J. Noncommut. Geom. 17 (2023), 163–188
DOI 10.4171/JNCG/462

HPD-invariance of the Tate conjecture(s)

Gonçalo Tabuada

Abstract. We prove that the Tate conjecture (and its variants) is invariant under *homological projective duality*. As an application, we obtain a proof, resp. an alternative proof, of the Tate conjecture (and of its variants) in the new case of linear sections of determinantal varieties, resp. in the old cases of Pfaffian cubic fourfolds and complete intersections of quadrics. In addition, we generalize the Tate conjecture (and its variants) from schemes to stacks and prove this generalized conjecture(s) for low-dimensional root stacks and low-dimensional (twisted) orbifolds.

1. Introduction

Let k be a base field of characteristic $p \geq 0$ and X a smooth projective k-scheme. Throughout the article, we will write $\mathcal{Z}^*(X)_{\mathbb{Q}} := \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}}$ for the graded $\mathbb{Q}$-vector space of algebraic cycles on X up to rational equivalence and $\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim \mathrm{num}}$ for its quotient with respect to the numerical equivalence relation. Given a prime number $l \neq p$, consider the classical cycle class map

$$\mathcal{Z}^*(X)_{\mathbb{Q}_l} \to H^{2*}_{\mathrm{et}}\big(X_{k_s}, \mathbb{Q}_l(*)\big)^{\mathrm{Gal}(k_s/k)}, \tag{1.1}$$

where k_s stands for a (chosen) separable closure of k, $\mathrm{Gal}(k_s/k)$ for the absolute Galois group of k, and $H^*_{\mathrm{et}}(-, \mathbb{Q}_l) := (\lim_\nu H^*_{\mathrm{et}}(-, \mathbb{Z}/l^\nu)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ for l-adic cohomology. In the sixties, Tate [40, 41] conjectured the following:

Conjecture $\mathrm{T}^l(X)$**.** *When k is finitely generated over its prime field, the above cycle class map* (1.1) *is surjective.*

Examples of fields which are finitely generated over their prime fields include finite fields, number fields, function fields, etc. In contrast, algebraically closed fields are *not* finitely generated over their prime fields. The Tate conjecture holds when $\dim(X) \leq 1$, when X is an abelian variety of dimension ≤ 3, and also when X is a $K3$-surface; consult the surveys [3, 28, 44]. Besides these cases (and some other cases scattered in the literature), the Tate conjecture remains wide open. In Theorem 2.7, resp. in Theorems 2.13 and 2.15, below we will provide a proof, resp. an alternative proof, of the Tate conjecture in some new, resp. old, cases.

2020 Mathematics Subject Classification. Primary 14C25; Secondary 14A22, 14F20.
Keywords. Tate conjecture, homological projective duality, noncommutative algebraic geometry.

The above conjecture of Tate admits several variants. For example, in the sixties, Tate [40, 41] conjectured moreover the following:

Conjecture $\overline{\mathrm{T}}^l(X)$**.** *When k is finitely generated over its prime field, the classical cycle class map is surjective*

$$\mathcal{Z}^*(X_{k_s})_{\mathbb{Q}_l} \to \bigcup_{k'/k} H^{2*}_{\mathrm{et}}\big(X_{k_s}, \mathbb{Q}_l(*)\big)^{\mathrm{Gal}(k_s/k')}, \tag{1.2}$$

where the union runs over all the finite field extensions k'/k inside k_s.

Conjecture $\mathrm{SS}^l(X)$**.** *When k is finitely generated over its prime field, the (continuous) l-adic representations $\{H^n_{\mathrm{et}}(X_{k_s}, \mathbb{Q}_l) \mid 0 \leq n \leq 2\dim(X)\}$ of the absolute Galois group* $\mathrm{Gal}(k_s/k)$ *are semi-simple.*

It is well known that $\overline{\mathrm{T}}^l(X) \Rightarrow \mathrm{T}^l(X)$. Moreover, when $p = 0$, we have the following implication $\{\overline{\mathrm{T}}^l(X) \mid X \in \mathrm{SmProj}(k)\} \Rightarrow \{\mathrm{SS}^l(X) \mid X \in \mathrm{SmProj}(k)\}$, where $\mathrm{SmProj}(k)$ stands for the category of smooth projective k-schemes; consult [29].

In the 2000s, Milne [28] formulated the following p-adic variant:

Conjecture $\mathrm{T}^p(X)$**.** *When k is finite, the classical cycle class map is surjective:*

$$\mathcal{Z}^*(X)_{\mathbb{Q}_p} \to H^{2*}_{\mathrm{crys}}(X)(*)^{\phi}, \tag{1.3}$$

where $H^*_{\mathrm{crys}}(-)$ stands for crystalline cohomology and ϕ is the crystalline Frobenius.

Finally, in the eighties, Beilinson (see [16, Conj. 50]) conjectured the following:

Conjecture $\mathrm{B}(X)$**.** *When k is finite, we have $\mathcal{Z}^*(X)_{\mathbb{Q}} = \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim \mathrm{num}}$.*

Note that combining Conjecture $\mathrm{B}(X)$ with Conjectures $\mathrm{T}^l(X)$ and $\mathrm{T}^p(X)$, we conclude that the classical cycle class maps are bijective:

$$\mathcal{Z}^*(X)_{\mathbb{Q}_l} \xrightarrow{\simeq} H^{2*}_{\mathrm{et}}\big(X_{k_s}, \mathbb{Q}_l(*)\big)^{\mathrm{Gal}(k_s/k)} \quad \mathcal{Z}^*(X)_{\mathbb{Q}_p} \xrightarrow{\simeq} H^{2*}_{\mathrm{crys}}(X)(*)^{\phi}.$$

This provides an optimal description of algebraic cycles using l-adic and crystalline cohomology. The conjectures $\mathrm{C}(X)$, with $\mathrm{C} \in \{\overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, also hold when $\dim(X) \leq 1$ and when X is an abelian variety of dimension ≤ 3; consult [9, 15, 16, 45]. Moreover, these conjectures, with $\mathrm{C} \in \{\overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p\}$, hold[1] for $K3$-surfaces.

2. Statement of results

A *differential graded (= dg) category* $\mathcal{A}$ is a category enriched over complexes of k-vector spaces; consult Section 3.1. As explained in Section 4, given a smooth proper dg

[1]When $\dim(X) \leq 3$, we have $\mathrm{T}^l(X) \Leftrightarrow \mathrm{T}^p(X)$ (for every $l \neq p$).

category $\mathcal{A}$ in the sense of Kontsevich, the Tate conjecture and its variants admit noncommutative analogues $\mathrm{C}_{\mathrm{nc}}(\mathcal{A})$ with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$. Examples of smooth proper dg categories include finite dimensional k-algebras of finite global dimension A as well as the canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$ of the category of perfect complexes $\mathrm{perf}(X)$ of every smooth proper k-scheme X (or, more generally, of every smooth proper algebraic stack $\mathcal{X}$); consult [18, 26].

Theorem 2.1. *Given a smooth projective k-scheme X, we have the equivalences:*

$$\mathrm{C}(X) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big) \quad \textit{with}\ \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

Intuitively speaking, Theorem 2.1 shows that the Tate conjecture and its variants belong not only to the realm of algebraic geometry but also to the broad noncommutative setting of smooth proper dg categories in the sense of Kontsevich.

HPD-invariance

For surveys on homological projective duality (HPD), we invite the reader to consult [25, 42]. Let X be a smooth projective k-scheme equipped with a line bundle $\mathcal{L}_X(1)$; we will write $X \to \mathbb{P}(V)$ for the associated morphism, where $V := H^0(X, \mathcal{L}_X(1))^\vee$. Assume that the triangulated category $\mathrm{perf}(X)$ admits a Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{i-1}(i-1)\rangle$ with respect to $\mathcal{L}_X(1)$ in the sense of [23, Def. 4.1]. Following [23, Def. 6.1], let Y be the HP-dual of X, $\mathcal{L}_Y(1)$ the HP-dual line bundle, and $Y \to \mathbb{P}(V^\vee)$ the morphism associated to $\mathcal{L}_Y(1)$. Given a linear subspace $L \subset V^\vee$, consider the linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L)$. As a first application of Theorem 2.1, we obtain the following result:

Theorem 2.2 (HPD-invariance). *Assume the following:*

(a) *the linear sections X_L and Y_L are smooth*[2]*;*

(b) *we have* $\dim(X_L) = \dim(X) - \dim(L)$ *and* $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$*;*

(c) *the conjectures $\mathrm{C}_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold, where $\mathbb{A}_0^{\mathrm{dg}}$ stands for the dg enhancement of $\mathbb{A}_0$ induced from* $\mathrm{perf}_{\mathrm{dg}}(X)$.

Under the assumptions (a)–(c), we have the following equivalences:

$$\mathrm{C}(X_L) \Leftrightarrow \mathrm{C}(Y_L) \quad \textit{with}\ \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

Remark 2.3 (Mild assumptions). Given a *generic* subspace $L \subset V^\vee$, the sections X_L and Y_L are smooth, and the equalities $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$ hold. Moreover, the conjectures $\mathrm{C}_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold whenever the triangulated category $\mathbb{A}_0$ admits a full exceptional collection. This is the case in all the examples in the literature. For these reasons, the assumptions (a)–(c) of Theorem 2.2 are quite mild.

[2] The linear section X_L is smooth if and only if the linear section Y_L is smooth; see [25, p. 9].

Remark 2.4 (Generalization). Theorem 2.2 holds more generally when Y is singular. In this case, we need to replace Y by a noncommutative resolution of singularities $\mathrm{perf}_{\mathrm{dg}}(Y;\mathcal{F})$, where $\mathcal{F}$ is a certain sheaf of noncommutative algebras (consult [25, §2.4]), and conjecture $\mathrm{C}(Y)$ by its noncommutative analogue $\mathrm{C}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y;\mathcal{F}))$.

To the best of the author's knowledge, Theorem 2.2 is new in the literature. In what follows, we illustrate its strength in three important examples:

Example 1: Determinantal duality

Let U_1 and U_2 be two k-vector spaces of dimensions d_1 and d_2, with $d_1 \leq d_2$, $V := U_1 \otimes U_2$, and $0 < r < d_1$ an integer. Consider the determinantal variety $\mathcal{Z}^r_{d_1,d_2} \subset \mathbb{P}(V)$ defined as the locus of those matrices $U_2 \to U_1^\vee$ with rank $\leq r$. Recall that the determinantal varieties with $r = 1$ are the classical Segre varieties. For example, $\mathcal{Z}^1_{2,2} \subset \mathbb{P}^3$ is the quadric hypersurface $\{[v_0 : v_1 : v_2 : v_3] \mid v_0 v_3 - v_1 v_2 = 0\}$. In contrast with the Segre varieties, the determinantal varieties $\mathcal{Z}^r_{d_1,d_2}$, with $r \geq 2$, are not smooth. The singular locus of $\mathcal{Z}^r_{d_1,d_2}$ consists of those matrices $U_2 \to U_1^\vee$ with rank $< r$, i.e., it agrees with the closed subvariety $\mathcal{Z}^{r-1}_{d_1,d_2}$. Nevertheless, it is well known that $\mathcal{Z}^r_{d_1,d_2}$ admits a canonical Springer resolution of singularities $\mathcal{X}^r_{d_1,d_2} \to \mathcal{Z}^r_{d_1,d_2}$, which comes equipped with a projection $q\colon \mathcal{X}^r_{d_1,d_2} \to \mathrm{Gr}(r, U_1)$ to the Grassmannian of r-dimensional subspaces in U_1. Following [4, §3.3], the category $\mathrm{perf}(X)$, with $X := \mathcal{X}^r_{d_1,d_2}$, admits a Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{d_2 r-1}(d_2 r - 1)\rangle$, where $\mathbb{A}_0 = \mathbb{A}_1 = \cdots = \mathbb{A}_{d_2 r-1} = q^*(\mathrm{perf}(\mathrm{Gr}(r, U_1))) \simeq \mathrm{perf}(\mathrm{Gr}(r, U_1))$.

