Tracing projective modules over noncommutative orbifolds

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Abstract. For an action of a finite cyclic group F on an n-dimensional noncommutative torus A_θ, we give sufficient conditions when the fundamental projective modules over A_θ, which determine the range of the canonical trace on A_θ, extend to projective modules over the crossed product C^*-algebra $A_\theta \rtimes F$. Our results allow us to understand the range of the canonical trace on $A_\theta \rtimes F$, and determine it completely for several examples including the crossed products of 2-dimensional noncommutative tori with finite cyclic groups and the flip action of $\mathbb{Z}_2$ on any n-dimensional noncommutative torus. As an application, for the flip action of $\mathbb{Z}_2$ on a simple n-dimensional torus A_θ, we determine the Morita equivalence class of $A_\theta \rtimes \mathbb{Z}_2$, in terms of the Morita equivalence class of A_θ.

1. Introduction

For $n \geq 2$, let $\mathcal{T}_n$ denote the space of all $n \times n$ real skew-symmetric matrices. The n-dimensional noncommutative torus A_θ is the universal C^*-algebra generated by unitaries $U_1, U_2, U_3, \ldots, U_n$ subject to the relations

$$U_k U_j = e^{2\pi i \theta_{jk}} U_j U_k \tag{1.1}$$

for $j, k = 1, 2, 3, \ldots, n$, where $\theta := (\theta_{jk}) \in \mathcal{T}_n$. For the 2-dimensional noncommutative tori, since θ is determined by only one real number, θ_{12}, we will denote θ_{12} by θ again and the corresponding 2-dimensional noncommutative torus by A_θ.

There is a canonical action of $\mathrm{SL}(2, \mathbb{Z})$ on 2-dimensional noncommutative tori, which is given by sending U_1 to $e^{\pi i a c \theta_{12}} U_1^a U_2^c$ and U_2 to $e^{\pi i b d \theta_{12}} U_1^b U_2^d$, for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. This action was further generalised to the higher dimensional noncommutative tori. It was pointed out in [14] that the right replacement of the group $\mathrm{SL}(2, \mathbb{Z})$ is

$$\mathrm{Sp}(n, \mathbb{Z}, \theta) := \{W \in \mathrm{GL}(n, \mathbb{Z}) : W^T \theta W = \theta\}.$$

Then there is a natural action of $\mathrm{Sp}(n, \mathbb{Z}, \theta)$ on the n-dimensional noncommutative torus A_θ. It is easy to see that $\mathrm{Sp}(2, \mathbb{Z}, \theta)$ is exactly $\mathrm{SL}(2, \mathbb{Z})$.

The study of crossed product C^*-algebras associated to finite group actions on non-commutative tori goes back to the work of Bratteli, Elliott, Evans, and Kishimoto [3]. However, they only looked at the action of $\mathbb{Z}_2$ on the C^*-algebra A_θ, for 2-dimensional tori. Recall that the action of $\mathbb{Z}_2$ on any n-dimensional A_θ, often called the *flip action*, is defined by sending U_i to U_i^{-1}. Note that the above action is basically given by the matrix $-\operatorname{id}_n \in \operatorname{Sp}(n, \mathbb{Z}, \theta)$, where id_n is the $n \times n$ unit matrix. Later various other authors studied actions of other finite cyclic subgroups of $\operatorname{SL}(2, \mathbb{Z})$ on 2-dimensional noncommutative tori; see [4, 9, 24, 25]. Motivated by the 2-dimensional results, it is also natural to consider a finite cyclic group F inside $\operatorname{Sp}(n, \mathbb{Z}, \theta)$ and consider the crossed product $A_\theta \rtimes F$, for an n-dimensional torus A_θ. We may call such a crossed product a *noncommutative orbifold*.

The authors in [2, 9] considered actions of cyclic subgroups of $\operatorname{SL}(2, \mathbb{Z})$ on 2-dimensional noncommutative tori. Along with K-theory computations of the corresponding crossed products, the authors computed the images of the canonical tracial states of such algebras. The recent development of the classification program of C^*-algebras allowed them to deduce results about isomorphism and Morita equivalence classes of such algebras, when the algebras are simple. One of the major facts they used is that the algebras are simple AH algebras when θ is irrational (for the finite group actions, the algebras are even AF). Then the algebras are classifiable in the sense of Elliot's classification program.

Jeong and Lee in [14] and He in [12] studied actions of finite subgroups of $\operatorname{Sp}(n, \mathbb{Z}, \theta)$ on an n-dimensional A_θ and found many of such crossed products to be classifiable, when θ is non-degenerate (see Definition 5.6) so that A_θ is simple. However, they did not discuss isomorphism and Morita equivalence classes of the crossed products. Our paper is a first attempt towards this kind of results for the higher dimensional cases.

To understand isomorphism and Morita equivalence classes of such noncommutative orbifolds, it is necessary to compute the K-theory of the orbifolds and understand the ranges of the canonical tracial states of the algebras. While the dimensions of the K-groups are known (from [15]), the tracial ranges are not understood. Our main results help to understand which numbers belong to the tracial ranges of the orbifolds and even determine the tracial ranges completely for several examples.

To understand the tracial range of an orbifold, one should first understand the same for the noncommutative torus itself. This was done by Elliott in [10]. To give an overview of our results, we recall the tracial range result from [10]. For an integer p with $1 \leq p \leq \frac{n}{2}$, if we denote the sub-matrix M_I^θ of θ consisting of rows and columns indexed by the numbers $i_1, i_2, \ldots, i_{2p}$ for some $i_1 < i_2 < \cdots < i_{2p}$, $I := (i_1, i_2, \ldots, i_{2p})$, then Elliott's result may be stated as

$$\operatorname{Tr}\left(\operatorname{K}_0(A_\theta)\right) = \mathbb{Z} + \sum_{0 < |I| \leq n} \operatorname{pf}(M_I^\theta)\mathbb{Z},$$

where $|I| := 2m$ for $I = (i_1, i_2, \ldots, i_{2m})$ and pf denotes the pfaffian. Here Tr denotes the canonical tracial state on A_θ.

It was observed in [5] that for each such I, there is a projective module $\mathcal{E}_I^\theta$ over A_θ, trace of which is exactly $\operatorname{pf}(M_I^\theta)$, assuming $\operatorname{pf}(M_I^\theta) \neq 0$. This module is governed by an

element $g_{I,\Sigma} \in \mathrm{SO}(n, n|\mathbb{Z})$. Here $\mathrm{SO}(n, n|\mathbb{Z})$ is a certain subgroup of the group of linear transformations of the space $\mathbb{R}^{2n}$ preserving the quadratic form $x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n}$ (see Section 3 for more details). The modules of such kind are called *fundamental projective modules*.

Now coming back to the crossed products of n-dimensional tori A_θ with a finite cyclic group $F \subset \mathrm{Sp}(n, \mathbb{Z}, \theta)$, if Tr^F denotes the canonical trace on $A_\theta \rtimes F$, the regular representation $A_\theta \rtimes F \hookrightarrow \mathrm{M}_N(A_\theta)$ gives

$$\mathrm{Tr}^F \left(\mathrm{K}_0(A_\theta \rtimes F) \right) \subseteq \frac{1}{N} \mathrm{Tr} \left(\mathrm{K}_0(A_\theta) \right) = \frac{1}{N} \left(\mathbb{Z} + \sum_{0 < |I| \leq n} \mathrm{pf}(M_I^\theta)\mathbb{Z} \right).$$

Our main theorem (Theorem 1.1) determines when the term $\frac{1}{N} \mathrm{pf}(M_I^\theta)$ lies in the left-hand side of the above equation. The proof of the theorem involves extending the modules $\mathcal{E}_I^\theta$ to modules over the crossed products using the so-called *metaplectic operators*, which were already used by the author (in a joint work with Luef) in [6] to extend a specific type of modules (*Bott classes*) to modules over the crossed products.

Let $\mathcal{R}$ denote the subgroup of $\mathrm{SO}(n, n|\mathbb{Z})$ generated by the elements of $\mathrm{GL}(n, \mathbb{Z})$.

Theorem 1.1 (Theorems 4.2 and 5.2). *With all the notations introduced above, assume that* $\mathrm{pf}(M_I^\theta) \neq 0$. *Let* $W \in \mathrm{GL}(n, \mathbb{Z})$ *be of finite order such that* $W^t \theta W = \theta$ *and* $F := \langle W \rangle$. *Suppose that* $g_{I,\Sigma} F (g_{I,\Sigma})^{-1} \subset \mathcal{R}$ *inside* $\mathrm{SO}(n, n|\mathbb{Z})$. *Then* $\mathcal{E}_I^\theta$ *becomes a finitely generated, projective module over* $A_\theta \rtimes F$ *and* $\frac{1}{N} \mathrm{pf}(M_I^\theta) \in \mathrm{Tr}^F (\mathrm{K}_0(A_\theta \rtimes F))$, *where* N *is the order of* W.

