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Callias-type operators associated to spectral triples

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Abstract. Callias-type (or Dirac–Schrödinger) operators associated to abstract semifinite spectral triples are introduced and their indices are computed in terms of an associated index pairing derived from the spectral triple. The result is then interpreted as an index theorem for a non-commutative analogue of spectral flow. Both even and odd spectral triples are considered, and both commutative and non-commutative examples are given.

1. Outline

The Callias index theorem [2, 6, 11, 23] and its even dimensional analogue [10, 22, 25] give formulas for the index of Dirac operators on non-compact manifolds which are perturbed by self-adjoint potentials that act on sections of a finite-dimensional vector bundle and are invertible at infinity. There are many possible generalizations, for example, one can allow infinite-dimensional vector bundles as in the Robbin–Salamon theorem [42] or Hilbert-module bundles of finite type [9, 19]. This paper generalizes in a different direction, namely the underlying manifold is replaced by a semifinite spectral triple and the perturbing potentials will be elements of a certain multiplier algebra. In this abstract setting, one can still express the index of a Callias-type operator in terms of an index pairing derived from the spectral triple. Concretely, if $(\mathcal{N}, D, \mathcal{A})$ is a semifinite spectral triple with trace $\mathcal{T}$ and a Callias potential H is a self-adjoint differentiable multiplier of $\mathscr{A} = C^*(\mathcal{A})$ which is invertible modulo $\mathscr{A}$ (in the sense of Definition 3 below), then the Callias-type operator $\kappa D + \imath H$ is $\mathcal{T}$-Fredholm for small enough $\kappa > 0$ and

$$\mathcal{T}\text{-Ind}(\kappa D + \imath H) = \langle [U]_1, [D] \rangle,$$

where $U = \exp(\imath\pi(G(H) + \mathbf{1}))$ is a unitary defining a K-theory class in $K_1(\mathscr{A})$ for G a suitable switch function, and $\imath = \sqrt{-1}$. The precise statement is given in Section 3, for the case of unbounded H in Section 6. Furthermore, Section 5 states and proves an even analogue for pairings of an even spectral triple with a potential H having a further symmetry. Section 6 then also covers the unbounded even case. Section 8 presents classical and new examples.

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In the commutative setting one would take for $\mathcal{A}$ the algebra of smooth compactly supported fiberwise compact multiplication operators on sections $\mathscr{H}$ of a vector bundle over a complete Riemannian manifold X, for D a weakly elliptic first-order differential operator and $\mathcal{T} = \mathrm{Tr}_{L^2(X)} \otimes \mathrm{Tr}_{\mathscr{H}}$. Results comparable to our main result in that commutative setting with infinite-dimensional vector bundles have been obtained by Kaad and Lesch [28] and more recently also with less regularity assumptions by van den Dungen [46]. These proofs rely on previous work on unbounded Kasparov products. In essence, the strategy in [28, 46] is to prove that a Callias-type operator is an unbounded representative of the product of a K-homology class defined by a first-order differential operator and a class in the K_1-group over the continuous functions on the manifold defined by a self-adjoint multiplication operator. As discussed in Section 7 this approach can also be made to work in the present more general non-commutative setting with some technical limitations. Instead, here we provide a new and rather elementary proof using semifinite spectral flow and explicit operator homotopies. Even for the classical case (e.g., [23, 25]), the argument constitutes a considerable simplification of the proof.

In the special case where X is the real line the potential H represents a path of self-adjoint Fredholm operators and the index coincides with the spectral flow of the family. This analogy has been carried further by [28, 46] who call the index pairing $\langle [U]_1, [D] \rangle$ a spectral flow also in higher dimensions, motivated by notions from KK-theory. We follow that interpretation and therefore consider the index pairing as a non-commutative analogy of spectral flow.

2. Callias-type operators with bounded potentials

Let $\mathcal{N}$ be a von Neumann algebra with semifinite normal faithful trace $\mathcal{T}$ acting on a Hilbert space $\mathscr{H}$. For the convenience of the reader, several facts about the trace $\mathcal{T}$, the set $\mathcal{K}_{\mathcal{T}}$ of $\mathcal{T}$-compact operators and the notion of Breuer–Fredholm or $\mathcal{T}$-Fredholm operators and their index $\mathcal{T}$-Ind are recalled in Appendix A. An (odd) semifinite spectral triple $(\mathcal{N}, D, \mathcal{A})$ [14, 15] consists of an unbounded self-adjoint operator D affiliated with $\mathcal{N}$ and a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ such that

(i) each $A \in \mathcal{A}$ preserves the domain of D and the hence densely defined operator $[D, A]$ extends to a bounded element of $\mathcal{N}$;

(ii) for each $A \in \mathcal{A}$ the product $A(1 + D^2)^{-\frac{1}{2}}$ is $\mathcal{T}$-compact.

Let $\mathscr{A} = C^*(\mathcal{A})$ be the C^*-algebra generated by $\mathcal{A}$. By default a spectral triple produces an index pairing with the K-theory group $K_1(\mathscr{A})$ through [15]

$$\langle [U]_1, [D] \rangle = \mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) \in \mathbb{R}, \tag{1}$$

where $P = \chi(D > 0)$ and the $\mathcal{T}$-index on the right-hand side is of a $\mathcal{T}$-operator, for any representative $U \in \mathbf{1} + \mathcal{A}$ defining a class in $K_1(\mathscr{A})$. The potentials for our Callias-type operators will be recruited from a larger algebra:

Definition 1. Let $(\mathcal{N}, D, \mathcal{A})$ be a semifinite spectral triple.

(i) The multiplier algebra $M(\mathcal{A}, \mathcal{N})$ is the idealizer of $\mathcal{A}$ in $\mathcal{N}$, i.e., the largest C^*-subalgebra of $\mathcal{N}$ such that $M(\mathcal{A}, \mathcal{N})\mathcal{A} \subset \mathcal{A}$ and $\mathcal{A}M(\mathcal{A}, \mathcal{N}) \subset \mathcal{A}$. Elements of $M(\mathcal{A}, \mathcal{N})$ are also called $\mathcal{A}$-multipliers.

(ii) An $\mathcal{A}$-multiplier $H \in M(\mathcal{A}, \mathcal{N})$ is differentiable with respect to D if H preserves $\mathrm{Dom}(D)$ and $[D, H]$ extends to a bounded operator.

