mathfrak{h}), \mathfrak{g})$. By straightforward computations, we have

$$\begin{aligned}
&\Big(\mathrm{d}T + \frac{1}{2}[\![T, T]\!]\Big)_p(v_1, \dots, v_p)\\
&\quad = -\sum_{1\le i<j\le p} (-1)^{\alpha} T_{p-1}\big(\lambda[v_i, v_j]_{\mathfrak{h}}, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_p\big)\\
&\qquad - \sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(l,1,p-l-1)}}} \varepsilon(\sigma) T_{k-1}\big(\rho\big(T_l(v_{\sigma(1)}, \dots, v_{\sigma(l)})\big)v_{\sigma(l+1)}, v_{\sigma(l+2)}, \dots, v_{\sigma(p)}\big)\\
&\qquad + \frac{1}{2}\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,l)}}} \varepsilon(\sigma)\big[T_{k-1}(v_{\sigma(1)}, \dots, v_{\sigma(k-1)}), T_l(v_{\sigma(k)}, \dots, v_{\sigma(p)})\big]_{\mathfrak{g}},
\end{aligned}$$

where $\alpha = v_i(v_1 + \dots + v_{i-1}) + v_j(v_1 + \dots + v_{j-1}) + v_i v_j$. Thus, $T = \sum_{i=0}^{+\infty} T_i \in \mathrm{Hom}(S(\mathfrak{h}), \mathfrak{g})$ is a homotopy $\mathcal{O}$-operator of weight λ on $\mathfrak{g}$ with respect to the action ρ if and only if $T = \sum_{i=0}^{+\infty} T_i$ is a Maurer–Cartan element of the $\mathrm{dgLa}(sC^*(\mathfrak{h}, \mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d})$. ∎

We note that a Lie algebra is an sgLa concentrated at degree -1. Moreover, a Lie algebra action is the same as an action of the sgLa on a graded vector space concentrated at degree -1. Therefore, we have the following corollary.

Corollary 2.17. *Let* $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ *be an action of a Lie algebra* $\mathfrak{g}$ *on a Lie algebra* $\mathfrak{h}$. *Then a linear map* $T : \mathfrak{h} \to \mathfrak{g}$ *is an* $\mathcal{O}$*-operator of weight* λ *on* $\mathfrak{g}$ *with respect to the action* ρ *if and only if* T *is a Maurer–Cartan element of the* $\mathrm{dgLa}(\oplus_{n=0}^{+\infty}\mathrm{Hom}(\wedge^n\mathfrak{h}, \mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d})$, *where the differential* $\mathrm{d} : \mathrm{Hom}(\wedge^n\mathfrak{h}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{n+1}\mathfrak{h}, \mathfrak{g})$ *is given by*

$$\begin{aligned}
&(\mathrm{d}g)(v_1, \dots, v_{n+1})\\
&\quad = \sum_{1\le i<j\le n+1} (-1)^{n+i+j-1} g\big(\lambda[v_i, v_j]_{\mathfrak{h}}, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1}\big),
\end{aligned}$$

for all $g \in \mathrm{Hom}(\wedge^n\mathfrak{h}, \mathfrak{g})$ *and* $v_1, \dots, v_{n+1} \in \mathfrak{h}$, *and the graded Lie bracket*

$$[\![\cdot,\cdot]\!] : \mathrm{Hom}(\wedge^n\mathfrak{h}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^m\mathfrak{h}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{m+n}\mathfrak{h}, \mathfrak{g})$$

is given by

$$\begin{aligned}
&[\![g_1, g_2]\!](v_1, \dots, v_{m+n})\\
&\quad = \sum_{\sigma\in\mathbb{S}_{(m,1,n-1)}} (-1)^{1+\sigma} g_1\big(\rho\big(g_2(v_{\sigma(1)}, \dots, v_{\sigma(m)})\big)v_{\sigma(m+1)}, v_{\sigma(m+2)}, \dots, v_{\sigma(m+n)}\big)\\
&\qquad + \sum_{\sigma\in\mathbb{S}_{(n,1,m-1)}} (-1)^{mn+\sigma} g_2\big(\rho\big(g_1(v_{\sigma(1)}, \dots, v_{\sigma(n)})\big)v_{\sigma(n+1)}, v_{\sigma(n+2)}, \dots, v_{\sigma(m+n)}\big)\\
&\qquad + \sum_{\sigma\in\mathbb{S}_{(n,m)}} (-1)^{1+mn+\sigma}\big[g_1(v_{\sigma(1)}, \dots, v_{\sigma(n)}), g_2(v_{\sigma(n+1)}, \dots, v_{\sigma(m+n)})\big]_{\mathfrak{g}},
\end{aligned}$$

for all $g_1 \in \mathrm{Hom}(\wedge^n\mathfrak{h}, \mathfrak{g})$, $g_2 \in \mathrm{Hom}(\wedge^m\mathfrak{h}, \mathfrak{g})$ *and* $v_1, \dots, v_{m+n} \in \mathfrak{h}$.

If the Lie algebra $\mathfrak{h}$ is abelian in the above corollary, we recover the gLa that controls the deformations of $\mathcal{O}$-operators of weight 0 given in [45, Proposition 2.3].

3. Operator homotopy post-Lie algebras

In this section, we first recall the notion of a post-Lie algebra, and then give the definition of an operator homotopy post-Lie algebra as a variation of a homotopy post-Lie algebra. We construct a dgLa and show that operator homotopy post-Lie algebras can be characterized as its Maurer–Cartan elements to justify the notion. We also show that operator homotopy post-Lie algebras naturally arise from homotopy $\mathcal{O}$-operators of weight 1.

Definition 3.1 ([47]). A *post-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ consists of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and a binary product $\rhd : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$x \rhd [y,z]_{\mathfrak{g}} = [x \rhd y, z]_{\mathfrak{g}} + [y, x \rhd z]_{\mathfrak{g}}, \tag{3.1}$$

$$[x,y]_{\mathfrak{g}} \rhd z = a_{\rhd}(x,y,z) - a_{\rhd}(y,x,z), \tag{3.2}$$

here $a_{\rhd}(x,y,z) := x \rhd (y \rhd z) - (x \rhd y) \rhd z$ and $x, y, z \in \mathfrak{g}$.

Define $L_{\rhd} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $L_{\rhd}(x)(y) = x \rhd y$. Then by (3.1), $L_{\rhd}$ is a linear map from $\mathfrak{g}$ to $\mathrm{Der}(\mathfrak{g})$. In the sequel, we will say that $L_{\rhd}$ is a post-Lie algebra structure on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

Remark 3.2. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ be a post-Lie algebra. If the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}} = 0$, then $(\mathfrak{g}, \rhd)$ becomes a pre-Lie algebra. Thus, a post-Lie algebra can be viewed as a nonabelian version of a pre-Lie algebra. See [10, 11] for the classifications of post-Lie algebras on certain Lie algebras, and [35] for applications of post-Lie algebras in numerical integration.

The following well-known result is a special case of splitting of operads [4, 39, 47].

Proposition 3.3. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ *be a post-Lie algebra. Then the bracket* $[\cdot,\cdot]_C$ *defined by*

$$[x,y]_C := x \rhd y - y \rhd x + [x,y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}, \tag{3.3}$$

is a Lie bracket.

We denote this Lie algebra by $\mathfrak{g}^C$ and call it the *sub-adjacent Lie algebra* of the post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$.

Definition 3.4. An *operator homotopy post-Lie algebra* is an sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ equipped with a collection $(k \geq 1)$ of linear maps $\theta_k : \otimes^k \mathfrak{g} \to \mathfrak{g}$ of degree 1 satisfying, for every collection of homogeneous elements $x_1, \ldots, x_n, x_{n+1} \in \mathfrak{g}$,

(i) (*graded symmetry*) for every $\sigma \in \mathbb{S}_{n-1}$, $n \geq 1$,

$$\theta_n(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}, x_n) = \varepsilon(\sigma)\theta_n(x_1, \ldots, x_{n-1}, x_n), \tag{3.4}$$

(ii) (*graded derivation*) for all $n \geq 1$,

$$\begin{aligned}&\theta_n\big(x_1, \ldots, x_{n-1}, [x_n, x_{n+1}]_{\mathfrak{g}}\big)\\&\quad = (-1)^{x_1+\cdots+x_{n-1}+1}\big[\theta_n(x_1, \ldots, x_{n-1}, x_n), x_{n+1}\big]_{\mathfrak{g}}\\&\qquad + (-1)^{(x_1+\cdots+x_{n-1}+1)(x_n+1)}\big[x_n, \theta_n(x_1, \ldots, x_{n-1}, x_{n+1})\big]_{\mathfrak{g}},\end{aligned} \tag{3.5}$$