The condition in the above theorem is easy to check for many examples. In fact, we provide some examples with explicit tracial range computations. These examples include the 2-dimensional cases and the flip action of $\mathbb{Z}_2$. It is worthwhile to explicitly state the consequences for the flip action here in the introduction, since the results were unknown to the author. For the tracial range, we get

$$\mathrm{Tr}^{\mathbb{Z}_2} \left(\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \right) = \frac{1}{2} \mathrm{Tr} \left(\mathrm{K}_0(A_\theta) \right),$$

for any θ in $\mathcal{T}_n$. And as a corollary we have the following:

Corollary 1.2 (Corollary 5.11). *Let* $\theta_1, \theta_2 \in \mathcal{T}_n$ *be non-degenerate. Let* $\mathbb{Z}_2$ *act on* A_{θ_1} *and* A_{θ_2} *by the flip actions. Then* $A_{\theta_1} \rtimes \mathbb{Z}_2$ *is strongly Morita equivalent to* $A_{\theta_2} \rtimes \mathbb{Z}_2$ *if and only if* A_{θ_1} *is strongly Morita equivalent to* A_{θ_2}.

It is worth mentioning that the only action of a finite cyclic subgroup of $\mathrm{Sp}(3, \mathbb{Z}, \theta)$ on a 3-dimensional torus A_θ, when θ is non-degenerate, is the flip action [14, Theorem 1.4].

Apart from the applications in classification of C^*-algebras, the computations of the ranges of tracial states turn out to be useful in physics (Bellisard's gap labelling theorem, in particular). Our results are similar to results which appeared in connection with the study of a twisted version of the gap labelling theorem, recently conjectured in [1]. We hope that our techniques will be helpful for a better understanding of the conjecture.

This article is organised as follows: in Section 2, we recall the definition of twisted group C^*-algebras and give relevant examples. In Section 3, we discuss the fundamental projective modules over noncommutative tori. Section 4 deals with extending the fundamental modules to modules over orbifolds, and proving Theorem 1.1 (Theorem 4.2). In the last section, Section 5, the proof of Theorem 1.1 (Theorem 5.2) about the ranges of the canonical traces on orbifolds is discussed along with various examples. We also discuss the results about Morita equivalence classes of orbifolds, along with Corollary 1.2 in Section 5.

Notation. $e(x)$ will always denote the number $e^{2\pi i x}$, and id_m will be the $m \times m$ unit matrix.

2. Twisted group C^*-algebras and noncommutative orbifolds

Let G be a discrete group. A map $\omega : G \times G \to \mathbb{T}$ is called a *2-cocycle* if

$$\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$$

whenever $x, y, z \in G$, and if

$$\omega(x, 1) = 1 = \omega(1, x)$$

for any $x \in G$.

The *ω-twisted left regular representation* of the group G is given by the formula

$$\bigl(L_\omega(x)f\bigr)(y) = \omega(x, x^{-1}y)f(x^{-1}y),$$

for $f \in l^2(G)$. The *reduced twisted group C^*-algebra $C^*(G, \omega)$* is defined as the sub-C^*-algebra of $B(l^2(G))$ generated by the ω-twisted left regular representation of the group G. Since we do not talk about full group C^*-algebras in this paper, we simply call $C^*(G, \omega)$ the twisted group C^*-algebra of G with respect to ω. When $\omega = 1$, $C^*(G, \omega) =: C^*(G)$ is the usual reduced group C^*-algebra of G. We refer to [9, Section 1] for more on twisted group C^*-algebras and the details of the above construction.

Example 2.1. Let G be the group $\mathbb{Z}^n$. For each $\theta \in \mathcal{T}_n$, construct a 2-cocycle on G by defining $\omega_\theta(x, y) = e(\langle -\theta x, y \rangle)$. The corresponding twisted group C^*-algebra $C^*(G, \omega_\theta)$ is isomorphic to the n-dimensional noncommutative torus A_θ, which was defined in the introduction.

Example 2.2. Suppose that W is an invertible $n \times n$ matrix of finite order with integer entries. Let $F := \langle W \rangle$ act on $\mathbb{Z}^n$ by usual matrix multiplication with vectors. Let us also take $\theta \in \mathcal{T}_n$. We assume in addition that W is a *θ-symplectic* matrix; i.e., $W^t \theta W = \theta$. Then we can define a 2-cocycle ω'_θ on $G := \mathbb{Z}^n \rtimes F$ by $\omega'_\theta((x, s), (y, t)) = \omega_\theta(x, s \cdot y)$. Sometimes one calls the corresponding twisted group C^*-algebra, $C^*(G, \omega'_\theta)$, a *noncommutative orbifold*. We will come back to this example in Section 4.

3. K-theory generators of noncommutative tori

3.1. Projective modules over noncommutative tori

In [22], Rieffel and Schwarz defined (densely) an action of the group $\mathrm{SO}(n, n|\mathbb{Z})$ on $\mathcal{T}_n$. Recall that $\mathrm{SO}(n, n|\mathbb{Z})$ is the subgroup of $\mathrm{GL}(2n, \mathbb{R})$, which contains matrices, with integer entries and of determinant 1, of the following 2×2 block form:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are arbitrary $n \times n$ matrices over $\mathbb{Z}$ satisfying

$$A^t C + C^t A = 0, \quad B^t D + D^t B = 0, \quad \text{and} \quad A^t D + C^t B = \mathrm{id}_n.$$

The action of $\mathrm{SO}(n, n|\mathbb{Z})$ on $\mathcal{T}_n$ is defined as

$$g\theta := (A\theta + B)(C\theta + D)^{-1}$$

whenever $C\theta + D$ is invertible. The subset of $\mathcal{T}_n$ on which the action of every $g \in \mathrm{SO}(n, n|\mathbb{Z})$ is defined, is dense in $\mathcal{T}_n$ (see [22, p. 291]). We have the following theorem due to Hanfeng Li.

Theorem 3.1 ([16, Theorem 1.1]). *For any $\theta \in \mathcal{T}_n$ and $g \in \mathrm{SO}(n, n|\mathbb{Z})$, if $g\theta$ is defined, then A_θ and $A_{g\theta}$ are strongly Morita equivalent.*

For any $R \in \mathrm{GL}(n, \mathbb{Z})$, let us denote by $\rho(R)$ the matrix $\begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix} \in \mathrm{SO}(n, n|\mathbb{Z})$, and for any $N \in \mathcal{T}_n \cap M_n(\mathbb{Z})$, we denote by $\mu(N)$ the matrix $\begin{pmatrix} \mathrm{id}_n & N \\ 0 & \mathrm{id}_n \end{pmatrix} \in \mathrm{SO}(n, n|\mathbb{Z})$. Notice that the noncommutative tori corresponding to the matrices $\rho(R)\theta = R\theta R^t$ and $\mu(N)\theta = \theta + N$ are both isomorphic to A_θ. Also define

$$\mathrm{SO}(n, n|\mathbb{Z}) \ni \sigma_{2p} := \begin{pmatrix} 0 & 0 & \mathrm{id}_{2p} & 0 \\ 0 & \mathrm{id}_{n-2p} & 0 & 0 \\ \mathrm{id}_{2p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{n-2p} \end{pmatrix}, \quad 1 \leqslant p \leqslant n/2.$$

We recall the approach of Rieffel [21] to find the $A_{\sigma_{2p}\theta} - A_\theta$ bimodule and follow the presentation in [16].

We fix $1 \leqslant p \leqslant n/2$ and let $q \in \mathbb{N}$ such that $n = 2p + q$. Let us write $\theta \in \mathcal{T}_n$ as $\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$, partitioned into four sub-matrices $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$, and assume θ_{11} to be an invertible $2p \times 2p$ matrix. Define a new cocycle $\omega_{\theta'}$ on $\mathbb{Z}^n$ by $\omega_{\theta'}(x, y) = e(\langle -\theta' x, y \rangle / 2)$, where

$$\theta' = \begin{pmatrix} \theta_{11}^{-1} & -\theta_{11}^{-1}\theta_{12} \\ \theta_{21}\theta_{11}^{-1} & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \end{pmatrix} = \sigma_{2p}\theta.$$

Set $\mathcal{A} = C^*(\mathbb{Z}^n, \omega_\theta)$ and $\mathcal{B} = C^*(\mathbb{Z}^n, \omega_{\theta'})$. Let M be the group $\mathbb{R}^p \times \mathbb{Z}^q$, let $G := M \times \widehat{M}$, and let $\langle \cdot, \cdot \rangle$ be the natural pairing between M and its dual group $\widehat{M}$ (our notation

does not distinguish between the pairing of a group and its dual group, and the standard inner product on a linear space). Consider the Schwartz space $\mathcal{E}^\infty := \mathcal{S}(M)$ consisting of smooth and rapidly decreasing complex-valued functions on M.

Denote by $\mathcal{A}^\infty = \mathcal{S}(\mathbb{Z}^n, \omega_\theta)$ and $\mathcal{B}^\infty = \mathcal{S}(\mathbb{Z}^n, \omega_{\theta'})$ the dense sub-algebras of $\mathcal{A}$ and $\mathcal{B}$, respectively, consisting of formal series (of the variables $\{U_i\}$) with rapidly decaying coefficients. Let us consider the following $(2p + 2q) \times (2p + q)$ real valued matrix:

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & \mathrm{id}_q \\ T_{31} & T_{32} \end{pmatrix}, \tag{3.1}$$

where T_{11} is an invertible matrix such that $T_{11}^t J_0 T_{11} = \theta_{11}$, $J_0 := \begin{pmatrix} 0 & \mathrm{id}_p \\ -\mathrm{id}_p & 0 \end{pmatrix}$, $T_{31} = \theta_{21}$, and T_{32} is any $q \times q$ matrix such that $\theta_{22} = T_{32} - T_{32}^t$. For our purposes, we take $T_{32} = \theta_{22}/2$.

