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Unitary Harish-Chandra representations of real supergroups

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Abstract. We give conditions for unitarizability of Harish-Chandra super modules for Lie supergroups and superalgebras.

1. Introduction

Let $\mathfrak{g}$ be a real Lie superalgebra. It is natural to ask how to define the concept of (infinitesimal) *unitarity or unitarizability* for a super module V for $\mathfrak{g}$ and how to obtain, starting from V, a unitary module for G a Lie supergroup with $\mathfrak{g} = \mathrm{Lie}(G)$, G_0 simply connected. This problem, of great interest also in physics, was studied in several papers from a mathematical perspective, see [1, 12, 13, 17, 24, 29] and the references therein. From the physical point of view, in [11,16,28] (see also references therein) the realization of unitary representations as holomorphic sections on harmonic superspace and, more generally, on superflags is instrumental for their applications in supergravity theories.

Let γ be the representation of $\mathfrak{g}$ in V. We say γ is *unitary* if V is equipped with a hermitian product i.e. a positive definite hermitian form in which V_0 and V_1 are orthogonal, and the following conditions are met (see [3, 17] and also [24, 29]):

(U1) For all $Z \in \mathfrak{g}_0$, $i\gamma(Z)$ is symmetric on V.

(U2) For all $X \in \mathfrak{g}_1$, $\rho(X) := e^{-i\pi/4}\gamma(X)$ is symmetric on V.

These are not enough in general to define in the completion $\mathcal{H}$ of V a unitary representation of a Lie supergroup G, (see [3, Sec. 2.3] for its definition), with $\mathfrak{g} = \mathrm{Lie}(G)$, whose infinitesimal form on V is γ. Indeed, as was remarked in Nelson [25], this is already not enough in the classical setting, that is when $\mathfrak{g}_1 = 0$. In general, we need an additional condition.

(U3) There is an even unitary representation π_0 of G_0, the simply connected group defined by $\mathfrak{g}_0$, on the completion $\mathcal{H}$ of V such that $d\pi_0(Z)$ is defined on V for all $Z \in \mathfrak{g}_0$ and coincides with $\gamma(Z)$ on V; in the usual notation $V \subset \mathcal{H}$ and $\gamma(Z) \prec d\pi_0(Z)$, $Z \in \mathfrak{g}_0$ (see [27, Vol. 1, Ch. 8, Sec. 1, 2, 5] for definitions and notation).

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We recall here that $d\pi_0(Z)$ is the unique self adjoint operator on $\mathcal{H}$ such that $\pi_0(\exp(tZ)) = e^{itd\pi_0(Z)}$. Then, [3, Proposition 3] leads to the following theorem.

Theorem 1. *Let V be a module for a real Lie superalgebra $\mathfrak{g}$, via the representation γ such that conditions* (U1)–(U3) *are satisfied. Suppose that $V \subset C^\omega(\pi_0)$. Then each $\rho(X)$ ($X \in \mathfrak{g}_1$) is essentially self adjoint on V with $C^\infty(\pi_0) \subset D(\overline{\rho(X)})$, and there is a unique unitary representation $(\pi_0, \rho, \mathcal{H})$ of the Lie supergroup $(G_0, \mathfrak{g})$ in $\mathcal{H}$ such that $\rho(X)$ is the restriction to $C^\infty(\pi_0)$ of $\overline{\rho(X)}$ for all $X \in \mathfrak{g}_1$.*

The shortcoming of this theorem is that it assumes the existence of π_0. As we shall see, when $\mathfrak{g}_0$ is reductive and the modules are Harish-Chandra modules of $(\mathfrak{g}, \mathfrak{k})$-type, then we can dispense with (U3) entirely. As a notational convention, when we say that some module is a Harish-Chandra module, we assume it is already of $(\mathfrak{g}, \mathfrak{k})$-type, i.e., each vector in the module lies in a direct sum of a finite number of irreducible $\mathfrak{k}$-modules, with $\mathfrak{k}$ the maximal compact subalgebra in $\mathfrak{g}$, $\mathfrak{k} = \mathfrak{k}_0$ (see [5]). We also recall that $\mathfrak{k}$ is not always semisimple and so not all of its finite dimensional modules are completely reducible; the condition for complete reducibility is that each element of the center of $\mathfrak{k}$ acts semisimply in the representation space. We also shall use freely the super Harish-Chandra pairs (SHCP) terminology; for all the notation and preliminaries on supergeometry refer to [4, Ch. 4 and 7], [32], besides the classical references [2, 20–22].

We shall prove the following.

Theorem 2. *Let $\mathfrak{g}$ be a real Lie superalgebra with $\mathfrak{g}_0$ reductive acting via γ on a complex vector superspace V. Assume the following:*

(U1) *For all $Z \in \mathfrak{g}_0$, $i\gamma(Z)$ is symmetric on V.*

(U2) *For all $X \in \mathfrak{g}_1$, $\rho(X) := e^{-i\pi/4}\gamma(X)$ is symmetric on V.*

Let G_0 be the simply connected Lie group defined by $\mathfrak{g}_0$ and let G be the supergroup whose SHCP is $(G_0, \mathfrak{g})$. If V is finitely generated, then there is a unique unitary representation of the Lie supergroup G on the completion $\mathcal{H}$ of V, say $(\pi_0, \rho, \mathcal{H})$, such that $V \subset C^\omega(\pi_0)$.

We turn then to infinitesimal unitarity, where we need to introduce some extra hypotheses. Let $\mathfrak{g}_\mathbb{C}$ the complexification of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the super Cartan decomposition (see [6,7] for its definition). Assume $\mathfrak{g}_\mathbb{C}$ contragredient. This is to say, $\mathfrak{g}_\mathbb{C}$ is the sum of complex simple Lie algebras and complex simple Lie superalgebras of type (see [18])

$$A(m,n),\ m \neq n,\ B(m,n), C(n), D(m,n), D(2,1;\alpha), F(4), G(3).$$

Assume further that $\mathfrak{g}$ is *equal rank*, that is $\mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g})$ and that $\mathfrak{k}_\mathbb{C}$ has a non trivial center (see [5, 7]). Fix $\mathfrak{h}_\mathbb{C}$ a Cartan subalgebra of $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$. Let Δ be the root system of $\mathfrak{g}_\mathbb{C}$, $\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ the root space decomposition. The equal rank condition allows us to decompose $\mathfrak{k}_\mathbb{C}$, $\mathfrak{p}_\mathbb{C}$ into root spaces; we say that a root α is compact (non compact) if $\mathfrak{g}_\alpha \subset \mathfrak{k}_\mathbb{C}$ ($\mathfrak{g}_\alpha \subset \mathfrak{p}_\mathbb{C}$). Let $\beta : Z(\mathfrak{g}_\mathbb{C}) \to S(\mathfrak{h}_\mathbb{C})^W$ denote the Harish-Chandra homomorphism (see [23, Sec. 13.1], [19, Thm. 3]). We prove the following result.