(iii) For a self-adjoint differentiable $\mathcal{A}$-multiplier H the associated Callias-type operator on the domain $\mathrm{Dom}(D)$ is defined by

$$D_{\kappa,H} = \kappa D + \imath H, \qquad \kappa > 0.$$

The parameter κ can be interpreted as the scale of the non-commutative space quanta. It plays a prominent role in the following. It will next be useful to pass to a self-adjoint supersymmetric operator $L_{\kappa,H}$ which, due to the prior works [36, 37, 43], will also be referred to as the spectral localizer.

Proposition 2. *For a self-adjoint $\mathcal{A}$-multiplier H, the adjoint of the Callias-type operator $D_{\kappa,H}$ is $(D_{\kappa,H})^* = \kappa D - \imath H$ and the spectral localizer*

$$L_{\kappa,H} = \begin{pmatrix} 0 & D^*_{\kappa,H} \\ D_{\kappa,H} & 0 \end{pmatrix}$$

is self-adjoint on the domain $\mathrm{Dom}(D)^{\times 2}$. *Moreover, $D_{\kappa,H}$ and $L_{\kappa,H}$ are affiliated to $\mathcal{N}$ and $M_2(\mathcal{N})$ respectively.*

Proof. Note that

$$L_{\kappa,H} = \kappa \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\imath H \\ \imath H & 0 \end{pmatrix}.$$

The first summand is self-adjoint, and the second is a bounded self-adjoint perturbation leaving the domain $\mathrm{Dom}(D)^{\times 2}$ invariant. Hence the self-adjointness of $L_{\kappa,H}$ immediately follows from the Kato–Rellich theorem and this also implies that $(D_{\kappa,H})^* = D_{\kappa,-H}$. As to the last claim, recall that an equivalent condition for the affiliation of an operator T is that each unitary $U \in \mathcal{N}'$ preserves the domain of T and commutes $UTU^* = T$ (see, e.g., [30]). ■

The following often deals with operators in or affiliated to the matrix algebras $M_2(\mathcal{N})$ and $M_2(\mathcal{A})$. The former is supplied with the natural trace $\mathcal{T} \circ \mathrm{Tr}$, but for notational convenience we will denote it by the same letter and speak of $\mathcal{T}$-compact and $\mathcal{T}$-Fredholm operators with no regard for the size of the matrices.

The following provides a criterion for Callias-type operators to be $\mathcal{T}$-Fredholm:

Definition 3. A self-adjoint $\mathcal{A}$-multiplier H is called asymptotically invertible with respect to $\mathcal{A}$ if there is a positive element $V \in \mathcal{A}$ such that $H^2 + V > 0$ is invertible. An asymptotically invertible and differentiable self-adjoint $\mathcal{A}$-multiplier H is called a Callias potential.

In the classical case of a Riemannian manifold X where $\mathcal{A} = C_0(X, \mathcal{K}(\mathcal{H}))$ and H is given by an operator-valued bounded function $x \in X \mapsto H_x \in \mathcal{B}(\mathcal{H})$, the asymptotic invertibility is indeed equivalent to the uniform invertibility of H_x outside a compact subset $K \subset X$, namely there is a positive constant c such that $H_x^2 \geq c\mathbf{1}$ for $x \in X \setminus K$. Proposition 5 shows that V in Definition 3 can always be chosen as a spectral function of H itself. This also implies that asymptotic invertibility of H is equivalent to the invertibility of $\pi(H)$ in the quotient algebra $M(\mathcal{A}, \mathcal{N})/\mathcal{A}$.

Proposition 4. *If H is a Callias potential, then there exists a $\kappa_0 > 0$ such that $L_{\kappa,H}$ and therefore $D_{\kappa,H}$, $D^*_{\kappa,H}$ are $\mathcal{T}$-Fredholm for all $\kappa \in (0, \kappa_0]$. Moreover, the $\mathcal{T}$-index of the Callias-type operator given by*

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = \mathcal{T}(\mathrm{Ker}(D_{\kappa,H})) - \mathcal{T}(\mathrm{Ker}(D^*_{\kappa,H})) \in \mathbb{R}$$

is independent of $\kappa \in (0, \kappa_0]$.

One may also view $\mathcal{T}$-Ind$(D_{\kappa,H})$ as the supersymmetric index of $L_{\kappa,H}$. The proof of Proposition 4 will use smooth functional calculus of a self-adjoint operator via the well-known Helffer–Sjöstrand or Dynkin formula [20]. For later use in Section 6, let us recall the details for a possibly unbounded self-adjoint operator H. For $\rho \in \mathbb{R}$, let $\mathcal{S}^\rho(\mathbb{R})$ denote the set of smooth functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|\partial^k f(x)| \leq C_k (1 + x^2)^{\frac{\rho-k}{2}}, \qquad k \in \mathbb{N}.$$

Then there exists for any $N > 0$ an almost analytic representation $\tilde{f}_N$ of f supported in a complex set G of the form $G = \{x + \imath y : |y| < 2\sqrt{1 + x^2}\}$ which coincides with f on $\mathbb{R}$ and

$$|\partial_{\overline{z}} \tilde{f}_N(z)| \leq c_N \widetilde{C}_{N+1} (1 + x^2)^{\frac{\rho-1-N}{2}} |\Im m(z)|^N \tag{2}$$

for a universal constant c_N and $\widetilde{C}_{N+1} = \sum_{k=1}^{N+1} C_k$, (see, e.g., [20, Lemma 2.2.1]). Provided $\rho < 0$, the Helffer–Sjöstrand representation

$$f(H) = \int_G (\partial_{\overline{z}} \tilde{f}_N(z))(H - z)^{-1} \, \mathrm{d}z \wedge \mathrm{d}\overline{z} \tag{3}$$

is a norm-convergent integral for any $N \geq 1$ and $\|f(H)\| \leq c_N \|f\|_N$ where

$$\|f\|_N = \sum_{n=0}^{N} \int_{\mathbb{R}} dx |f^{(n)}(x)| (1 + x^2)^{\frac{n-1}{2}}.$$

For a complex-valued function $f : \mathbb{R} \to \mathbb{C}$ this can be done for real and imaginary part separately.