We also define the following $(2p + 2q) \times (2p + q)$ real valued matrix:

$$S = \begin{pmatrix} J_0(T_{11}^t)^{-1} & -J_0(T_{11}^t)^{-1}T_{31}^t \\ 0 & \mathrm{id}_q \\ 0 & T_{32}^t \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_q \\ 0 & -\mathrm{id}_q & 0 \end{pmatrix}$$

and J' be the matrix obtained from J by replacing the negative entries of it by zeroes. Note that T and S can be thought as maps from $(\mathbb{R}^n)^*$ to $\mathbb{R}^p \times (\mathbb{R}^p)^* \times \mathbb{R}^q \times (\mathbb{R}^q)^*$ (see the definition of an embedding map in [16, Definition 2.1]), and $S(\mathbb{Z}^n), T(\mathbb{Z}^n) \subseteq \mathbb{R}^p \times (\mathbb{R}^p)^* \times \mathbb{Z}^q \times (\mathbb{R}^q)^*$. Then we can think of $S(\mathbb{Z}^n), T(\mathbb{Z}^n)$ as in G via composing $S|_{\mathbb{Z}^n}, T|_{\mathbb{Z}^n}$ with the natural covering map $\mathbb{R}^p \times (\mathbb{R}^p)^* \times \mathbb{Z}^q \times (\mathbb{R}^q)^* \to G$. Let P' and P'' be the canonical projections of G to M and $\widehat{M}$, respectively, and let

$$T' := P' \circ T, \quad T'' := P'' \circ T, \quad S' := P' \circ S, \quad S'' := P'' \circ S.$$

Then the following formulas define a $\mathcal{B}^\infty$-$\mathcal{A}^\infty$ bimodule structure on $\mathcal{E}^\infty$:

$$(f U_l^\theta)(x) = e(\langle -T(l), J'T(l)/2 \rangle)\langle x, T''(l) \rangle f(x - T'(l)), \tag{3.2}$$

$$\langle f, g \rangle_{\mathcal{A}^\infty}(l) = e(\langle -T(l), J'T(l)/2 \rangle) \int_G \langle x, -T''(l) \rangle g(x + T'(l)) \bar{f}(x) dx, \tag{3.3}$$

$$(U_l^{\sigma_{2p}\theta} f)(x) = e(\langle -S(l), J'S(l)/2 \rangle)\langle x, -S''(l) \rangle f(x + S'(l)), \tag{3.4}$$

$${}_{\mathcal{B}^\infty}\langle f, g \rangle(l) = e(\langle S(l), J'S(l)/2 \rangle) \int_G \langle x, S''(l) \rangle \bar{g}(x + S'(l)) f(x) dx, \tag{3.5}$$

where U_l^θ, $U_l^{\sigma_{2p}\theta}$ denote the canonical unitaries with respect to the group element $l \in \mathbb{Z}^n$ in $\mathcal{A}^\infty$ and $\mathcal{B}^\infty$, respectively. See [16, Proposition 2.2] for the following well-known result.

Theorem 3.2 (Rieffel). *The smooth module* $\mathcal{E}^{\infty}$, *with the above structures, is an* $\mathcal{B}^{\infty}$-$\mathcal{A}^{\infty}$ *Morita equivalence bimodule which can be extended to a strong Morita equivalence between* $\mathcal{B}$ *and* $\mathcal{A}$.

Let $\mathcal{E}$ denote the completion of $\mathcal{E}^{\infty}$ with respect to the C^*-valued inner products given above. Now, $\mathcal{E}$ becomes a right projective A-module which is also finitely generated (see the discussion preceding [9, Proposition 4.6]). Note that $\mathcal{E}$ is a Morita equivalence bimodule between $\mathcal{B} = A_{\sigma_{2p}\theta}$ and $\mathcal{A} = A_{\theta}$.

3.2. Fundamental projective modules

For a definition of the pfaffian of a skew-symmetric matrix A, pf(A), we refer to [5, Definition 3.1]. We start with the following remark.

Remark 3.3. The trace of the module $\mathcal{E}$, which was computed by Rieffel [21], is exactly the absolute value of the pfaffian of the upper left $2p \times 2p$ corner of the matrix θ, which is θ_{11}. Indeed, as [21, Proposition 4.3, p. 289] says that the trace of $\mathcal{E}$ is $|\det \widetilde{T}|$, where

$$\widetilde{T} = \begin{pmatrix} T_{11} & 0 \\ 0 & \mathrm{id}_q \end{pmatrix},$$

the relation $T_{11}^t J_0 T_{11} = \theta_{11}$ and the fact $\det(J_0) = 1$ give the claim.

Let p be an integer such that $1 \leq p \leq \frac{n}{2}$.

Definition 3.4. A *$2p$-pfaffian minor* (or just pfaffian minor) of a skew-symmetric matrix A is the pfaffian of a sub-matrix M_I^A of A consisting of rows and columns indexed by $i_1, i_2, \ldots, i_{2p}$ for some numbers $i_1 < i_2 < \cdots < i_{2p}$ and $I := (i_1, i_2, \ldots, i_{2p})$.

Note that the number of $2p$-pfaffian minors is $\binom{n}{2p}$ and the number of all pfaffian minors is $2^{n-1} - 1$.

Let $\theta \in \mathcal{T}_n$. We will now see that for each non-zero pfaffian minor of θ, we can construct a projective module over A_{θ} such that the trace of this module is exactly the pfaffian minor. Fix $1 \leq p \leq \frac{n}{2}$. Choose $I := (i_1, i_2, \ldots, i_{2p})$ for $i_1 < i_2 < \cdots < i_{2p}$, and assume that the pfaffian minor pf(M_I^{θ}) is non-zero (so that M_I^{θ} is invertible). Choose a permutation $\Sigma \in \mathcal{S}_n$ such that $\Sigma(1) = i_1, \Sigma(2) = i_2, \ldots, \Sigma(2p) = i_{2p}$. If $U_1, U_2, \ldots, U_n$ are generators of A_{θ}, there exists an $n \times n$ skew-symmetric matrix, denoted by $\Sigma(\theta)$, such that $U_{\Sigma(1)}, U_{\Sigma(2)}, \ldots, U_{\Sigma(n)}$ are generators of $A_{\Sigma(\theta)}$ and $A_{\Sigma(\theta)} \cong A_{\theta}$. Note that the upper left $2p \times 2p$ block $\Sigma(\theta)$ is exactly M_I^{θ}, which is invertible. Now, consider the projective module constructed as completion of $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^{n-2p})$ over $A_{\Sigma(\theta)}$ as in the previous subsection and denote it by $\mathcal{E}_I^{\theta}$. The trace of this module is the pfaffian of M_I^{θ} by the remark above, which is $\sum_{\xi \in \Pi} (-1)^{|\xi|} \prod_{s=1}^{p} \theta_{i_{\xi(2s-1)} i_{\xi(2s)}}$. Varying p, and assuming that all the pfaffian minors are non-zero, we get $2^{n-1} - 1$ projective modules. We call these $2^{n-1} - 1$ elements the *fundamental projective modules*.

We recall the following fact due to Elliott which will play a key role.

Theorem 3.5 (Elliott). *Let θ be a skew-symmetric real $n \times n$ matrix. Then $\mathrm{Tr}(\mathrm{K}_0(A_\theta))$ is the range of the exterior exponential*

$$\exp(\theta) : \Lambda^{\mathrm{even}}\mathbb{Z}^n \to \mathbb{R}.$$

We refer to [10, Theorem 3.1] for the definition of exterior exponential and the proof of the above theorem. The range of the exterior exponential is well known and is given below as a corollary of the above theorem:

Corollary 3.6. $\mathrm{Tr}(\mathrm{K}_0(A_\theta))$ *is the subgroup of $\mathbb{R}$ generated by* 1 *and the numbers*

$$\sum_\xi (-1)^{|\xi|} \prod_{s=1}^m \theta_{j_{\xi(2s-1)} j_{\xi(2s)}} \quad \text{for } 1 \leq j_1 < j_2 < \cdots < j_{2m} \leq n,$$

where the sum is taken over all elements ξ of the permutation group $\mathcal{S}_{2m}$ such that

$$\xi(2s-1) < \xi(2s) \text{ for all } 1 \leq s \leq m \quad \text{and} \quad \xi(1) < \xi(3) < \cdots < \xi(2m-1).$$

Noting that $\sum_\xi (-1)^{|\xi|} \prod_{s=1}^m \theta_{j_{\xi(2s-1)} j_{\xi(2s)}}$ is exactly the pfaffian of M_I^θ, where $I = (i_1, i_2, \ldots, i_{2m})$, we have

$$\mathrm{Tr}\left(\mathrm{K}_0(A_\theta)\right) = \mathbb{Z} + \sum_{0 < |I| \leq n} \mathrm{pf}(M_I^\theta)\mathbb{Z}, \tag{3.6}$$

where $|I| := 2m$ for $I = (i_1, i_2, \ldots, i_{2m})$.