Theorem 3. *Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ and let π_λ be the irreducible highest weight representation of highest weight λ. Then π_λ is unitary if and only if $(-i)^{|a|}\beta(a^*a)(\lambda) > 0$ for all $a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. In particular, it is necessary that $\lambda(H_\alpha) \geq 0$ for α compact and $\lambda(H_\alpha) \leq 0$ for α non compact even roots.*

In the end, we give an explicit example regarding $\mathfrak{g}_{\mathbb{C}} = \mathrm{osp}_{\mathbb{C}}(1|2)$ and its real form $\mathrm{osp}_{\mathbb{R}}(1|2)$ ([4, Appendix A]) proving the following.

Theorem 4. *Let V_t be the universal (Verma) $\mathrm{osp}_{\mathbb{C}}(1|2)$-module of highest weight t.*

(1) *Then V_t is irreducible. It is a unitary module for $\mathrm{osp}_{\mathbb{R}}(1|2)$ if and only if t is real and negative.*

(2) *All unitary representation of the real Lie supergroup*

$$\mathrm{Osp}_{\mathbb{R}}(1|2) = (\mathrm{SL}_2(\mathbb{R}), \mathrm{osp}(1|2))$$

are given on the completion $\mathcal{H}$ of V_t, and are such that $V_t \subset C^\omega(\pi_0)$, π_0 unitary representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ in $\mathcal{H}$ integrating $(V_t)_0$.

2. $(\mathfrak{g}, \mathfrak{k})$-supermodules and their unitarity

2.1. Harish-Chandra modules for reductive Lie algebras

Let $\mathfrak{g}$ be a real reductive Lie algebra. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. Let $\mathfrak{k} \subset \mathfrak{g}'$ be a maximal subalgebra of compact type, which means that it is the set of fixed points of a Cartan involution of $\mathfrak{g}'$. Let V be a $(\mathfrak{g}', \mathfrak{k})$-module. We recall that this means that V is a $\mathfrak{g}'$-module which, as a $\mathfrak{k}$-module, is a direct sum of finite dimensional irreducible $\mathfrak{k}$-modules. Recall also that $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Note that if V is irreducible, then $\mathfrak{c}$ acts through an additive character on V and so V is an irreducible $(\mathfrak{g},\mathfrak{k})$-module. This allows a reduction to the case when $\mathfrak{g}$ is itself semisimple. One knows from Harish-Chandra's work (slightly modified to include the reductive case) that if V is irreducible, or more generally, is finitely generated as a $\mathcal{U}(\mathfrak{g}')$-module on which $\mathfrak{c}$ acts semisimply through a finite number of additive characters, then the $\mathfrak{k}$-isotypical subspaces V_θ are all finite dimensional, where θ runs through the set $\hat{\mathfrak{k}}$ of equivalence classes of irreducible finite dimensional representations of $\mathfrak{k}$ ($\mathfrak{k}$ is not in general semisimple, see [30, Ch. 5] for details on representations of $(\mathfrak{g}, \mathfrak{k})$-modules). A basic question in the theory of $(\mathfrak{g}, \mathfrak{k})$-modules is whether such a module is the module of $\mathfrak{k}$-finite vectors of a Hilbert space representation (not necessarily unitary) of the simply connected group G defined by $\mathfrak{g}$. In his paper [14], Harish-Chandra proved this for irreducible $(\mathfrak{g}, \mathfrak{k})$-modules which satisfy a certain condition. He later verified that this condition is satisfied for highest weight $(\mathfrak{g}, \mathfrak{k})$-modules and so all such modules can be realized as the $\mathfrak{k}$-finite vectors of Hilbert space representations of G. This is actually sufficient for our purposes. However, it is possible to remove the special condition imposed by Harish-Chandra in [14, Thm. 4]. The general result is as follows (see [33, Ch. 8]).

Theorem 5. *Any $(\mathfrak{g}, \mathfrak{k})$-module V which is a direct sum of a finite number of irreducible $\mathfrak{g}$-submodules is identifiable as the module of $\mathfrak{k}$-finite vectors of a Hilbert space (not necessarily unitary) representation π of G. Moreover, $V \subset C^\omega(\pi)$.*

As mentioned earlier, when we deal with irreducible highest weight Harish-Chandra modules, the above general result is not needed, and Harish-Chandra already proves the above theorem for these modules. The question arises if an irreducible $(\mathfrak{g}, \mathfrak{k})$-module V, which is infinitesimally unitary, is the module of $\mathfrak{k}$-finite vectors of an irreducible unitary representation of G. In [14], Harish-Chandra proves that such a unitary representation exists and is unique up to equivalence, provided V is the module of $\mathfrak{k}$-finite vectors of a Banach space representation of G ([14, Thm. 9]). In view of the above remarks and results, we can now state the following theorem.

Theorem 6. *Let V be an irreducible $(\mathfrak{g}, \mathfrak{k})$-module defined by the representation γ of $\mathcal{U}(\mathfrak{g})$, which is unitary in the sense that there is a hermitian product $(\ ,\ )$ on V such that $i\gamma(X)$ is symmetric for all $X \in \mathfrak{g}$. Then, there is a unitary representation of G (unique up to unitary equivalence) in the completion $\mathcal{H}$ of V with respect to the norm defined by the hermitian product, such that V is the module of $\mathfrak{k}$-finite vectors in $\mathcal{H}$.*

2.2. Unitarity of super modules

We shall now present a proof that for a Lie superalgebra $\mathfrak{g}$ with $\mathfrak{g}_0$ reductive, conditions (U1) and (U2) of Section 1 are enough to guarantee the existence of a unitary representation of the Lie supergroup. Let $\mathfrak{g}$ be a Lie superalgebra with $\mathfrak{g}_0$ reductive. Write, as in Section 2.1, $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathfrak{c}_0$. $\mathfrak{k}_0$ is the subalgebra of $\mathfrak{g}_0'$ fixed by a Cartan involution.

