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Nonnoetherian singularities and their noncommutative blowups

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Abstract. We establish a new fundamental class of varieties in nonnoetherian algebraic geometry related to the central geometry of dimer algebras. Specifically, given an affine algebraic variety X and a finite collection of non-intersecting positive dimensional algebraic sets $Y_i \subset X$, we construct a nonnoetherian coordinate ring whose variety coincides with X except that each Y_i is identified as a distinct positive dimensional closed point. We then show that the noncommutative blowup of such a singularity is a noncommutative desingularization, in a suitable geometric sense.

1. Introduction

The primary objectives of this article are (i) to extend the framework of depictions, introduced in [6], to a much larger class of varieties with nonnoetherian coordinate rings; (ii) to show that noncommutative blowups of these varieties are noncommutative desingularizations, in a suitable sense. This framework was originally developed to provide the geometric tools needed to understand the representation theory of a class of quiver algebras called non-cancellative dimer algebras (e.g., [5, 7, 9]). Dimer algebras arose in string theory [14, 15], and have found wide application to many areas of mathematics (e.g., [3, 10, 13, 17–19, 22]). Depictions have enabled various notions in noncommutative algebraic geometry, such as noncommutative crepant resolutions [26], homological homogeneity [12], and Azumaya loci, to be generalized to tiled matrix algebras that are not finitely generated modules over their centers [5,7]; we will consider some of these generalizations here. The underlying ideas of nonnoetherian algebraic geometry also suggest possible directions towards a new theory of quantum gravity [4, 8].

Throughout, let k be an algebraically closed field, and let R be a subalgebra of an affine coordinate ring S over k. It is generally believed that nonnoetherian algebras do not admit concrete geometric descriptions. For example, consider the subalgebras of the polynomial rings $S_1 = k[x, y]$ and $S_2 = k[x, y, z]$,

$$R_1 = k[x] + x(x-1)(x-2)S_1,$$
$$R_2 = k[x^2 - y - z^2] + (x^2 - y, z - 5)(x - z, y)S_2.$$

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We may ask, informally, what their maximal spectra Max R "look like", but such a question initially appears hopeless, at least in terms of geometries we can visualize.

We could instead consider the simpler subalgebras

$$R_1' = k + x(x-1)(x-2)S_1,$$
$$R_2' = k + (x^2 - y, z-5)(x-z, y)S_2.$$

Both are of the form $R = k + I$, where I is an ideal of S. A geometric description of such subalgebras was introduced in [6]: the maximal spectrum Max R of R coincides with the algebraic variety Max S, except that the zero locus $\mathcal{Z}(I) \subset \operatorname{Max} S$ is identified as a single "smeared-out" point.

In particular, we may view the variety Max R_1' as $\mathbb{A}_k^2$, with the union of the three lines

$$\mathcal{Z}(x) = \{x = 0\}, \quad \mathcal{Z}(x-1) = \{x = 1\}, \quad \mathcal{Z}(x-2) = \{x = 2\} \tag{1}$$

identified as a single 1-dimensional point. Similarly, we may view the variety Max R_2' as $\mathbb{A}_k^3$, with the union of the two curves

$$\mathcal{Z}(x^2 - y, z - 5) \quad \text{and} \quad \mathcal{Z}(x - z, y) \tag{2}$$

identified as a single 1-dimensional point.

These geometric pictures are made precise using depictions and geometric dimension. A *depiction* of a nonnoetherian domain R is a finitely generated k-algebra S that is as close to R as possible, in a suitable geometric sense (Definition 2.1). In particular, if R is depicted by S, then R and S have equal Krull dimension, and their maximal spectra are birationally equivalent [6, Theorem 2.5]. Furthermore, the locus where R and S locally coincide,

$$U_{S/R} := \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}\},$$

is open dense in Max S [6, Proposition 2.4].

Algebras of the form $R = k + I$, with $\dim S/I \geq 1$, comprise an elementary class of examples in nonnoetherian algebraic geometry. Two ideals $I_1, I_2 \subset S$ are said to be coprime if $I_1 + I_2 = S$; equivalently, their zero loci in Max S do not intersect,

$$\mathcal{Z}(I_1) \cap \mathcal{Z}(I_2) = \emptyset.$$

In this article, we consider the question: *given a collection of pairwise coprime ideals*

$$I_1, \ldots, I_n \subset S,$$

is there a nonnoetherian ring R for which Max R *coincides with* Max S*, except that each $\mathcal{Z}(I_i)$ is identified as a distinct closed point of* Max R*?* We will show that this question has a positive answer, with R given by the intersection

$$R = \bigcap_i (k + I_i).$$

Our first main theorem is the following.

Theorem A (Propositions 3.3, 3.4, and Theorem 3.14). *Let X be an affine algebraic variety over k with coordinate ring S. Consider a collection of pairwise non-intersecting algebraic sets $Y_1, \ldots, Y_n$ of X, where each ideal $I(Y_i)$ is proper, nonzero, and non-maximal. Then the maximal spectrum of the ring*

$$R := \bigcap_i \left(k + I(Y_i)\right) \tag{3}$$

coincides with X except that each Y_i is identified as a distinct closed point. In particular, the locus $U_{S/R} \subset X$ is given by the intersection of the complements Y_i^c,

$$U_{S/R} = \bigcap_i Y_i^c.$$

Furthermore, we have the following:

(i) *R is nonnoetherian if and only if there is some i for which* $\dim Y_i \geq 1$*;*

(ii) *R is depicted by S if and only if for each i,* $\dim Y_i \geq 1$.

Theorem A answers our initial question in a surprisingly simple way: observe that the subalgebras R_1 and R_2 are of the form (3):

$$\begin{aligned} R_1 &= k[x] + x(x-1)(x-2)S_1 \\ &= (k + xS_1) \cap \left(k + (x-1)S_1\right) \cap \left(k + (x-2)S_1\right) \end{aligned}$$

and

$$\begin{aligned} R_2 &= k[x^2 - y - z^2] + (x^2 - y, z - 5)(x - z, y)S_2 \\ &= \left(k + (x^2 - y, z - 5)S_2\right) \cap \left(k + (x - z, y)S_2\right). \end{aligned}$$

The variety Max R_1 therefore looks exactly like $\mathbb{A}_k^2$, except that each of the three lines in (1) is identified as a distinct 1-dimensional point. Similarly, Max R_2 looks exactly like $\mathbb{A}_k^3$, except that each of the curves in (2) is identified as a distinct 1-dimensional point.

To note, it is peculiar that by adjoining to R_2' the polynomial $x^2 - y - z^2$,

$$R_2 = R_2'[x^2 - y - z^2];$$

the single 1-dimensional point of Max R_2' separates into two distinct 1-dimensional points, while all other points of Max R_2' are left unchanged.

Theorem A also implies the following generalization of the fact that, given any maximal ideal $\mathfrak{n}$ of S, S decomposes as the sum $S = k + \mathfrak{n}$.

Corollary B. *Let I be a proper non-maximal nonzero radical ideal of S, and set $R = k + I$. The following are equivalent:*

(i) $\dim S/I \geq 1$*;*

(ii) *R is nonnoetherian;*

(iii) *R is depicted by S.*

In particular, $R = k + I$ is noetherian if and only if $\dim S/I = 0$, that is,

$$I = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_\ell$$

for some maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_\ell \in \operatorname{Max} S$. The implication (ii) $\Rightarrow$ (i) was also shown by Stafford in [25, Lemma 1.4] using different methods.

In Section 4, we define a sheaf of depictions on an affine scheme X to be a sheaf of algebras that is a depiction on each principal open set of X. We show that the sheafification of a depiction S of R is a sheaf of depictions on $\operatorname{Spec} R$.

In Section 5, we consider nonnoetherian coordinate rings in the setting of noncommutative algebraic geometry. Let S be a finite type normal integral domain, let $Y_1, \ldots, Y_n$ be positive dimensional proper subvarieties of $\operatorname{Max} S$ that intersect the smooth locus, and denote by $I_i := I(Y_i)$ their radical ideals in S. By Theorem A, $R := \bigcap_i (k + I_i)$ is a nonnoetherian coordinate ring with n positive dimensional closed points,

$$\mathfrak{m}_i := I_i \cap R \in \operatorname{Spec} R.$$

Following [20, Section R], we call the endomorphism ring

$$A := \operatorname{End}_R({}_R R \oplus \bigoplus_i \mathfrak{m}_i) \tag{4}$$

the "noncommutative blowup" of $\operatorname{Max} R$ at the points $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. We would like to know whether A is a desingularization of its center R.

A resolution of a singularity X is a proper birational morphism of schemes $Y \to X$ such that Y is smooth. If we omit the requirement of properness, then we may say that $Y \to X$ is a desingularization of X. We note the following:

(a) birationality implies that X and Y have isomorphic function fields,

$$\operatorname{Frac} k[X] \cong \operatorname{Frac} k[Y];$$

(b) let $\operatorname{Spec} S$ be an affine open subset of Y. Then $\operatorname{Spec} S$ is smooth over $\operatorname{Spec} k$ at a closed point $\mathfrak{n} \in \operatorname{Spec} S$ if and only if[1] the global dimension of $S_\mathfrak{n}$, the projective dimension of the residue field $S_\mathfrak{n}/\mathfrak{n} \cong k$, and the Krull dimension of $S_\mathfrak{n}$ all coincide [1, 2, 24],

$$\operatorname{gldim} S_\mathfrak{n} = \operatorname{pd}_{S_\mathfrak{n}}(S_\mathfrak{n}/\mathfrak{n}) = \dim S_\mathfrak{n}.$$

Following Brown and Hajarnavis's notion of a homologically homogeneous ring [12], and Van den Bergh's notion of a noncommutative crepant resolution [26], we say that a noncommutative algebra A, module-finite over its noetherian center R, is a noncommutative desingularization of R if the following two conditions hold:

(a′) $\operatorname{Frac} R$ and $A \otimes_R \operatorname{Frac} R$ are Morita equivalent,

[1] Since we are assuming k algebraically closed, $\operatorname{Spec} S$ is smooth at $\mathfrak{n}$ if and only if $S_\mathfrak{n}$ is regular [16, Example III.10.0.3].

(b′) for each closed point $\mathfrak{m} \in \operatorname{Spec} R$, the central localization $A_{\mathfrak{m}} := A \otimes_R R_{\mathfrak{m}}$ satisfies

$$\operatorname{gldim} A_{\mathfrak{m}} = \operatorname{pd}_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/\mathfrak{m}) = \dim R_{\mathfrak{m}}.$$

However, the singularities we are considering here are nonnoetherian, and their noncommutative blowups are not module-finite over their centers (just as the case for noncancellative dimer algebras). Condition (b′) must therefore be modified to allow for this generality. Such a modification is possible for tiled matrix algebras using the notions of "cycle algebra" and "cyclic localization", introduced in [6, 7] (Definition 2.2). In cases of interest, if the center R is noetherian, then the cycle algebra and center coincide, and cyclic localization is the same as central localization [6, Theorem 4.1]. We thus replace (b′) with the following condition:

(b″) Let S be the cycle algebra of A. For each closed point $\mathfrak{m} \in \operatorname{Spec} R$ and each minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}$, the cyclic localization $A_{\mathfrak{q}}$ satisfies

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}/\mathfrak{q}) = \dim S_{\mathfrak{q}}.$$

Our second main theorem is the following.

Theorem C (Theorem 5.17). *Let A be the endomorphism ring in (4) and let S be its cycle algebra. If each Y_i is irreducible, or $n = 1$, then A is a noncommutative desingularation of its center R:*

- $\operatorname{Frac} R$ *and* $A \otimes_R \operatorname{Frac} R$ *are Morita equivalent, and*
- *for each $i \in [1, n]$ and minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}_i$, we have*

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}/\mathfrak{q}) = \dim S_{\mathfrak{q}}.$$

Furthermore, the Azumaya locus of A and the noetherian locus $U_{S/R}$ of R coincide.

