J. Noncommut. Geom. 17 (2023), 189–210
DOI 10.4171/JNCG/464
© 2023 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

A short proof of the localization formula for the loop space Chern character of spin manifolds

Matthias Ludewig and Zelin Yi

Abstract. In this note, we give a short proof of the localization formula for the loop space Chern character of a compact Riemannian spin manifold M, using the rescaled spinor bundle on the tangent groupoid associated to M.

1. Introduction

In supersymmetric quantum mechanics, one is interested in the path integral corresponding to the $N = 1/2$ supersymmetric σ-model, which mathematically can be viewed as an integration functional for differential forms ξ on the loop space $\mathsf{L}M$ of a compact Riemannian spin manifold M. Formally, this should be given by the expression

$$I[\xi] = \int_{\mathsf{L}M} e^{-S-\omega} \wedge \xi, \tag{1.1}$$

where S is the classical energy and ω is the canonical pre-symplectic 2-form on the loop space. This formula and its close relation to the Atiyah–Singer index theorem have been inspiring research for more than 30 years (see [1,2] and the introduction of [10] for further references).

The path integral formula (1.1) has been finally given a rigorous interpretation for a certain class of differential forms θ in [10, 12]; see also [16]. Essentially, this is the class of *iterated integrals*, first considered by Chen [7] and later extended by Getzler, Jones, and Petrack [9] in order to contain the Bismut–Chern character forms first introduced by Bismut [4]. Iterated integrals are the image of the *iterated integral map*, which maps the cyclic chain complex associated to the algebra $\Omega(M)$ of differential forms on M to the algebra $\Omega(\mathsf{L}M)$ of differential forms on the loop space.

Pulling back the integration functional I of [12] with the iterated integral map, one obtains a coclosed functional on the cyclic chain complex of $\Omega(M)$, which we denote by Ch_D; namely, it has then been observed in [10] that this functional can be viewed as a non-commutative *Chern character* associated to a Fredholm module over $\Omega(M)$ determined by the Dirac operator D on M, with a combinatorial formula similar to the JLO cocycle [14].

2020 Mathematics Subject Classification. Primary 46L87; Secondary 58B34.
Keywords. Tangent groupoid, Bismut–Chern character, Chern character, JLO cocycle, cyclic cohomology, entire cohomology, Atiyah–Singer index theorem.

The most remarkable feature of the loop space path integral I and its combinatorial counterpart Ch_D is that they satisfy a *localization formula* of Duistermaat–Heckmann type, as though the loop space $\mathsf{L}M$ was a finite-dimensional manifold; see [12, Theorem 3.19] and [10, Theorem E]. This in particular facilitates the proof of the Atiyah Singer index theorem using *loop space localization* envisioned by Atiyah [2].

Theorem. *Let M be a compact Riemannian spin manifold of even dimension n and let Θ be an entire cyclic chain over $\Omega(M)$. If Θ is closed, then*

$$\mathrm{Ch}_D(\Theta) = \frac{1}{(2\pi i)^{n/2}} \int_M \widehat{A}(M) \wedge i(\Theta). \tag{1.2}$$

Here $\widehat{A}(M)$ denotes the $\widehat{A}$-genus-form, and i is the combinatorial analog of the map that restricts a differential form on the loop space to the fixed point set $M \subset \mathsf{L}M$ (see (2.14)). In fact, we prove formula (1.2) more generally for entire cyclic chains over the *acyclic extension* $\Omega_{\mathbb{T}}(M)$ of $\Omega(M)$ (see Section 2.1), which is necessary in order to encompass the Bismut–Chern characters.

The purpose of this note is to give a short proof of the above theorem using Connes' tangent groupoid and its extension to the spinor bundle introduced by Higson and the second author [13,20]. The strategy is as follows: rescaling of the Fredholm module to which Ch_D is associated yields a one-parameter family of Chern-characters $\{\mathrm{Ch}_t\}_{t>0}$, which are all cohomologous on suitable complexes by the homotopy invariance of the Chern character. Therefore, for *closed* chains Θ, the value $\mathrm{Ch}_t(\Theta)$ is therefore independent of t, hence can be obtained from calculating its limit as $t \to 0$. The result is Theorem 2.10 below, which can be viewed as a "loop space version" of a corresponding result of Block-Fox on the JLO cocycle [5, Theorem 4.1].

It is the calculation of this limit for which the machinery of the tangent groupoid is particularly useful. In brief, the Chern character is defined as the supertrace of a certain family of operators, the kernels of which turn out to patch together to define a smooth section of the *rescaled spinor bundle* $\mathbb{S}$ over the tangent groupoid $\mathbb{T}M$ that *can be extended down to zero*. Here one then has a one-parameter family of supertraces at disposal (defined in [13]), which allow to easily calculate the value at zero.

Below, we briefly explain the construction of the Chern characters Ch_t and reduce the proof of formula (1.2) to the calculation of the short time limit of Ch_t. This is essentially algebraic. In the second part of this note, we briefly recount the theory of the rescaled spinor bundle and prove the relevant Theorem 2.10 below.

2. The Chern character

In this section, we give a short review of the construction of the Chern character associated to the Fredholm module over $\Omega(M)$ determined by the Dirac operator over a compact Riemannian spin manifold. We focus on the algebraic construction, leaving out many analytical details; for these, we refer to [10].

2.1. The bar complex and cyclic chains

Let M be a manifold and let $\Omega(M)$ be its complex of (complex-valued) differential forms. The *acyclic extension* of $\Omega(M)$ is the algebra $\Omega_{\mathbb{T}}(M) := \Omega(M)[\sigma]$, where σ is a formal variable of degree -1 satisfying $\sigma^2 = 0$. Elements of $\Omega_{\mathbb{T}}(M)$ will be written as $\theta = \theta' + \sigma\theta''$ with $\theta', \theta'' \in \Omega(M)$. $\Omega_{\mathbb{T}}(M)$ has a differential $d_{\mathbb{T}} = d - \iota$, where d is the de Rham differential and ι is defined by $\iota(\theta' + \sigma\theta'') = \theta''$.

Remark 2.1. In [9], the algebra $\Omega(M \times \mathbb{T})^{\mathbb{T}}$ of $\mathbb{T}$-invariant differential forms on $M \times \mathbb{T}$ is used (where $\mathbb{T} = S^1$). This corresponds to our setup through setting $\sigma = \mathbf{t}^2 d\varphi$, where φ is the coordinate of $\mathbb{T}$ and $\mathbf{t}$ is a formal variable of degree -1. Carrying around the formal variable $\mathbf{t}$ would allow us to stay $\mathbb{Z}$-graded throughout, but we feel that for this presentation, it is simpler to leave this variable out and to just take the grading mod 2.

The *bar complex* of $\Omega_{\mathbb{T}}(M)$ is

$$\mathsf{B}\big(\Omega_{\mathbb{T}}(M)\big) = \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}}(M)\langle 1\rangle^{\otimes N},$$

where $\Omega_{\mathbb{T}}(M)\langle 1\rangle$ denotes a grading shift, i.e., $(\Omega_{\mathbb{T}}(M)\langle 1\rangle)^{\ell} = \Omega_{\mathbb{T}}^{\ell+1}(M)$. There are two differentials on $\mathsf{B}(\Omega_{\mathbb{T}}(M))$, given on homogeneous elements by

$$b_0(\theta_1, \dots, \theta_N) = -\sum_{k=1}^{N} (-1)^{n_{k-1}} (\theta_1, \dots, \theta_{k-1}, d_{\mathbb{T}}\theta_k, \dots, \theta_N),$$
$$b_1(\theta_1, \dots, \theta_N) = -\sum_{k=1}^{N-1} (-1)^{n_k} (\theta_1, \dots, \theta_{k-1}, \theta_k \wedge \theta_{k+1}, \theta_{k+2}, \dots, \theta_N),$$

where $n_k = |\theta_1| + \cdots + |\theta_k| - k$. Here elements of $\mathsf{B}(\Omega_{\mathbb{T}}(M))$ are written as $(\theta_1, \dots, \theta_N)$, omitting the tensor signs for brevity. The above differentials satisfy $b_0 b_1 + b_1 b_0 = 0$, hence turn $\mathsf{B}(\Omega_{\mathbb{T}}(M))$ into a ($\mathbb{Z}_2$-graded) bi-complex with total differential $b := b_0 + b_1$. The differentials descend to the subspace

$$\mathsf{B}^{\natural}\big(\Omega_{\mathbb{T}}(M)\big) = \operatorname{span}\left\{ \sum_{k=0}^{N} (-1)^{n_k(n_N - n_k)} (\theta_{k+1}, \dots, \theta_N, \theta_1, \dots, \theta_k) \right\} \tag{2.1}$$

of *cyclic chains*, making it a subcomplex.

Remark 2.2. The significance of the space $\mathsf{B}(\Omega_{\mathbb{T}}(M))$ for loop space geometry is that via Chen's iterated integral map, it provides a combinatorial model for the space of equivariant differential forms on the loop space of M via the *(extended) iterated integral map* ρ [7,9]. Explicitly, it is given by the formula

$$\rho(\theta_1, \dots, \theta_N) = \int_{\Delta_N} \big(\iota_K \theta_1'(\tau_1) - \theta''(\tau_1)\big) \wedge \cdots \wedge \big(\iota_K \theta_N'(\tau_N) - \theta''(\tau_N)\big) d\tau, \tag{2.2}$$

where we wrote $\theta(\tau)$ for the pullback of $\theta \in \Omega(M)$ by the evaluation-at-τ-map $\mathrm{ev}_\tau : \mathsf{L}M \to M$, $\gamma \mapsto \gamma(\tau)$ and ι_K denotes insertion of the velocity vector field $K(\gamma) = \dot{\gamma}$ on $\mathsf{L}M$. The iterated integral map can be viewed as a differential form counterpart of the Jones isomorphism [15], which connects the bar complex of the dg algebra of singular chains on M to chains on the loop space.