(iii) for all $n \geq 1$,

$$\begin{aligned}&\sum_{\substack{i+j=n+1, i\geq 1, j\geq 2\\ \sigma\in\mathbb{S}_{(i-1,1,j-2)}}} \varepsilon(\sigma)\theta_j\big(\theta_i(v_{\sigma(1)}, \ldots, v_{\sigma(i-1)}, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n-1)}, v_n\big)\\&+ \sum_{\substack{i+j=n+1, i\geq 1, j\geq 1\\ \sigma\in\mathbb{S}_{(j-1,i-1)}}} (-1)^{\alpha}\varepsilon(\sigma)\theta_j\big(v_{\sigma(1)}, \ldots, v_{\sigma(j-1)}, \theta_i(v_{\sigma(j)}, \ldots, v_{\sigma(n-1)}, v_n)\big)\\&= \sum_{1\leq i<j\leq n-1} (-1)^{\beta}\theta_{n-1}\big([x_i, x_j]_{\mathfrak{g}}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n\big),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\alpha &= x_{\sigma(1)} + x_{\sigma(2)} + \cdots + x_{\sigma(j-1)},\\ \beta &= x_i(x_1 + \cdots + x_{i-1}) + x_j(x_1 + \cdots + x_{j-1}) + x_i x_j + 1.\end{aligned}$$

The notion of a pre-Lie$_\infty$-algebra was introduced in [13]. Recall that a *pre-Lie$_\infty$-algebra* is a graded vector space V equipped with a collection of linear maps

$$\theta_k : \otimes^k V \to V, \quad k \geq 1,$$

of degree 1 with the property that, for any homogeneous elements $v_1, \ldots, v_n \in V$, we have

(i) (*graded symmetry*) for every $\sigma \in \mathbb{S}_{n-1}$,

$$\theta_n(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}, v_n) = \varepsilon(\sigma)\theta_n(v_1, \ldots, v_{n-1}, v_n),$$

(ii) for all $n \geq 1$,

$$\begin{aligned}&\sum_{\substack{i+j=n+1,\, i\geq 1,\, j\geq 2\\ \sigma\in\mathbb{S}_{(i-1,1,j-2)}}} \varepsilon(\sigma)\theta_j\big(\theta_i(v_{\sigma(1)}, \ldots, v_{\sigma(i-1)}, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n-1)}, v_n\big)\\&+ \sum_{\substack{i+j=n+1,\, i\geq 1,\, j\geq 1\\ \sigma\in\mathbb{S}_{(j-1,i-1)}}} (-1)^{\alpha}\varepsilon(\sigma)\theta_j\big(v_{\sigma(1)}, \ldots, v_{\sigma(j-1)}, \theta_i(v_{\sigma(j)}, \ldots, v_{\sigma(n-1)}, v_n)\big) = 0,\end{aligned}$$

where $\alpha = v_{\sigma(1)} + v_{\sigma(2)} + \cdots + v_{\sigma(j-1)}$.

Obviously, an operator homotopy post-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \{\theta_k\}_{k=1}^{+\infty})$ reduces to a pre-Lie$_\infty$-algebra when $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is an abelian sgLa.

Remark 3.5. Since a Rota–Baxter Lie algebra of weight 1 gives a post-Lie algebra, we expect that a Rota–Baxter homotopy Lie algebra of weight 1 (a homotopy Lie algebra with a Rota–Baxter operator of weight 1) induces a homotopy post-Lie algebra.

Now we construct the dgLa that characterizes operator homotopy post-Lie algebras as Maurer–Cartan elements. Let $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ be an sgLa. Denote

$$\bar{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}) := \mathrm{Hom}^n\big(S(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h})\big), \quad \bar{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}) := \oplus_{n\in\mathbb{Z}}\bar{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}).$$

Define a graded linear map $\partial : \bar{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}) \to \bar{\mathcal{C}}^{n+1}(\mathfrak{h},\mathfrak{h})$ by

$$(\partial\beta)_p(v_1,\dots,v_p) = \sum_{1\le i<j\le p} (-1)^{\alpha}\beta_{p-1}\big([v_i,v_j]_{\mathfrak{h}}, v_1,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_p\big),$$

where $\alpha = n+1+v_i(v_1+\dots+v_{i-1})+v_j(v_1+\dots+v_{j-1})+v_iv_j$. Then define a graded bracket operation

$$[\cdot,\cdot]^c : \bar{\mathcal{C}}^m(\mathfrak{h},\mathfrak{h}) \times \bar{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}) \to \bar{\mathcal{C}}^{m+n+1}(\mathfrak{h},\mathfrak{h})$$

by

$$\begin{aligned}
&[\alpha,\beta]^c_p(v_1,\dots,v_p)\\
&= -\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(l,1,p-l-1)}}} \varepsilon(\sigma)\alpha_{k-1}\big(s\big(\beta_l(v_{\sigma(1)},\dots,v_{\sigma(l)})\big)v_{\sigma(l+1)}, v_{\sigma(l+2)},\dots,v_{\sigma(p)}\big)\\
&\quad + (-1)^{\xi}\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,1,p-k)}}} \varepsilon(\sigma)\beta_l\big(s\big(\alpha_{k-1}(v_{\sigma(1)},\dots,v_{\sigma(k-1)})\big)v_{\sigma(k)}, v_{\sigma(k+1)},\dots,v_{\sigma(p)}\big)\\
&\quad - \sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,l)}}} (-1)^{\eta}\varepsilon(\sigma)s^{-1}\big[s\alpha_{k-1}(v_{\sigma(1)},\dots,v_{\sigma(k-1)}), s\beta_l(v_{\sigma(k)},\dots,v_{\sigma(p)})\big],
\end{aligned}$$

where $\xi = (m+1)(n+1)$, $\eta = (n+1)(v_{\sigma(1)}+\dots+v_{\sigma(k-1)})$ and $\alpha = \sum_i \alpha_i \in \bar{\mathcal{C}}^m(\mathfrak{h},\mathfrak{h})$, $\beta = \sum_i \beta_i \in \bar{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h})$ with $\alpha_i, \beta_i \in \mathrm{Hom}(S^i(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$ and $v_1,\dots,v_p \in \mathfrak{h}$. Here we write $\partial\beta = \sum_i(\partial\beta)_i$ with $(\partial\beta)_i \in \mathrm{Hom}(S^i(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$ and $[\alpha,\beta]^c = \sum_i[\alpha,\beta]^c_i$ with $[\alpha,\beta]^c_i \in \mathrm{Hom}(S^i(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$.

Theorem 3.6. *With the above notations,* $(s\bar{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}), [\cdot,\cdot]^c, \partial)$ *is a* dgLa. *Moreover, its Maurer–Cartan elements are precisely the operator homotopy post-Lie algebra structures on the* sgLa$(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$.

Proof. Setting $\lambda = 1$, $\mathfrak{g} = s^{-1}\,\mathrm{Der}(\mathfrak{h})$ and $\rho = s$ in Theorem 2.15, we obtain that $(s\bar{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}), [\cdot,\cdot]^c, \partial)$ is a dgLa.

Let $L = \sum_{i=0}^{+\infty} L_i \in \mathrm{Hom}^0(S(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$ with $L_i \in \mathrm{Hom}(S^i(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$ be a Maurer–Cartan element of the dgLa$(s\bar{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}), [\cdot,\cdot]^c, \partial)$. We define a collection of linear maps $\theta_k : \otimes^k\mathfrak{h} \to \mathfrak{h}$ $(k \ge 1)$ of degree 1 by

$$\theta_k(v_1,\dots,v_k) := \big(sL_{k-1}(v_1,\dots,v_{k-1})\big)v_k, \quad \forall v_1\dots,v_k \in \mathfrak{h}. \tag{3.7}$$

By $L_{n-1} \in \mathrm{Hom}^0(S^{n-1}(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$, we obtain (3.4) and (3.5). Moreover, write $\partial L + \frac{1}{2}[L,L]^c = \sum_i (\partial L + \frac{1}{2}[L,L]^c)_i$ with $(\partial L + \frac{1}{2}[L,L]^c)_i \in \mathrm{Hom}(S^i(\mathfrak{h}), s^{-1}\,\mathrm{Der}(\mathfrak{h}))$. Then for all $v_1,\dots,v_n \in \mathfrak{h}$, by a similar computation as in the proof of Theorem 2.16, we have

$$
\begin{aligned}
&\Big(s\Big(\Big(\partial L + \frac{1}{2}[L,L]^c\Big)_{n-1}\Big)(v_1,\dots,v_{n-1})\Big)v_n \\
&= -\sum_{1\le i<j\le n-1} \theta_{n-1}(-1)^{\alpha}\big([v_i,v_j]_{\mathfrak{h}}, v_1,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_{n-1},v_n\big) \\
&\quad - \sum_{\substack{k+l=n\\ \sigma\in\mathbb{S}_{(l,1,n-l-2)}}} \varepsilon(\sigma)\theta_k\big(\theta_{l+1}(v_{\sigma(1)},\dots,v_{\sigma(l)},v_{\sigma(l+1)}), v_{\sigma(l+2)},\dots,v_{\sigma(n-1)},v_n\big) \\
&\quad - \sum_{\substack{k+l=n\\ \sigma\in\mathbb{S}_{(k-1,l)}}} (-1)^{\beta}\varepsilon(\sigma)\theta_k\big(v_{\sigma(1)},\dots,v_{\sigma(k-1)},\theta_{l+1}(v_{\sigma(k)},\dots,v_{\sigma(n-1)},v_n)\big),
\end{aligned}
$$

where $\alpha = v_i(v_1+\cdots+v_{i-1}) + v_j(v_1+\cdots+v_{j-1}) + v_iv_j$ and $\beta = v_{\sigma(1)}+\cdots+v_{\sigma(k-1)}$, which implies (3.6). Thus, $\{\theta_k\}_{k=1}^{+\infty}$ is an operator homotopy post-Lie algebra structure on the sgLa$(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$. ■

When the sgLa $\mathfrak{h}$ reduces to a usual Lie algebra, we characterize post-Lie algebra structures on $\mathfrak{h}$ as Maurer–Cartan elements. See [10, 11, 20, 38] for classifications of post-Lie algebras on some specific Lie algebras.