2. Preliminary definitions

Given an integral domain k-algebra S, denote by $\operatorname{Max} S$, $\operatorname{Spec} S$, $\operatorname{Frac} S$, and $\dim S$ the maximal spectrum (or variety), prime spectrum (or affine scheme), fraction field, and Krull dimension of S, respectively. For a subset $I \subset S$, set $\mathcal{Z}(I) := \{\mathfrak{n} \in \operatorname{Max} S \mid \mathfrak{n} \supseteq I\}$.

Given a (not-necessarily-commutative) k-algebra A and an A-module V, denote by $\operatorname{gldim} A$ and $\operatorname{pd}_A(V)$ the left global dimension of A and projective dimension of V, respectively. By module we mean left module, unless stated otherwise.

The following definitions have been instrumental in studying dimer algebras (e.g., [5, 7, 9]).

Definition 2.1 ([6, Definition 3.1]). Let S be an integral domain and a finitely generated k-algebra, and let R be a subalgebra of S.

- We say S is a *depiction* of R if the morphism

$$\iota_{S/R} : \operatorname{Spec} S \to \operatorname{Spec} R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R$$

is surjective and

$$U_{S/R} := \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}\} = \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} \text{ is noetherian}\} \neq \emptyset.$$

- The *geometric height* of $\mathfrak{p} \in \operatorname{Spec} R$ is the minimum

$$\operatorname{ght}(\mathfrak{p}) := \min\left\{\operatorname{ht}_S(\mathfrak{q}) \mid \mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p}),\ S \text{ a depiction of } R\right\}.$$

The *geometric dimension* of $\mathfrak{p}$ is[2]

$$\operatorname{gdim} \mathfrak{p} := \dim R - \operatorname{ght}(\mathfrak{p}).$$

For brevity, we will often write ι for $\iota_{S/R}$. To note, if R is depicted by S, then R is noetherian if and only if $R = S$ [6, Theorem 3.12].

Now let B be an integral domain and k-algebra, and let

$$A = [A^{ij}] \subset M_n(B)$$

be a tiled matrix ring, that is, each diagonal $A^i := A^{ii}$ is a unital subalgebra of B. The following definitions, with the exception of residue module, were introduced in [7]; the notion of residue module we are considering here is new.

Definition 2.2 ([7, Definition 3.1]). Set

$$R := k\Big[\bigcap_i A^i\Big] \quad \text{and} \quad S := k\Big[\bigcup_i A^i\Big].$$

We call S the *cycle algebra* of A, and in cases of interest, R is the center of A [6, Theorem 4.1]. The *cyclic localization* of A at a prime $\mathfrak{q} \in \operatorname{Spec} S$ is the algebra

$$A_{\mathfrak{q}} := \left\langle \begin{bmatrix} A^1_{\mathfrak{q}\cap A^1} & A^{12} & \cdots & A^{1n} \\ A^{21} & A^2_{\mathfrak{q}\cap A^2} & & A^{2n} \\ \vdots & & \ddots & \vdots \\ A^{n1} & A^{n2} & \cdots & A^n_{\mathfrak{q}\cap A^n} \end{bmatrix} \right\rangle \subset M_n(\operatorname{Frac} B).$$

The *residue module* $A_{\mathfrak{q}}/\mathfrak{q}$ of A at $\mathfrak{q}$ is the quotient of $A_{\mathfrak{q}}$ by the ideal

$$A_{\mathfrak{q}} \begin{bmatrix} \mathfrak{q}\cap A^1 & 0 & \cdots & 0 \\ 0 & \mathfrak{q}\cap A^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{q}\cap A^n \end{bmatrix} A_{\mathfrak{q}}.$$

[2] Recall that if S is an integral domain and a finitely generated k-algebra, then for each $\mathfrak{q} \in \operatorname{Spec} S$, we have $\dim S/\mathfrak{q} = \dim S - \operatorname{ht}(\mathfrak{q})$.

Remark 2.3. If $R = S$, that is, $A^i = A^j$ for each i, j, then cyclic localization coincides with the usual notion of central localization:

$$A_{\mathfrak{q}} \cong A \otimes_R R_{\mathfrak{q}} \quad \text{and} \quad A_{\mathfrak{q}}/\mathfrak{q} \cong A \otimes_R R_{\mathfrak{q}}/\mathfrak{q}.$$

Definition 2.4 ([7, Definition 3.2]). We say A is *cycle regular* at $\mathfrak{m} \in \operatorname{Max} R$ if for each minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}$, we have[3]

$$\operatorname{gldim}(A_{\mathfrak{q}}) = \operatorname{pd}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}/\mathfrak{q}) = \dim S_{\mathfrak{q}}.$$

If, in addition, $\operatorname{Frac} R$ and $A \otimes_R \operatorname{Frac} R$ are Morita equivalent, then we say A is a *noncommutative desingularization* of R.

3. Nonnoetherian coordinate rings with multiple positive dimensional points

Let S be an integral domain and a finitely generated k-algebra. Let $I_1, \ldots, I_n$ be a collection of proper non-maximal nonzero radical ideals of S such that, for each $i \neq j$, $\mathcal{Z}(I_i) \cap \mathcal{Z}(I_j) = \emptyset$; equivalently, I_i and I_j are coprime: $I_i + I_j = S$. Unless stated otherwise, we denote by R the algebra

$$R := \bigcap_i (k + I_i).$$

Remark 3.1. If some I_j were a maximal ideal of S, then $k + I_j = S$, whence $R = \bigcap_{i \neq j}(k + I_i)$. The assumption that each I_i is proper and nonzero implies that $\dim S \geq 1$.

Lemma 3.2. *Suppose $n \geq 2$. For each $i \in [1, n]$, there are elements $a, b \in R$ satisfying*

$$a \in I_i \setminus \Big(\bigcup_{j \neq i} I_j\Big), \quad b \in \Big(\bigcap_{j \neq i} I_j\Big) \setminus I_i,$$

and which sum to unity, $a + b = 1$.

Proof. Fix $i \in [1, n]$. By assumption, we have

$$\mathcal{Z}(1) = \emptyset = \bigcup_{j \neq i} \big(\mathcal{Z}(I_i) \cap \mathcal{Z}(I_j)\big) = \mathcal{Z}(I_i) \cap \Big(\bigcup_{j \neq i} \mathcal{Z}(I_j)\Big) = \mathcal{Z}\Big(I_i + \bigcap_{j \neq i} I_j\Big).$$

Whence

$$1 \in I_i + \bigcap_{j \neq i} I_j.$$

[3] In [7], we defined A to be cycle regular at $\mathfrak{m} \in \operatorname{Max} R$ if, for each minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}$ and each simple $A_{\mathfrak{q}}$-module V, we have $\operatorname{gldim}(A_{\mathfrak{q}}) = \operatorname{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}}$. In this article, we replace the set of simple $A_{\mathfrak{q}}$-modules V with the residue module $A_{\mathfrak{q}}/\mathfrak{q}$, which, in our case, is a direct sum of all such simples (see Propositions 5.14 and 5.15).

Thus there is some $a \in I_i$ and $b \in \bigcap_{j \neq i} I_j$ such that $a + b = 1$. In particular,

$$a = 1 - b \in I_i \cap \big(\bigcap_{j \neq i} (k + I_j) \big) \subset R.$$

It also follows that for each $j \neq i$,

$$a = 1 - b \in I_i \setminus I_j \quad \text{and} \quad b = 1 - a \in I_j \setminus I_i. \qquad \blacksquare$$

Proposition 3.3. *Each ideal $I_i \cap R$ is a distinct closed point of* Spec R.

Proof. Fix i. For each $a \in R \subseteq (k + I_i)$, there is some $\alpha_i \in k$ and $b_i \in I_i$ such that $a = \alpha_i + b_i$. In particular, there is an algebra epimorphism

$$R \to k, \quad a \mapsto \alpha_i,$$

with kernel $I_i \cap R$; whence an algebra isomorphism $R/(I_i \cap R) \cong k$. Furthermore, there exists some $a \in (I_i \cap R) \setminus (\bigcup_{j \neq i} I_j)$, by Lemma 3.2. Thus, for each $j \neq i$,

$$I_j \cap R \neq I_i \cap R.$$

Therefore each $I_i \cap R$ is a distinct maximal ideal of R. $\blacksquare$

Proposition 3.4. *The locus $U_{S/R} := \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}\}$ is given by*

$$U_{S/R} = \big(\bigcup_i \mathcal{Z}(I_i) \big)^c.$$

Proof. (i) We first claim that $U_{S/R} \subseteq (\bigcup_i \mathcal{Z}(I_i))^c$. Indeed, let $\mathfrak{n} \in \bigcup_i \mathcal{Z}(I_i)$. Then $\mathfrak{n}$ contains some I_i. By assumption, I_i is a non-maximal radical ideal of S. Thus there is another maximal ideal $\mathfrak{n}' \neq \mathfrak{n}$ of S which contains I_i. Whence

$$I_i \cap R \subseteq \mathfrak{n} \cap R \neq R \quad \text{and} \quad I_i \cap R \subseteq \mathfrak{n}' \cap R \neq R.$$

But $I_i \cap R$ is a maximal ideal of R by Proposition 3.3. Therefore

$$\mathfrak{n} \cap R = I_i \cap R = \mathfrak{n}' \cap R.$$

Now fix $c \in \mathfrak{n} \setminus \mathfrak{n}'$. Assume to the contrary that $c \in R_{\mathfrak{n} \cap R}$. Then there is some $a \in R$ and $b \in R \setminus (\mathfrak{n} \cap R)$ such that $c = \frac{a}{b}$. Whence

$$a = bc \in \mathfrak{n} \cap R = \mathfrak{n}' \cap R.$$

In particular, $bc \in \mathfrak{n}'$ with $b, c \in S$. Therefore

$$b \in \mathfrak{n}', \tag{5}$$

since $c \notin \mathfrak{n}'$ and $\mathfrak{n}'$ is a prime ideal of S. But $b \in R$ and

$$b \notin \mathfrak{n} \cap R = \mathfrak{n}' \cap R.$$

Whence $b \notin \mathfrak{n}'$, a contradiction to (5). Thus $c \in S_{\mathfrak{n}} \setminus R_{\mathfrak{n} \cap R}$. Therefore $\mathfrak{n} \in U^c_{S/R}$.

