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# Symmetries of simple $A\mathbb{T}$-algebras

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**Abstract.** Let $A$ be a unital simple $A\mathbb{T}$-algebra of real rank zero. Given an order two automorphism $h : K_1(A) \to K_1(A)$, we show that there is an order two automorphism $\alpha: A \to A$ such that $\alpha_{*0} =$ id, $\alpha_{*1} = h$ and the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property. Consequently, $C^*(A, \mathbb{Z}_2, \alpha)$ is a simple unital AH-algebra with no dimension growth, and with tracial rank zero. Thus, our main result can be considered the $\mathbb{Z}_2$-action analogue of the Lin–Osaka theorem. As a consequence, a positive answer to a lifting problem of Blackadar is also given for certain split case.

## 1. Introduction

It has been an important issue to find and classify all (or some particular) finite-order automorphisms of a given $C^*$-algebra. Historically, partly because of their intrinsic interest and partly because of their applications in $C^*$-dynamical systems, these kinds of problems have attracted considerable attention in the literature (see [4, 7, 16, 17, 21–23, 25, 26, 28, 29, 34, 41]). One landmark among them is Blackadar's famous construction of symmetries (automorphisms of order 2) on the CAR algebra whose fixed point algebras have nontrivial $K_1$-group [4], hence giving a negative answer to one of two questions about AF-algebras posed by him in [3, 10.11.3]. The other one is a lifting question, which is as follows.

**Question 1.1.** Let $A$ be an AF-algebra and $\sigma$ an automorphism of the scaled ordered group $K_0(A)$ with $\sigma^n =$ id. Is there an automorphism $\alpha$ of $A$ with $\alpha_{*0} = \sigma$ and $\alpha^n =$ id?

More generally, if a certain class of $C^*$-algebras is well understood, one could seek whether every finite group action on the level of $K$-theory of a $C^*$-algebra in this class can be lifted to a group action on the $C^*$-algebra. To be precise, one can consider the following folklore question [1, 39].

**Question 1.2.** If $A$ belongs to a class of unital simple $C^*$-algebras that is classifiable by Elliott invariant and $\sigma: G \to \mathrm{Ell}(A)$ is an action of a finite group on the Elliott invariant of $A$, does there exist an action $\alpha: G \to A$ with $\mathrm{Ell}(\alpha) = \sigma$?

For the case that $A$ is a unital universal coefficient theorem (UCT) Kirchberg algebra, satisfactory answers to this question have been given. Firstly, Benson, Kumjian, and Phillips solved this question affirmatively for $G = \mathbb{Z}_2$ and for unital UCT Kirchberg alge-

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bras $A$ in the Cuntz standard form [2]. Later, this result was extended by Spielberg who showed this question has an affirmative answer for $G = \mathbb{Z}_p$, where $p$ is a prime number, and for an arbitrary unital UCT Kirchberg algebra [38]. Finally, in [27], this was further extended by Katsura to actions of finite groups whose Sylow subgroups are cyclic.

Within the setting of $A$ being a unital simple stably finite $C^*$-algebra, compared to the recent progress in the Elliott classification programme (see, e.g., [19,20,40]), this question is still at an early stage, and there is much to do. We note that even Question 1.1 appears to be still open. Recently, Barlak and Szabó showed that if $A$ is a separable, unital, simple and nuclear $C^*$-algebra with tracial rank zero which satisfies the UCT, then any action of a finite group $G$-action on the Elliott invariant could be lifted to a Rokhlin action of $G$ on $A$, provided that $A$ absorbs the UHF-algebra $M_{|G|^\infty}$ [1, Corollary 2.13].

Our main goal in the present article is to examine Question 1.2 with the setting of $G = \mathbb{Z}_2$ and $A$ being a unital, simple $A\mathbb{T}$-algebra of real rank zero. Recall that an $A\mathbb{T}$-algebra is a $C^*$-algebra which is an inductive limit of $C^*$-algebras that are finite direct sums of matrix algebras over continuous functions on the circle $\mathbb{T}$. Unital simple $A\mathbb{T}$-algebras of real rank zero are classified by Elliott using scaled ordered $K_0$-groups and $K_1$-groups in [13]. Many $C^*$-algebras of interest (e.g., [6, 8, 35]), including irrational rotation algebras [14], are in this class.

As we shall see below (Theorem 3.3), for $A$ being a unital simple $A\mathbb{T}$-algebra of real rank zero, given an order two automorphism $h : K_1(A) \to K_1(A)$, we show that there is a symmetry $\alpha\colon A \to A$ such that $\alpha_{*0} = \mathrm{id}$, $\alpha_{*1} = h$ and the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property. Consequently, $C^*(A, \mathbb{Z}_2, \alpha)$ is a simple unital AH-algebra with no dimension growth, and with tracial rank zero. In fact, in the $\mathbb{Z}$-action setting, Lin and Osaka show that any unital simple $A\mathbb{T}$-algebra $A$ admits an automorphism $\alpha$ with the tracial (cyclic) Rokhlin property such that the induced homomorphism $\alpha_{*1}$ on $K_1(A)$ is equal to any given isomorphism of $K_1(A)$, and the induced homomorphism $\alpha_{*0}$ on $K_0(A)$ is the identity [32, Theorem 3.5]. Therefore, our aforementioned result could be viewed as a $\mathbb{Z}_2$-action analogue of the Lin–Osaka theorem.

In Section 4, as a variation of Theorem 3.3, we obtain the following: Let $\mathfrak{A}$ be the AF-algebra whose scaled ordered group $K_0(\mathfrak{A})$ is $(G \oplus H, (G_+ \setminus \{0\}) \oplus H \cup \{(0,0)\}, \tilde{g} \oplus \tilde{h})$, where $(G, G_+, \tilde{g})$ is the scaled ordered group $K_0(B)$ of a unital simple AF-algebra $B$, and $H$ is a countable torsion-free abelian group, $\tilde{h} \in H$. Let $\sigma$ be an order two automorphism of $K_0(\mathfrak{A})$, defined by $\sigma(g \oplus h) = g \oplus \eta(h)$, where $g \oplus h \in G \oplus H$, and $\eta$ is an order two automorphism of $H$. Then, there is a symmetry $\alpha$ of $\mathfrak{A}$ such that $\alpha_{*0} = \sigma$ and the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property (Theorem 4.1). It is a generalization of [43, Theorem 4.1], where $\eta$ is further assumed to be of type I and hence provides a partial affirmative answer to Question 1.1.

## 2. Preliminaries

In this section, we will review definitions, elementary facts and important results which we need in later sections.

We use the notation $\mathbb{Z}_2$ for $\mathbb{Z}/2\mathbb{Z}$. If $A$ is a $C^*$-algebra and $\alpha : A \to A$ is an automorphism of order two, then we write $C^*(\mathbb{Z}_2, A, \alpha)$, $A^\alpha$ for the crossed product and the fixed point subalgebra of $A$ by the action of $\mathbb{Z}_2$ generated by $\alpha$, respectively. Given a $C^*$-algebra $A$, and a unitary $u \in A$, we denote by $\mathrm{Ad}\, u : A \to A$ the continuous linear map $\mathrm{Ad}\, u(a) = u^* a u$. Let RR denote the real rank. We take $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$. Throughout this paper, an AF-algebra is always assumed to be a non-elementary one.

We shall assume that the reader is familiar with the notions and fundamental properties of the inductive limits of $C^*$-algebras and those of abelian groups. We shall also assume that the reader is familiar with $K$-theory, especially the functionalities of $K_0$ and $K_1$, as found in [30, 36]. We also assume that the reader is familiar with approximately finite algebras, or AF-algebras, as inductive limits of finite-dimensional $C^*$-algebras and their classification [12] in terms of $K$-theory. The reader may refer to [30] for more details if required.

**2.1.** We shall use $a^{\sim k}$ to denote $\overbrace{a, \ldots, a}^{k \text{ copies}}$ as used in [18, 1.1.7 (b)]. For example,

$$\{a^{\sim 2}, b^{\sim 3}\} = \{a, a, b, b, b\}.$$

**2.2.** Under the canonical bases of $\mathbb{Z}^n$ and $\mathbb{Z}^m$, we shall identify a group homomorphism $T$ from $\mathbb{Z}^n \to \mathbb{Z}^m$ with its matrix representation $T = (t_{i,j}) \in M_{m \times n}(\mathbb{Z})$. Set

$$|T|_{\max} = \max\left\{|t_{i,j}|,\ 1 \le i \le m,\ 1 \le j \le n\right\},$$

and

$$|T|_{\min} = \min\left\{|t_{i,j}|,\ 1 \le i \le m,\ 1 \le j \le n\right\}.$$

**2.3.** Define

$$\nu(t) = \begin{cases} 0, & t > 0; \\ 1, & t < 0. \end{cases}$$

Let $\lambda$ be a homeomorphism of $X$. For $n \ge 0$, define $\lambda^n$ as the power $n$ iteration of $\lambda$; in particular, $\lambda^0 = \mathrm{id}_X$.

**2.4.** Let $A$, $B$ be unital $C^*$-algebras and $\Phi : A \to B$ a unital homomorphism. Let us denote by $\Phi_{*i} : K_i(A) \to K_i(B)$ the map induced by $\Phi$, $i = 0, 1$.

**2.5.** Here are some basic $K$-theory properties of the reflection maps of $S^n$, $n = 1, 2$.

(1) Let $(w_1, w_2) \in S^1$, and let $\lambda$ be the reflection map defined by

$$\lambda(w_1, w_2) = (w_1, -w_2).$$

It is well known that $K_0(C(S^1)) = \mathbb{Z}$, $K_1(C(S^1)) = \mathbb{Z}$, and $\lambda_{*0}(m) = m$, $\lambda_{*1}(n) = -n$, for $m \in K_0(C(S^1))$, $n \in K_1(C(S^1))$.

(2) Let $(w_1, w_2, w_3) \in S^2$, and let $\lambda$ be the reflection map defined by

$$\lambda(w_1, w_2, w_3) = (w_1, w_2, -w_3).$$

It is well known that $K_0(C(S^2)) = \mathbb{Z} \oplus \mathbb{Z}$, and $\lambda_{*0}(m, n) = (m, -n)$, for $(m, n) \in K_0(C(S^2))$, where the first coordinate of $\mathbb{Z} \oplus \mathbb{Z}$ denotes the rank part.

**2.6.** Let $A = \lim_{k\to\infty}(A_k, \Phi_k)$ be an inductive limit of $C^*$-algebras. Here, $\Phi_k$ is a homomorphism from $A_k$ to $A_{k+1}$. We will use $\Phi_{k,\infty} : A_k \to A$ to denote the homomorphism induced by the inductive limit system. Similarly, the notions could be defined *mutatis mutandis* to the setting of an inductive limit of abelian groups.

**Definition 2.7.** Let $G_1 = \oplus_{i=1}^{m_1} G_{1,i}$ and $G_2 = \oplus_{j=1}^{m_2} G_{2,j}$, where $G_{1,i} \cong G_{2,j} \cong \mathbb{Z}$. Let $\pi_j : G_2 \to G_{2,j}$ be the quotient maps. Suppose that $\varphi : G_1 \to G_2$. A partial map of $\varphi$ from $G_{1,i}$ to $G_{2,j}$ is the map $\varphi^{(i,j)} = \pi_j \circ \varphi|_{G_{1,i}}$ induced by $\varphi$. If $\varphi^{(i,j)}(1) = l$, then we say the multiplicity of partial map $\varphi^{(i,j)}$, denoted by $|\varphi^{(i,j)}|$, is $l$.

**Definition 2.8.** Let $\Phi$ be a unital homomorphism from $A_1 = \oplus_{i=1}^{m_1} M_{l(1,i)}(C(X_i))$ to $A_2 = \oplus_{j=1}^{m_2} M_{l(2,j)}(C(Y_j))$. For any $i, j$, if the partial map $\Phi^{(i,j)}$, the restriction of the map $\Phi$ to any direct summands $M_{l(1,i)}(C(X_i))$ and $M_{l(2,j)}(C(Y_j))$, has the form

$$f \mapsto \begin{bmatrix} f\circ\lambda_1 & & \\ & \ddots & \\ & & f\circ\lambda_{d_{(i,j)}} \end{bmatrix}$$

for some positive integer $d(i,j)$ and some continuous maps $\lambda_1, \dots, \lambda_{d_{i,j}} : Y_j \to X_i$, then $\Phi$ is called a diagonal map, and $|\Phi^{(i,j)}| := d(i,j)$ is called the multiplicity of $\Phi^{(i,j)}$.