Proof of Proposition 4. Let $\chi : [0,\infty) \to [0,1]$ be a smooth function with $\chi(0) = 1$ and vanishing outside $[0, \frac{g^2}{2}]$ We have to show that $\chi(L^2_{\kappa,H})$ is $\mathcal{T}$-compact for which formula (3) is used with a quasianalytic extension $\tilde{\chi}_N$ of χ. To control the resolvent of $L^2_{\kappa,H}$, let us note that

$$L^2_{\kappa,H} = \begin{pmatrix} \kappa^2 D^2 + H^2 & 0 \\ 0 & \kappa^2 D^2 + H^2 \end{pmatrix} + \imath\kappa \begin{pmatrix} [D,H] & 0 \\ 0 & -[D,H] \end{pmatrix}.$$

Now let V and $g > 0$ be such that $H^2 + V \geq g^2\mathbf{1}$. Then with $\widetilde{V} = V \oplus V$

$$L^2_{\kappa,H} + \widetilde{V} \geq \big(g^2 - \kappa\|[D,H]\|\big)\mathbf{1}_2, \tag{4}$$

which shows that $L^2_{\kappa,H} + \widetilde{V}$ is invertible for κ sufficiently small. Now replace the resolvent identity into the Helffer–Sjöstrand formula

$$\chi(L^2_{\kappa,H}) = \int_G (\partial_{\overline{z}}\tilde{\chi}_N(z))\big[(L^2_{\kappa,H} + \widetilde{V} - z)^{-1} + (L^2_{\kappa,H} + \widetilde{V} - z)^{-1}\widetilde{V}(L^2_{\kappa,H} - z)^{-1}\big]\,\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

The first summand is $\chi(L^2_{\kappa,H} + \widetilde{V})$ and thus vanishes if $\kappa \leq \kappa_0$ where $\kappa_0 = \frac{g^2}{2}\|[D,H]\|^{-1}$. For the remaining term, applying the resolvent identity again shows

$$\widetilde{V}(L^2_{\kappa,H} - z)^{-1} = \widetilde{V}(\kappa^2 D^2 \otimes \mathbf{1}_2 - z)^{-1}\big[\mathbf{1} + (H^2 \otimes \mathbf{1}_2 + \imath\kappa[D,H] \otimes \sigma_3)(L^2_{\kappa,H} - z)^{-1}\big].$$

As $\widetilde{V} \in M_2(\mathcal{A})$, the factor $\widetilde{V}(\kappa^2 D^2 \otimes \mathbf{1}_2 - z)^{-1}$ is $\mathcal{T}$-compact due to the definition of the spectral triple. As $\mathcal{K}_{\mathcal{T}}$ is a norm closed ideal and the integral in the Helffer–Sjöstrand formula is norm convergent, this implies that $\chi(L^2_{\kappa,H})$ is $\mathcal{T}$-compact.

The final claim follows from the fact that $\kappa \mapsto D_{\kappa,H}$ is continuous in the gap topology so that the $\mathcal{T}$-index is constant along this path. ■

The next result is the last technical preparation.

Proposition 5. *Let H be a self-adjoint $\mathcal{A}$-multiplier satisfying $H^2 + V > g^2\mathbf{1}$ for some $V = V^* \in \mathcal{A}$. Then for every function $f : \mathbb{R} \to \mathbb{C}$ supported by $[-\frac{g}{2}, \frac{g}{2}]$ one has $f(H) \in \mathcal{A}$.*

Proof. Let $\chi : [0,\infty) \to [0,1]$ be a smooth function satisfying $\chi(\lambda) = 1$ for $\lambda \leq \frac{g^2}{4}$ and $\chi(\lambda) = 0$ for $\lambda \geq g^2$. Then $\chi(H^2 + V) = 0$ by hypothesis so that the Helffer–Sjöstrand formula and the resolvent identity imply

$$\chi(H^2) = \int_G (\partial_{\overline{z}}\tilde{\chi}_N(z))(H^2 + V - z)^{-1}V(H^2 - z)^{-1}\,\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

As $V \in \mathcal{A}$, this implies $\chi(H^2) \in \mathcal{A}$ because $\mathcal{A}$ is a norm-closed ideal in $M(\mathcal{A},\mathcal{N})$. Now by construction, $f(H) = \chi(H^2)f(H)$ and therefore invoking the ideal property once again leads to $f(H) \in \mathcal{A}$. ■

3. Main result for bounded Callias potentials

Proposition 6. *Let H be a self-adjoint $\mathcal{A}$-multiplier which is asymptotically invertible with respect to $\mathcal{A}$ and satisfies $H^2 + V > g^2\mathbf{1}$ for some $g > 0$ and self-adjoint $V \in \mathcal{A}$. Let $G : \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing odd function taking values -1 below $-\frac{g}{2}$ and 1 above $\frac{g}{2}$. Then the unitary operator*

$$U = e^{\imath\pi(G(H)+\mathbf{1})}$$

defines a class in $K_1(\mathcal{A})$ which does not depend on the function G. It represents the image of the spectral projection $[\chi(H + \mathcal{A} < 0)]_0 \in K_0(M(\mathcal{A}, \mathcal{N})/\mathcal{A})$ under the exponential map in K-theory $\partial_0 : K_0(M(\mathcal{A}, \mathcal{N})/\mathcal{A}) \to K_1(\mathcal{A})$ associated to the short exact sequence

$$0 \to \mathcal{A} \to M(\mathcal{A}, \mathcal{N}) \to M(\mathcal{A}, \mathcal{N})/\mathcal{A} \to 0, \tag{5}$$

namely

$$[U]_1 = \partial_0[\chi(H + \mathcal{A} < 0)]_0.$$

Proof. By construction, $e^{\imath\pi(G(\lambda)+1)} - 1$ is supported in $[-\frac{g}{2}, \frac{g}{2}]$ and one therefore has $U - \mathbf{1} \in \mathcal{A}$ by Proposition 5. Naturally it defines the class $[U]_1$ in $K_1(\mathcal{A})$. The second claim results from the fact that $\frac{1}{2}(G(H) + \mathbf{1})$ is a lift of the projection $\mathbf{1} - \chi(H + \mathcal{A} < 0)$ into $M(\mathcal{A}, \mathcal{N})$. ■

Clearly one can also replace $M(\mathcal{A}, \mathcal{N})$ by any smaller C^*-algebra that contains H and has $\mathcal{A}$ as an ideal, since by naturality all computations pull back to the connecting map of $0 \to \mathcal{A} \to C^*(H, \mathcal{A}) \to C^*(H, \mathcal{A})/\mathcal{A} \to 0$ where $C^*(H, \mathcal{A}) = C^*(H) + \mathcal{A}$.