(ii) We now claim that $U_{S/R} \supseteq (\bigcup_i \mathcal{Z}(I_i))^c$.[4] Let $\mathfrak{n} \in (\bigcup_i \mathcal{Z}(I_i))^c$. For each i, $\mathfrak{n} \not\supseteq I_i$. In particular, for each i there is some $c_i \in I_i \setminus \mathfrak{n}$. Furthermore, since $\mathfrak{n}$ is prime, we have

$$c := c_1 \cdots c_n \in \Big(\bigcap_i I_i\Big) \setminus \mathfrak{n}. \tag{6}$$

Now let $\frac{a}{b} \in S_{\mathfrak{n}}$, with $a \in S$ and $b \in S \setminus \mathfrak{n}$. Then by (6),

$$ac \in R \quad \text{and} \quad bc \in R \setminus (\mathfrak{n} \cap R).$$

Thus

$$\frac{a}{b} = \frac{ac}{bc} \in R_{\mathfrak{n} \cap R}.$$

Whence

$$S_{\mathfrak{n}} \subseteq R_{\mathfrak{n} \cap R} \subseteq S_{\mathfrak{n}}.$$

Therefore $R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}$. ∎

Lemma 3.5. *If J is a proper ideal of R and $\mathcal{Z}(J) \cap U_{S/R} = \emptyset$, then J is contained in some I_i.*

Proof. Suppose the hypotheses hold, and let $\mathfrak{n} \in \mathcal{Z}(J)$. Then $\mathfrak{n} \in U^c_{S/R}$. Whence $\mathfrak{n} \in \bigcup_j \mathcal{Z}(I_j)$ by Proposition 3.4. Thus $\mathfrak{n}$ contains some I_i. Consequently,

$$I_i \cap R \subseteq \mathfrak{n} \cap R \neq R.$$

Whence $I_i \cap R = \mathfrak{n} \cap R$ since $I_i \cap R \in \operatorname{Max} R$ by Proposition 3.3. Therefore

$$J = J \cap R \subseteq \mathfrak{n} \cap R = I_i \cap R \subseteq I_i. \qquad ∎$$

Lemma 3.6. *For each i,*

$$R_{I_i \cap R} = (k + I_i)_{I_i}. \tag{7}$$

Proof. The lemma is trivial if $n = 1$, so suppose that $n \geq 2$. Fix $i \in [1, n]$. By Lemma 3.2, there is some

$$c \in \Big(\bigcap_{j \neq i} I_j\Big) \cap R \setminus I_i.$$

Let $\frac{a}{b} \in (k + I_i)_{I_i}$, with $a \in k + I_i$ and $b \in (k + I_i) \setminus I_i$. Since c is in R, c is in $k + I_i$. Thus, since a is also in $k + I_i$, the product ac is in $k + I_i$. Furthermore, since c

[4] This claim was proven in the special case $n = 1$ in [6, Proposition 2.8].

is in $\bigcap_{j \neq i} I_j$, ac is in $\bigcap_{j \neq i} I_j$. Whence, ac is in R. Similarly, bc is in R. But bc is not in I_i since I_i is a maximal, hence prime, ideal of $k + I_i$. Consequently,

$$\frac{a}{b} = \frac{ac}{bc} \in R_{I_i \cap R}.$$

It follows that

$$(k + I_i)_{I_i} \subseteq R_{I_i \cap R}.$$

Conversely,

$$R_{I_i \cap R} = \Big(\bigcap_j (k + I_j)\Big)_{I_i \cap R} \subseteq \bigcap_j (k + I_j)_{I_i \cap (k+I_j)} \subseteq (k + I_i)_{I_i \cap (k+I_i)} = (k + I_i)_{I_i}.$$

Therefore (7) holds. ■

For the following, note that if $\mathfrak{n}_1, \ldots, \mathfrak{n}_\ell$ are maximal ideals of S, then

$$I = \mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_\ell = \sqrt{\mathfrak{n}_1 \cdots \mathfrak{n}_\ell}$$

is a radical ideal of S satisfying $\dim S/I = 0$.

Lemma 3.7. *Suppose that I is a radical ideal of S satisfying* $\dim S/I = 0$. *Then the ring $R = k + I$ is noetherian.*

Proof. Suppose that R is nonnoetherian. We claim that

$$\dim S/I = \dim \mathcal{Z}(I) \overset{(\mathrm{I})}{=} \dim U^c_{S/R} \overset{(\mathrm{II})}{\geq} 1.$$

Indeed, (I) holds since by Proposition 3.4,

$$\mathcal{Z}(I) = U^c_{S/R}. \tag{8}$$

To show (II), recall [6, Theorem 3.13.2]:[5] if R is a nonnoetherian subalgebra of a finitely generated k-algebra S, and there is some $\mathfrak{m} \in \iota(U^c_{S/R})$ satisfying $\sqrt{\mathfrak{m}S} = \mathfrak{m}$, then

$$\dim U^c_{S/R} \geq 1.$$

In our case, $R = k + I$ is nonnoetherian and $\sqrt{IS} = I$. Moreover, I is in $\iota(U^c_{S/R})$: for $\mathfrak{n} \in \mathcal{Z}(I)$, we have

$$I \overset{(\mathrm{A})}{=} \mathfrak{n} \cap R = \iota(\mathfrak{n}) \in \iota\big(\mathcal{Z}(I)\big) \overset{(\mathrm{B})}{=} \iota(U^c_{S/R}),$$

where (A) holds since I is maximal in R, and (B) holds by (8). Therefore (II) holds. ■

[5] In the published version of [6, Theorem 3.13.2], S is assumed to be a depiction of R, but this is not used in the proof of the theorem.

Proposition 3.8. *Suppose that each I_i is a radical ideal of S.*

(1) *If* $\dim S/I_i = 0$ *for each i, then R is noetherian.*

(2) *If* $\dim S/I_i = 0$, *then the localization $R_{I_i \cap R}$ is noetherian.*

Proof. (1) Suppose that $\dim S/I_i = 0$ for each i. Set

$$R^m := \bigcap_{i=1}^{m} (k + I_i).$$

We proceed by induction on m.

By Lemma 3.7, R^1 is noetherian. So suppose that R^m is noetherian; we claim that R^{m+1} is noetherian.

Indeed, recall that a ring T is noetherian if there is a finite set of elements $a_1, \ldots, a_m \in T$ such that $(a_1, \ldots, a_m)T = T$, and each localization $T_{a_i} := T[a_i^{-1}]$ is noetherian (e.g., [16, Proposition III.3.2]).

By Lemma 3.2, R^{m+1} contains elements

$$a \in I_{m+1} \setminus \Big(\bigcup_{i=1}^{m} I_i \Big) \quad \text{and} \quad b \in \Big(\bigcap_{i=1}^{m} I_i \Big) \setminus I_{m+1} \tag{9}$$

satisfying $a + b = 1$. In particular,

$$(a, b)R^{m+1} = R^{m+1}.$$

Furthermore, (9) implies that

$$R_a^{m+1} = R_a^m \quad \text{and} \quad R_b^{m+1} = (k + I_{m+1})_b. \tag{10}$$

But R^m is noetherian by assumption, and $(k + I_{m+1})$ is noetherian by Lemma 3.7. Thus the localizations (10) are noetherian. Therefore R^{m+1} is noetherian, proving our claim.

(2) Now suppose that $\dim S/I_i = 0$. Then the ring $k + I_i$ is noetherian by Lemma 3.7. Thus the localization $(k + I_i)_{I_i}$ is noetherian. But $R_{I_i \cap R} = (k + I_i)_{I_i}$ by Lemma 3.6. Therefore $R_{I_i \cap R}$ is noetherian. ∎

Proposition 3.9. *Suppose that I is a nonzero radical ideal of S satisfying* $\dim S/I \geq 1$. *Then the ring $R = k + I$ is nonnoetherian and I contains a strict infinite ascending chain of ideals of R.*[6]

Proof. Since $\dim S/I \geq 1$, I is a non-maximal ideal of S. Thus there is a maximal ideal $\mathfrak{n}$ of S for which $\mathfrak{n} \supset I$. Since I is a maximal ideal of R and $I \subset \mathfrak{n}$, we have

$$\mathfrak{n} \cap R = I. \tag{11}$$

[6]This proposition is erroneously claimed as a corollary to [6, Theorem 3.13, published version]. [6, Theorem 3.13] assumes that S is a depiction of R, but if R is noetherian, then S will not be a depiction of R. Indeed, in this case the only depiction of R will be itself [6, Theorem 3.12], and $R \neq S$ if I is a non-maximal ideal of S.

Furthermore, since I is a radical of S, there are primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of S such that $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, by the Lasker–Noether theorem. Fix $h \in \mathfrak{n} \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n)$. Then for $f \in S$, we have

$$fh \in I \Rightarrow f \in I. \tag{12}$$

Indeed, if $fh \in I$, then $fh \in \mathfrak{p}_i$ for each i. Whence $f \in \mathfrak{p}_i$ for each i, and therefore $f \in I$.

By assumption, $I \neq 0$. Fix $g \in I \setminus 0$, and consider the chain of ideals of R,

$$0 \subset gR \subseteq (g, gh)R \subseteq (g, gh, gh^2)R \subseteq \cdots \subseteq I.$$

We claim that each inclusion is proper. Indeed, assume to the contrary that there is some $\ell \geq 0$ and $r_0, \ldots, r_\ell \in R$ such that

$$gh^{\ell+1} = \sum_{j=0}^{\ell} r_j gh^j.$$

Then since S is an integral domain,

$$h^{\ell+1} = \sum_{j=0}^{\ell} r_j h^j.$$

Whence

$$h^{\ell+1} - \sum_{j=1}^{\ell} r_j h^j = r_0 \in R. \tag{13}$$

But $h \in \mathfrak{n}$. Therefore $r_0 \in \mathfrak{n} \cap R = I$ by (11). Furthermore, since $R = k + I$, for each $j \in [0, \ell]$ there is some $\beta_j \in k$ and $t_j \in I$ such that $r_j = \beta_j + t_j$. Since r_0 and each $t_j h^j$ are in I, (13) yields

$$t := h^{\ell+1} - \sum_{j=1}^{\ell} \beta_j h^j = r_0 + \sum_{j=1}^{\ell} t_j h^j \in I \subset \mathfrak{n}. \tag{14}$$

The left-hand side implies that t is a polynomial in $k[h]$. Therefore, since k is algebraically closed, t splits

$$t = h^{\ell+1} - \sum_{j=1}^{\ell} \beta_j h^j = h^m (h - \alpha_1) \cdots (h - \alpha_{\ell-m}),$$

where $m \geq 1$ and $\alpha_1, \ldots, \alpha_{\ell-m} \in k \setminus 0$. Set $f := (h - \alpha_1) \cdots (h - \alpha_\ell)$. By (14) we have $hf = t \in I$. Thus, by (12), $f \in I$. Consequently, $f \in \mathfrak{n}$. But this is not possible by Hilbert's Nullstellensatz, since $\alpha_1, \ldots, \alpha_{\ell-m}$ are nonzero scalars, h is in $\mathfrak{n}$, and $\mathfrak{n}$ is a maximal ideal of S. ■

Proposition 3.10. *Suppose that* $\dim S/I_i \geq 1$ *for some* i. *Then* $R = \bigcap_i (k + I_i)$ *is nonnoetherian.*

Proof. Suppose that $\dim S/I_i \geq 1$. By Proposition 3.9, I_i contains a strict infinite ascending chain of ideals of $k + I_i$,

$$J_1 \subset J_2 \subset J_3 \subset \cdots \subset I_i.$$

(i) We claim that each J_ℓ is an R-module. Let $r \in R$. Then $r \in k + I_i$. Whence $J_\ell r \subseteq J_\ell$ since J_ℓ is an ideal of $k + I_i$, proving our claim.

(ii) Now let $a \in \bigcap_j I_j$. Then each aJ_ℓ is in $\bigcap_j I_j \subset R$. Thus each aJ_ℓ is an ideal of R by Claim (i).

Consider the chain of ideals of R,

$$aJ_1 \subseteq aJ_2 \subseteq aJ_3 \subseteq \cdots . \tag{15}$$

Assume to the contrary that for some ℓ,

$$aJ_\ell = aJ_{\ell+1}.$$

Then for each $b \in J_{\ell+1} \setminus J_\ell$, there is some $c \in J_\ell$ such that

$$ab = ac.$$

But S is an integral domain. Whence

$$b = c \in J_\ell,$$

a contradiction to our choice of b. Thus the chain (15) is strict. Therefore R is nonnoetherian. ∎

We recall the following elementary facts.

Lemma 3.11. *Let* R *be an integral domain, and let* $\mathfrak{p}, \mathfrak{m} \in \operatorname{Spec} R$ *be ideals satisfying* $\mathfrak{p} \subseteq \mathfrak{m}$. *Then*[7]

(1) $\mathfrak{p}R_\mathfrak{m} \cap R = \mathfrak{p}$,

(2) $\mathfrak{p}R_\mathfrak{m} \in \operatorname{Spec} R_\mathfrak{m}$.