The following is the Elliott–Gong classification theorem.

**Theorem 2.9** ([15]). *Let $A$ and $B$ be two unital simple AH-algebras with slow dimension growth and with real rank zero. Then, $A \cong B$ if and only if*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Lastly, we introduce a special case of a useful criterion for an action of $\mathbb{Z}_2$ to have the tracial Rokhlin property obtained by Phillips. The reader is referred to Phillips' seminal paper [33] for details and more background information about tracial Rokhlin property.

**Lemma 2.10** ([34, Lemma 1.8]). *Let $A$ be a separable infinite-dimensional simple unital $C^*$-algebra with tracial rank zero. Let $\alpha \in \mathrm{Aut}(A)$ satisfy $\alpha^2 = \mathrm{id}_A$. Suppose that for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_0$, $e_1$ such that*

(1) $\|\alpha(e_0) - e_1\| < \varepsilon$;

(2) $\|e_j a - a e_j\| < \varepsilon$ *for all* $a \in F$ *and* $j = 0, 1$;

(3) *with* $e = e_0 + e_1$, $\tau(1-e) < \varepsilon$ *for each tracial state* $\tau$ *on* $A$.

*The action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property.*

## 3. $\mathbb{Z}_2$-action analogue of the Lin–Osaka theorem

The purpose of this section is to present Theorem 3.3. We will start with a construction about the equivalence of two Bratteli diagrams, which proves very useful in our later constructions.

**Lemma 3.1.** *Let $B$ be a unital simple* AF*-algebra. Let $\{n_k\}, \{c_k\}$ be two increasing sequences of positive integers. Then, $B$ could be written as*

$$B = \lim_{k\to\infty} (B_k, \Psi_k)$$

*such that for $k \in \mathbb{N}$,*

(1) $B_k = M_{l(k,1)} \oplus \cdots \oplus M_{l(k,m_k)}$, *where $m_k \geq 3n_k$;*

(2) $\Psi_k$ *is diagonal, and each partial map of $\Psi_k$ has a positive multiplicity of at least $(2k+3)c_k$;*

(3) *for $1 \leq i \leq n_k$, $1 \leq j \leq m_{k+1}$, $|\Psi_k^{(i,j)}| = (2k+3)c_k$; for $n_k + 1 \leq i \leq m_k$, $1 \leq j_1, j_2 \leq n_{k+1}$, $|\Psi_k^{(i,j_1)}| = |\Psi_k^{(i,j_2)}|$;*

(4) $l(k,1) = \cdots = l(k,n_k)$.

*Proof.* By [30, Proposition 4.7.2 and Lemma 4.7.3], $B$ can be written so that

$$B = \lim_{k\to\infty} (C_k, \Phi_k),$$

where $C_k = \oplus_{r=1}^{l_k} M_{h(k,r)}$, $G_k' := K_0(C_k)$ is a finite direct sum of $l_k$ copies of $\mathbb{Z}$ with $l_k \geq 2n_k$ and each partial map of $\Phi_k$ has a positive multiplicity of at least 2, $k \in \mathbb{N}$. Without loss of generality, we may assume that the connecting map $\Phi_k$ is a diagonal map, $k \in \mathbb{N}$. Since $B$ is simple, by passing to a subsequence if necessary, we may further assume that

$$|\Phi_k^{(r,s)}| \geq 2n_k(2k+3)c_k,$$

where $1 \leq r \leq l_k, 1 \leq s \leq l_{k+1}$. For $k \in \mathbb{N}$, set $\varphi_k = (\Phi_k)_{*0}$.

We will first microscope the Bratteli diagram in a suitable way and then telescope it. To have a quick understanding about the construction, it is useful to draw the corresponding Bratteli diagrams.

Fix $k \in \mathbb{N}$, and let $G_k = \mathbb{Z}^{m_k}$, where $m_k := l_k + n_k \geq 3n_k$.

(a) For $r = 1$, define $\delta_k^{(1,i)} = 1$, for $1 \leq i \leq n_k + 1$; define $\delta_k^{(1,i)} = 0$, for $2 + n_k \leq i \leq m_k$. For each $2 \leq r \leq l_k$, define $\delta_k^{(r,i)} = 1$, for $i = r + n_k$; define $\delta_k^{(r,i)} = 0$, otherwise. Hence, we define a homomorphism $\delta_k$ from $G_k'$ to $G_k$.

(b) For $1 \leq i \leq n_k$, for $1 \leq s \leq l_{k+1}$, define $\theta_k^{(i,s)} = (2k+3)c_k$. For $i = n_k + 1$, for $1 \leq s \leq l_{k+1}$, define $\theta_k^{(i,s)} = |\Phi_k^{(1,s)}| - n_k(2k+3)c_k$. For $n_k + 2 \leq i \leq m_k$, for $1 \leq s \leq l_{k+1}$, define $\theta_k^{(i,s)} = |\Phi_k^{(i-n_k,s)}|$. Consequently, we define a homomorphism $\theta_k$ from $G_k$ to $G_{k+1}'$ similarly.

Fix $k \in \mathbb{N}$. It is routine to verify that $\varphi_k = \theta_k \circ \delta_k$. In fact, for $r = 1$, $1 \leq s \leq l_{k+1}$,

$$\begin{aligned}\big|(\theta_k \circ \delta_k)^{(1,s)}\big| &= \sum_{i=1}^{m_k} \delta_k^{(1,i)} \theta_k^{(i,s)} = \sum_{i=1}^{n_k+1} \theta_k^{(i,s)} \\ &= n_k(2k+3)c_k + \big[|\Phi_k^{(1,s)}| - n_k(2k+3)c_k\big] = |\Phi_k^{(1,s)}| = |\varphi_k^{(1,s)}|;\end{aligned}$$

for $2 \le r \le l_k$, $1 \le s \le l_{k+1}$,

$$\begin{aligned}\left|(\theta_k \circ \delta_k)^{(r,s)}\right| &= \sum_{i=1}^{m_k} \delta_k^{(r,i)} \theta_k^{(i,s)} = \delta_k^{(r,r+n_k)} \theta_k^{(r+n_k,s)} \\ &= |\Phi_k^{(r+n_k-n_k,s)}| = |\Phi_k^{(r,s)}| = |\varphi_k^{(r,s)}|.\end{aligned}$$

Therefore,

$$G_1' \xrightarrow{\delta_1} G_1 \xrightarrow{\theta_1} G_2' \xrightarrow{\delta_2} G_2 \xrightarrow{\theta_2} \cdots G_k' \xrightarrow{\delta_k} G_k \xrightarrow{\theta_k} G_{k+1}' \xrightarrow{\delta_{k+1}} \cdots \to K_0(B).$$

For $k \in \mathbb{N}$, define $\psi_k := \delta_{k+1} \circ \theta_k$ as a homomorphism from $G_k$ to $G_{k+1}$. For $1 \le i \le n_k$, $1 \le j \le n_{k+1} + 1$,

$$|\psi_k^{(i,j)}| = \sum_{s=1}^{l_{k+1}} \theta_k^{(i,s)} \delta_{k+1}^{(s,j)} = (2k+3)c_k \sum_{s=1}^{l_{k+1}} \delta_{k+1}^{(s,j)} = (2k+3)c_k \delta_{k+1}^{(1,j)} = (2k+3)c_k;$$

for $1 \le i \le n_k$, $n_{k+1} + 2 \le j \le m_{k+1}$,

$$|\psi_k^{(i,j)}| = \sum_{s=1}^{l_{k+1}} \theta_k^{(i,s)} \delta_{k+1}^{(s,j)} = (2k+3)c_k \sum_{s=1}^{l_{k+1}} \delta_{k+1}^{(s,j)} = (2k+3)c_k \delta_{k+1}^{(j-n_{k+1},j)} = (2k+3)c_k.$$

For $i = n_k + 1$, $1 \le j \le n_{k+1} + 1$,

$$|\psi_k^{(i,j)}| = \sum_{s=1}^{l_{k+1}} \theta_k^{(i,s)} \delta_{k+1}^{(s,j)} = \sum_{s=1}^{l_{k+1}} \left[|\Phi_k^{(1,s)}| - n_k(2k+3)c_k\right] \delta_{k+1}^{(s,j)} = |\Phi_k^{(1,1)}| - n_k(2k+3)c_k,$$

which is independent of $j$; for $i = n_k + 1$, $n_{k+1} + 2 \le j \le m_{k+1}$,

$$\begin{aligned}|\psi_k^{(i,j)}| &= \sum_{s=1}^{l_{k+1}} \theta_k^{(i,s)} \delta_{k+1}^{(s,j)} = \sum_{s=1}^{l_{k+1}} \left[|\Phi_k^{(1,s)}| - n_k(2k+3)c_k\right] \delta_{k+1}^{(s,j)} \\ &= |\Phi_k^{(1,j-n_{k+1})}| - n_k(2k+3)c_k.\end{aligned}$$

For $n_k + 2 \le i \le m_k$, $1 \le j \le n_{k+1} + 1$,

$$|\psi_k^{(i,j)}| = \sum_{s=1}^{l_{k+1}} \theta_k^{(i,s)} \delta_{k+1}^{(s,j)} = \sum_{s=1}^{l_{k+1}} |\Phi_k^{(i-n_k,s)}| \delta_{k+1}^{(s,j)} = |\Phi_k^{(i-n_k,1)}|,$$

which is independent of $j$; for $n_k + 2 \le i \le m_k$, $n_{k+1} + 2 \le j \le m_{k+1}$,

$$|\psi_k^{(i,j)}| = \sum_{s=1}^{l_{k+1}} \theta_k^{(i,s)} \delta_{k+1}^{(s,j)} = \sum_{s=1}^{l_{k+1}} |\Phi_k^{(i-n_k,s)}| \delta_{k+1}^{(s,j)} = |\Phi_k^{(i-n_k,j-n_{k+1})}|.$$

Summarizing, for $1 \le i \le n_k$, $1 \le j \le m_{k+1}$, $|\psi_k^{(i,j)}| = (2k+3)c_k$; for $n_k + 1 \le i \le m_k$, $1 \le j_1, j_2 \le n_{k+1}$, $|\psi_k^{(i,j_1)}| = |\psi_k^{(i,j_2)}|$. Since the multiplicity of each part map of $\Phi_k$ is at least $2n_k(2k+3)c_k$, it follows that

$$|\psi_k^{(i,j)}| \ge (2k+3)c_k, \quad 1 \le i \le m_k,\ 1 \le j \le m_{k+1},\ k \in \mathbb{N}.$$

Let $B_1 = \oplus_{i=1}^{m_1} M_{l(1,i)}$, where $l(1,i) = h(1,1)$, for $1 \le i \le n_1$, and $l(1,i) = h(1, i-n_1)$, for $1 + n_1 \le i \le m_1$. Define

$$l(k+1, j) = \sum_{i=1}^{m_k} |\psi_k^{(i,j)}| l(k,i)$$

inductively. Since for $1 \le j_1, j_2 \le n_{k+1}$,

$$|\psi_k^{(i,j_1)}| = |\psi_k^{(i,j_2)}|, \quad \forall 1 \le i \le m_k,$$

it follows that

$$l(k+1, j_1) = \sum_{i=1}^{m_k} |\psi_k^{(i,j_1)}| l(k,i) = \sum_{i=1}^{m_k} |\psi_k^{(i,j_2)}| l(k,i) = l(k+1, j_2).$$

Set $B_k = \oplus_{i=1}^{m_k} M_{l(k,i)}$, $k \ge 2$. Fix $k \in \mathbb{N}$, and define

$$\Delta_k^{(r,i)}(a) = \operatorname{diag}\{a^{\sim \delta_k^{(r,i)}}\},$$

where $a \in M_{h(k,r)}$, $1 \le r \le l_k$, $1 \le i \le m_k$. Set $\Delta_k = \oplus_{r,i} \Delta_k^{(r,i)} : C_k \to B_k$. Similarly, define

$$\Theta_k^{(i,s)}(a) = \operatorname{diag}\{a^{\sim \theta_k^{(i,s)}}\},$$

where $a \in M_{l(k,i)}$, $1 \le i \le m_k$, $1 \le s \le l_{k+1}$. Set $\Theta_k = \oplus_{i,s} \Theta_k^{(i,s)} : B_k \to C_{k+1}$.