Lemma 7. *Let V be a $(\mathfrak{g}_0, \mathfrak{k}_0)$-module. If V admits a hermitian product which is $\mathfrak{k}_0$-invariant, namely, elements of $\mathfrak{k}_0$ are skew symmetric with respect to it, then the isotypical subspaces V_θ are mutually orthogonal.*

Proof. Let V_1, V_2 be two irreducible $\mathfrak{k}_0$ -stable finite dimensional subspaces such that they carry inequivalent representations of $\mathfrak{k}_0$. We want to prove that $V_1 \perp V_2$. Let $W = V_1 \oplus V_2$. Let P be the orthogonal projection $V_1 \to V_2$ in V. We claim that P is a $\mathfrak{k}_0$-map. Let $u \in V_1$, write $u = x + y$ where $x \in V_2$, $y \in W$, $y \perp V_2$, or $y \in V_2^\perp \cap W$. Then $Pu = x$ and for $X \in \mathfrak{k}_0$, $XPu = Xx$. On the other hand, $Xu = Xx + Xy$ and we know that $Xx \in V_2$, $Xy \in W \cap V_2^\perp$. Hence $PXu = Xx = XPu$, proving the claim. This implies that $P = 0$, as otherwise P will be a non zero $\mathfrak{k}_0$-map between V_1 and V_2. ∎

Lemma 8. *Suppose that V is a unitary $(\mathfrak{g}_0, \mathfrak{k}_0)$-module such that the V_θ are all finite dimensional. Then for any submodule $W \subset V$, $W^\perp$ is also a submodule, and $V = W \oplus W^\perp$.*

Proof. It is only a question of proving that $V = W \oplus W^\perp$. The point is that V is in general not complete. Now $W = \oplus_\theta W_\theta$ where the W_θ are finite dimensional and mutually orthogonal, and $W_\theta \subset V_\theta$. Let W_θ' be the orthogonal complement of W_θ in V_θ. Since the isotypical subspaces of V are mutually orthogonal, it is clear that W_θ' is $\perp$ to all $V_{\theta'}$ for

$\theta' \neq \theta$. Thus $W'_\theta \subset W^\perp$. Since this is true for all θ, we see that $W \oplus W^\perp \supset W_\theta \oplus W'_\theta = V_\theta$ for all θ (as the V_θ are finite dimensional). So $W \oplus W^\perp = V$. ∎

Lemma 9. *If V is as in the previous lemma, then V is the orthogonal direct sum of irreducible submodules.*

Proof. We shall first show that if $W \subset V$ is any submodule, then W has an irreducible submodule. This is a standard argument of Harish-Chandra. Consider pairs (W', θ) for submodules $W' \subset W$ and θ such that $W'_\theta \neq 0$. Among these choose one for which W'_θ has the smallest dimension; let (W', θ) be the corresponding pair. Let W'' be the cyclic submodule of W' generated by W'_θ. If L is a proper submodule of W'', then we claim that $L \cap W'_\theta$ is either 0 or equal to W'_θ. Otherwise $\dim(L_\theta)$ is positive and strictly less than $\dim(W'_\theta)$, a contradiction. It cannot equal W'_θ, as then $L = W''$. So $L \cap W'_\theta = 0$, hence $L \perp W'_\theta$. In other words, all proper submodules of W'' are orthogonal to W'_θ, showing that their sum is still proper. Let Z denote this sum. Then $W'' \cap Z^\perp$ is an irreducible submodule of W''.

This proves the existence of irreducible submodules of V. Let (V_i) be a maximal family of mutually orthogonal irreducible submodules of V. If $Y := \bigoplus_i V_i \neq V$, then $Y^\perp$ will contain an irreducible submodule, contradicting the maximality of (V_i). Hence the lemma. ∎

Corollary 10. *Let the notation be as above. If V is finitely generated, then V is an orthogonal direct sum of finitely many irreducible submodules.*

Proof. Each generator lies in a finite sum of the V_i. Since there are only finitely many generators, the corollary follows. ∎

We are now ready to prove our main result for this section.

Proof of Theorem 2. Since $\mathfrak{g}_0$ leaves invariant V_0, V_1 separately, π_0 can be constructed separately on the closures of V_0 and V_1, by Theorem 5 in Section 2.1, and so the full π_0 is even. We know that $V \subset C^\omega(\pi_0)$, again by Theorem 5. Theorem 1 of Section 1 now proves the present theorem. ∎

Remark 1. In the special case of highest weight modules, the proof of unitarizability is simpler. In view of our corollary to Lemma 9, it is enough to show, besides (U1) and (U2), only that the V_i are highest weight modules for $\mathfrak{g}_0$, because the conditions in Corollary 10 are automatically verified (see [23, Ch. 8]).

2.3. Construction of Harish-Chandra modules for $(\mathfrak{g}, \mathfrak{k})$

Apart from the highest weight modules we have not produced any other Harish-Chandra modules (see [5]). In this section, we do precisely this. First notice that V_0, V_1 are Harish-Chandra modules for $(\mathfrak{g}_0, \mathfrak{k})$, $(\mathfrak{k} = \mathfrak{k}_0)$. We now need some preliminary remarks.

Let M be a Harish-Chandra module for $\mathfrak{g}_0$ and define $V := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M$.

