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The Arens-Michael envelopes of the Jordan plane and $U_q(\mathfrak{sl}(2))$

Dmitrii Pedchenko

Abstract. The Arens-Michael functor in noncommutative geometry is an analogue of the analytification functor in algebraic geometry: out of the ring of "algebraic functions" on a noncommutative affine scheme, it constructs the ring of "holomorphic functions" on it when viewed as a noncommutative complex analytic space. In this paper, we explicitly compute the Arens-Michael envelopes of the Jordan plane and the quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$ for $|q| = 1$.

1. Introduction

The basic idea of noncommutative geometry is to view an arbitrary (noncommutative) algebra as an "algebra of functions on a noncommutative space". This idea is based on an observation that many important geometric concepts and constructions stated in algebraic terms remain meaningful for noncommutative algebras providing us with the tools and intuition for studying these algebras.

Noncommutative geometry currently includes several mathematical disciplines which have different research objects but are unified by that aforementioned idea. Noncommutative measure theory studies von Neumann algebras; noncommutative topology studies C*-algebras; noncommutative affine algebraic geometry studies finitely generated algebras; noncommutative differential geometry studies dense subalgebras of C*-algebras equipped with a special "differential" structure.

Notice that an important discipline is missing from this list – noncommutative complex analytic geometry which in the commutative case bridges differential and algebraic geometry. One of the main reasons of why this theory is underdeveloped is that it is unclear which algebras should be considered the noncommutative generalizations of the algebras of holomorphic functions on complex analytic spaces.

The ideal starting point for the development of any kind of noncommutative geometry is having a category $\mathcal{A}$ of associative algebras such that the full subcategory of $\mathcal{A}$ consisting of commutative algebras is anti-equivalent to a certain category $\mathcal{C}$ of "spaces". In such a situation, we may think of algebras belonging to $\mathcal{A}$ as the noncommutative analogues of the spaces belonging to $\mathcal{C}$. Therefore, in the case of noncommutative complex analytic geometry we would like to start with some category consisting of algebras such that the

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commutative ones are exactly the algebras of holomorphic functions on complex analytic spaces. As Pirkovskii states in [5], apparently in full generality such a class of algebras has not yet been introduced.

However, we can simplify our problem as follows. Any affine algebraic variety over $\mathbb{C}$ is certainly a complex analytic space. So we can narrow down the category $\mathcal{C}$ of complex analytic spaces to the category of affine algebraic varieties over $\mathbb{C}$ viewed as complex analytic spaces. It turns out that in this case we have a construction that for each finitely generated $\mathbb{C}$-algebra A (deemed as the "algebra of regular algebraic functions on a noncommutative affine scheme of finite type") assigns a new algebra $\widehat{A}$ (deemed as the "algebra of holomorphic functions on that scheme") such that if our algebra was a commutative algebra of regular algebraic functions $A = \mathcal{O}^{\mathrm{alg}}(X)$ on an affine scheme X of finite type, then we obtain the algebra of holomorphic functions on X: $\widehat{A} = \mathcal{O}^{\mathrm{hol}}(X)$.

The resulting algebra $\widehat{A}$ is known as the Arens-Michael envelope of the algebra A. Therefore, we view the Arens-Michael envelopes of finitely generated algebras as the algebras of holomorphic functions on noncommutative affine schemes of finite type. This way, the Arens-Michael functor can be viewed as a generalization of the classic analytification functor in complex algebraic geometry to the noncommutative setting.

So far, the Arens-Michael envelopes are explicitly known only for a handful of noncommutative algebras (see Section 3 and [6, Section 5]), and it is important for the development of the theory and its scope of applicability to grow the body of examples.

In this paper, we add two algebras to the list of examples: we explicitly compute the Arens-Michael envelopes of the Jordan plane (see Theorem 5.4) and the quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ for $|q| = 1$ (see Theorem 6.3) following the recently developed techniques in this area. We find that the Arens-Michael envelopes of the Jordan plane and the quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ for $|q| = 1$, $q \neq 1, -1$ are given as convergent power series with a "twisted" multiplication, and in this sense are seen as a direct generalization of holomorphic functions in the noncommutative setup. In contrast, the Arens-Michael envelope of the quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ for $q = 1$ comes in a representation-theoretic form (see Example 3.8 and Theorem 6.3 (1)).

Organization of the paper. In Section 2, we review the basic definitions pertaining to the Arens-Michael functor. In Section 3, we recall some known examples of algebras for which the Arens-Michael envelope is explicitly known. In Section 4, we review the theoretical constructions necessary for our computations following [6].

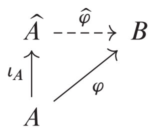
Finally, we present our key results in Sections 5 and 6. In Section 5, we explicitly compute the Arens-Michael envelope of the Jordan plane. In Section 6, we explicitly compute the Arens-Michael envelope of the quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ for $|q| = 1$.

2. The Arens-Michael functor

In what follows all vector spaces and algebras are taken over the field of complex numbers $\mathbb{C}$; all algebras are assumed to be associative and unital. A seminorm $\|\cdot\|$ on an algebra

A is called *submultiplicative* if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. A complete topological algebra with a topology generated by a family of submultiplicative seminorms is called an *Arens-Michael algebra*. We start with our main definitions.

Definition 2.1. Let A be a topological algebra. The pair $(\widehat{A}, \iota_A)$, consisting of an Arens-Michael algebra $\widehat{A}$ and a continuous homomorphism $\iota_A : A \to \widehat{A}$, is called *the Arens-Michael envelope* of algebra A if for an arbitrary Arens-Michael algebra B and an arbitrary continuous homomorphism $\varphi : A \to B$, there exists a unique continuous homomorphism $\widehat{\varphi} : \widehat{A} \to B$ such that the following diagram commutes:



Arens-Michael envelopes were introduced by Taylor [10], but here we are using the terminology of Helemskii [1] and Pirkovskii [6, Section 3].

It is clear from the definition that the Arens-Michael envelope is unique up to a unique isomorphism of topological algebras over A. Moreover, it always exists [10], and it can be obtained as the completion of A with respect to all continuous submultiplicative seminorms on A. Note that the topology induced by submultiplicative seminorms might be non-Hausdorff so that before taking the completion we should take the quotient by the closure of $\{0\}$. Therefore, the canonical homomorphism $\iota_A : A \to \widehat{A}$ might have a nontrivial kernel.

Definition 2.2. Let A be an algebra without a topology. The *Arens-Michael envelope* of A is the Arens-Michael envelope of A endowed with the strongest locally convex topology in the sense of Definition 2.1.

Finally, let us note that the association $A \mapsto \widehat{A}$ extends to algebra homomorphisms $A \to B$ so that we obtain a functor from the category of algebras to the category of Arens-Michael algebras (*the Arens-Michael functor*).

3. Examples of Arens-Michael envelopes

Next we discuss some known examples of the Arens-Michael envelopes.

The next example and proposition justify our assertions from the introduction about the Arens-Michael functor being a noncommutative analogue of the analytification functor in algebraic geometry.

Example 3.1. As was noted by Taylor [10], the Arens-Michael envelope of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n] = \mathcal{O}^{\mathrm{alg}}(\mathbb{C}^n)$ is the algebra of holomorphic functions $\mathcal{O}^{\mathrm{hol}}(\mathbb{C}^n)$ with compact-open topology.

Pirkovskii generalized this statement to affine algebraic varieties.