Then, it is standard to check $\Phi_k = \Theta_k \circ \Delta_k$, $k \in \mathbb{N}$; hence,

$$C_1 \xrightarrow{\Delta_1} B_1 \xrightarrow{\Theta_1} C_2 \xrightarrow{\Delta_2} B_2 \xrightarrow{\Theta_2} \cdots \xrightarrow{\Delta_k} B_k \xrightarrow{\Theta_k} C_{k+1} \xrightarrow{\Delta_{k+1}} B_{k+1} \cdots \to B.$$

For $k \in \mathbb{N}$, define

$$\Psi_k = \Delta_{k+1} \circ \Theta_k : B_k \to B_{k+1}.$$

Then, it is obvious that

$$B = \lim_{k \to \infty} (B_k, \Psi_k).$$

Finally, the lemma now follows from the constructions. ■

The following proposition could be found in [43]. It states that any order two automorphism of a countable torsion-free abelian group is actually an inductive limit action. For the reader's convenience, we give a detailed proof here (comparing with the original proof in [43]).

**Proposition 3.2.** *Let $H$ be a nonzero countable torsion-free abelian group and $\eta$ an order two automorphism of $H$. Then, there are a nondecreasing sequence of positive integers $\{n_k : k \in \mathbb{N}\}$ and monomorphisms $\beta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$, order two automorphisms $\eta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k}$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & H \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \downarrow & & \downarrow{\scriptstyle \eta} \\
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & H
\end{array}$$

*and hence $H = \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \beta_k)$, $\eta = \lim_{k\to\infty} \eta_k$. Moreover, it can be required that under the canonical basis of $\mathbb{Z}^{n_k}$, $\eta_k$ has the form*

$$\eta_k = \operatorname{diag}\left\{ \overbrace{1,\dots,1}^{p_k}; \overbrace{-1,\dots,-1}^{q_k}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r_k} \right\}$$

*for suitable nonnegative integers $p_k$, $q_k$, $r_k$ such that $p_k + q_k + 2r_k = n_k$, $k \in \mathbb{N}$.*

*Proof.* Since $H$ is countable, we can write $H$ as $H = \{e_1, e_2, \dots\}$. Define

$$H_k = \mathbb{Z}\big[e_1, \dots, e_k; \eta(e_1), \dots, \eta(e_k)\big],$$

so for $k \in \mathbb{N}$, $\eta(H_k) = H_k$, $H = \lim_{k\to\infty}(H_k, \iota_k)$, where $\iota_k$ is the embedding map from $H_k$ to $H_{k+1}$. Since $H_k$ is finitely generated and torsion-free, there exist a positive integer $n_k$ and an isomorphism $\chi_k$ such that $\chi_k(\mathbb{Z}^{n_k}) = H_k$. Since $H_k \subset H_{k+1}$, we have $n_k \le n_{k+1}$. For $k \in \mathbb{N}$, define $\psi_k = \chi_{k+1}^{-1} \circ \iota_k \circ \chi_k$. Thus, it is easy to check that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\psi_1} & \mathbb{Z}^{n_2} & \xrightarrow{\psi_2} & \mathbb{Z}^{n_3} & \xrightarrow{\psi_3} & \cdots & \longrightarrow & \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \psi_k) \\
\downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} & & \downarrow{\scriptstyle \chi_3} & & & & \\
H_1 & \xrightarrow{\iota_1} & H_2 & \xrightarrow{\iota_2} & H_3 & \xrightarrow{\iota_3} & \cdots & \longrightarrow & H \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & & & \\
H_1 & \xrightarrow{\iota_1} & H_2 & \xrightarrow{\iota_2} & H_3 & \xrightarrow{\iota_3} & \cdots & \longrightarrow & H \\
\downarrow{\scriptstyle \chi_1^{-1}} & & \downarrow{\scriptstyle \chi_2^{-1}} & & \downarrow{\scriptstyle \chi_3^{-1}} & & & & \\
\mathbb{Z}^{n_1} & \xrightarrow{\psi_1} & \mathbb{Z}^{n_2} & \xrightarrow{\psi_2} & \mathbb{Z}^{n_3} & \xrightarrow{\psi_3} & \cdots & \longrightarrow & \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \psi_k).
\end{array}$$

Thus,

$$\lim_{k\to\infty}(\mathbb{Z}^{n_k}, \psi_k) = H.$$

Moreover, $\theta_k := \chi_k^{-1} \circ \eta \circ \chi_k$ is an order two automorphism of $\mathbb{Z}^{n_k}$, $k \in \mathbb{N}$, or equivalently, $\theta_k$ is an involution matrix in $M_{n_k}(\mathbb{Z})$. By [24, Lemma 1] (or [2, Lemma 2.1]), for

$k \in \mathbb{N}$, there are an invertible matrix $S \in M_{n_k}(\mathbb{Z})$ and nonnegative integers $p_k$, $q_k$, $r_k$ with

$$p_k + q_k + 2r_k = n_k$$

such that

$$\theta_k = S_k^{-1}\eta_k S_k,$$

where

$$\eta_k := \operatorname{diag}\left\{\overbrace{1,\dots,1}^{p_k};\overbrace{-1,\dots,-1}^{q_k};\overbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix},\dots,\begin{bmatrix}0&1\\1&0\end{bmatrix}}^{r_k}\right\}.$$

For $k \in \mathbb{N}$, set $\beta_k = S_{k+1}\psi_k S_k^{-1}$. Thus, it is easy to check that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & \lim_{k\to\infty}(\mathbb{Z}^{n_k},\beta_k) \\
\downarrow{\scriptstyle S_1^{-1}} & & \downarrow{\scriptstyle S_2^{-1}} & & \downarrow{\scriptstyle S_3^{-1}} & & & & \\
\mathbb{Z}^{n_1} & \xrightarrow{\psi_1} & \mathbb{Z}^{n_2} & \xrightarrow{\psi_2} & \mathbb{Z}^{n_3} & \xrightarrow{\psi_3} & \cdots & \longrightarrow & H \\
\downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} & & & & \\
\mathbb{Z}^{n_1} & \xrightarrow{\psi_1} & \mathbb{Z}^{n_2} & \xrightarrow{\psi_2} & \mathbb{Z}^{n_3} & \xrightarrow{\psi_3} & \cdots & \longrightarrow & H \\
\downarrow{\scriptstyle S_1} & & \downarrow{\scriptstyle S_2} & & \downarrow{\scriptstyle S_3} & & & & \\
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & \lim_{k\to\infty}(\mathbb{Z}^{n_k},\beta_k).
\end{array}$$

Therefore,

$$H = \lim_{k\to\infty}(\mathbb{Z}^{n_k},\beta_k).$$

Note that $\iota_k$ is injective, so is $\psi_k$, and then, so is $\beta_k$. The proposition follows immediately from the constructions. ∎

We are now in a position to prove the main result.

**Theorem 3.3.** *Let $A$ be a unital simple $A\mathbb{T}$-algebra with real rank zero. Let $h : K_1(A) \to K_1(A)$ be an automorphism with $h^2 = \mathrm{id}_{K_1(A)}$. Then, there exists an automorphism $\alpha : A \to A$ such that $\alpha^2 = \mathrm{id}$, $\alpha_{*0} = \mathrm{id}$, $\alpha_{*1} = h$, and the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property. In this case, $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple AH-algebra with no dimension growth, and with tracial rank zero.*

*Proof.* Set $X = S^1$, and let $\lambda$ be the homeomorphism of $X$ defined by $\lambda(w_1, w_2) = (w_1, -w_2)$, where $(w_1, w_2) \in S^1$. It is evident that $\lambda^2 = \mathrm{id}$. Let $z_0$ be a fixed point of $\lambda$ and $\{x_i : i \in \mathbb{N}\}$ a dense set of $X$. Suppose that $K_1(A) \neq 0$. (We will make a short remark for the trivial case $K_1(A) = 0$ afterwards.)

We divide the proof into four steps.

**Step 1.** By Proposition 3.2, there are an increasing sequence of positive integers $\{n_k : k \in \mathbb{N}\}$ and monomorphisms $\beta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$, order two automorphisms $\eta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & K_1(A) \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \downarrow & & \downarrow{\scriptstyle h} \\
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & K_1(A)
\end{array}$$

and

$$K_1(A) = \lim_{k\to\infty} (\mathbb{Z}^{n_k}, \beta_k), \quad h = \lim_{k\to\infty} \eta_k.$$

Moreover, under the canonical basis of $\mathbb{Z}^{n_k}$, $\eta_k$ has the form

$$\eta_k = \mathrm{diag}\left\{ \overbrace{1, \ldots, 1}^{p_k}; \overbrace{-1, \ldots, -1}^{q_k}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r_k} \right\}$$

for suitable nonnegative integers $p_k, q_k, r_k$ such that $p_k + q_k + 2r_k = n_k$, $k \in \mathbb{N}$.

Since $\beta_k, \gamma_k := \beta_k \circ \eta_k = \eta_{k+1} \circ \beta_k$ are monomorphisms from $\mathbb{Z}^{n_k}$ to $\mathbb{Z}^{n_{k+1}}$, write $\beta_k$ and $\gamma_k$ as

$$\beta_k = \begin{bmatrix} b_{1,1}^{(k)} & b_{1,2}^{(k)} & \cdots & b_{1,n_k}^{(k)} \\ b_{2,1}^{(k)} & b_{2,2}^{(k)} & \cdots & b_{2,n_k}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_{k+1},1}^{(k)} & b_{n_{k+1},2}^{(k)} & \cdots & b_{n_{k+1},n_k}^{(k)} \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} r_{1,1}^{(k)} & r_{1,2}^{(k)} & \cdots & r_{1,n_k}^{(k)} \\ r_{2,1}^{(k)} & r_{2,2}^{(k)} & \cdots & r_{2,n_k}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n_{k+1},1}^{(k)} & r_{n_{k+1},2}^{(k)} & \cdots & r_{n_{k+1},n_k}^{(k)} \end{bmatrix}.$$

Note that $A$ is a unital simple $A\mathbb{T}$-algebra. Let $B$ be a unital simple AF-algebra such that

$$\big(K_0(B), K_0(B)_+, [1_B]\big) = \big(K_0(A), K_0(A)_+, [1_A]\big).$$

Set $c_k = |\beta_k|_{\max} + k$, $k \in \mathbb{N}$. By Lemma 3.1, we may assume that

$$B = \lim_{k\to\infty} (B_k, \Psi_k),$$

where $B_k = M_{l(k,1)} \oplus \cdots \oplus M_{l(k,m_k)}$ is a finite-dimensional $C^*$-algebra such that

(1) $m_k \geq 3n_k$, $l(k,1) = \cdots = l(k,n_k)$;

(2) $|\psi_k|_{\min} \geq (2k+3)c_k$, where $\psi_k := (\Psi_k)_{*0}$;

(3) for $1 \leq i \leq n_k$, $1 \leq j \leq m_{k+1}$, $|\Psi_k^{(i,j)}| = (2k+3)c_k$; for $n_k + 1 \leq i \leq m_k$, $1 \leq j_1, j_2 \leq n_{k+1}$, $|\Psi_k^{(i,j_1)}| = |\Psi_k^{(i,j_2)}|$.

Similarly, write $\psi_k$ as

$$\psi_k = \begin{bmatrix} s^{(k)}_{1,1} & s^{(k)}_{1,2} & \cdots & s^{(k)}_{1,n_k} & s^{(k)}_{1,n_k+1} & \cdots & s^{(k)}_{1,m_k} \\ s^{(k)}_{2,1} & s^{(k)}_{2,2} & \cdots & s^{(k)}_{2,n_k} & s^{(k)}_{2,n_k+1} & \cdots & s^{(k)}_{2,m_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s^{(k)}_{n_{k+1},1} & s^{(k)}_{n_{k+1},2} & \cdots & s^{(k)}_{n_{k+1},n_k} & s^{(k)}_{n_{k+1},n_k+1} & \cdots & s^{(k)}_{n_{k+1},m_k} \\ s^{(k)}_{n_{k+1}+1,1} & s^{(k)}_{n_{k+1}+1,2} & \cdots & s^{(k)}_{n_{k+1}+1,n_k} & s^{(k)}_{n_{k+1}+1,n_k+1} & \cdots & s^{(k)}_{n_{k+1}+1,m_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s^{(k)}_{m_{k+1},1} & s^{(k)}_{m_{k+1},2} & \cdots & s^{(k)}_{m_{k+1},n_k} & s^{(k)}_{m_{k+1},n_k+1} & \cdots & s^{(k)}_{m_{k+1},m_k} \end{bmatrix}.$$

Then, it follows that

(1) for $1 \le i \le n_k$, $1 \le j \le m_{k+1}$, $s^{(k)}_{j,i} = (2k+3)c_k$;

(2) for $n_k + 1 \le i \le m_k$, $1 \le j_1, j_2 \le n_{k+1}$, $s^{(k)}_{j_1,i} = s^{(k)}_{j_2,i}$;

(3) $|\psi_k|_{\min} \ge (2k+3)c_k$.