So for a non-zero $\mathrm{pf}(M_I^\theta)$, $I = (i_1, i_2, \ldots, i_{2p})$, we have constructed a projective module $\mathcal{E}_I^\theta$ over A_θ, whose trace is $\mathrm{pf}(M_I^\theta)$. A quick thought shows that $\mathcal{E}_I^\theta$ is an equivalence bimodule between A_θ and $A_{g_{I,\Sigma}\theta}$ for some $g_{I,\Sigma} \in \mathrm{SO}(n, n|\mathbb{Z})$. Indeed, let R_I^Σ be the permutation matrix corresponding to the permutation Σ. Note that $\Sigma(\theta) = \rho(R_I^\Sigma)\theta$. Then clearly $g_{I,\Sigma} = \sigma_{2p}\rho(R_I^\Sigma)$.

3.3. Explicit generators of $\mathrm{K}_0(A_\theta)$ for a general $\theta \in \mathcal{T}_n$

Consider the matrix $Z \in \mathcal{T}_n$ whose entries above the diagonal are all 1:

$$Z = \begin{pmatrix} 0 & 1 & \cdots & & \cdots & 1 \\ -1 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & & & \\ & & & & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 1 \\ -1 & \cdots & & & \cdots & -1 & 0 \end{pmatrix}.$$

Now, for any $\theta \in \mathcal{T}_n$, there exists some positive integer t, such that all the pfaffian minors of $\mu(tZ)\theta = \theta + tZ$ are positive (see [5, Proposition 4.6]). Note that $A_{\theta+tZ}$ and A_θ define the same noncommutative torus. We then have the following theorem.

Theorem 3.7. *The K-theory classes of the fundamental projective modules $\mathcal{E}_I^{\theta+tZ}$, along with* [1], *generate* $\mathrm{K}_0(A_{\theta+tZ})$ *and hence* $\mathrm{K}_0(A_\theta)$.

Proof. See [5, Theorem 4.7]. ∎

4. Noncommutative orbifolds and projective modules

Let us recall Example 2.2. Let $W := (a_{ij})$ be an invertible $n \times n$ matrix of finite order with integer entries and let F be the finite cyclic group generated by W. In addition, we assume that $W^t \theta W = \theta$. Hence F is a finite subgroup of $\mathrm{Sp}(n, \mathbb{Z}, \theta) := \{A \in \mathrm{GL}(n, \mathbb{Z}) : A^T \theta A = \theta\}$. By [9, Lemma 2.1], we have $C^*(\mathbb{Z}^n \rtimes F, \omega'_\theta) = A_\theta \rtimes_\alpha F$, where the action of F on A_θ is given by (see [14, Equation (2.6)]):

$$\alpha(U_i) = e\left(\sum_{k=2}^{n} \sum_{j=1}^{k-1} a_{ki} a_{ji} \theta_{jk}\right) U_1^{a_{1i}} \cdots U_n^{a_{ni}}, \tag{4.1}$$

where $U_1, \ldots, U_n$ are the generators of A_θ. Sometimes we just write the crossed product as $A_\theta \rtimes F$, without the "α" decoration.

Let us look into the case where $n = 2$. Note that $\mathrm{Sp}(2, \mathbb{Z}, \theta) = \mathrm{SL}(2, \mathbb{Z})$. Finite cyclic subgroups of $\mathrm{SL}(2, \mathbb{Z})$ are up to conjugacy generated by the following four matrices:

$$W_{(2)} := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W_{(3)} := \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$W_{(4)} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad W_{(6)} := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

where the notation $W_{(r)}$ indicates that it is a matrix of order r. The actions of the cyclic groups generated by these matrices are considered already in [9], where the authors constructed projective modules over the corresponding crossed products using the fundamental projective modules.

For $n \geq 3$, finding a finite order matrix $W \in \mathrm{Sp}(n, \mathbb{Z}, \theta)$ is non-trivial. In [14], and in [12], the authors found some of the matrices for $n \geq 3$ and studied the associated actions. Note that, for all n, there will always be a matrix W of order 2, i.e., $-\mathrm{id}_n$. The action by $\mathbb{Z}_2 = \langle W \rangle$ is the flip action, which was already defined in the introduction.

One natural question is how does one extend the fundamental projective modules, $\mathcal{E}_I^\theta$, over noncommutative tori A_θ to the aforementioned crossed products. In [6], this was answered when the module $\mathcal{E}_I^\theta$ is a completion of $\mathcal{S}(\mathbb{R}^p)$, i.e., when the dimension of the torus is even $(= 2p)$, and θ is invertible so that $\mathcal{E}_I^\theta$ is defined. This module is called the *Bott class*. In this section, we do this extension for a general $\mathcal{E}_I^\theta$. We need the following proposition for such extensions.

Proposition 4.1. *Suppose that F is a finite group acting on a C^*-algebra A by the action α. Also suppose that $\mathcal{E}$ is a finitely generated projective (right) A-module with*

a right action $T : F \to \mathrm{Aut}(\mathcal{E})$, written $(\xi, g) \rightarrowtail \xi T_g$, such that $\xi(T_g)a = (\xi \alpha_g(a))T_g$ for all $\xi \in \mathcal{E}$, $a \in A$, and $g \in F$. Then $\mathcal{E}$ becomes a finitely generated projective $A \rtimes F$ module with action defined by

$$\xi \cdot \left(\sum_{g \in F} a_g \delta_g \right) = \sum_{g \in F} (\xi a_g) T_g.$$

Also, if we restrict the new module to A, we get the original A-module $\mathcal{E}$, with the action of F forgotten.

Proof. This is exactly the construction of the Green–Julg map. See [9, Proposition 4.5]. ∎

Let us first recall the approach of [6], where the authors define the necessary action of F on the Bott class which allows them to conclude that the Bott class is a projective module over the crossed product $A_\theta \rtimes F$, using Proposition 4.1. Hence assume $n \, (= 2p)$ to be even for the moment. Since $F = \langle W \rangle$ acts on $\mathbb{Z}^n$ as before, we have $W^t \theta W = \theta$. In order to define an action of F on the Bott class, the authors (in [6]) used the so-called *metaplectic representation* of the symplectic matrix $T W T^{-1}$, where $T^t J_0 T = \theta$ as in equation (3.1). Note that, in this case, $q = 0$ and hence $T = T_{11}$. The main idea is to use the following metaplectic extension:

$$0 \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{Mp}^c(n) \longrightarrow \mathrm{Sp}(n) \longrightarrow 0, \qquad (4.2)$$

where $\mathrm{Sp}(n)$ is the usual symplectic group, and $\mathrm{Mp}^c(n)$ is the complex metaplectic group (see [6, Section 5]). For our purposes, we do not need much details about the metaplectic group, but we need to know that it has a (metaplectic) representation on $\mathcal{S}(\mathbb{R}^p)$ ([6, Definition 5.1], also see [8, Chapter 7]). Now, $F \cong \langle T W T^{-1} \rangle$ sits inside $\mathrm{Sp}(n)$. But also we have the following lift (since $\mathrm{H}^2(F, \mathbb{S}^1)$ is trivial, see [6, p. 158]) possible:

$$
\begin{array}{ccccccccc}
 & & & & & & & F & \\
 & & & & & & \swarrow & \downarrow & \\
0 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & \mathrm{Mp}^c(n) & \longrightarrow & \mathrm{Sp}(n) & \longrightarrow & 0.
\end{array}
$$

The above defines an action of F on $\mathcal{S}(\mathbb{R}^p)$ which extends to the necessary completion (Bott class) of $\mathcal{S}(\mathbb{R}^p)$ and it satisfies the conditions of Proposition 4.1 (see [6, Theorem 5.4]). Hence the Bott class becomes a projective module over $A_\theta \rtimes F$. In the following, we shall often write $f W$ for the above action of W on $\mathcal{S}(\mathbb{R}^p)$, for $f \in \mathcal{S}(\mathbb{R}^p)$. So from [6, Equation (5.12)] we have

$$(f W)U_l = \big(f \alpha_W(U_l) \big)W, \quad f \in \mathcal{S}(\mathbb{R}^p), \, l \in \mathbb{Z}^{2p}, \qquad (4.3)$$

which is the condition in Proposition 4.1.

Now, we take a general n, not necessarily even. We have $\mathcal{R} := \langle \rho(R), R \in \mathrm{GL}(n, \mathbb{Z}) \rangle \subseteq \mathrm{SO}(n, n | \mathbb{Z})$. Also for $W \in F$, we have $\rho(W^t) \in \mathrm{SO}(n, n | \mathbb{Z})$. In this way, $F \subseteq \mathrm{SO}(n, n | \mathbb{Z})$. Recall that $g_{I, \Sigma} = \sigma_{2p} \rho(R_I^\Sigma)$.

Theorem 4.2. *With all the notations introduced before, assume that* $\mathrm{pf}(M_I^\theta) \neq 0$. *Let* $W \in \mathrm{GL}(n, \mathbb{Z})$ *be of finite order such that* $W^t \theta W = \theta$ *and* $F := \langle W \rangle$. *Suppose that* $g_{I,\Sigma} F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$ *inside* $\mathrm{SO}(n, n | \mathbb{Z})$. *Then* $\mathcal{E}_I^\theta$ *becomes a finitely generated, projective module over* $A_\theta \rtimes F$.