By the Poincaré–Birkhoff–Witt theorem, if $X_1, X_2, \ldots, X_r$ is a basis for $\mathfrak{g}_1$, and L is the span of all the $X_{i_1} \cdots X_{i_m}$ where $i_1 < \cdots < i_m, m \leq r$, then $\mathcal{U}(\mathfrak{g}) = L\mathcal{U}(\mathfrak{g}_0)$. Although $\mathfrak{g}_1$ is stable under $\mathrm{ad}(\mathfrak{g}_0)$, this is not true for L. Let R be the linear span of all monomials $X_{i_1} \cdots X_{i_m}$ where the i's are not ordered and satisfy only $1 \leq i_1, i_2, \ldots, i_m \leq r$, $m \leq r$. Then R is finite dimensional, stable under $\mathrm{ad}(\mathfrak{g}_0)$, graded, and $R\mathcal{U}(\mathfrak{g}_0) = \mathcal{U}(\mathfrak{g})$. Hence

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M = R \otimes_{\mathcal{U}(\mathfrak{g}_0)} M.$$

The action of $\mathcal{U}(\mathfrak{g})$ on V is by the left of the first factor. Since R is $\mathrm{ad}(\mathfrak{g}_0)$-stable, the action of $\mathfrak{g}_0$ is the tensor product of the adjoint action on R and the action on M. We recall a well-known result. If p, q, r are three irreducible representations of $\mathfrak{k}$, we write $p < q \otimes r$ if p occurs in $q \otimes r$. Then

$$p < q \otimes r \quad \Longleftrightarrow \quad r^* < q \otimes p^*$$

where a^* is the dual representation of a. This follows from the fact that $p < q \otimes r$ if and only if $q \otimes r \otimes p^*$ contains the trivial representation, and hence if and only if $r^* < q \otimes p^*$.

Proposition 11. *Let M be a Harish-Chandra module for $\mathfrak{g}_0$. Then*

$$V := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M$$

is a Harish-Chandra module for $(\mathfrak{g}, \mathfrak{k})$.

Proof. We must show that for any irreducible class p of $\mathfrak{k}$, $\dim(V_p) < \infty$. Let $r_1, \ldots, r_t$ be the irreducible classes in R. M is the direct sum of the M_q for the various irreducible classes q of $\mathfrak{k}$, and we know that $\dim(M_q) < \infty$ for all q. Now, by our remark above, p occurs in $r \otimes q$ if and only if $q^* < r \otimes p^*$. Taking $r = r_1, \ldots, r_t$ and fixing p, this gives only finitely many choices for q. Let Q be the finite set of q such that $q^* < r_j \otimes p^*$ for some $j = 1, 2, \ldots, t$. Hence

$$V_p \subset R \otimes \bigoplus_{q \in Q} M_q,$$

showing that $\dim(V_p) < \infty$. ∎

Remark 2. By a slight variation of the argument in Lemma 9, we can show that V has subquotients which are irreducible. Starting with a module M for which the weight spaces are not all finite dimensional, one of the subquotients of a finite composition series for V will have this property and so will not be a highest weight module. These modules were studied in [8, 9] for ordinary Lie algebras and the above theory allows us to build non highest weight Harish-Chandra modules for Lie superalgebras. We plan to explore this further in a forthcoming paper.

3. Infinitesimal unitarity

3.1. Harish-Chandra homomorphism

Let $\mathfrak{g}_{\mathbb{C}}$ be a contragredient complex Lie superalgebra, $\mathfrak{h}_{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (see [18]). The Harish-Chandra homomorphism

$$\beta\colon Z(\mathfrak{g}_{\mathbb{C}}) \to S(\mathfrak{h}_{\mathbb{C}})^W \tag{1}$$

identifies the center $Z(\mathfrak{g}_{\mathbb{C}})$ of the universal enveloping algebra with the subalgebra $I(\mathfrak{h}_{\mathbb{C}})$ of $S(\mathfrak{h}_{\mathbb{C}})^W$ (see [19]),

$$I(\mathfrak{h}_{\mathbb{C}}) = \left\{\phi \in S(\mathfrak{h}_{\mathbb{C}})^W \mid \phi(\lambda + t\alpha) = \phi(\lambda), \forall \lambda \in \langle\alpha\rangle^{\perp}, \alpha \text{ isotropic}, \forall t \in \mathbb{C}\right\}.$$

For any $\mu \in \mathfrak{h}_{\mathbb{C}}^*$, let $\mathcal{U}[\mu]$ be the subspace of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ given by

$$\mathcal{U}[\mu] = \left\{a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \mid [H, a] = \mu(H)a, \forall H \in \mathfrak{h}_{\mathbb{C}}\right\}.$$

Then $\mathcal{U}[0]$ is a subalgebra, $Z(\mathfrak{g}_{\mathbb{C}}) \subset \mathcal{U}[0]$, and $(\mathcal{U}[\mu])$ is a grading of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$; moreover, $\mathcal{U}[\mu] \neq 0$ if and only if μ is in the $\mathbf{Z}$-span of the roots. If $\gamma_1, \ldots, \gamma_t$ is an enumeration of the positive roots, $\Delta^+ = \{\gamma_1, \ldots, \gamma_t\}$ and (H_i) is a basis for $\mathfrak{h}_{\mathbb{C}}$, then elements of $\mathcal{U}[0]$ are linear combinations of monomials

$$X_{-\gamma_1}^{p_1} \cdots H_1^{c_1} \cdots X_{\gamma_1}^{n_1} \tag{2}$$

with $(p_1 - n_1)\gamma_1 + \cdots = 0$. It is then clear that every term occurring in such a linear combination must necessarily have some $p_i > 0$ except those that are just monomials in the H_i alone. So for any $u \in \mathcal{U}[0]$ we have an element $\beta(u) \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ such that

$$u \cong \beta(u) (\mathrm{mod}\, \mathcal{P}), \qquad \mathcal{P} = \sum_{\gamma > 0} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}_{\gamma}, \quad \gamma \in \Delta^+. \tag{3}$$

Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. The action of u on the Verma module V_λ must leave the weight spaces stable since it commutes with $\mathfrak{h}_{\mathbb{C}}$, and so applying it to the highest weight vector v_λ we see that $uv_\lambda = \beta(u)(\lambda)v_\lambda$ where we are identifying $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ with the algebra of all polynomials on $\mathfrak{h}_{\mathbb{C}}^*$, so that $\beta(u)(\lambda)$ makes sense. It follows from this that if $u \in U(\mathfrak{h}_{\mathbb{C}}) \cap \mathcal{P}$ then $u(\lambda) = 0$ for all λ and so $u = 0$, i.e., $\mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \cap \mathcal{P} = 0$. Hence $\beta(u)$ is uniquely determined by equation (3).