These will be used again and again.

**Step 2.** For $k \in \mathbb{N}$, define

$$A_k = \big(M_{l(k,1)}\big(C(X)\big) \oplus \cdots \oplus M_{l(k,n_k)}\big(C(X)\big)\big) \oplus (M_{l(k,n_k+1)} \oplus \cdots \oplus M_{l(k,m_k)}).$$

Then,

$$K_0(A_k) = \mathbb{Z}^{m_k}, \quad K_0(A_k)_+ = \mathbb{Z}^{m_k}_+, \quad K_1(A_k) = \mathbb{Z}^{n_k},$$

where

$$\mathbb{Z}^{m_k}_+ := \big\{(\lambda_1, \ldots, \lambda_{m_k}) \in \mathbb{Z}^{m_k} : \lambda_i \ge 0,\ 1 \le i \le m_k\big\}.$$

We next define two unital monomorphisms $\Phi_k, \Theta_k : A_k \to A_{k+1}$. Here, we may recall the notation $\nu(t)$ defined in 2.3.

(1) For $1 \le i \le n_k$, and $1 \le j \le n_{k+1}$, if $b^{(k)}_{j,i} \ne 0$, define

$$\Phi^{(i,j)}_k(f) = \operatorname{diag}\big\{\big(f \circ \lambda^{(\nu(b^{(k)}_{j,i}))}\big)^{\sim|b^{(k)}_{j,i}|}; f(z_0)^{\sim(c_k-|b^{(k)}_{j,i}|)+2kc_k}; \\ f(x_1), f\big(\lambda(x_1)\big); \ldots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\big\};$$

if $b^{(k)}_{j,i} = 0$, define

$$\Phi^{(i,j)}_k(f) = \operatorname{diag}\big\{f(z_0)^{\sim c_k}, f(z_0)^{\sim 2kc_k}; f(x_1), f\big(\lambda(x_1)\big); \ldots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\big\};$$

similarly, if $r^{(k)}_{j,i} \ne 0$, define

$$\Theta^{(i,j)}_k(f) = \operatorname{diag}\big\{\big(f \circ \lambda^{(\nu(r^{(k)}_{j,i}))}\big)^{\sim|r^{(k)}_{j,i}|}; f(z_0)^{\sim(c_k-|r^{(k)}_{j,i}|)+2kc_k}; \\ f(x_1), f\big(\lambda(x_1)\big); \ldots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\big\};$$

if $r_{j,i}^{(k)} = 0$, define

$$\Theta_k^{(i,j)}(f) = \operatorname{diag}\big\{f(z_0)^{\sim c_k}, f(z_0)^{\sim 2kc_k}; f(x_1), f\big(\lambda(x_1)\big); \dots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\big\}.$$

(2) For $1 \le i \le n_k$, and $n_{k+1} + 1 \le j \le m_{k+1}$, define

$$\Phi_k^{(i,j)}(f) = \Theta_k^{(i,j)}(f) = \operatorname{diag}\big\{f(z_0)^{\sim c_k}; f(z_0)^{\sim 2kc_k}; f(z_0)^{\sim 2c_k}\big\}.$$

(3) For $n_k + 1 \le i \le m_k$, and $1 \le j \le m_{k+1}$, define

$$\Phi_k^{(i,j)}(a) = \Theta_k^{(i,j)}(a) = \operatorname{diag}\big\{a^{\sim\gamma(k,i,j)}; a^{\sim 2\xi(k,i,j)}\big\},$$

where

$$\gamma(k,i,j) = 1, \quad \xi(k,i,j) = \frac{s_{j,i}^{(k)} - 1}{2}, \quad \text{if } s_{j,i}^{(k)} \text{ is odd;}$$

$$\gamma(k,i,j) = 2, \quad \xi(k,i,j) = \frac{s_{j,i}^{(k)} - 2}{2}, \quad \text{if } s_{j,i}^{(k)} \text{ is even.}$$

Since for $n_k + 1 \le i \le m_k$, $1 \le j_1, j_2 \le n_{k+1}$, $s_{j_1,i}^{(k)} = s_{j_2,i}^{(k)}$, one could correspondingly show that $\Phi_k^{(i,j_1)} = \Phi_k^{(i,j_2)}$.

Set $\Phi_k = \oplus_{1\le i\le m_k,\ 1\le j\le m_{k+1}} \Phi_k^{(i,j)}$, $\Theta_k = \oplus_{1\le i\le m_k,\ 1\le j\le m_{k+1}} \Theta_k^{(i,j)}$, where $\Theta_k$ will be used as an auxiliary homomorphism in Step 3. Set $C = \lim_{k\to\infty}(A_k, \Phi_k)$. It is a standard matter to check that

$$\varphi_k := (\Phi_k)_{*0} = \psi_k, \quad (\Phi_k)_{*1} = \beta_k.$$

Hence,

$$\big(K_0(C), K_0(C)_+, [1_C], K_1(C)\big) \cong \big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big).$$

Define

$$\pi_i^{(k)} : A_k \to M_{l(k,i)}\big(C(X)\big) \quad \text{for } 1 \le i \le n_k$$

and

$$\pi_i^{(k)} : A_k \to M_{l(k,i)} \quad \text{for } n_k + 1 \le i \le m_k$$

to be the quotient maps.

We next show that $C$ is simple. Fix $j \in \mathbb{N}$, for a nonzero element $f \in A_j$.

*Case 1: $\pi_i^{(j)}(f) \neq 0$ for some $i \ge n_j + 1$.* As each partial map of $\Phi_j$ has positive multiplicity, $\Phi_j(f)$ is full in $A_{j+1}$.

*Case 2: $\pi_i^{(j)}(f) \neq 0$ for some $1 \le i \le n_j$.* Choose $x^* \in X$ such that

$$\pi_i^{(j)}(f)(x^*) \neq 0.$$

Since $\pi_i^{(j)}(f)$ is a continuous map on the compact metric space $X$, there exists a $\delta > 0$ such that

$$\left\|\pi_i^{(j)}(f)(x) - \pi_i^{(j)}(f)(x^*)\right\| < \frac{\left\|\pi_i^{(j)}(f)(x^*)\right\|}{2}$$

provided that $\operatorname{dist}(x, x^*) < \delta$, where $x \in X$. Choose $k \in \mathbb{N}$, $k > j$, such that

$$\operatorname{dist}\left(x^*, \{x_l : 1 \le l \le c_k\}\right) < \frac{\delta}{2}.$$

Now, $\Phi_{j,k+2}(f)$ is full in $A_{k+2}$, where $\Phi_{j,k+2} := \Phi_{k+1} \circ \cdots \circ \Phi_j$.

By [9, Proposition 2.1 (iii)], it follows that $C$ is simple.

Next, we will show that $\mathrm{RR}(C) = 0$. Fix $j \in \mathbb{N}$, and a unit norm element $f \in A_j$. For $\varepsilon > 0$, choose $k \in \mathbb{N}$, $k > j + 1$, such that

$$\frac{1}{k} < \frac{\varepsilon}{2}.$$

Thus, for any $y_1, y_2 \in X$,

$$\left|\operatorname{tr}\left(\Phi_{j,k}(f)(y_1)\right) - \operatorname{tr}\left(\Phi_{j,k}(f)(y_2)\right)\right| \le \frac{2}{2k+3} < \varepsilon.$$

Therefore, by [36, Proposition 3.1.4], $\mathrm{RR}(C) = 0$. By Theorem 2.9,

$$C \cong A.$$

**Step 3.** For $k \in \mathbb{N}$, recalling that $l(k, 1) = \cdots = l(k, n_k)$, we could define an order two automorphism $\rho_k : A_k \to A_k$ by

$$\rho_k(f) = (f_1, \ldots, f_{p_k}; g_1 \circ \lambda, \ldots, g_{q_k} \circ \lambda; \hat{h}_1, h_1, \ldots, \hat{h}_{r_k}, h_{r_k}; a_{n_k+1}, \ldots, a_{m_k}),$$

where

$$f = (f_1, \ldots, f_{p_k}; g_1, \ldots, g_{q_k}; h_1, \hat{h}_1, \ldots, h_{r_k}, \hat{h}_{r_k}; a_{n_k+1}, \ldots, a_{m_k}) \in A_k.$$

In fact, under the canonical basis of $\mathbb{Z}^{m_k}$, it is routine – if tedious – to verify that

$$(\rho_k)_{*0} = \operatorname{diag}\left\{\overbrace{1, \ldots, 1}^{p_k+q_k}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r_k}; \overbrace{1, \ldots, 1}^{m_k-n_k}\right\} := \sigma_k,$$

and

$$(\rho_k)_{*1} = \eta_k, \quad k \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. We claim that

$$\sigma_{k+1} \circ \psi_k = \psi_k,$$

or equivalently,

$$(\rho_{k+1})_{*0}|_{(\Phi_k)_{*0}(K_0(A_k))} = \mathrm{id}|_{(\Phi_k)_{*0}(K_0(A_k))}.$$

This essentially follows from the fact that the first $n_{k+1}$ rows of the matrix $\psi_k$ are the same. More precisely, fix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_k} \\ x_{n_k+1} \\ \vdots \\ x_{m_k} \end{bmatrix} \in K_0(A_k) = \mathbb{Z}^{m_k}.$$

Define

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_{k+1}} \\ y_{n_{k+1}+1} \\ \vdots \\ y_{m_{k+1}} \end{bmatrix} := \varphi_k(x) = \psi_k(x) \in K_0(A_{k+1}) = \mathbb{Z}^{m_{k+1}}.$$

One has that

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_{k+1}} \\ y_{n_{k+1}+1} \\ \vdots \\ y_{m_{k+1}} \end{bmatrix} = \psi_k \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_k} \\ x_{n_k+1} \\ \vdots \\ x_{m_k} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m_k} s_{1,i}^{(k)} x_i \\ \sum_{i=1}^{m_k} s_{2,i}^{(k)} x_i \\ \vdots \\ \sum_{i=1}^{m_k} s_{n_{k+1},i}^{(k)} x_i \\ \sum_{i=1}^{m_k} s_{n_{k+1}+1,i}^{(k)} x_i \\ \vdots \\ \sum_{i=1}^{m_k} s_{m_{k+1},i}^{(k)} x_i \end{bmatrix}.$$

Recall that for $1 \leq i \leq n_k$, $1 \leq j \leq m_{k+1}$,

$$s_{j,i}^{(k)} = (2k+3)c_k,$$

and for $n_k + 1 \leq i \leq m_k$, $1 \leq j_1, j_2 \leq n_{k+1}$,

$$s_{j_1,i}^{(k)} = s_{j_2,i}^{(k)}.$$

It follows that

$$y_1 = \cdots = y_{n_{k+1}},$$

whence

$$\sigma_{k+1}(y) = y.$$

Thus,

$$(\rho_{k+1})_{*0}\big((\Phi_k)_{*0}(x)\big) = \sigma_{k+1}\big(\varphi_k(x)\big) = \sigma_{k+1}\big(\psi_k(x)\big) = \sigma_{k+1}(y) = y \\ = \varphi_k(x) = (\Phi_k)_{*0}(x).$$

Similarly, by observing that the first $n_k$ columns of the matrix $\psi_k$ are the same, one could show that

$$\psi_k \circ \sigma_k = \psi_k, \quad k \in \mathbb{N}.$$

Consider that $\rho_{k+1} \circ \Phi_k$. Recall that $\gamma_k = \eta_{k+1} \circ \beta_k$.