---

[7]We prove Lemma 3.11 for completeness.

(1) It suffices to show that $\mathfrak{p}R_\mathfrak{m} \cap R \subseteq \mathfrak{p}$. Let $\frac{a}{b} \in \mathfrak{p}R_\mathfrak{m} \cap R$, with $a \in \mathfrak{p}$ and $b \in R \setminus \mathfrak{m}$. Then

$$b \cdot \frac{a}{b} = a \in \mathfrak{p}.$$

Thus, since $b, \frac{a}{b} \in R$ and $\mathfrak{p}$ is prime in R, we have $b \in \mathfrak{p}$ or $\frac{a}{b} \in \mathfrak{p}$. But $b \notin \mathfrak{p}$ since $b \notin \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$. Therefore $\frac{a}{b} \in \mathfrak{p}$.

(2) Let $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in R_\mathfrak{m}$, with $a_1, a_2 \in R$ and $b_1, b_2 \in R \setminus \mathfrak{m}$. Suppose that

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \in \mathfrak{p}R_\mathfrak{m}.$$

Again let $R = \bigcap_i (k + I_i)$.

Lemma 3.12. *If $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{p} \subseteq I_i$ for some i, then*

$$\mathfrak{p}S \cap R = \mathfrak{p}.$$

Proof. Suppose the hypotheses hold. Let $ab \in \mathfrak{p}S \cap R$, with $a \in \mathfrak{p}$ and $b \in S$. We claim that $ab \in \mathfrak{p}$. Indeed, by Lemma 3.2 there is some

$$c \in \Big(\bigcap_{j \neq i} I_j\Big) \cap R \setminus I_i.$$

Then $ac \in \bigcap_j I_j$ since $a \in \mathfrak{p} \subseteq I_i$. Thus for any $s \in S$,

$$acs \in \bigcap_j I_j \subset R.$$

In particular,

$$acb^2 \in R.$$

Thus, since $a \in \mathfrak{p}$,

$$(ab)^2 \cdot c = a \cdot (acb^2) \in \mathfrak{p}.$$

But $c \in R \setminus \mathfrak{p}$ and $(ab)^2 \in R$. Thus $(ab)^2 \in \mathfrak{p}$ since $\mathfrak{p}$ is prime in R. Therefore $ab \in \mathfrak{p}$, again since $\mathfrak{p}$ is prime in R. ∎

Proposition 3.13. *The morphism*

$$\iota : \operatorname{Spec} S \to \operatorname{Spec} R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is surjective.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$. We claim that there is some $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.

(i) First suppose that $\mathcal{Z}(\mathfrak{p}) \cap U_{S/R} = \emptyset$. Then there is some i for which $\mathfrak{p} \subseteq I_i$, by Lemma 3.5. Set

$$\mathfrak{t} := \mathfrak{p}(k + I_i)_{I_i} \cap (k + I_i).$$

Recall that $I_i \cap R \in \operatorname{Spec} R$ by Proposition 3.3.

We claim that $\frac{a_1}{b_1}$ or $\frac{a_2}{b_2}$ is in $\mathfrak{p}R_{\mathfrak{m}}$. Indeed, there is some $c \in \mathfrak{p}$ and $d \in R \setminus \mathfrak{m}$ such that

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{c}{d}.$$

Whence

$$a_1 a_2 d = b_1 b_2 c \in \mathfrak{p}.$$

Now $d \notin \mathfrak{p}$ since $d \notin \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$. Thus $a_1 a_2 \in \mathfrak{p}$ since $\mathfrak{p}$ is prime in R. In particular, $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$; say $a_1 \in \mathfrak{p}$. Then $\frac{a_1}{b_1} \in \mathfrak{p}R_{\mathfrak{m}}$, proving our claim.

(i.a) We have $\mathfrak{p} = \mathfrak{t} \cap R$ since

$$\mathfrak{p} \overset{\text{(I)}}{=} \mathfrak{p}R_{I_i \cap R} \cap R \overset{\text{(II)}}{=} \mathfrak{p}(k + I_i)_{I_i} \cap R = \mathfrak{p}(k + I_i)_{I_i} \cap (k + I_i) \cap R = \mathfrak{t} \cap R,$$

where (I) holds by Lemma 3.11 (1), and (II) holds by Lemma 3.6.

(i.b) We claim that

$$\mathfrak{t} \in \operatorname{Spec}(k + I_i) \quad \text{and} \quad \mathfrak{t} \subseteq I_i.$$

By Lemma 3.11 (2),

$$\mathfrak{p}R_{I_i \cap R} \in \operatorname{Spec} R_{I_i \cap R}.$$

Thus by Lemma 3.6,

$$\mathfrak{p}(k + I_i)_{I_i} \in \operatorname{Spec}(k + I_i)_{I_i}.$$

Therefore $\mathfrak{t} \in \operatorname{Spec}(k + I_i)$, since the intersection of a prime ideal with a subalgebra is a prime ideal of the subalgebra.

Furthermore,

$$\mathfrak{t} = \mathfrak{p}(k + I_i)_{I_i} \cap (k + I_i) \subseteq I_i(k + I_i)_{I_i} \cap (k + I_i) \overset{\text{(I)}}{=} I_i,$$

where (I) holds by Lemma 3.11 (1) since $I_i \in \operatorname{Spec}(k + I_i)$.

(i.c) We claim that

$$\mathfrak{p} = \sqrt[S]{\mathfrak{t}S} \cap R.$$

Indeed,

$$\begin{aligned}\mathfrak{p} &\overset{\text{(I)}}{=} \mathfrak{t} \cap R \subseteq \sqrt[S]{\mathfrak{t}S} \cap R \overset{\text{(II)}}{\subseteq} \sqrt[R]{\mathfrak{t}S \cap R} \\ &= \sqrt[R]{\mathfrak{t}S \cap (k + I_i) \cap R} \overset{\text{(III)}}{=} \sqrt[R]{\mathfrak{t} \cap R} \overset{\text{(IV)}}{=} \sqrt[R]{\mathfrak{p}} = \mathfrak{p},\end{aligned}$$

where (I) and (IV) hold by Claim (i.a); (II) holds since if $s^n \in tS$ and $s \in R$, then $s \in \sqrt[R]{tS \cap R}$; and (III) holds by Claim (i.b) together with Lemma 3.12 (with $k + I_i$ in place of R).

(i.d) Since S is noetherian, the Lasker–Noether theorem implies that there are ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m \in \operatorname{Spec} S$, minimal over $\sqrt[S]{\mathfrak{t}S}$, such that

$$\sqrt[S]{\mathfrak{t}S} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m.$$

Thus

$$\mathfrak{p} \overset{\text{(I)}}{=} \sqrt[S]{\mathfrak{t}S} \cap R = (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m) \cap R = (\mathfrak{q}_1 \cap R) \cap \cdots \cap (\mathfrak{q}_m \cap R), \tag{16}$$

where (I) holds by Claim (i.c). Furthermore, each $\mathfrak{q}_j \cap R$ is a prime ideal of R since $\mathfrak{q}_j \in \operatorname{Spec} S$ and $R \subset S$ (e.g., [6, Lemma 2.1]).

Assume to the contrary that for each $j \in [1, m]$,

$$\mathfrak{q}_j \cap R \neq \mathfrak{p}.$$

Then for each j there is some
$$a_j \in (\mathfrak{q}_j \cap R) \setminus \mathfrak{p}.$$
Whence
$$a_1 \cdots a_m \in \bigcap_j (\mathfrak{q}_j \cap R) \overset{\text{(I)}}{=} \mathfrak{p},$$
where (I) holds by (16). But $\mathfrak{p}$ is prime in R, a contradiction. Thus there is some j for which
$$\mathfrak{q}_j \cap R = \mathfrak{p}.$$
Our desired ideal is therefore $\mathfrak{q} := \mathfrak{q}_j \in \operatorname{Spec} S$.

(ii) Now suppose that $\mathcal{Z}(\mathfrak{p}) \cap U_{S/R} \neq \emptyset$; say $\mathfrak{n} \in \mathcal{Z}(\mathfrak{p}) \cap U_{S/R}$. Set
$$\mathfrak{q} := \mathfrak{p}S_{\mathfrak{n}} \cap S.$$
We claim that
$$\mathfrak{q} \cap R = \mathfrak{p} \quad \text{and} \quad \mathfrak{q} \in \operatorname{Spec} S.$$

First observe that
$$\mathfrak{p} \overset{\text{(I)}}{=} \mathfrak{p}R_{\mathfrak{n}\cap R} \cap R \overset{\text{(II)}}{=} \mathfrak{p}S_{\mathfrak{n}} \cap R = \mathfrak{p}S_{\mathfrak{n}} \cap S \cap R = \mathfrak{q} \cap R,$$
where (I) holds by Lemma 3.11 (1), and (II) holds since $\mathfrak{n} \in U_{S/R}$. Furthermore, since $\mathfrak{p} \in \operatorname{Spec} R$, we have $\mathfrak{p}R_{\mathfrak{n}\cap R} \in \operatorname{Spec}(R_{\mathfrak{n}\cap R})$ by Lemma 3.11 (2). Whence $\mathfrak{p}S_{\mathfrak{n}} \in \operatorname{Spec} S_{\mathfrak{n}}$ since $\mathfrak{n} \in U_{S/R}$. Therefore $\mathfrak{q} = \mathfrak{p}S_{\mathfrak{n}} \cap S \in \operatorname{Spec} S$. ■

Theorem 3.14. *Let $I_1, \ldots, I_n$ be a set of proper non-maximal nonzero radical ideals of S which are pairwise coprime, and set $R := \bigcap_i (k + I_i)$. Then*

(1) *R is nonnoetherian if and only if there is some i for which* $\dim S/I_i \geq 1$,

(2) *R is depicted by S if and only if for each i,* $\dim S/I_i \geq 1$.

Proof. (1) The implications $\Rightarrow$ and $\Leftarrow$ are respectively Propositions 3.8 (1) and 3.10.

(2) The morphism $\iota : \operatorname{Spec} S \to \operatorname{Spec} R$ is surjective by Proposition 3.13. Furthermore, $U_{S/R}$ is nonempty since $U_{S/R} = (\bigcup_i \mathcal{Z}(I_i))^c$ is an open dense subset of $\operatorname{Max} S$, by Proposition 3.4. It thus suffices to show that
$$U^c_{S/R} = \bigcup_i \mathcal{Z}(I_i) \subseteq \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} \text{ is nonnoetherian}\}, \tag{17}$$
where the inclusion holds if and only if $\dim S/I_i \geq 1$ for each i.

Suppose that $\mathfrak{n} \in \bigcup_i \mathcal{Z}(I_i)$. Then $\mathfrak{n}$ contains some I_j. Whence $\mathfrak{n} \cap R = I_j \cap R$ by Proposition 3.3. Thus by Lemma 3.6,
$$R_{\mathfrak{n}\cap R} = R_{I_j \cap R} = (k + I_j)_{I_j}.$$

• First suppose that $\dim S/I_j = 0$. Then $R_{\mathfrak{n}\cap R} = R_{I_j\cap R}$ is noetherian by Proposition 3.8 (2). Therefore the inclusion in (17) does not hold.

• Now suppose that $\dim S/I_j \geq 1$. Then I_j contains a strict infinite ascending chain of ideals of $k + I_j$, by Proposition 3.9. Therefore the localization $R_{\mathfrak{n} \cap R} = (k + I_j)_{I_j}$ is nonnoetherian. In particular, if $\dim S/I_i \geq 1$ for each i, then the inclusion in (17) holds. ■

Corollary 3.15. *If* $\dim S/I_i \geq 1$ *for each* i*, then each of the closed points* $I_i \cap R$ *of* $\operatorname{Spec} R$ *has positive geometric dimension.*

Proof. By Theorem 3.14, S is a depiction of R. Therefore for each i,

$$\operatorname{gdim}(I_i \cap R) \geq \dim S/I_i \geq 1.$$ ■

4. Sheaves of depictions

Let $(X, \mathcal{O})$ be an affine scheme, and set $R := \mathcal{O}(X)$. We introduce the following definition.