Proposition 3.2 ([5, Proposition 1] and [6, Example 3.6]). *Let X be an affine algebraic variety over $\mathbb{C}$ and let $A = \mathcal{O}^{\mathrm{alg}}(X)$ be the algebra of regular algebraic functions on X. The Arens-Michael envelope of A is the algebra $\mathcal{O}^{\mathrm{hol}}(X_{\mathrm{an}})$ of holomorphic functions on X when we view X as a complex analytic space, with compact-open topology. The same is true for the affine schemes of finite type over $\mathbb{C}$.*

From this proposition, we see that the geometric analytification functor associating to an affine algebraic scheme X a complex analytic space X_{an} corresponds to the algebraic or functional-analytic Arens-Michael functor when we instead work with functions on those spaces. As we noted in the introduction, the finitely generated noncommutative algebras are the natural candidates for the noncommutative affine schemes of finite type, so we view the Arens-Michael envelopes of the finitely generated noncommutative algebras as the algebras of holomorphic functions on the noncommutative affine schemes of finite type.

Here is the "most noncommutative" example:

Example 3.3 (The free algebra). Let $F_n = \mathbb{C}\langle x_1, \ldots, x_n\rangle$ be a free algebra with n generators. For each k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ of integers, $1 \le \alpha_i \le n$, set $x_\alpha = x_{\alpha_1} \cdots x_{\alpha_k}$ and $|\alpha| = k$. Then each element of F_n is written as a noncommutative polynomial $\sum_{|\alpha|\le N} c_\alpha x_\alpha$. Denote the set of all α by W_n.

Taylor [10] showed that

$$\widehat{F_n} = \left\{ a = \sum_{\alpha\in W_n} c_\alpha x_\alpha : \|a\|_\rho = \sum_{\alpha\in W_n} |c_\alpha|\rho^{|\alpha|} < \infty \text{ for any } \rho > 0 \right\}.$$

The topology on $\widehat{F_n}$ is defined by the family of seminorms $\{\|\cdot\|_\rho : \rho \in \mathbb{R}_{>0}\}$.

Example 3.4 (The quantum plane). Fix a complex number $q \in \mathbb{C}\setminus\{0\}$. *The quantum plane* is an algebra (denoted by $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^2)$) with two generators x, y, subject to a relation $xy = qyx$. The monomials $x^i y^j$ $(i, j \ge 0)$ form a basis of $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^2)$ so that this algebra can be viewed as an algebra of polynomials with a "twisted" multiplication.

Denote the Arens-Michael envelope of $\mathcal{O}_q^{\mathrm{alg}}(\mathbb{C}^2)$ by $\mathcal{O}_q^{\mathrm{hol}}(\mathbb{C}^2)$, and view it as an algebra of holomorphic functions on the quantum plane. The next result is due to Pirkovskii.

Proposition 3.5 ([6, Corollary 5.14]). *Let $q \in \mathbb{C}\setminus\{0\}$.*

(1) *If $|q| \ge 1$, then*

$$\mathcal{O}_q^{\mathrm{hol}}(\mathbb{C}^2) = \left\{ a = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j : \|a\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}|\rho^{i+j} < \infty \textit{ for any } \rho > 0 \right\}.$$

(2) *If $|q| \le 1$, then*

$$\mathcal{O}_q^{\mathrm{hol}}(\mathbb{C}^2) = \left\{ a = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j : \|a\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}||q|^{ij}\rho^{i+j} < \infty \textit{ for any } \rho > 0 \right\}.$$

In both cases the topology on $\mathcal{O}_q^{\mathrm{hol}}(\mathbb{C}^2)$ is generated by the family of seminorms $\{\|\cdot\|_\rho : \rho \in \mathbb{R}_{>0}\}$ and the multiplication is defined by the relation $xy = qyx$.

Example 3.6. Consider the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ with basis $\{x, y\}$ and the commuting relation $[x, y] = y$. Due to the Poincaré–Birkhoff–Witt theorem, the universal enveloping algebra $U(\mathfrak{g})$ can be viewed as a polynomial algebra with a "twisted" multiplication. As shown in [6, Example 5.1],

$$\widehat{U}(\mathfrak{g}) = \left\{a = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j : \sum_{i=0}^{\infty} |c_{ij}|\rho^i < \infty, \forall j \in \mathbb{Z}_+, \forall \rho > 0\right\}.$$

The topology of $\widehat{U}(\mathfrak{g})$ is generated by the family of seminorms

$$\left\{\|\cdot\|_{n,\rho} : \left\|\sum_{i,j=0}^{\infty} c_{ij} x^i y^j\right\|_{n,\rho} = \sum_{j=0}^{n}\sum_{i=0}^{\infty} |c_{ij}|\rho^i < \infty,\ n \in \mathbb{Z}_+,\ \rho \in \mathbb{R}_{>0}\right\}.$$

Remark 3.7. In Examples 3.3–3.6 above, one could instead generate the topology by a *countable* family of submultiplicative seminorms. Such complete locally m-convex algebras whose topology can be generated by a countable family of submultiplicative seminorms are called *Fréchet–Arens-Michael algebras* or *m-convex Fréchet algebras*.

In the previous examples, the Arens-Michael envelopes of polynomial algebras with a "twisted" multiplication happened to be algebras of "noncommutative (convergent) power series", as one would expect by comparing to the commutative case. Interestingly, the next example (due to Taylor [10]) shows that this is not always the case.

Example 3.8 (The universal enveloping algebra of a semisimple Lie algebra). Suppose that $\mathfrak{g}$ is a semisimple Lie algebra. Every finite-dimensional irreducible representation π_λ of the algebra $\mathfrak{g}$ extends to a homomorphism

$$\pi_\lambda : U(\mathfrak{g}) \to M_{d_\lambda}(\mathbb{C}) \quad (d_\lambda = \dim \pi_\lambda).$$

If we denote the set of equivalence classes of irreducible finite-dimensional representations of $\mathfrak{g}$ by $\hat{\mathfrak{g}}$, we get a homomorphism

$$\prod_{\lambda \in \hat{\mathfrak{g}}} \pi_\lambda : U(\mathfrak{g}) \to \prod_{\lambda \in \hat{\mathfrak{g}}} M_{d_\lambda}(\mathbb{C}).$$

The algebra $\prod_{\lambda \in \hat{\mathfrak{g}}} M_{d_\lambda}(\mathbb{C})$ with the product topology and the homomorphism $\prod_{\lambda \in \hat{\mathfrak{g}}} \pi_\lambda$ form the Arens-Michael envelope of $U(\mathfrak{g})$.

This example is a bit discouraging since contrary to the above examples, this time the Arens-Michael envelope looks completely different from the initial algebra (for example, $U(\mathfrak{g})$ is an integral domain but $\prod_{\lambda \in \hat{\mathfrak{g}}} M_{d_\lambda}(\mathbb{C})$ is not). Nonetheless, the canonical homomorphism $A \to \widehat{A}$ is injective (as it was in all other examples so far).

The next example shows the worst possible situation.

Example 3.9 (Weyl's algebra). The Weyl algebra A is an algebra with two generators x, ∂ with the commutation relation $[\partial, x] = 1$. It is well known that in a non-zero normed algebra there are no elements with this commutation relation. Therefore, $\widehat{A} = 0$ and the canonical homomorphism is not injective.