(a) For $1 \le i \le n_k$, $1 \le j \le p_{k+1}$, as $b_{j,i}^{(k)} = r_{j,i}^{(k)}$,

$$(\rho_{k+1} \circ \Phi_k)^{(i,j)} = \Phi_k^{(i,j)}, \quad \Theta_k^{(i,j)} = \Phi_k^{(i,j)}.$$

(b) For $1 \le i \le n_k$, $p_{k+1} + 1 \le j \le p_{k+1} + q_{k+1}$. If $b_{j,i}^{(k)} \neq 0$, as $-b_{j,i}^{(k)} = r_{j,i}^{(k)}$,

$$\begin{aligned}\Theta_k^{(i,j)}(f) &= \operatorname{diag}\Big\{\big(f \circ \lambda^{(\nu(r_{j,i}^{(k)}))}\big)^{\sim|r_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|r_{j,i}^{(k)}|)+2kc_k};\\ &\qquad f(x_1), f\big(\lambda(x_1)\big); \dots; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\}\\ &= \operatorname{diag}\Big\{\big(f \circ \lambda^{(\nu(-b_{j,i}^{(k)}))}\big)^{\sim|b_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|b_{j,i}^{(k)}|)+2kc_k};\\ &\qquad f(x_1), f\big(\lambda(x_1)\big); \dots; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\}\\ &= \operatorname{diag}\Big\{\big(f \circ \lambda^{(\nu(b_{j,i}^{(k)})+1)}\big)^{\sim|b_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|b_{j,i}^{(k)}|)+2kc_k};\\ &\qquad f(x_1), f\big(\lambda(x_1)\big); \dots; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\}\\ &= (\rho_{k+1} \circ \Phi_k)^{(i,j)}(f).\end{aligned}$$

If $b_{j,i}^{(k)} = 0$, then $r_{j,i}^{(k)} = -b_{j,i}^{(k)} = 0$,

$$\begin{aligned}&(\rho_{k+1} \circ \Phi_k)^{(i,j)}(f)\\ &\quad = \operatorname{diag}\big\{f(z_0)^{\sim c_k}, f(z_0)^{\sim 2kc_k}; f(x_1), f\big(\lambda(x_1)\big); \dots; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\big\}\\ &\quad = \Theta_k^{(i,j)}(f).\end{aligned}$$

(c) For $1 \le i \le n_k$, $p_{k+1} + q_{k+1} + 1 \le j \le n_{k+1}$. If $2 \nmid (j - p_{k+1} - q_{k+1})$, then $r_{j,i}^{(k)} = b_{j+1,i}^{(k)}$; hence,

$$(\rho_{k+1} \circ \Phi_k)^{(i,j)} = \Phi_k^{(i,j+1)} = \Theta_k^{(i,j)};$$

if $2 | (j - p_{k+1} - q_{k+1})$, then $r_{j,i}^{(k)} = b_{j-1,i}^{(k)}$; hence,

$$(\rho_{k+1} \circ \Phi_k)^{(i,j)} = \Phi_k^{(i,j-1)} = \Theta_k^{(i,j)}.$$

(d) For $1 \le i \le n_k$, $n_{k+1} + 1 \le j \le m_{k+1}$,

$$(\rho_{k+1} \circ \Phi_k)^{(i,j)} = \Phi_k^{(i,j)} = \Theta_k^{(i,j)}.$$

(e) For $n_k + 1 \le i \le m_k$, $1 \le j \le m_{k+1}$,

$$(\rho_{k+1} \circ \Phi_k)^{(i,j)} = \mathrm{diag}\{a^{\sim\gamma(k,i,j)}; a^{\sim 2\xi(k,i,j)}\} = \Theta_k^{(i,j)}.$$

In summary,

$$\rho_{k+1} \circ \Phi_k = \Theta_k.$$

In a similar manner, consider that $\Phi_k \circ \rho_k$. Recall that $\gamma_k = \beta_k \circ \eta_k$.

(a′) For $1 \le i \le p_k$, $1 \le j \le n_{k+1}$, as $b_{j,i}^{(k)} = r_{j,i}^{(k)}$,

$$(\Phi_k \circ \rho_k)^{(i,j)} = \Phi_k^{(i,j)}, \quad \Theta_k^{(i,j)} = \Phi_k^{(i,j)}.$$

(b′) For $p_k + 1 \le i \le p_k + q_k$, $1 \le j \le n_{k+1}$. If $b_{j,i}^{(k)} \ne 0$, as $-b_{j,i}^{(k)} = r_{j,i}^{(k)}$,

$$\begin{aligned}(\Phi_k \circ \rho_k)^{(i,j)}(f) = \mathrm{diag}\,\Big\{&\big(f \circ \lambda^{(\nu(b_{j,i}^{(k)})+1)}\big)^{\sim|b_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|b_{j,i}^{(k)}|)+2kc_k}; \\ &f\big(\lambda(x_1)\big), f(x_1); \dots ; f\big(\lambda(x_{c_k})\big), f(x_{c_k})\Big\},\end{aligned}$$

and

$$\begin{aligned}\Theta_k^{(i,j)}(f) &= \mathrm{diag}\,\Big\{\big(f \circ \lambda^{(\nu(r_{j,i}^{(k)}))}\big)^{\sim|r_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|r_{j,i}^{(k)}|)+2kc_k}; \\ &\qquad f(x_1), f\big(\lambda(x_1)\big); \dots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\} \\ &= \mathrm{diag}\,\Big\{\big(f \circ \lambda^{(\nu(-b_{j,i}^{(k)}))}\big)^{\sim|b_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|b_{j,i}^{(k)}|)+2kc_k}; \\ &\qquad f(x_1), f\big(\lambda(x_1)\big); \dots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\} \\ &= \mathrm{diag}\,\Big\{\big(f \circ \lambda^{(\nu(b_{j,i}^{(k)})+1)}\big)^{\sim|b_{j,i}^{(k)}|}; f(z_0)^{\sim(c_k-|b_{j,i}^{(k)}|)+2kc_k}; \\ &\qquad f(x_1), f\big(\lambda(x_1)\big); \dots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\}.\end{aligned}$$

If $b_{j,i}^{(k)} = 0$, then $r_{j,i}^{(k)} = -b_{j,i}^{(k)} = 0$,

$$\begin{aligned}&(\Phi_k \circ \rho_k)^{(i,j)}(f) \\ &= \mathrm{diag}\,\Big\{f(z_0)^{\sim c_k}, f(z_0)^{\sim 2kc_k}; f\big(\lambda(x_1)\big), f(x_1); \dots ; f\big(\lambda(x_{c_k})\big), f(x_{c_k})\Big\},\end{aligned}$$

while

$$\begin{aligned}&\Theta_k^{(i,j)}(f) \\ &\quad = \mathrm{diag}\,\Big\{f(z_0)^{\sim c_k}, f(z_0)^{\sim 2kc_k}; f(x_1), f\big(\lambda(x_1)\big); \dots ; f(x_{c_k}), f\big(\lambda(x_{c_k})\big)\Big\}.\end{aligned}$$

(c′) For $p_k + q_k + 1 \le i \le n_k$, $1 \le j \le n_{k+1}$. If $2 \nmid (i - p_k - q_k)$, as $r_{j,i}^{(k)} = b_{j,i+1}^{(k)}$,

$$(\Phi_k \circ \rho_k)^{(i,j)} = \Phi_k^{(i+1,j)} = \Theta_k^{(i,j)};$$

if $2|(i - p_k - q_k)$, as $r_{j,i}^{(k)} = b_{j,i-1}^{(k)}$,

$$(\Phi_k \circ \rho_k)^{(i,j)} = \Phi_k^{(i-1,j)} = \Theta_k^{(i,j)}.$$

(d′) For $1 \le i \le n_k, n_{k+1}+1 \le j \le m_{k+1}$,

$$(\Phi_k \circ \rho_k)^{(i,j)} = \operatorname{diag}\left\{f(z_0)^{\sim c_k}; f(z_0)^{\sim 2kc_k}; f(z_0)^{\sim 2c_k}\right\} = \Theta_k^{(i,j)}.$$

(e′) For $n_k+1 \le i \le m_k, 1 \le j \le m_{k+1}$,

$$(\Phi_k \circ \rho_k)^{(i,j)} = \Phi_k^{(i,j)} = \Theta_k^{(i,j)}.$$

Summarizing, except for the case $(b')$,

$$(\Phi_k \circ \rho_k)^{(i,j)} = \Theta_k^{(i,j)}.$$

Set $u_1 = 1_{A_1}$. For $1 \le i \le p_k$, and $p_k+q_k+1 \le i \le n_k, 1 \le j \le m_{k+1}$, define

$$u_{k+1,i,j} = \operatorname{diag}\left\{1^{\sim c_k}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\sim kc_k}; 1^{\sim 2c_k}\right\};$$

for $p_k+1 \le i \le p_k+q_k, 1 \le j \le n_{k+1}$, define

$$u_{k+1,i,j} = \operatorname{diag}\left\{1^{\sim c_k}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\sim kc_k}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\sim c_k}\right\};$$

for $p_k+1 \le i \le p_k+q_k, n_{k+1}+1 \le j \le m_{k+1}$, define

$$u_{k+1,i,j} = \operatorname{diag}\left\{1^{\sim c_k}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\sim kc_k}; 1^{\sim 2c_k}\right\};$$

for $n_k+1 \le i \le m_k, 1 \le j \le m_{k+1}$,

$$u_{k+1,i,j} = \operatorname{diag}\left\{1^{\sim \gamma(k,i,j)}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\sim \xi(k,i,j)}\right\}.$$

Next, define

$$\begin{aligned} u_{k+1} = {} & \left( \oplus_{j=1}^{n_{k+1}} \left( (\oplus_{i=1}^{m_k} u_{k+1,i,j} \otimes I_{l(k,i)}) \otimes 1_{C(X)} \right)\right) \\ & \oplus \left( \oplus_{j=n_{k+1}+1}^{m_{k+1}} (\oplus_{i=1}^{m_k} u_{k+1,i,j} \otimes I_{l(k,i)})\right) \in A_{k+1}, \quad k \in \mathbb{N}. \end{aligned}$$

Then,

$$u_{k+1}^* = u_{k+1}, \quad u_{k+1}^2 = 1_{A_{k+1}}, \quad \rho_{k+1}(u_{k+1}) = u_{k+1}.$$

By comparing $(a) \sim (e)$ and $(a') \sim (e')$, it is routine – if tedious – to verify that

$$\Phi_k \circ \rho_k = \operatorname{Ad} u_{k+1} \circ \Theta_k = \operatorname{Ad} u_{k+1} \circ (\rho_{k+1} \circ \Phi_k), \quad k \in \mathbb{N}.$$

Set $v_1 = 1_{A_1}$, and define $v_{k+1} = u_{k+1}\Phi_k(v_k) \in A_{k+1}$ inductively, $k = 1, 2, \ldots$. Define

$$\alpha_k = \operatorname{Ad} v_k \circ \rho_k.$$

Clearly,

$$\alpha_{k+1} \circ \Phi_k = \Phi_k \circ \alpha_k, \quad k \in \mathbb{N}.$$

To check that $\alpha_k^2 = \mathrm{id}_{A_k}$, since $\rho_k^2 = \mathrm{id}_{A_k}$, it is sufficient to show that $\rho_k(v_k)v_k = 1_{A_k}$. Since $v_1 = 1_{A_1}$, $\rho_1(v_1)v_1 = 1_{A_1}$. Assume that $\rho_k(v_k)v_k = 1_{A_k}$. We will show that

$$\rho_{k+1}(v_{k+1})v_{k+1} = 1_{A_{k+1}}.$$

Indeed,

$$\begin{aligned}
\rho_{k+1}(v_{k+1})v_{k+1} &= \rho_{k+1}\big(u_{k+1}\Phi_k(v_k)\big)u_{k+1}\Phi_k(v_k) \\
&= \rho_{k+1}(u_{k+1})\rho_{k+1}\big(\Phi_k(v_k)\big)u_{k+1}\Phi_k(v_k) \\
&= u_{k+1}u_{k+1}\Phi_k\big(\rho_k(v_k)\big)u_{k+1}^* u_{k+1}\Phi_k(v_k) \\
&= \Phi_k\big(\rho_k(v_k)v_k\big) \\
&= \Phi_k(1_{A_k}) \\
&= 1_{A_{k+1}}.
\end{aligned}$$

Then, one can easily construct the following commutative diagram:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & & & \\
A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots & \longrightarrow & A.
\end{array}$$

Hence, the sequence of order two automorphisms $\alpha_k$ defines

$$\alpha : A = \lim_{k\to\infty}(A_k, \Phi_k) \to A = \lim_{k\to\infty}(A_k, \Phi_k)$$

which is also a symmetry.