Definition 4.1. A *sheaf of depictions* $\widetilde{S}$ on $(X, \mathcal{O})$ is a sheaf of algebras such that on each principal open set $D(a) \subset X$, $a \in R$, the algebra $\widetilde{S}(D(a))$ is a depiction of $\mathcal{O}(D(a))$.

A sheaf $\mathcal{M}$ on X is said to be a sheaf of modules if, on each open set $U \subset X$, $\mathcal{M}(U)$ is an $\mathcal{O}(U)$-module, and for each inclusion of open sets $U \subset V$, the restriction $\mathcal{M}(V) \to \mathcal{M}(U)$ is an $\mathcal{O}(V)$-module homomorphism. The sheafification of an R-module M is the sheaf of modules $\widetilde{M}$ defined on each principal open set $D(a)$ by

$$\widetilde{M}\big(D(a)\big) := M \otimes_{\mathcal{O}(X)} \mathcal{O}\big(D(a)\big) = M \otimes_R R[a^{-1}],$$

and on a general open set U by the inverse limit

$$\widetilde{M}(U) := \varprojlim_{D(a) \subset U} \widetilde{M}\big(D(a)\big).$$

In this section, we show that the sheafification of a depiction is a sheaf of depictions.

Let S be an integral domain and k-algebra. For an element $a \in S$ and ideal $I \subset S$, set $S_a := S[a^{-1}]$ and $I_a := IS[a^{-1}]$.

Lemma 4.2. *Fix* $a \in S$.

(1) *If* $\mathfrak{q} \in \operatorname{Spec} S$ *and* $a \notin \mathfrak{q}$*, then* $\mathfrak{q}_a \in \operatorname{Spec} S_a$.

(2) *If* $\mathfrak{n} \in \operatorname{Max} S$ *and* $a \notin \mathfrak{n}$*, then* $\mathfrak{n}_a \in \operatorname{Max} S_a$.

Proof. (1) Suppose that $\mathfrak{q} \in \operatorname{Spec} S$ and $a \notin \mathfrak{q}$. Since S_a is a flat S-module, the short exact sequence $0 \to \mathfrak{q} \to S \to S/\mathfrak{q} \to 0$ induces the short exact sequence

$$0 \to \mathfrak{q} \otimes_S S_a \to S \otimes_S S_a \cong S_a \to S/\mathfrak{q} \otimes_S S_a \to 0.$$

Whence

$$S/\mathfrak{q} \otimes_S S_a \cong S_a/\mathfrak{q}_a. \tag{18}$$

But $S/\mathfrak{q}$ is an integral domain since $\mathfrak{q}$ is prime. Furthermore, $S/\mathfrak{q} \otimes S_a$ is not the zero ring since $a^n \notin \mathfrak{q}$ for all $n \geq 0$. Thus $S/\mathfrak{q} \otimes S_a$ is also an integral domain. Therefore $\mathfrak{q}_a$ is a prime of S_a, by (18).

(2) Suppose $\mathfrak{n} \in \operatorname{Max} S$ and $a \notin \mathfrak{n}$. By Claim (1), we have

$$S/\mathfrak{n} \otimes_S S_a \cong S_a/\mathfrak{n}_a \neq 0.$$

Furthermore, $S/\mathfrak{n} \otimes S_a$ is a field since $\mathfrak{n}$ is a maximal ideal of S. Consequently, $\mathfrak{n}_a$ is a maximal ideal of S_a. ■

Let R be a subalgebra of S.

Lemma 4.3. *Fix $a \in R$. If*

$$\iota_{S/R} : \operatorname{Spec} S \to \operatorname{Spec} R$$

is surjective, then so is

$$\iota_{S_a/R_a} : \operatorname{Spec} S_a \to \operatorname{Spec} R_a.$$

Proof. Suppose that $\iota_{S/R}$ is surjective. Let $\tilde{\mathfrak{p}} \in \operatorname{Spec} R_a$, and set $\mathfrak{p} := \tilde{\mathfrak{p}} \cap R$. Then $\mathfrak{p}$ is in $\operatorname{Spec} R$. Thus there is a prime $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, by the surjectivity of $\iota_{S/R}$. Furthermore, the ideal $\mathfrak{q}_a$ is in $\operatorname{Spec} S_a$, by Lemma 4.2 (1).

We want to show that $\mathfrak{q}_a \cap R_a = \tilde{\mathfrak{p}}$, from which the lemma follows.

(i) We first claim that $\mathfrak{q}_a \cap R_a \supseteq \tilde{\mathfrak{p}}$.

Let $g \in \tilde{\mathfrak{p}}$. Then for $\ell \geq 0$ sufficiently large, $a^\ell g$ is in R. Whence $a^\ell g \in \tilde{\mathfrak{p}} \cap R = \mathfrak{p}$. Thus $a^\ell g \in \mathfrak{q}$. Therefore $g = a^{-\ell} a^\ell g \in \mathfrak{q}_a$.

(ii) We now claim that $\mathfrak{q}_a \cap R_a \subseteq \tilde{\mathfrak{p}}$.

Let $g \in \mathfrak{q}_a \cap R_a$. Then again for $\ell \geq 0$ sufficiently large, $a^\ell g$ is in $\mathfrak{q}$ and R. Thus,

$$a^\ell g \in \mathfrak{q} \cap R = \mathfrak{p} = \tilde{\mathfrak{p}} \cap R.$$

Consequently, $g = a^{-\ell} a^\ell g \in \tilde{\mathfrak{p}}$. ■

Proposition 4.4. *Fix $a \in R$. If S is a depiction of R, then S_a is a depiction of R_a.*

Proof. Suppose that S is a depiction of R.

(i) The morphism $\iota_{S_a/R_a} : \operatorname{Spec} S_a \to \operatorname{Spec} R_a$ is surjective by Lemma 4.3.

(ii) Let $\mathfrak{n} \in \operatorname{Max} S_a$, and suppose that $(R_a)_{\mathfrak{n} \cap R_a}$ is noetherian. We claim that

$$(R_a)_{\mathfrak{n} \cap R_a} = (S_a)_{\mathfrak{n}}.$$

Since $\mathfrak{n}$ is a proper ideal of S_a, we have $\mathfrak{n} \not\ni a$. Therefore

$$(R_a)_{\mathfrak{n} \cap R_a} \overset{\text{(I)}}{=} R_{\mathfrak{n} \cap R} \overset{\text{(II)}}{=} S_{\mathfrak{n} \cap S} \overset{\text{(III)}}{=} (S_a)_{\mathfrak{n}},$$

where (I) and (III) hold since $a \in R \setminus \mathfrak{n}$; and (II) holds since $R_{\mathfrak{n} \cap R} = (R_a)_{\mathfrak{n} \cap R_a}$ is noetherian and S is a depiction of R.

(iii) Finally, we claim that the locus U_{S_a/R_a} is nonempty.

Let $D_S(a) := \{\mathfrak{n} \in \operatorname{Max} S \mid \mathfrak{n} \not\ni a\}$ denote the complement of the vanishing locus of a in $\operatorname{Max} S$. Then

$$U_{S_a/R_a} = U_{S/R} \cap D_S(a) \neq \emptyset$$

since $U_{S/R}$ and $D_S(a)$ are open dense sets of $\operatorname{Max} S$. ∎

Corollary 4.5. *Suppose that S is a depiction of R. Then the sheafification $\widetilde{S}$ of the R-module S on* Spec R *is a sheaf of depictions on* Spec R.

5. Noncommutative blowups of nonnoetherian singularities

Let S be a normal integral domain and a finitely generated k-algebra. Let $Y_1, \dots, Y_n$ be positive dimensional proper subvarieties of $\operatorname{Max} S$ that intersect the smooth locus. For each $i \in [1, n]$, denote by $I_i := I(Y_i)$ the corresponding radical ideal of S. Consider the nonnoetherian coordinate ring $R := \bigcap_i (k + I_i)$ and its set of positive dimensional closed points (Proposition 3.3),

$$\mathfrak{m}_i := I_i \cap R \in \operatorname{Spec} R.$$

Following [20, Section R], we call the endomorphism ring

$$A := \operatorname{End}_R({}_R R \oplus \bigoplus_i \mathfrak{m}_i)$$

the "noncommutative blowup" of $\operatorname{Max} R$ at the points $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. These points are precisely the nonnoetherian points of R (that is, the points $\mathfrak{m} \in \operatorname{Max} R$ for which $R_\mathfrak{m}$ is nonnoetherian), by Theorem 3.14 and Proposition 3.4. Our main theorem in this section is that if either (i) each Y_i is irreducible, or (ii) $n = 1$, then A is a noncommutative desingularization of its center R. Furthermore, S is the cycle algebra of A, and thus A provides a means to retrieve S from the knowledge of R alone. In particular, R is depicted by the cycle algebra of A.

In the following lemma, we do not assume S is normal.

Lemma 5.1. *Let I be a nonzero ideal of a noetherian integral domain S, and suppose that I is also an ideal of an overring $T \subset \operatorname{Frac} S$ of S. Then T is contained in the integral closure $\overline{S}$ of S.*

Proof. Let $s \in I \setminus \{0\}$ and $t \in T$. By assumption, $t^\ell s \in I$ for each $\ell \geq 0$. Consider the ascending chain of ideals of S

$$sS \subseteq (s, ts)S \subseteq (s, ts, t^2 s)S \subseteq (s, ts, t^2 s, t^3 s) \subseteq \cdots .$$

Since S is noetherian, there is some $m \geq 1$ and $\sigma_0, \dots, \sigma_{m-1} \in S$ such that

$$t^m s = \sum_{j=0}^{m-1} \sigma_j t^j s.$$

Thus, since S is an integral domain and $s \neq 0$, we have

$$t^m - \sum_{j=0}^{m-1} \sigma_j t^j = 0.$$

Consequently, t is in the integral closure $\overline{S}$ of S. ∎

Again let S be a normal finitely generated domain. For brevity, set

$$R^i := S \cap \Big(\bigcap_{j \neq i} (k + I_j) \Big).$$

We include S in the intersection for the case $n = 1$.

Lemma 5.2. *For each $i \in [1, n]$, we have*

$$\operatorname{Hom}_R(\mathfrak{m}_i, \mathfrak{m}_i) = \operatorname{Hom}_R(\mathfrak{m}_i, R) = R^i.$$

Proof. (i) We first claim that $\operatorname{Hom}_R(\mathfrak{m}_i, R) \subseteq S$.

Indeed, $\operatorname{Hom}_S(I_i, I_i)$ is the largest overring of S for which I_i is an ideal. Thus, since S is normal, Lemma 5.1 implies that

$$\operatorname{Hom}_S(I_i, I_i) \subseteq S. \tag{19}$$

Let $x \in \operatorname{Hom}_R(\mathfrak{m}_i, R)$ and $w \in I_1 I_2 \cdots I_n$. Then $x^\ell w$ is in $\operatorname{Hom}_S(I_i, I_i)$ for each $\ell \geq 1$, since $w I_i \subseteq \mathfrak{m}_i$. Whence $x^\ell w$ is in S by (19). But since S is a normal noetherian domain, the same argument given in the proof of Lemma 5.1, with x and w in place of t and s, shows that x itself is in S.

(ii) We now claim that $\operatorname{Hom}_R(\mathfrak{m}_i, R) \subseteq R$.