Proposition 2.5. *The conjectures* $\mathrm{C}_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$*, with* $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$*, hold.*

Dually, consider the variety $\mathcal{W}^r_{d_1,d_2} \subset \mathbb{P}(V^\vee)$ of those matrices $U_2^\vee \to U_1$ with corank $\geq r$, and the associated resolution of singularities $Y := \mathcal{Y}^r_{d_1,d_2} \to \mathcal{W}^r_{d_1,d_2}$. As proved[3] in [4, Prop. 3.4 and Thm. 3.5], X and Y are HP-dual to each other. Given a generic linear subspace $L \subseteq V^\vee$, consider the associated smooth linear sections X_L and Y_L; note that whenever $\mathbb{P}(L^\perp)$ does not intersect the singular locus of $\mathcal{Z}^r_{d_1,d_2}$, we have $X_L = \mathbb{P}(L^\perp) \cap \mathcal{Z}^r_{d_1,d_2}$. Theorem 2.2 yields the following result:

Corollary 2.6. *We have the following equivalences:*

$$\mathrm{C}(X_L) \Leftrightarrow \mathrm{C}(Y_L) \quad \textit{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

By construction,

$$\dim(X) = r(d_1 + d_2 - r) - 1 \quad \text{and} \quad \dim(Y) = r(d_1 - d_2 - r) + d_1 d_2 - 1.$$

[3] In [4, Prop. 3.4 and Thm. 3.5], the authors worked over an algebraically closed field of characteristic zero. However, the same proof holds *mutatis mutandis* over any field k. Simply replace the reference [17] concerning the existence of a full strong exceptional collection on $\mathrm{perf}(\mathrm{Gr}(r, U_1))$ by the reference [7, Thm. 1.3] concerning the existence of a tilting bundle on $\mathrm{perf}(\mathrm{Gr}(r, U_1))$. The author is grateful to Marcello Bernardara for an e-mail exchange on this issue.

Consequently, we have

$$\dim(X_L) = r(d_1 + d_2 - r) - 1 - \dim(L) \quad \text{and} \quad \dim(Y_L) = r(d_1 - d_2 - r) - 1 + \dim(L).$$

Since the Tate conjecture and its variants hold in dimensions ≤ 1, we hence obtain from Corollary 2.6 the following result:

Theorem 2.7 (Linear sections of determinantal varieties). *Let X_L and Y_L be as in Corollary 2.6 and $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$.*

(i) *When $r(d_1 + d_2 - r) - 1 - \dim(L) \leq 1$, the conjectures $\mathrm{C}(X_L)$ hold.*

(ii) *When $r(d_1 - d_2 - r) - 1 + \dim(L) \leq 1$, the conjectures $\mathrm{C}(Y_L)$ hold.*

To the best of the author's knowledge, Theorem 2.7 is new in the literature. It proves the Tate conjecture and its variants in new cases. Here are two examples:

Example 2.8 (Segre varieties). Let $r = 1$. Thanks to Theorem 2.7 (i), the conjectures $\mathrm{C}(X_L)$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold when $d_1 - d_2 - 2 + \dim(L) \leq 1$. In all these cases, X_L is a linear section of the Segre variety $\mathcal{Z}^1_{d_1,d_2}$ and its dimension is $2(d_2 - \dim(L))$ or $2(d_2 - \dim(L)) + 1$. Therefore, by letting $d_2 \to \infty$ and by keeping $\dim(L)$ fixed, we obtain infinitely many new examples of smooth projective k-schemes X_L, of arbitrary dimension, satisfying the Tate conjecture and its variants.

Subexample 2.9. Let $r = 1$, $d_1 = 4$, and $d_2 = 2$. In this particular case, the Segre variety $\mathcal{Z}^1_{4,2} \subset \mathbb{P}^7$ agrees with the rational normal 4-fold scroll $S_{1,1,1,1}$; see [13, Ex. 8.27]. Choose a generic linear subspace $L \subset V^\vee$ of dimension 1 such that the hyperplane $\mathbb{P}(L^\perp) \subset \mathbb{P}^7$ does not contain any 3-plane of the ruling of $S_{1,1,1,1}$. By combining Example 2.8 with [8, Prop. 2.5], we hence conclude that the rational normal 3-fold scroll $X_L = S_{1,1,2}$ satisfies the Tate conjecture and its variants.

Example 2.10 (Square matrices). Let $d_1 = d_2 = d$. Thanks to Theorem 2.7 (i), the conjectures $\mathrm{C}(X_L)$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold when $-r^2 - 1 + \dim(L) \leq 1$. In all these cases, X_L is of dimension $2(dr - \dim(L))$ or $2(dr - \dim(L)) + 1$. Therefore, by letting $d \to \infty$ and by keeping r and $\dim(L)$ fixed, we obtain infinitely many new examples of smooth projective k-schemes X_L, of arbitrary dimension, satisfying the Tate conjecture and its variants.

Subexample 2.11. Let $d_1 = d_2 = 3$ and $r = 2$. In this particular case, the determinantal variety $\mathcal{Z}^2_{3,3} \subset \mathbb{P}^8$ has dimension 7 and its singular locus is the 4-dimensional Segre variety $\mathcal{Z}^1_{3,3} \subset \mathcal{Z}^2_{3,3}$. Given a generic linear subspace $L \subset V^\vee$ of dimension 5, the associated smooth linear section X_L is 2-dimensional and, thanks to Example 2.10, it satisfies the Tate conjecture and its variants. Note that since $\mathrm{codim}(L^\perp) = 5 > 4 = \dim(\mathcal{Z}^1_{3,3})$, the subspace $\mathbb{P}(L^\perp) \subset \mathbb{P}^8$ does not intersect the singular locus $\mathcal{Z}^1_{3,3}$ of $\mathcal{Z}^2_{3,3}$. Therefore, for all the above choices of L, the associated surface X_L is a linear section of the determinantal variety $\mathcal{Z}^2_{3,3}$.

Example 2: Grassmannian–Pfaffian duality

Assume that $p = 0$. Let W be a k-vector space of dimension 6 and $X = \mathrm{Gr}(2, W)$ the Grassmannian variety equipped with the Plücker embedding

$$\mathrm{Gr}(2, W) \to \mathbb{P}\big(\wedge^2 (W)\big), (w_1, w_2) \mapsto [w_1 \wedge w_2].$$

Following [22] and [25, §4.4], the category $\mathrm{perf}(X)$ admits a Lefschetz decomposition $\langle \mathbb{A}_0, \ldots, \mathbb{A}_5(5)\rangle$ with $\mathbb{A}_0 = \mathbb{A}_1 = \mathbb{A}_2 = \langle \mathcal{O}_X, \mathcal{U}_X^\vee, S^2(\mathcal{U}_X^\vee)\rangle$ and $\mathbb{A}_3 = \mathbb{A}_4 = \mathbb{A}_5 = \langle \mathcal{O}_X, \mathcal{U}_X^\vee\rangle$, where $\mathcal{U}_X^\vee$ stands for the dual of the tautological bundle on X and $S^2(\mathcal{U}_X^\vee)$ for the symmetric power of $\mathcal{U}_X^\vee$. Following Remark 2.3, these full exceptional collections imply that the conjectures $\mathrm{C}_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold. As proved in [22, Thm. 1] and [25, §4.4], the HP-dual Y of X is given by $\mathrm{perf}(\mathrm{Pf}(4, W^\vee); \mathcal{F})$, where $\mathrm{Pf}(4, W^\vee) \subset \mathbb{P}(\wedge^2(W^\vee))$ is the (singular) Pfaffian variety and $\mathcal{F}$ is a certain sheaf of non-commutative algebras; consult Remark 2.4. Let $L \subset \wedge^2(W^\vee)$ be a generic linear subspace. When $\dim(L) \leq 6$, the associated subspace $\mathbb{P}(L) \subset \mathbb{P}(\wedge^2(W^\vee))$ does *not* intersect the singular locus of $\mathrm{Pf}(4, W^\vee)$. Consequently, the linear section Y_L agrees with the smooth section $\mathrm{Pf}(4, W^\vee)_L := \mathrm{Pf}(4, W^\vee) \cap \mathbb{P}(L)$. Theorem 2.2 yields the following result:

Corollary 2.12. *When* $\dim(L) \leq 6$, *we have the following equivalences:*

$$\mathrm{C}(X_L) \Leftrightarrow \mathrm{C}\big(\mathrm{Pf}(4, W^\vee)_L\big) \quad \textit{with}\, \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l\}.$$

By construction, $\dim(X_L) = 8 - \dim(L)$ and $\dim(Y_L) = \dim(L) - 2$. Moreover, when $\dim(L) = 6$, X_L is a $K3$-surface and Y_L a cubic fourfold; consult [22, §10]. Since the Tate conjecture and its variants hold for $K3$-surfaces, we hence obtain from Corollary 2.12 the following result:

Theorem 2.13 (Pfaffian cubic fourfolds). *Let* $\mathrm{Pf}(4, W^\vee)_L$ *be as in Corollary* 2.12. *When* $\dim(L) = 6$, *the conjectures* $\mathrm{C}(\mathrm{Pf}(4, W^\vee)_L)$, *with* $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l\}$, *hold.*

An alternative (geometric) proof of Theorem 2.13, based on the Kuga–Satake correspondence, was obtained by André in the mid nineties; consult [1, Thm. 1.6.1].

Example 3: Veronese–Clifford duality

Let W be a k-vector space of dimension d and X the projective space $\mathbb{P}(W)$ equipped with the double Veronese embedding $\mathbb{P}(W) \to \mathbb{P}(S^2 W)$, $[w] \mapsto [w \otimes w]$. Consider the Beilinson full exceptional collection

$$\mathrm{perf}(X) = \big\langle \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(d-2)\big\rangle$$

(see [5]) and set $i := \lceil d/2 \rceil$ and

$$\mathbb{A}_0 = \mathbb{A}_1 = \cdots = \mathbb{A}_{i-2} := \big\langle \mathcal{O}_X(-1), \mathcal{O}_X\big\rangle, \quad \mathbb{A}_{i-1} := \begin{cases} \big\langle \mathcal{O}_X(-1), \mathcal{O}_X\big\rangle & \text{if } d = 2i, \\ \big\langle \mathcal{O}_X(-1)\big\rangle & \text{if } d = 2i - 1. \end{cases}$$

Under the preceding notations, the category $\mathrm{perf}(X)$ admits the Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{i-1}(i-1)\rangle$ with respect to the line bundle $\mathcal{L}_X(1) = \mathcal{O}_X(2)$. Following Remark 2.3, these full exceptional collections imply that the conjectures $\mathrm{C}_{\mathrm{nc}}(\mathbb{A}_0^{\mathrm{dg}})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold.

Let $\mathcal{H} := X \times_{\mathbb{P}(S^2 W)} \mathcal{Q} \subset X \times \mathbb{P}(S^2(W^\vee))$ be the universal hyperplane section, where $\mathcal{Q} \subset \mathbb{P}(S^2(W)) \times \mathbb{P}(S^2(W^\vee))$ stands for the incidence quadric. By construction, the projection $q\colon \mathcal{H} \to \mathbb{P}(S^2(W^\vee))$ is a flat quadric fibration. As proved in [24, Thm. 5.4] (see also [2, Thm. 2.3.6]) the HP-dual Y of X is given by $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(S^2(W^\vee)); \mathcal{C}l_0(q))$, where $\mathcal{C}l_0(q)$ stands for the sheaf of even Clifford algebras associated to q; consult Remark 2.4. Let $L \subset S^2(W^\vee)$ be a generic linear subspace. On the one hand, X_L corresponds to the smooth complete intersection of the $\dim(L)$ quadric hypersurfaces in $\mathbb{P}(W)$ parametrized by L. On the other hand, Y_L is given by $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)_{|L})$. Theorem 2.2 yields the following result:

Corollary 2.14. *We have the following equivalences:*

$$\mathrm{C}(X_L) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}\left(\mathbb{P}(L); \mathcal{C}l_0(q)_{|L}\right)\big) \quad \textit{with}\ \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

Recall that the space of quadrics $\mathbb{P}(S^2(W^\vee))$ comes equipped with a canonical filtration $\Delta_d \subset \cdots \subset \Delta_2 \subset \Delta_1 \subset \mathbb{P}(S^2(W^\vee))$, where Δ_i stands for the closed subscheme of those singular quadrics of corank $\geq i$.

Theorem 2.15 (Intersection of two quadrics). *Let X_L be as in Corollary* 2.14. *Assume that* $\dim(L) = 2$ *and that* $\mathbb{P}(L) \cap \Delta_2 = \emptyset$.