Proof. $g_{I,\Sigma} F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$ means $\sigma_{2p} \rho(R_I^\Sigma W^t (R_I^\Sigma)^{-1}) \sigma_{2p} \in \mathcal{R}$, noting that the inverse of σ_{2p} is σ_{2p} again. Now, $(R_I^\Sigma W^t (R_I^\Sigma)^{-1})^t$ is a $\rho(R_I^\Sigma)\theta$-symplectic matrix and the algebras $A_{\rho(R_I^\Sigma)\theta}$ and A_θ are F-equivariantly isomorphic, where the action of F on $A_{\rho(R_I^\Sigma)\theta}$ is given by identifying F with $\langle (R_I^\Sigma W^t (R_I^\Sigma)^{-1})^t \rangle$. So by passing from W^t to $R_I^\Sigma W^t (R_I^\Sigma)^{-1}$ if necessary, we may assume without loss of generality that $\mathcal{E}_I^\theta$ is an A_θ-module.

From $\sigma_{2p} \rho(W^t) \sigma_{2p} \in \mathcal{R}$, we have

$$\sigma_{2p} \rho(W^t) \sigma_{2p} = \begin{pmatrix} S & 0 \\ 0 & (S^{-1})^t \end{pmatrix}, \tag{4.4}$$

for some $S \in \mathrm{GL}(n, \mathbb{Z})$. Writing $W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$, where W_1 is the $2p \times 2p$ block, a simple computation shows that $W_2 = W_3 = 0$ and $S = \begin{pmatrix} W_1^{-1} & 0 \\ 0 & W_4^t \end{pmatrix}$. So we have

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_4 \end{pmatrix}. \tag{**}$$

Writing $\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ as before, $W^t \theta W = \theta$ gives the following *compatibility* relations:

$$\begin{cases} W_1^t \theta_{11} W_1 = \theta_{11}, \\ W_1^t \theta_{12} W_4 = \theta_{12}, \\ W_4^t \theta_{21} W_1 = \theta_{21}, \\ W_4^t \theta_{22} W_4 = \theta_{22}. \end{cases} \tag{4.5}$$

Let us first write down equation (3.2), which is

$$(f U_l)(x) = e\big(\langle -T(l), J'T(l)/2 \rangle\big)\langle x, T''(l) \rangle f\big(x - T'(l)\big), \tag{4.6}$$

more explicitly. Writing $l = (l_1, l_2) \in \mathbb{Z}^n$, for $l_1 \in \mathbb{Z}^{2p}$, $l_2 \in Z^q$, we have

$$T(l) = \begin{pmatrix} T_{11} & 0 \\ 0 & \mathrm{id}_q \\ \theta_{21} & \frac{\theta_{22}}{2} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} T_{11} l_1 \\ l_2 \\ \theta_{21} l_1 + \frac{\theta_{22}}{2} l_2 \end{pmatrix}.$$

Let J_0' be the matrix obtained by replacing the negative entries of J_0 by zeroes. Also, if Q' and Q'' are the canonical projections of $\mathbb{R}^p \times \widehat{\mathbb{R}^p}$ to $\mathbb{R}^p$ and $\widehat{\mathbb{R}^p}$, respectively, denote

$$T_{11}' := Q' \circ T_{11}, \quad T_{11}'' := Q'' \circ T_{11}.$$

Then

$$e\left(\left\langle -T(l), J'T(l)/2\right\rangle\right)$$

$$= e\left(-\begin{pmatrix} T_{11}l_1 \\ l_2 \\ \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2 \end{pmatrix} \cdot \begin{pmatrix} J_0' & 0 & 0 \\ 0 & 0 & \mathrm{id}_q \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_{11}l_1 \\ l_2 \\ \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2 \end{pmatrix}/2\right)$$

$$= e\left(-\begin{pmatrix} T_{11}l_1 \\ l_2 \\ \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2 \end{pmatrix} \cdot \begin{pmatrix} J_0'T_{11}l_1 \\ \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2 \\ 0 \end{pmatrix}/2\right)$$

$$= e(-T_{11}l_1 \cdot J_0'T_{11}l_1/2)e\left(-l_2 \cdot \theta_{21}l_1 - l_2 \cdot \frac{\theta_{22}}{2}l_2/2\right)$$

$$= C_1(l_1)A(l_1, l_2),$$

where $C_1(l_1) := e(-T_{11}l_1 \cdot J_0'T_{11}l_1/2)$, $A(l_1, l_2) := e(-l_2 \cdot \theta_{21}l_1 - l_2 \cdot \frac{\theta_{22}}{2}l_2/2)$. Also,

$$\langle x, T''(l)\rangle = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} T_{11}''l_1 \\ \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2 \end{pmatrix} \right\rangle$$

$$= \langle x_1, T_{11}''l_1\rangle\left\langle x_2, \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2\right\rangle$$

$$= C_2(x_1, l_1)B(x_2, l_1, l_2),$$

$$f\left(x - T'(l)\right) = f\begin{pmatrix} x_1 - T_{11}'l_1 \\ x_2 - l_2 \end{pmatrix},$$

where $C_2(x_1, l_1) := \langle x_1, T_{11}''l_1\rangle$, $B(x_2, l_1, l_2) := \langle x_2, \theta_{21}l_1 + \frac{\theta_{22}}{2}l_2\rangle$.

Now, for $f \in \mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$, we define

$$(fW)(x_1, x_2) := \sqrt{\det(W_4)}(f^\sharp W_1)(x_1), \tag{4.7}$$

where $f^\sharp \in \mathcal{S}(\mathbb{R}^p)$ defined as $f^\sharp(x') = f(x', W_4x_2)$. Note that here we have used the metaplectic action of W_1 on $f^\sharp$. We first want to show that $f \to fW$ extends to a unitary operator on $L^2(\mathbb{R}^p \times \mathbb{Z}^q)$ using the fact that the metaplectic operators are unitary. To this end, we check that

$$\langle fW, g\rangle_{L^2} = \langle f, gW^{-1}\rangle_{L^2}, \tag{4.8}$$

which follows from the following computation:

$$\langle fW, g\rangle_{L^2}$$

$$= \int_G (fW)(x_1, x_2)\overline{g(x_1, x_2)}dx$$

$$= \sqrt{\det(W_4)} \int_G (f^\sharp W_1)(x_1)\overline{g(x_1, x_2)}dx$$

$$= \sqrt{\det(W_4)} \int_G f^\sharp(x_1)\overline{(g'W_1^{-1})(x_1)}dx_1dx_2 \quad \left(\text{where } g'(x_1) = g(x_1, x_2)\right)$$

$$= \sqrt{\det(W_4)}\,\det(W_4)^{-1} \int_G f(x_1, x_2)\overline{g^\sharp W_1^{-1}(x_1)}dx_1 dx_2 \quad \text{(change } x_2 \text{ to } W_4^{-1}x_2\text{)}$$

$$= \int_G f(x_1, x_2)\overline{g W^{-1}(x_1, x_2)}dx_1 dx_2$$

$$= \langle f, g W^{-1}\rangle_{L^2}.$$

We want to show

$$(f W)U_l = \big(f\alpha_W(U_l)\big)W = (f U_{Wl})W. \tag{4.9}$$

From equation (4.3), we already have

$$(f^\sharp W_1)U_{l_1} = (f^\sharp U_{W_1 l_1})W_1. \tag{4.10}$$

Now,

$$(f U_{Wl})^\sharp(x_1)$$

$$= (f U_{Wl})(x_1, W_4 x_2),$$

$$= C_1(W_1 l_1)A(W_1 l_1, W_4 l_2)C_2(x_1, W_1 l_1)B(W_4 x_2, W_1 l_1, W_4 l_2)f\begin{pmatrix} x_1 - T'_{11}W_1 l_1 \\ W_4 x_2 - W_4 l_2 \end{pmatrix}$$

$$\overset{(4.5)}{=} C_1(W_1 l_1)C_2(x_1, W_1 l_1)A(l_1, l_2)B(x_2, l_1, l_2)f\begin{pmatrix} x_1 - T'_{11}W_1 l_1 \\ W_4(x_2 - l_2) \end{pmatrix}$$

$$= A(l_1, l_2)B(x_2, l_1, l_2)(g^\sharp U_{W_1 l_1})(x_1), \quad \text{where } g(x_1) = f(x_1, x_2 - l_2).$$

So the RHS of (4.9) becomes

$$\sqrt{\det(W_4)}\big((f U_{Wl})^\sharp\big)W_1(x_1) = \sqrt{\det(W_4)}A(l_1, l_2)B(x_2, l_1, l_2)\big((g^\sharp U_{W_1 l_1})W_1\big)(x_1)$$

$$\overset{(4.10)}{=} \sqrt{\det(W_4)}A(l_1, l_2)B(x_2, l_1, l_2)(g^\sharp W_1)U_{l_1}(x_1).$$

Now, the LHS becomes

$$(f W)U_l(x) = C_1(l_1)C_2(x_1, l_1)A(l_1, l_2)B(x_2, l_1, l_2)f W\begin{pmatrix} x_1 - T'_{11}l_1 \\ x_2 - l_2 \end{pmatrix}$$

$$= \sqrt{\det(W_4)}C_1(l_1)C_2(x_1, l_1)A(l_1, l_2)B(x_2, l_1, l_2)(g^\sharp W_1)(x_1 - T'_{11}l_1)$$

$$= \sqrt{\det(W_4)}A(l_1, l_2)B(x_2, l_1, l_2)(g^\sharp W_1)U_{l_1}(x_1).$$

Thus we have proved equation (4.9). We finish the proof with the compatibility of the action with the inner product $\langle\cdot, \cdot\rangle_{\mathcal{A}^\infty}$ as defined in (3.3):

$$\langle f W, g W\rangle_{\mathcal{A}^\infty} = \alpha_{W^{-1}}\big(\langle f, g\rangle_{\mathcal{A}^\infty}\big).$$