We extend the homomorphism $\beta : \mathcal{U}[0] \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ to a linear map $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ by making it 0 on $\mathcal{U}[\mu]$ for $\mu \neq 0$.

3.2. Hermitian forms

Let V be a complex super vector space. A *hermitian form* on V is a complex valued sesquilinear form $(\,,\,)$ (linear in the first, antilinear in the second argument) such that

$$(u, v) = (-1)^{|u||v|}\overline{(v, u)}, \quad \forall u, v \in V \tag{4}$$

and $(u, v) = 0$ for $|u| \neq |v|$, where $|u|$ denotes the parity of an homogeneous element $u \in V$ (see [32, p. 111] and [10, Sec. 4]). If X is an endomorphism of V, we define its *adjoint* X^* as

$$(Xu, v) = (-1)^{|u||X|}(u, X^*v). \tag{5}$$

One can immediately verify that (see [32, p. 110])

$$\langle u, v\rangle = \begin{cases} i(u, v) & |u| = |v| = 1, \\ (u, v) & \text{otherwise}, \end{cases} \tag{6}$$

is an ordinary hermitian form. If $X^\dagger$ is the adjoint with respect to this ordinary hermitian form, we have that $X^* = i^{|X|}X^\dagger$. In fact, taking $|u| = |X| = 1$, $|v| = 0$, we have $(Xu, v) = -(u, X^*v)$ and

$$(Xu, v) = -i\langle Xu, v\rangle = -i\langle u, X^\dagger v\rangle. \tag{7}$$

A similar calculation is done if $|u| = 0$ and $|X| = |v| = 1$.

The form $(\,,\,)$ is a *hermitian product* on V if $(\,,\,)$ and $i(\,,\,)$ are positive definite on V_0 and V_1 respectively, i.e. if the ordinary form $\langle\,,\,\rangle$ is positive definite on V.

Let V be a $\mathfrak{g}$-module, $\mathfrak{g}$ a real Lie algebra, via the representation π. V (or π) is said to be *unitary* if there is a hermitian product $(\,,\,)$ for V such that

$$(\pi(X)u, v) = -(-1)^{|u||X|}(u, \pi(X)v), \qquad (u, v \in V, X \in \mathfrak{g}), \tag{8}$$

(see [32, p. 111]). This implies

$$\pi(X)^* = \begin{cases} -\pi(X), & |X| = 0, \\ +\pi(X), & |X| = 1, \end{cases} \qquad \pi(X)^\dagger = \begin{cases} -\pi(X), & |X| = 0, \\ -i\pi(X), & |X| = 1. \end{cases} \tag{9}$$

As one can readily check, this is equivalent to (U1), (U2) in Section 1, with (6) as hermitian product there. In fact, while condition (U1) regards the ordinary case, condition (U2) is expressed for $|X| = |u| = 1$ (similarly for $|X| = |v| = 1$) as

$$\langle e^{-i\pi/4}\pi(X)u, v\rangle = \langle u, e^{-i\pi/4}\pi(X)v\rangle,$$

that is

$$\langle \pi(X)u, v\rangle = i\langle u, \pi(X)v\rangle.$$

This implies the condition of unitarity to be $\pi(X)^\dagger = -i\pi(X)$, in agreement with (9).

Let $\mathfrak{g}$ be a real form of the contragredient complex superalgebra $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition. We assume $\mathfrak{g}_\mathbb{C}$ to satisfy the *equal rank condition*

$$\mathfrak{h}_\mathbb{C} \subset \mathfrak{k}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$$

for a fixed Cartan subalgebra $\mathfrak{h}_\mathbb{C} = (\mathfrak{h}_\mathbb{C})_0$. Assume also that $\mathfrak{k}_\mathbb{C}$ has a non trivial center. Then $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{p}_\mathbb{C}$ decompose into the sum of root spaces and the root system of $\mathfrak{g}_\mathbb{C}$ has admissible positive systems and we fix P one of such (see [7]). We can extend the

antiautomorphism $X \to -(-1)^{|X|}X$ on $\mathfrak{g}$ to an antiautomorphism of $\mathcal{U}(\mathfrak{g})$. Then, this antiautomorphism can be furtherly extended to a conjugate linear antiautomorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, that we denote by $a \to a^\star$ and call the *adjoint*. It has the following properties:

(1) $a^{\star\star} = a$,

(2) $(ab)^\star = (-1)^{|a||b|}b^\star a^\star$,

(3) $a^\star$ is conjugate linear in a,

(4) $X^\star = -(-1)^{|X|}X$ for all $X \in \mathfrak{g}$.

It is uniquely determined by these requirements. We then can extend the unitary condition for a representation expressed in (8),

$$(\pi(X)u, v) = (u, \pi(X)^\star v), \qquad X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}).$$

3.3. Unitary highest weight representations

We now wish to give a criterion for a highest weight representation of $\mathfrak{g}_{\mathbb{C}}$ to be unitary. We shall follow closely [15].

Lemma 12. *Let π_λ be a unitary highest weight representation of $\mathfrak{g}_{\mathbb{C}}$ of highest weight λ. Then $(-i)^{|a|}\beta(a^*a)(\lambda) > 0$ for all $a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.*

Proof. Let v be the highest weight vector. It is not restrictive to assume v to be even. By definition of β we have

$$(av, v) = \beta(a)(\lambda)(v, v), \qquad a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}).$$

Hence,

$$0 < i^{|a|}(av, av) = i^{|a|}(-1)^{|a|}(a^*av, v) = (-i)^{|a|}\beta(a^*a)(\lambda)(v, v),$$

which gives our claim, since $(v, v) = \langle v, v\rangle > 0$. ∎

To ease the notation let $\beta_\lambda(a) := \beta(a)(\lambda)$, $a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

Lemma 13. *Assume $(-i)^{|a|}\beta_\lambda(a^\star a) \geq 0$ for all $a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. Then*

$$(w, z) = (-i)^{|z||w|}\beta_\lambda(z^\star w), \qquad w, z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), \tag{10}$$

defines a semipositive definite supersymmetric sesquilinear form on $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, whose radical $\mathcal{R}$ is a left ideal.