(i) *When d is even, the conjectures* $\mathrm{C}(X_L)$, *with* $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, *hold.*

(ii) *When d is odd and $k = \mathbb{F}_q$ is a finite field of characteristic $p \geq 3$, the conjectures* $\mathrm{C}(X_L)$, *with* $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, *hold.*

The proof of Theorem 2.15 is based on the solution of the corresponding noncommutative conjectures of Corollary 2.14; consult Section 8 for details. In what concerns the Tate conjecture, an alternative (geometric) proof, based on the notion of variety of maximal planes, was obtained by Reid[4] in the early seventies; consult [31, Thms. 3.14 and 4.14]. Therein, Reid proved the Hodge conjecture but, as Bruno Kahn informed me, a similar proof works for the Tate conjecture. In what concerns the variants of the Tate conjecture, Theorem 2.15 is, to the best of the author's knowledge, new in the literature.

Remark 2.16 (Intersection of even-dimensional quadrics). As explained in Remark 8.3 below, when d is even and $\dim(L) \geq 2$, i.e., when X_L is the intersection of $\dim(L)$ even-dimensional quadric hypersurfaces, a proof similar to the one of Theorem 2.15 shows that the conjectures $\mathrm{C}(X_L)$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, are equivalent to the corresponding conjectures for the discriminant 2-fold cover of the projective space $\mathbb{P}(L)$. This result is also new in the literature.

[4] Reid also assumed in *loc. cit.* that $\mathbb{P}(L) \cap \Delta_2 = \emptyset$; consult [31, Def. 1.9].

Tate conjecture(s) for stacks

As a second application of Theorem 2.1, we obtain the following extension of the Tate conjecture and of its variants to the broad setting of smooth proper algebraic k-stacks:

$$\mathrm{C}(\mathcal{X}) := \mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X})\big) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}. \tag{2.17}$$

The next results prove these extended conjectures in several different cases:

Theorem 2.18 (Root stacks). *Let X be a smooth projective k-scheme, $\mathcal{L}$ a line bundle on X, $\varsigma \in \Gamma(X, \mathcal{L})$ a global section, $n \geq 1$ an integer, and $\mathcal{X} := \sqrt[n]{(\mathcal{L}, \varsigma)/X}$ the associated root stack. When the zero locus $Z \hookrightarrow X$ of the global section ς is smooth, we have the following equivalences:*

$$\mathrm{C}(X) + \mathrm{C}(Z) \Leftrightarrow \mathrm{C}(\mathcal{X}) \quad \textit{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}. \tag{2.19}$$

Corollary 2.20 (Low-dimensional root stacks). *When $\dim(X) \leq 1$ or X is an abelian surface, the conjectures $\mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold. The conjectures $\mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p\}$, also hold when X is a $K3$-surface.*

Theorem 2.21 (Orbifolds). *Let G be a finite group of order n, X a smooth projective k-scheme equipped with a G-action, and $\mathcal{X} := [X/G]$ the associated global orbifold. When $p \nmid n$, we have the following implications:*

$$\sum_{\sigma \subseteq G} \mathrm{C}\big(X^\sigma \times \mathrm{Spec}\,\big(k[\sigma]\big)\big) \Rightarrow \mathrm{C}(\mathcal{X}) \quad \textit{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}, \tag{2.22}$$

where σ is a cyclic subgroup of G. Moreover, when k contains the nth roots of unity, the k-schemes $X^\sigma \times \mathrm{Spec}(k[\sigma])$ in (2.22) can be replaced by the k-schemes X^σ.

Corollary 2.23 (Low-dimensional orbifolds). *Assume that $p \nmid n$.*

(i) *When $\dim(X) \leq 1$, the conjectures $\mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold.*

(ii) *Assume that k contains the nth roots of unity. When X is an abelian surface, the conjectures $\mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold. The conjectures $\mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p\}$, hold also when X is a $K3$-surface.*

(iii) *Assume that k contains the nth roots of unity. When X is an abelian variety of dimension 3 and G acts by group homomorphisms[5], the conjectures $\mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold.*

Let G, X, and $\mathcal{X} := [X/G]$ be as above in Theorem 2.21. Suppose that $\mathcal{X}$ is equipped with a sheaf of Azumaya algebras[6] $\mathcal{F}$ of rank r. In this case, similarly to (2.17), we can write $\mathrm{C}(\mathcal{X}; \mathcal{F}) := \mathrm{C}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}; \mathcal{F}))$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, where $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}; \mathcal{F})$ stands for the dg category of perfect complexes of $\mathcal{F}$-modules. The following result is the "twisted" version of Theorem 2.21.

[5]For example, in the case where $G = \mathbb{Z}/2$, we can consider the canonical involution $x \mapsto -x$.

[6]In other words, $\mathcal{F}$ is a G-equivariant sheaf of Azumaya algebras of rank r over X.

Theorem 2.24 (Twisted orbifolds). *When $p \nmid nr$ and k contains the nth roots of unity, we have the following implications:*

$$\sum_{\sigma \subseteq G} \mathrm{C}(Y_\sigma) \Rightarrow \mathrm{C}(\mathcal{X};\mathcal{F}) \quad \textit{with}\ \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}, \tag{2.25}$$

where σ is a cyclic subgroup of G and Y_σ is a certain $\sigma^\vee$-Galois cover of X^σ induced by the restriction of $\mathcal{F}$ to X^σ.

Corollary 2.26 (Low-dimensional twisted orbifolds). *When* $\dim(X) \leq 1$ *or X is an abelian surface, the conjectures* $\mathrm{C}(\mathcal{X};\mathcal{F})$*, with* $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$*, hold. The conjectures* $\mathrm{C}(\mathcal{X};\mathcal{F})$*, with* $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p\}$*, also hold when X is a $K3$-surface.*

3. Preliminaries

Throughout the article, k denotes a base field of characteristic $p \geq 0$.

3.1. Dg categories

For a survey on dg categories, we invite the reader to consult [18]. A *differential graded (= dg) category* $\mathcal{A}$ is a category enriched over complexes of k-vector spaces. Let us write $\mathrm{dgcat}(k)$ for the category of (small) dg categories. Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and $\mathcal{A}^{\mathrm{op}}(x,y) := \mathcal{A}(y,x)$. A *right dg $\mathcal{A}$-module* is a dg functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-vector spaces. Following [18, §3.2], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of the category of right dg $\mathcal{A}$-modules $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. Let us write $\mathcal{D}_c(\mathcal{A})$ for the triangulated subcategory of compact objects.

A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence on derived categories $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$; see [18, §4.6]. As explained in [35, §1.6], the category $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by $\mathrm{Hmo}(k)$ the associated homotopy category.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is $\mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{B})$ and $(\mathcal{A} \otimes \mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y) \otimes \mathcal{B}(w,z)$. As explained in [18, §2.3], this construction gives rise to a symmetric monoidal structure $- \otimes -$ on $\mathrm{dgcat}(k)$ which descends to the homotopy category $\mathrm{Hmo}(k)$.

A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B}\colon \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ or, equivalently, a right dg $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. A standard example is the dg $\mathcal{A}$-$\mathcal{B}$-bimodule

$${}_F\mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x,z) \mapsto \mathcal{B}\big(z, F(x)\big) \tag{3.1}$$

associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$. Following Kontsevich [19–21], a dg category $\mathcal{A}$ is called *smooth* if the dg $\mathcal{A}$-$\mathcal{A}$-bimodule ${}_{\mathrm{id}}\mathcal{A}$ belongs to the category $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$ and *proper* if $\sum_n \dim H^n\mathcal{A}(x,y) < \infty$ for any pair of objects (x,y).

3.2. Additive invariants

Given dg categories $\mathcal{A}$ and $\mathcal{B}$ and a dg $\mathcal{A}$-$\mathcal{B}$-bimodule B, consider the following dg category $T(\mathcal{A}, \mathcal{B}; \mathrm{B})$: the set of objects is $\mathrm{obj}(\mathcal{A}) \amalg \mathrm{obj}(\mathcal{B})$; the complexes of k-vector spaces of morphisms $T(\mathcal{A}, \mathcal{B}; \mathrm{B})(x, y)$ are equal to $\mathcal{A}(x, y)$ when $x, y \in \mathcal{A}$, to $\mathcal{B}(x, y)$ when $x, y \in \mathcal{B}$, to $\mathrm{B}(x, y)$ when $x \in \mathcal{A}$ and $y \in \mathcal{B}$, and to 0 when $x \in \mathcal{B}$ and $y \in \mathcal{A}$; and the composition law is induced by the composition law of $\mathcal{A}$ and $\mathcal{B}$ and by the dg $\mathcal{A}$-$\mathcal{B}$-bimodule structure of B. By construction, we have canonical dg functors $\iota_{\mathcal{A}} \colon \mathcal{A} \to T(\mathcal{A}, \mathcal{B}; \mathrm{B})$ and $\iota_{\mathcal{B}} \colon \mathcal{B} \to T(\mathcal{A}, \mathcal{B}; \mathrm{B})$.

Recall from [35, Def. 2.1] that a functor $E \colon \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it satisfies the following conditions:

(i) it sends the Morita equivalences to isomorphisms;

(ii) given $\mathcal{A}$, $\mathcal{B}$, and B, as above, the dg functors $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ induce an isomorphism

$$E(\mathcal{A}) \oplus E(\mathcal{B}) \xrightarrow{\simeq} E\big(T(\mathcal{A}, \mathcal{B}; \mathrm{B})\big).$$

Let us write $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-modules B such that for every object $x \in \mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathrm{B}(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. As explained in [35, §1.6.3], there is a natural bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$ and the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ corresponds to the (derived) tensor product of bimodules. Therefore, since the dg $\mathcal{A}$-$\mathcal{B}$-bimodules (3.1) belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, we have the following functor:

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad \big(\mathcal{A} \xrightarrow{F} \mathcal{B}\big) \mapsto {}_F\mathcal{B}. \tag{3.2}$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects as $\mathrm{Hmo}(k)$ and with abelian groups of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B})$ given by the Grothendieck group $K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B})$ of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. As explained in [35, §2.3], the following composition is the *universal* additive invariant:

$$U \colon \mathrm{dgcat}(k) \xrightarrow{(3.2)} \mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad \big(\mathcal{A} \xrightarrow{F} \mathcal{B}\big) \mapsto [{}_F\mathcal{B}]. \tag{3.3}$$

3.3. Noncommutative motives

For a book on noncommutative motives, we invite the reader to consult [35]. Recall from [35, §4.1] that the category of *noncommutative Chow motives* $\mathrm{NChow}(k)_{\mathbb{Q}}$ (with $\mathbb{Q}$-coefficients) is defined as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)_{\mathbb{Q}}$ consisting of those objects $U(\mathcal{A})_{\mathbb{Q}}$ with $\mathcal{A}$ a smooth proper dg category. This category $\mathrm{NChow}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive, and rigid symmetric monoidal. Moreover, we have natural isomorphisms:

$$\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}\big(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}}\big) := K_0\big(\mathrm{rep}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})\big)_{\mathbb{Q}} \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})_{\mathbb{Q}}. \tag{3.4}$$

Given a $\mathbb{Q}$-linear, additive, rigid symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, its $\mathcal{N}$-ideal is defined as follows ($\mathrm{tr}(g \circ f)$ stands for the categorical trace of $g \circ f$):

$$\mathcal{N}(a,b) := \big\{ f \in \mathrm{Hom}_{\mathcal{C}}(a,b) \mid \forall g \in \mathrm{Hom}_{\mathcal{C}}(b,a) \text{ we have } \mathrm{tr}(g \circ f) = 0 \big\}.$$

Under these notations, recall from [35, §4.6] that the category of *noncommutative numerical motives* $\mathrm{NNum}(k)_{\mathbb{Q}}$ (with $\mathbb{Q}$-coefficients) is defined as the idempotent completion of the quotient category $\mathrm{NChow}(k)_{\mathbb{Q}}/\mathcal{N}$.

4. Noncommutative conjectures

Let $\mathcal{A}$ be a smooth proper (k-linear) dg category.