Note that

$$(\alpha_k)_{*1} = (\rho_k)_{*1} = \eta_k;$$

hence,

$$\alpha_{*1} = \lim_{k\to\infty}(\alpha_k)_{*1} = \lim_{k\to\infty}\eta_k = h.$$

Since

$$(\rho_{k+1})_{*0}|_{(\Phi_k)_{*0}(K_0(A_k))} = \mathrm{id}\,|_{(\Phi_k)_{*0}(K_0(A_k))}, \quad (\alpha_{k+1})_{*0} = (\rho_{k+1})_{*0},$$

it follows that

$$(\alpha_{k+1})_{*0}|_{(\Phi_k)_{*0}(K_0(A_k))} = \mathrm{id}\,|_{(\Phi_k)_{*0}(K_0(A_k))}.$$

Consequently,

$$\alpha_{*0}|_{(\Phi_{k,\infty})_{*0}(K_0(A_k))} = \mathrm{id}\,|_{(\Phi_{k,\infty})_{*0}(K_0(A_k))}, \quad k \in \mathbb{N}.$$

Noting that

$$\bigcup_{k=1}^{\infty} (\Phi_{k,\infty})_{*0}\big(K_0(A_k)\big) = K_0(A),$$

we have

$$\alpha_{*0} = \mathrm{id}\,.$$

**Step 4.** We are now in a position to show that the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property. To this end, we fix a finite set $F \subset A^1 := \{a : a \in A,\ \|a\| \le 1\}$, and an $\varepsilon > 0$. Then, there exists a positive integer $k$ such that

$$\mathrm{dist}\big(f, \Phi_{k,\infty}(A_k)\big) \le \frac{\varepsilon}{2} \quad \text{for any } f \in F,$$

and

$$\frac{3}{2k+3} < \frac{\varepsilon}{2}.$$

For $1 \le i \le n_k$, $1 \le j \le m_{k+1}$, define

$$e_{i,j}^{(0)} = \mathrm{diag}\left\{0^{\sim c_k}; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{\sim k c_k}; 0^{\sim 2c_k}\right\}, \quad e_{i,j}^{(1)} = \mathrm{diag}\left\{0^{\sim c_k}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{\sim k c_k}; 0^{\sim 2c_k}\right\};$$

for $n_k + 1 \le i \le m_k$, $1 \le j \le m_{k+1}$, define

$$e_{i,j}^{(0)} = \mathrm{diag}\left\{0^{\sim \gamma(k,i,j)}; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{\sim \xi(k,i,j)}\right\}, \quad e_{i,j}^{(1)} = \mathrm{diag}\left\{0^{\sim \gamma(k,i,j)}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{\sim \xi(k,i,j)}\right\}.$$

Define

$$e_l = \big(\oplus_{j=1}^{n_{k+1}} \big((\oplus_{i=1}^{m_k} e_{i,j}^{(l)} \otimes I_{l(k,i)}) \otimes 1_{C(X)}\big)\big) \oplus \big(\oplus_{j=n_{k+1}+1}^{m_{k+1}} (\oplus_{i=1}^{m_k} e_{i,j}^{(l)} \otimes I_{l(k,i)})\big),$$
$$l = 0, 1.$$

Then, it is easy to check that $e_0, e_1$ are mutually orthogonal projections of $A_{k+1}$ with

$$u_{k+1}^* e_0 u_{k+1} = e_1.$$

In addition, it is easily seen that

$$\rho_{k+1}(e_l) = e_l,$$

and $e_l$ commutes with $\Phi_k(A_k)$, $l = 0, 1$.

Note that

$$\begin{aligned}
\alpha_{k+1}(e_0) &= v_{k+1}^* \rho_{k+1}(e_0) v_{k+1} \\
&= \big(u_{k+1}\Phi_k(v_k)\big)^* \rho_{k+1}(e_0) u_{k+1} \Phi_k(v_k) \\
&= \Phi_k(v_k)^* u_{k+1}^* \rho_{k+1}(e_0) u_{k+1} \Phi_k(v_k) \\
&= \Phi_k(v_k)^* u_{k+1}^* e_0 u_{k+1} \Phi_k(v_k) \\
&= \Phi_k(v_k)^* e_1 \Phi_k(v_k) \\
&= \Phi_k(v_k)^* \Phi_k(v_k) e_1 \\
&= e_1.
\end{aligned}$$

Moreover, for any tracial state $\tau$ on $A_{k+1}$,

$$\tau\big(1_{A_{k+1}} - e_0 - \alpha_{k+1}(e_0)\big) = \tau(1_{A_{k+1}} - e_0 - e_1) \le \frac{3}{2k+3} < \frac{\varepsilon}{2}.$$

Then, by Lemma 2.10, the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property.

Since $A$ is a unital simple $A\mathbb{T}$-algebra with $\mathrm{RR}(A) = 0$, by [30, Theorem 4.3.5], $A$ has tracial rank zero. Hence, by [33, Corollary 1.6 and Theorem 2.6], $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple, separable $C^*$-algebra with tracial rank zero. Since $A_k$ is nuclear and $\mathbb{Z}_2$ is amenable and compact, it follows from [42, Corollary 7.18] and [10, Proposition 6.1] that $C^*(\mathbb{Z}_2, A_k, \alpha_k)$ is nuclear and satisfies the UCT. Hence, by [36, Proposition 2.4.7 (ii)],

$$C^*(\mathbb{Z}_2, A, \alpha) = \lim_{k\to\infty} \big(C^*(\mathbb{Z}_2, A_k, \alpha_k), \Phi_k\big)$$

is also nuclear and satisfies the UCT. Therefore, by [31, Theorem 5.2] and its proof, $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple AH-algebra with no dimension growth. ∎

**Remark 3.4.** If we do not insist on the fact that the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property, the unitary $u_{k+1}$ could be constructed in a simpler form which has less flips, comparing with that we used in Step 3 of the proof of Theorem 3.3.

**Remark 3.5.** Let $\widetilde{A}$ be a unital simple $A\mathbb{T}$-algebra with real rank zero and $K_1(\widetilde{A}) = 0$. Pick a *companion* algebra $A$, which is a unital simple $A\mathbb{T}$-algebra with real rank zero, and

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(\widetilde{A}), K_0(\widetilde{A})_+, [1_{\widetilde{A}}], \mathbb{Z}\big).$$

Let $h$ be the identity map of $K_1(A)$. By examining the proof of Theorem 3.3 carefully, it is straightforward to check that $B \cong \widetilde{A}$, and $B$ is a unital subalgebra of $A$ by embedding $B_k$ into $A_k$. Moreover, one could easily check that $\alpha$ is also a symmetry of $B$, and $\alpha_{*0}|_{K_0(B)} = \mathrm{id}\,|_{K_0(B)}$. Finally, exactly as that in Step 4, the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property, and $C^*(\mathbb{Z}_2, B, \alpha)$ is a unital simple AH-algebra with no dimension growth, tracial rank zero (actually an AF-algebra) as desired. In other words, we finish the proof of the trivial case of Theorem 3.3.

## 4. An application to a lifting problem of Blackadar

In this section, we will modify the procedure of the proof of Theorem 3.3 to give a positive answer to a lifting problem of Blackadar for the split case.

**Theorem 4.1.** *Let $\mathfrak{A}$ be the* AF*-algebra whose scaled ordered group $K_0(\mathfrak{A})$ is (isomorphic to)*

$$\big(G \oplus H, \big(G_+ \setminus \{0\}\big) \oplus H \cup \{0 \oplus 0\},\ \tilde{g} \oplus \tilde{h}\big),$$

*where $(G, G_+, \tilde{g})$ is the scaled ordered group $K_0(B)$ of a unital simple AF-algebra $B$, and $H$ is a countable torsion-free abelian group, $\tilde{h} \in H$. Let $\sigma$ be an order two automorphism of $K_0(\mathfrak{A})$, defined by $\sigma(g \oplus h) = g \oplus \eta(h)$, where $g \oplus h \in G \oplus H$, and $\eta$ is an order* *two automorphism of $H$. Then, there is an automorphism $\alpha$ of $\mathfrak{A}$ with $\alpha_{*0} = \sigma$, $\alpha^2 = \mathrm{id}$ and the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property.*

*Consequently, $C^*(\mathbb{Z}_2, \mathfrak{A}, \alpha)$ is a unital simple AH-algebra with no dimension growth, and with tracial rank zero.*

*Proof.* By Theorem 3.3, we assume that $H \neq \{0\}$. Set $X = S^2$, and let $\lambda$ be the homeomorphism of $X$ defined by $\lambda(w_1, w_2, w_3) = (w_1, w_2, -w_3)$, where $(w_1, w_2, w_3) \in S^2$. Let $z_0$ be a fixed point of $\lambda$ and $\{x_i : i \in \mathbb{N}\}$ a dense set of $X$. Paralleled with that of Theorem 3.3, we will divide the whole proof into four steps.

**Step 1.** By Proposition 3.2, there are a sequence of positive integers $\{n_k : k \in \mathbb{N}\}$ and monomorphisms $\beta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$, order two automorphisms $\eta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & H \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \downarrow & & \downarrow{\scriptstyle \eta} \\
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots & \longrightarrow & H
\end{array}$$

and $H = \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \beta_k)$, $\eta = \lim_{k\to\infty} \eta_k$. Moreover, under the canonical basis of $\mathbb{Z}^{n_k}$, $\eta_k$ has the form

$$\eta_k = \mathrm{diag}\left\{ \overbrace{1, \ldots, 1}^{p_k}; \overbrace{-1, \ldots, -1}^{q_k}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r_k} \right\}$$

for suitable nonnegative integers $p_k, q_k, r_k$ such that $p_k + q_k + 2r_k = n_k$, $k \in \mathbb{N}$.

By Lemma 3.1, we may assume that

$$B = \lim_{k\to\infty} (B_k, \Psi_k),$$

where $B_k = M_{l(k,1)} \oplus \cdots \oplus M_{l(k,m_k)}$ is a finite-dimensional $C^*$-algebra and (1)–(4) in Lemma 3.1 hold. Let $\tilde{g}_k$ denote $(l(k,1), \ldots, l(k, m_k)) \in \mathbb{Z}^{m_k}$, $k \in \mathbb{N}$. Then,

$$\big(K_0(B_k), K_0(B_k)_+, [1_{B_k}]\big) = (\mathbb{Z}^{m_k}, \mathbb{Z}^{m_k}_+, \tilde{g}_k).$$

Set $\psi_k := (\Psi_k)_{*0}$, $k \in \mathbb{N}$. Note that $\psi_{k,\infty}(\tilde{g}_k) = \tilde{g}$, $k \in \mathbb{N}$. Consider the ordered group $\mathbb{Z}^{m_k} \oplus \mathbb{Z}^{n_k}$ with the positive cone

$$\left\{(\lambda_1, \ldots, \lambda_{m_k}; \mu_1, \ldots, \mu_{n_k}) \in \mathbb{Z}^{m_k} \oplus \mathbb{Z}^{n_k} : \lambda_i \geq 0,\ 1 \leq i \leq m_k, \sum_{i=1}^{m_k} \lambda_i > 0\right\}$$
$$\cup \left\{(0^{\sim m_k}; 0^{\sim n_k})\right\}.$$

Then, it is evident that

$$\mathbb{Z}^{m_1} \oplus \mathbb{Z}^{n_1} \xrightarrow{\psi_1 \oplus \beta_1} \mathbb{Z}^{m_1} \oplus \mathbb{Z}^{n_2} \xrightarrow{\psi_2 \oplus \beta_2} \mathbb{Z}^{m_3} \oplus \mathbb{Z}^{n_3} \xrightarrow{\psi_3 \oplus \beta_3} \cdots \to G \oplus H,$$

which forms an inductive limit system of ordered groups.

**Step 2.** Define

$$A_k = \big(M_{2l(k,1)}\big(C(X)\big) \oplus \cdots \oplus M_{2l(k,n_k)}\big(C(X)\big)\big) \oplus (M_{2l(k,n_k+1)} \oplus \cdots \oplus M_{2l(k,m_k)}).$$

Then,

- $K_0(A_k) = \overbrace{(\mathbb{Z} \oplus \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z})}^{n_k} \oplus \overbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{m_k - n_k}$;
- $K_0(A_k)_+ = \overbrace{(\mathbb{Z} \oplus \mathbb{Z})_+ \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z})_+}^{n_k} \oplus \overbrace{\mathbb{Z}_+ \oplus \cdots \oplus \mathbb{Z}_+}^{m_k - n_k}$, where
$$(\mathbb{Z} \oplus \mathbb{Z})_+ = \big\{(\lambda, \mu) : \lambda \in \mathbb{N},\ \mu \in \mathbb{Z},\ \text{or } \lambda = \mu = 0\big\};$$
- $[1_{A_k}] = (2l(k,1), 0, \ldots, 2l(k,n_k), 0; 2l(k,n_k+1), \ldots, 2l(k,m_k))$.