It is interesting to note that if we quantize the Weyl algebra by taking the commutation relation to be

$$\partial x - qx\partial = 1 \quad (q \neq 0, 1),$$

the resulting Arens-Michael envelope would again be the algebra of "noncommutative" power series (see [6, Corollary 5.19]). In the case $0 < q < 1$, this can also be explained by noting that the above q-Weyl algebra admits a different presentation as a quantum disk whose Arens-Michael envelope is an algebra of "noncommutative" power series; see [7, Sections 4 and 5]. The connection between quantum Weyl algebras and quantum balls and polydisks can be traced back to the work of Vaksman [11].

For more examples of Arens-Michael envelopes, see [5, 6].

4. Theoretical constructions

In this section, we collect the theoretical facts necessary for our computations following [4, 6]. We will be referring to a complete, Hausdorff, locally convex topological algebra with jointly continuous multiplication as an $\widehat{\otimes}$-*algebra*.

4.1. Arens-Michael envelopes of tensor products

First, we recall how to describe the topology on the projective tensor product of two $\widehat{\otimes}$-modules.

Proposition 4.1 ([6, Proposition 2.3 (vi)]). *Suppose that A is an $\widehat{\otimes}$-algebra, X is a right A-$\widehat{\otimes}$-module, Y is left A-$\widehat{\otimes}$-module. Furthermore, suppose that both X and Y have countable or finite dimension and the topology on X and Y is the strongest locally convex topology. Then the algebraic tensor product $X \otimes_A Y$ with the strongest locally convex topology coincides with the projective tensor product $X \widehat{\otimes}_A Y$.*

The next proposition shows that the Arens-Michael envelope of the projective tensor product of two $\widehat{\otimes}$-algebras can be computed as the projective tensor product of their Arens-Michael envelopes.

Proposition 4.2 ([4, Proposition 6.4]). *Let A, B be $\widehat{\otimes}$-algebras. Then there exists a topological algebra isomorphism*

$$(A \widehat{\otimes} B)\widehat{\ } \cong \widehat{A} \widehat{\otimes} \widehat{B}.$$

In other words, the operations of taking the Arens-Michael envelope and taking the projective tensor product can be interchanged.

4.2. Arens-Michael envelopes of Ore extensions

The computation of the Arens-Michael envelopes of many polynomial algebras (including Examples 3.4 and 3.6) is greatly facilitated by a theoretical construction known as an *Ore extension*.

4.2.1. Algebraic Ore extensions. First, we consider a purely algebraic construction.

Definition 4.3. Let R be an associative $\mathbb{C}$-algebra (without a topology) and $\alpha : R \to R$ an algebra endomorphism. A $\mathbb{C}$-linear map $\delta : R \to R$ is called *α-derivation* if

$$\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$$

for any $a, b \in R$. The *Ore extension* $R[z; \alpha, \delta]$ is a noncommutative algebra obtained by endowing the left R-module of polynomials $\sum_{i=0}^{n} r_i z^i$ with a "twisted" multiplication with a relation

$$zr = \alpha(r)z + \delta(r)$$

for $r \in R$. Note that the natural inclusions $R \hookrightarrow R[z; \alpha, \delta]$ and $\mathbb{C}[z] \hookrightarrow R[z; \alpha, \delta]$ become algebra homomorphisms.

Let us also recall a useful formula describing multiplication in $R[z; \alpha, \delta]$. For any $k, n \in \mathbb{Z}_{>0}$ with $k \leq n$, let $S_{n,k} : R \to R$ denote an operator defined as the sum of all $\binom{n}{k}$ different compositions of k derivations δ and $n - k$ homomorphisms α. Then for any $r \in R$, we have the following formula for how to commute z^n and r (see [6, Section 4.1]):

$$z^n r = \sum_{k=0}^{n} S_{n,k}(r) z^{n-k}. \tag{4.3.1}$$

Turning back to our examples, we see that the quantum plane from Example 3.4 is the Ore extension $\mathbb{C}[x][y; \alpha, 0]$, where $\alpha(x) = q^{-1}x$ and the commutation relation becomes $yx = q^{-1}xy$. The universal enveloping algebra $U(\mathfrak{g})$ from Example 3.6 is the Ore extension $\mathbb{C}[y][x; \mathrm{id}, y\frac{d}{dy}]$, and the commutation relation becomes $xy = yx + y$.

4.2.2. Analytic Ore extensions. Next we consider a locally convex counterpart of the algebra $R[z; \alpha, \delta]$ – an analytical Ore extension $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ – and state the theorem telling us the conditions under which the algebra $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ (or some variant of it) becomes the Arens-Michael envelope of $R[z; \alpha, \delta]$. Below we explain the key steps in the construction of $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$. This theoretical framework is explained in detail in [6].

First, we recall the following two technical definitions.

Definition 4.4 ([6, Definition 4.1]). Let E be a vector space and let $\mathcal{T}$ be a family of linear operators on E. A seminorm on E is *$\mathcal{T}$-stable* if for any $T \in \mathcal{T}$ there exists $C > 0$ such that

$$\|Tv\| \leq C\|v\|$$

for every $v \in E$.

Definition 4.5 ([6, Definition 4.2]). Let E be a locally convex topological space. A family $\mathcal{T}$ of linear operators on E is called *localizable* if the topology on E can be defined by a family of $\mathcal{T}$-stable seminorms. A single operator T is called *localizable* if the singleton family $\mathcal{T} = \{T\}$ is localizable.

Let now R be an $\widehat{\otimes}$-algebra equipped with a localizable endomorphism $\alpha : R \to R$ and a localizable derivation $\delta : R \to R$. The next two lemmas will show that we can equip the space $\mathcal{O}(\mathbb{C}, R)$ of R-valued entire functions with a "twisted" multiplication which coincides with the multiplication on the Ore extension $R[z; \alpha, \delta]$ when we restrict to the polynomial subspace in $\mathcal{O}(\mathbb{C}, R)$. Recall that $\mathcal{O}(\mathbb{C}, R)$ is isomorphic to the projective tensor product $R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$ both as a locally convex topological space and as a left R-$\widehat{\otimes}$-module. Explicitly, for any family of seminorms $\{\| \cdot \|_\lambda\}_{\lambda \in \Lambda}$ defining the topology on R, the space $\mathcal{O}(\mathbb{C}, R)$ is described as convergent Taylor series

$$\left\{ f(z) = \sum_n c_n z^n : c_n \in R,\ \|f\|_{\lambda,\rho} < \infty \text{ for any } \lambda \in \Lambda,\ \rho > 0 \right\},$$

where $\|f\|_{\lambda,\rho} = \sum_{n=0}^{\infty} \|c_n\|_\lambda \, \rho^n$. The topology on $\mathcal{O}(\mathbb{C}, R)$ is defined by the family of seminorms

$$\left\{ \| \cdot \|_{\lambda,\rho} : \lambda \in \Lambda, \rho \in \mathbb{R}_{>0} \right\}.$$

Consider the inclusion of locally convex topological vector spaces

$$R[z; \alpha, \delta] \hookrightarrow \mathcal{O}(\mathbb{C}, R),$$

where $R[z; \alpha, \delta]$ is equipped with the "twisted multiplication" from Definition 4.3 and the induced topology from $\mathcal{O}(\mathbb{C}, R)$. The next lemma shows that we can extend (4.3.1) from the dense subspace $R[z; \alpha, \delta]$ to the whole space $\mathcal{O}(\mathbb{C}, R)$. We use the shorthand $\mathcal{O}(\mathbb{C}) = \mathcal{O}(\mathbb{C}, \mathbb{C})$ to denote complex-valued entire functions.