Definition 7. For an asymptotically invertible $\mathcal{A}$-multiplier H, the D-spectral flow is defined as an index pairing in the sense of (1) by

$$\mathrm{Sf}_D(H) = \langle [e^{\imath\pi(G(H)+\mathbf{1})}]_1, [D] \rangle \in \mathbb{R}$$

for any admissible function G as specified in Proposition 6 above.

Let us briefly justify why this definition applied to a particular set-up indeed reduces to the standard notion of semifinite spectral flow. Let $(\mathfrak{n}, \tau)$ be a semifinite von Neumann algebra and $\mathfrak{n}_{\mathcal{T}}$ the traceclass elements. Then a differentiable path $x \in \mathbb{R} \mapsto H_x \in \mathfrak{n}$ of self-adjoint Fredholm operators with invertible limits can be paired with a winding number 1-cocycle to give the spectral flow in the formulation of Wahl [48], see Definition 41 in the appendix where this is spelled out for a finite interval. This latter spectral flow coincides with Definition 7 if one chooses

$$(\mathcal{A}, \mathcal{N}, \mathcal{T}, D) = \Big(C_c^\infty(\mathbb{R}, \mathfrak{n}_{\mathcal{T}}), L^\infty(\mathbb{R}, \mathfrak{n}), \mathcal{T} = \int \mathrm{d}x \otimes \tau, \imath\partial_x \otimes \mathbf{1}\Big).$$

The classical case is obtained when $\mathfrak{n} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{Tr}$. More generally, for differentiable families $x \in \mathbb{R}^d \mapsto H_x \in \mathfrak{n}$ with odd d, the above definition reduces to a volume integral version of a generalized multiparameter spectral flow, see the discussion in Section 8.1. Definition 7 further conceptualizes these special cases, and is in the spirit of [28, Definition 8.9] and [46, Definition 2.18] where a K-theory valued spectral flow is introduced. The following main result shows that the index of a Callias-type operator is equal to the spectral flow in the sense of Definition 7:

Theorem 8. *Let H be a Callias potential for the semifinite spectral triple $(\mathcal{N}, D, \mathcal{A})$. Set*

$$\kappa_0 = \frac{g^2}{2\|[D,H]\|}, \qquad g^2 = \min \sigma(H^2 + \mathcal{A}). \tag{6}$$

Then for all $\kappa \in (0, \kappa_0)$

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = \mathrm{Sf}_D(H).$$

Let us comment that the equality of index and spectral flow in general does not hold for large values of κ. Indeed, a counterexample (with $X = \mathbb{R}$ and $\mathcal{A} = C_0(X, \mathcal{K}(\mathcal{H}))$ with an infinite-dimensional fiber Hilbert space $\mathcal{H}$) can be found in the work of Abbondandolo and Majer [1, Section 7]. Theorem 8 only concerns the semiclassical regime of small κ, or otherwise stated the limiting index for small κ. For unbounded H the situation may be different. Indeed, the Robbin–Salamon theorem states that for one-dimensional potentials growing at infinity, all values of κ are allowed. This case is covered by Theorem 31 below.

4. Proof of the main result

The first step is to make the Dirac operator invertible which can be achieved by a standard doubling trick. More precisely, set

$$\widetilde{D} = \begin{pmatrix} D & \mu \\ \mu & -D \end{pmatrix}, \qquad \widetilde{H} = \begin{pmatrix} H & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{7}$$

for some $\mu > 0$ and also define

$$\widetilde{D}_{\kappa,\widetilde{H}} = \kappa \widetilde{D} + \imath \widetilde{H}, \qquad \tilde{L}_{\kappa,\widetilde{H}} = \begin{pmatrix} 0 & \widetilde{D}^*_{\kappa,\widetilde{H}} \\ \widetilde{D}_{\kappa,\widetilde{H}} & 0 \end{pmatrix}.$$

Self-adjointness of $\tilde{L}_{\kappa,\widetilde{H}}$ follows again from the Kato–Rellich theorem, but the Fredholm property can depend non-trivially on κ and μ. In Lemma 11 below, we show that one may choose $\mu = \mathcal{O}(1)$ and then $\kappa \leq \mathcal{O}(\mu)$. For $\mu = 0$ the additivity of the Fredholm index gives

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = \mathcal{T}\text{-Ind}(\widetilde{D}_{\kappa,\widetilde{H}}) \tag{8}$$

and then the index stays unchanged for non-vanishing μ as long as the Fredholm-property is not violated. It is sufficient to prove the index formula for some sufficiently small κ because it then immediately holds for all κ as stated in Theorem 8. The next step is to express the index pairing as a spectral flow and to separate $\widetilde{D}$ and $\widetilde{H}$ in the 2×2 matrix.

Lemma 9. *For any $m > 0$ and κ sufficiently small, the $\mathcal{T}$-index of $\widetilde{D}_{\kappa,\widetilde{H}}$ can be computed as spectral flow along a straight-line path*

$$\mathcal{T}\text{-Ind}(\widetilde{D}_{\kappa,\widetilde{H}}) = -\,\mathrm{Sf}\left(\begin{pmatrix} \kappa\widetilde{D} & \widetilde{H} - \imath m \\ \widetilde{H} + \imath m & -\kappa\widetilde{D} \end{pmatrix}, \begin{pmatrix} \kappa\widetilde{D} & \widetilde{H} + \imath m \\ \widetilde{H} - \imath m & -\kappa\widetilde{D} \end{pmatrix}\right).$$

Proof. By conjugation with the unitary matrix $C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \imath \\ 1 & -\imath \end{pmatrix}$, one has

$$\mathrm{Sf}\left(\begin{pmatrix} \kappa\widetilde{D} & \widetilde{H} - \imath m \\ \widetilde{H} + \imath m & -\kappa\widetilde{D} \end{pmatrix}, \begin{pmatrix} \kappa\widetilde{D} & \widetilde{H} + \imath m \\ \widetilde{H} - \imath m & -\kappa\widetilde{D} \end{pmatrix}\right)$$
$$= \mathrm{Sf}\left(\begin{pmatrix} -m & \kappa\widetilde{D} + \imath\widetilde{H} \\ \kappa\widetilde{D} - \imath\widetilde{H} & m \end{pmatrix}, \begin{pmatrix} m & \kappa\widetilde{D} + \imath\widetilde{H} \\ \kappa\widetilde{D} - \imath\widetilde{H} & -m \end{pmatrix}\right)$$

and so the claim follows from Proposition 44. ■

The next step will be to deform the off-diagonal entries (more precisely the path in the lower left corner from $\widetilde{H} + \imath m$ to $\widetilde{H} - \imath m$) in the matrices on the right-hand side of Lemma 9 into a unitary, without changing the spectral flow. This will be done by functional calculus in $\widetilde{H}$ (so for every spectral value $\lambda \in \mathbb{R}$) by homotopically deforming the function

$$t \in [0, 1] \mapsto f_{t,0}(\lambda) = (1 - t)(\lambda + \imath m) + t(\lambda - \imath m) \tag{9}$$

into

$$t \in [0, 1] \mapsto f_{t,1}(\lambda) = (1 - t)\imath e^{-\imath\frac{\pi}{2}G(\lambda)} + t(-\imath)e^{\imath\frac{\pi}{2}G(\lambda)}. \tag{10}$$