Corollary 3.7. *Let* $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ *be a Lie algebra. Denote* $\overline{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}) := \mathrm{Hom}(\wedge^n\mathfrak{h}, \mathrm{Der}(\mathfrak{h}))$ *and* $\overline{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}) := \oplus_{n\ge 0}\overline{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h})$. *Then* $(\overline{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}),[\cdot,\cdot]^c,\partial)$ *is a* dgLa, *where the differential* $\partial : \overline{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}) \to \overline{\mathcal{C}}^{n+1}(\mathfrak{h},\mathfrak{h})$ *is given by*

$$
\begin{aligned}
&(\partial\alpha)(u_1,\dots,u_{n+1}) \\
&\quad = \sum_{1\le i<j\le n+1} (-1)^{n+i+j-1}\alpha\big([u_i,u_j]_{\mathfrak{h}}, u_1,\dots,\hat{u}_i,\dots,\hat{u}_j,\dots,u_{n+1}\big),
\end{aligned}
$$

and the graded Lie bracket $[\cdot,\cdot]^c : \overline{\mathcal{C}}^n(\mathfrak{h},\mathfrak{h}) \times \overline{\mathcal{C}}^m(\mathfrak{h},\mathfrak{h}) \to \overline{\mathcal{C}}^{m+n}(\mathfrak{h},\mathfrak{h})$ *is given by*

$$
\begin{aligned}
&[\alpha,\beta]^c(u_1,\dots,u_{m+n}) \\
&= -\sum_{\sigma\in\mathbb{S}_{(m,1,n-1)}} (-1)^{\sigma}\alpha\big(\beta(u_{\sigma(1)},\dots,u_{\sigma(m)})u_{\sigma(m+1)}, u_{\sigma(m+2)},\dots,u_{\sigma(m+n)}\big) \\
&\quad + (-1)^{mn}\sum_{\sigma\in\mathbb{S}_{(n,1,m-1)}} (-1)^{\sigma}\beta\big(\alpha(u_{\sigma(1)},\dots,u_{\sigma(n)})u_{\sigma(n+1)}, u_{\sigma(n+2)},\dots,u_{\sigma(m+n)}\big) \\
&\quad - (-1)^{mn}\sum_{\sigma\in\mathbb{S}_{(n,m)}} (-1)^{\sigma}\big[\alpha(u_{\sigma(1)},\dots,u_{\sigma(n)}), \beta(u_{\sigma(n+1)},\dots,u_{\sigma(m+n)})\big],
\end{aligned}
$$

for all $\alpha \in \mathrm{Hom}(\wedge^n\mathfrak{h},\mathrm{Der}(\mathfrak{h}))$, $\beta \in \mathrm{Hom}(\wedge^m\mathfrak{h},\mathrm{Der}(\mathfrak{h}))$, *and* $u_1,\dots,u_{m+n} \in \mathfrak{h}$.

Moreover, $L_{\triangleright} : \mathfrak{h} \to \mathrm{Der}(\mathfrak{h})$ *defines a post-Lie algebra structure on the Lie algebra* $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ *if and only if* $L_{\triangleright}$ *is a Maurer–Cartan element of the* dgLa$(\overline{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}),[\cdot,\cdot]^c,\partial)$.

When the sgLa $\mathfrak{h}$ is abelian, we characterize pre-Lie$_\infty$-algebras as Maurer–Cartan elements, which was originally given in [13].

Corollary 3.8. *Let V be a $\mathbb{Z}$-graded vector space. Denote*

$$\bar{\mathcal{C}}^n(V,V) := \mathrm{Hom}^n\big(S(V), \mathfrak{gl}(V)\big), \quad \bar{\mathcal{C}}^*(V,V) := \oplus_{n\in\mathbb{Z}}\bar{\mathcal{C}}^n(V,V).$$

Then $(\bar{\mathcal{C}}^*(V,V), [\cdot,\cdot]^c)$ is a gLa*, with the graded Lie bracket*

$$\begin{aligned}
&[\cdot,\cdot]^c : \bar{\mathcal{C}}^m(V,V)\times\bar{\mathcal{C}}^n(V,V)\to\bar{\mathcal{C}}^{m+n}(V,V),\\
&[\alpha,\beta]^c_p(v_1,\ldots,v_p)\\
&\quad = -\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(l,1,p-l-1)}}} \varepsilon(\sigma)\alpha_{k-1}\big(\beta_l(v_{\sigma(1)},\ldots,v_{\sigma(l)})v_{\sigma(l+1)}, v_{\sigma(l+2)},\ldots,v_{\sigma(p)}\big)\\
&\qquad + (-1)^{mn}\sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,1,p-k)}}} \varepsilon(\sigma)\beta_l\big(\alpha_{k-1}(v_{\sigma(1)},\ldots,v_{\sigma(k-1)})v_{\sigma(k)}, v_{\sigma(k+1)},\ldots,v_{\sigma(p)}\big)\\
&\qquad - \sum_{\substack{k+l=p+1\\ \sigma\in\mathbb{S}_{(k-1,l)}}} (-1)^{\xi}\varepsilon(\sigma)\big[\alpha_{k-1}(v_{\sigma(1)},\ldots,v_{\sigma(k-1)}), \beta_l(v_{\sigma(k)},\ldots,v_{\sigma(p)})\big],
\end{aligned}$$

where $\xi = n(v_{\sigma(1)}+\cdots+v_{\sigma(k-1)})$ and $\alpha=\sum_i\alpha_i\in\bar{\mathcal{C}}^m(V,V)$, $\beta=\sum\beta_i\in\bar{\mathcal{C}}^n(V,V)$ with $\alpha_i,\beta_i\in\mathrm{Hom}(S^i(V),V)$.

Moreover,

$$L=\sum_{i=0}^{+\infty}L_i\in\mathrm{Hom}^1\big(S(V),\mathfrak{gl}(V)\big)$$

defines a pre-Lie$_\infty$-algebra structure by

$$\theta_k(v_1,\ldots,v_k) := L_{k-1}(v_1,\ldots,v_{k-1})v_k, \quad \forall v_1\ldots,v_k\in V, \tag{3.8}$$

on the graded vector space V if and only if $L=\sum_{i=0}^{+\infty}L_i$ is a Maurer–Cartan element of the gLa$(\bar{\mathcal{C}}^*(V,V),[\cdot,\cdot]^c)$.

In the above corollary, if the graded vector space V reduces to a usual vector space, we characterize pre-Lie algebra structures as Maurer–Cartan elements. See [13, 36] for more details.

It is known that post-Lie algebras naturally arise from $\mathcal{O}$-operators of weight 1 as follows.

Proposition 3.9 ([5]). *Let $T:\mathfrak{h}\to\mathfrak{g}$ be an $\mathcal{O}$-operator of weight* 1*. Then $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},\rhd)$ is a post-Lie algebra with the multiplication $u\rhd v := \rho(Tu)v$.*

In the sequel, we generalize the above relation to homotopy $\mathcal{O}$-operators of weight 1 and operator homotopy post-Lie algebras.

Define a graded linear map $\Psi : C^*(\mathfrak{h},\mathfrak{g}) \to \overline{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h})$ of degree 0 by

$$\Psi(f) = s^{-1} \circ \rho \circ f, \quad \forall f \in \mathrm{Hom}^m(S(\mathfrak{h}),\mathfrak{g}).$$

Therefore, we have $\Psi(f)_k = s^{-1} \circ \rho \circ f_k$. In the following, we set $\lambda = 1$ in Theorem 2.15 for notational simplicity.

Theorem 3.10. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *and* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ *be* sgLas *and* $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ *an* sgLa *action. Then* Ψ *is a homomorphism of* dgLas *from* $(sC^*(\mathfrak{h},\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d})$ *to* $(s\overline{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}), [\cdot,\cdot]^c, \partial)$.

Proof. It follows from a direct but tedious computation. We omit details. ■

Now we are ready to show that the homotopy $\mathcal{O}$-operators of weight 1 induce operator homotopy post-Lie algebras.