Consider $x \in \operatorname{Hom}_R(\mathfrak{m}_i, R)$ and $y \in \mathfrak{m}_i$. Then for each $j \in [1, n]$, xy is in $k + I_j$. Furthermore, since y is in R, there is a $c \in k$ and $z \in I_j$ such that $y = c + z \in k + I_j$. In particular, xz is in I_j, since x is in S by Claim (i). Thus x itself is in $k + I_j$, since $cx + xz = xy$ is in $k + I_j$. But j was arbitrary, and therefore x is in R.

(iii) Finally, we claim that $R^i \subseteq \operatorname{Hom}_R(\mathfrak{m}_i, \mathfrak{m}_i)$.

Since $\mathfrak{m}_i \subset R \subseteq k + I_j$ for each j, and $R^i \subseteq S$, we have $R^i \mathfrak{m}_i \subseteq R$. Furthermore, $R^i \subseteq S$ implies that $R^i \mathfrak{m}_i \subseteq I_i$. Therefore $R^i \mathfrak{m}_i \subseteq I_i \cap R = \mathfrak{m}_i$.

(iv) We have

$$R^i \overset{(\mathrm{I})}{\subseteq} \operatorname{Hom}_R(\mathfrak{m}_i, \mathfrak{m}_i) \subseteq \operatorname{Hom}_R(\mathfrak{m}_i, R) \overset{(\mathrm{II})}{\subseteq} R \subseteq R^i,$$

where (I) holds by Claim (iii), and (II) holds by Claim (ii). ∎

Lemma 5.3. *Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ be coprime ideals. Then*

$$\operatorname{Hom}_R(\mathfrak{p}, \mathfrak{q}) = \mathfrak{q}.$$

Proof. Since $\mathfrak{p}$, $\mathfrak{q}$ are ideals of R, $\operatorname{Hom}_R(\mathfrak{p}, \mathfrak{q})$ is isomorphic as an R-module to the maximum R-module $C \subseteq \operatorname{Frac} R$ satisfying $C\mathfrak{p} \subseteq \mathfrak{q}$. In particular, $C \supseteq \mathfrak{q}$.

To show the reverse inclusion, let $c \in C$. Since $\mathfrak{p}$, $\mathfrak{q}$ are coprime, there is an $a \in \mathfrak{p}$ and $b \in \mathfrak{q}$ such that $a + b = 1$. Whence

$$c(1 - b) = ca \in C\mathfrak{p} \subseteq \mathfrak{q}.$$

But $\mathfrak{q}$ is prime and $1 - b \notin \mathfrak{q}$. Thus $c \in \mathfrak{q}$. Therefore $C = \mathfrak{q}$. ∎

Proposition 5.4. *There is an algebra isomorphism*

$$A = \operatorname{End}_R({}_R R \oplus \bigoplus_i \mathfrak{m}_i) \cong \begin{bmatrix} R & \mathfrak{m}_1 & \mathfrak{m}_2 & \cdots & \mathfrak{m}_n \\ R^1 & R^1 & \mathfrak{m}_2 & \cdots & \mathfrak{m}_n \\ R^2 & \mathfrak{m}_1 & R^2 & \cdots & \mathfrak{m}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^n & \mathfrak{m}_1 & \mathfrak{m}_2 & \cdots & R^n \end{bmatrix}. \tag{20}$$

Proof. Each $\mathfrak{m}_i$ is a prime ideal of R, by Proposition 3.3. Furthermore, for each $i \neq j$, there is some

$$a \in I_i \cap R = \mathfrak{m}_i \quad \text{and} \quad b \in I_j \cap R = \mathfrak{m}_j$$

such that $a + b = 1$, by Lemma 3.2. Thus the set of ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are pairwise coprime. The isomorphism (20) therefore holds by Lemmas 5.2 and 5.3. ∎

Remark 5.5. The endomorphism ring of the *right* R-module $R \oplus \bigoplus_i \mathfrak{m}_i$ is the transpose of the matrix ring given in (20), and it is not known whether it is cycle regular. (As a right (resp. left) R-module, $R \oplus \bigoplus_i \mathfrak{m}_i$ may be viewed as an $n + 1$ column (resp. row) vector.)

Remark 5.6. In the case $n = 1$, we have $\mathfrak{m} = I$ (omitting the subscript i), and the tiled matrix ring (20) simplifies to

$$A = \operatorname{End}_R({}_R R \oplus I) \cong \begin{bmatrix} R & I \\ S & S \end{bmatrix}.$$

Proposition 5.7. *The cycle algebra of A is S.*

Proof. By Proposition 5.4, the cycle algebra of A is $\tilde{S} := k[R + R^1 + \cdots + R^n]$. By Remark 5.6, it suffices to suppose that $n \geq 2$.

We first claim that for any subset $K \subseteq \{1, \ldots, n\}$ with $|K| \geq 2$, we have

$$\sum_{i \in K} \bigcap_{j \in K \setminus \{i\}} I_j = S. \tag{21}$$

We proceed by induction on $|K|$. Let $K \ni 1, 2$.

First suppose $|K| = 2$. Then (21) reduces to $I_1 + I_2 = S$, and this holds since I_1 and I_2 are coprime ideals of S.

Now suppose that (21) holds for $|K| \leq N$, and let $|K| = N+1$. Set $K_1 := K \setminus \{1\}$ and $K_2 := K \setminus \{2\}$. Then

$$\begin{aligned} S &\overset{\text{(I)}}{=} I_1 + I_2 = I_1 \cap S + I_2 \cap S \\ &\overset{\text{(II)}}{=} I_1 \cap \Big(\sum_{i \in K_1} \bigcap_{j \in K_1 \setminus \{i\}} I_j \Big) + I_2 \cap \Big(\sum_{i \in K_2} \bigcap_{j \in K_2 \setminus \{i\}} I_j \Big) \\ &\subseteq \sum_{i \in K} \bigcap_{j \in K \setminus \{i\}} I_j \subseteq S, \end{aligned}$$

where (I) holds since I_1 and I_2 are coprime, and (II) holds by induction. This proves our claim.

Thus,

$$S = \sum_{i=1}^{n} \bigcap_{j \neq i} I_j \subseteq \sum_{i=1}^{n} R^i \subseteq \widetilde{S} \subseteq S.$$

Therefore $\widetilde{S} = S$. ∎

Fix $1 \leq i \leq n$, and let $\mathfrak{q} \in \operatorname{Spec} S$ be a minimal prime over $\mathfrak{m}_i$. Since $\mathfrak{m}_i$ is a maximal ideal of R, we have

$$\mathfrak{m}_i = I_i \cap R = \mathfrak{q} \cap R. \tag{22}$$

Lemma 5.8. *Suppose that $\mathfrak{q} \in \operatorname{Spec} S$ is a minimal prime over $\mathfrak{m}_i$. Then $I_i \subseteq \mathfrak{q}$. Consequently, if I_i is prime in S, then $I_i = \mathfrak{q}$.*

Proof. We first claim that $I_i \subseteq \mathfrak{q}$. Let $a \in S \setminus \mathfrak{q}$; we want to show that $a \notin I_i$.

Assume to the contrary that $a \in I_i$. By Lemma 3.2, there is some $b \in R$ that is in $\bigcap_{j \neq i} I_j \setminus I_i$. Whence, $ab \in \bigcap_j I_j \subset R$. Furthermore, since $b \in R \setminus I_i$, we have $b \notin \mathfrak{q} \cap R$ by (22). In particular, $b \notin \mathfrak{q}$. Since a and b are not in $\mathfrak{q}$ and $\mathfrak{q}$ is prime, their product ab is not in $\mathfrak{q}$. Thus

$$a^{-1} = b(ab)^{-1} \in R_{\mathfrak{q} \cap R} \overset{\text{(I)}}{=} (k + I_i)_{I_i},$$

where (I) holds by Lemma 3.6. Whence $a^{-1} \in (k + I_i)_{I_i}$. But $a \in I_i$, and thus a is not invertible in $(k + I_i)_{I_i}$, a contradiction. Therefore $I_i \subseteq \mathfrak{q}$. ∎

Lemma 5.9. *For each minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}_i$ and $j \neq i$, the following hold:*

$$R_{\mathfrak{q} \cap R} = R^j_{\mathfrak{q} \cap R^j} = \mathfrak{m}_j (k + I_i)_{I_i} = (k + I_i)_{I_i} \quad \textit{and} \quad \mathfrak{m}_j S_{\mathfrak{q}} = S_{\mathfrak{q}}.$$

Furthermore, if either I_i is prime in S or $n = 1$, then

$$R^i_{\mathfrak{q} \cap R^i} = S_{\mathfrak{q}} \quad \textit{and} \quad \mathfrak{m}_i S_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}}.$$

Proof. (i) By Lemma 3.6, we have $R_{\mathfrak{q} \cap R} = (k + I_i)_{I_i}$, and for $j \neq i$, $R^j_{\mathfrak{q} \cap R^j} = (k + I_i)_{I_i}$.

(ii) Let $j \neq i$. We claim that $\mathfrak{m}_j(k + I_i)_{I_i} = (k + I_i)_{I_i}$. Fix $b \in \mathfrak{m}_j \setminus I_i$. Then $b \in k + I_i$ since $b \in R$. Whence, $b^{-1} \in (k + I_i)_{I_i}$. Therefore

$$1 = bb^{-1} \in \mathfrak{m}_j(k + I_i)_{I_i}.$$

(iii) Let $j \neq i$. We claim that $\mathfrak{m}_j S_{\mathfrak{q}} = S_{\mathfrak{q}}$. By Lemma 3.2, there is some

$$b \in (I_j \cap R) \setminus I_i = \mathfrak{m}_j \setminus \mathfrak{m}_i.$$

Whence $b \notin \mathfrak{q}$ by (22). Therefore

$$1 = bb^{-1} \in \mathfrak{m}_j S_{\mathfrak{q}}.$$

(iv) Suppose that I_i is prime in S. We claim that $R^i_{\mathfrak{q} \cap R^i} = S_{\mathfrak{q}}$. Clearly, $R^i_{\mathfrak{q} \cap R^i} \subseteq S_{\mathfrak{q}}$.

To show the reverse inclusion, suppose that $\frac{a}{b} \in S_{\mathfrak{q}}$ with $a \in S$ and $b \in S \setminus \mathfrak{q}$. By Lemma 3.2, there is some $c \in (\bigcap_{j \neq i} I_j) \setminus I_i$. Thus, ac and bc are in $\bigcap_{j \neq i} I_j \subset R^i$. Furthermore, $c \notin \mathfrak{q}$ since $\mathfrak{q} = I_i$, by Lemma 5.8. Whence $bc \notin \mathfrak{q}$ since $\mathfrak{q}$ is prime. Therefore

$$\frac{a}{b} = \frac{ac}{bc} \in R^i_{\mathfrak{q} \cap R^i},$$

proving our claim.

(v) Again suppose that I_i is prime in S. We claim that $\mathfrak{m}_i S_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}}$. Clearly, $\mathfrak{m}_i S_{\mathfrak{q}} \subseteq \mathfrak{q} S_{\mathfrak{q}}$.

To show the reverse inclusion, let $a \in \mathfrak{q} = I_i$. Fix $b \in \bigcap_{j \neq i} I_j \setminus I_i$. Then

$$ab \in \bigcap_j I_j \subset R.$$

Whence, $ab \in I_i \cap R = \mathfrak{m}_i$. Furthermore, $b \in S \setminus \mathfrak{q}$ since $\mathfrak{q} = I_i$. Therefore

$$a = abb^{-1} \in \mathfrak{m}_i S_{\mathfrak{q}}.$$

(vi) Finally, suppose that $n = 1$, in which case $\mathfrak{m} = I$ (we omit the subscript i). We claim that $IS_{\mathfrak{q}} = \mathfrak{q} S_{\mathfrak{q}}$. The inclusion $IS_{\mathfrak{q}} \subseteq \mathfrak{q} S_{\mathfrak{q}}$ follows from Lemma 5.8.