The second path is constructed to contain a square root of the unitary $U = e^{\imath\pi(G(H)+\mathbf{1})}$ appearing in the image of the exponential map in Proposition 6. The main analytical difficulty that has to be addressed next is that along such a deformation the Fredholm property has to be maintained. For this purpose, let use the odd spectral localizer

$$L^o_{\kappa,f} = \begin{pmatrix} \kappa\widetilde{D} & f(\widetilde{H})^* \\ f(\widetilde{H}) & -\kappa\widetilde{D} \end{pmatrix}$$

for an arbitrary differentiable function $f : \mathbb{R} \to \mathbb{C}$. The Fredholm property of $L^o_{\kappa,f}$ can again be checked by formally squaring

$$(L^o_{\kappa,f})^2 = \begin{pmatrix} \kappa\widetilde{D}^2 + |f(\widetilde{H})|^2 & \kappa[\widetilde{D}, f(\widetilde{H})^*] \\ \kappa[f(\widetilde{H}), \widetilde{D}] & \kappa^2\widetilde{D}^2 + |f(\widetilde{H})|^2 \end{pmatrix},$$

with the modified commutator by the doubling given by

$$[f(\widetilde{H}), \widetilde{D}] = \begin{pmatrix} [f(H), D] & \mu(f(\mathbf{1}) - f(H)) \\ -\mu(f(\mathbf{1}) - f(H)) & 0 \end{pmatrix}.$$

For the control of $[f(H), D]$, let us recall:

Lemma 10. *For every smooth function f, the commutator $[D, f(H)]$ extends from* $\mathrm{Dom}(D)$ *to a bounded operator with norm bound*

$$\|[D, f(H)]\| \leq \widetilde{C}_3 \|H\| \|[D, H]\|$$

where $\widetilde{C}_3 = C \sum_{i=0}^{3} \|f^{(i)}\|_\infty$ is a constant.

Proof. Using the Helffer–Sjöstrand formula (3) for $N = 2$, one has the norm convergent integral

$$[D, f(H)] = -\int_G (\partial_{\overline{z}} f_2(z))(H - z)^{-1}[D, H](H - z)^{-1}\, \mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

and the claimed bound on the commutator follows immediately. An alternative proof can also be given using $\|[D, f(H)]\| \leq \|(\mathcal{F} f')\|_{L^1(\mathbb{R})} \|[D, H]\|$ where $\mathcal{F}$ is the Fourier transform [24, Lemma 10.15]. ∎

Using Lemma 10, one obtains

$$\begin{aligned}(L^o_{\kappa,f})^2 &\geq \kappa^2\mu^2 + |f(\widetilde{H})|^2 - \kappa\|[f(\widetilde{H}), \widetilde{D}]\| \\ &\geq \kappa^2\mu^2 + |f(\widetilde{H})|^2 - \kappa\big(\widetilde{C}_3\|H\|\|[D, H]\| + \mu\|f(H) - f(\mathbf{1})\|\big).\end{aligned} \tag{11}$$

From this, we will now need to derive a quantitative lower bound on the essential spectrum of $(L^o_{\kappa,f})^2$, namely a lower bound on $(L^o_{\kappa,f})^2 + M_2(\mathcal{K}_\mathcal{T})$. For that purpose and the remainder of the section let us now assume that $|H| > \mathbf{1}$ mod $\mathcal{A}$ which can be achieved without loss of generality by rescaling H and all other parameters (κ, μ, m, etc.).

Lemma 11. *Associated to a smooth function $f \in C(\sigma(\widetilde{H}), \mathbb{C})$, let $\widetilde{C}_3$ be as in Lemma 10 and set*

$$c_1 = \min_{|\lambda|\geq 1} |f(\lambda)|, \qquad c_2 = 2\|f\|_\infty,$$

and κ such that

$$\tfrac{1}{2}c_1^2 + \mu^2\kappa^2 - \kappa\big(\widetilde{C}_3\|H\|\|[D, H]\| + \mu c_2\big) > 0, \tag{12}$$

then $(L^o_{\kappa,f})^2$ is a self-adjoint $\mathcal{T}$-Fredholm operator with spectrum mod $M_2(\mathcal{K}_\mathcal{T})$ *bounded from below by $\frac{1}{2}c_1^2 + \mu^2\kappa^2 - \kappa(Cc_3\|H\|\|[D, H]\| + \mu c_2)$.*

Proof. Due to Lemma 10, there is a constant $C > 0$ such that

$$\|[D, f(H)]\| \leq \widetilde{C}_3\|H\|\|[D, H]\|.$$

Moreover, $|f(H)| \geq c_1$ mod $\mathcal{A}$ holds by functional calculus due to the normalization assumption $\sigma(H + \mathcal{A}) \cap (-1, 1) = \emptyset$. Adding a spectral function $V = \tilde{\chi}(H^2) \in \mathcal{A}$ for $\tilde{\chi}$

a smooth positive function supported in the interval $[0, c_1^2)$ and equal to $\frac{c_1^2}{2}$ in $[0, \frac{c_1^2}{2}]$, it follows from (11)

$$\begin{aligned}(L^o_{\kappa,f})^2 &\geq c_1^2 - \tfrac{1}{2}c_1^2 + \mu^2\kappa^2 - \kappa\|[D, \widetilde{H}]\| - \mu\kappa c_2 &&\mod M_2(\mathcal{A})\\ &\geq \tfrac{1}{2}c_1^2 + \mu^2\kappa^2 - \kappa(\widetilde{C}_3\|H\|\|[D,H]\| + \mu c_2) &&\mod M_2(\mathcal{A}).\end{aligned}$$

Since $(L^o_{\kappa,f})^2$ is a bounded perturbation of $\kappa^2\widetilde{D}^2$ it follows that $\mathcal{A}$ is relatively $\mathcal{T}$-compact with respect to $(L^o_{\kappa,f})^2$. Hence, arguing as in the proof of Proposition 4, the same lower bound also holds modulo $M_2(\mathcal{K}_{\mathcal{T}})$. ■

Lemma 12. *Let G be any switch function as in Proposition 6 with $g = 1$, namely G' supported in $(-1, 1)$. The straight-line paths in (9) and (10) are homotopic via*

$$s \in [0,1] \mapsto f_{t,s}(\lambda) = (1-s)f_{t,0}(\lambda) + sf_{t,1}(\lambda)$$

in such a way that (12) computed for $f_{s,t}$ is uniformly bounded from below by a strictly positive number for any small enough $\kappa > 0$. Moreover, the functions $s \in [0,1] \mapsto |f_{0,s}|$ and $s \in [0,1] \mapsto |f_{1,s}|$ are invertible.