A straightforward calculation shows that the iterated integral map sends the subspace $\mathsf{B}^\natural(\Omega_\mathbb{T}(M))$ of cyclic chains to the space $\Omega(\mathsf{L}M)^\mathbb{T} \subset \Omega(\mathsf{L}M)$ of equivariant differential forms (where $\mathbb{T}$ acts by rotation) and on this domain intertwines the differential $b = b_0 + b_1$ with the equivariant differential $d - \iota_K$ on $\mathsf{L}M$.

The Bismut–Chern characters alluded to above in fact do not live in $\mathsf{B}(\Omega_\mathbb{T}(M))$ but in the larger complex of *entire chains* $\mathsf{B}_\epsilon(\Omega_\mathbb{T}(M))$, which allows certain infinite sums of chains. It is defined as the closure of $\mathsf{B}(\Omega_\mathbb{T}(M))$ with respect to the seminorms

$$\epsilon_k(\Theta) := \sum_{N=0}^{\infty} \frac{\|\Theta_N\|_{k,N}}{\lfloor N/2 \rfloor !} \quad \text{for } \Theta = \sum_{N=0}^{\infty} \Theta_N. \tag{2.3}$$

Here $c_N \in \Omega_\mathbb{T}(M)\langle 1\rangle^{\otimes N} \subset \Omega(M^N)[\sigma_1, \dots, \sigma_N]\langle N\rangle$ and $\| - \|_{k,N}$ denotes the C^k norm on $\Omega(M^N)$; see [10, Section 3.3] for details. The differential b extends to the entire complex and, in total, we have the hierarchy of chain complexes

$$\begin{array}{cccccc} \text{(entire cyclic chains)} & \mathsf{B}^\natural_\epsilon\big(\Omega_\mathbb{T}(M)\big) & \subset & \mathsf{B}_\epsilon\big(\Omega_\mathbb{T}(M)\big) & \text{(entire chains)} \\ & \cup & & \cup & \\ \text{(cyclic chains)} & \mathsf{B}^\natural\big(\Omega_\mathbb{T}(M)\big) & \subset & \mathsf{B}\big(\Omega_\mathbb{T}(M)\big) & \text{(bar chains).} \end{array}$$

The Bismut–Chern character $\mathrm{Ch}(p)$ defined below is an entire cyclic chain. In contrast, the Chern character Ch_D (to be defined in the next section) is a linear functional, defined a priori on $\mathsf{B}(\Omega_\mathbb{T}(M))$, which turns out to satisfy the necessary estimates to extend to the space of entire chains. In particular, Ch_D can be evaluated on $\mathrm{Ch}(p)$.

Example 2.3. The Bismut–Chern characters forms on the loop space $\mathsf{L}M$ were first defined in [4], while their combinatorial versions, to be reviewed now, were introduced by Getzler–Jones–Petrack [9, Section 6]. Let p be a smooth function on M with values in $M_m(\mathbb{C})$ such that $p^2 = p$ and let $p^\perp = 1 - p$. Then $E := \mathrm{im}(p)$ is a vector bundle on M, which inherits a natural metric and connection from the trivial $\mathbb{C}^m$ bundle over M (in fact, any vector bundle with connection on M can be realized this way [17, Theorem 1]). We set

$$\mathcal{R} := (2p - 1)dp + \sigma(dp)^2, \tag{2.4}$$

which is an odd element of $\Omega_\mathbb{T}(M)$. The *(cyclic) Bismut Chern character* of p is then defined by

$$\mathrm{Ch}(p) := \sum_{N=0}^{\infty} \sum_{k=0}^{N} \mathrm{tr}\big(\underbrace{\mathcal{R}, \dots, \mathcal{R}}_{k}, \sigma p, \underbrace{\mathcal{R}, \dots, \mathcal{R}}_{N-k}\big), \tag{2.5}$$

where tr is the functional defined for elements $\boldsymbol{\theta}_i = ((\theta_i)^a_b)_{1\leq a,b\leq m} \in \Omega_{\mathbb{T}}(M) \otimes M_m(\mathbb{C})$ by

$$\mathrm{tr}(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N) = \sum_{a_1,\ldots,a_N=1}^{m} \big((\boldsymbol{\theta}_1)^{a_1}_{a_N}, (\boldsymbol{\theta}_2)^{a_2}_{a_1}, \ldots, (\boldsymbol{\theta}_N)^{a_N}_{a_{N-1}}\big).$$

Due to the grading shift in the definition of $\mathsf{B}(\Omega_{\mathbb{T}}(M))$, it is an even chain, which is clearly cyclic, i.e., contained in the subcomplex (2.1). It was shown by [9, Section 6] that the iterated integral map (2.2) sends $\mathrm{Ch}(p)$ to the Bismut–Chern character form on $\Omega(\mathsf{L}M)$, defined in [4]. A complication of the theory is that $\mathrm{Ch}(p)$ is *not* closed with respect to the differential b; instead $b(\mathrm{Ch}(p))$ is contained in a certain subcomplex of $\mathsf{B}^{\natural}_{\epsilon}(\Omega_{\mathbb{T}}(M))$. One says that $\mathrm{Ch}(p)$ is closed as a *Chen normalized cochain*; see [10, Section 7]. A construction of *odd* Bismut–Chern characters (depending on a smooth map to a unitary group) has been given in [6], providing further examples of interesting chains.

A subtle point in this theory is that the Bismut–Chern characters are *not* directly closed in $\mathsf{B}^{\natural}_{\epsilon}(\Omega_{\mathbb{T}}(X))$, i.e., $b\,\mathrm{Ch}(p) \neq 0$. However, we have $b\,\mathrm{Ch}(p) \in \ker(\rho)$, in other words, $\mathrm{Ch}(p)$ is closed in the quotient complex

$$\mathsf{N}^{\natural}_{\epsilon}\big(\Omega_{\mathbb{T}}(X)\big) = \mathsf{B}^{\natural}_{\epsilon}\big(\Omega_{\mathbb{T}}(X)\big)\big/\big(\ker(\rho) \cap \mathsf{B}^{\natural}_{\epsilon}\big(\Omega_{\mathbb{T}}(X)\big)\big), \tag{2.6}$$

called the *Chen-normalized complex*. Since b is a chain map when restricted to $\mathsf{B}^{\natural}_{\epsilon}(\Omega_{\mathbb{T}}(X))$, $\ker(\rho) \cap \mathsf{B}^{\natural}_{\epsilon}(\Omega_{\mathbb{T}}(X))$ is indeed a subcomplex of $\mathsf{B}^{\natural}_{\epsilon}(\Omega_{\mathbb{T}}(X))$.

2.2. Cochains and the Chern character

Given a $\mathbb{Z}_2$-graded algebra[1] $\mathscr{L}$, an *$\mathscr{L}$-valued bar cochain* over $\Omega_{\mathbb{T}}(M)$ is a linear map $\ell : \mathsf{B}(\Omega_{\mathbb{T}}(M)) \to \mathscr{L}$. Such a cochain can be viewed as a sequence of multilinear maps:

$$\ell : \underbrace{\Omega_{\mathbb{T}}(M) \times \cdots \times \Omega_{\mathbb{T}}(M)}_{N} \to \mathscr{L},$$

again denoted by the same letter. In particular, for $N = 0$, this is just an element of $\mathscr{L}$, which we denote by $\ell(\emptyset)$, by abuse of notation. We say that ℓ is *even* if it preserves parity and *odd* if it reverses parity. The standard coalgebra structure on the tensor algebra $\mathsf{B}(\Omega_{\mathbb{T}}(M))$ induces a product on bar cochains, given by

$$(\ell_1\ell_2)(\theta_1, \ldots, \theta_N) = \sum_{k=0}^{N} (-1)^{n_k|\ell_2|}\ell_1(\theta_1, \ldots, \theta_k)\ell_2(\theta_{k+1}, \ldots, \theta_N), \tag{2.7}$$

where $n_k = |\theta_1| + \cdots + |\theta_k| - k$. This product is compatible with the codifferential β on cochains defined by

$$(\beta\ell)(\theta_1, \ldots, \theta_N) = -(-1)^{|\ell|}\ell\big(b(\theta_1, \ldots, \theta_N)\big), \tag{2.8}$$

[1] In the case that $\mathscr{L} = \mathbb{C}$, we endow $\mathbb{C}$ with the trivial grading rendering it purely even and just speak of *bar cochains*.

in the sense that $\beta(\ell_1\ell_2) = \beta(\ell_1)\ell_2 + (-1)^{|\ell_1|}\ell_1\beta(\ell_2)$ for all homogeneous cochains ℓ_1, ℓ_2; in other words, β is a derivation on the cochain algebra.

To obtain interesting cochains on $\mathsf{B}(\Omega_{\mathbb{T}}(M))$, one starts with a *Fredholm module* over $\Omega(M)$, which is a triple (H, Q, c), consisting of a $\mathbb{Z}_2$-graded Hilbert space H, an odd operator Q on H, and an even linear map $c : \Omega(M) \to B(H)$, which are assumed to satisfy

$$\big[Q, c(f)\big] = c(df) \quad \text{and} \quad c(f\theta) = c(f)c(\theta) \tag{2.9}$$

for $f \in \Omega^0(M)$ and $\theta \in \Omega(M)$, together with some further analytic conditions; see [10, Section 2] for details.