To show the reverse inclusion, let $a \in \mathfrak{q}$. Consider the set of minimal primes over I,

$$\mathfrak{q}_1 := \mathfrak{q}, \mathfrak{q}_2, \ldots, \mathfrak{q}_m \in \operatorname{Spec} S.$$

In particular, $I = \bigcap_j \mathfrak{q}_j$ since I is radical.

For each $2 \leq j \leq m$, fix $b_j \in \mathfrak{q}_j \setminus \mathfrak{q}$. Then $b_2 \cdots b_m \in S \setminus \mathfrak{q}$ since $\mathfrak{q}$ is prime. Therefore

$$a = (ab_2 \cdots b_m)(b_2 \cdots b_m)^{-1} \in \Big(\bigcap_j \mathfrak{q}_j\Big) S_{\mathfrak{q}} = IS_{\mathfrak{q}}. \qquad \blacksquare$$

Set

$$\tilde{R} := (k + I_i)_{I_i} + \mathfrak{q} S_{\mathfrak{q}}.$$

If I_i is prime in S, then by Lemma 5.8 this reduces to

$$\tilde{R} = (k + \mathfrak{q})_{\mathfrak{q}} + \mathfrak{q} S_{\mathfrak{q}}.$$

Lemma 5.10. *$\widetilde{R}$ is a subalgebra of $S_{\mathfrak{q}}$.*

Proof. Since $k + I_i \subset S$, it suffices to show that if a is invertible in $(k + I_i)_{I_i}$, then a is also invertible in $S_{\mathfrak{q}}$. So suppose that $a \in (k + I_i) \setminus I_i$. Then $a = c + \alpha$, where $c \in k^{\times}$ and $\alpha \in I_i$. But $I_i \subseteq \mathfrak{q}$ by Lemma 5.8. Whence $a \in S \setminus \mathfrak{q}$. ∎

Index the rows and columns of $A_{\mathfrak{q}}$ by $0, 1, \ldots, n$. Denote by $e_{ij} \in M_{n+1}(\operatorname{Frac} S)$ the matrix with a 1 in the ij-th slot and zeros elsewhere, and set $e_i := e_{ii}$.

Proposition 5.11. *Suppose that each $I_i \subset S$ is prime, or $n = 1$. Fix $\mathfrak{p} \in \operatorname{Spec} R$, and let $\mathfrak{q} \in \operatorname{Spec} S$ be a minimal prime over $\mathfrak{p}$.*

(1) *If $R_{\mathfrak{p}}$ is noetherian, then the cyclic localization $A_{\mathfrak{q}}$ at $\mathfrak{q}$ is the full matrix ring*

$$A_{\mathfrak{q}} = M_{n+1}(R_{\mathfrak{p}}) \cong A \otimes_R R_{\mathfrak{p}}.$$

(2) *If $R_{\mathfrak{p}}$ is nonnoetherian, then $\mathfrak{p} = \mathfrak{m}_i$ for some i, and*

$$A_{\mathfrak{q}} = \begin{bmatrix} \widetilde{R} & \cdots & \widetilde{R} & \mathfrak{q}S_{\mathfrak{q}} & \widetilde{R} & \cdots & \widetilde{R} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widetilde{R} & \cdots & \widetilde{R} & \mathfrak{q}S_{\mathfrak{q}} & \widetilde{R} & \cdots & \widetilde{R} \\ S_{\mathfrak{q}} & \cdots & S_{\mathfrak{q}} & S_{\mathfrak{q}} & S_{\mathfrak{q}} & \cdots & S_{\mathfrak{q}} \\ \widetilde{R} & \cdots & \widetilde{R} & \mathfrak{q}S_{\mathfrak{q}} & \widetilde{R} & \cdots & \widetilde{R} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widetilde{R} & \cdots & \widetilde{R} & \mathfrak{q}S_{\mathfrak{q}} & \widetilde{R} & \cdots & \widetilde{R} \end{bmatrix} \subset M_{n+1}(\operatorname{Frac} S),$$

where the ith row and column are, respectively,

$$e_i A_{\mathfrak{q}} = \begin{bmatrix} S_{\mathfrak{q}} & \cdots & S_{\mathfrak{q}} \end{bmatrix} = S_{\mathfrak{q}}^{\oplus n+1},$$
$$A_{\mathfrak{q}} e_i = \begin{bmatrix} \mathfrak{q}S_{\mathfrak{q}} & \cdots & \mathfrak{q}S_{\mathfrak{q}} & S_{\mathfrak{q}} & \mathfrak{q}S_{\mathfrak{q}} & \cdots & \mathfrak{q}S_{\mathfrak{q}} \end{bmatrix}^{\mathrm{t}},$$

and all other entries are $\widetilde{R}$.

Proof. (1) Suppose that $R_{\mathfrak{p}}$ is noetherian. Then $R_{\mathfrak{p}} = S_{\mathfrak{q}}$ since S is a depiction of R.

(1.i) We first claim that the diagonal entries of $A_{\mathfrak{q}}$ are all $S_{\mathfrak{q}}$. Fix $i \in [1, n]$. We have

$$S_{\mathfrak{q}} \overset{\text{(I)}}{=} R_{\mathfrak{q} \cap R} \overset{\text{(II)}}{\subseteq} R^i_{\mathfrak{q} \cap R^i} \overset{\text{(III)}}{\subseteq} S_{\mathfrak{q}},$$

where (I) holds since S is a depiction of R; (II) holds since $R \subset R^i$; and (III) holds since $R^i \subseteq S$. Therefore $R^i_{\mathfrak{q} \cap R^i} = S_{\mathfrak{q}}$.

(1.ii) We now claim that the off-diagonal entries of $A_{\mathfrak{q}}$ are also all $S_{\mathfrak{q}}$.

Fix $i \in [1, n]$, and assume to the contrary that $\mathfrak{m}_i \subseteq \mathfrak{q}$. Then

$$\mathfrak{m}_i = \mathfrak{m}_i \cap R \subseteq \mathfrak{q} \cap R = \mathfrak{p}.$$

Whence $\mathfrak{m}_i = \mathfrak{p}$, since $\mathfrak{m}_i$ is maximal. But $R_{\mathfrak{m}_i}$ is nonnoetherian by Theorem 3.14 and Proposition 3.4, contrary to our choice of $\mathfrak{p}$. Thus $\mathfrak{m}_i \not\subseteq \mathfrak{q}$. Hence, there is some $a \in \mathfrak{m}_i \setminus \mathfrak{q}$.

Consequently, $1 = aa^{-1} \in \mathfrak{m}_i S_{\mathfrak{q}}$. Therefore $\mathfrak{m}_i S_{\mathfrak{q}} = S_{\mathfrak{q}}$. Together with (1.i), this implies that the off-diagonal entries of $A_{\mathfrak{q}}$ in columns $1, \ldots, n$ are $S_{\mathfrak{q}}$.

Finally, the off-diagonal entries in column 0 are also $S_{\mathfrak{q}}$: since $1 \in R^i$ and $R^i \subseteq S$, we have $R^i S_{\mathfrak{q}} = S_{\mathfrak{q}}$.

(2) Follows from Lemmas 5.9 and 5.10. ∎

Fix $i \in [1, n]$ and a minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}_i$.

Lemma 5.12. *Let $j \in [0, n]$, let P be a projective $A_{\mathfrak{q}}$-module, and let*

$$\delta : A_{\mathfrak{q}} e_j \to P$$

be an $A_{\mathfrak{q}}$-module homomorphism. Suppose that $e_{ij} \in A_{\mathfrak{q}}$. If $\delta(e_{ij}) = 0$, then $\delta \equiv 0$.

Proof. Set $\Lambda := A_{\mathfrak{q}}$, and suppose that $\delta(e_{ij}) = 0$. Let $\ell \geq 1$ be minimal such that P is a direct summand of $\Lambda^{\oplus \ell}$. Let $a_1, \ldots, a_\ell \in \Lambda$ be such that

$$\delta(e_j) = (a_1, \ldots, a_\ell) \in \Lambda^{\oplus \ell}.$$

Each a_k is in $e_j \Lambda$ since

$$(a_1, \ldots, a_\ell) = \delta(e_j) = \delta(e_j^2) = e_j \delta(e_j) \in e_j \Lambda^{\oplus \ell}.$$

Furthermore, each product $e_{ij} a_k$ is zero since

$$(e_{ij} a_1, \ldots, e_{ij} a_\ell) = e_{ij}(a_1, \ldots, a_\ell) = e_{ij} \delta(e_j) = \delta(e_{ij}) = 0.$$

But $e_{ij} \alpha \neq 0$ for all nonzero α in $e_j \Lambda$. Therefore each a_k is zero. ∎

Proposition 5.13. *The left global dimension of $A_{\mathfrak{q}}$ is bounded above by the Krull dimension of $S_{\mathfrak{q}}$,*

$$\operatorname{gldim} A_{\mathfrak{q}} \leq \dim S_{\mathfrak{q}}.$$

Proof. Set $\Lambda := A_{\mathfrak{q}}$ and $d := \dim S_{\mathfrak{q}} - 1$. Let V be a Λ-module. We claim that

$$\operatorname{pd}_\Lambda(V) \leq d + 1. \tag{23}$$

It suffices to show that there is a projective resolution $P_\bullet$ of V,

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} V \to 0,$$

for which the $(d+1)$th syzygy $\ker \delta_d$ is a projective Λ-module [23, Proposition 8.6.iv].

Since $\{e_0, \ldots, e_n\}$ is a complete set of orthogonal idempotents of Λ, we may assume that for each $\ell \geq 0$ and $j \in [0, n]$, there is some $m_{\ell j} \geq 0$ such that

$$P_\ell = \bigoplus_{j : m_{\ell j} \geq 1} (\Lambda e_j)^{\oplus m_{\ell j}} = \bigoplus_{j : m_{\ell j} \geq 1} \bigoplus_{t \in [1, m_{\ell j}]} \Lambda e_j \varepsilon_t,$$

where $e_j \varepsilon_t$ generates the t-th Λe_j summand of $(\Lambda e_j)^{\oplus m_{\ell j}}$ over Λ.