This will make sure that the action of F on $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ defined through equation (4.7) has a unique extension to $\mathcal{E}_I^\theta$, and hence we can use Proposition 4.1. Now, replacing f by $f W^{-1}$, it suffices to check that

$$\langle f, g W\rangle_{\mathcal{A}^\infty} = \alpha_{W^{-1}}\big(\langle f W^{-1}, g\rangle_{\mathcal{A}^\infty}\big). \tag{4.11}$$

Note that

$$\langle f, g\rangle_{\mathcal{A}^\infty}(l) = \langle g U_{-l}, f\rangle_{L^2} \tag{4.12}$$

for $\langle f, g \rangle_{L^2} = \int_G f(x)\overline{g(x)}dx$, and hence

$$\alpha_{W^{-1}}\big(\langle f, g \rangle_{\mathcal{A}^\infty}\big)(l) = \big\langle g\alpha_W(U^\theta_{-l}), f\big\rangle_{L^2}. \tag{4.13}$$

Now,

$$\begin{aligned}
\langle f, gW \rangle_{\mathcal{A}^\infty}(l) &\overset{(4.12)}{=} \big\langle (gW)U_{-l}, f\big\rangle_{L^2} \\
&\overset{(4.9)}{=} \int_{\mathbb{R}^p \times \mathbb{Z}^q} \big(g\alpha_W(U_{-l})W(x)\big)\overline{f(x)}dx, \\
&\overset{(4.8)}{=} \int_{\mathbb{R}^p \times \mathbb{Z}^q} \big(g\alpha_W(U_{-l})\big)(x)\overline{(fW^{-1})(x)}dx, \\
&\overset{(4.13)}{=} \alpha_{W^{-1}}\big(\langle fW^{-1}, g\rangle_{\mathcal{A}^\infty}\big)(l),
\end{aligned}$$

which is the desired identity. ∎

Remark 4.3. The condition of the above theorem holds if and only if $R_I^\Sigma W^t (R_I^\Sigma)^{-1}$ is of the form (**). However, the condition reveals more information: equation (4.4) really means that one can define an action of $\langle S^t \rangle$ on $A_{\sigma_{2p}\theta}$, and the Morita equivalence between A_θ and $A_{\sigma_{2p}\theta}$ can be lifted to an equivalence between the corresponding orbifolds. This will appear in a joint work with Ullisch [7].

As an immediate corollary, we have the following:

Corollary 4.4. *Let Z and t be as in Section 3.3. With all the notations introduced before, assume that $\mathrm{pf}(M_I^{\theta+tZ}) \neq 0$. Let $W \in \mathrm{GL}(n, \mathbb{Z})$ be of finite order such that $W^t \theta W = \theta$, $W^t Z W = Z$ and let $F := \langle W \rangle$. Suppose that $g_{I,\Sigma} F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$ inside $\mathrm{SO}(n, n|\mathbb{Z})$. Then $\mathcal{E}_I^{\theta+tZ}$ becomes a finitely generated, projective module over $A_\theta \rtimes F$.*

Proof. It follows immediately from the preceding theorem (Theorem 4.2) and Theorem 3.7, noting that the isomorphism between $A_{\theta+tZ}$ and A_θ is F-equivariant. ∎

The above corollary shows that under the extra assumption $W^t Z W = Z$, all the fundamental projective modules over $A_{\theta+tZ} \cong A_\theta$ become finitely generated, projective modules over the crossed product.

5. Some applications: Morita equivalence of noncommutative orbifolds

5.1. Trace of the extended module

Let F be a finite group acting on a C^*-algebra A. Also suppose that τ is an F-invariant trace on A. Then we can define a trace τ^F on $A \rtimes F$ by

$$\tau^F\Big(\sum_{g \in F} a_g \delta_g \Big) := \tau(a_e).$$

Let $\mathcal{E}^F$ denote the finitely generated, projective $A \rtimes F$-module, which is obtained from a finitely generated, projective A-module $\mathcal{E}$, as in Proposition 4.1.

Lemma 5.1. $\tau^F([\mathcal{E}^F]) = \frac{\tau([\mathcal{E}])}{|F|}$, *where* $[\mathcal{E}^F]$ *and* $[\mathcal{E}]$ *denote the K-theory classes of* $[\mathcal{E}^F]$ *and* $[\mathcal{E}]$, *respectively.*

Proof. Let p^F denote the projection corresponding to $\mathcal{E}^F$, and p the projection corresponding to $\mathcal{E}$. Define the canonical injection (regular representation) Ψ from $A \rtimes F$ to $A \otimes \mathcal{B}(l^2(F))$ by mapping a to $\sum_{g \in F} g \cdot a \otimes p_g$ (where p_g is the projection onto the functions supported on $\{g\}$) and by mapping δ_g to $1 \otimes \rho(g)$, where ρ is the right regular representation. It is well known that the above map defines an inverse to the Green–Julg map in F-equivariant K-theory (see e.g. [13, p. 191]). If p^F is in $A \rtimes F$, let us write $p^F = \sum_{g \in F} a_g \delta_g$. Then $\tau^F([p^F]) = \tau(a_e)$. On the other hand, $[\Psi(p^F)] = [p]$ in $K(A \otimes \mathcal{B}(l^2(F)))$, and hence, $\tau([\Psi(p^F)]) = \tau([p])$. But $\tau([\Psi(p^F)]) = |F|\tau(a_e)$, using the above formula of Ψ and the fact that τ is F-invariant. Hence $\tau^F([p^F]) = \frac{\tau([p])}{|F|}$. A similar computation holds when p^F is in some matrix algebra over $A \rtimes F$. ∎

5.2. Images of the canonical traces of noncommutative orbifolds

Let us come back to the noncommutative orbifolds. As in Example 2.2, take a finite order matrix $W \in \mathrm{GL}(n, \mathbb{Z})$ such that $W^t \theta W = \theta$. Assume that the order of W is N. We then have $C^*(\mathbb{Z}^n \rtimes F, \omega'_\theta) = A_\theta \rtimes_\alpha F$, $F := \langle W \rangle$.

For $A_\theta \rtimes_\alpha F$, the regular representation $\Psi : A_\theta \rtimes_\alpha F \hookrightarrow \mathrm{M}_N(A_\theta)$ is given by the following:

$$\Psi\left(\sum_{i=0}^{N-1} a_i W^i\right) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ \alpha(a_{N-1}) & \alpha(a_0) & \alpha(a_1) & \cdots & \alpha(a_{N-2}) \\ \alpha^2(a_{N-2}) & \alpha^2(a_{N-1}) & \alpha^2(a_0) & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha^{N-2}(a_1) \\ \alpha^{n-1}(a_1) & \alpha^{N-1}(a_2) & \cdots & \alpha^{N-1}(a_{N-1}) & \alpha^{N-1}(a_0) \end{pmatrix}. \quad (5.1)$$

The canonical trace Tr on A_θ is clearly F-invariant. Now, the canonical trace on $A_\theta \rtimes F$ is given by

$$\mathrm{Tr}^F\left(\sum_{i=0}^{N-1} a_i W^i\right) := \mathrm{Tr}(a_0).$$

If we identify $A_\theta \rtimes F$ inside $\mathrm{M}_N(A_\theta)$ via the map Ψ, the trace Tr^F is the normalised trace on $\mathrm{M}_N(A_\theta)$. This immediately gives

$$\mathrm{Tr}^F\left(\mathrm{K}_0(A_\theta \rtimes F)\right) \subseteq \frac{1}{N} \mathrm{Tr}\left(\mathrm{K}_0(A_\theta)\right). \quad (5.2)$$

So from equation (3.6) we have

$$\mathrm{Tr}^F\left(\mathrm{K}_0(A_\theta \rtimes F)\right) \subseteq \frac{1}{N}\left(\mathbb{Z} + \sum_{0 < |I| \leq n} \mathrm{pf}(M_I^\theta)\mathbb{Z}\right).$$

Our main theorem (Theorem 4.2) gives sufficient conditions on W so that $\frac{1}{N}\operatorname{pf}(M_I^\theta) \in \operatorname{Tr}^F(K_0(A_\theta \rtimes F))$ as we have the following theorem.

Theorem 5.2. *With all the notations introduced before, let $W \in \operatorname{GL}(n, \mathbb{Z})$ be of finite order such that $W^t \theta W = \theta$ and $F := \langle W \rangle$. Suppose that $g_{I,\Sigma} F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$ inside $\operatorname{SO}(n, n | \mathbb{Z})$. Then $\frac{1}{N}\operatorname{pf}(M_I^\theta) \in \operatorname{Tr}^F(K_0(A_\theta \rtimes F))$, where N is the order of W.*

Proof. If $\operatorname{pf}(M_I^\theta) \neq 0$, using Theorem 4.2, $\mathcal{E}_I^\theta$ becomes a projective $A_\theta \rtimes F$-module. Since the trace of $\mathcal{E}_I^\theta$ is $\operatorname{pf}(M_I^\theta)$, use Lemma 5.1. If $\operatorname{pf}(M_I^\theta) = 0$, the statement is obvious. ∎

We now discuss various examples. We observe that the trace of the projection $p_0 := \frac{1}{N}(1 + W + W^2 + \cdots + W^{N-1}) \in K_0(A_\theta \rtimes F)$ is $\frac{1}{N}$. Also for an even n with $n = 2p$, and $I := (1, 2, \ldots, 2p)$, Σ must be trivial. In this case, $g_{I,\Sigma} F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$, since $g_{I,\Sigma}\rho(W^t)(g_{I,\Sigma})^{-1} = \rho(W^{-1})$. Hence

$$\frac{1}{N}\operatorname{pf}(M_I^\theta) = \frac{1}{N}\operatorname{pf}(\theta) \in \operatorname{Tr}^F\left(K_0(A_\theta \rtimes F)\right),$$

for $|I| = n = 2p$. For an odd n, we of course have $\frac{1}{N}\operatorname{pf}(\theta) = 0 \in \operatorname{Tr}^F(K_0(A_\theta \rtimes F))$.