Proof. By (10) and by the definition of β and $\star$ we immediately have that $(\,,\,)$ is sesquilinear: linear in the first and antilinear in the second argument. Notice that $(w, z) = 0$ if $|z| \neq |w|$. In fact, if $|z| \neq |w|$, $|z^\star w| = 1$, hence $z^\star w \notin \mathcal{U}[0]$, which consists of even elements only and β_λ is zero on $\mathcal{U}[\mu]$, for $\mu \neq 0$. Moreover, since $(-i)^{|a|}\beta(a^\star a)(\lambda) \geq 0$, we

have that $\langle a, a\rangle \geq 0$, by (6). By a standard argument in ordinary linear algebra (see [26]), this implies that $\langle a, b\rangle = \overline{\langle b, a\rangle}$, $|a| = |b|$ and this concludes the first part of the proof.

Let $\mathcal{R}$ be the set of $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ with $\|z\| := \sqrt{\langle z, z\rangle} = 0$. If $z, z' \in \mathcal{R}$, that is $\|z\| = \|z'\| = 0$, by $\|z + z'\| \leq \|z\| + \|z'\|$, we immediately have that $\mathcal{R}$ is a subspace. Furthermore, if $b^\star = b'$, for $b, b' \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, we have

$$(w, bz) = (-i)^{|w||bz|}\beta_\lambda((bz)^\star w) = \pm(-i)^{|w||b||z|}\beta_\lambda(z^\star b^\star w) = \pm(b'w, z).$$

Hence $\|(w, bz)\| \leq \|b'w\|\|z\|$. If $z \in \mathcal{R}$, i.e. $\|z\| = 0$ and $w = bz$, we get $\|bz\| = 0$, hence $\mathcal{R}$ is a left ideal. ∎

Lemma 14. *Let the notation be as above. Assume* $(-i)^{|a|}\beta_\lambda(a^\star a) \geq 0$ *for all* $a \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. *Then* $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathcal{R}$ *is a unitary representation of* $\mathfrak{g}_{\mathbb{C}}$ *with highest weight* λ.

Proof. One can check right away that $(\,,\,)$ is well defined on $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathcal{R}$. To prove our claim, we need to show $\mathcal{R} = \mathcal{M}_\lambda$, the (unique) maximal (left) ideal containing

$$\mathcal{P}_\lambda = \sum_{\gamma \in P} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}_{\mathbb{C}\gamma} + \sum_{\gamma \in P} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})(H_\gamma - \lambda(H_\gamma)1).$$

We have $\mathcal{P}_\lambda \subset \mathcal{R}$. They are both left ideals, so it is enough to show $X_\alpha \in \mathcal{R}$ for $\alpha > 0$ and $H_\gamma - \lambda(H_\gamma)1 \in \mathcal{R}$, the latter being an ordinary statement, so true for the ordinary theory. Notice that $(X_\alpha, v) = \beta_\lambda(v^* X_\alpha) = 0$ because of (3), hence $X_\alpha \in \mathcal{R}$.

Now, let $\mathcal{M}'$ be a proper maximal ideal containing $\mathcal{R}$. We want to show $\mathcal{M}' = \mathcal{R}$. We first notice that it is stable under the $\mathfrak{h}_{\mathbb{C}}$ action, in fact

$$[H, m] = Hm - m(H - \lambda(H)) - \lambda(H)m \in \mathcal{M}', \qquad H \in \mathfrak{h}_{\mathbb{C}}, m \in \mathcal{M}'.$$

By a standard fact, then also m_0, the $\mathcal{U}[0]$ component of m is in $\mathcal{M}'$. Then, by (3), $m_0 \equiv h \bmod (\mathcal{P})$ and $h \equiv \beta_\lambda(h) \bmod (\mathcal{P}_\lambda)$, so that $m_0 \equiv \beta_\lambda(h) \bmod (\mathcal{P}_\lambda)$ for some $h \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ ($\mathcal{P} \subset \mathcal{P}_\lambda$). Since $\mathcal{P}_\lambda \subset \mathcal{R} \subset \mathcal{M}'$, we have $\beta_\lambda(h) \in \mathcal{M}'$, and being a complex number, this tells that $\beta_\lambda(h) = 0$, otherwise $\mathcal{M}'$ would not be a proper ideal. Hence, also $\beta_\lambda(m_0) = \beta_\lambda(h) = 0$. Now, let $z \in \mathcal{M}'$. Since $X_\alpha^\star = c_\alpha X_{-\alpha}$ for any root vector X_α, α a root of $\mathfrak{g}_{\mathbb{C}}$ (see [10, Sec. 4]), we have $z^\star z \in \mathcal{U}[0]$. Taking $m_0 = z^\star z$, for any $z \in \mathcal{M}'$, this gives $(-i)^{|z|}\beta_\lambda(z^\star z) = \langle z, z\rangle = 0$, so $\mathcal{M}' = \mathcal{R}$. ∎

We are ready for the main result of this section.

Proof of Theorem 3. The first statement is immediate from the previous lemmas. The second statement comes from the ordinary result in [15] and easy calculations. ∎

4. Irreducible representations of $\mathrm{osp}_{\mathbb{R}}(1|2)$

4.1. Introductory remarks

We present here some calculations on highest weight Harish-Chandra modules for $\mathfrak{g}_{\mathbb{C}} = \mathrm{osp}_{\mathbb{C}}(1|2)$ and the unitary ones of $\mathfrak{g} = \mathrm{osp}_{\mathbb{R}}(1|2)$.

Let $\mathbf{N} = \{0, 1, 2, \ldots\}$. We assume that $t \notin \mathbf{N}$. The Lie superalgebra $\mathfrak{g}_{\mathbb{C}}$ consists of matrices

$$\begin{pmatrix} 0 & \xi & \eta \\ \eta & a & b \\ -\xi & c & -a \end{pmatrix}$$

where ξ, η are complex odd variables, a, b, c are complex even variables. The real form $\mathfrak{g}$ consists of the real Lie superalgebra of matrices

$$\begin{pmatrix} 0 & \xi & -i\overline{\xi} \\ -i\overline{\xi} & ia & b \\ -\xi & -\overline{b} & -ia \end{pmatrix}$$

where the variables ξ, b are still complex, a is real, and bar denotes complex conjugation (see [4, Appendix A] for notation).