Lemma 4.6 ([6, Lemma 4.2]). *Suppose that R is an $\widehat{\otimes}$-algebra, $\alpha : R \to R$ – a localizable endomorphism, and $\delta : R \to R$ – a localizable derivation. Then there exists a unique continuous linear map*

$$\tau : \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \to R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}),$$

such that

$$\tau(z^n \otimes r) = \sum_{k=0}^{n} S_{n,k}(r) \otimes z^{n-k} \quad \textit{for all } r \in R \textit{ and } n \in \mathbb{Z}_{\geq 0}.$$

Now set $A = \mathcal{O}(\mathbb{C}, R) \cong R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$. Define the multiplication map $m_A : A \mathbin{\widehat{\otimes}} A \to A$ as the composition

$$\begin{aligned} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) &\xrightarrow{\mathbf{1}_R \otimes \tau \otimes \mathbf{1}_{\mathcal{O}(\mathbb{C})}} R \mathbin{\widehat{\otimes}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}) \\ &\xrightarrow{m_R \widehat{\otimes} m_{\mathcal{O}(\mathbb{C})}} R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}). \end{aligned}$$

Proposition 4.7 ([6, Proposition 4.3]). *The map* $m_A : A \mathbin{\widehat{\otimes}} A \to A$ *turns* $A = \mathcal{O}(\mathbb{C}, R)$ *into an* $\widehat{\otimes}$*-algebra, such that the inclusion map* $i : R[z; \alpha, \delta] \hookrightarrow \mathcal{O}(\mathbb{C}, R)$ *is an algebra homomorphism.*

The last proposition allows us to give the following definition.

Definition 4.8 ([6, Definition 4.3]). The algebra $A = R \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C})$ with the above multiplication map will be denoted by $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ and called *the analytical Ore extension* of the algebra R.

Note that $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ contains R as a closed subalgebra and is therefore an R-$\widehat{\otimes}$-algebra.

Next, we strengthen the above result in the case when R is moreover an Arens-Michael algebra. First, we have the following refinement of Definition 4.5.

Definition 4.9 ([6, Definition 4.4]). Let R be an Arens-Michael algebra. A family $\mathcal{T}$ of linear operators on R is called *m-localizable* if the topology on R can be defined by a family of $\mathcal{T}$-stable submultiplicative seminorms. A single operator T is called *m-localizable* if the singleton family $\mathcal{T} = \{T\}$ is m-localizable.

The next proposition shows that if R is an Arens-Michael algebra and operators α and δ form an m-localizable family, then the analytic Ore extension $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ is itself an Arens-Michael algebra.

Proposition 4.10 ([6, Proposition 4.5]). *Let R be an Arens-Michael algebra,* $\alpha : R \to R$ *– an algebra endomorphism, and* $\delta : R \to R$ *– an* α*-derivation. Suppose that the set* $\{\alpha, \delta\}$ *is m-localizable. Then* $\mathcal{O}(\mathbb{C}, R; \alpha, \delta)$ *is an Arens-Michael algebra.*

Now suppose that R is an algebra (without a topology), $\alpha = \mathrm{id} : R \to R$ is the identity map, and $\delta : R \to R$ is a derivation. Denote by R_δ the Arens-Michael algebra obtained as the completion of R by the system of all δ-stable submultiplicative seminorms. Let j be the canonical homomorphism $j : R \to R_\delta$. Clearly, δ defines a unique m-localizable derivation $\widehat{\delta}$ of R_δ with the property $\widehat{\delta} \circ j = j \circ \delta$. Therefore, we get homomorphisms

$$R[z; \mathrm{id}, \delta] \to R_\delta[z; \mathrm{id}, \widehat{\delta}] \hookrightarrow \mathcal{O}(\mathbb{C}, R_\delta; \mathrm{id}, \widehat{\delta}),$$

where the first one coincides with j on R and maps z to z, and the second one is a canonical inclusion. Let $\iota_{R[z;\mathrm{id},\delta]}$ be the composition homomorphism. The next result describes the Arens-Michael envelope of $R[z; \mathrm{id}, \delta]$ as the analytic Ore extension $\mathcal{O}(\mathbb{C}, R_\delta; \mathrm{id}, \widehat{\delta})$.

Theorem 4.11 ([6, Theorem 5.1]). *The pair* $(\mathcal{O}(\mathbb{C}, R_\delta; \mathrm{id}, \widehat{\delta}), \iota_{R[z;\mathrm{id},\delta]})$ *is the Arens-Michael envelope of the algebra* $R[z; \mathrm{id}, \delta]$.

The situation when $\alpha \neq \mathrm{id}$ is harder to handle. Let R be an algebra, let $\alpha : R \to R$ be an endomorphism, and let X be an R-bimodule. We will denote by ${}_\alpha X$ the R-bimodule obtained by endowing the underlying abelian group of X with a new R-multiplication rule $\bullet$:

$$r \bullet x = \alpha(r)x, \quad x \bullet r = xr, \quad r \in R,\ x \in X.$$

Let now $\delta : R \to R$ be an α-derivation. By applying the Arens-Michael functor to α, we obtain an endomorphism

$$\widehat{\alpha} : \widehat{R} \to \widehat{R}$$

of the Arens-Michael envelope of R satisfying

$$\widehat{\alpha} \circ \iota_R = \iota_R \circ \alpha.$$

Since we can view $\iota_R : R \to \widehat{R}$ as a morphism ${}_\alpha R \to {}_{\widehat{\alpha}} \widehat{R}$ of R-bimodules, we get that the composition

$$R \xrightarrow{\delta} R \xrightarrow{\iota_R}{}_{\widehat{\alpha}} \widehat{R}$$

is a derivation. By applying the universal property of the Arens-Michael envelopes (or rather its version for R-modules, see [6, Definition 3.2]), we obtain a unique derivation

$$\widehat{\delta} : \widehat{R} \to_{\widehat{\alpha}} \widehat{R},$$

i.e., a unique $\widehat{\alpha}$-derivation of $\widehat{R}$ satisfying

$$\widehat{\delta} \circ \iota_R = \iota_R \circ \delta.$$

We have the following general result describing the Arens-Michael envelope of the algebraic Ore extension as an analytic Ore extension when certain technical conditions are met.

Theorem 4.12 ([6, Theorem 5.17]). *In the above setup, if the family $\{\widehat{\alpha}, \widehat{\delta}\}$ is m-localizable, then there exists a unique R-homomorphism*

$$\iota_{R[z;\alpha,\delta]} : R[z;\alpha,\delta] \to \mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta}) \quad \textit{such that } z \mapsto z.$$

The algebra $\mathcal{O}(\mathbb{C}, \widehat{R}; \widehat{\alpha}, \widehat{\delta})$ with the homomorphism $\iota_{R[z;\alpha,\delta]}$ is the Arens-Michael envelope of $R[z;\alpha,\delta]$.

5. The Arens-Michael envelope of the Jordan plane

Now we turn to the main results of this paper. We first compute the Arens-Michael envelope of the Jordan plane.