Proof. By construction, the parameter t merely flips the imaginary part

$$f_{t,s} = \Re e(f_{0,s}) + \frac{\imath}{2}(1-2t)\Im m(f_{0,s}).$$

We consider the homotopy for $|\lambda| \geq 1$ where the functions simplify to

$$f_{t,s}(\lambda) = \mathrm{sgn}(\lambda)(1 + (1-s)|\lambda|) + \imath m(1-s)(1-2t)$$

and hence $c_1 \geq 1$ and $c_2 \leq 2(\|H\| + m)$ uniformly in $s, t \in [0,1]$. For μ and κ small enough, the quantity (12) is therefore obviously bounded from below. Checking pointwise invertibility for $t = 0$ and $t = 1$ is also simple: the imaginary part never changes sign and only ever vanishes when $|\lambda| \geq 1$ where one always has a non-vanishing real part. ■

Let us now fix μ, without restriction, to the value $\mu = 1$. Smallness of κ is such that (12) in Lemma 12 holds.

Corollary 13. *Let us introduce the unitary $\widetilde{W} = -\imath e^{\imath\frac{\pi}{2}G(\widetilde{H})}$. Then for κ small enough*

$$\mathcal{T}\text{-Ind}(\widetilde{D}_{\kappa,\widetilde{H}}) = -\,\mathrm{Sf}\left(\begin{pmatrix}\kappa\widetilde{D} & \widetilde{W}\\ \widetilde{W}^* & -\kappa\widetilde{D}\end{pmatrix}, \begin{pmatrix}\kappa\widetilde{D} & \widetilde{W}^*\\ \widetilde{W} & -\kappa\widetilde{D}\end{pmatrix}\right).$$

Proof. Start out with Lemma 9 and note that this straight line path there is given in (9). As the Fredholm property holds throughout the square $(t,s) \in [0,1]^2$ by Lemma 11, the homotopy invariance of the spectral flow as stated in Proposition 43 (ii) allows to deform the path (9) into (10) by respecting the invertibility of the end points, see Lemma 12. ■

Proof of Theorem 8. As already stated above, it is sufficient to prove the equality for some $\kappa > 0$ because then the constancy of the $\mathcal{T}$-index along paths of Fredholm operators allows to conclude, due to the bound (4). Now let us start out with (8) and then invoke Corollary 13,

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = -\,\mathrm{Sf}\left(\begin{pmatrix}\kappa\widetilde{D} & \widetilde{W}\\ \widetilde{W}^* & -\kappa\widetilde{D}\end{pmatrix}, \begin{pmatrix}\kappa\widetilde{D} & \widetilde{W}^*\\ \widetilde{W} & -\kappa\widetilde{D}\end{pmatrix}\right).$$

Set $\widetilde{U} = e^{\imath\pi(G(\widetilde{H})+\mathbf{1})} = -e^{\imath\pi G(\widetilde{H})}$. Then by construction, $\widetilde{U} = \widetilde{W}^2$ and therefore the adjoint action of the unitary $\mathrm{diag}(\mathbf{1}, \widetilde{W})$ transforms the spectral flow to

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = -\,\mathrm{Sf}\left(\begin{pmatrix}\kappa\widetilde{D} & 1\\ 1 & -\kappa\widetilde{W}\widetilde{D}\widetilde{W}^*\end{pmatrix}, \begin{pmatrix}\kappa\widetilde{D} & \widetilde{U}^*\\ \widetilde{U} & -\kappa\widetilde{W}\widetilde{D}\widetilde{W}^*\end{pmatrix}\right).$$

The next aim is to replace $\widetilde{W}\widetilde{D}\widetilde{W}^*$ by $\widetilde{D}$ by a homotopy

$$s \in [0,1] \mapsto (1-s)\widetilde{W}\widetilde{D}\widetilde{W}^* + s\widetilde{D}$$

leading to a homotopy of straight line paths. The difference $\widetilde{W}\widetilde{D}\widetilde{W}^* - \widetilde{D} = \widetilde{W}[\widetilde{D}, \widetilde{W}^*]$ is a bounded operator by Lemma 10, and therefore, for κ small enough, the invertibility of the end points remains valid along the homotopy as does the lower bound on the essential spectrum so that the Fredholm property is conserved throughout. Therefore

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = -\,\mathrm{Sf}\left(\begin{pmatrix}\kappa\widetilde{D} & 1\\ 1 & -\kappa\widetilde{D}\end{pmatrix}, \begin{pmatrix}\kappa\widetilde{D} & \widetilde{U}^*\\ \widetilde{U} & -\kappa\widetilde{D}\end{pmatrix}\right).$$

Now we are in the situation to apply Proposition 47 which gives

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = \mathcal{T}\text{-Ind}\big(\chi(\widetilde{D} > 0)\widetilde{U}\chi(\widetilde{D} > 0) + 1 - \chi(\widetilde{D} > 0)\big)$$

where we took into account that compared to the formulation of Proposition 47 the spectral projection is flipped, which cancels the factor of -1. The last expression is the index pairing between $\widetilde{D}$ and $\widetilde{U}$ which is equal to the pairing between the undoubled Dirac operator D and $\widetilde{U} \ominus \mathbf{1}$, see [17]. Since $\widetilde{U} \ominus \mathbf{1} = U = \exp(\imath\pi(G(H) + \mathbf{1}))$, the expression is equal to the spectral flow $\mathrm{Sf}_D(H)$. ∎

5. Even version

A spectral triple is called even if there is a proper self-adjoint unitary $\gamma \in \mathcal{N}$ that anti-commutes with D, but commutes with all elements of $\mathcal{A}$. As matrices with respect to the projections $\pi_\pm = \frac{1}{2}(\gamma \pm \mathbf{1})$ induced by the grading γ one then has the decompositions

$$D = \begin{pmatrix}0 & D_0^*\\ D_0 & 0\end{pmatrix}, \qquad \mathrm{sgn}(D) = \begin{pmatrix}0 & F^*\\ F & 0\end{pmatrix}, \qquad A = \begin{pmatrix}A_+ & 0\\ 0 & A_-\end{pmatrix},$$

for each $A \in \mathcal{A}$ and a partial isometry F. In the following, we denote $T_\pm = \pi_\pm T \pi_\pm$ for any operator where such a decomposition makes sense, and also set $\Sigma = \mathrm{sgn}(D)$.