Theorem 3.11. *Let* $T = \sum_{i=0}^{+\infty} T_i \in \mathrm{Hom}^0(S(\mathfrak{h}),\mathfrak{g})$ *be a homotopy* $\mathcal{O}$*-operator of weight* 1 *on an* sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *with respect to an action* $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$. *Then* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \{\theta_k\}_{k=1}^{+\infty})$ *is an operator homotopy post-Lie algebra, where* $\theta_k : \otimes^k \mathfrak{h} \to \mathfrak{h}$ $(k \geq 1)$ *are linear maps of degree* 1 *defined by*

$$\theta_k(v_1,\ldots,v_k) := \rho\big(T_{k-1}(v_1,\ldots,v_{k-1})\big)v_k, \quad \forall v_1 \ldots, v_k \in \mathfrak{h}. \tag{3.9}$$

Proof. By Theorems 2.16 and 3.10, we deduce that $\Psi(T)$ is a Maurer–Cartan element of the dgLa$(s\overline{\mathcal{C}}^*(\mathfrak{h},\mathfrak{h}), [\cdot,\cdot]^c, \partial)$. Moreover, by Theorem 3.6, we obtain that $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \{\theta_k\}_{k=1}^{+\infty})$ is an operator homotopy post-Lie algebra. ■

Corollary 3.12. *Let* $T = \sum_{i=0}^{+\infty} T_i \in \mathrm{Hom}^0(S(V),\mathfrak{g})$ *be a homotopy* $\mathcal{O}$*-operator of weight* 0 *on an* sgLa$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *with respect to an action* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. *Then* $(V, \{\theta_k\}_{k=1}^{+\infty})$ *is a pre-Lie*$_\infty$*-algebra, where* $\theta_k : \otimes^k V \to V$ $(k \geq 1)$ *are linear maps of degree* 1 *defined by*

$$\theta_k(v_1,\ldots,v_k) := \rho\big(T_{k-1}(v_1,\ldots,v_{k-1})\big)v_k, \quad \forall v_1 \ldots, v_k \in V. \tag{3.10}$$

It is straightforward to obtain the following result.

Proposition 3.13. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \{\theta_k\}_{k=1}^{+\infty})$ *be an operator homotopy post-Lie algebra. Then* $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ *is an* L_∞*-algebra, where* $l_1 = \theta_1$, l_2 *is defined by*

$$l_2(x,y) = \theta_2(x,y) + (-1)^{xy}\theta_2(y,x) + [x,y]_{\mathfrak{g}}, \tag{3.11}$$

and for $k \geq 3$, l_k *is defined by*

$$l_k(x_1,\ldots,x_k) = \sum_{i=1}^{k} (-1)^{x_i(x_{i+1}+\cdots+x_k)}\theta_k(x_1,\ldots,\hat{x}_i,\ldots,x_k,x_i). \tag{3.12}$$

Definition 3.14 ([25]). Let $(V, \{l_k\}_{k=1}^{+\infty})$ be an L_∞-algebra and $(V', \{l'_k\}_{k=1}^{+\infty})$ an L_∞-algebra in which $l'_k = 0$ for all $k \geq 1$ except $k = 2$. A curved L_∞-algebra *homomorphism* from $(V, \{l_k\}_{k=1}^{+\infty})$ to (V', l'_2) consists of a collection of degree 0 graded multilinear maps

$f_k : V^{\otimes k} \to V'$, $k \geq 0$, with the property that $f_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \varepsilon(\sigma) f_n(v_1, \ldots, v_n)$, for any $n \geq 0$ and homogeneous elements $v_1, \ldots, v_n \in V$, and

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma) f_{n-i+1}\big(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}\big)$$
$$= \frac{1}{2} \sum_{i=0}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma) l'_2\big(f_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), f_{n-i}(v_{\sigma(i+1)}, \ldots, v_{\sigma(n)})\big).$$

Then combining Theorems 3.11 and 3.13, we obtain the following one.

Theorem 3.15. *Let* $T = \sum_{i=0}^{+\infty} T_i \in \mathrm{Hom}(S(\mathfrak{h}), \mathfrak{g})$ *be a homotopy* $\mathcal{O}$*-operator of weight* 1 *on an* $\mathrm{sgLa}(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *with respect to the action* $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$. *Then* T *is a curved* L_∞*-algebra homomorphism from the* L_∞*-algebra* $(\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ *to* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

Proof. For all $v_1, \ldots, v_n \in \mathfrak{h}$, by straightforward computations, we have

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma) T_{n-i+1}\big(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}\big)$$
$$= \sum_{1 \leq i < j \leq n} (-1)^{\alpha} T_{n-1}\big([v_i, v_j]_{\mathfrak{h}}, v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n\big),$$
$$\sum_{i=1}^{n} \sum_{\sigma \in \mathbb{S}_{(i-1,1,n-i)}} \varepsilon(\sigma) T_{n-i+1}\big(\rho\big(T_{i-1}(v_{\sigma(1)}, \ldots, v_{\sigma(i-1)})\big) v_{\sigma(i)}, v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}\big)$$
$$\overset{(2.5)}{=} \frac{1}{2} \sum_{i=0}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma) \big[T_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), T_{n-i}(v_{\sigma(i+1)}, \ldots, v_{\sigma(n)})\big]_{\mathfrak{g}},$$

where $\alpha = v_i(v_1 + \cdots + v_{i-1}) + v_j(v_1 + \cdots + v_{j-1}) + v_i v_j$, which implies that T is a curved L_∞-algebra homomorphism. ∎

Similarly, the above result also holds for homotopy $\mathcal{O}$-operators of weight 0.

4. Classification of 2-term skeletal operator homotopy post-Lie algebras

In general, it is expected that the 2-term homotopy of an algebra structure is equivalent to the categorification of this algebraic structure, and the 2-term homotopy algebras are quasi-isomorphic to the 2-term skeletal homotopy algebras, which are classified by the third cohomological group. Baez and Crans [2] accomplished these for Lie algebras. In this spirit, we classify 2-term skeletal operator homotopy post-Lie algebras by the third cohomology group of a post-Lie algebra. For this purpose, we first define representations of post-Lie algebras and then develop the corresponding cohomology theory.

4.1. Representations of post-Lie algebras

Here we introduce the notion of a representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ on a vector space V. We show that there is naturally an induced representation of the sub-adjacent Lie algebra $\mathfrak{g}^C$ on $\mathrm{Der}(\mathfrak{g}, V)$. This fact plays a crucial role in our study of cohomology groups of post-Lie algebras in the next subsection.

Definition 4.1. A *representation* of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ on a vector space V is a triple (ρ, μ, ν), where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ on V, and $\mu, \nu : \mathfrak{g} \to \mathfrak{gl}(V)$ are linear maps satisfying that for all $x, y \in \mathfrak{g}$,

$$\rho(x \rhd y) = \mu(x) \circ \rho(y) - \rho(y) \circ \mu(x), \tag{4.1}$$

$$\nu\big([x, y]_{\mathfrak{g}}\big) = \rho(x) \circ \nu(y) - \rho(y) \circ \nu(x), \tag{4.2}$$

$$\mu\big([x, y]_{\mathfrak{g}}\big) = \mu(x) \circ \mu(y) - \mu(x \rhd y) - \mu(y) \circ \mu(x) + \mu(y \rhd x), \tag{4.3}$$

$$\nu(y) \circ \rho(x) = \mu(x) \circ \nu(y) - \nu(y) \circ \mu(x) - \nu(x \rhd y) + \nu(y) \circ \nu(x). \tag{4.4}$$

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ be a post-Lie algebra and $(V; \rho, \mu, \nu)$ a representation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$. By Proposition 3.3 and (4.3), we deduce that $(V; \mu)$ is a representation of the sub-adjacent Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$. It is obvious that $(\mathfrak{g}; \mathsf{ad}, L_\rhd, R_\rhd)$ is a representation of a post-Lie algebra on itself, which is called the *regular representation*.

Let $(V; \rho, \mu, \nu)$ be a representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$. Define a Lie bracket $[\cdot,\cdot]_\rho : \otimes^2(\mathfrak{g} \oplus V) \to \mathfrak{g} \oplus V$ by

$$[x_1 + v_1, x_2 + v_2]_\rho := [x_1, x_2]_{\mathfrak{g}} + \rho(x_1)v_2 - \rho(x_2)v_1, \tag{4.5}$$

and a bilinear operation $\rhd_{\mu,\nu}$: $\otimes^2(\mathfrak{g} \oplus V) \to \mathfrak{g} \oplus V$ by

$$(x_1 + v_1) \rhd_{\mu,\nu} (x_2 + v_2) := x_1 \rhd x_2 + \mu(x_1)v_2 + \nu(x_2)v_1. \tag{4.6}$$

By straightforward computations, we have the following theorem.

Theorem 4.2. *With the above notations,* $(\mathfrak{g} \oplus V, [\cdot,\cdot]_\rho, \rhd_{\mu,\nu})$ *is a post-Lie algebra.*

The post-Lie algebra given above is called the *semidirect product* of the post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ and the representation $(V; \rho, \mu, \nu)$, and is denoted by $\mathfrak{g} \ltimes_{\rho,\mu,\nu} V$.