Example 2.4. Our main example here is defined in case that M is an even-dimensional compact spin manifold with spinor bundle S; in that case, $H = L^2(M, S)$, $Q = D$, the Dirac operator, and c is the quantization map (see Section 3.1).

Inspired by Quillen [18], a Fredholm module induces a *connection* ω, which is a cochain on $\mathsf{B}(\Omega_{\mathbb{T}}(M))$ with values in linear operators on H, given by

$$\omega(\emptyset) = Q, \quad \omega(\theta) = c(\theta'), \quad \omega(\theta_1, \dots, \theta_k) = 0 \ (k \geq 2).$$

Due to the grading shift in the definition of $\mathsf{B}(\Omega_{\mathbb{T}}(M))$, this is an odd cochain. Its *curvature* is defined by the formula $F := \beta\omega + \omega^2$, where β is the codifferential (2.8). Explicitly, its components can be easily calculated to be $F(\emptyset) = Q^2$,

$$\begin{aligned} F(\theta) &= \big[Q, c(\theta')\big] - c(d\theta') + c(\theta''), \\ F(\theta_1, \theta_2) &= (-1)^{|\theta_1|}\big(c(\theta_1' \wedge \theta_2') - c(\theta_1')c(\theta_2')\big) \end{aligned} \tag{2.10}$$

and $F(\theta_1, \dots, \theta_k) = 0$ for $k \geq 3$; here $[Q, c(\theta')]$ denotes the graded commutator. Motivated by the definition of the Chern-character from Chern-Weil theory, one now makes the following definition.

Definition 2.5. The *Chern character* of a Fredholm module (H, Q, c) is the bar cochain

$$\mathrm{Ch}_Q = \mathrm{Str}\big(e^{-F}\big). \tag{2.11}$$

Here, because F takes values in *unbounded* operators on H, some care is needed to make sense of the exponential. This is dealt with by writing $F = Q^2 + F'$ and expanding $e^{-F} = e^{-Q^2 - F'}$ as a perturbation series. This results in the formula

$$\mathrm{Ch}_Q = \sum_{N=0}^{\infty} (-1)^N \int_{\Delta_N} \mathrm{Str}\left(e^{-\tau_1 Q^2} \prod_{p=1}^{N} F' e^{-(\tau_{p+1} - \tau_p)Q^2}\right) d\tau, \tag{2.12}$$

where $\Delta_N = \{0 \leq \tau_1 \leq \cdots \leq \tau_N \leq \tau_{N+1} := 1\}$ is the N-simplex, giving an explanation to the right-hand side of (2.11).

Proposition 2.6. *Restricted to the space of cyclic chains, we have* $\beta(\mathrm{Ch}_Q) = 0$.

Proof (Sketch). Since β is a derivation with respect to the product (2.7), the curvature F satisfies the Bianchi identity

$$\beta F = \beta^2\omega + \beta(\omega^2) = (\beta\omega)\omega - \omega(\beta\omega) = F\omega - \omega F = [F,\omega].$$

Hence

$$\beta(\mathrm{Ch}) = \mathrm{Str}\left(\beta(e^{-F})\right) = -\mathrm{Str}\left(e^{-F}\beta F\right) = -\mathrm{Str}\left(e^{-F}[F,\omega]\right) = -\mathrm{Str}\left([e^{-F},\omega]\right).$$

One now verifies that the right-hand side is zero when restricted to cyclic chains to finish the proof. The above calculations are somewhat formal and the proof remains a sketch here due to the analytical difficulties involved in defining e^{-F}; a complete treatment is given in [10, Section 4]. ∎

In order to evaluate Ch_Q at the Bismut–Chern characters (2.5), one needs the following proposition; see [10, Theorem B].

Proposition 2.7. *The Chern character is analytic, i.e., continuous with respect to the seminorms* (2.3) *and hence extends to a cochain on the entire complex* $\mathsf{B}_\epsilon(\Omega_{\mathbb{T}}(M))$.

We now consider the Fredholm module given by the Dirac operator; see Example 2.4. Before we give an explicit formula for the components of Ch_D, we slightly generalize our Example 2.4. Namely, given a parameter $t > 0$, one can define a new Fredholm module (H, Q_t, c_t) by setting $Q_t = tD$ and $c_t(\theta) = t^{|\theta|}c(\theta)$. Observing that the relations (2.9) still hold, one obtains a one-parameter family $\{\mathrm{Ch}_t\}$ of Chern characters. Plugging the product formula (2.7) into the perturbation series (2.12), one obtains the explicit but somewhat cumbersome combinatorial formula

$$\mathrm{Ch}_t(\theta_1,\ldots,\theta_N) = \sum_{\substack{k=1 \\ 1\le i_1<\cdots<i_k\le N}}^{N} (-1)^k t^{|\theta|-N+2k}\int_{\Delta_k}\mathrm{Str}\left(e^{-t^2\tau_1 D^2}\prod_{p=1}^{k}F(\theta_{i_{p-1}+1},\ldots,\theta_{i_p})e^{-t^2(\tau_{p+1}-\tau_p)D^2}\right)d\tau, \tag{2.13}$$

for homogeneous elements $\theta_1,\ldots,\theta_N \in \Omega_{\mathbb{T}}(M)$, where $|\theta| = |\theta_1| + \cdots + |\theta_N|$ is the total degree. This formula can be understood as a certain time-ordered expectation value, where, since the operators F vanish when more than two entries are filled, only neighboring θ_i "interact".

Proposition 2.8. *For each* $t > 0$*, the Chern characters* Ch_t *are Chen normalized, meaning that they vanish on the subcomplex* $\ker(\rho) \cap \mathsf{B}^\natural_\epsilon(\Omega_{\mathbb{T}}(X)) \subset \mathsf{B}^\natural_\epsilon(\Omega_{\mathbb{T}}(X))$.

This proposition can be found as [10, Theorem 5.5]. It implies that the Chern characters descend to linear functionals on the Chen normalized quotient complex $\mathsf{N}^\natural_\epsilon(\Omega_{\mathbb{T}}(X))$ defined in (2.6). Since the Chern characters $\mathrm{Ch}(p)$ of Example 2.3 are only closed in this quotient complex, but not in $\mathsf{B}^\natural_\epsilon(\Omega_{\mathbb{T}}(X))$, this property of Ch_t is crucial when calculating the pairings $\mathrm{Ch}_t(\mathrm{Ch}(p))$.

2.3. The localization formula

We now aim to prove the localization formula (1.2) for closed entire cyclic chains Θ, where the *restriction map* is the map $i : \mathsf{B}(\Omega_{\mathbb{T}}(M)) \to \Omega(M)$ given by

$$i(\theta_1, \dots, \theta_N) = \frac{(-1)^N}{N!} \theta_1'' \wedge \cdots \wedge \theta_N''; \tag{2.14}$$

here, as always, $\theta_i = \theta_i' + \sigma\theta_i'' \in \Omega_{\mathbb{T}}(M)$. By the following lemma, this map is an "algebraic version" of the restriction to map to the subset of constant loops $M \subset \mathsf{L}M$.

Lemma 2.9. *The above map satisfies $i = j^*\rho$, where ρ is the iterated integral map* (2.2) *and j^* denotes the pullback with respect to the inclusion $j : M \to \mathsf{L}M$ as the subset of constant loops.*

Proof. For any $\tau \in \mathbb{T}$, we have $j^*\theta(\tau) = \theta$, while $j^*(\iota_K\theta(\tau)) = 0$ (as $K \equiv 0$ on constant loops). The result therefore follows directly from the formula (2.2) for the iterated integral map, after observing that the integral over Δ_N in (2.2) is constant and integration contributes a factor of $\mathrm{vol}(\Delta_N) = 1/N!$. ∎

Let M be a compact Riemannian spin manifold of even dimension n, so that the family $\{(H, Q_t, c_t)\}_{t>0}$ of Fredholm modules together with the corresponding family of Chern characters $\{\mathrm{Ch}_t\}_{t>0}$ introduced in Section 2.2 is defined. By homotopy invariance of the Chern character [10, Theorem 6.2], for any $s, t > 0$, there exists an analytic bar cochain $\mathrm{CS}_{s,t}$ such that, when restricted to $\mathsf{B}^\natural_\epsilon(\Omega_{\mathbb{T}}(M))$,

$$\mathrm{Ch}_s - \mathrm{Ch}_t = \beta\, \mathrm{CS}_{s,t}; \tag{2.15}$$

in other words Ch_s, Ch_t are cohomologous as cyclic cochains. Explicitly, this means that for all entire cyclic chains $\Theta \in \mathsf{B}^\natural_\epsilon(\Omega_{\mathbb{T}}(M))$, we have

$$\mathrm{Ch}_D(\Theta) - \mathrm{Ch}_t(\Theta) = \beta\, \mathrm{CS}_{1,t}(\Theta) = \mathrm{CS}_{1,t}\big(b(\Theta)\big).$$

Therefore, if Θ is additionally *closed*, i.e., $b(\Theta) = 0$, then $\mathrm{Ch}_D(\Theta) = \mathrm{Ch}_1(\Theta) = \mathrm{Ch}_t(\Theta)$ for all $t > 0$. This discussion shows that we can compute the value of $\mathrm{Ch}_Q(\Theta)$ by taking the limit of $\mathrm{Ch}_t(\Theta)$, as $t \to 0$. The localization formula (1.2) therefore follows from the following theorem, which will be proved in Section 3.