Noncommutative conjecture $\mathrm{T}^l_{\mathrm{nc}}(\mathcal{A})$

Let $l \neq p$ be a prime number. Following Thomason [43], consider the l-adic étale K-theory groups

$$K_n^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s) := \pi_n\big(\mathrm{holim}_\nu L_{KU} I\!K(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^\nu)\big) \quad n \in \mathbb{Z}, \tag{4.1}$$

where $I\!K(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^\nu)$ stands for the (non-connective) algebraic K-theory spectrum with $\mathbb{Z}/l^\nu$-coefficients of the dg category $\mathcal{A} \otimes_k k_s$ and $L_{KU} I\!K(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^\nu)$ for the Bousfield localization of $I\!K(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^\nu)$ with respect to complex topological K-theory KU. Thanks to the work of Suslin [33, 34] and Gabber [12], we have a $\mathbb{Z}$-graded ring isomorphism $K_*^{\hat{l},\mathrm{et}}(k_s) \simeq \mathbb{Z}_l[t, t^{-1}]$, where t is of degree 2. Note that this implies that the above K-theory groups (4.1) are $\mathbb{Z}_l$-modules and that

$$K_n^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s) \simeq K_{n+2}^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)$$

for every $n \in \mathbb{Z}$. Note also that, by construction, the absolute Galois group $\mathrm{Gal}(k_s/k)$ acts on the above $\mathbb{Z}_l$-modules (4.1) and that we have a canonical $\mathbb{Q}_l$-linear homomorphism:

$$K_0(\mathcal{A})_{\mathbb{Q}_l} \to K_0^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k)}. \tag{4.2}$$

Conjecture $\mathrm{T}^l_{\mathrm{nc}}(\mathcal{A})$**.** *When k is finitely generated over its prime field, the above homomorphism* (4.2) *is surjective.*

Noncommutative conjecture $\overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathcal{A})$

Let $l \neq p$ be a prime number. Similarly to (4.2), we have a canonical $\mathbb{Q}_l$-linear homomorphism

$$K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l} \to K_0^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k')} \tag{4.3}$$

for every finite field extension k'/k inside k_s. Moreover, since $k_s = \bigcup_{k'/k} k'$, where the union runs over all the finite field extensions k'/k inside k_s, the functors $\mathcal{A} \otimes_k -$

and $K_0(-)_{\mathbb{Q}_l}$ preserve filtered colimits, and the $\mathbb{Q}_l$-linearized Grothendieck group $K_0(\mathcal{A} \otimes_k k_s)_{\mathbb{Q}_l}$ identifies with $\mathrm{colim}_{k'/k} K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l}$. Consequently, we obtain a canonical $\mathbb{Q}_l$-linear homomorphism:

$$K_0(\mathcal{A} \otimes_k k_s)_{\mathbb{Q}_l} \to \bigcup_{k'/k} K_0^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k')}. \tag{4.4}$$

Conjecture $\overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathcal{A})$**.** *When k is finitely generated over its prime field, the above homomorphism* (4.4) *is surjective.*

Lemma 4.5. *We have the implication* $\overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathcal{A}) \Rightarrow \mathrm{T}^l_{\mathrm{nc}}(\mathcal{A})$.

Proof. Let β be an element of $K_0^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)_{1/l}$ which is fixed under the $\mathrm{Gal}(k_s/k)$-action. We need to construct an element α of $K_0(\mathcal{A})_{\mathbb{Q}_l}$ which is mapped to β by the homomorphism (4.2). Since the conjecture $\overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathcal{A})$ holds, there exists a finite field extension k'/k inside k_s (which we can assume without loss of generality to be Galois[7]) and an element α' of $K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l}$ which is mapped to β by the homomorphism (4.3). Recall from [27, §7] that we have well-defined $\mathbb{Q}_l$-homomorphisms

$$- \otimes_k k' \colon K_0(\mathcal{A})_{\mathbb{Q}_l} \to K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l}, \quad \mathrm{res}_{k'/k} \colon K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l} \to K_0(\mathcal{A})_{\mathbb{Q}_l}$$

such that

$$\mathrm{res}_{k'/k}(\alpha') \otimes_k k' = \sum_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(\alpha').$$

Let us take $\alpha := \frac{1}{[k':k]} \mathrm{res}_{k'/k}(\alpha')$. Since β is fixed by the $\mathrm{Gal}(k_s/k)$-action and (4.3) is $\mathrm{Gal}(k'/k)$-equivariant, we hence conclude that α is mapped to β by the homomorphism (4.2). ∎

Noncommutative conjecture $\mathrm{SS}^l_{\mathrm{nc}}(\mathcal{A})$

Recall from above that, up to isomorphism, we have two (continuous) l-adic representations

$$K_0^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)_{1/l}, \quad K_1^{\hat{l},\mathrm{et}}(\mathcal{A} \otimes_k k_s)_{1/l} \tag{4.6}$$

of the absolute Galois group $\mathrm{Gal}(k_s/k)$.

Conjecture $\mathrm{SS}^l_{\mathrm{nc}}(\mathcal{A})$**.** *When k is finitely generated over its prime field, the above l-adic representations* (4.6) *are semi-simple.*

Noncommutative conjecture $\mathrm{T}^p(\mathcal{A})$

Let $k := \mathbb{F}_q$ be a finite field of characteristic $p > 0$, $W(k)$ the associated ring of p-typical Witt vectors, $K := W(k)_{1/p}$ the fraction field of $W(k)$, and $\sigma \colon K \xrightarrow{\simeq} K$ the automorphism induced by the Frobenius map $\lambda \mapsto \lambda^p$ on k. Recall that the associated field extension $K/\mathbb{Q}_p$ is finite.

[7]If the field extension k'/k is not Galois, take the normal closure of k' inside k_s.

Consider the topological Hochschild homology $\mathrm{THH}(\mathcal{A})$ of the dg category $\mathcal{A}$. The canonical S^1-action on $\mathrm{THH}(\mathcal{A})$ gives rise to the spectrum of homotopy orbits $\mathrm{THH}(\mathcal{A})_{hS^1}$, to the spectrum of homotopy fixed-points $TC^-(\mathcal{A}) := \mathrm{THH}(\mathcal{A})^{hS^1}$, and also to the Tate construction $TP(\mathcal{A}) := \mathrm{THH}(\mathcal{A})^{tS^1}$. As explained in [30, Cor. I.4.3], these spectra are related by the following cofiber sequence:

$$\Sigma\,\mathrm{THH}(\mathcal{A})_{hS^1} \xrightarrow{\mathrm{N}} \mathrm{THH}(\mathcal{A})^{hS^1} \xrightarrow{\mathrm{can}} \mathrm{THH}(\mathcal{A})^{tS^1}, \tag{4.7}$$

where N is the norm map. It is well known that the abelian groups $\mathrm{THH}_*(\mathcal{A})$ are k-linear. Hence, the spectrum $\Sigma\,\mathrm{THH}(\mathcal{A})_{hS^1}$ becomes trivial after inverting p. Consequently, the cofiber sequence (4.7) leads to a canonical isomorphism:

$$\mathrm{can}\colon TC_0^-(\mathcal{A})_{1/p} \xrightarrow{\simeq} TP_0(\mathcal{A})_{1/p}. \tag{4.8}$$

Recall that $TP_0(\mathcal{A})_{1/p}$ is a finitely-generated module over $TP_0(k)_{1/p} \simeq K$, i.e., a finite-dimensional K-vector space. Since $\mathcal{A}$ is smooth and proper, it is also well known that the spectrum $\mathrm{THH}(\mathcal{A})$ is bounded below and p-complete. Making use of [30, Lem. II 4.2], we hence obtain moreover a "cyclotomic Frobenius":

$$\varphi_p\colon TC_0^-(\mathcal{A})_{1/p} \to TP_0(\mathcal{A})_{1/p}. \tag{4.9}$$

Let us write $\varphi := \varphi_p \circ \mathrm{can}^{-1}$ for the associated endomorphism of $TP_0(\mathcal{A})_{1/p}$. This endomorphism φ is *not* K-linear but only σ-semilinear.

Now, recall from [36, Prop. 4.2] that the assignment $\mathcal{A} \mapsto TP_0(\mathcal{A})_{1/p}$ gives rise to a $\mathbb{Q}_p$-linear functor from $\mathrm{NChow}(k)_{\mathbb{Q}_p}$ to the category of K-vector spaces $\mathrm{Vect}(K)$. By compositing it with the forgetful functor from K-vector spaces to $\mathbb{Q}_p$-vector spaces, we hence obtain the following $\mathbb{Q}_p$-linear functor:

$$TP_0(-)_{1/p}\colon \mathrm{NChow}(k)_{\mathbb{Q}_p} \to \mathrm{Vect}(\mathbb{Q}_p). \tag{4.10}$$

This leads, in particular, to the induced $\mathbb{Q}_p$-linear homomorphism

$$\begin{array}{c} K_0(\mathcal{A})_{\mathbb{Q}_p} \simeq \mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}_p}}\big(U(k)_{\mathbb{Q}_p}, U(\mathcal{A})_{\mathbb{Q}_p}\big) \\ \big\downarrow \\ TP_0(\mathcal{A})_{1/p} \simeq \mathrm{Hom}_{\mathrm{Vect}(\mathbb{Q}_p)}\big(TP_0(k)_{1/p}, TP_0(\mathcal{A})_{1/p}\big). \end{array} \tag{4.11}$$

Lemma 4.12. *The preceding homomorphism* (4.11) *takes values in the $\mathbb{Q}_p$-linear subspace $TP_0(\mathcal{A})_{1/p}^{\varphi}$ of those elements that are fixed by φ.*

Proof. On the one hand, the $\mathbb{Q}_p$-linear endomorphisms $\varphi\colon TP_0(\mathcal{A})_{1/p} \to TP_0(\mathcal{A})_{1/p}$ (parametrized by the smooth proper dg categories $\mathcal{A}$) give rise to a natural transformation from (4.10) to itself. On the other hand, thanks to the enriched Yoneda lemma, the $\mathbb{Q}_p$-linear natural transformations from the following functor

$$K_0(-)_{\mathbb{Q}_p} \simeq \mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}_p}}\big(U(k)_{\mathbb{Q}_p}, -\big)\colon \mathrm{NChow}(k)_{\mathbb{Q}_p} \to \mathrm{Vect}(\mathbb{Q}_p)$$

to the above functor (4.10) are in one-to-one correspondence with the elements of the $\mathbb{Q}_p$-vector space $TP_0(k)_{1/p} \simeq K$. Under this bijection, the unit element $1 \in K$ corresponds to the above homomorphism (4.11). Therefore, in order to prove Lemma 4.12, it suffices to show that the endomorphism $\varphi\colon TP_0(k)_{1/p} \to TP_0(k)_{1/p}$ sends 1 to 1. This follows from the explicit descriptions

$$\begin{aligned} \mathrm{can}\colon W(k)[u,v]/(uv-p) &\to W(k)[\delta,\delta^{-1}], \quad u \mapsto p\delta, \quad v \mapsto \delta^{-1}, \\ \varphi_p\colon W(k)[u,v]/(uv-p) &\to W(k)[\delta,\delta^{-1}], \quad u \mapsto \delta, \quad\ \ v \mapsto p\delta^{-1} \end{aligned}$$

of the homomorphisms can, $\varphi_p\colon TC_*^-(k) \to TP_*(k)$, where the variables u and δ have degree 2 and the variable v has degree -2; consult [6, Props. 6.2-6.3]. ■

Thanks to Lemma 4.12, we have an induced $\mathbb{Q}_p$-linear homomorphism:

$$K_0(\mathcal{A})_{\mathbb{Q}_p} \to TP_0(\mathcal{A})^{\varphi}_{1/p}. \tag{4.13}$$

Conjecture $\mathrm{T}^p_{\mathrm{nc}}(\mathcal{A})$**.** *The above homomorphism* (4.13) *is surjective.*

Noncommutative conjecture $\mathbf{B}_{\mathrm{nc}}(\mathcal{A})$

Recall from [35, §4.7] that the Grothendieck group $K_0(\mathcal{A}) := K_0(\mathcal{D}_c(\mathcal{A}))$ of the dg category $\mathcal{A}$ is equipped with the Euler pairing $\chi\colon K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$ defined as

$$\big([M],[N]\big) \mapsto \sum_n (-1)^n \dim \mathrm{Hom}_{\mathcal{D}_c(\mathcal{A})}\big(M, N[-n]\big).$$

This bilinear pairing is not symmetric neither skew-symmetric. Nevertheless, as proved in [35, Prop. 4.24], the left and right kernels of χ agree. Consequently, we have a well-defined numerical Grothendieck group $K_0(\mathcal{A})/_{\sim\mathrm{num}} := K_0(\mathcal{A})/\operatorname{Ker}(\chi)$. In what follows, we will write $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim\mathrm{num}}$ for the associated $\mathbb{Q}$-vector space $(K_0(\mathcal{A})/_{\sim\mathrm{num}})_{\mathbb{Q}}$, which is isomorphic to the quotient $K_0(\mathcal{A})_{\mathbb{Q}}/\operatorname{Ker}(\chi_{\mathbb{Q}})$.