Proposition 4.3. *Let* $(V; \rho, \mu, \nu)$ *be a representation of a post-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$. *Then* $(V; \rho + \mu - \nu)$ *is a representation of the sub-adjacent Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_C)$.

Proof. By Theorem 4.2, we have the semidirect product post-Lie algebra $\mathfrak{g} \ltimes_{\rho,\mu,\nu} V$. Considering its sub-adjacent Lie algebra structure $[\cdot,\cdot]_C$, we have

$$\begin{aligned}
&\big[(x_1 + v_1), (x_2 + v_2)\big]_C \\
&\quad = x_1 \rhd x_2 + \mu(x_1)v_2 + \nu(x_2)v_1 - x_2 \rhd x_1 - \mu(x_2)v_1 - \nu(x_1)v_2 \\
&\qquad + [x_1, x_2]_{\mathfrak{g}} + \rho(x_1)v_2 - \rho(x_2)v_1 \\
&\quad = [x_1, x_2]_C + (\rho + \mu - \nu)(x_1)v_2 - (\rho + \mu - \nu)(x_2)v_1.
\end{aligned} \tag{4.7}$$

Thus $(V; \rho + \mu - \nu)$ is a representation of the sub-adjacent Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$. ■

In particular, if $(\rho, \mu, \nu) = (\mathsf{ad}, L_\triangleright, R_\triangleright)$ is the regular representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \triangleright)$, then $\mathsf{ad} + L_\triangleright - R_\triangleright$ is the adjoint representation of the sub-adjacent Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$.

Corollary 4.4. *Let $(V;\rho,\mu,\nu)$ be a representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \triangleright)$. Then the semidirect product post-Lie algebras $\mathfrak{g} \ltimes_{\rho,\mu,\nu} V$ and $\mathfrak{g} \ltimes_{0,\rho+\mu-\nu,0} V$ given by the representations $(V;\rho,\mu,\nu)$ and $(V;0,\rho+\mu-\nu,0)$ respectively have the same sub-adjacent Lie algebra $\mathfrak{g}^C \ltimes_{\rho+\mu-\nu} V$ given by* (4.7)*, which is the semidirect product of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$ and its representation $(V;\rho+\mu-\nu)$.*

Let $(V;\rho,\mu,\nu)$ be a representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \triangleright)$. We set

$$\mathrm{Der}(\mathfrak{g}, V) := \big\{ f \in \mathrm{Hom}(\mathfrak{g}, V) \mid f\big([x,y]_\mathfrak{g}\big) = \rho(x)f(y) - \rho(y)f(x) \big\} \tag{4.8}$$

and define $\widehat{\rho} : \mathfrak{g} \to \mathrm{Hom}(\mathrm{Der}(\mathfrak{g}, V), \mathrm{Hom}(\mathfrak{g}, V))$ by

$$\big(\widehat{\rho}(x)(f)\big)y := \mu(x)f(y) + \nu(y)f(x) - f(x \triangleright y), \tag{4.9}$$

where $x, y \in \mathfrak{g}$, $f \in \mathrm{Der}(\mathfrak{g}, V)$. By a straightforward computation, we deduce the following lemma.

Lemma 4.5. *For all $x \in \mathfrak{g}$, we have $\widehat{\rho}(x) \in \mathfrak{gl}(\mathrm{Der}(\mathfrak{g}, V))$.*

Theorem 4.6. *Let $(V;\rho,\mu,\nu)$ be a representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \triangleright)$. Then $(\mathrm{Der}(\mathfrak{g}, V);\widehat{\rho})$ is a representation of the sub-adjacent Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$, where $\widehat{\rho}$ is given by* (4.9)*.*

Proof. By (4.9), for all $x, y, z \in \mathfrak{g}$ and $f \in \mathrm{Der}(\mathfrak{g}, V)$, we have

$$\begin{aligned}
&\Big(\big([\widehat{\rho}(x), \widehat{\rho}(y)] - \widehat{\rho}([x,y]_C)\big)(f)\Big)z \\
&\quad = \mu(x)\big(\mu(y)f(z) + \nu(z)f(y) - f(y \triangleright z)\big) \\
&\qquad + \nu(z)\big(\mu(y)f(x) + \nu(x)f(y) - f(y \triangleright x)\big) \\
&\qquad - \big(\mu(y)f(x \triangleright z) + \nu(x \triangleright z)f(y) - f\big(y \triangleright (x \triangleright z)\big)\big) \\
&\qquad - \mu(y)\big(\mu(x)f(z) + \nu(z)f(x) - f(x \triangleright z)\big) \\
&\qquad - \nu(z)\big(\mu(x)f(y) + \nu(y)f(x) - f(x \triangleright y)\big) \\
&\qquad + \big(\mu(x)f(y \triangleright z) + \nu(y \triangleright z)f(x) - f\big(x \triangleright (y \triangleright z)\big)\big) \\
&\qquad - \mu\big(x \triangleright y - y \triangleright x + [x,y]_\mathfrak{g}\big)f(z) - \nu(z)f\big(x \triangleright y - y \triangleright x + [x,y]_\mathfrak{g}\big) \\
&\qquad + f\big(\big(x \triangleright y - y \triangleright x + [x,y]_\mathfrak{g}\big) \triangleright z\big) \\
&\overset{(3.2),(4.3)}{=} \mu(x)\nu(z)f(y) + \nu(z)\mu(y)f(x) + \nu(z)\nu(x)f(y) - \nu(x \triangleright z)f(y) \\
&\qquad - \mu(y)\nu(z)f(x) - \nu(z)\mu(x)f(y) - \nu(z)\nu(y)f(x) \\
&\qquad + \nu(y \triangleright z)f(x) - \nu(z)f\big([x,y]_\mathfrak{g}\big)
\end{aligned}$$

$$\stackrel{(4.8)}{=} \underline{\mu(x)\nu(z)f(y)}\,\underbrace{+\nu(z)\mu(y)f(x)}\,\underline{+\nu(z)\nu(x)f(y) - \nu(x \rhd z)f(y)}$$
$$\underbrace{\underline{-\mu(y)\nu(z)f(x)}}\,\underline{-\nu(z)\mu(x)f(y)}\,\underbrace{\underline{-\nu(z)\nu(y)f(x) + \nu(y \rhd z)f(x)}}$$
$$\underline{-\nu(z)\rho(x)f(y)}\,\underbrace{\underline{+\nu(z)\rho(y)f(x)}}$$
$$\stackrel{(4.4)}{=} 0.$$

Thus $(\mathrm{Der}(\mathfrak{g}, V); \widehat{\rho})$ is a representation of the sub-adjacent Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$. ■

4.2. Cohomology groups of post-Lie algebras

In this subsection, we define the cohomology groups of a post-Lie algebra with coefficients in an arbitrary representation. Furthermore, we establish a precise relationship between the cohomology groups of a post-Lie algebra and those of its sub-adjacent Lie algebra.

Let $(V; \rho, \mu, \nu)$ be a representation of a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$. We have the natural isomorphism

$$\Phi : \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V) \to \mathrm{Hom}\left(\wedge^{n-1}\mathfrak{g}, \mathrm{Hom}(\mathfrak{g}, V)\right)$$

defined by

$$\big(\Phi(\omega)(x_1, \ldots, x_{n-1})\big)x_n := \omega(x_1, \ldots, x_{n-1}, x_n), \tag{4.10}$$

where $x_1, \ldots, x_{n-1}, x_n \in \mathfrak{g}$. Define the set of 0-cochains to be 0. For $n \geq 1$, we define the set of n-cochains $C^n_{\mathrm{Der}}(\mathfrak{g}, V)$ by

$$C^n_{\mathrm{Der}}(\mathfrak{g}, V) = \left\{f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V) \mid \Phi(f)(x_1, \ldots, x_{n-1}) \in \mathrm{Der}(\mathfrak{g}, V)\right\}.$$

For $f \in C^n_{\mathrm{Der}}(\mathfrak{g}, V)$, $x_1, \ldots, x_{n+1} \in \mathfrak{g}$, define an operator

$$\delta : C^n_{\mathrm{Der}}(\mathfrak{g}, V) \to \mathrm{Hom}(\wedge^n\mathfrak{g} \otimes \mathfrak{g}, V)$$

by

$$\begin{aligned}
&(\delta f)(x_1, \ldots, x_{n+1}) \\
&\quad = \sum_{i=1}^{n}(-1)^{i+1}\mu(x_i)f(x_1, \ldots, \hat{x}_i, \ldots, x_n, x_{n+1}) \\
&\qquad + \sum_{i=1}^{n}(-1)^{i+1}\nu(x_{n+1})f(x_1, \ldots, \hat{x}_i, \ldots, x_n, x_i) \\
&\qquad - \sum_{i=1}^{n}(-1)^{i+1}f(x_1, \ldots, \hat{x}_i, \ldots, x_n, x_i \rhd x_{n+1}) \\
&\qquad + \sum_{1\leq i<j\leq n}(-1)^{i+j}f\big([x_i, x_j]_C, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n, x_{n+1}\big).
\end{aligned} \tag{4.11}$$