Theorem 2.10. *For all $\theta_1, \dots, \theta_N \in \Omega_{\mathbb{T}}(M)$, we have*

$$\lim_{t\to 0} \mathrm{Ch}_t(\theta_1, \dots, \theta_N) = \frac{}{(2\pi i)^{n/2}} \int_M \hat{A}(M) \wedge i(\theta_1, \dots, \theta_N), \tag{2.16}$$

where the characteristic form $\hat{A}(M)$ is given by

$$\hat{A}(M) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right). \tag{2.17}$$

Here R is the Riemannian curvature tensor, interpreted as a skew-adjoint matrix of differential 2-forms in a local frame.

2.4. An application

We finish this section with an application of the localization formula, which features the Bismut–Chern characters from Example 2.3; compare [10, Section 8]. An issue here is that we cannot directly apply the localization formula (1.2), since $\mathrm{Ch}(p)$ is not closed, but only satisfies $b\,\mathrm{Ch}(p) \in \ker(\rho)$. This problem can be remedied as follows. By Proposition 2.8, each of the Chern characters Ch_t is Chen-normalized, i.e., vanishes on $\ker(\rho)$. The same result holds for $\mathrm{CS}_{s,t}$, so that

$$\mathrm{Ch}_D\big(\mathrm{Ch}(p)\big) - \mathrm{Ch}_t\big(\mathrm{Ch}(p)\big) = \mathrm{CS}_{1,t}\big(b\,\mathrm{Ch}(p)\big) = 0.$$

Hence the localization formula (1.2) applies for the Bismut–Chern characters, so that

$$\mathrm{Ch}_D\big(\mathrm{Ch}(p)\big) = \lim_{t\to 0}\mathrm{Ch}_t\big(\mathrm{Ch}(p)\big) = (2\pi i)^{-n/2}\int_M \hat{A}(M)\wedge i\big(\mathrm{Ch}(p)\big). \tag{2.18}$$

We have

$$i\big(\mathrm{Ch}(p)\big) = \sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\,\mathrm{tr}\big(p(dp)^{2N}\big) = \mathrm{tr}\big(p\exp\big(-(dp)^2\big)\big).$$

Since $(dp)^2$ is the curvature of the bundle $E = \mathrm{im}(p)$ with respect to the connection pdp, this is precisely the Chern character form $\mathrm{ch}(E)$ for the bundle E.

On the other hand, we have the following proposition, which calculates the value $\mathrm{Ch}_D(\mathrm{Ch}(p))$ independently of the localization formula.

Proposition 2.11. *Let $p \in \Omega(M)\otimes M_n(\mathbb{C})$ with $p^2 = p$ and define*

$$D_p = pDp + (1-p)D(1-p),$$

where, by abuse of notation, D denotes the Dirac operator on $S\otimes\underline{\mathbb{C}}^n$. Then

$$\mathrm{Ch}_{\mathsf{D}}\big(\mathrm{Ch}(p)\big) = \mathrm{Str}\big(pe^{-D_p^2}\big).$$

By the McKean–Singer formula, the supertrace $\mathrm{Str}(pe^{-D_p})$ is just the index of the Dirac operator twisted with the vector bundle $E = \mathrm{im}(p)$.

Proof of Proposition 2.11. Observe that $D_p = D + c((2p-1)dp)$ and with $\mathcal{R}$ defined as in (2.4),

$$F(\mathcal{R}) = \big[D, c\big((2p-1)dp\big)\big] - c\big((dp)^2\big),$$
$$F(\mathcal{R},\mathcal{R}) = c\big((2p-1)dp\big)^2 + c\big((dp)^2\big).$$

Put together,

$$D_p^2 = D^2 + \big[D, c\big((2p-1)dp\big)\big] + c\big((2p-1)dp\big)^2 = D^2 + F(\mathcal{R}) + F(\mathcal{R},\mathcal{R}).$$

Writing e^{-D_p} as a perturbation series, we therefore obtain

$$e^{-D_p^2} = \sum_{N=0}^{\infty}(-1)^N \int_{\Delta_N} e^{-\tau_1 D^2} \prod_{k=1}^{N} \big(F(\mathcal{R}) + F(\mathcal{R},\mathcal{R})\big) e^{-(\tau_{k+1}-\tau_k)D^2} d\tau.$$

By the cyclic permutation property of the supertrace, multiplying this by p and taking the supertrace yields $\mathrm{Ch}_D(\mathrm{Ch}(p))$. ■

Combining the localization result (2.18) with the one from Proposition 2.11 (and the McKean–Singer formula) now gives the twisted Atiyah–Singer index theorem:

Corollary 2.12. *Let $E = \mathrm{im}(p)$, with its connection induced from viewing it as a vector subbundle of the trivial $\mathbb{C}^m$-bundle. Let D_E be the Dirac operator twisted by E and* $\mathrm{ch}(E)$ *its Chern character form. Then*

$$\mathrm{ind}(D_E) = \int_M \widehat{A}(M) \wedge \mathrm{ch}(E).$$

3. The tangent groupoid and the localization formula

In this section, we first give a brief introduction to the tangent groupoid and the rescaled spinor bundle and then use the techniques introduced to prove Theorem 2.10.

3.1. The scaling order

Let M be a spin manifold of even dimension n, with spinor bundle S. In this section, we briefly review the notion of scaling order for sections of the bundle $S \boxtimes S^*$ over $M \times M$, the fiber of which over (m_1, m_2) is $S_{m_1} \otimes S^*_{m_2}$. For a more detailed treatment, we refer to [13, Section 3.3].

To begin with, denote by $\mathrm{Cliff}(T_m M)$ the Clifford algebra of $T_m M$ and let

$$c : \Lambda^* T_m M \to \mathrm{Cliff}(T_m M)$$

be the *quantization map*, defined by $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}$ in terms of an orthonormal basis $e_1, \dots, e_n \in T_m M$; see [3, Section 3.1] for details. c is not an algebra homomorphism, but defining $\mathrm{Cliff}_k(T_m M)$ to be the image of $\Lambda^{\leq k} T_m M$ under c defines a filtration on the Clifford algebra. An element a of the Clifford algebra is said to have *Clifford order* k or less if it is contained in $\mathrm{Cliff}_k(T_m M)$. For $a \in \mathrm{Cliff}(T_m M)$, we denote by $[a] \in \Lambda^* T_m M$ its inverse image under the quantization map (often called the *Clifford symbol*) and if $a \in \mathrm{Cliff}_k(T_m M)$, we let $[a]_k$ be the k-form component of $[a] \in \Lambda^{\leq k} T_m M$.

A differential operator P has *Getzler order* p or less if, locally, it can be written as

$$P = f D_1 \cdots D_p,$$

where f is a smooth function, and each D_i is either a covariant derivative ∇_X, a Clifford multiplication $c(X)$, or the identity operator. The definition of scaling order now uses the fact that on the diagonal of $M \times M$, we have the identification

$$(S \boxtimes S^*)_{(m,m)} \cong S_m \otimes S_m^* \cong \mathrm{End}(S_m) \cong \mathrm{Cliff}(T_m M).$$

Definition 3.1 ([13, Section 3.4]). Let $p \in \mathbb{Z}$. We say that a section s of $S \boxtimes S^*$ has *scaling order p or more* if for every $m \in M$,

$$\text{Clifford-order}\big(Ds(-,m)|_m\big) \leq q - p$$

for every differential operator D of Getzler order q or less, acting on the first component of s.

3.2. The tangent groupoid and the rescaled spinor bundle

The tangent groupoid was introduced by Alain Connes to give a simple and elegant proof of the Atiyah–Singer index theorem [8, Chapter 2, Section 5]. Given smooth manifold M, the tangent groupoid $\mathbb{T}M$ is a smooth manifold whose underlying set is

$$\mathbb{T}M = (TM \times \{0\}) \sqcup (M \times M \times \mathbb{R}^\times).$$

If $M \supset U \xrightarrow{\varphi} \mathbb{R}^n$ is a local coordinate chart, then $\mathbb{T}U \subset \mathbb{T}M$ is an open subset and there is a local coordinate chart

$$\mathbb{T}U \xrightarrow{\phi} \mathbb{R}^{2n+1}$$

given by

$$\begin{cases} (x, m, t) \to \left(\dfrac{\varphi(x) - \varphi(m)}{t}, \varphi(m), t\right), \\ (X, m, 0) \to \big(\varphi_* X, \varphi(m), 0\big). \end{cases} \tag{3.1}$$

In [11], the authors adopt a more algebraic way towards the tangent groupoid, namely it is built as spectrum of the following algebra.

Definition 3.2. Denote by $\mathcal{A}(\mathbb{T}M) \subseteq C^\infty(M \times M)[t^{-1}, t]$ the $\mathbb{R}$-algebra of those Laurent polynomials

$$\sum_{p \in \mathbb{Z}} f_p t^{-p} \tag{3.2}$$

for which each coefficient f_p is a smooth, real-valued function on $M \times M$ that vanishes to order $\geq p$ on M (and all but finitely many f_p are zero).

In general, the spectrum of an algebra comes naturally with a topology, the Zariski topology. In this particular case, the spectrum of $\mathcal{A}(\mathbb{T}M)$ turns out to have a smooth manifold structure that coincides with the manifold structure on $\mathbb{T}M$ defined above. A Laurent polynomial of the form (3.2) naturally defines a smooth function on the tangent groupoid $\mathbb{T}M$, and the evaluation maps are given by

$$\varepsilon_{(x,m,\lambda)} : \sum_{p\in\mathbb{Z}} f_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} f_p(x,m)\lambda^{-p},$$

$$\varepsilon_{(X,m)} : \sum_{p\in\mathbb{Z}} f_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} \frac{1}{p!} X^p(f_p).$$

The set of smooth functions on $\mathbb{T}M$ is locally smoothly generated by these functions (see [11, Lemma 2.4]).