Conjecture $\mathrm{B}_{\mathrm{nc}}(\mathcal{A})$**.** *When k is finite, we have* $K_0(\mathcal{A})_{\mathbb{Q}} = K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim\mathrm{num}}$.

5. Proof of Theorem 2.1

We start by proving the equivalence $\mathrm{T}^l(X) \Leftrightarrow \mathrm{T}^l_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. On the one hand, recall from [11, §18.3] that since $\mathbb{Q} \subset \mathbb{Q}_l$, we have a natural isomorphism between the $\mathbb{Q}_l$-linearized Grothendieck group

$$K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}_l} \simeq K_0(X)_{\mathbb{Q}_l}$$

and the direct sum $\bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}_l}$. On the other hand, it follows from the canonical Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X) \otimes_k k_s \to \mathrm{perf}_{\mathrm{dg}}(X_{k_s})$ and from the work of Thomason

[43, Thm. 4.1] and Soulé [32, §3.3.2] that we have a natural isomorphism between the $\mathbb{Z}[1/l]$-linearized l-adic étale K-theory group $K_0^{\hat{l},\mathrm{et}}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_s)_{1/l}$ and the direct sum $\bigoplus_{i=0}^{\dim(X)} H^{2i}_{\mathrm{et}}(X_{k_s},\mathbb{Q}_l(i))$. Moreover, as explained by Friedlander in [10, §5], under the preceding isomorphisms the homomorphism (4.2) (with $\mathcal{A}=\mathrm{perf}_{\mathrm{dg}}(X)$) corresponds to the graded homomorphism (1.1). Consequently, (1.1) is surjective if and only if (4.2) is surjective.

We now prove the equivalence $\overline{\mathrm{T}}^l(X)\Leftrightarrow\overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. Recall from Section 4 that the canonical $\mathbb{Q}_l$-linear homomorphism (4.4) (with $\mathcal{A}=\mathrm{perf}_{\mathrm{dg}}(X)$) may be described as

$$\mathrm{colim}_{k'/k}\Big(K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k'\big)_{\mathbb{Q}_l}\xrightarrow{(4.3)}K_0^{\hat{l},\mathrm{et}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_s\big)_{1/l}^{\mathrm{Gal}(k_s/k')}\Big),$$

where k'/k is a finite field extension inside k_s. Similarly, the canonical $\mathbb{Q}_l$-linear homomorphism (1.2) may be described as the filtered colimit

$$\mathrm{colim}_{k'/k}\big(\mathcal{Z}^*(X_{k'})_{\mathbb{Q}_l}\to H^{2*}_{\mathrm{et}}\big(X_{k_s},\mathbb{Q}_l(*)\big)^{\mathrm{Gal}(k_s/k')}\big).$$

Consequently, the proof is now similar to the proof of $\mathrm{T}^l(X)\Leftrightarrow\mathrm{T}^l_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ with k replaced by a finite field extension k' of k inside k_s.

We now prove the equivalence $\mathrm{SS}^l(X)\Leftrightarrow\mathrm{SS}^l_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. As above, by combining the canonical Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_s\to\mathrm{perf}_{\mathrm{dg}}(X_{k_s})$ with the work of Thomason [43, Thm. 4.1] and Soulé [32, §3.3.2], we obtain natural isomorphisms between the l-adic (continuous) representations

$$K_0^{\hat{l},\mathrm{et}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_s\big)_{1/l},\quad K_1^{\hat{l},\mathrm{et}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_s\big)_{1/l}$$

of the absolute Galois group $\mathrm{Gal}(k_s/k)$ and the direct sums $\bigoplus_{i=0}^{\dim(X)} H^{2i}_{\mathrm{et}}(X_{k_s},\mathbb{Q}_l(i))$ and $\bigoplus_{i=0}^{\dim(X)} H^{2i+1}_{\mathrm{et}}(X_{k_s},\mathbb{Q}_l(i))$, respectively. Note that these direct sums are semi-simple if and only if the l-adic representations

$$\big\{H^{2i}_{\mathrm{et}}(X_{k_s},\mathbb{Q}_l)\mid 0\leq i\leq\dim(X)\big\}\quad\text{and}\quad\big\{H^{2i+1}_{\mathrm{et}}(X_{k_s},\mathbb{Q}_l)\mid 0\leq i\leq\dim(X)\big\}$$

are semi-simple. Therefore, we conclude that the l-adic representations (4.6) (with $\mathcal{A}=\mathrm{perf}_{\mathrm{dg}}(X)$) are semi-simple if and only if the l-adic representations $\{H^n_{\mathrm{et}}(X_{k_s},\mathbb{Q}_l)\mid 0\leq n\leq 2\dim(X)\}$ are semi-simple.

We now prove the equivalence $\mathrm{T}^p(X)\Leftrightarrow\mathrm{T}^p_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. Note first that we have the following equality of graded $\mathbb{Q}_l$-vector spaces:

$$H^{2*}_{\mathrm{crys}}(X)(*)^{\phi}=H^{2*}_{\mathrm{crys}}(X)^{\frac{1}{p^*}\phi},$$

where ϕ stands for the crystalline Frobenius. Therefore, the conjecture $\mathrm{T}^p(X)$ may be re-formulated as the surjectivity of the classical cycle class map

$$\mathcal{Z}^*(X)_{\mathbb{Q}_p}\to H^{2*}_{\mathrm{crys}}(X)^{\frac{1}{p^*}\phi}.\qquad(5.1)$$

On the one hand, recall from [11, §18.3] that since $\mathbb{Q} \subset \mathbb{Q}_l$, we have a natural isomorphism between the $\mathbb{Q}_p$-linearized Grothendieck group $K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}_p} \simeq K_0(X)_{\mathbb{Q}_p}$ and the direct sum $\bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}_p}$. On the other hand, recall from [37, Thm. 5.2] that we have a natural isomorphism between the $\mathbb{Z}[1/p]$-linearized topological periodic cyclic homology group $TP_0(\mathrm{perf}_{\mathrm{dg}}(X))_{1/p}$ and the direct sum $\bigoplus_{i=0}^{\dim(X)} H^{2i}_{\mathrm{crys}}(X)$, which identifies φ with $\frac{1}{p^*}\phi$. Under the preceding isomorphisms, the homomorphism (4.13) (with $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(X)$) corresponds to the graded homomorphism (5.1). Consequently, (5.1) is surjective if and only if (4.13) is surjective.

Finally, we prove the equivalence $\mathrm{B}(X) \Leftrightarrow \mathrm{B}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$. Note first that since

$$\mathcal{D}_c\big(\mathrm{perf}_{\mathrm{dg}}(X)\big) \simeq \mathrm{perf}(X),$$

the Euler pairing $\chi\colon K_0(X) \times K_0(X) \to \mathbb{Z}$ may be re-written as

$$\big([\mathcal{F}], [\mathcal{G}]\big) \mapsto \sum_n (-1)^n \dim \mathrm{Hom}_{\mathrm{perf}(X)}\big(\mathcal{F}, \mathcal{G}[-n]\big).$$

Recall from [11, §19] that a graded algebraic cycle $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$ is *numerically equivalent to zero* if $\int_X \beta \cdot \alpha = 0$ for every $\beta \in \mathcal{Z}^*(X)_{\mathbb{Q}}$. Recall also that we have the isomorphism

$$\tau\colon K_0(X)_{\mathbb{Q}} \xrightarrow{\simeq} \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}}, \quad [\mathcal{F}] \mapsto \mathrm{ch}(\mathcal{F}) \cdot \sqrt{\mathrm{Td}_X}, \tag{5.2}$$

where $\mathrm{ch}(\mathcal{F})$ stands for the Chern character of $\mathcal{F}$ and $\sqrt{\mathrm{Td}_X}$ for the square root of the Todd class; see [11, §18.3]. Given any two perfect complexes $\mathcal{F}, \mathcal{G} \in \mathrm{perf}(X)$, the Hirzebruch–Riemann–Roch theorem (see [11, Cor. 18.3.1]) yields the equality

$$\mathrm{Eu}\big(\pi_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})\big) = \int_X \tau\big([\mathcal{F}^\vee]\big) \cdot \tau\big([\mathcal{G}]\big),$$

where Eu denotes the Euler characteristic and $\pi\colon X \to \mathrm{Spec}(k)$ is the structural morphism of X. Since $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$, where $\underline{\mathrm{Hom}}(-,-)$ stands for the internal Hom of the rigid symmetric monoidal category $\mathrm{perf}(X)$, we hence conclude that $\mathrm{Eu}(\pi_*(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})))$ agrees with $\chi([\mathcal{F}], [\mathcal{G}])$. This implies that (5.2) descends to the numerical quotients

$$\begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow[\simeq]{\tau} & \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ K_0(X)_{\mathbb{Q}}/_{\sim \mathrm{num}} & \xrightarrow[\tau]{\simeq} & \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}}/_{\sim \mathrm{num}}. \end{array} \tag{5.3}$$

Consequently, the proof follows now from the fact that the conjecture $\mathrm{B}(X)$, resp. $\mathrm{B}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$, is equivalent to the injectivity of the vertical graded homomorphism on the right-hand side of (5.3), resp. to the injectivity of the vertical homomorphism on the left-hand side of (5.3).

6. Proof of Theorem 2.2

By definition of the Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{i-1}(i-1)\rangle$, we have a chain of admissible triangulated subcategories $\mathbb{A}_{i-1} \subseteq \cdots \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_0$ with $\mathbb{A}_r(r) := \mathbb{A}_r \otimes \mathcal{L}_X(r)$. Note that $\mathbb{A}_r(r) \simeq \mathbb{A}_r$. Let $\mathfrak{a}_r$ be the right orthogonal complement to $\mathbb{A}_{r+1}$ in $\mathbb{A}_r$; these are called the *primitive subcategories* in [23, §4]. By construction, we have the semi-orthogonal decompositions

$$\mathbb{A}_r = \langle \mathfrak{a}_r, \mathfrak{a}_{r+1}, \dots, \mathfrak{a}_{i-1}\rangle, \quad 0 \le r \le i-1. \tag{6.1}$$

As proved in [23, Thm. 6.3] (see also [2, Thm. 2.3.4]), the category perf(Y) admits an HP-dual Lefschetz decomposition $\langle \mathbb{B}_{j-1}(1-j), \mathbb{B}_{j-2}(2-j), \dots, \mathbb{B}_0\rangle$ with respect to $\mathcal{L}_Y(1)$; as above, we have a chain of admissible triangulated subcategories $\mathbb{B}_{j-1} \subseteq \mathbb{B}_{j-2} \subseteq \cdots \subseteq \mathbb{B}_0$. Moreover, the primitive subcategories coincide (via a Fourier–Mukai-type functor) with those of perf(X) and we have the semi-orthogonal decompositions

$$\mathbb{B}_r = \langle \mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_{\dim(V)-r-2}\rangle, \quad 0 \le r \le j-1. \tag{6.2}$$

Furthermore, the assumptions (a)–(b) of Theorem 2.2 imply the existence of the semi-orthogonal decompositions

$$\operatorname{perf}(X_L) = \big\langle \mathbb{C}_L, \mathbb{A}_{\dim(V)}(1), \dots, \mathbb{A}_{i-1}\big(i-\dim(V)\big)\big\rangle, \tag{6.3}$$
$$\operatorname{perf}(Y_L) = \big\langle \mathbb{B}_{j-1}\big(\dim(L^\perp)-j\big), \dots, \mathbb{B}_{\dim(L^\perp)}(-1), \mathbb{C}_L\big\rangle, \tag{6.4}$$

where $\mathbb{C}_L$ is a common (triangulated) category. Let us denote by $\mathbb{C}_L^{\mathrm{dg}}$, $\mathbb{A}_r^{\mathrm{dg}}$, and $\mathfrak{a}_r^{\mathrm{dg}}$, the dg enhancement of $\mathbb{C}_L$, $\mathbb{A}_r$, and $\mathfrak{a}_r$, induced from the dg category $\operatorname{perf}_{\mathrm{dg}}(X_L)$. Similarly, let us denote by $\mathbb{C}_L^{\mathrm{dg}'}$ and $\mathbb{B}_r^{\mathrm{dg}}$ the dg enhancement of $\mathbb{C}_L$ and $\mathbb{B}_r$ induced from the dg category $\operatorname{perf}_{\mathrm{dg}}(Y_L)$. Note that thanks to assumption (a) of Theorem 2.2, all the above dg categories are smooth and proper.