The complex basis of $(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{sl}(2)$ is the standard one, $H = E_{22} - E_{33}$, $X = E_{23}$, $Y = E_{32}$, E_{ij} denoting the elementary matrices (see [31] for notation). The complex basis for the odd part $(\mathfrak{g}_{\mathbb{C}})_1$ is $\{x, y\}$ where

$$x = E_{13} + E_{21}, \qquad y = E_{12} - E_{31}.$$

For the real form, the even part has real basis $\{iH, X + Y, i(X - Y)\}$ and the odd part has real basis

$$\tilde{x} = -ix + y, \qquad \tilde{y} = -x + iy.$$

4.2. Verma modules for $(\mathfrak{g}_{\mathbb{C}})_0$ with highest weight $t \notin \mathbf{N}$.

We recall here the ordinary theory. Let W_t be the Verma module for $(\mathfrak{g}_{\mathbb{C}})_0$ of highest weight t. Then W_t has basis $\{v_t, v_{t-2}, \ldots\}$ where $v_t \neq 0$, $Xv_t = 0$, $v_{t-2r} = Y_r v_t$. One knows that all the v_{t-2r} are non zero, because of the identity

$$XY^{r+1} = Y^{r+1}X + (r+1)Y^r(H - r)$$

in $\mathcal{U}((\mathfrak{g}_{\mathbb{C}})_0)$, established easily by induction on r. This shows that

$$Xv_{t-2(r+1)} = (r+1)(t-r)v_{t-2r}. \tag{11}$$

Since $t \notin \mathbf{N}$, the factor $(r+1)(t-r)$ is not zero for any integer $r \geq 0$, it follows that if some $v_{t-2(r+1)} = 0$, then $v_{t-2r} = 0$, so that we eventually get $v_t = 0$. That this is irreducible already is seen because of (11). Indeed, (11) shows that starting with any v_{t-2r},

we can reach v_t by applying X repeatedly. Thus the Verma modules W_t are already irreducible. We now want to determine when the W_t are unitary. By unitary we mean the existence of a hermitian product such that

$$(Zu, v) = -(u, Zv)$$

for all Z in the real form of $\mathfrak{sl}(2)$, and $u, v \in W_t$, i.e., for $Z = iH$, $X + Y$, $i(X - Y)$. The main idea is to transfer the condition for unitarity to the complex Lie algebra $\mathfrak{sl}(2)$. For the Verma modules W_t, unitarity is equivalent to assuming that the v_{t-2r} are mutually orthogonal and $X^* = -Y$ or $Y^* = -X$ or both. In fact, the condition is that $H^* = H$, $(X + Y)^* = -(X + Y)$, $(X - Y)^* = X - Y$.

Proposition 15. *W_t is unitary if and only if t is real and $t < 0$.*

Proof. Recall that $t \notin \mathbf{N}$. Let W_t be unitary. Then

$$(v_{t-2}, v_{t-2}) = (Yv_t, v_{t-2}) = -(v_t, Xv_{t-2}).$$

But

$$Xv_{t-2} = XYv_t = YXv_t + Hv_t = tv_t.$$

Hence $(v_{t-2}, v_{t-2}) = -t > 0$ if we normalize $(v_t, v_t) = 1$ (possible). Hence $-t > 0$. For the converse we must, when $t < 0$, define a unique hermitian product such that $(v_t, v_t) = 1$ and $X^* = -Y$. The v_{t-2r} are to be mutually orthogonal and so we need to determine the $N(r) := (v_{t-2r}, v_{t-2r})$ inductively so that $X^* = -Y$ and all the $N(r) > 0$. The requirement $X^* = -Y$ forces the relation (by (11))

$$(v_{t-2r}, v_{t-2r}) = (Yv_{t-2(r-1)}, v_{t-2r}) = -r(t - r + 1)(v_{t-2(r-1)}, v_{t-2(r-1)})$$

or

$$N(r) = -r(t - r + 1)N(r - 1), \qquad N(1) = 1,$$

the second being the normalization $(v_t, v_t) = 1$. We define $N(r)$ inductively by this and note that for $t < 0$ we have $N(r) > 0$ for all r, since the factor $-r(t - r + 1)$ is always > 0 for $r \geq 1$, as $t < 0$. The hermitian product is now well defined and positive definite. It is now only a question of verifying that $X^* = -Y$. For this we only need to check $(Yv_{t-2(r-1)}, v_{t-2r}) = -(v_{t-2(r-1)}, Xv_{t-2r})$ as all other hermitian products needed are zero. But the left-hand side is $N(r)$ while the right-hand side is

$$-r(t - r + 1)(v_{t-2(r-1)}, v_{t-2(r-1)}),$$

which is $-r(t - r + 1)N(r - 1)$, and these are equal by definition of $N(r)$. ∎

4.3. Super Verma modules for $\mathfrak{g}_{\mathbb{C}}$

Now, we report here for completeness some preliminary results. Let $t \notin \mathbf{N}$ where $\mathbf{N} = \{0, 1, 2, \ldots\}$ and let V_t be the $\mathfrak{g}_{\mathbb{C}}$-module with highest weight t and highest weight vector v_t.