Let M be an even dimensional spinor manifold with spinor bundle $S \to M$. In order to introduce Getzler's rescaling technique in the context of the tangent groupoid, by deforming S, we build a vector bundle $\mathbb{S} \to \mathbb{T}M$ over the tangent groupoid, following the construction in [13]. This bundle is called the rescaled bundle and it is built from the following $\mathcal{A}(\mathbb{T}M)$-module.

Definition 3.3. Denote by $S(\mathbb{T}M)$ the complex vector space of Laurent polynomials

$$\sum_{p\in\mathbb{Z}} s_p t^{-p}, \tag{3.3}$$

where each s_p is a smooth section of $S \boxtimes S^*$ of scaling order at least p.

The complex vector space $S(\mathbb{T}M)$ so constructed is indeed an $\mathcal{A}(M)$-module; the module structure is given by the Laurent polynomial multiplication. It turns out that the module $S(\mathbb{T}M)$ can be made into a sheaf of locally free modules over the sheaf of smooth functions on $\mathbb{T}M$, thus giving rise to the rescaled bundle $\mathbb{S} \to \mathbb{T}M$.

A Laurent polynomial of the form (3.3) naturally defines a smooth section of $\mathbb{S}$ whose evaluation map is given by

$$\varepsilon_{(x,m,\lambda)} : \sum_{p\in\mathbb{Z}} s_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} s_p(x,m)\lambda^{-p}, \tag{3.4}$$

$$\varepsilon_{(X,m)} : \sum_{p\in\mathbb{Z}} s_p t^{-p} \mapsto \sum_{q,p} \frac{1}{q!} \big[\nabla_X^q s_p(-,m)|_m\big]_{q-p}, \tag{3.5}$$

where ∇_X is the covariant derivative acting on the first variable of $S \boxtimes S^*$ and $[\cdot]_k$. Observe here that since s_p has scaling order at least p, $\nabla_X^q s_p(-,m)|_m \in \mathrm{Cliff}(T_m M)$ has Clifford order at most $q-p$ at each $m \in M$, and hence its $(q-p)$-th Clifford symbol is well defined. A general smooth section f of $\mathbb{S}$ can locally be written as a finite sum

$$f = \sum_j f_j \cdot s_j, \tag{3.6}$$

where $f_j \in C^\infty(\mathbb{T}M)$ and the s_j are Laurent polynomials of the form (3.3), which determine smooth sections of $\mathbb{S}$ denoted by the same symbol.

Set theoretically, over $M \times M \times \{t\}$, the rescaled bundle is the tensor product bundle $S \boxtimes S^*$ while over $TM \times \{0\}$ are the pullback of the exterior bundle $\wedge^* T^* M \to M$ along $\pi : TM \to M$:

$$\begin{array}{ccc} \mathbb{S} & \pi^* \wedge^* T^* M & S \boxtimes S^* \\ \downarrow & \downarrow & \downarrow \\ \mathbb{T}M & TM \times \{0\} \quad \sqcup & M \times M \times \mathbb{R}^\times . \end{array}$$

The space of compactly supported smooth sections of the rescaled bundle has an algebra structure in the following way: for $f, g \in C_c^\infty(\mathbb{T}M, \mathbb{S})$, $f * g \in C_c^\infty(\mathbb{T}M, \mathbb{S})$ is defined by

$$\begin{aligned} (f * g)(x, m, t) &= \int_M f(x, y, t) g(y, m, t) t^{-n} dy, \\ (f * g)(X, m, 0) &= \int_{T_m M} f(X - Y, m, 0) g(Y, m, 0) e^{-\frac{1}{2}[\kappa(X,Y)]} dY, \end{aligned} \tag{3.7}$$

where $(x, m, t) \in M \times M \times \mathbb{R}^\times$ and $(X, m, 0) \in TM \times \{0\}$ and where κ is the curvature tensor of the spinor bundle (so that $\kappa(X, Y) \in \mathrm{Cliff}(T_m M)$ for $X, Y \in T_m M$) and $[\kappa(X, Y)] \in \Lambda T_m M$ is the inverse image of the Clifford algebra element $\kappa(X, Y)$ under the quantization map (see [13, Section 5.2]). Crucially, we will use the following result.

Theorem 3.4 ([13, Theorem 5.4.2]). *For each $t \in \mathbb{R}$, the formula*

$$\begin{aligned} \mathrm{Str}_t(f) &= \int_M \mathrm{str}\big(f(m, m, t)\big) t^{-n} dm \quad \text{for } t \neq 0, \\ \mathrm{Str}_0(f) &= \left(\frac{2}{i}\right)^{n/2} \int_M f(0, -, 0) \end{aligned} \tag{3.8}$$

defines a supertrace on $C_c^\infty(\mathbb{T}M, \mathbb{S})$; here in the second integral, $f(0, -, 0)$ is a differential form on M, which is integrated using the orientation on M. Moreover, the map $t \mapsto \mathrm{Str}_t(f)$ is smooth.

Remark 3.5. For $t \neq 0$, the traces Str_t can be viewed as an integral over the t-fiber of $\mathbb{T}M$, when the fibers are equipped with the rescaled metric $t^{-2}g$. The theorem then asserts that the formula for Str_0 is the continuous (even smooth) extension of this family to the fiber over $t = 0$.

3.3. Rapidly decaying sections

A disadvantage of the algebra $C_c^\infty(\mathbb{T}M, \mathbb{S})$ considered above is that it is too small to contain the "heat kernel element" $e^{-t^2 D^2}$. In this section, we shall construct an enlargement $\mathcal{S}(\mathbb{T}M, \mathbb{S})$ of this algebra consisting of sections of rapid decay, in particular $e^{-t^2 D^2}$, and this still supports the family of supertraces (3.8).

Definition 3.6. Let f be a compactly supported smooth section of the rescaled bundle. Define a family of norms $\{N_k\}$, $k \in \mathbb{N}$, on $C_c^\infty(\mathbb{T}M, \mathbb{S})$ by

$$N_k(f) = \sup_{(x,m,t)\in M\times M\times\mathbb{R}^\times} \left(1 + \frac{d(x,m)^2}{t^2}\right)^{k/2} \big|f(x,m,t)\big|, \tag{3.9}$$

where $d(x,m)$ is the Riemannian distance between x and m and let

$$\mathcal{S}(\mathbb{T}M, \mathbb{S}) := \big\{f \in C(\mathbb{T}M, \mathbb{S}) \mid \forall k \in \mathbb{N} : N_k(f) < \infty\big\}.$$

Lemma 3.7. *The following holds:*

(1) $\mathcal{S}(\mathbb{T}M, \mathbb{S})$ *is complete and* $C_c^\infty(\mathbb{T}M, \mathbb{S}) \subset \mathcal{S}(\mathbb{T}M, \mathbb{S})$ *is dense;*

(2) *the convolution product extends continuously to a product on* $\mathcal{S}(\mathbb{T}M, \mathbb{S})$*;*

(3) *each of the supertraces* (3.8) *extends continuously to* $\mathcal{S}(\mathbb{T}M, \mathbb{S})$ *and for each* $f \in \mathcal{S}(\mathbb{T}M, \mathbb{S})$, $\mathrm{Str}_t(f)$ *is continuous in* t.

Proof. (1) Let $\{\varphi_\alpha : M \supset U_\alpha \to \mathbb{R}^n\}$ be a collection of coordinate charts on M and let $\{\phi_\alpha : \mathbb{T}U_\alpha \to \mathbb{R}^{2n+1}\}$ be the induced coordinate charts on $\mathbb{T}M$, as given in (3.1). One then easily shows that restricted to $\mathbb{T}U_\alpha$, the seminorm N_k is equivalent to the seminorm

$$\sup_{(a,b,t)\in\phi_\alpha(\mathbb{T}U_\alpha)} \big(1 + |a|^2\big)^{k/2}\big|f \circ \phi_\alpha^{-1}(a,b,t)\big|.$$

The rest follows from routine arguments.

(2) We calculate

$$\begin{aligned}
N_k(f * g) &= \sup_{(x,m,t)} \left(1 + \frac{d(x,m)^2}{t^2}\right)^{k/2} \left|\int_M f(x,y,t)g(y,m,t)t^{-n}dy\right| \\
&\le C_k \sup_{(x,m,t)} \left|\int_M \left(1 + \frac{d(x,y)^2}{t^2}\right)^{k/2} f(x,y,t)g(y,m,t)t^{-n}dy\right| \\
&\quad + C_k \sup_{(x,m,t)} \left|\int_M f(x,y,t)\left(1 + \frac{d(y,m)^2}{t^2}\right)^{k/2} g(y,m,t)t^{-n}dy\right| \\
&\le C_k N_{k+n+1}(f)N_0(g) \sup_{(x,t)} \int_M \left(1 + \frac{d(x,y)^2}{t^2}\right)^{-(n+1)/2} t^{-n}dy \\
&\quad + C_k N_{n+1}(f)N_k(g) \sup_{(x,t)} \int_M \left(1 + \frac{d(x,y)^2}{t^2}\right)^{-(n+1)/2} t^{-n}dy,
\end{aligned}$$

where C_k is a constant such that for all $a, b \ge 0$,

$$(1 + a + b)^{k/2} \le C_k(1 + a)^{k/2} + C_k(1 + b)^{k/2}.$$

It remains to show that the integral

$$\int_M \left(1 + \frac{d(x,y)^2}{t^2}\right)^{-(n+1)/2} t^{-n}dy \tag{3.10}$$

is uniformly bounded with respect to $t \in \mathbb{R}^\times$. For $\varepsilon > 0$, split the integral up in one integral over $M \setminus B_\varepsilon(x)$ and one over $B_\varepsilon(x)$. The first of these is clearly bounded, and the second one can be estimated by the integral

$$\int_{B_\varepsilon(0)} \left(1 + \frac{|v|^2}{t^2}\right)^{-(n+1)/2} t^{-n} dv = \int_{B_{\varepsilon/t}(0)} \left(1 + |\xi|^2\right)^{-(n+1)/2} d\xi$$

over Euclidean space. In the second step, we replaced $\xi = v/t$ to obtain an expression which is clearly bounded.