We start by proving the equivalence $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$. As explained in [35, Prop. 2.2], since the functor (3.3) is an additive invariant, the above semi-orthogonal decomposition (6.3) gives rise to the direct sum decomposition

$$U\big(\operatorname{perf}_{\mathrm{dg}}(X_L)\big)_{\mathbb{Q}_l} \simeq U(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}_l} \oplus U(\mathbb{A}^{\mathrm{dg}}_{\dim(V)})_{\mathbb{Q}_l} \oplus \cdots \oplus U(\mathbb{A}^{\mathrm{dg}}_{i-1})_{\mathbb{Q}_l} \tag{6.5}$$

in the $\mathbb{Q}_l$-linearized category $\mathrm{NChow}(k)_{\mathbb{Q}_l}$. Consider the functors

$$K_n^{\hat{l},\mathrm{et}}(-\otimes_k k_s)_{1/l} \colon \mathrm{dgcat}(k) \to \mathrm{Vect}(\mathbb{Q}_l), \quad n \in \mathbb{Z} \tag{6.6}$$

with values in the category of $\mathbb{Q}_l$-vector spaces.

Proposition 6.7. *The above functors* (6.6) *are additive invariants.*

Proof. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a Morita equivalence. As proved in [27, Prop. 7.1], the induced dg functor $F \otimes_k k_s\colon \mathcal{A} \otimes_k k_s \to \mathcal{B} \otimes_k k_s$ is also a Morita equivalence. Therefore, since (non-connective) algebraic K-theory with $\mathbb{Z}/l^\nu$-coefficients sends Morita equivalences to equivalences of spectra (consult [35, §2.2.2]), we conclude that the above functors (6.6) send Morita equivalences to isomorphisms. Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories and B a dg $\mathcal{A}$-$\mathcal{B}$-bimodule. Following Section 3.2, we need to show that the dg functors $\iota_{\mathcal{A}}$ and ι_B induce an isomorphism:

$$K_n^{\hat{l},\mathrm{et}}(\mathcal{A}\otimes_k k_s)_{1/l}\oplus K_n^{\hat{l},\mathrm{et}}(\mathcal{A}\otimes_k k_s)_{1/l}\to K_n^{\hat{l},\mathrm{et}}\big(T(\mathcal{A},\mathcal{B};\mathrm{B})\big)_{1/l}. \tag{6.8}$$

Consider the dg categories $\mathcal{A}\otimes_k k_s$ and $\mathcal{B}\otimes_k k_s$ and the dg bimodule $\mathrm{B}\otimes_k k_s$. Since algebraic K-theory with $\mathbb{Z}/l^\nu$-coefficients is an additive invariant, the dg functors $\iota_{\mathcal{A}\otimes_k k_s}$ and $\iota_{\mathcal{B}\otimes_k k_s}$ induce an equivalence of spectra between the wedge sum $I\!K(\mathcal{A}\otimes_k k_s;\mathbb{Z}/l^\nu)\vee I\!K(\mathcal{B}\otimes_k k_s;\mathbb{Z}/l^\nu)$ and $I\!K(T(\mathcal{A}\otimes_k k_s,\mathcal{B}\otimes_k k_s;\mathrm{B}\otimes_k k_s);\mathbb{Z}/l^\nu)$. Therefore, using the fact that the dg category $T(\mathcal{A}\otimes_k k_s,\mathcal{B}\otimes_k k_s;\mathrm{B}\otimes_k k_s)$ is equal to the dg category $T(\mathcal{A},\mathcal{B};\mathrm{B})\otimes_k k_s$, we conclude from the definition of the above functors (6.6) that (6.8) is indeed an isomorphism. ■

Proposition 6.7 yields, in particular, the $\mathbb{Q}_l$-linear functor

$$K_0^{\hat{l},\mathrm{et}}(-\otimes_k k_s)_{1/l}\colon \mathrm{NChow}(k)_{\mathbb{Q}_l}\to \mathrm{Vect}(\mathbb{Q}_l). \tag{6.9}$$

Making use of the functor (6.9), the canonical $\mathbb{Q}_l$-linear homomorphism (4.2) (with $\mathcal{A}=\mathrm{perf}_{\mathrm{dg}}(X_L)$) may be described as the induced homomorphism

$$\begin{array}{c}\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}_l}}\big(U(k)_{\mathbb{Q}_l},U\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\big)_{\mathbb{Q}_l}\big)\\ \downarrow\\ \mathrm{Hom}_{\mathrm{Vect}(\mathbb{Q}_l)}\big(K_0^{\hat{l},\mathrm{et}}(k_s)_{1/l},K_0^{\hat{l},\mathrm{et}}\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s\big)_{1/l}\big);\end{array} \tag{6.10}$$

note that (6.10) takes values in $K_0^{\hat{l},\mathrm{et}}(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k)}$. Therefore, thanks to the above direct sum decomposition (6.5), the induced homomorphism (6.10) identifies with the (diagonal) $\mathbb{Q}_l$-linear homomorphism:

$$\begin{array}{c}K_0(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}_l}\oplus\bigoplus_{r=\dim(V)}^{i-1}K_0(\mathbb{A}_r^{\mathrm{dg}})_{\mathbb{Q}_l}\\ \downarrow\\ K_0^{\hat{l},\mathrm{et}}(\mathbb{C}_L^{\mathrm{dg}}\otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k)}\oplus\bigoplus_{r=\dim(V)}^{i-1}K_0^{\hat{l},\mathrm{et}}(\mathbb{A}_r^{\mathrm{dg}}\otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k)}.\end{array}$$

This implies the following equivalence:

$$\mathrm{T}^l_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\big)\Leftrightarrow \mathrm{T}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}})+\mathrm{T}^l_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{\dim(V)})+\cdots+\mathrm{T}^l_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{i-1}). \tag{6.11}$$

Note that all the above holds *mutatis mutandis* with X_L replaced by Y_L. Hence, the semi-orthogonal decomposition (6.4) leads to the equivalence

$$\mathrm{T}^l_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(Y_L)\big)\Leftrightarrow \mathrm{T}^l_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{j-1})+\cdots+\mathrm{T}^l_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{\dim(L^\perp)})+\mathrm{T}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}'}). \tag{6.12}$$

The assumption (c) of Theorem 2.2 and the semi-orthogonal decompositions (6.1)–(6.2) imply that the conjectures $\mathrm{T}^l_{\mathrm{nc}}(\mathbb{A}_r^{\mathrm{dg}})$ and $\mathrm{T}^l_{\mathrm{nc}}(\mathbb{B}_r^{\mathrm{dg}})$, with $0 \leq r \leq i-1$, hold. Consequently, the right-hand side of (6.11), resp. (6.12), reduces to the conjecture $\mathrm{T}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}})$, resp. $\mathrm{T}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}'})$. Moreover, since the functor $\mathrm{perf}(X_L) \to \mathbb{C}_L \to \mathrm{perf}(Y_L)$ is of Fourier–Mukai type, the dg categories $\mathbb{C}_L^{\mathrm{dg}}$ and $\mathbb{C}_L^{\mathrm{dg}'}$ are Morita equivalent. This implies that the conjectures $\mathrm{T}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}})$ and $\mathrm{T}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}'})$ are equivalent. Consequently, the proof of Theorem 2.2 follows now from the equivalences $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X_L))$ and $\mathrm{T}^l(Y_L) \Leftrightarrow \mathrm{T}^l_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L))$ established in Theorem 2.1.

We now prove the equivalence $\overline{\mathrm{T}}^l(X_L) \Leftrightarrow \overline{\mathrm{T}}^l(Y_L)$. Recall from [27, Thm. 7.1] that the functor $-\otimes_k k_s\colon \mathrm{dgcat}(k) \to \mathrm{dgcat}(k_s)$ preserves smooth proper dg categories and gives rise to a $\mathbb{Q}_l$-linear functor $-\otimes_k k_s\colon \mathrm{NChow}(k)_{\mathbb{Q}_l} \to \mathrm{NChow}(k_s)_{\mathbb{Q}_l}$. Consequently, the above direct sum decomposition (6.5) in the $\mathbb{Q}_l$-linearized category $\mathrm{NChow}(k)_{\mathbb{Q}_l}$ gives rise to the direct sum decomposition

$$\begin{aligned} &U\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s\big)_{\mathbb{Q}_l} \\ &\qquad \simeq U(\mathbb{C}_L^{\mathrm{dg}}\otimes_k k_s)_{\mathbb{Q}_l} \oplus U(\mathbb{A}^{\mathrm{dg}}_{\dim(V)}\otimes_k k_s)_{\mathbb{Q}_l} \oplus \cdots \oplus U(\mathbb{A}^{\mathrm{dg}}_{i-1}\otimes_k k_s)_{\mathbb{Q}_l}\end{aligned}$$

in the $\mathbb{Q}_l$-linearized category $\mathrm{NChow}(k_s)_{\mathbb{Q}_l}$. Similarly to (the proof of) Proposition 6.7, the functors

$$K_n^{\hat{l},\mathrm{et}}(-)_{1/l}\colon \mathrm{dgcat}(k_s) \to \mathrm{Vect}(\mathbb{Q}_l)$$

are additive invariants. This yields, in particular, the $\mathbb{Q}_l$-linear functor

$$K_0^{\hat{l},\mathrm{et}}(-)_{1/l}\colon \mathrm{NChow}(k_s)_{\mathbb{Q}_l} \to \mathrm{Vect}(\mathbb{Q}_l). \tag{6.13}$$

Making use of the functor (6.13), the canonical $\mathbb{Q}_l$-linear homomorphism (4.4) (with $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(X_L)$) may be described as the induced homomorphism

$$\begin{array}{c} \mathrm{Hom}_{\mathrm{NChow}(k_s)_{\mathbb{Q}_l}}\big(U(k_s)_{\mathbb{Q}_l}, U\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s\big)_{\mathbb{Q}_l}\big) \\ \downarrow \\ \mathrm{Hom}_{\mathrm{Vect}(\mathbb{Q}_l)}\big(K_0^{\hat{l},\mathrm{et}}(k_s)_{1/l}, K_0^{\hat{l},\mathrm{et}}\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s\big)_{1/l}\big); \end{array} \tag{6.14}$$

note that (6.14) takes values in $\bigcup_{k'/k} K_0^{\hat{l},\mathrm{et}}(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k')}$. Therefore, thanks to the above direct sum decomposition of $U(\mathrm{perf}_{\mathrm{dg}}(X_L)\otimes_k k_s)_{\mathbb{Q}_l}$, the induced homomorphism (6.14) identifies with the (diagonal) $\mathbb{Q}_l$-linear homomorphism

$$\begin{array}{c} K_0(\mathbb{C}_L^{\mathrm{dg}}\otimes_k k_s)_{\mathbb{Q}_l} \oplus \bigoplus_{r=\dim(V)}^{i-1} K_0(\mathbb{A}_r^{\mathrm{dg}}\otimes_k k_s)_{\mathbb{Q}_l} \\ \downarrow \\ \bigcup_{k'/k} K_0^{\hat{l},\mathrm{et}}(\mathbb{C}_L^{\mathrm{dg}}\otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k')} \oplus \bigoplus_{r=\dim(V)}^{i-1} \bigcup_{k'/k} K_0^{\hat{l},\mathrm{et}}(\mathbb{A}_r^{\mathrm{dg}}\otimes_k k_s)_{1/l}^{\mathrm{Gal}(k_s/k')}. \end{array}$$

This implies the equivalence

$$\overline{\mathrm{T}}^l_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\big) \Leftrightarrow \overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}}) + \overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{\dim(V)}) + \cdots + \overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{i-1}). \tag{6.15}$$

Note that all the above holds *mutatis mutandis* with X_L replaced by Y_L. Hence, the semi-orthogonal decomposition (6.4) leads to the equivalence:

$$\overline{\mathrm{T}}^l_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(Y_L)\big) \Leftrightarrow \overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{j-1}) + \cdots + \overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{\dim(L^\perp)}) + \overline{\mathrm{T}}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}'}). \tag{6.16}$$

The remainder of the proof is now similar to the proof of $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$.