For even spectral triples the index pairing with any unitary vanishes, but instead there is a pairing with $K_0(\mathcal{A})$ given by the skew-corner index [17, 29],

$$\langle [P]_0, [D_0] \rangle = \mathcal{T}\text{-Ind}_{P_+ \cdot P_-}(P_+ F^* P_-) \tag{13}$$

where $P \in \mathcal{A}^\sim$ is a projection representing the class $[P]_0 - [s(P)]_0 \in K_0(\mathcal{A})$ (and the formulas adapt to matrices in the obvious manner).

Any Callias-type operator in the sense of Definition 1 has a vanishing index since one has $D_{\kappa,H} = -\gamma D^*_{\kappa,H} \gamma$. To obtain an even analogue for the index theorem, let us therefore shift to non-self-adjoint potentials T, which form the off-diagonal part of a doubled potential $H \in M(M_2(\mathcal{A}), M_2(\mathcal{N}))$, or, alternatively and more in the spirit of physical systems, having an extra (so-called chiral) symmetry, the self-adjoint potential H is required to be a 2×2 matrix that is off-diagonal with respect to a natural extra grading by the third Pauli matrix $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$.

Definition 14. An $\mathcal{A}$-multiplier $T \in M(\mathcal{A}, \mathcal{N})$ is an even Callias potential if

$$H = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \in M(M_2(\mathcal{A}), M_2(\mathcal{N}))$$

is a Callias potential for the spectral triple $(M_2(\mathcal{N}), D \otimes \mathbf{1}_2, \mathcal{A} \otimes \mathbf{1}_2)$. The associated even Callias-type operator is

$$D^e_{\kappa,T} = \begin{pmatrix} T_+ & \kappa D_0^* \\ \kappa D_0 & -T_-^* \end{pmatrix},$$

acting on the domain $\mathrm{Dom}(D_0) \oplus \mathrm{Dom}(D_0^*)$.

Let us note the differentiability of H with respect to $D \otimes \mathbf{1}$ is equivalent to the differentiability of T with respect to D. The asymptotic invertibility of H (contained in the notion of Callias potential) requires that there is a self-adjoint operator $V \in M_2(\mathcal{A})$ and a $g > 0$ such that $H^2 + V \geq g^2 \mathbf{1}_2$. Moreover, the off-diagonal nature of J is equivalent to the (chiral) symmetry

$$JHJ = -H, \qquad J = \mathrm{diag}(\mathbf{1}, -\mathbf{1}).$$

Note that also $J = J_+ \oplus J_-$ and then $J_\pm H_\pm J_\pm = -H_\pm$.

Next let us show how $D^e_{\kappa,T}$ naturally arises from the Callias operator $D_{\kappa,H}$ as given in Definition 1. In fact, one readily checks

$$D_{\kappa,H} = \begin{pmatrix} 0 & \imath T_+^* & \kappa D_0^* & 0 \\ \imath T_+ & 0 & 0 & \kappa D_0^* \\ \kappa D_0 & 0 & 0 & \imath T_-^* \\ 0 & \kappa D_0 & \imath T_- & 0 \end{pmatrix} = \Pi^*_{\frac{3\pi}{2}} \begin{pmatrix} 0 & -(D^e_{\kappa,T^*}) \\ D^e_{\kappa,T} & 0 \end{pmatrix} \Pi_{\frac{3\pi}{2}}, \tag{14}$$

where

$$\Pi_\varphi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\imath\varphi} \\ 0 & e^{\imath\varphi} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{15}$$

Hence, $D_{\kappa,H}$ is block off-diagonal in an appropriate basis and one of the off-diagonal entries is indeed $D^e_{\kappa,T}$, hence motivating Definition 14. The above identity also allows to deduce several analytic properties of $D^e_{\kappa,T}$ from corresponding statements for the odd case. Proposition 2 implies that $(D^e_{\kappa,T})^* = D^e_{\kappa,T^*}$, while Proposition 4 shows that $D^e_{\kappa,T}$ is a $\mathcal{T}$-Fredholm operator. The following K-theoretic result now corresponds to Proposition 6.

Proposition 15. *Let T be an even Callias potential such that the associated $H \in M_2(M(\mathcal{A},\mathcal{N}))$ satisfies $H^2 + V > g^2\mathbf{1}_2$ for some $g > 0$ and self-adjoint $V \in M_2(\mathcal{A})$. For an odd switch function $G : \mathbb{R} \to \mathbb{R}$ as in Proposition 6, define the following self-adjoint unitary $S \in M_2(\mathcal{A}^\sim)$ and projection $P \in M_2(\mathcal{A}^\sim)$*

$$S = e^{-\imath\frac{\pi}{2}G(H)} J e^{\imath\frac{\pi}{2}G(H)}, \qquad P = \frac{1}{2}(\mathbf{1}_2 - S).$$

Then the index map $\partial_1 : K_1(M(\mathcal{A},\mathcal{N})/\mathcal{A}) \to K_0(\mathcal{A})$ associated to the exact sequence (5) gives

$$[P]_0 - [\mathrm{diag}(\mathbf{1},0)]_0 = \partial_0([T]_1).$$

Proof. This is exactly the definition of the index map. ∎

Note that $JHJ = -H$ implies

$$S = Je^{\imath\pi G(H)} = e^{-\imath\pi G(H)}J.$$

Definition 16. For an even Callias potential $H = -JHJ$ with off-diagonal entry T, the D-spectral flow is defined as a skew-corner index pairing (13) by

$$\mathrm{Sf}_D(T) = \langle [P]_0, [D_0] \rangle \in \mathbb{R},$$

where P is as in Proposition 15.