(3) For $t \neq 0$, the formula (3.8) clearly extends to a continuous linear functional on $\mathcal{S}(\mathbb{T}M, \mathbb{S})$. In the case $t = 0$, we observe that by (1) above, elements $f \in \mathcal{S}(\mathbb{T}M, \mathbb{S})$ satisfy $|f(X, m, 0)| \leq C_k(1 + |X|^2)^{k/2}$ for any $k \in \mathbb{N}$. Hence also the second formula in (3.8) extends to a continuous linear functional on $\mathcal{S}(\mathbb{T}M, \mathbb{S})$. To show continuity at $t = 0$, let $f_i \in C_c(\mathbb{T}M, \mathbb{S})$ be compactly support sections with $f_i \to f$ in $\mathcal{S}(\mathbb{T}M, \mathbb{S})$. Then

$$\begin{aligned}\left|\mathrm{Str}_t(f) - \mathrm{Str}_0(f)\right| \leq \left|\mathrm{Str}_t(f) - \mathrm{Str}_t(f_i)\right| + \left|\mathrm{Str}_t(f_i) - \mathrm{Str}_0(f_i)\right| \\ + \left|\mathrm{Str}_0(f_i) - \mathrm{Str}_0(f)\right|.\end{aligned}$$

The second term converges to zero by Theorem 3.4, and the first and the third by continuity of Str_t. This finishes the proof. ■

3.4. The heat kernel element

We now show that the space $\mathcal{S}(\mathbb{T}M, \mathbb{S})$ of rapidly decaying sections of $\mathbb{S}$ contains the "heat kernel element" $e^{-t^2 D^2}$.

Lemma 3.8. *Let f be a smooth section of $S \boxtimes S^* \to M \times M \times \mathbb{R}$. Then $t^{n+1} f$ defines a smooth section of the rescaled bundle $\mathbb{S} \to \mathbb{T}M$ such that*

$$\left(t^{n+1} f\right)(\gamma) = \begin{cases} t^{n+1} f(x, y, t) & \gamma = (x, y, t), \\ 0 & \gamma = (X, m, 0). \end{cases}$$

Proof. As a $C^\infty(M \times M \times \mathbb{R})$ module, $C^\infty(M \times M \times \mathbb{R}, S \boxtimes S^*)$ is locally finitely generated and free. We could choose $s_1, s_2, \ldots, s_p$ as a sequence of local sections of $S \boxtimes S^* \to M \times M$ such that f can be locally written as combination of $s_1, \ldots, s_p$. That is

$$t^{n+1} f(x, y, t) = f_1(x, y, t) t^{n+1} s_1(x, y) + \cdots + f_p(x, y, t) t^{n+1} s_p(x, y)$$

locally, for some smooth functions $f_1, f_2, \ldots, f_p$ on $M \times M \times \mathbb{R}$. Here $t^{n+1} s_i$ defines local section of the rescaled bundle and its value at $(X, m, 0)$ which can be evaluated by (3.5) is clearly zero. ■

Proposition 3.9. *For each $\tau > 0$, there is $H_\tau \in \mathcal{S}(\mathbb{T}M, \mathbb{S})$ such that for $t > 0$,*

$$H_\tau(x, m, t) = t^n e^{-t^2 \tau D^2}(x, m), \tag{3.11}$$

where $e^{-t^2\tau D^2}(x,m)$ *is the heat kernel of* D*. Moreover, this element satisfies*

$$
\begin{aligned}
&H_\tau(X,m,0)\\
&\quad = \frac{1}{(4\pi\tau)^{n/2}} \det{}^{1/2}\left(\frac{\tau R/2}{\sinh(\tau R/2)}\right) \exp\left(-\frac{1}{4\tau}\left\langle X, \frac{\tau R}{2}\coth\left(\frac{\tau R}{2}\right)X\right\rangle\right), \qquad (3.12)
\end{aligned}
$$

where R *is the Riemannian curvature tensor, interpreted as a skew-adjoint matrix of differential 2-forms in a local frame.*

Proof. We need a slight refinement of [3, Theorem 4.1]. By the asymptotic expansion of the heat kernel, near the diagonal in $M \times M$, we have

$$
t^n \cdot e^{-t^2\tau \mathrm{D}^2}(x,m) = \frac{1}{(4\pi\tau)^{n/2}} e^{\frac{-d(x,m)^2}{4t^2\tau}} \sum_{i=0}^{n/2} t^{2i}\tau^i \Phi_i(x,m) + \mathcal{O}(t^{n+1}), \qquad (3.13)
$$

where the $\Phi_i(-,m)$ are determined by a system of differential equations:

$$
\begin{cases} \nabla_{\mathcal{R}}\Phi_0(-,m) = 0, \\ (\nabla_{\mathcal{R}} + i)\Phi_i(-,m) = -B\Phi_{i-1}(-,m) \end{cases}
$$

with initial condition $\Phi_0(m,m) = 1$, where $\mathcal{R}$ is the radial vector field associated with a Riemannian normal coordinate system around m, B is a differential operator on S of Getzler order 2 (see [3, Section 2.5] for details). We claim that Φ_i has scaling order $2i$. The first equation $\nabla_{\mathcal{R}}\Phi_0 = 0$ says that $\Phi_0(x,m) = P(x,m)$, the parallel translation operator which has scaling order 0 according to [13, Proposition 3.3.10]. The rest can be shown by an induction argument: assume that Φ_{i-1} has scaling order $2i-2$; then $B\Phi_{i-1}$ has scaling order $2i$. Since $\mathcal{R}$ vanishes at m, $\nabla_{\mathcal{R}}$ does not change the scaling order so that by the differential equation, Φ_i also has scaling order $2i$. According to (3.6), the sum of the first n terms in the asymptotic expansion defines a smooth section of $\mathbb{S}$. The remainder term $\mathcal{O}(t^{n+1})$ is a section of $S \boxtimes S^* \to M \times M \times \mathbb{R}$ which satisfies the condition of Lemma 3.8. Overall, H_τ defines an element in $C^\infty(\mathbb{T}M, \mathbb{S})$.

Next we shall show that $N_k(H_\tau) < \infty$ for all k. If $x \neq y$, it is well known that the heat kernel is rapidly decreasing as $t \to 0$. We only have to consider the case when (x,y) is very close to the diagonal. In that case, the estimate is done by using the asymptotic expansion (3.13). Indeed, $\Phi_i(x,y)$ are all bounded near the diagonal, and

$$
\left(1 + \frac{d(x,m)^2}{t^2}\right)^{k/2} e^{\frac{-d(x,m)^2}{4t^2\tau}}
$$

is uniformly bounded in (x,m,t) for any given k and $\tau > 0$. Therefore, $H_\tau \in \mathcal{S}(\mathbb{T}M, \mathbb{S})$.

The value of $H_\tau(X,m,0)$ is calculated for example in [3, Theorem 4.20] or [19, Proposition 12.25] with other means. However, (3.12) can also be obtained within the framework of [13], as we explain now. Because D^2 has Getzler order 2, the results of [13, Section 3.6] imply that t^2D^2 extends to an operator $\mathbf{D}^2$ on $\mathbb{T}M$, acting on sections of $\mathbb{S}$; over

$t = 0$, it is given by its Getzler symbol, as computed, e.g., in [19, Proposition 12.17]. For $X \in T_m M \subset \mathbb{T}M$, the formula is

$$\varepsilon_X(\mathbf{D}^2 s) = L \cdot \varepsilon_X(s) \quad \text{with } L = \sum_{i=1}^{d} \left(\frac{\partial}{\partial X_i} - \frac{1}{4} \sum_{j=1}^{d} R_{ij} X_j \right)^2. \tag{3.14}$$

Here $\varepsilon_X = \varepsilon_{(X,m)}$ is the point evaluation map (3.5) and the components X_i of X and the $R_{ij} \in \Lambda^2 T_m M$ are the components of the curvature tensor of M defined with respect to some orthonormal basis of $T_m M$.