Proposition 4.7. *For all $f \in C^n_{\mathrm{Der}}(\mathfrak{g}, V)$, we have $\delta f \in C^{n+1}_{\mathrm{Der}}(\mathfrak{g}, V)$.*

Proof. By the definition of $C^{n+1}_{\mathrm{Der}}(\mathfrak{g}, V)$, we just need to prove that $\Phi(\delta f)(x_1, \dots, x_n)$ is in $\mathrm{Der}(\mathfrak{g}, V)$ for any $x_1, \dots, x_n \in \mathfrak{g}$. For all $x, y \in \mathfrak{g}$, we have

$$
\begin{aligned}
&\Phi(\delta f)(x_1, \dots, x_n)\big([x, y]_{\mathfrak{g}}\big)\\
&\overset{(4.11)}{=} \sum_{i=1}^{n}(-1)^{i+1}\mu(x_i) f\big(x_1, \dots, \hat{x}_i, \dots, x_n, [x, y]_{\mathfrak{g}}\big)\\
&+\sum_{i=1}^{n}(-1)^{i+1}\nu\big([x, y]_{\mathfrak{g}}\big) f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i)\\
&-\sum_{i=1}^{n}(-1)^{i+1} f\big(x_1, \dots, \hat{x}_i, \dots, x_n, x_i \rhd [x, y]_{\mathfrak{g}}\big)\\
&+\sum_{1\le i<j\le n}(-1)^{i+j} f\big([x_i, x_j]_C, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n, [x, y]_{\mathfrak{g}}\big)\\
&\overset{(3.1),(4.2)}{=} \sum_{i=1}^{n}(-1)^{i+1}\mu(x_i)\big(\rho(x) f(x_1, \dots, \hat{x}_i, \dots, x_n, y)\big)\\
&-\sum_{i=1}^{n}(-1)^{i+1}\mu(x_i)\big(\rho(y) f(x_1, \dots, \hat{x}_i, \dots, x_n, x)\big)\\
&+\sum_{i=1}^{n}(-1)^{i+1}\big(\rho(x)\nu(y)-\rho(y)\nu(x)\big) f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i)\\
&-\sum_{i=1}^{n}(-1)^{i+1}\rho(x_i \rhd x) f(x_1, \dots, \hat{x}_i, \dots, x_n, y)\\
&+\sum_{i=1}^{n}(-1)^{i+1}\rho(y) f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i \rhd x)\\
&-\sum_{i=1}^{n}(-1)^{i+1}\rho(x) f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i \rhd y)\\
&+\sum_{i=1}^{n}(-1)^{i+1}\rho(x_i \rhd y) f(x_1, \dots, \hat{x}_i, \dots, x_n, x)\\
&+\sum_{1\le i<j\le n}(-1)^{i+j}\rho(x) f\big([x_i, x_j]_C, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n, y\big)\\
&-\sum_{1\le i<j\le n}(-1)^{i+j}\rho(y) f\big([x_i, x_j]_C, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n, x\big)\\
&\overset{(4.1)}{=} \sum_{i=1}^{n}(-1)^{i+1}\big(\rho(x)\nu(y)-\rho(y)\nu(x)\big) f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i)\\
&+\sum_{i=1}^{n}(-1)^{i+1}\rho(x)\big(\mu(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_n, y)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}(-1)^{i+1}\rho(y)f(x_1,\dots,\hat{x}_i,\dots,x_n,x_i\rhd x)\\
&-\sum_{i=1}^{n}(-1)^{i+1}\rho(x)f(x_1,\dots,\hat{x}_i,\dots,x_n,x_i\rhd y)\\
&-\sum_{i=1}^{n}(-1)^{i+1}\rho(y)\big(\mu(x_i)f(x_1,\dots,\hat{x}_i,\dots,x_n,x)\big)\\
&+\sum_{1\le i<j\le n}(-1)^{i+j}\rho(x)f\left([x_i,x_j]_C,x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n,y\right)\\
&-\sum_{1\le i<j\le n}(-1)^{i+j}\rho(y)f\left([x_i,x_j]_C,x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n,x\right)\\
&=\rho(x)\big(\Phi(\delta f)(x_1,\dots,x_n)y\big)-\rho(y)\big(\Phi(\delta f)(x_1,\dots,x_n)x\big).
\end{aligned}
$$

Thus we deduce that $\Phi(\delta f)(x_1,\dots,x_n)$ is in $\mathrm{Der}(\mathfrak{g},V)$. ■

Proving that the operator δ is indeed a coboundary operator needs some preparation.

Proposition 4.8. *Let* $(V;\rho,\mu,\nu)$ *be a representation of a post-Lie algebra* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\rhd)$. *Then we have* $\Phi\circ\delta=d_{\widehat{\rho}}\circ\Phi$, *where* $d_{\widehat{\rho}}$ *is the coboundary operator of the sub-adjacent Lie algebra* $\mathfrak{g}^C$ *with coefficients in the representation* $(\mathrm{Der}(\mathfrak{g},V);\widehat{\rho})$ *given in Theorem* 4.6 *and* δ *is defined by* (4.11).

Proof. For all $f\in C^n_{\mathrm{Der}}(\mathfrak{g},V)$ and $x_1,\dots,x_{n+1}\in\mathfrak{g}$, we have

$$
\begin{aligned}
&\big(d_{\widehat{\rho}}\big(\Phi(f)\big)(x_1,\dots,x_n)\big)x_{n+1}\\
&=\sum_{i=1}^{n}(-1)^{i+1}\big(\widehat{\rho}(x_i)\Phi(f)(x_1,\dots,\hat{x}_i,\dots,x_n)\big)x_{n+1}\\
&+\sum_{1\le i<j\le n}(-1)^{i+j}\left(\Phi(f)\big([x_i,x_j]_C,x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n\big)\right)x_{n+1}\\
&\overset{(4.9)}{=}\sum_{i=1}^{n}(-1)^{i+1}\mu(x_i)\Phi(f)(x_1,\dots,\hat{x}_i,\dots,x_n)x_{n+1}\\
&+\sum_{i=1}^{n}(-1)^{i+1}\nu(x_{n+1})\Phi(f)(x_1,\dots,\hat{x}_i,\dots,x_n)x_i\\
&-\sum_{i=1}^{n}(-1)^{i+1}\Phi(f)(x_1,\dots,\hat{x}_i,\dots,x_n)(x_i\rhd x_{n+1})\\
&+\sum_{1\le i<j\le n}(-1)^{i+j}\left(\Phi(f)\big([x_i,x_j]_C,x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n\big)\right)x_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(4.10)}{=} \sum_{i=1}^{n}(-1)^{i+1}\mu(x_i)f(x_1,\ldots,\hat{x}_i,\ldots,x_n,x_{n+1})\\
&+\sum_{i=1}^{n}(-1)^{i+1}\nu(x_{n+1})f(x_1,\ldots,\hat{x}_i,\ldots,x_n,x_i)\\
&-\sum_{i=1}^{n}(-1)^{i+1}f(x_1,\ldots,\hat{x}_i,\ldots,x_n,x_i\rhd x_{n+1})\\
&+\sum_{1\le i<j\le n}(-1)^{i+j}f\big([x_i,x_j]_C,x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_n,x_{n+1}\big)\\
&\overset{(4.11)}{=} (\delta f)(x_1,\ldots,x_{n+1})\\
&\overset{(4.10)}{=} \big(\Phi(\delta f)(x_1,\ldots,x_n)\big)x_{n+1},
\end{aligned}
$$

which implies that $d_{\widehat{\rho}}\circ\Phi=\Phi\circ\delta$. ■

Theorem 4.9. *The operator* $\delta: C^n_{\mathrm{Der}}(\mathfrak{g},V)\to C^{n+1}_{\mathrm{Der}}(\mathfrak{g},V)$ *defined by* (4.11) *satisfies* $\delta\circ\delta=0$.

Proof. By Proposition 4.8, we have $\delta=\Phi^{-1}\circ d_{\widehat{\rho}}\circ\Phi$. Thus, by the fact that $d_{\widehat{\rho}}\circ d_{\widehat{\rho}}=0$, we obtain $\delta\circ\delta=\Phi^{-1}\circ d_{\widehat{\rho}}\circ d_{\widehat{\rho}}\circ\Phi=0$. ■

Let $(V;\rho,\mu,\nu)$ be a representation of a post-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\rhd)$. Denote

$$C^*_{\mathrm{Der}}(\mathfrak{g},V):=\bigoplus_{n\ge 1}C^n_{\mathrm{Der}}(\mathfrak{g},V).$$

Then we have the cochain complex $(C^*_{\mathrm{Der}}(\mathfrak{g},V),\delta)$. Denote the set of closed n-cochains by $Z^n(\mathfrak{g},V)$ and the set of exact n-cochains by $B^n(\mathfrak{g},V)$. We denote

$$H^n(\mathfrak{g},V)=Z^n(\mathfrak{g},V)/B^n(\mathfrak{g},V)$$

and call them the *cohomology groups of the post-Lie algebra* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\rhd)$ with coefficients in the representation $(V;\rho,\mu,\nu)$.