We now prove the equivalence $\mathrm{SS}^l(X_L) \Leftrightarrow \mathrm{SS}^l(Y_L)$. Note that similarly to (the proof of) Proposition 6.7, the following functors are additive invariants:

$$K_n^{\hat{l},\mathrm{et}}(-\otimes_k k_s)_{1/l}\colon \mathrm{dgcat}(k) \to \mathrm{Rep}_{\mathrm{Gal}(k_s/k)}(\mathbb{Q}_l) \quad n \in \mathbb{Z},$$

where $\mathrm{Rep}_{\mathrm{Gal}(k_s/k)}(\mathbb{Q}_l)$ stands for the category of (continuous) l-adic representations of $\mathrm{Gal}(k_s/k)$. Hence, they yield, in particular, the $\mathbb{Q}_l$-linear functors

$$K_0^{\hat{l},\mathrm{et}}(-\otimes_k k_s)_{1/l}, K_1^{\hat{l},\mathrm{et}}(-\otimes_k k_s)_{1/l}\colon \mathrm{NChow}(k)_{\mathbb{Q}_l} \to \mathrm{Rep}_{\mathrm{Gal}(k_s/k)}(\mathbb{Q}_l). \tag{6.17}$$

Therefore, by applying the $\mathbb{Q}_l$-linear functors (6.17) to the above direct sum decomposition (6.5), we obtain the equivalence

$$\mathrm{SS}^l_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\big) \Leftrightarrow \mathrm{SS}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}}) + \mathrm{SS}^l_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{\dim(V)}) + \cdots + \mathrm{SS}^l_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{i-1}).$$

All the above holds *mutatis mutandis* with X_L replaced by Y_L. Consequently, the above semi-orthogonal decomposition (6.4) leads to the equivalence

$$\mathrm{SS}^l_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(Y_L)\big) \Leftrightarrow \mathrm{SS}^l_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{j-1}) + \cdots + \mathrm{SS}^l_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{\dim(L^\perp)}) + \mathrm{SS}^l_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}'}).$$

The remainder of the proof is now similar to the proof of $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$.

The proof of the equivalence $\mathrm{T}^p(X_L) \Leftrightarrow \mathrm{T}^p(Y_L)$ is similar to the proof of $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$: simply replace $\mathbb{Q}_l$ by $\mathbb{Q}_p$ and the $\mathbb{Q}_l$-linear functor (6.9) by the $\mathbb{Q}_p$-linear functor (4.10).

Finally, we prove the equivalence $\mathrm{B}(X_L) \Leftrightarrow \mathrm{B}(Y_L)$. As above, the semi-orthogonal decomposition (6.3) gives rise to the direct sum decomposition

$$U\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\big)_{\mathbb{Q}} \simeq U(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}} \oplus U(\mathbb{A}^{\mathrm{dg}}_{\dim(V)})_{\mathbb{Q}} \oplus \cdots \oplus U(\mathbb{A}^{\mathrm{dg}}_{i-1})_{\mathbb{Q}} \tag{6.18}$$

in the category $\mathrm{NChow}(k)_{\mathbb{Q}}$. As proved in [36, §6], given any smooth proper dg category $\mathcal{A}$, we have a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}\big(U(k)_{\mathbb{Q}}, U(\mathcal{A})_{\mathbb{Q}}\big) \simeq K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{num}}. \tag{6.19}$$

Therefore, by applying $\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, -)$ and $\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, -)$ to the direct sum decomposition (6.18), we obtain the equivalence of

$$\mathrm{B}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(X_L)\big) \Leftrightarrow \mathrm{B}_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}}) + \mathrm{B}_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{\dim(V)}) + \cdots + \mathrm{B}_{\mathrm{nc}}(\mathbb{A}^{\mathrm{dg}}_{i-1}).x$$

All the above holds *mutatis mutandis* with X_L replaced by Y_L. Consequently, the above semi-orthogonal decomposition (6.4) leads to the equivalence

$$\mathrm{B}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}(Y_L)\big) \Leftrightarrow \mathrm{B}_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{j-1}) + \cdots + \mathrm{B}_{\mathrm{nc}}(\mathbb{B}^{\mathrm{dg}}_{\dim(L^\perp)}) + \mathrm{B}_{\mathrm{nc}}(\mathbb{C}^{\mathrm{dg}'}_L).$$

The remainder of the proof is now similar to the proof of $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$.

7. Proof of Proposition 2.5

As proved in [7, Thms. 1.3 and 1.7], the dg category $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr}(r, U_1))$ is Morita equivalent to a finite dimensional k-algebra of finite global dimension A. Let us write $J(A)$ for the Jacobson radical of A, $V_1, \ldots, V_s$ for the simple (right) $A/J(A)$-modules, and $D_1 := \mathrm{End}_{A/J(A)}(V_1), \ldots, D_s := \mathrm{End}_{A/J(A)}(V_s)$ for the associated division k-algebras. Thanks to the Artin–Wedderburn theorem, the quotient $A/J(A)$ is Morita equivalent to the product $D_1 \times \cdots \times D_s$. Moreover, the center of D_i is a finite field extension l_i of k and D_i is a central simple l_i-algebra. As proved in [38, Thm. 3.15], we have the direct sum decomposition

$$U(\mathbb{A}^{\mathrm{dg}}_0)_{\mathbb{Q}} = U\big(\mathrm{perf}_{\mathrm{dg}}\big(\mathrm{Gr}(r, U_1)\big)\big)_{\mathbb{Q}} \simeq U\big(A/J(A)\big)_{\mathbb{Q}} \simeq U(l_1)_{\mathbb{Q}} \oplus \cdots \oplus U(l_s)_{\mathbb{Q}}$$

in the category $\mathrm{NChow}(k)_{\mathbb{Q}}$. Similarly to the proof of Theorem 2.2, we hence obtain the equivalences

$$\mathrm{C}(\mathbb{A}^{\mathrm{dg}}_0) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}(l_1) + \cdots + \mathrm{C}_{\mathrm{nc}}(l_s) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

Note that since the k-schemes $\mathrm{Spec}(l_i)$ are 0-dimensional, the conjectures $\mathrm{C}(\mathrm{Spec}(l_i))$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold. Consequently, the proof follows now from the equivalences $\mathrm{C}(\mathrm{Spec}(l_i)) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}(l_i)$ established in Theorem 2.1.

8. Proof of Theorem 2.15

Item (i). Following [24, §3.5] (see also [2, §1.6]), let us write $\mathcal{Z}$ for the center of $\mathcal{C}l_0(q)_{|L}$ and $\mathrm{Spec}(\mathcal{Z}) =: \widetilde{\mathbb{P}}(L) \to \mathbb{P}(L)$ for the *discriminant cover* of $\mathbb{P}(L)$. As explained in *loc. cit.*, $\widetilde{\mathbb{P}}(L) \to \mathbb{P}(L)$ is a 2-fold cover which is ramified over the divisor $D := \mathbb{P}(L) \cap \Delta_1$. Since by assumption $\dim(L) = 2$, we have $\dim(D) = 0$. Consequently, since D is smooth, $\widetilde{\mathbb{P}}(L)$ is also smooth. Let us write $\mathcal{F}$ for the sheaf of noncommutative algebras $\mathcal{C}l_0(q)_{|L}$ considered as a sheaf of noncommutative algebras over $\widetilde{\mathbb{P}}(L)$. As proved in *loc. cit.*, since by assumption we have $\mathbb{P}(L) \cap \Delta_2 = \emptyset$, $\mathcal{F}$ is a sheaf of Azumaya algebras over $\widetilde{\mathbb{P}}(L)$ of rank $2^{(d/2)-1}$. Moreover, the category $\mathrm{perf}(\mathbb{P}(L); \mathcal{C}l_0(q)_{|L})$ is equivalent (via a Fourier–Mukai-type functor) to $\mathrm{perf}(\widetilde{\mathbb{P}}(L); \mathcal{F})$. This leads to a Morita equivalence between the dg categories $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)_{|L})$ and $\mathrm{perf}_{\mathrm{dg}}(\widetilde{\mathbb{P}}(L); \mathcal{F})$. Making use of [38, Thm. 2.1], we hence obtain the isomorphisms

$$U\big(\mathrm{perf}_{\mathrm{dg}}\big(\mathbb{P}(L); \mathcal{C}l_0(q)_{|L}\big)\big)_{\mathbb{Q}} \simeq U\big(\mathrm{perf}_{\mathrm{dg}}\big(\widetilde{\mathbb{P}}(L); \mathcal{F}\big)\big)_{\mathbb{Q}} \simeq U\big(\mathrm{perf}_{\mathrm{dg}}\big(\widetilde{\mathbb{P}}(L)\big)\big)_{\mathbb{Q}}$$

in the category $\mathrm{NChow}(k)_{\mathbb{Q}}$. Similarly to the proof of Theorem 2.2, this leads to the equivalences

$$\mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}\,(\mathbb{P}(L);\mathcal{C}l_0(q)_{|L})\big) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}\,(\widetilde{\mathbb{P}}(L))\big) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

Since $\widetilde{\mathbb{P}}(L)$ is a curve, the conjectures $\mathrm{C}(\widetilde{\mathbb{P}}(L))$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$, hold. Consequently, thanks to Corollary 2.6, the proof follows now from the equivalence

$$\mathrm{C}\big(\widetilde{\mathbb{P}}(L)\big) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}\,(\widetilde{\mathbb{P}}(L))\big)$$

established in Theorem 2.1.

Item (ii). Following [24, §3.6] (see also [2, §1.7]), let us write $\widehat{\mathbb{P}}(L)$ for the *discriminant stack* associated to the pull-back $q_{|L}$ along $\mathbb{P}(L) \subset \mathbb{P}(S^2(W^\vee))$ of the flat quadric fibration $q\colon \mathcal{H} \to \mathbb{P}(S^2(W^\vee))$. As explained in *loc. cit.*, since by assumption $1/2 \in k$, $\widehat{\mathbb{P}}(L)$ is a smooth Deligne–Mumford stack. Moreover, using the fact that $\widehat{\mathbb{P}}(L)$ is a square root stack and that the critical locus of the flat quadric fibration $q_{|L}$ is the divisor D, we conclude from [14, Thm. 1.6] that $\mathrm{perf}(\widehat{\mathbb{P}}(L)) = \langle \mathrm{perf}(D), \mathrm{perf}(\mathbb{P}(L))\rangle$. Hence, similarly to the proof of Theorem 2.2, we obtain the equivalences

$$\mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}\,(\widehat{\mathbb{P}}(L))\big) \Leftrightarrow \mathrm{C}(D) + \mathrm{C}\big(\mathbb{P}(L)\big) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}. \tag{8.1}$$

Let us write $\mathcal{F}$ for the sheaf of noncommutative algebras $\mathcal{C}l_0(q)_{|L}$ considered as a sheaf of noncommutative algebras over $\widehat{\mathbb{P}}(L)$. As proved in [24, §3.6] (see also [2, §1.7]), since by assumption we have $\mathbb{P}(L) \cap \Delta_2 = \emptyset$, $\mathcal{F}$ is a sheaf of Azumaya algebras over $\widehat{\mathbb{P}}(L)$. Moreover, the category $\mathrm{perf}(\mathbb{P}(L);\mathcal{C}l_0(q)_{|L})$ is equivalent (via a Fourier–Mukai-type functor) to $\mathrm{perf}(\widehat{\mathbb{P}}(L);\mathcal{F})$. This leads to a Morita equivalence between the dg categories $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}(L);\mathcal{C}l_0(q)_{|L})$ and $\mathrm{perf}_{\mathrm{dg}}(\widehat{\mathbb{P}}(L);\mathcal{F})$. Making use of Corollary 2.14, we hence obtain the equivalences

$$\mathrm{C}(X_L) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}\big(\mathrm{perf}_{\mathrm{dg}}\,(\widehat{\mathbb{P}}(L);\mathcal{F})\big) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}. \tag{8.2}$$

Since by assumption $\dim(L) = 2$, we have $\dim(\mathbb{P}(L)) = 1$. Using the fact that the Brauer group of every smooth curve over a finite field k is trivial, we hence conclude that in (8.2) we can replace the dg category $\mathrm{perf}_{\mathrm{dg}}(\widehat{\mathbb{P}}(L);\mathcal{F})$ by the dg category $\mathrm{perf}_{\mathrm{dg}}(\widehat{\mathbb{P}}(L))$. Consequently, since $\dim(D) = 0$, the proof follows now from the combination of the above equivalences (8.1)–(8.2) with the fact that the Tate conjecture and its variants hold in dimensions ≤ 1.