We want to show that for any $\tau > 0$, we have

$$\varepsilon_X\big(\exp(-\tau \mathbf{D}^2)s\big) = \exp(-\tau L) \cdot \varepsilon_X(s). \tag{3.15}$$

It suffices to verify this for all s in the $\mathcal{A}(\mathbb{T}M)$-module $S(\mathbb{T}M)$ (remember Definition 3.3), as the general section is a linear combination over $C^\infty(\mathbb{T}M)$ of elements of $S(\mathbb{T}M)$ and one easily checks that the formula (3.15) still holds when replacing s by $f \cdot s$ for $f \in C^\infty(\mathbb{T}M)$. On $S(\mathbb{T}M)$, $\mathbf{D}^2$ acts as

$$\mathbf{D}^2 : \sum_p s_p t^{-p} \mapsto \sum_p D^2(s_p) t^{-p+2}.$$

Let $S_0(\mathbb{T}M)$ be the quotient of $S(\mathbb{T}M)$ by the subspace $t \cdot S(\mathbb{T}M)$. Since the point evaluations ε_X are zero on $t \cdot S(\mathbb{T}M)$, they descend to $S_0(\mathbb{T}M)$ and it suffices to verify (3.15) for $s \in S_0(\mathbb{T}M)$. However, since any section of $S \boxtimes S^*$ has scaling order at least $-n$, we see that the action of $(\tau \mathbf{D}^2)^N$ is zero on $S_0(\mathbb{T}M)$ for N sufficiently large. Hence both sides of (3.15) are actually given by an exponential series truncated at some finite N, so that (3.15) follows from (3.14).

On the other hand, by (3.7),

$$\varepsilon_X\big(\exp(-\tau \mathbf{D}^2)s\big) = \int_{T_m M} \varepsilon_{X-Y}(H_\tau) e^{-\frac{1}{2}[\kappa(X,Y)]} \varepsilon_Y(s) dY. \tag{3.16}$$

Equations (3.15) and (3.16) together imply

$$\exp(-\tau L)(X, Y) = \varepsilon_{X-Y}(H_\tau) e^{-\frac{1}{2}\kappa(X,Y)} \tag{3.17}$$

in particular, $\exp(-\tau L)(X, 0) = \varepsilon_X(H_\tau)$ which combined with Mehler's formula (see, e.g., [3, Section 4.2]) verifies (3.12). ■

Remark 3.10. Fix $m \in M$. The full integral kernel $\widetilde{H}_\tau(X, Y) := e^{-\tau L}(X, Y)$ of the heat operator $e^{-\tau L}$ on $T_m M$ is given by *Mehler's formula*,

$$\widetilde{H}_\tau(X, Y) = (4\pi)^{-n/2} \cdot \det\left(\frac{\tau R/2}{\sinh(\tau R/2)} \right)^{1/2}$$
$$\times \exp\left(-\left\langle X, \frac{R}{8} \coth\left(\frac{\tau R}{2}\right) X \right\rangle + \left\langle X, e^{\tau R/2} \frac{R}{4} \operatorname{cosech}\left(\frac{\tau R}{2}\right) Y \right\rangle - \left\langle Y, \frac{R}{8} \coth\left(\frac{\tau R}{2}\right) Y \right\rangle \right);$$

see [3, Section 4.2], which satisfies the convolution identity

$$\widetilde{H}_{\tau+\tau'}(X, Z) = \int_{T_m M} \widetilde{H}_\tau(X, Y)\widetilde{H}_{\tau'}(Y, Z)dY.$$

Now one can check that $\widetilde{H}_\tau(X, Y) = H_\tau(X - Y, m, 0)e^{-\frac{1}{2}[\kappa(X,Y)]}$, hence the element H_τ from above satisfies the *twisted* convolution identity

$$\begin{aligned} H_{\tau+\tau'}(X, m, 0) &= (H_\tau * H_{\tau'})(X, m, 0) \\ &= \int_{T_m M} H_\tau(X - Y, m, 0)H_{\tau'}(Y, m, 0)e^{-\frac{1}{2}[\kappa(X,Y)]}dY. \end{aligned}$$

Of course, this twisted convolution identity $H_{\tau+\tau'} = H_\tau * H_{\tau'}$ also follows from the semigroup property of $e^{-t^2\tau D^2}$, which holds for $t \neq 0$ and by continuity must continue to hold at zero. However, the above calculations show that the factor of $e^{-\frac{1}{2}[\kappa(X,Y)]}$ appearing in the formula (3.7) for the twisted convolution precisely accounts for the failure of the Mehler kernel to be translation invariant.

3.5. Proof of Theorem 2.10

Let M be a compact Riemannian spin manifold of even dimension n and $t > 0$. Given $\theta_1, \ldots, \theta_N$, the explicit formula (2.13) reveals that the corresponding Chern character $\mathrm{Ch}_t(\theta_1, \ldots, \theta_N)$ is a sum of terms of the form

$$(-1)^k t^{|\theta|+2k-N} \int_{\Delta_k} \mathrm{Str}\Big(e^{-t^2\tau_1 D^2} \prod_{p=1}^{k} F(\theta_{i_{p-1}+1}, \ldots, \theta_{i_p})e^{-t^2(\tau_{p+1}-\tau_p)D^2}\Big)d\tau, \quad (3.18)$$

where $k \leq N$ and $1 \leq i_1 < \cdots i_k \leq N$ are given, and where $|\theta| = |\theta_0| + \cdots + |\theta_N|$; we assume each θ_i to be homogeneous throughout. Recall moreover that

$$\begin{aligned} F(\theta) &= \big[D, c(\theta')\big] - c(d\theta') + c(\theta''), \\ F(\theta_1, \theta_2) &= (-1)^{|\theta_1|}\big(c(\theta_1' \wedge \theta_2') - c(\theta_1')c(\theta_2')\big). \end{aligned}$$

To prove Theorem 2.10, the goal is now to calculate the limit as $t \to 0$ of these terms. In fact, we will show that if $k < N$, the result is zero, while if $k = N$, we have

$$\begin{aligned} &\lim_{t\to 0}(-1)^N t^{|\theta|+N} \int_{\Delta_k} \mathrm{Str}\left(e^{-t^2\tau_1 D^2} \prod_{i=1}^{N} F(\theta_i)e^{-t^2(\tau_{i+1}-\tau_i)D^2}\right)d\tau \\ &\quad = \frac{(-1)^N}{(2\pi i)^{-n/2}N!} \int_X \widehat{A}(X) \wedge \theta_1'' \wedge \cdots \wedge \theta_N''. \end{aligned} \quad (3.19)$$

The proof of this will occupy the rest of this section.

Lemma 3.11. *Let $\theta, \theta_1, \theta_2 \in \Omega_{\mathbb{T}}(M)$ be homogeneous. Then each of the operators*

$$t^{|\theta|+1}F(\theta), \quad t^{|\theta_1|+|\theta_2|}F(\theta_1, \theta_2),$$

acting on sections of $S \boxtimes S^*$ *over* $M \times M \times \mathbb{R}^\times$ *with respect to the first variable, extends smoothly to an operator acting on sections of* $\mathbb{S}$ *over* $\mathbb{T}M$. *Moreover, over* $TM \times \{0\} \subset \mathbb{T}M$, *these extensions are given by*

$$\theta'' \wedge (-), \quad \textit{respectively } 0.$$

Proof. Each of the operators $F(\theta)$, $F(\theta_1, \theta_2)$ can (locally) be written as a composition of Clifford multiplication and covariant derivatives, therefore it follows from [13, Lemmas 3.6.2 and 3.6.3] that when multiplied by t^ℓ for ℓ less than or equal to their Getzler order, they extend smoothly to all of $\mathbb{T}M$ and their action over the $t = 0$ slice is given by their Getzler symbol. We deal with them in turn.

(a) Regarding the operator $F(\theta)$, suppose that $\theta = \theta' + \sigma\theta''$ has total degree $|\theta| = \ell$, meaning that $\theta' \in \Omega^\ell(M)$ and $\theta'' \in \Omega^{\ell+1}(M)$. A local calculation shows that in terms of a local orthonormal basis $e_1, \dots, e_n$, one has the formula

$$\big[D, c(\theta')\big] - c(d\theta') = -2\sum_{i=1}^{n} c(e_i \lrcorner \theta')\nabla_{e_i} + c(d^*\theta'),$$

where $\lrcorner$ denotes insertion of vectors into differential forms and d^* is the L^2-adjoint of the de-Rham differential; compare [3, Proposition 3.45]. Since for each i, $c(e_i \lrcorner \theta)$ can be written as a sum of composites of $\ell - 1$ Clifford multiplications, the right hand has Getzler order at most ℓ. We obtain that $[D, c(\theta')] - c(d\theta')$ is of lower order compared to $c(\theta'')$, which has Getzler order $\ell + 1$ (as $c(\theta'')$ can be written as a sum of composites of $\ell + 1$ Clifford multiplications). Hence $t^{\ell+1}F(\theta)$ extends smoothly to all of $\mathbb{T}M$, and over $t = 0$, we have $t^{\ell+1}F(\theta) = t^{\ell+1}c(\theta'')$. It then follows from [13, Lemma 3.6.2] that over $t = 0$, $t^{\ell+1}c(\theta'')$ is given by wedging with θ''.

(b) Looking at the formula for $F(\theta_1, \theta_2)$, it is clear that it has Getzler order at most $\ell_1 + \ell_2$ (where $\ell_i = |\theta_i|$), hence $t^{\ell_1+\ell_2}$ extends continuously to all of $\mathbb{T}M$, and over $t = 0$, it is given by wedging with

$$(-1)^{\ell_1}\big(\big[c(\theta_1' \wedge \theta_2')\big]_{\ell_1+\ell_2} - \big[c(\theta_1')\big]_{\ell_1} \wedge \big[c(\theta_2')\big]_{\ell_2}\big) = 0,$$

where $[-]_\ell$ denotes the ℓ-th order Clifford symbol (see Section 3.1). ∎

We are now in the position to prove the following result, which implies Theorem 2.10 and hence finishes the proof of the localization formula.

Proposition 3.12. *If* $k < N$, *the expression* (3.18) *converges to zero, as* $t \to 0$. *In the case* $k = N$, *the limit is given by the right-hand side of* (2.16).