It is obvious that $f\in C^1_{\mathrm{Der}}(\mathfrak{g},V)$ is closed if and only if $f\in \mathrm{Der}(\mathfrak{g},V)$ and

$$\nu(y)f(x)+\mu(x)f(y)-f(x\rhd y)=0,\quad \forall x,y\in\mathfrak{g}.$$

Also $f\in C^2_{\mathrm{Der}}(\mathfrak{g},V)$ is closed if and only if $\Phi(f)\in\mathrm{Hom}(\mathfrak{g},\mathrm{Der}(\mathfrak{g},V))$ and

$$
\begin{aligned}
&\nu(x_3)f(x_2,x_1)+\mu(x_1)f(x_2,x_3)-f(x_2,x_1\rhd x_3)-\nu(x_3)f(x_1,x_2)\\
&-\mu(x_2)f(x_1,x_3)+f(x_1,x_2\rhd x_3)-f(x_1\rhd x_2,x_3)+f(x_2\rhd x_1,x_3)\\
&-f\big([x_1,x_2]_{\mathfrak{g}},x_3\big)=0,\quad \forall x_1,x_2,x_3\in\mathfrak{g}.
\end{aligned}
$$

There is a close relationship between the cohomology groups of post-Lie algebras and those of the corresponding sub-adjacent Lie algebras.

Theorem 4.10. *Let $(V;\rho,\mu,\nu)$ be a representation of a post-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\rhd)$. Then the cohomology group $H^n(\mathfrak{g},V)$ of the post-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\rhd)$ and the cohomology group $H^{n-1}(\mathfrak{g}^C,\mathrm{Der}(\mathfrak{g},V))$ of the sub-adjacent Lie algebra $\mathfrak{g}^C$ are isomorphic for all $n\geq 1$.*

Proof. By Proposition 4.8, we deduce that Φ is an isomorphism from the cochain complex $(C^*_{\mathrm{Der}}(\mathfrak{g},V),\delta)$ to the cochain complex $(\mathcal{C}^{*-1}(\mathfrak{g},\mathrm{Der}(\mathfrak{g},V)),d_{\widehat{\rho}})$. Thus, Φ induces an isomorphism Φ_* from $H^*(\mathfrak{g},V)$ to $H^{*-1}(\mathfrak{g}^C,\mathrm{Der}(\mathfrak{g},V))$. ■

In the above theorem, if $[\cdot,\cdot]_{\mathfrak{g}}$ and ρ are zero, then the post-Lie algebra is a pre-Lie algebra and we recover the result of [16] as follows.

Corollary 4.11. *Let $(V;\mu,\nu)$ be a representation of a pre-Lie algebra $(\mathfrak{g},\rhd)$. Then the cohomology group $H^n(\mathfrak{g},V)$ of the pre-Lie algebra $(\mathfrak{g},\rhd)$ and the cohomology group $H^{n-1}(\mathfrak{g}^C,\mathrm{Hom}(\mathfrak{g},V))$ of the sub-adjacent Lie algebra $\mathfrak{g}^C$ are isomorphic for all $n\geq 1$.*

4.3. Classification of 2-term skeletal operator homotopy post-Lie algebras

In this subsection, we first give an equivalent definition of an operator homotopy post-Lie algebra and then classify 2-term skeletal operator homotopy post-Lie algebras using the third cohomology group given in Section 4.2.

For all $i\geq 1$, let $\Theta_i:\wedge^{i-1}\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ be a graded linear map of degree $2-i$. Define

$$D(\Theta_i):\odot^{i-1}s^{-1}\mathfrak{g}\otimes s^{-1}\mathfrak{g}\to s^{-1}\mathfrak{g}$$

by

$$D(\Theta_i)=(-1)^{\frac{i(i-1)}{2}}s^{-1}\circ\Theta_i\circ s^{\otimes i},$$

which is a graded linear map of degree 1.

Using this process, we can give an equivalent definition of an operator homotopy post-Lie algebra as follows.

Definition 4.12. An *operator homotopy post-Lie algebra* is a graded Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ equipped with a collection of linear maps $\Theta_k:\otimes^k\mathfrak{g}\to\mathfrak{g}$, $k\geq 1$, of degree $2-k$ satisfying that, for any homogeneous elements $x_1,\dots,x_n,x_{n+1}\in\mathfrak{g}$, the following conditions hold:

(i) (*graded antisymmetry*) for every $\sigma\in\mathbb{S}_{n-1}$, $n\geq 1$,

$$\Theta_n(x_{\sigma(1)},\dots,x_{\sigma(n-1)},x_n)=\chi(\sigma)\Theta_n(x_1,\dots,x_{n-1},x_n),\tag{4.12}$$

(ii) (*graded derivation*) for all $n\geq 1$,

$$\begin{aligned}\Theta_n\big(x_1,\dots,x_{n-1},[x_n,x_{n+1}]_{\mathfrak{g}}\big)&\\ =\big[\Theta_n(x_1,\dots,x_{n-1},x_n),x_{n+1}\big]_{\mathfrak{g}}&\\ +(-1)^{x_n(x_1+\cdots+x_{n-1}+n)}\big[x_n,\Theta_n(x_1,\dots,x_{n-1},x_{n+1})\big]_{\mathfrak{g}},&\end{aligned}\tag{4.13}$$

(iii) for all $n \geq 1$,

$$\sum_{1\leq i<j\leq n-1} (-1)^{\beta}\Theta_{n-1}\big([x_i,x_j]_{\mathfrak{g}},x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n\big)$$
$$= \sum_{\substack{i+j=n+1,\, i\geq 1,\, j\geq 2\\ \sigma\in\mathbb{S}_{(i-1,1,j-2)}}} (-1)^{i(j-1)}\chi(\sigma) \times \Theta_j\big(\Theta_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),x_{\sigma(i+1)},\dots,x_{\sigma(n-1)},x_n\big)$$
$$+ \sum_{\substack{i+j=n+1,\, i\geq 1,\, j\geq 1\\ \sigma\in\mathbb{S}_{(j-1,i-1)}}} (-1)^{j-1}(-1)^{\alpha}\chi(\sigma) \times \Theta_j\big(x_{\sigma(1)},\dots,x_{\sigma(j-1)},\Theta_i(x_{\sigma(j)},\dots,x_{\sigma(n-1)},x_n)\big), \tag{4.14}$$

where $\beta = x_i(x_1+\cdots+x_{i-1}) + x_j(x_1+\cdots+x_{j-1}) + x_ix_j + i + j$ and $\alpha = i(x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(j-1)})$.

By (4.13), for $n = 1$, we have

$$\Theta_1\big([x_1,x_2]_{\mathfrak{g}}\big) = \big[\Theta_1(x_1),x_2\big]_{\mathfrak{g}} + (-1)^{x_1}\big[x_1,\Theta_1(x_2)\big]_{\mathfrak{g}}. \tag{4.15}$$

By (4.14), for $n = 1$, we have $\Theta_1^2 = 0$ and so $(\mathfrak{g},\Theta_1)$ is a complex. By (4.14), for $n = 2$, we have

$$0 = -\Theta_2\big(\Theta_1(x_1),x_2\big) - (-1)^{x_1}\Theta_2\big(x_1,\Theta_1(x_2)\big) + \Theta_1\big(\Theta_2(x_1,x_2)\big). \tag{4.16}$$

Now we will show that the corresponding cohomology space $H^*(\mathfrak{g})$ of the complex $(\mathfrak{g},\Theta_1)$ enjoys a graded post-Lie algebra structure and this justifies our definition of an "operator homotopy post-Lie algebra".

Theorem 4.13. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\{\Theta_k\}_{k=1}^{+\infty})$ be an operator homotopy post-Lie algebra. Then the cohomology space $H^*(\mathfrak{g})$ is a graded post-Lie algebra.*

Proof. For any homogeneous element $x \in \ker(\Theta_1)$, we denote by $\bar{x} \in H^*(\mathfrak{g})$ its cohomological class. First, we define a graded bracket operation $[\cdot,\cdot]$ on the graded vector space $H^*(\mathfrak{g})$ by

$$[\bar{x},\bar{y}] := \overline{[x,y]_{\mathfrak{g}}}, \quad \forall \bar{x},\bar{y} \in H^*(\mathfrak{g}).$$

If $\bar{x} = \bar{x}'$, then there exists $X \in \mathfrak{g}$ such that $x' = x + \Theta_1(X)$. Hence by (4.15), we have

$$[\bar{x}',\bar{y}] = \overline{\big[x+\Theta_1(X),y\big]_{\mathfrak{g}}} = \overline{[x,y]_{\mathfrak{g}}} + \overline{\Theta_1\big([X,y]_{\mathfrak{g}}\big)} = \overline{[x,y]_{\mathfrak{g}}} = [\bar{x},\bar{y}],$$

which implies that $[\cdot,\cdot]$ is well defined. It is straightforward to obtain that $(H^*(\mathfrak{g}),[\cdot,\cdot])$ is a graded Lie algebra.