Remark 8.3 (Intersection of even-dimensional quadrics). Let X_L be as in Corollary 2.14. Assume that d is even, that $\dim(L) \geq 2$, that $\mathbb{P}(L) \cap \Delta_2 = \emptyset$, and that the divisor $\mathbb{P}(L) \cap \Delta_1$ is smooth. Under these assumptions, a proof similar to the one of Theorem 2.15 gives rise to the equivalences

$$\mathrm{C}(X_L) \Leftrightarrow \mathrm{C}\big(\widetilde{\mathbb{P}}(L)\big) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}, \tag{8.4}$$

where $\widetilde{\mathbb{P}}(L)$ is the discriminant 2-fold cover of $\mathbb{P}(L)$.

9. Proof of Theorem 2.18

Let us write $f\colon \mathcal{X} \to X$ for the canonical morphism. As proved by Ishii–Ueda in [14, Thm. 1.6], we have a semi-orthogonal decomposition:

$$\mathrm{perf}(\mathcal{X}) = \big\langle \mathrm{perf}(Z)_{n-1}, \dots, \mathrm{perf}(Z)_1, f^*\big(\mathrm{perf}(X)\big)\big\rangle,$$

where all the categories $\mathrm{perf}(Z)_i$ are (Fourier–Mukai) equivalent to $\mathrm{perf}(Z)$ and moreover $f^*(\mathrm{perf}(X))$ is (Fourier–Mukai) equivalent to $\mathrm{perf}(X)$. Therefore, arguments similar to those used in the proof of Theorem 2.2 yield the searched equivalences (2.19).

10. Proof of Theorem 2.21

Note first that since $p \nmid n$, the integer n is invertible in k. Making use of [39, Thm. 1.1 and Rk. 1.3], we hence conclude that the noncommutative Chow motive $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{\sigma \subseteq G} U(\mathrm{perf}_{\mathrm{dg}}(X^\sigma \times \mathrm{Spec}(k[\sigma])))_{\mathbb{Q}}$. Therefore, arguments similar to those used in the proof of Theorem 2.2 yield the searched implications (2.22). Assume now moreover that k contains the nth roots of unity. In this case, [39, Cor. 1.5 (i)] implies that $U(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{\sigma \subseteq G} U(\mathrm{perf}_{\mathrm{dg}}(X^\sigma))_{\mathbb{Q}}^{\oplus r_\sigma}$, where r_σ are certain positive integers. Hence, as above, arguments similar to those used in the proof of Theorem 2.2 yield the implications $\sum_{\sigma \subseteq G} \mathrm{C}(X^\sigma) \Rightarrow \mathrm{C}(\mathcal{X})$, with $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$.

11. Proof of Theorem 2.24

The proof of Theorem 2.24 is similar to the proof of Theorem 2.21 (in the case where $p \nmid n$ and k contains the nth roots of unity). Simply replace $\mathrm{perf}(\mathcal{X})$ by $\mathrm{perf}(\mathcal{X}; \mathcal{F})$, $\mathrm{perf}_{\mathrm{dg}}(X^\sigma)$ by $\mathrm{perf}_{\mathrm{dg}}(Y_\sigma)$, and [39, Cor. 1.5 (i)] by [39, Cor. 1.28 (ii)].

Acknowledgments. The author is grateful to the Institut des Hautes Études Scientifiques (IHÉS) for its hospitality and excellent working conditions.

Funding. The author was partially supported by the Huawei-IHÉS research funds and by the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

References

[1] Y. André, On the Shafarevich and Tate conjectures for hyper-Kähler varieties. *Math. Ann.* **305** (1996), no. 2, 205–248 Zbl 0942.14018 MR 1391213

[2] A. Auel, M. Bernardara, and M. Bolognesi, Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems. *J. Math. Pures Appl. (9)* **102** (2014), no. 1, 249–291 Zbl 1327.14078 MR 3212256

[3] O. Benoist, Construction de courbes sur les surfaces K3 (d'après Bogomolov–Hassett–Tschinkel, Charles, Li–Liedtke, Madapusi Pera, Maulik. . .). *Astérisque* **367-368** (2015), Exp. No. 1081, viii, 219–253 Zbl 1356.14001 MR 3363592
[4] M. Bernardara, M. Bolognesi, and D. Faenzi, Homological projective duality for determinantal varieties. *Adv. Math.* **296** (2016), 181–209 Zbl 1362.14020 MR 3490767
[5] A. A. Beĭlinson, Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Funktsional. Anal. i Prilozhen.* **12** (1978), no. 3, 68–69 MR 509388
[6] B. Bhatt, M. Morrow, and P. Scholze, Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 199–310 Zbl 07059677 MR 3949030
[7] R.-O. Buchweitz, G. J. Leuschke, and M. Van den Bergh, On the derived category of Grassmannians in arbitrary characteristic. *Compos. Math.* **151** (2015), no. 7, 1242–1264 Zbl 1333.14017 MR 3371493
[8] A. Conca and D. Faenzi, A remark on hyperplane sections of rational normal scrolls. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **60(108)** (2017), no. 4, 337–349 Zbl 1399.14020 MR 3727393
[9] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.* **73** (1983), no. 3, 349–366 Zbl 0588.14026 MR 718935
[10] E. M. Friedlander, Étale K-theory. I. Connections with etale cohomology and algebraic vector bundles. *Invent. Math.* **60** (1980), no. 2, 105–134 Zbl 0519.14010 MR 586424
[11] W. Fulton, *Intersection theory*. Ergeb. Math. Grenzgeb. (3) 2, Springer, Berlin, 1984 Zbl 0541.14005 MR 732620
[12] O. Gabber, K-theory of Henselian local rings and Henselian pairs. In *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, pp. 59–70, Contemp. Math. 126, Amer. Math. Soc., Providence, RI, 1992 Zbl 0791.19002 MR 1156502
[13] J. Harris, *Algebraic geometry. A first course*. Grad. Texts in Math. 133, Springer, New York, 1992 Zbl 0779.14001 MR 1182558
[14] A. Ishii and K. Ueda, The special McKay correspondence and exceptional collections. *Tohoku Math. J. (2)* **67** (2015), no. 4, 585–609 Zbl 1335.14005 MR 3436544
[15] B. Kahn, Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini. *Ann. Sci. École Norm. Sup. (4)* **36** (2003), no. 6, 977–1002 Zbl 1073.14034 MR 2032532
[16] B. Kahn, Algebraic K-theory, algebraic cycles and arithmetic geometry. In *Handbook of K-theory. Vol. 1, 2*, pp. 351–428, Springer, Berlin, 2005 Zbl 1115.19003 MR 2181827
[17] M. M. Kapranov, On the derived categories of coherent sheaves on some homogeneous spaces. *Invent. Math.* **92** (1988), no. 3, 479–508 Zbl 0651.18008 MR 939472
[18] B. Keller, On differential graded categories. In *International congress of mathematicians. Vol. II*, pp. 151–190, EMS Press, Zürich, 2006 Zbl 1140.18008 MR 2275593
[19] M. Kontsevich, Non-commutative motives. 2005, talk at the IAS on the occasion of the 61st birthday of Pierre Deligne, available at http://video.ias.edu/Geometry-and-Arithmetic
[20] M. Kontsevich, Notes on motives in finite characteristic. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, pp. 213–247, Progr. Math. 270, Birkhäuser, Boston, MA, 2009 Zbl 1279.11065 MR 2641191
[21] M. Kontsevich, Mixed noncommutative motives. 2010, talk at the Workshop on Homological Mirror Symmetry, Miami, available at https://people.math.harvard.edu/∼auroux/frg/miami10-notes/

[22] A. Kuznetsov, Homological projective duality for Grassmannians of lines. 2006, available at http://www.mi-ras.ru/∼akuznet/publications/publications.html

[23] A. Kuznetsov, Homological projective duality. *Publ. Math. Inst. Hautes Études Sci.* (2007), no. 105, 157–220 MR 2354207

[24] A. Kuznetsov, Derived categories of quadric fibrations and intersections of quadrics. *Adv. Math.* **218** (2008), no. 5, 1340–1369 Zbl 1168.14012 MR 2419925

[25] A. Kuznetsov, Semiorthogonal decompositions in algebraic geometry. In *Proceedings of the international congress of mathematicians—Seoul 2014. Vol. II*, pp. 635–660, Kyung Moon Sa, Seoul, 2014 Zbl 1373.18009 MR 3728631

[26] V. A. Lunts and D. O. Orlov, Uniqueness of enhancement for triangulated categories. *J. Amer. Math. Soc.* **23** (2010), no. 3, 853–908 Zbl 1197.14014 MR 2629991

[27] M. Marcolli and G. Tabuada, Noncommutative Artin motives. *Selecta Math. (N.S.)* **20** (2014), no. 1, 315–358 Zbl 1296.14002 MR 3147418

[28] J. Milne, The Tate conjecture over finite fields. 2007, american Institute of Mathematics talk, available at http://www.jmilne.org/math/articles/2007e.pdf

[29] B. Moonen, A remark on the Tate conjecture. *J. Algebraic Geom.* **28** (2019), no. 3, 599–603 Zbl 1448.11100 MR 3959072

[30] T. Nikolaus and P. Scholze, On topological cyclic homology. *Acta Math.* **221** (2018), no. 2, 203–409 Zbl 2018 MR 3904731

[31] M. A. Reid, *The complete intersection of two or more quadrics*. Ph.D. thesis, University of Cambridge, 1972, available at https://homepages.warwick.ac.uk/∼masda/3folds/qu.pdf

[32] C. Soulé, Operations on étale K-theory. Applications. In *Algebraic K-theory, Part I (Oberwolfach, 1980)*, pp. 271–303, Lecture Notes in Math. 966, Springer, Berlin, 1982 Zbl 0507.14013 MR 689380

[33] A. Suslin, On the K-theory of algebraically closed fields. *Invent. Math.* **73** (1983), no. 2, 241–245 Zbl 0514.18008 MR 714090

[34] A. A. Suslin, On the K-theory of local fields. Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983). *J. Pure Appl. Algebra* **34** (1984), no. 2-3, 301–318 Zbl 0548.12009 MR 772065

[35] G. Tabuada, *Noncommutative motives. With a preface by Yuri I. Manin*. Univ. Lecture Ser. 63, Amer. Math. Soc., Providence, RI, 2015 Zbl 1333.14002 MR 3379910

[36] G. Tabuada, Noncommutative motives in positive characteristic and their applications. *Adv. Math.* **349** (2019), 648–681 Zbl 1441.14009 MR 3942181

[37] G. Tabuada, A note on Grothendieck's standard conjectures of type C^+ and D in positive characteristic. *Proc. Amer. Math. Soc.* **147** (2019), no. 12, 5039–5054 Zbl 1436.14004 MR 4021067

[38] G. Tabuada and M. Van den Bergh, Noncommutative motives of Azumaya algebras. *J. Inst. Math. Jussieu* **14** (2015), no. 2, 379–403 Zbl 1356.14007 MR 3315059

[39] G. Tabuada and M. Van den Bergh, Additive invariants of orbifolds. *Geom. Topol.* **22** (2018), no. 5, 3003–3048 Zbl 1387.14018 MR 3811776

[40] J. Tate, Conjectures on algebraic cycles in l-adic cohomology. In *Motives (Seattle, WA, 1991)*, pp. 71–83, Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, RI, 1994 Zbl 0814.14009 MR 1265523

[41] J. T. Tate, Algebraic cycles and poles of zeta functions. In *Arithmetical algebraic geometry (Proc. Conf. Purdue Univ., 1963)*, pp. 93–110, Harper & Row, New York, 1965 Zbl 0213.22804 MR 0225778

[42] R. P. Thomas, Notes on homological projective duality. In *Algebraic geometry: Salt Lake City 2015*, pp. 585–609, Proc. Sympos. Pure Math. 97, Amer. Math. Soc., Providence, RI, 2018 Zbl 1451.14052 MR 3821163

[43] R. W. Thomason, Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)* **18** (1985), no. 3, 437–552 Zbl 0596.14012 MR 826102

[44] B. Totaro, Recent progress on the Tate conjecture. *Bull. Amer. Math. Soc. (N.S.)* **54** (2017), no. 4, 575–590 Zbl 1375.14037 MR 3683625

[45] J. G. Zarhin, Endomorphisms of Abelian varieties over fields of finite characteristic. *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975), no. 2, 272–277, 471 MR 0371897

Received 10 October 2020.

Gonçalo Tabuada
Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK;
and Center for Mathematics and Applications (NOVA Math) and Department of Mathematics, NOVA School of Science and Technology, NOVA University Lisbon, 2829-516 Caparica, Portugal;
goncalo.tabuada@warwick.ac.uk