Definition 5.1. *The Jordan plane* over $\Bbbk$ is the $\Bbbk$-algebra $\Lambda_2(\Bbbk)$ given by generators x and y and a commutation relation $yx = xy + y^2$, i.e.,

$$\Lambda_2(\Bbbk) = \Bbbk\langle x, y\rangle/(yx - xy - y^2).$$

We are interested in the case when $\Bbbk = \mathbb{C}$ so we will use the following shorthand notation $\Lambda_2 = \Lambda_2(\mathbb{C})$.

Following a simple induction argument, it is easy to check that the monomials $\{x^i y^j \mid i, j \in \mathbb{Z}_+\}$ span Λ_2. It is shown in [9] by a direct computation that they are also linearly independent and, therefore, form the basis of Λ_2. Alternatively, this follows from the theory of quadratic algebras (see [8, Section 4, Theorem 2.1]). As a result, we can again view Λ_2 as a polynomial algebra with a "twisted" multiplication.

Comparing Definition 5.1 to Definition 4.3, we see that the Jordan plane is the Ore extension $\mathbb{C}[y][x; \mathrm{id}, -y^2\frac{d}{dy}]$. Therefore, in order to apply Theorem 4.11 we need to describe the system of all submultiplicative δ-stable seminorms on $\mathbb{C}[y]$, where $\delta = -y^2\frac{d}{dy}$.

Write an element $a \in \mathbb{C}[y]$ as a polynomial $\sum_{i=0}^n a_i y^i$. We first have the following result.

Proposition 5.2. *The family*

$$\left\{\|\cdot\|_\rho : \|a\|_\rho = \sum_{i=0}^{n} |a_i| \frac{1}{(i-1)!}\rho^i < \infty,\ \rho \in \mathbb{R}_{>0}\right\}, \quad (-1! := 1,\ 0! := 1),$$

is equivalent to the family of all submultiplicative δ-stable seminorms on $\mathbb{C}[y]$ with $\delta = -y^2\frac{d}{dy}$.

Proof. We split the proof of the proposition into steps for the reader's convenience.

Step 1. First note that $\|\cdot\|_\rho$ is indeed a seminorm. Moreover, it is submultiplicative:

$$\|y^k y^l\|_\rho = \|y^{k+l}\|_\rho = \frac{\rho^{k+l}}{(k+l-1)!} \le \frac{\rho^k}{(k-1)!}\frac{\rho^l}{(l-1)!} = \|y^k\|_\rho \|y^l\|_\rho, \quad \forall k, l \ge 0.$$

Now for $a, b \in \mathbb{C}[y]$ we have

$$\begin{aligned}
\|ab\|_\rho &= \left\|\sum_{i=0}^n a_i y^i \cdot \sum_{j=0}^m b_j y^j\right\|_\rho = \left\|\sum_{i,j=0}^{n,m} a_i b_j y^i y^j\right\|_\rho \\
&\le \sum_{i,j=0}^{n,m} |a_i||b_j|\|y^i y^j\|_\rho \le \sum_{i,j=0}^{n,m} |a_i||b_j|\|y^i\|_\rho\|y^j\|_\rho \\
&= \sum_{i=0}^n |a_i|\|y^i\|_\rho \cdot \sum_{j=0}^m |b_j|\|y^j\|_\rho = \left\|\sum_{i=0}^n a_i y^i\right\|_\rho \cdot \left\|\sum_{j=0}^m b_j y^j\right\|_\rho = \|a\|_\rho\|b\|_\rho.
\end{aligned}$$

Step 2. Next, we show that $\|\cdot\|_\rho$ is δ-stable. Note that

$$\delta(y^i) = -y^2\frac{d}{dy}(y^i) = -iy^{i+1}, \quad \forall i \ge 1,$$

and

$$\delta(y^0) = \delta(1) = -y^2\frac{d}{dy}(1) = 0.$$

We have

$$
\begin{aligned}
\left\|\delta(a)\right\|_\rho &= \left\|\delta\Big(\sum_{i=0}^{n} a_i y^i\Big)\right\|_\rho = \left\|\sum_{i=0}^{n} a_i \delta(y^i)\right\|_\rho = \left\|\sum_{i=1}^{n} a_i(-iy^{i+1})\right\|_\rho \\
&= \left\|\sum_{i=1}^{n} a_i i y^{i+1}\right\|_\rho = \left\|\sum_{j=2}^{n+1} a_{j-1}(j-1)y^j\right\|_\rho = \sum_{j=2}^{n+1} |a_{j-1}| \frac{j-1}{(j-1)!}\rho^j \\
&= \rho \sum_{j=2}^{n+1} |a_{j-1}| \frac{1}{(j-2)!}\rho^{j-1} \le \rho \sum_{i=1}^{n} |a_i| \frac{1}{(i-1)!}\rho^i \\
&\le \rho \sum_{i=0}^{n} |a_i| \frac{1}{(i-1)!}\rho^i = \rho\|a\|_\rho.
\end{aligned}
$$

Step 3. Finally, we show that any submultiplicative δ-stable seminorm $\|\cdot\|$ is dominated by $\|\cdot\|_\rho$ for some $\rho > 0$.

Note that by induction we get

$$\delta^j(y) = (-1)^j\, j!\, y^{j+1}, \quad j \ge 1.$$

From δ-stability, we get

$$\left\|\delta^j(a)\right\| \le C\left\|\delta^{j-1}(a)\right\| \le \cdots \le C^j\|a\|.$$

Now setting $a = y$, we have

$$\|\delta^j(y)\| = j!\|y^{j+1}\| \le C^j\|y\|,$$

so that

$$\|y^{j+1}\| \le \frac{C^j\|y\|}{j!}, \quad j \ge 1.$$

Note that if we pick $C \ge 1$ and set $\rho := C\max\{\|y\|, 1\}$, we have

$$C^j\|y\| \le C^j \max\left\{\|y\|, 1\right\} \le C^j\big(\max\left\{\|y\|, 1\right\}\big)^j = \rho^j,$$

and therefore

$$\|y^{j+1}\| \le \frac{C^j\|y\|}{j!} \le \frac{\rho^j}{j!} \le \frac{\rho^{j+1}}{j!} = \|y^{j+1}\|_\rho, \quad j \ge 1.$$

For $j = 0$, we have

$$\|y\| = \frac{\|y\|}{\rho}\rho \le D\|y\|_\rho,$$

where

$$D = \max\left\{\frac{\|y\|}{\rho}, 1\right\}.$$

Finally,

$$\|a\| = \left\| \sum_{i=0}^{n} a_i y^i \right\| \le \sum_{i=0}^{n} |a_i| \|y^i\| \le |a_0| + |a_1| D \|y\|_\rho + \sum_{i=2}^{n} |a_i| \|y^i\|_\rho$$
$$\le D \sum_{i=0}^{n} |a_i| \|y^i\|_\rho = D\|a\|_\rho. \qquad \blacksquare$$

Next, we pass to a simpler family of seminorms.

Lemma 5.3. *The family of seminorms on* $\mathbb{C}[y]$

$$P = \left\{ \|\cdot\|_\rho : \|a\|_\rho = \sum_{i=0}^{n} |a_i| \frac{1}{(i-1)!} \rho^i < \infty,\ \rho \in \mathbb{R}_{>0}, \right\}, \quad (-1! := 1,\ 0! := 1),$$

is equivalent to the family

$$Q = \left\{ \|\cdot\|_q : \|a\|_q = \sum_{i=0}^{n} |a_i| \frac{1}{i!} q^i < \infty,\ q \in \mathbb{R}_{>0} \right\}, \quad (0! := 1),$$

where

$$a = \sum_{i=0}^{n} a_i y^i \in \mathbb{C}[y].$$

Proof. First, we observe that

$$\sum_{i=0}^{n} |a_i| \frac{1}{i!} q^i \le \sum_{i=0}^{n} |a_i| \frac{1}{(i-1)!} q^i,$$

and, therefore, $Q \prec P$.