Lemma 16. *Let $t \notin \mathbf{N}$ where $\mathbf{N} = \{0, 1, 2, \ldots\}$. Then the super Verma module V_t with highest weight t is irreducible. Let $W_t = \mathcal{U}((\mathfrak{g}_{\mathbb{C}})_0)v_t$ and $W_{t-1} = \mathcal{U}((\mathfrak{g}_{\mathbb{C}})_0)v_{t-1}$ with $v_{t-1} := yv_t$. Then W_t and W_{t-1} are irreducible Verma modules for $(\mathfrak{g}_{\mathbb{C}})_0$ and $V_t = W_t \oplus W_{t-1}$.*

Proof. By the Poincaré–Birkhoff–Witt theorem, $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}((\mathfrak{g}_{\mathbb{C}})_0)\{1, x, y, yx\}$. Hence

$$V_t = \mathcal{U}((\mathfrak{g}_{\mathbb{C}})_0)v_t + \mathcal{U}((\mathfrak{g}_{\mathbb{C}})_0)v_{t-1}.$$

If $v_{t-1} = 0$ then $xv_t = yv_t = 0$, hence, as $H = xy + yx$, we have $Hv_t = tv_t = 0$, showing that $t = 0$. Also, if $Xv_{t-1} \neq 0$, then it has weight $t + 1$ which is impossible. The modules $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})v_t$, $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})v_{t-1}$ are then highest weight non zero modules, of highest weights t, $t - 1$. Hence, by our assumption that $t \notin \mathbf{N}$, they are Verma modules and irreducible. Note the sum is direct since H has disjoint spectra in the two pieces. Hence the result follows. ∎

Corollary 17. *Let the notation be as above. V_t has basis $\{v_t, v_{t-1}, \ldots\}$ where*

$$v_{t-r} = y^r v_t.$$

Proof. Recall that $y^2 = -Y$. Given $t \notin \mathbf{N}$ there is only one structure of a super $\mathfrak{g}_{\mathbb{C}}$-module for $W_t \oplus W_{t-1}$, namely the super Verma with highest weight t. ∎

Lemma 18. *Let the notation be as above. In $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ we have*

$$xy^{2m} = y^{2m}x - my^{2m-1}, \qquad xy^{2m+1} = -y^{2m+1}x + y^{2m}(H - m).$$

In particular, in V_t,

$$xv_{t-m} = c_m v_{t-m+1}, \qquad c_{2m} = -m, \qquad c_{2m+1} = t - m.$$

Proof. Since $xy = -yx + H$ in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, we have, by direct calculation, $xy^2 = y^2x - y$ and $xy^3 = -y^3x + y^2(H - 1)$. Hence the results are true for $m = 1$. We use induction on m. We have

$$xy^{2m+2} = xyy^{2m+1} = (-yx + H)y^{2m+1} = y^{2m+2}x - (m + 1)y^{2m+1}$$

and

$$xy^{2m+3} = xyy^{2m+2} = (-yx + H)y^{2m+2} = -y^{2m+3}x + y^{2m+2}(H - m - 1)$$

by direct calculation. The induction is complete. The formulae for V_t are immediate consequences by applying them to $v_{t-m} = y^{t-m}v_t$. ∎

4.4. Unitary super Verma modules for $\mathfrak{g}_{\mathbb{C}}$

The main idea now is to transform the condition for unitarity to the complex setting. For the ordinary Verma modules W_t, unitarity is equivalent to $X^* = -Y$ or $Y^* = -X$, or both. In fact, the condition is that $H^* = H$, $(X+Y)^* = -(X+Y)$, $(X-Y)^* = X-Y$. For supermodules we impose, following [3] and (U2) as in Section 1, the condition ζZ is symmetric in the hermitian product where $\zeta = e^{-i\pi/4}$ (see [3]). As in the Verma case, we must convert this definition into a condition on the complex basis for $\mathfrak{g}_{\mathbb{C}}$. The condition is that $\zeta\tilde{x}$, $\zeta\tilde{y}$ are symmetric (acting on the module) where $\tilde{x} = -ix + y$, $\tilde{y} = -x + iy$. This is the same as (see (9))

$$\tilde{x}^\dagger = -i\tilde{x}, \qquad \tilde{y}^\dagger = -i\tilde{y}$$

or

$$ix^\dagger + y^\dagger = -x - iy, \qquad -x^\dagger - iy^\dagger = ix + y.$$

These are the same as

$$x^\dagger = -y, \qquad y^\dagger = -x$$

or even just one of these relations, as the other follows by taking adjoints. Notice that for u, v even $u^\dagger = u^*$ and $(u, v) = \langle u, v\rangle$, see Section 3.2 for the notation.

Theorem 19. *V_t is unitary if and only if t is real and $t < 0$.*

Proof. Recall that $t \notin \mathbf{N}$ and Proposition 15. Let V_t be unitary. Then

$$xv_{t-1} = xyv_t = -yxv_t + Hv_t = tv_t.$$

So

$$\langle v_{t-1}, v_{t-1}\rangle = \langle yv_t, v_{t-1}\rangle = -\langle v_t, xv_{t-1}\rangle = -t\langle v_t, v_t\rangle.$$

We can normalize $\langle v_t, v_t\rangle = 1$ so that we get $\langle v_{t-1}, v_{t-1}\rangle = -t$. Thus we must have $t < 0$. We now prove the converse. If $t < 0$ we must define a hermitian product on V_t such that $x^\dagger = -y$.

The definition of the hermitian product goes as in the Verma case. The formulae for c_m of Lemma 18 show that for $m \geq 1$, we see that $c_m < 0$ always. Let $N(r) = \langle v_{t-r}, v_{t-r}\rangle$. The relation $x^\dagger = -y$ forces the relation

$$\langle v_{t-r}, v_{t-r}\rangle = \langle yv_{t-r+1}, v_{t-r}\rangle = -c_r\langle v_{t-r+1}, v_{t-r+1}\rangle$$

or

$$N(r) = -c_r N(r-1).$$

We define $N(r)$ inductively with $N(0) = 1$. Then, as $-c_r > 0$, the $N(r)$ are defined and > 0 for all r. With the orthogonality of the v_{t-r} this defines a hermitian product for v_t. To prove that $x^\dagger = -y$ in this hermitian product we only need to check that $\langle xyv_{t-r+1}, v_{t-r}\rangle = -\langle v_{t-r+1}, xv_{t-r}\rangle$. The left side is $N(r)$ while the right side is $-c_r N(r-1)$ and so we are done. ∎

Remark 3. We observe that the necessary condition of Theorem 3 for V, chosen as in Theorem 19, to be unitary is satisfied, since we have only a non compact even root α and $\lambda(H_\alpha) = \lambda(H) = t < 0$.

Proof of Theorem 4. The first statement is Theorem 19. The second statement is an immediate consequence of the first statement and Theorem 2. ∎

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