Exactly as that in the proof of Theorem 3.3, we define two unital monomorphisms $\Phi_k, \Theta_k : A_k \to A_{k+1}$. Set $A = \lim_{k\to\infty}(A_k, \Phi_k)$.

For $k \in \mathbb{N}$, define $\varphi_k = (\Phi_k)_{*0}$, and

$$\omega_k : (\mathbb{Z} \oplus \mathbb{Z})^{n_k} \oplus \mathbb{Z}^{m_k - n_k} \to \mathbb{Z}^{m_k} \oplus \mathbb{Z}^{n_k}$$

by

$$\omega_k(\lambda_1, \mu_1; \ldots; \lambda_{n_k}, \mu_{n_k}; \lambda_{n_k+1}, \ldots, \lambda_{m_k}) = (\lambda_1, \ldots, \lambda_{m_k}; \mu_1, \ldots, \mu_{n_k}).$$

Then, it is routine to check that $\omega_k$ is a positive homomorphism and bijection, while $w_k^{-1}$ is not necessarily positive.

For $k \in \mathbb{N}$, consider

$$\chi_k := \omega_{k+1}^{-1} \circ (\psi_k \oplus \beta_k) : \mathbb{Z}^{m_k} \oplus \mathbb{Z}^{n_k} \to (\mathbb{Z} \oplus \mathbb{Z})^{n_{k+1}} \oplus \mathbb{Z}^{m_{k+1} - n_{k+1}}.$$

According to each partial map of $\psi_k$ having positive multiplicity, it is routine to check that $\chi_k$ is a positive homomorphism. Moreover,

$$\psi_k \oplus \beta_k = \omega_{k+1} \circ \chi_k.$$

On the other hand, by checking each basis vector of $(\mathbb{Z} \oplus \mathbb{Z})^{n_k} \oplus \mathbb{Z}^{m_k - n_k}$, it is straightforward – if tedious – to show that

$$\omega_{k+1} \circ \varphi_k = (\psi_k \oplus \beta_k) \circ \omega_k; \tag{$\star$}$$

hence,

$$\varphi_k = \chi_k \circ \omega_k, \quad k \in \mathbb{N}.$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccc}
(\mathbb{Z} \oplus \mathbb{Z})^{(n_1)} \oplus \mathbb{Z}^{(m_1 - n_1)} & \xrightarrow{\varphi_1} & (\mathbb{Z} \oplus \mathbb{Z})^{(n_2)} \oplus \mathbb{Z}^{(m_2 - n_2)} & \xrightarrow{\varphi_2} & \cdots & \longrightarrow & K_0(A) \\
\downarrow{\scriptstyle \omega_1} & \nearrow{\scriptstyle \chi_1} & \downarrow{\scriptstyle \omega_2} & \nearrow{\scriptstyle \chi_2} & & & \downarrow{\scriptstyle \omega} \\
\mathbb{Z}^{m_1} \oplus \mathbb{Z}^{n_1} & \xrightarrow[\psi_1 \oplus \beta_1]{} & \mathbb{Z}^{m_2} \oplus \mathbb{Z}^{n_2} & \xrightarrow[\psi_2 \oplus \beta_2]{} & \cdots & \longrightarrow & G \oplus H.
\end{array}$$

Since each homomorphism in the above diagram is positive monomorphism, by the standard intertwining argument, it follows that, via $\omega$,

$$\big(K_0(A), K_0(A)_+\big) \cong \big(G \oplus H, \big(G_+ \setminus \{0\}\big) \oplus H \cup \{0 \oplus 0\}\big) \cong \big(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+\big),$$

and

$$\begin{aligned}\omega\big([1_A]\big) &= (\psi_{1,\infty} \oplus \beta_{1,\infty})\big(\omega_1\big([1_{A_1}]\big)\big) = (\psi_{1,\infty} \oplus \beta_{1,\infty})\big(\omega_1(2\nu_k)\big)\\ &= (\psi_{1,\infty} \oplus \beta_{1,\infty})(2\tilde{g}_k \oplus 0_{n_k}) = 2\tilde{g} \oplus 0.\end{aligned}$$

Since $K_1(A_k) = 0$, $\forall k \in \mathbb{N}$, one has $K_1(A) = 0$.

Using the same argument where appropriate in the proof of Theorem 3.3, we conclude that $A$ is simple and $\mathrm{RR}(A) = 0$. By Theorem 2.9, $A$ is a simple AF-algebra.

**Step 3.** For $k \in \mathbb{N}$, define order two automorphisms $\rho_k : A_k \to A_k$ by

$$\rho_k(f) = (f_1, \dots, f_{p_k}; g_1 \circ \lambda, \dots, g_{q_k} \circ \lambda; \hat{h}_1, h_1, \dots, \hat{h}_{r_k}, h_{r_k}; a_{n_k+1}, \dots, a_{m_k}),$$

where

$$f = (f_1, \dots, f_{p_k}; g_1, \dots, g_{q_k}; h_1, \hat{h}_1, \dots, h_{r_k}, \hat{h}_{r_k}; a_{n_k+1}, \dots, a_{m_k}) \in A_k.$$

In fact, it is routine – if tedious – to verify that

$$\begin{aligned}\varsigma_k := (\rho_k)_{*0} = \operatorname{diag}\Bigg\{&\overbrace{1, \dots, 1}^{2p_k}; \overbrace{\begin{bmatrix}1 & \\ & -1\end{bmatrix}, \dots, \begin{bmatrix}1 & \\ & -1\end{bmatrix}}^{q_k};\\ &\overbrace{\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}, \dots, \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}}^{r_k}; \overbrace{1, \dots, 1}^{m_k-n_k}\Bigg\}.\end{aligned}$$

For $k \in \mathbb{N}$, let

$$\begin{aligned}\hat{\sigma}_k := \operatorname{diag}\Bigg\{&\overbrace{1, \dots, 1}^{p_k+q_k}; \overbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \dots, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}^{r_k}; \overbrace{1, \dots, 1}^{m_k-n_k}; \overbrace{1, \dots, 1}^{p_k}; \overbrace{-1, \dots, -1}^{q_k};\\ &\overbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \dots, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}^{r_k}\Bigg\}\\ = \sigma_k \oplus \eta_k,&\end{aligned}$$

where

$$\sigma_k := \operatorname{diag}\left\{\overbrace{1,\ldots,1}^{p_k+q_k};\overbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix},\ldots,\begin{bmatrix}0&1\\1&0\end{bmatrix}}^{r_k};\overbrace{1,\ldots,1}^{m_k-n_k}\right\}.$$

Then, $\widehat{\sigma}_k$ is an order two automorphism of the ordered group $(\mathbb{Z}^{m_k}\oplus\mathbb{Z}^{n_k},\mathbb{Z}_+^{m_k}\oplus\mathbb{Z}^{n_k})$. As in Step 3 of the proof of Theorem 3.3, one could find that

$$\sigma_{k+1}\circ\psi_k=\psi_k=\psi_k\circ\sigma_k.$$

It is standard to check that

$$\widehat{\sigma}_k\circ\omega_k=\omega_k\circ\varsigma_k,$$

so

$$\begin{aligned}\varsigma_{k+1}\circ\chi_k&=\varsigma_{k+1}\circ\omega_{k+1}^{-1}\circ(\psi_k\oplus\beta_k)\\&=\omega_{k+1}^{-1}\circ\widehat{\sigma}_{k+1}\circ(\psi_k\oplus\beta_k)\\&=\omega_{k+1}^{-1}\circ(\sigma_{k+1}\oplus\eta_{k+1})\circ(\psi_k\oplus\beta_k)\\&=\omega_{k+1}^{-1}\circ\big((\sigma_{k+1}\circ\psi_k)\oplus(\eta_{k+1}\circ\beta_k)\big)\\&=\omega_{k+1}^{-1}\circ\big((\psi_k\circ\sigma_k)\oplus(\beta_k\circ\eta_k)\big)\\&=\omega_{k+1}^{-1}\circ(\psi_k\oplus\beta_k)\circ(\sigma_k\oplus\eta_k)\\&=\chi_k\circ\widehat{\sigma}_k.\end{aligned}$$

Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc}(\mathbb{Z}\oplus\mathbb{Z})^{n_1}\oplus\mathbb{Z}^{m_1-n_1}&\xrightarrow{\omega_1}&\mathbb{Z}^{m_1}\oplus\mathbb{Z}^{n_1}&\xrightarrow{\chi_1}&(\mathbb{Z}\oplus\mathbb{Z})^{n_2}\oplus\mathbb{Z}^{m_2-n_2}&\xrightarrow{\omega_2}&\mathbb{Z}^{m_2}\oplus\mathbb{Z}^{n_2}&\xrightarrow{\chi_2}&\cdots\\ \downarrow{\scriptstyle\varsigma_1}&&\downarrow{\scriptstyle\widehat{\sigma}_1}&&\downarrow{\scriptstyle\varsigma_2}&&\downarrow{\scriptstyle\widehat{\sigma}_2}&&\\(\mathbb{Z}\oplus\mathbb{Z})^{n_1}\oplus\mathbb{Z}^{m_1-n_1}&\xrightarrow{\omega_1}&\mathbb{Z}^{m_1}\oplus\mathbb{Z}^{n_1}&\xrightarrow{\chi_1}&(\mathbb{Z}\oplus\mathbb{Z})^{n_2}\oplus\mathbb{Z}^{m_2-n_2}&\xrightarrow{\omega_2}&\mathbb{Z}^{m_2}\oplus\mathbb{Z}^{n_2}&\xrightarrow{\chi_2}&\cdots\end{array}$$

Note that for each $k\in\mathbb{N}$,

$$\omega_{k+1}\circ\chi_k=\psi_k\oplus\beta_k.$$

Thus, after telescoping the aforementioned commutative diagram, there exists an order two automorphism $\widehat{\sigma}$ of the ordered group $G\oplus H$ such that for each $k\in\mathbb{N}$, the following diagram is commutative:

$$\begin{array}{ccccc}\mathbb{Z}^{m_k}\oplus\mathbb{Z}^{n_k}&\xrightarrow{\psi_k\oplus\beta_k}&\mathbb{Z}^{m_{k+1}}\oplus\mathbb{Z}^{n_{k+1}}&\xrightarrow{\psi_{k+1,\infty}\oplus\beta_{k+1,\infty}}&G\oplus H\\ \downarrow{\scriptstyle\widehat{\sigma}_k}&&\downarrow{\scriptstyle\widehat{\sigma}_{k+1}}&&\downarrow{\scriptstyle\widehat{\sigma}}\\ \mathbb{Z}^{m_k}\oplus\mathbb{Z}^{n_k}&\xrightarrow{\psi_k\oplus\beta_k}&\mathbb{Z}^{m_{k+1}}\oplus\mathbb{Z}^{n_{k+1}}&\xrightarrow{\psi_{k+1,\infty}\oplus\beta_{k+1,\infty}}&G\oplus H.\end{array}$$

Fix $k \in \mathbb{N}$. By $(\star)$,

$$\begin{aligned}\widehat{\sigma}_{k+1}\big((\psi_k \oplus \beta_k)(g_k \oplus h_k)\big) &= \sigma_{k+1}\big(\psi_k(g_k)\big) \oplus \eta_{k+1}\big(\beta_k(h_k)\big)\\ &= \psi_k(g_k) \oplus \beta_k\big(\eta_k(h_k)\big).\end{aligned}$$

It follows that

$$\begin{aligned}\widehat{\sigma}\big(\psi_{k,\infty}(g_k) \oplus \beta_{k,\infty}(h_k)\big) &= \widehat{\sigma}\big((\psi_{k,\infty} \oplus \beta_{k,\infty})(g_k \oplus h_k)\big)\\ &= (\psi_{k,\infty} \oplus \beta_{k,\infty})\big(\widehat{\sigma}_k(g_k \oplus h_k)\big)\\ &= (\psi_{k+1,\infty} \oplus \beta_{k+1,\infty})\big(\big((\psi_k \oplus \beta_k) \circ \widehat{\sigma}_k\big)(g_k \oplus h_k)\big)\\ &= (\psi_{k+1,\infty} \oplus \beta_{k+1,\infty})\big(\big(\widehat{\sigma}_{k+1} \circ (\psi_k \oplus \beta_k)\big)(g_k \oplus h_k)\big)\\ &= (\psi_{k+1,\infty} \oplus \beta_{k+1,\infty})\big(\psi_k(g_k) \oplus \beta_k\big(\eta_k(h_k)\big)\big)\\ &= \psi_{k,\infty}(g_k) \oplus \beta_{k,\infty}\big(\eta_k(h_k)\big)\\ &= \psi_{k,\infty}(g_k) \oplus \eta\big(\beta_{k,\infty}(h_k)\big)\\ &= \sigma\big(\psi_{k,\infty}(g_k) \oplus \beta_{k,\infty}(h_k)\big).\end{aligned}$$