Since for $i \ge 0$

$$i \le 2^i \Leftrightarrow \frac{1}{(i-1)!} \le \frac{2^i}{i!},$$

we have

$$\sum_{i=0}^{n} |a_i| \frac{1}{(i-1)!} \rho^i \le \sum_{i=0}^{n} |a_i| \frac{2^i}{i!} \rho^i = \sum_{i=0}^{n} |a_i| \frac{1}{i!} q^i, \quad q = 2\rho,$$

and $P \prec Q$. ■

Finally, we can apply Theorem 4.11 to get the description of the Arens-Michael envelope of the Jordan plane Λ_2.

Theorem 5.4. *The Arens-Michael envelope of the Jordan plane* Λ_2 *is*

$$\widehat{\Lambda}_2 := \left\{ a = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j : \|a\|_\rho < \infty \text{ for any } \rho > 0 \right\},$$

where

$$\|a\|_\rho = \sum_{i,j=0}^{\infty} |a_{ij}| \frac{1}{j!} \rho^{i+j}.$$

The topology on $\widehat{\Lambda}_2$ is generated by the system $\{\| \cdot \|_\rho : \rho \in \mathbb{R}_{>0}\}$, and multiplication is characterized by the relation $yx = xy + y^2$.

Proof. By Theorem 4.11, the Arens-Michael envelope of $\Lambda_2 = \mathbb{C}[y][x; \mathrm{id}, -y^2 \frac{d}{dy}]$ is given by the analytic Ore extension $\mathcal{O}(\mathbb{C}, \mathbb{C}[y]_\delta; \mathrm{id}, \widehat{\delta})$, where $\mathbb{C}[y]_\delta$ is the completion of $\mathbb{C}[y]$ with respect to the family Q of seminorms by Lemma 5.3.

By the discussion preceding Theorem 4.11, the space $\mathcal{O}(\mathbb{C}, \mathbb{C}[y]_\delta; \mathrm{id}, \widehat{\delta})$ can be described as the set

$$\left\{ a = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i y^j : \|a\|_{q,\rho} < \infty \text{ for any } q, \rho > 0 \right\},$$

where

$$\|a\|_{q,\rho} = \sum_{i=0}^{\infty} \left\| \sum_{j=0}^{\infty} a_{ij} y^j \right\|_q \rho^i = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{ij}| \frac{1}{j!} q^j \rho^i,$$

with topology given by the family of seminorms $\{\| \cdot \|_{q,\rho} : q, \rho \in \mathbb{R}_{>0}\}$. It is easy to check that this description is equivalent to the description in the statement of the theorem. ∎

We see that again the Arens-Michael envelope of a polynomial algebra with a "twisted" multiplication is a power series algebra with the same multiplication rule.

6. The Arens-Michael envelope of $U_q(\mathfrak{sl}(2))$, $|q| = 1$

In this section, we turn to the second main result of this paper.

The quantum enveloping algebra $U_q(\mathfrak{sl}(2))$ of the Lie algebra $\mathfrak{sl}(2)$ is an important basic example of a Hopf algebra and a quantum group. It can be defined in two slightly different ways.

Definition 6.1. For $q \in \mathbb{C} \setminus \{1, -1\}$, consider an algebra $U_q(\mathfrak{sl}(2))$ on four generators E, F, K, K^{-1} subject to the following relations:

(1) $KK^{-1} = K^{-1}K = 1$,

(2) $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-2} F$,

(3) $[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$.

Definition 6.2. For $q \in \mathbb{C} \setminus \{1, -1\}$, consider an algebra $U_q'(\mathfrak{sl}(2))$ on five generators E, F, K, K^{-1}, L with the relations

(1) $KK^{-1} = K^{-1}K = 1$,

(2) $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-2} F$,

(3) $[E, F] = L$, $(q - q^{-1})L = K - K^{-1}$,

(4) $[L, E] = q(EK + K^{-1}E)$, $[L, F] = -q^{-1}(FK + K^{-1}F)$.

Clearly, $U'_q(\mathfrak{sl}(2))$ is isomorphic to $U_q(\mathfrak{sl}(2))$ via a map that sends $L \in U'_q(\mathfrak{sl}(2))$ to $[E, F] \in U_q(\mathfrak{sl}(2))$ and leaves other generators intact. Note that the second definition allows us to consider the limiting case $q = 1$ where our quantum enveloping algebra almost becomes the usual enveloping algebra $U(\mathfrak{sl}(2))$. In fact, we have the following isomorphisms (see [2]):

$$U'_1\big(\mathfrak{sl}(2)\big) \cong \frac{U(\mathfrak{sl}(2))[K]}{(K^2 - 1)} \cong U\big(\mathfrak{sl}(2)\big) \otimes \frac{\mathbb{C}[K]}{(K^2 - 1)}. \tag{6.2.1}$$

For more information on $U_q(\mathfrak{sl}(2))$, see [2].

We obtain the following description of the Arens-Michael envelope of $U_q(\mathfrak{sl}(2))$ for $|q| = 1$.

Theorem 6.3. *Let $U_q(\mathfrak{sl}(2))$ be the quantum enveloping algebra of $\mathfrak{sl}(2)$ and $\widehat{U_q(\mathfrak{sl}(2))}$ its Arens-Michael envelope.*

(1) *If $q = 1$, then*

$$\widehat{U_1\big(\mathfrak{sl}(2)\big)} \cong \widehat{U\big(\mathfrak{sl}(2)\big)} \mathbin{\widehat{\otimes}} \frac{\mathbb{C}[K]}{(K^2 - 1)}.$$

(2) $|q| = 1$, $q \neq 1, -1$, *then*

$$\widehat{U_q\big(\mathfrak{sl}(2)\big)} = \Big\{c = \sum_{i \in \mathbb{Z}, n, m \geq 0} c_{i,n,m} K^i F^n E^m : \|c\|_\rho < \infty \text{ for any } \rho > 0\Big\},$$

where

$$\|c\|_\rho = \sum_{i \in \mathbb{Z}, n, m \geq 0} |c_{i,n,m}| \rho^{i+n+m}.$$

The topology on $\widehat{U_q(\mathfrak{sl}(2))}$ is generated by the system $\{\| \cdot \|_\rho : \rho \in \mathbb{R}_{>0}\}$.

Proof. We split the proof of the theorem into two cases according to how the theorem is stated.

Proof of (1). Definition 6.2 allows us to consider the limiting case $q = 1$, where our quantum enveloping algebra almost becomes the usual enveloping algebra $U(\mathfrak{sl}(2))$ as we mentioned in (6.2.1).