Now $e_i\Lambda$ is a left $S_{\mathfrak{q}}$-module since $e_i\Lambda e_i = e_i S_{\mathfrak{q}}$. Furthermore, $e_i\Lambda$ is a projective, hence flat, right Λ-module. Thus, setting $\otimes := \otimes_\Lambda$, the sequence of $S_{\mathfrak{q}}$-modules

$$\cdots \to e_i\Lambda \otimes P_2 \xrightarrow{1\otimes\delta_2} e_i\Lambda \otimes P_1 \xrightarrow{1\otimes\delta_1} e_i\Lambda \otimes P_0 \xrightarrow{1\otimes\delta_0} e_i\Lambda \otimes V \to 0$$

is exact. Moreover, each term $e_i\Lambda \otimes P_\ell$ is a free $S_{\mathfrak{q}}$-module since

$$e_i\Lambda \otimes P_\ell = \bigoplus_{j\,:\,m_{\ell j}\geq 1} (e_i\Lambda \otimes \Lambda e_j)^{\oplus m_{\ell j}} \cong \bigoplus_j (e_i\Lambda e_j)^{\oplus m_{\ell j}} = \bigoplus_j (e_{ij} S_{\mathfrak{q}})^{\oplus m_{\ell j}}. \tag{24}$$

It follows that $e_i\Lambda \otimes P_\bullet$ is a free resolution of the $S_{\mathfrak{q}}$-module $e_i\Lambda \otimes V \cong e_i V$. Thus, since $S_{\mathfrak{q}}$ is a regular local ring of dimension $d+1$, the $(d+1)$th syzygy $\ker(1\otimes\delta_d)$ of $e_i\Lambda \otimes P_\bullet$ is a free $S_{\mathfrak{q}}$-module. Therefore, since $\ker(1\otimes\delta_d)$ is a submodule of $\bigoplus_j e_{ij} S_{\mathfrak{q}}^{\oplus m_{dj}}$, for each $j \in [0,n]$ there is some $r_j \in [0, m_{dj}]$ such that

$$\ker(1\otimes\delta_d) \cong \bigoplus_{j\,:\,r_j\geq 1} (e_{ij} S_{\mathfrak{q}})^{\oplus r_j}. \tag{25}$$

Again since $e_i\Lambda$ is a flat right Λ-module, the sequence

$$0 \to e_i\Lambda \otimes \ker\delta_d \to e_i\Lambda \otimes P_d \xrightarrow{1\otimes\delta_d} e_i\Lambda \otimes P_{d-1}$$

is exact. Whence

$$e_i\Lambda \otimes \ker\delta_d = \ker(1\otimes\delta_d). \tag{26}$$

But (25) and (26) together imply that

$$e_i \ker\delta_d = \bigoplus_{j\,:\,r_j\geq 1} (e_{ij} S_{\mathfrak{q}})^{\oplus r_j} = e_i \bigoplus_j (\Lambda e_j)^{\oplus r_j} = e_i \bigoplus_{j\,:\,r_j\geq 1} \bigoplus_{t\in[1,r_j]} \Lambda e_j \varepsilon_t. \tag{27}$$

In particular, for each $t \in [1, r_j]$ we have $\delta_d(e_{ij}\varepsilon_t) = 0$. Thus by Lemma 5.12,

$$\delta_d(\Lambda e_j \varepsilon_t) = 0.$$

Therefore

$$\ker\delta_d \supseteq \bigoplus_{j\,:\,r_j\geq 1} \bigoplus_{t\in[1,r_j]} \Lambda e_j \varepsilon_t. \tag{28}$$

To show the reverse inclusion, fix $j \in [0,n]$ satisfying $m_{dj} \geq 1$, and let $t \in [1, m_{dj}]$. Suppose that $e_{kj}\varepsilon_t \in \ker\delta_d$. Then, since $1 \in \Lambda^{ik}$ for each $k \in [0,n]$, we have

$$\delta_d(e_{ij}\varepsilon_t) = \delta_d(e_{ik}e_{kj}\varepsilon_t) = e_{ik}\delta_d(e_{kj}\varepsilon_t) = 0.$$

Whence $e_{ij}\varepsilon_t \in e_i \ker\delta_d$. Thus $t \in [1, r_j]$, by (27). Therefore

$$\ker\delta_d \subseteq \bigoplus_{j\,:\,r_j\geq 1} \bigoplus_{t\in[1,r_j]} \Lambda e_j \varepsilon_t. \tag{29}$$

(28) and (29) together imply that $\ker\delta_d = \bigoplus(\Lambda e_j)^{\oplus r_j}$. Consequently, (23) holds. ∎

Proposition 5.14. *The cyclic localization $A_{\mathfrak{q}}$ has precisely two simple modules up to isomorphism:*

$$\begin{aligned} V &:= A_{\mathfrak{q}}e_i/A_{\mathfrak{q}}(1-e_i)A_{\mathfrak{q}}e_i \cong S_{\mathfrak{q}}/\mathfrak{q}, \\ W &:= A_{\mathfrak{q}}e_0/A_{\mathfrak{q}}e_iA_{\mathfrak{q}}e_0 \cong \bigoplus_{j\neq i} ke_{j0} \cong (\tilde{R}/\mathfrak{q})^{\oplus n}. \end{aligned} \tag{30}$$

Their projective dimensions are, respectively,

$$\operatorname{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}} \quad \textit{and} \quad \operatorname{pd}_{A_{\mathfrak{q}}}(W) = 1.$$

Proof. Set $\Lambda := A_{\mathfrak{q}}$.

(i) We claim that the simple Λ-modules are precisely the modules V and W in (30). For each $j \in [0,n] \setminus \{i\}$, there is a (left) Λ-module isomorphism

$$\Lambda e_0 \xrightarrow{\cdot e_{0j}} \Lambda e_j.$$

Furthermore, W is simple since for each $j,k \in [0,n] \setminus \{i\}$, the matrix entry A^{jk} contains $1 \in R$; whence

$$e_{jk}e_{k0} = e_{j0} \quad \text{and} \quad e_{kj}e_{j0} = e_{k0}.$$

(ii) We claim that $\operatorname{pd}_{\Lambda}(V) = \dim S_{\mathfrak{q}}$. Indeed, we have

$$\dim S_{\mathfrak{q}} \overset{\text{(I)}}{=} \operatorname{pd}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q}) \overset{\text{(II)}}{\leq} \operatorname{pd}_{\Lambda}(V) \overset{\text{(III)}}{\leq} \dim S_{\mathfrak{q}},$$

where (I) holds since $S_{\mathfrak{q}}$ is a regular local ring, and (III) holds by Proposition 5.13. (II) holds since if $P_\bullet$ is a projective resolution of V over Λ, then $e_i\Lambda \otimes_\Lambda P_\bullet$ is a free resolution of $V \cong S_{\mathfrak{q}}/\mathfrak{q}$ over $e_i\Lambda e_i \cong S_{\mathfrak{q}}$, as shown in (24).

(iii) We claim that $\operatorname{pd}_{\Lambda}(W) = 1$. Consider the complex

$$0 \to \Lambda e_i \xrightarrow{\cdot e_{i0}} \Lambda e_0 \to W \to 0. \tag{31}$$

The module homomorphism $\Lambda e_i \xrightarrow{\cdot e_{i0}} \Lambda e_0$ maps onto the kernel of $\Lambda e_0 \to W$, namely $\Lambda e_i \Lambda e_0$, since $\Lambda^{ii} = S_{\mathfrak{q}} = \Lambda^{i0}$. Thus the complex (31) is exact. ∎

Proposition 5.15. *The residue module at $\mathfrak{q}$ decomposes as*

$$A_{\mathfrak{q}}/\mathfrak{q} = V \oplus W^{\oplus n} \tag{32}$$

and has projective dimension

$$\operatorname{pd}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}/\mathfrak{q}) = \dim S_{\mathfrak{q}}.$$

Proof. Set $\Lambda := A_{\mathfrak{q}}$.

The direct sum decomposition (32) follows from Proposition 5.14, where we view V and W as columns of the $(n+1) \times (n+1)$ tiled matrix ring $\Lambda/(\mathfrak{q} \cap \Lambda)$.

The projective dimension of $\Lambda/(\mathfrak{q} \cap \Lambda)$ equals the Krull dimension of $S_{\mathfrak{q}}$ since

$$\dim S_{\mathfrak{q}} \overset{(\text{I})}{=} \operatorname{pd}_{\Lambda}(V) \overset{(\text{II})}{\leq} \operatorname{pd}_{\Lambda}\left(\Lambda/(\mathfrak{q} \cap \Lambda)\right) \leq \operatorname{gldim} \Lambda \overset{(\text{III})}{\leq} \dim S_{\mathfrak{q}}.$$

Indeed, (I) holds by Proposition 5.14; (II) holds by (32), since the projective dimension of a module M is greater than or equal to the projective dimension of any direct summand of M; and (III) holds by Proposition 5.13. ∎

Let A be a k-algebra with prime center Z, and let $\mathfrak{m} \in \operatorname{Max} Z$. Then $A_{\mathfrak{m}} = A \otimes_Z Z_{\mathfrak{m}}$ is said to be Azumaya over its center $Z_{\mathfrak{m}}$ if $A_{\mathfrak{m}}$ is a free $Z_{\mathfrak{m}}$-module of finite rank, and the algebra homomorphism

$$\begin{aligned} A_{\mathfrak{m}} \otimes_{Z_{\mathfrak{m}}} A_{\mathfrak{m}}^{\text{op}} &\to \operatorname{End}_{Z_{\mathfrak{m}}}(A_{\mathfrak{m}}) \\ a \otimes b &\mapsto (x \mapsto axb) \end{aligned}$$

is an isomorphism [21, Section 13.7.6], [11, Definition III.1.3]. The Azumaya locus of A is the set of points $\mathfrak{m} \in \operatorname{Max} Z$ for which $A_{\mathfrak{m}}$ is an Azumaya algebra.

Remark 5.16. It is well known that if $A_{\mathfrak{m}}$ is free of finite rank over $Z_{\mathfrak{m}}$, then $A_{\mathfrak{m}}$ is Azumaya if and only if $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$ is a central simple algebra over k, if and only if $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \cong M_n(k)$ for some $n \geq 1$ (assuming k is algebraically closed).

Theorem 5.17. *Let S be a finite type normal integral domain. Let $I_1, \ldots, I_n$ be a set of proper non-maximal nonzero radical ideals of S such that either $n = 1$, or the closed sets $\mathcal{Z}(I_i) \subset \operatorname{Max} S$ are irreducible and pairwise non-intersecting.*

Set $R := \bigcap_{i=1}^{n}(k + I_i)$ and consider its nonnoetherian points $\mathfrak{m}_i := I_i \cap R \in \operatorname{Max} R$. Then

(1) *S can be retrieved from R as the cycle algebra of the endomorphism ring*

$$A = \operatorname{End}_R(R \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n).$$

Furthermore, the center of A is R.

(2) *The Azumaya locus of A and the noetherian locus $U_{S/R}$ of R coincide.*

(3) *If each $\mathcal{Z}(I_i)$ intersects the smooth locus of $\operatorname{Max} S$, then A is a noncommutative desingularization of its center R:*

   (a) $\operatorname{Frac} R$ *and* $A \otimes_R \operatorname{Frac} R$ *are Morita equivalent, and*

   (b) *for each $i \in [1, n]$ and minimal prime $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}_i$, we have*

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}/\mathfrak{q}) = \dim S_{\mathfrak{q}}.$$

Proof. (1) The algebra A has center R by Proposition 5.4, and cycle algebra S by Proposition 5.7.

(2) The noetherian locus $U_{S/R}$ of R is contained in the Azumaya locus of A by Proposition 5.11 (1) and Remark 5.16. Conversely, if $\mathfrak{n} \in \operatorname{Max} S \setminus U_{S/R}$, then $A \otimes_R R_{\mathfrak{n} \cap R}$ is not

a free $R_{\mathfrak{n}\cap R}$-module by Proposition 5.4. Whence $\mathfrak{n} \cap R \in \operatorname{Max} R$ is not in the Azumaya locus of A.

(3.a) We claim that Frac R and $A \otimes_R \operatorname{Frac} R$ are Morita equivalent. By Theorem 3.14, S is a depiction of R; in particular, $U_{S/R} \neq \emptyset$. Thus Frac $R =$ Frac S. Therefore

$$A \otimes_R \operatorname{Frac} R = A \otimes_R \operatorname{Frac} S \overset{(\mathrm{I})}{=} M_{n+1}(\operatorname{Frac} S) = M_{n+1}(\operatorname{Frac} R),$$

where (I) holds by Proposition 5.4. The claim follows.

(3.b) We have gldim $A_{\mathfrak{q}} \leq \dim S_{\mathfrak{q}}$ by Proposition 5.13, and gldim $A_{\mathfrak{q}} \geq \dim S_{\mathfrak{q}}$ by Proposition 5.14. Furthermore, $\operatorname{pd}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}/\mathfrak{q}) = \dim S_{\mathfrak{q}}$ by Proposition 5.15. ■

Remark 5.18. The advantage of the noncommutative blowup A over the depiction S is given by Theorem 5.17 (2): the noetherian locus $U_{S/R}$ of R is intrinsic to A since it is encoded in the representation theory of A. However, in the absence of R, the noetherian locus is invisible to S. Furthermore, A "sees" both R and S: they appear as the center and cycle algebra of A, respectively.

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