Example 5.3 (2-dimensional cases). Let θ be a real number. For the 2-dimensional torus A_θ, we have actions of $F := \langle W \rangle$, where $W = W_{(2)}, W_{(3)}, W_{(4)}, W_{(6)} \in \operatorname{SL}(2, \mathbb{Z})$, on A_θ as in Section 4. From the above observations, we have $\frac{1}{N}$ and $\frac{1}{N}\operatorname{pf}\left(\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)\right) \in \operatorname{Tr}^F(K_0(A_\theta \rtimes F))$, where $N = 2, 3, 4, 6$ for $W = W_{(2)}, W_{(3)}, W_{(4)}, W_{(6)}$, respectively. But $\operatorname{pf}\left(\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)\right) = \theta$. Hence

$$\operatorname{Tr}^F\left(K_0(A_\theta \rtimes F)\right) = \frac{1}{N}(\mathbb{Z} + \theta\mathbb{Z}) = \frac{1}{N}\operatorname{Tr}\left(K_0(A_\theta)\right). \tag{5.3}$$

In these 2-dimensional cases, the above ranges $\operatorname{Tr}^F(K_0(A_\theta \rtimes F))$ have already been computed in [9].

Example 5.4 (diagonal actions on 4-dimensional tori). First take $\theta_1 \in \mathcal{T}_{n_1}(\mathbb{R})$ and $\theta_2 \in \mathcal{T}_{n_2}(\mathbb{R})$. Let W_1, W_2 be θ_1-symplectic and θ_2-symplectic matrices of order N_1 and N_2, respectively. Then clearly $W := \left(\begin{smallmatrix} W_1 & 0 \\ 0 & W_2 \end{smallmatrix}\right)$ is a $\theta := \left(\begin{smallmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{smallmatrix}\right)$-symplectic matrix of order $N := \operatorname{lcm}(N_1, N_2)$. Hence $F := \langle W \rangle$ acts on A_θ. Clearly, $\operatorname{pf}(\theta), \operatorname{pf}(\theta_1), \operatorname{pf}(\theta_2)$ belong to $\operatorname{Tr}(K_0(A_\theta))$. Now, assume that these three terms are non-zero. So n_1 and n_2 must be even. Then $\frac{\operatorname{pf}(\theta)}{N}$ is in $\operatorname{Tr}^F(K_0(A_\theta \rtimes F))$, from the previous observation. For $I = (1, 2, \ldots, n_1)$ and $I = (n_1 + 1, n_1 + 1, \ldots, n_1 + n_2)$, one can choose $R_I^\Sigma = \operatorname{id}_{n_1 + n_2}$ and $\left(\begin{smallmatrix} 0 & \operatorname{id}_{n_1} \\ \operatorname{id}_{n_2} & 0 \end{smallmatrix}\right)$, respectively. Then one easily checks that in both cases, $g_{I,\Sigma} F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$. Hence $\frac{\operatorname{pf}(\theta_1)}{N}$, $\frac{\operatorname{pf}(\theta_2)}{N}$ are in $\operatorname{Tr}^F(K_0(A_\theta \rtimes F))$.

Let us specialise this example to $n_1 = n_2 = 2$. The 4×4 matrix θ is then given by

$$\theta = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}.$$

In this case,

$$\mathrm{Tr}\left(\mathrm{K}_0(A_\theta)\right) = \mathbb{Z} + \sum_{0 < |I| \le 4} \mathrm{pf}(M_I^\theta)\mathbb{Z}$$
$$= \mathbb{Z} + \mathrm{pf}(M_{(1,2)}^\theta)\mathbb{Z} + \mathrm{pf}(M_{(3,4)}^\theta)\mathbb{Z} + \mathrm{pf}(M_{(1,2,3,4)}^\theta)\mathbb{Z}$$
$$= \mathbb{Z} + \theta_{12}\mathbb{Z} + \theta_{34}\mathbb{Z} + \theta_{12}\theta_{34}\mathbb{Z}.$$

Let us also take $W_1, W_2 \in \mathrm{SL}(2, \mathbb{Z})$ of finite order (say N_1 and N_2, respectively) as in the previous example. Then $F := \langle W \rangle$ acts on A_θ, where $W := \left(\begin{smallmatrix} W_1 & 0 \\ 0 & W_2 \end{smallmatrix}\right)$. Using the above,

$$\mathrm{Tr}^F\left(\mathrm{K}_0(A_\theta \rtimes F)\right) = \frac{1}{N}\mathrm{Tr}\left(\mathrm{K}_0(A_\theta)\right),$$

for $N := \mathrm{lcm}(N_1, N_2)$. One may look at [12] for more examples of a similar kind, where one can compute the ranges of the traces explicitly just like the above.

Example 5.5 (flip actions on n-dimensional noncommutative tori). Let us consider the flip action ($W = -\mathrm{id}_n$) of $\mathbb{Z}_2$ on an n-dimensional noncommutative torus A_θ. In this case, $g_{I,\Sigma}\rho(W^t)(g_{I,\Sigma})^{-1} = \rho(W^t)$. Hence $g_{I,\Sigma}F(g_{I,\Sigma})^{-1} \subset \mathcal{R}$ trivially, for every I and Σ. Hence

$$\mathrm{Tr}^{\mathbb{Z}_2}\left(\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)\right) = \frac{1}{2}\left(\mathbb{Z} + \sum_{0 < |I| \le n} \mathrm{pf}(M_I^\theta)\mathbb{Z}\right) = \frac{1}{2}\mathrm{Tr}\left(\mathrm{K}_0(A_\theta)\right). \tag{5.4}$$

Note that Examples 5.3 and 5.5 also satisfy the conditions of Corollary 4.4.

5.3. Morita equivalence of noncommutative tori and orbifolds

To obtain results about classification, we will restrict ourselves to simple C^*-algebras. We start with the following definition.

Definition 5.6. A skew symmetric real $n \times n$ matrix θ is called *non-degenerate* if, whenever $x \in \mathbb{Z}^n$ satisfies $e(\langle x, \theta y \rangle) = 1$ for all $y \in \mathbb{Z}^n$, $x = 0$.

Let us denote the canonical trace of A_θ by Tr_θ. We want to prove the following theorem.

Theorem 5.7. *Let θ_1 and θ_2 be non-degenerate inside $\mathcal{T}_n$. Let $W \in \mathrm{GL}(n, \mathbb{Z})$ be of finite order such that $W^t\theta_1 W = \theta_1$ and $W^t\theta_2 W = \theta_2$. Also assume that the action of $F := \langle W \rangle$ on $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$. Then $A_{\theta_1} \rtimes F$ is strongly Morita equivalent to $A_{\theta_2} \rtimes F$ if and only if there exists a $\lambda > 0$ such that $\mathrm{Tr}_{\theta_1}^F$ and $\lambda \mathrm{Tr}_{\theta_2}^F$ have the same range.*

It is clear that the actions in Example 5.3, Example 5.5, and the 4-dimensional example in Example 5.4 are free outside the origin $0 \in \mathbb{Z}^n$. Also in [12, 14], various examples of W are constructed which have the same property.

The proof of Theorem 5.7 needs some preparation. Let us first recall the following proposition.

Proposition 5.8 ([18, Proposition 3.7]). *Let A be a simple infinite dimensional separable unital nuclear C^*-algebra with tracial rank zero and which satisfies the universal coefficient theorem. Then A is a simple AH algebra with real rank zero and no dimension growth. If $K_*(A)$ is torsion free, A is an AT algebra. If, in addition, $K_1(A) = 0$, then A is an AF algebra.*

Let $\theta \in \mathcal{T}_n$ be non-degenerate. Then the following are known.

- A_θ is a simple C^*-algebra (even the converse is true: simplicity of A_θ implies that θ must be non-degenerate) with a unique tracial state [18, Theorem 1.9].
- A_θ has tracial rank zero [18, Theorem 3.6].
- If β is an action of a finite group on A_θ which has the tracial Rokhlin property (see [9, Section 5]), $A_\theta \rtimes_\beta F$ is a simple C^*-algebra with tracial rank zero [19, Corollary 1.6, Theorem 2.6]. Also, $A_\theta \rtimes_\beta F$ has a unique tracial state [9, Proposition 5.7].
- Let $W \in GL(n, \mathbb{Z})$ be of finite order such that $W^t \theta W = \theta$. Then the action α of $F := \langle W \rangle$ on A_θ has the tracial Rokhlin property [9, Lemma 5.10 and Theorem 5.5].
- For the action α, $A_\theta \rtimes_\alpha F$ satisfies the universal coefficient theorem [14, Proposition 3.1].
- $K_*(A_\theta \rtimes_\alpha F) \cong K_*(C^*(\mathbb{Z}^n \rtimes F, \omega'_\theta)) \cong K_*(C^*(\mathbb{Z}^n \rtimes F))$ [9, Theorem 0.3].