The representation of $U'_1(\mathfrak{sl}(2))$ as the tensor product of two algebras for which the Arens-Michael envelope is already known allows us to quickly compute the Arens-Michael envelope of $U'_1(\mathfrak{sl}(2))$ using the results of Section 4.1. Specifically, endow $U'_1(\mathfrak{sl}(2))$ with the strongest locally convex topology τ_{str}. By Proposition 4.1, we have the following isomorphism of $\widehat{\otimes}$-algebras:

$$\big(U'_1\big(\mathfrak{sl}(2)\big), \tau_{\text{str}}\big) \cong \big(U\big(\mathfrak{sl}(2)\big), \tau_{\text{str}}\big) \mathbin{\widehat{\otimes}} \left(\frac{\mathbb{C}[K]}{K^2 - 1}, \tau_{\text{str}}\right).$$

Next, we apply Proposition 4.2 (along with Definition 2.2) to the above projective tensor product to get

$$\widehat{U_1'\big(\mathfrak{sl}(2)\big)} \cong \widehat{U\big(\mathfrak{sl}(2)\big)} \mathbin{\widehat{\otimes}} \widehat{\left(\frac{\mathbb{C}[K]}{K^2-1}\right)}.$$

Since $\frac{\mathbb{C}[K]}{(K^2-1)}$ is a finite-dimensional vector space, its Arens-Michael envelope coincides with $\frac{\mathbb{C}[K]}{(K^2-1)}$. Therefore, we finally get

$$\widehat{U_1'\big(\mathfrak{sl}(2)\big)} \cong \widehat{U\big(\mathfrak{sl}(2)\big)} \mathbin{\widehat{\otimes}} \frac{\mathbb{C}[K]}{(K^2-1)}.$$

The Arens-Michael envelope $\widehat{U(\mathfrak{sl}(2))}$ was shown to be the direct product of matrix algebras in Example 3.8.

Proof of (2). For $q \neq 1, -1$, note that the algebra $U_q(\mathfrak{sl}(2))$ is an iterated Ore extension (see Section 4.2). Namely, start with

$$A_0 = \mathbb{C}[K, K^{-1}]$$

along with a $\mathbb{C}$-linear algebra homomorphism $\alpha_0 : A_0 \to A_0$ defined by

$$\alpha_0(K) = q^2 K.$$

Next, consider the Ore extension

$$A_1 = A_0[F, \alpha_0, \delta_0 = 0]$$

equipped with a $\mathbb{C}$-linear algebra homomorphism α_1 and a $\mathbb{C}$-linear derivation δ defined by

$$\alpha_1(F^j K^l) = q^{-2l} F^j K^l, \quad \delta(K) = 0, \quad \delta(F^j K^l) = \sum_{i=0}^{j-1} F^{j-1}\, \delta_{q^{-2i}K}(F) K^l,$$

where $\delta_K(F) = \frac{K-K^1}{q-q^{-1}}$ is a Laurent polynomial in K. Finally, consider the Ore extension

$$A_2 = A_1[E, \alpha_1, \delta].$$

One easily checks by comparing the above sequence of Ore extensions to Definition 6.1 that

$$A_2 \cong U_q\big(\mathfrak{sl}(2)\big).$$

Therefore, we can apply the results of Section 4.2 to each consecutive Ore extension to calculate the Arens-Michael envelope of A_2.

The Arens-Michael envelope of the algebra $A_0 = \mathbb{C}[K, K^{-1}]$ is very simple:

$$\widehat{A_0} = \left\{ a = \sum_{i \in \mathbb{Z}} a_i K^i : \|a\|_\rho = \sum_{i \in \mathbb{Z}} |a_i| \rho^i < \infty \text{ for all } \rho > 0 \right\},$$

where $\|a\|_\rho = \sum_{i \in \mathbb{Z}} |a_i| \rho^i$, and the topology is generated by the family $\{\| \cdot \|_\rho : \rho \in \mathbb{R}_{>0}\}$.

After extending $\alpha_0 : A_0 \to A_0$ to $\widehat{\alpha_0} : \widehat{A_0} \to \widehat{A_0}$, we check that $\widehat{\alpha_0}$ is indeed m-localizable (use $|q| = 1$):

$$\Big\|\widehat{\alpha_0}\Big(\sum_{i\in\mathbb{Z}} a_i K^i\Big)\Big\|_\rho = \Big\|\sum_{i\in\mathbb{Z}} a_i q^{2i} K^i\Big\|_\rho = \sum_{i\in\mathbb{Z}} |a_i||q^{2i}|\rho^i = \sum_{i\in\mathbb{Z}} |a_i|\rho^i = \Big\|\sum_{i\in\mathbb{Z}} a_i K^i\Big\|_\rho.$$

Applying Theorem 4.12, we obtain

$$\widehat{A_1} = \mathcal{O}(\mathbb{C}, \widehat{A_0}; \widehat{\alpha_0}, 0) = \Big\{b = \sum_{i\in\mathbb{Z}, n\geq 0} b_{i,n} K^i F^n : \|b\|_\rho < \infty \text{ for any } \rho > 0\Big\},$$

where $\|b\|_\rho = \sum_{i\in\mathbb{Z},n\geq 0} |b_{i,n}|\rho^{i+n}$, and the topology is generated by the family $\{\|\cdot\|_\rho : \rho \in \mathbb{R}_{>0}\}$.

Finally, the operators α_1 and δ simply extend to $\widehat{A_1}$ by their action on the generators K and F. We check that $\{\widehat{\alpha_1}, \widehat{\delta}\}$ is an m-localizable family. In the calculations below, we make extensive use of the relations between the generators $K^i F^n = q^{-2in} F^n K^l$.

$\widehat{\alpha_1}$ is m-localizable:

$$\begin{aligned}
&\Big\|\widehat{\alpha_1}\Big(\sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} K^i F^n\Big)\Big\|_\rho \\
&= \Big\|\widehat{\alpha_1}\Big(\sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} q^{-2in} F^n K^i\Big)\Big\|_\rho \\
&= \Big\|\sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} q^{-2in} q^{-2i} F^n K^i\Big\|_\rho = \sum_{i\in\mathbb{Z},n\geq 0} |b_{i,n}||q^{-2in}||q^{-2i}|\rho^{i+n} \\
&= \sum_{i\in\mathbb{Z},n\geq 0} |b_{i,n}||q^{-2in}|\rho^{i+n} = \Big\|\sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} q^{-2in} F^n K^i\Big\|_\rho = \Big\|\sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} K^i F^n\Big\|_\rho.
\end{aligned}$$

Before proving that $\widehat{\delta}$ is m-localizable, let us perform one auxiliary computation:

$$\begin{aligned}
\delta(K^i F^n) &= \delta(q^{-2in} F^n K^i) \\
&= q^{-2in} \sum_{j=0}^{n-1} F^{n-1} \delta_{q^{-2j}K}(F) K^i \\
&= q^{-2in} F^{n-1} K^i \frac{1}{q - q^{-1}} \\
&\quad \times \big[(K - K^{-1}) + (q^{-2}K - q^2 K) + \cdots + (q^{-2(n-1)}K - q^{2(n-1)}K)\big] \\
&= q^{-2in} F^{n-1} K^i \frac{1}{q - q^{-1}} \\
&\quad \times \big[K(1 + q^{-2} + \cdots + q^{-2(n-1)}) - K^{-1}(1 + q^2 + \cdots + q^{2(n-1)})\big] \\
&= q^{-2in} F^{n-1} K^i \frac{1}{q - q^{-1}} \Big[K\Big(\frac{1 - q^{-2n}}{1 - q^{-2}}\Big) - K^{-1}\Big(\frac{1 - q^{2n}}{1 - q^2}\Big)\Big].
\end{aligned}$$