Proof. Observe that since F vanishes whenever if more than two θ_i are inserted, the expression (3.18) can be non-zero only if $i_p - i_{p-1} \leq 2$ for all $p = 1, \dots, k$; we assume throughout that this is the case. We set

$$\ell_p = \begin{cases} |\theta_{i_p}| + 1 & \text{if } i_p - i_{p-1} = 1, \\ |\theta_{i_p-1}| + |\theta_{i_p}| & \text{if } i_p - i_{p-1} = 2. \end{cases}$$

Observe that ℓ_p is precisely the Getzler order of $F(\theta_{i_{p-1}+1}, \dots, \theta_{i_p})$, as seen in the proof of Lemma 3.11. Hence if we set

$$K_\tau^p = t^{n+\ell_p} F(\theta_{i_{p-1}+1}, \dots, \theta_{i_p}) e^{-t^2 \tau D^2},$$

then by Lemma 3.11 and Proposition 3.9, each K_τ^p (a priori defined only over $M \times M \times \mathbb{R}^\times$) extends smoothly to a section of the bundle $\mathbb{S} \to \mathbb{T}M$; in fact an element of $\mathcal{S}(\mathbb{T}M, \mathbb{S})$.

Because necessarily $i_p - i_{p-1} = 1$ for $2k - N$ many p and $i_p - i_{p-1} = 2$ for $N - k$ many p ($p \geq 1$), we have

$$\ell_0 + \cdots + \ell_k = |\theta| + 2k - N.$$

Therefore, using the factors of t in formula (3.18) together with the additional factors of t present in the formulas (3.7) for the twisted convolution and definition (3.8) for the t-supertrace, the expression from formula (3.18) can be written as

$$(-1)^k \int_{\Delta_k} \mathrm{Str}_t \left(H_{\tau_1} * K^1_{\tau_2 - \tau_1} * \cdots * K^k_{1-\tau_k} \right) d\tau, \tag{3.20}$$

where $*$ denotes the twisted convolution. By Lemma 3.7 (2), the integrand $H_{\tau_1} * K^1_{\tau_2-\tau_1} * \cdots * K^k_{1-\tau_k}$ is now an element of $\mathcal{S}(\mathbb{T}M, \mathbb{S})$, hence it can be evaluated at $t = 0$. But if $k < N$, then necessarily one of the K^p_τ contains a factor of $F(\theta_{i_p-1}, \theta_{i_p})$, which evaluates to zero over $TM \times \{0\}$, by Lemma 3.11. This shows that the term (3.18) converges to zero as $t \to 0$ (if $k < N$), as $\mathrm{Str}_t(A)$ only depends on the restriction of A to $TM \times \{0\}$ (see Theorem 3.4) and convolution preserves the fibers of $\mathbb{T}M \to \mathbb{R}$.

It is left to consider the case $k = N$. In this case, it follows from Lemma 3.11 and Proposition 3.9 that over $t = 0$,

$$K^i_\tau(X, m, 0) = t^{|\theta_i|+1} F(\theta_i) H_\tau(X, m, t)\big|_{t=0} = \theta''_p \wedge H_\tau(X, m, 0),$$

hence

$$\begin{aligned} H_{\tau_1} * K^1_{\tau_2-\tau_1} * \cdots * K^N_{1-\tau_N} &= H_{\tau_1} * \left(\theta''_1 \wedge H_{\tau_2-\tau_1}\right) * \cdots * \left(\theta''_N \wedge H_{1-\tau_N}\right) \\ &= \theta''_1 \wedge \cdots \wedge \theta''_N \wedge \left(H_{\tau_1} * H_{\tau_2-\tau_1} * \cdots * H_{1-\tau_N}\right). \end{aligned}$$

Here in the second step, we used that the θ_i can be pulled out of the convolution product since they are constant as functions on $T_m M$ and the H_τ are even. The twisted convolution identity of H_τ (see Remark 3.10) now implies that

$$H_{\tau_1} * H_{\tau_2-\tau_1} * \cdots * H_{1-\tau_N} = H_1.$$

By continuity of the t-supertraces (compare Lemma 3.7 (3)), the expression (3.20) is continuous in t. With a view on (3.8), evaluating at $t = 0$ therefore yields

$$\mathrm{Str}_0 \left(H_{\tau_1} * K^1_{\tau_2-\tau_1} * \cdots * K^N_{1-\tau_N} \right) = \left(\frac{2}{i}\right)^{n/2} \left(\int_M \theta''_1 \wedge \cdots \wedge \theta''_N \wedge H_1(0, -, 0) \right).$$

This shows that the integrand of (3.20) is in fact constant in τ, hence the integral over Δ_N just contributes a factor of $\mathrm{vol}(\Delta_N) = 1/N!$. Finally, comparing formula (3.12) with (2.17), one observes that $H_1(0, -, 0)$ is precisely $(4\pi)^{-n/2}$ times the $\hat{A}$-form on M. In total, we obtain (3.19), which finishes the proof of the theorem. ■

Acknowledgments. We are pleased to thank Nigel Higson for helpful discussions regarding this paper.

Funding. M. L. acknowledges funding from ARC Discovery Project grant FL170100020 under Chief Investigator and Australian Laureate Fellow Mathai Varghese.

References

[1] L. Alvarez-Gaumé, Supersymmetry and index theory. In *Supersymmetry (Bonn, 1984)*, pp. 1–44, NATO Adv. Sci. Inst. Ser. B: Phys. 125, Plenum, New York, 1985 MR 820487

[2] M. F. Atiyah, Circular symmetry and stationary-phase approximation. *Astérisque* **131** (1985), 43–59 Zbl 0578.58039 MR 816738

[3] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*. Grundlehren Math. Wiss. 298, Springer, Berlin, 1992 Zbl 0744.58001 MR 1215720

[4] J.-M. Bismut, Index theorem and equivariant cohomology on the loop space. *Comm. Math. Phys.* **98** (1985), no. 2, 213–237 Zbl 0591.58027 MR 786574

[5] J. Block and J. Fox, Asymptotic pseudodifferential operators and index theory. In *Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988)*, pp. 1–32, Contemp. Math. 105, Amer. Math. Soc., Providence, RI, 1990 Zbl 0704.58048 MR 1047274

[6] S. L. Cacciatori and B. Güneysu, Odd characteristic classes in entire cyclic homology and equivariant loop space homology. *J. Noncommut. Geom.* **15** (2021), no. 2, 615–642 Zbl 1494.55013 MR 4325717

[7] K.-t. Chen, Iterated integrals of differential forms and loop space homology. *Ann. of Math. (2)* **97** (1973), 217–246 Zbl 0227.58003 MR 380859

[8] A. Connes, *Noncommutative geometry*. Academic Press, San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[9] E. Getzler, J. D. S. Jones, and S. Petrack, Differential forms on loop spaces and the cyclic bar complex. *Topology* **30** (1991), no. 3, 339–371 Zbl 0729.58004 MR 1113683

[10] B. Güneysu and M. Ludewig, The Chern character of ϑ-summable Fredholm modules over dg algebras and localization on loop space. *Adv. Math.* **395** (2022), Paper No. 108143 Zbl 1486.58001 MR 4356812

[11] A. R. Haj Saeedi Sadegh and N. Higson, Euler-like vector fields, deformation spaces and manifolds with filtered structure. *Doc. Math.* **23** (2018), 293–325 Zbl 1393.57018 MR 3846057

[12] F. Hanisch and M. Ludewig, A rigorous construction of the supersymmetric path integral associated to a compact spin manifold. *Comm. Math. Phys.* **391** (2022), no. 3, 1209–1239 Zbl 1490.58011 MR 4405571

[13] N. Higson and Z. Yi, Spinors and the tangent groupoid. *Doc. Math.* **24** (2019), 1677–1720 Zbl 1429.58018 MR 4033830

[14] A. Jaffe, A. Lesniewski, and K. Osterwalder, Quantum K-theory. I. The Chern character. *Comm. Math. Phys.* **118** (1988), no. 1, 1–14 Zbl 0656.58048 MR 954672

[15] J. D. S. Jones, Cyclic homology and equivariant homology. *Invent. Math.* **87** (1987), no. 2, 403–423 Zbl 0644.55005 MR 870737

[16] M. Ludewig, Construction of the supersymmetric path integral: a survey. In *Differential geometry in the large*, pp. 241–259, London Math. Soc. Lecture Note Ser. 463, Cambridge Univ. Press, Cambridge, 2021 MR 4420793

[17] M. S. Narasimhan and S. Ramanan, Existence of universal connections. *Amer. J. Math.* **83** (1961), 563–572 Zbl 0114.38203 MR 133772

[18] D. Quillen, Algebra cochains and cyclic cohomology. *Inst. Hautes Études Sci. Publ. Math.* (1988), no. 68, 139–174 (1989) Zbl 0689.57026 MR 1001452

[19] J. Roe, *Elliptic operators, topology and asymptotic methods*. Pitman Research Notes in Mathematics Series 179, Longman Scientific & Technical, Harlow, 1988 Zbl 0654.58031 MR 960889

[20] Z. Yi, *Spinors and the tangent groupoid*. Ph.D. thesis, The Pennsylvania State University, 2019, https://etda.libraries.psu.edu/catalog/16662zuy106

Received 3 November 2020; revised 4 April 2021.

Matthias Ludewig
Fakultät für Mathematik, Universität Regensburg, Universitätsstraße 31, 93053 Regensburg, Germany; matthias.ludewig@mathematik.uni-regensburg.de

Zelin Yi
School of Mathematical Sciences, Tongji University, Shanghai 200092, P. R. China; zelin@tongji.edu.cn