Noting that

$$\bigcup_{k=1}^{\infty} \psi_{k,\infty}(G_k) \oplus \beta_{k,\infty}(H_k) = G \oplus H,$$

we have

$$\widehat{\sigma} = \sigma.$$

Set $u_1 = \mathrm{id}_{A_1}$. For $k \in \mathbb{N}$, $1 \le i \le m_k$, $1 \le j \le m_{k+1}$, define $u_{k+1,i,j}$ exactly as that in the proof of Theorem 3.3. Next, define

$$\begin{aligned}u_{k+1} = &\big(\oplus_{j=1}^{n_{k+1}} \big((\oplus_{i=1}^{m_k} u_{k+1,i,j} \otimes I_{2l(k,i)}) \otimes 1_{C(X)}\big)\big)\\ &\oplus \big(\oplus_{j=n_{k+1}+1}^{m_{k+1}} (\oplus_{i=1}^{m_k} u_{k+1,i,j} \otimes I_{2l(k,i)})\big), \quad k \in \mathbb{N}.\end{aligned}$$

Exactly as that in Section 3,

$$\Phi_k \circ \rho_k = \mathrm{Ad}\, u_{k+1} \circ \Theta_k = \mathrm{Ad}\, u_{k+1} \circ \rho_{k+1} \circ \Phi_k \quad \text{and} \quad u_{k+1}^2 = 1_{A_{k+1}}, \quad k \in \mathbb{N}.$$

Set $v_1 = \mathrm{id}_{A_1}$, and define $v_{k+1} = u_{k+1}\Phi_k(v_k)$ inductively, $k = 1, 2, \ldots$. Define

$$\alpha_k = \mathrm{Ad}\, v_k \circ \rho_k, \quad k \in \mathbb{N}.$$

So one can easily construct the following commutative diagram:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots & \longrightarrow & A\\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & & & \\ A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots & \longrightarrow & A. \end{array}$$

As in the corresponding part of the proof of Theorem 3.3, $\alpha_k^2 = \mathrm{id}_{A_k}$, $k \in \mathbb{N}$. Hence, the automorphisms $\alpha_k$ define

$$\alpha : A = \lim_{k\to\infty} (A_k, \Phi_k) \to A = \lim_{k\to\infty} (A_k, \Phi_k)$$

which is a symmetry. Moreover,

$$\alpha_{*0} = \lim_{k\to\infty} (\alpha_k)_{*0} = \lim_{k\to\infty} (\rho_k)_{*0} = \lim_{k\to\infty} \varsigma_k = \lim_{k\to\infty} \widehat{\sigma}_k = \widehat{\sigma} = \sigma.$$

**Step 4.** Also as in Section 3, the action of $\mathbb{Z}_2$ generated by $\alpha$ has the tracial Rokhlin property. Note that the unit of $\mathfrak{A}$ may not come from trivial projections in the AH inductive limit procedure. Hence, one could note that we have enlarged the algebra in each finite stage during the construction. Next, we will cut it down.

Since $\sigma$ preserves the scale,

$$\sigma(\tilde{g} \oplus \tilde{h}) = \tilde{g} \oplus \eta(\tilde{h}) = \tilde{g} \oplus \tilde{h}.$$

Therefore, $\eta(\tilde{h}) = \tilde{h}$. Without loss of generality, we may assume that there exists $h_1 \in \mathbb{Z}^{n_1}$ such that $\beta_{1,\infty}(h_1) = \tilde{h}$. Since $\beta_{1,\infty}$ is injective and

$$\beta_{1,\infty}\big(\eta_1(\tilde{h}_1)\big) = \eta\big(\beta_{1,\infty}(\tilde{h}_1)\big) = \eta(\tilde{h}) = \tilde{h},$$

it follows that

$$\eta_1(\tilde{h}_1) = \tilde{h}_1.$$

Denote

$$\tilde{h}_1 = (\lambda_1, \ldots, \lambda_{p_1}; \zeta_1, \ldots, \zeta_{q_1}; \mu_1, \nu_1, \ldots, \mu_{r_1}, \nu_{r_1}) \in \mathbb{Z}^{n_1}.$$

Since $\eta_1(\tilde{h}_1) = \tilde{h}_1$, we have

$$\zeta_1 = \cdots = \zeta_{q_1} = 0, \quad \text{and} \quad \mu_j = \nu_j,\ j = 1, \ldots, r_1.$$

By the elementary fact of $K$-theory (e.g., [37, Exercise 11.2]), there exist rank one projections $P_i \in M_2(C(X))$ such that

$$[P_{1,i}] = (1, \lambda_i) \in \mathbb{Z} \oplus \mathbb{Z} = K_0\big(C(X)\big)$$

for $1 \le i \le p_1$ and

$$[P_{1,i}] = (1, \mu_i) \in \mathbb{Z} \oplus \mathbb{Z} = K_0\big(C(X)\big)$$

for $i = p_1 + q_1 + 1, p_1 + q_1 + 3, \ldots, p_1 + q_1 + 2r_1 - 1$, where the first coordinate of $\mathbb{Z} \oplus \mathbb{Z}$ denotes the rank part. For $i = p_1 + q_1 + 2, p_1 + q_1 + 4, \ldots, p_1 + q_1 + 2r_1$, set $P_{1,i} = P_{1,i-1}$.

Set

$$\begin{aligned} P_1 = \big(\oplus_{i=1}^{p_1} (P_{1,i} \oplus 1_{M_{l(1,1)-1}})\big) \oplus \big(\oplus_{i=p_1+1}^{p_1+q_1} 1_{M_{l(1,i)}}\big) \\ \oplus \big(\oplus_{i=p_1+q_1+1}^{p_1+q_1+2r_1} P_{1,i} \oplus 1_{M_{l(1,i)-1}}\big) \oplus \big(\oplus_{i=n_1+1}^{m_1} 1_{M_{l(1,i)}}\big). \end{aligned}$$

Then,

$$\varphi_{1,\infty}\big([P_1]\big) = (\psi_{1,\infty} \oplus \beta_{1,\infty})(\tilde{g}_1 \oplus \tilde{h}_1) = \tilde{g} \oplus \tilde{h},$$

and

$$\alpha_1(P_1) = \rho_1(P_1) = P_1.$$

Inductively, we define $P_{k+1}$ as

$$P_{k+1} = \Phi_k(P_k), \quad k \in \mathbb{N}.$$

Noting that

$$P_1 \in A_1^{\alpha_1}, \quad \alpha_2 \circ \Phi_1 = \Phi_1 \circ \alpha_1,$$

it follows that $P_2 \in A_2^{\alpha_2}$; similarly, one has $P_k \in A_k^{\alpha_k}$, $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, let $\mathfrak{A}_k = P_k A_k P_k$. Since $P_k \in A_k^{\alpha_k}$, it is routine to check that $\Phi_k$ maps $\mathfrak{A}_k$ onto $\mathfrak{A}_{k+1}$ and $\alpha_k$ maps $\mathfrak{A}_k$ onto $\mathfrak{A}_k$ and is also an order two automorphism of $\mathfrak{A}_k$. Let

$$\mathcal{A} = \lim_{k\to\infty} (\mathfrak{A}_k, \Phi_k).$$

Define $P = \Phi_{1,\infty}(P_1)$. Then, it is obvious that $\mathcal{A} = PAP$; hence, $\mathcal{A}$ is an AF-algebra.

According to the construction of $P$, it is easy to check that

$$\big(K_0(\mathcal{A}), K_0(\mathcal{A})_+, [1_{\mathcal{A}}]\big) = \big(G \oplus H, \big(G_+ \setminus \{0\}\big) \oplus H \cup \{0 \oplus 0\}, \tilde{g} \oplus \tilde{h}\big).$$

Therefore, by Elliott's classification theorem of AF-algebras (see, e.g., [30, Theorem 3.4.8]), $\mathcal{A} = \mathfrak{A}$. Again, since

$$P_1 \in A_1^{\alpha_1}, \quad \alpha \circ \Phi_{1,\infty} = \Phi_\infty \circ \alpha_1,$$

it follows that $P \in A^\alpha$. Therefore, $\alpha$ is an order two automorphism of $\mathfrak{A}$ with $\alpha_{*0} = \sigma$.

Finally, noting that $P \in A^\alpha$, by [33, Lemma 3.7], we deduce that the $\mathbb{Z}_2$ action $\alpha$ of $\mathfrak{A}$ has the tracial Rokhlin property. Hence, by [33, Corollary 1.6 and Theorem 2.6], $C^*(\mathbb{Z}_2, \mathfrak{A}, \alpha)$ is a unital simple, separable $C^*$-algebra with tracial rank zero. Since $A_k$ is nuclear, $\mathfrak{A}_k = P_k A_k P_k$ is a hereditary subalgebra of $A_k$; hence, $\mathfrak{A}_k$ is nuclear. Also, as the appropriate part in Step 4 of the proof of Theorem 3.3, $C^*(\mathbb{Z}_2, \mathfrak{A}, \alpha)$ is a unital simple AH-algebra with no dimension growth. ∎

**Remark 4.2.** Let $(G, G_+)$ be a simple dimension group, that is, the ordered group of a simple AF-algebra $B$, and let $H$ be a countable torsion-free abelian group. It is not hard to check that $(G \oplus H, (G_+ \setminus \{0\}) \oplus H \cup \{(0,0)\})$ is unperforated and satisfies the Riesz interpolation property. Hence, by the Effros–Handelman–Shen theorem [11], it is a simple dimension group.

**Remark 4.3.** Some examples which satisfy the $K$-theory setup of Theorem 4.1 could be found in [3, 10.11.3], [7, Section 1], [34, Examples 4.1 and 4.5].

**Corollary 4.4.** *Let* $\mathfrak{A}$ *be a unital simple* AF*-algebra with a unique tracial state* $\tau$*, and assume that the following short exact sequence is split:*

$$0 \to \inf K_0(\mathfrak{A}) \to K_0(\mathfrak{A}) \to \tau_*\big(K_0(\mathfrak{A})\big) \to 0,$$

*where* $\inf K_0(\mathfrak{A}) = \{x : x \in K_0(\mathfrak{A}),\ \tau_*(x) = 0\}$*. Let* $\sigma$ *be an order two automorphism of the scaled ordered* $K_0(\mathfrak{A})$*. Then, there is an automorphism* $\alpha$ *of* $\mathfrak{A}$ *with* $\alpha_{*0} = \sigma$ *and* $\alpha^2 = \mathrm{id}$.

*Proof.* Set

$$H := \inf K_0(\mathfrak{A}), \quad (G, G_+, \tilde{g}) := \big(\tau_*\big(K_0(\mathfrak{A})\big), \tau_*\big(K_0(\mathfrak{A})\big) \cap \mathbb{R}_+, 1\big).$$

As the short exact sequence is split, $K_0(\mathfrak{A}) = G \oplus H$. By [5, Theorem 3.9],

$$K_0(\mathfrak{A})_+ = \{g \oplus h : g \in G,\ g > 0,\ h \in H\} \cup \{0 \oplus 0\}.$$

For $g \oplus h \in G \oplus H$, set $g_1 \oplus h_1 := \sigma(g \oplus h)$. We claim that $g_1 = g$. Otherwise, since $G$ is totally ordered, either $g_1 > g$ or $g > g_1$. If $g_1 > g$, then $g_1 \oplus h_1 > g \oplus h$. Since $\sigma$ is order preserving, it follows that

$$g \oplus h = \sigma(g_1 \oplus h_1) \geq \sigma(g \oplus h) = g_1 \oplus h_1.$$

This is a contradiction! If $g > g_1$, a similar argument yields the contradiction.

Set $\eta := \sigma|_H$ which is an order two automorphism of $H$. Therefore, for $g \oplus h \in G \oplus H$,

$$\sigma(g \oplus h) = g \oplus \eta(h).$$

Therefore, by Theorem 4.1, there is an automorphism $\alpha$ of $\mathfrak{A}$ with $\alpha_{*0} = \sigma$ and $\alpha^2 = \mathrm{id}$. ∎

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