Now we can show that $\widehat{\delta}$ is m-localizable:

$$
\begin{aligned}
&\left\| \widehat{\delta}\Bigg(\sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} K^i F^n \Bigg) \right\|_\rho \\
&= \left\| \sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} \delta(K^i F^n) \right\|_\rho \\
&= \left\| \sum_{i\in\mathbb{Z},n\geq 1} b_{i,n} q^{-2in} F^{n-1} K^i \frac{1}{q-q^{-1}} \left[K\left(\frac{1-q^{-2n}}{1-q^{-2}}\right) - K^{-1}\left(\frac{1-q^{2n}}{1-q^{2}}\right)\right] \right\|_\rho \\
&= \left\| \sum_{i\in\mathbb{Z},n\geq 1} b_{i,n} q^{-2in} \left[F^{n-1} K^{i+1} \frac{1-q^{-2n}}{(q-q^{-1})(1-q^{-2})} - F^{n-1} K^{i-1} \frac{1-q^{2n}}{(q-q^{-1})(1-q^{2})}\right] \right\|_\rho \\
&= \left\| \sum_{i\in\mathbb{Z},n\geq 0} \left[b_{i-1,n+1} q^{-2(i-1)(n+1)} \frac{1-q^{-2(n+1)}}{(q-q^{-1})(1-q^{-2})} \right.\right. \\
&\qquad\qquad \left.\left. - b_{i+1,n+1} q^{-2(i+1)(n+1)} \frac{1-q^{2(n+1)}}{(q-q^{-1})(1-q^{2})} \right] F^n K^i \right\|_\rho \\
&= \sum_{i\in\mathbb{Z},n\geq 0} \left| \left[b_{i-1,n+1} q^{-2(i-1)(n+1)} \frac{1-q^{-2(n+1)}}{(q-q^{-1})(1-q^{-2})} \right.\right. \\
&\qquad\qquad \left.\left. - b_{i+1,n+1} q^{-2(i+1)(n+1)} \frac{1-q^{2(n+1)}}{(q-q^{-1})(1-q^{2})} \right] \right| \rho^{i+n} \\
&\leq \sum_{i\in\mathbb{Z},n\geq 0} \left[\left| b_{i-1,n+1} q^{-2(i-1)(n+1)} \frac{1-q^{-2(n+1)}}{(q-q^{-1})(1-q^{-2})} \right| \right. \\
&\qquad\qquad \left. + \left| b_{i+1,n+1} q^{-2(i+1)(n+1)} \frac{1-q^{2(n+1)}}{(q-q^{-1})(1-q^{2})} \right| \right] \rho^{i+n} \\
&\leq \sum_{i\in\mathbb{Z},n\geq 0} \left[|b_{i-1,n+1}||q^{-2(i-1)(n+1)}| \frac{2}{|(q-q^{-1})(1-q^{-2})|} \right. \\
&\qquad\qquad \left. + |b_{i+1,n+1}||q^{-2(i+1)(n+1)}| \frac{2}{|(q-q^{-1})(1-q^{2})|} \right] \rho^{i+n} \\
&\leq \frac{2}{|(q-q^{-1})(1-q^{-2})|} \sum_{i\in\mathbb{Z},n\geq 0} |b_{i,n}| \rho^{i+n} \\
&\quad + \frac{2}{|(q-q^{-1})(1-q^{-2})|} \sum_{i\in\mathbb{Z},n\geq 0} |b_{i+1,n+1}| \frac{\rho^{i+n+2}}{\rho^2} \\
&\leq C \sum_{i\in\mathbb{Z},n\geq 0} |b_{i,n}| \rho^{i+n} + \frac{C}{\rho^2} \sum_{i\in\mathbb{Z},n\geq 0} |b_{i,n}| \rho^{i+n} \leq \left(C + \frac{C}{\rho^2}\right) \left\| \sum_{i\in\mathbb{Z},n\geq 0} b_{i,n} K^i F^n \right\|_\rho .
\end{aligned}
$$

Applying Theorem 4.12, we conclude that the Arens-Michael envelope of $U_q(\mathfrak{sl}(2))$ for $|q| = 1$, $q \neq 1, -1$ is

$$\begin{aligned}\widehat{U_q\big(\mathfrak{sl}(2)\big)} &= \widehat{A_2} = \mathcal{O}(\mathbb{C}, \widehat{A_1}; \widehat{\alpha_1}, \widehat{\delta}) \\ &= \Big\{c = \sum_{i\in\mathbb{Z}, n,m\geq 0} c_{i,n,m} K^i F^n E^m : \|c\|_\rho < \infty \text{ for any } \rho > 0\Big\},\end{aligned}$$

where

$$\|c\|_\rho = \sum_{i\in\mathbb{Z},\, n,m\geq 0} |c_{i,n,m}|\rho^{i+n+m},$$

and the topology is generated by the family

$$\big\{\|\cdot\|_\rho : \rho \in \mathbb{R}_{>0}\big\}. \qquad \blacksquare$$

Remark 6.4. To the best knowledge of the author, the case $|q| \neq 1$ remains an open question. However, see [3, Section 4] for a recent discussion of the possible approaches for this case.

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References

[1] A. Helemskii, *Banach and locally convex algebras. Translated from the Russian by A. West*. Oxford Science Publications, Oxford University Press, New York, 1993 Zbl 0785.46045 MR 1231796

[2] C. Kassel, *Quantum groups*. Grad. Texts in Math. 155, Springer, New York, 1995 Zbl 0808.17003 MR 1321145

[3] P. Kosenko, The Arens-Michael envelopes of Laurent Ore extensions. *Turkish J. Math.* **46** (2022), no. 3, 839–863 Zbl 07582017 MR 4406731

[4] A. Pirkovskii, Stably flat completions of universal enveloping algebras. *Dissertationes Math.* **441** (2006), 60 Zbl 1140.46031 MR 2283621

[5] A. Pirkovskii, Algebras of holomorphic functions on noncommutative affine varieties. In *Mathematical methods and applications, part 2. Proc. of the 17th RSSU mathematical lectures*, pp. 102–119, RSSU, Moscow, 2008

[6] A. Pirkovskii, Arens-Michael envelopes, homological epimorphisms, and relatively quasi-free algebras. *Trans. Mosc. Math. Soc.* **2008** (2008), 27–104; translated from *Tr. Mosk. Mat. Obs.* **69** (2008), 34–125 Zbl 1207.46064 MR 2549445

[7] A. Pirkovskii, Quantum polydisk, quantum ball, and q-analog of Poincaré's theorem. *J. Phys.: Conf. Ser.* **474** (2013), no. 1, 012026

[8] A. Polishchuk and L. Positselski, *Quadratic algebras*. Univ. Lecture Ser. 37, American Mathematical Society, Providence, RI, 2005 Zbl 1145.16009 MR 2177131

[9] E. N. Shirikov, Two-generated graded algebras. *Algebra Discrete Math.* **2005** (2005), no. 3, 60–84 Zbl 1093.16034 MR 2237896
[10] J. Taylor, Homology and cohomology for topological algebras. *Advances in Math.* **9** (1972), no. 2, 137–182 Zbl 0271.46040 MR 328624
[11] L. L. Vaksman, Integral intertwining operators and quantum homogeneous spaces. *Theoret. and Math. Phys.* **105** (1995), no. 3, 1476–1483 Zbl 0936.17019 MR 1605615

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Dmitrii Pedchenko
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA; dpedchenko@math.ucla.edu