For the K-groups of $C^*(\mathbb{Z}^n \rtimes F)$, the following result is known.

Theorem 5.9 ([15, Theorem 0.1]). *Let $n, m \in \mathbb{N}$. Consider the extension of groups*

$$1 \to \mathbb{Z}^n \to \mathbb{Z}^n \rtimes \mathbb{Z}_m \to \mathbb{Z}_m \to 1$$

such that the action of $\mathbb{Z}_m$ on $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$. Then

$$K_0\left(C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_m)\right) \cong \mathbb{Z}^{s_0}$$

for some $s_0 \in \mathbb{Z}$ and $K_1(C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_m)) \cong \mathbb{Z}^{s_1}$ for $s_1 \in \mathbb{Z}$. If m is even, $s_1 = 0$.

We are now ready to prove Theorem 5.7.

Proof of Theorem 5.7. The freeness condition and the fact that

$$K_*(A_\theta \rtimes F) \cong K_*\left(C^*(\mathbb{Z}^n \rtimes F)\right)$$

show that the K-groups are torsion free, using Theorem 5.9. Then the above list of results along with Proposition 5.8 shows that $A_{\theta_1} \rtimes F$ and $A_{\theta_2} \rtimes F$ are AT algebras.

Assume that there is a $\lambda > 0$ such that $Tr_{\theta_1}^F$ and $\lambda Tr_{\theta_2}^F$ have the same range. Now, it is enough to find an isomorphism

$$g : K_0(A_{\theta_1} \rtimes F) \to K_0(A_{\theta_2} \rtimes F)$$

such that $\lambda Tr_{\theta_2}^F \circ g = Tr_{\theta_1}^F$. Indeed, g is then an order isomorphism by [2, Proposition 3.7], and $g([1]) \in K_0(A_{\theta_2} \rtimes F)^+$. So there is a $q \in \mathbb{N}$ and a projection $p \in M_q(A_{\theta_2} \rtimes F)$ such

that $[p] = g([1])$. Since $A_{\theta_2} \rtimes F$ is simple, p is full. Then $A_{\theta_1} \rtimes F$ and $p \mathrm{M}_q(A_{\theta_2} \rtimes F)p$ have isomorphic Elliott invariants of AT algebras, so $A_{\theta_1} \rtimes F \cong p \mathrm{M}_q(A_{\theta_2} \rtimes F)p$ by classification ([11], [17, Theorem 5.2]). Clearly, the right-hand side algebra is Morita equivalent to $A_{\theta_2} \rtimes F$.

Let us now see the existence of the isomorphism g. Denote the ranges of $\mathrm{Tr}^F_{\theta_1}$ and $\mathrm{Tr}^F_{\theta_2}$ by R_1 and R_2, respectively. Since R_1 and R_2 are finitely generated subgroups of $\mathbb{R}$, they are free. Also $R_1 = \lambda R_2$ implies that they have the same rank. Now, we have the following exact sequences:

$$0 \to \ker(\mathrm{Tr}^F_{\theta_1}) \to \mathrm{K}_0(A_{\theta_1} \rtimes F) \xrightarrow{\mathrm{Tr}^F_{\theta_1}} R_1 \to 0,$$

$$0 \to \ker(\mathrm{Tr}^F_{\theta_2}) \to \mathrm{K}_0(A_{\theta_2} \rtimes F) \xrightarrow[\mathrm{Tr}^F_{\theta_2}]{} R_2 \to 0.$$

Note that the above sequences split since the K-groups are torsion free. Now, $\ker(\mathrm{Tr}^F_{\theta_1})$ and $\ker(\mathrm{Tr}^F_{\theta_2})$ are finitely generated abelian groups of the same rank. So there exists an isomorphism ψ between them. Now, g is defined as $\psi \oplus \phi$, where ϕ is the map between R_1 and R_2 given by multiplication with $\frac{1}{\lambda}$. Clearly, $\lambda \, \mathrm{Tr}^F_{\theta_2} \circ g = \mathrm{Tr}^F_{\theta_1}$, since the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{K}_0(A_{\theta_1} \rtimes F) & \xrightarrow{\mathrm{Tr}^F_{\theta_1}} & R_1 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \phi} \\
\mathrm{K}_0(A_{\theta_2} \rtimes F) & \xrightarrow[\mathrm{Tr}^F_{\theta_2}]{} & R_2.
\end{array}
$$

For the *only if* part, assume that $\mathcal{A} := A_{\theta_1} \rtimes F$ is strongly Morita equivalent to $\mathcal{B} := A_{\theta_2} \rtimes F$. Let X be an $\mathcal{A} - \mathcal{B}$-imprimitivity bimodule. Define a positive tracial functional τ_X on $\mathcal{B}$ by

$$\tau_X\big(\langle x, y \rangle_{\mathcal{B}}\big) := \mathrm{Tr}^F_{\theta_1}\big({}_{\mathcal{A}}\langle y, x \rangle\big), \quad x, y \in X.$$

By [20, Corollary 2.6], $\mathrm{Tr}^F_{\theta_1}$ and τ_X have the same range. Since $\mathcal{B}$ has a unique trace $\mathrm{Tr}^F_{\theta_2}$, τ_X must be a scalar multiple of that trace. ∎

Now, if $\mathrm{Tr}^F_{\theta_1}$ and Tr_{θ_1} have the same range up to a factor $\frac{1}{|F|}$, and if the same holds for $\mathrm{Tr}^F_{\theta_2}$ and Tr_{θ_2}, we have that $\mathrm{Tr}^F_{\theta_1}$ and $\lambda \, \mathrm{Tr}^F_{\theta_2}$ have the same range if and only if Tr_{θ_1} and $\lambda \, \mathrm{Tr}_{\theta_2}$ have the same range, for some $\lambda > 0$. But the last condition holds if and only if A_{θ_1} is strongly Morita equivalent to A_{θ_2}, using Theorem 5.7. This observation gives us the following corollaries.

Corollary 5.10. *Let θ_1 and θ_2 be irrational numbers and W one of the matrices $W_{(2)}$, $W_{(3)}$, $W_{(4)}$, $W_{(6)}$ as in Section 4 (see also Example 5.3). If $F := \langle W \rangle$, then $A_{\theta_1} \rtimes F$ is strongly Morita equivalent to $A_{\theta_2} \rtimes F$ if and only if A_{θ_1} is strongly Morita equivalent to A_{θ_2}.*

Proof. Since the action of F on $\mathbb{Z}^2$ is free outside the origin $0 \in \mathbb{Z}^2$, the result follows from the tracial range computation in Example 5.3 and Theorem 5.7. ∎

The above corollary is not new; see [2, Theorem 5.3].

Corollary 5.11. *Let $\theta_1, \theta_2 \in \mathcal{T}_n$ be non-degenerate. Let $\mathbb{Z}_2$ act on A_{θ_1} and A_{θ_2} by the flip actions. Then $A_{\theta_1} \rtimes \mathbb{Z}_2$ is strongly Morita equivalent to $A_{\theta_2} \rtimes \mathbb{Z}_2$ if and only if A_{θ_1} is strongly Morita equivalent to A_{θ_2}.*

Proof. It follows similarly as in Corollary 5.10 from the tracial range computation in Example 5.5, noting that the action of $\mathbb{Z}_2$ is free outside the origin $0 \in \mathbb{Z}^n$. ∎

One can definitely build more examples for which similar results can be stated. For example, a similar result is true for the 4-dimensional example in Example 5.4.

From Theorem 3.1, A_θ and $A_{g\theta}$ are Morita equivalent if $g\theta := \frac{A\theta + B}{C\theta + D}$ is well defined for $g \in \mathrm{SO}(n, n|\mathbb{Z})$. Note that if θ is non-degenerate, then A_θ is simple, and hence $A_{g\theta}$ has to be simple so that $g\theta$ is non-degenerate. In the case of 2-dimensional tori, we have even a stronger result due to Marc Rieffel. Rieffel (in [20]) showed that A_θ and $A_{\theta'}$ are Morita equivalent if and only if θ and θ' are in the same $\mathrm{GL}(2, \mathbb{Z})$ orbit; that is, $\theta' = \frac{a\theta + b}{c\theta + d}$ for some matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}(2, \mathbb{Z})$.

It is also known that for a non-degenerate θ, the fixed point algebra A_θ^F is Morita equivalent to the crossed product algebra $A_\theta \rtimes F$ (see [23, Proposition]). Hence as a consequence of Corollary 5.11, we have the following.

Corollary 5.12. *Let θ_1, θ_2, and $\mathbb{Z}_2$ be as in Corollary 5.11. Then $A_{\theta_1}^{\mathbb{Z}_2}$ is strongly Morita equivalent to $A_{\theta_2}^{\mathbb{Z}_2}$ if and only if A_{θ_1} is strongly Morita equivalent to A_{θ_2}.*

A similar statement is true for the 2-dimensional cases (Example 5.3), which follows from Corollary 5.10.

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