Then we define a multiplication $\rhd$ on the graded vector space $H^*(\mathfrak{g})$ by

$$\bar{x} \rhd \bar{y} := \overline{\Theta_2(x,y)}, \quad \forall \bar{x},\bar{y} \in H^*(\mathfrak{g}).$$

Similarly, by (4.16), we can deduce that $\rhd$ is well defined. By (4.13) for $n = 2$, we have

$$\begin{aligned}\bar{x} \rhd [\bar{y}, \bar{z}] &= \overline{\Theta_2(x, [y, z]_{\mathfrak{g}})} = \overline{[\Theta_2(x, y), z]_{\mathfrak{g}}} + (-1)^{xy}\overline{[y, \Theta_2(x, z)]_{\mathfrak{g}}} \\ &= [\bar{x} \rhd \bar{y}, \bar{z}] + +(-1)^{xy}[\bar{y}, \bar{x} \rhd \bar{z}].\end{aligned}$$

In the same way, by (4.14) for $n = 3$, we have

$$[\bar{x}, \bar{y}] \rhd \bar{z} = a_{\rhd}(\bar{x}, \bar{y}, \bar{z}) - a_{\rhd}(\bar{y}, \bar{x}, \bar{z}).$$

Therefore, $(H^*(\mathfrak{g}), [\cdot, \cdot], \rhd)$ is a graded post-Lie algebra. ■

By truncation, we obtain the definition of a 2-term operator homotopy post-Lie algebra.

Definition 4.14. A 2-*term operator homotopy post-Lie algebra* is a 2-term graded Lie algebra $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, [\cdot, \cdot]_{\mathfrak{g}})$ equipped with

- a linear map $\Theta_1 : \mathfrak{g}_{-1} \to \mathfrak{g}_0$,
- a linear map $\Theta_2 : \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}, -1 \le i + j \le 0$,
- a linear map $\Theta_3 : \wedge^2\mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_{-1}$

such that for all $x, y, z, w \in \mathfrak{g}_0$ and $a, b \in \mathfrak{g}_{-1}$, the following equalities hold:

(a_1) $\Theta_1([x, a]_{\mathfrak{g}}) = [x, \Theta_1(a)]_{\mathfrak{g}}$;

(a_2) $[\Theta_1(a), b]_{\mathfrak{g}} = [a, \Theta_1(b)]_{\mathfrak{g}}$;

(b_1) $\Theta_2(x, [y, z]_{\mathfrak{g}}) = [\Theta_2(x, y), z]_{\mathfrak{g}} + [y, \Theta_2(x, z)]_{\mathfrak{g}}$;

(b_2) $\Theta_2(x, [y, a]_{\mathfrak{g}}) = [\Theta_2(x, y), a]_{\mathfrak{g}} + [y, \Theta_2(x, a)]_{\mathfrak{g}}$;

(b_3) $\Theta_2(a, [x, y]_{\mathfrak{g}}) = [\Theta_2(a, x), y]_{\mathfrak{g}} + [x, \Theta_2(a, y)]_{\mathfrak{g}}$;

(c) $\Theta_3(x, y, [z, w]_{\mathfrak{g}}) = [\Theta_3(x, y, z), w]_{\mathfrak{g}} + [z, \Theta_3(x, y, w)]_{\mathfrak{g}}$;

(d_1) $\Theta_1\Theta_2(x, a) = \Theta_2(x, \Theta_1(a))$;

(d_2) $\Theta_1\Theta_2(a, x) = \Theta_2(\Theta_1(a), x)$;

(d_3) $\Theta_2(\Theta_1(a), b) = \Theta_2(a, \Theta_1(b))$;

(e_1) $\Theta_2(x, \Theta_2(y, z)) - \Theta_2(\Theta_2(x, y), z) - \Theta_2(y, \Theta_2(x, z)) + \Theta_2(\Theta_2(y, x), z)$
$= \Theta_2([x, y]_{\mathfrak{g}}, z) + \Theta_1\Theta_3(x, y, z)$;

(e_2) $\Theta_2(x, \Theta_2(y, a)) - \Theta_2(\Theta_2(x, y), a) - \Theta_2(y, \Theta_2(x, a)) + \Theta_2(\Theta_2(y, x), a)$
$= \Theta_2([x, y]_{\mathfrak{g}}, a) + \Theta_3(x, y, \Theta_1(a))$;

(e_3) $\Theta_2(a, \Theta_2(y, z)) - \Theta_2(\Theta_2(a, y), z) - \Theta_2(y, \Theta_2(a, z)) + \Theta_2(\Theta_2(y, a), z)$
$= \Theta_2([a, y]_{\mathfrak{g}}, z) + \Theta_3(\Theta_1(a), y, z)$;

(f) $\Theta_2(x, \Theta_3(y, z, w)) - \Theta_2(y, \Theta_3(x, z, w)) + \Theta_2(z, \Theta_3(x, y, w))$
$+\Theta_2(\Theta_3(y, z, x), w) - \Theta_2(\Theta_3(x, z, y), w) + \Theta_2(\Theta_3(x, y, z), w)$
$-\Theta_3(\Theta_2(x, y) - \Theta_2(y, x) + [x, y]_{\mathfrak{g}}, z, w) - \Theta_3(\Theta_2(y, z) - \Theta_2(z, y)$
$+[y, z]_{\mathfrak{g}}, x, w) + \Theta_3(\Theta_2(x, z) - \Theta_2(z, x) + [x, z]_{\mathfrak{g}}, y, w) - \Theta_3(y, z, \Theta_2(x, w))$
$+\Theta_3(x, z, \Theta_2(y, w)) - \Theta_3(x, y, \Theta_2(z, w)) = 0.$

A 2-term operator homotopy post-Lie algebra $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, [\cdot,\cdot]_{\mathfrak{g}}, \Theta_1, \Theta_2, \Theta_3)$ is said to be *skeletal* if $\Theta_1 = 0$.

Remark 4.15. If the underlying graded Lie algebra $(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, [\cdot,\cdot]_{\mathfrak{g}})$ in a 2-term operator homotopy post-Lie algebra $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, [\cdot,\cdot]_{\mathfrak{g}}, \Theta_1, \Theta_2, \Theta_3)$ is abelian, then $(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \Theta_1, \Theta_2, \Theta_3)$ reduces to a 2-term pre-Lie$_\infty$-algebra or equivalently a pre-Lie 2-algebra.

In [42], the author showed that skeletal pre-Lie 2-algebras are classified by the third cohomology group of pre-Lie algebras. See [2] also for more details of the classification of skeletal Lie 2-algebras. Similarly, we have the following theorem.

Theorem 4.16. *There is a one-to-one correspondence between* 2*-term skeletal operator homotopy post-Lie algebras and triples* $((\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd), (V;\rho,\mu,\nu), \omega)$*, where* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ *is a post-Lie algebra,* $(V;\rho,\mu,\nu)$ *is a representation of the post-Lie algebra* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$*, and* $\omega \in \mathrm{Hom}(\wedge^2\mathfrak{h} \otimes \mathfrak{h}, V)$ *is a* 3*-cocycle of* $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \rhd)$ *with coefficients in* $(V;\rho,\mu,\nu)$*.*

Proof. Let $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, [\cdot,\cdot]_{\mathfrak{g}}, \Theta_1, \Theta_2, \Theta_3)$ be a 2-term skeletal operator homotopy post-Lie algebra, i.e., $\Theta_1 = 0$. Then by condition (e_1) in Definition 4.14, we deduce that $(\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}}, \Theta_2)$ is a post-Lie algebra. Define linear maps ρ, μ, ν from $\mathfrak{g}_0$ to $\mathfrak{gl}(\mathfrak{g}_{-1})$ by

$$\rho(x)(a) := [x,a]_{\mathfrak{g}}, \quad \mu(x)(a) := \Theta_2(x,a), \quad \nu(x)(a) := \Theta_2(a,x),$$

where $x \in \mathfrak{g}_0, a \in \mathfrak{g}_{-1}$. Obviously, $(\mathfrak{g}_{-1};\rho)$ is a representation of the Lie algebra $(\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}})$. Then by (b_2), (b_3), (e_2), and (e_3) in Definition 4.14, we deduce (4.1)–(4.4), respectively. Thus $(\mathfrak{g}_{-1};\rho,\mu,\nu)$ is a representation of the post-Lie algebra $(\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}}, \Theta_2)$. Finally, by (c) and (f) in Definition 4.14, we deduce that Θ_3 is a 3-cocycle of the post-Lie algebra $(\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}}, \Theta_2)$ with coefficients in the representation $(\mathfrak{g}_{-1};\rho,\mu,\nu)$.

The proof of the other direction is similar. So the details will be omitted. ■

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