Now the main result of this section can be stated.

Theorem 17. *Let H be an even Callias potential with off-diagonal entry T and let κ_0 be as in (6). Then for all $\kappa \in (0,\kappa_0)$*

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) = \mathrm{Sf}_D(T).$$

The left-hand side can also be understood as the supersymmetric index of the odd self-adjoint operator $\kappa(D \otimes \mathbf{1}_2) + \gamma H$ (in the sense of [10]), though one may prefer the formulation in terms of T due to the homomorphism property:

Corollary 18. *If T_1, T_2 are even Callias potentials then $T_1 T_2$ is an even Callias potential with*

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T_1T_2}) = \mathcal{T}\text{-Ind}(D^e_{\kappa,T_1}) + \mathcal{T}\text{-Ind}(D^e_{\kappa,T_2})$$

for small enough κ and therefore

$$\mathrm{Sf}_D(T_1T_2) = \mathrm{Sf}_D(T_1) + \mathrm{Sf}_D(T_2).$$

Proof. There is a standard homotopy between $\begin{pmatrix} T_1T_2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ and it is not difficult to check that differentiability and asymptotic invertibility are satisfied along such a path. ∎

If the Dirac operator is not invertible, it is again necessary to regularize it by adding a mass term μ. This can in principle be done by the usual doubling procedure (7), but it is more convenient to work with a unitarily equivalent representation in which the regularized Dirac operator is again off-diagonal, namely by setting

$$\widetilde{D} = \begin{pmatrix} 0 & \widetilde{D}_0^* \\ \widetilde{D}_0 & 0 \end{pmatrix}, \qquad \widetilde{D}_0 = \begin{pmatrix} \mu & -D_0^* \\ D_0 & \mu \end{pmatrix}, \qquad \widetilde{\gamma} = \begin{pmatrix} \pi_+ \oplus \pi_- & 0 \\ 0 & -\pi_- \oplus \pi_+ \end{pmatrix},$$

and

$$\widetilde{T} = \begin{pmatrix} \mathrm{diag}(T_+, 1) & 0 \\ 0 & \mathrm{diag}(T_-, 1) \end{pmatrix}, \qquad \widetilde{H} = \begin{pmatrix} 0 & \widetilde{T}^* \\ \widetilde{T} & 0 \end{pmatrix}.$$

It is then again possible to decompose $\widetilde{H} \in M_2(\mathcal{A}^\sim)$ as $\widetilde{H} = \widetilde{H}_+ \oplus \widetilde{H}_-$ by applying $\widetilde{\pi}_\pm = \pi_\pm \otimes \mathbf{1}_2$ to each matrix entry. As before, the index of the Callias operators does not depend on μ unless the mass term is too large and breaks the Fredholm property. In order to avoid clumsy notations, let us from now on simply suppose without loss of generality that D is invertible with a lower bound $|D| \geq \mu$. This also leads to some minor simplification in Lemma 19 below compared to Lemma 11. From now on, we thus suppress all tildes on D, H, etc. Moreover, we will assume that a scaling as in Section 4 has been carried out, assuring that $H \geq 1 \mod \mathcal{A}$.

The proof of Theorem 17 starts out again by applying Proposition 44 which allows to compute the index of $D^e_{\kappa,A}$ as a spectral flow

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) = \mathrm{Sf}\left(\begin{pmatrix} -m & (D^e_{\kappa,T})^* \\ D^e_{\kappa,T} & m \end{pmatrix}, \begin{pmatrix} m & (D^e_{\kappa,T})^* \\ D^e_{\kappa,T} & -m \end{pmatrix}\right).$$

A permutation Π_0 defined via (15) mixing the spectral eigenspaces of γ and J leads to

$$\Pi_0^* \begin{pmatrix} m & (D^e_{\kappa,T})^* \\ D^e_{\kappa,T} & -m \end{pmatrix} \Pi_0 = \begin{pmatrix} m & T_+^* & \kappa D_0^* & 0 \\ T_+ & -m & 0 & \kappa D_0^* \\ \kappa D_0 & 0 & -m & -T_-^* \\ 0 & \kappa D_0 & -T_- & m \end{pmatrix}.$$

Using $J_\pm = \text{diag}(\mathbf{1}, -\mathbf{1})$ as a matrix also in the mixed eigenspaces, this can be written as

$$\Pi_0^* \begin{pmatrix} m & (D^e_{\kappa,T})^* \\ D^e_{\kappa,T} & -m \end{pmatrix} \Pi_0 = \begin{pmatrix} H_+ + mJ_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -H_- - mJ_- \end{pmatrix},$$

$$H_\pm = \begin{pmatrix} 0 & T_\pm^* \\ T_\pm & 0 \end{pmatrix}.$$

Hence by the unitary invariance of the spectral flow

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,H}) = \text{Sf}\left(\begin{pmatrix} H_+ - mJ_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -H_- + mJ_- \end{pmatrix}, \begin{pmatrix} H_+ + mJ_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -H_- - mJ_- \end{pmatrix}\right). \tag{16}$$

This turns out to be a better starting point for the homotopy arguments. More precisely, the operators $H_\pm + J_\pm m$ will be deformed within the set of operators of the form $f(H_\pm) + J_\pm g(H_\pm)$ to the operator $-J_\pm e^{\imath \frac{\pi}{2} G(H_\pm)}$, along a path that conserves the Fredholm property, for details see Lemma 20 below. For that purpose, one needs a Fredholm criterion for the homotopy of paths which is the next result, a modification of Lemma 11. For a smooth odd function $f : \mathbb{R} \to \mathbb{R}$ and a smooth function $g : \mathbb{R} \to \mathbb{C}$ satisfying $g(-\lambda) = \overline{g(\lambda)}$, both compactly supported, let us introduce the associated even spectral localizer

$$L^e_{\kappa,f,g} = \begin{pmatrix} f(H_+) + J_+ g(H_+) & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -f(H_-) - J_- g(H_-) \end{pmatrix}.$$

Due to $J_\pm H_\pm J_\pm = -H_\pm$ and the symmetry of g, one has $(J_\pm g(H_\pm))^* = J_\pm g(H_\pm)$ and therefore by the Kato–Rellich theorem also $L^e_{\kappa,f,g}$ is a self-adjoint operator with domain $\text{Dom}(D_0)^{\times 2} \oplus \text{Dom}(D_0^*)^{\times 2}$.

Lemma 19. *Let T be an even Callias potential such that $H^2 + V \geq 1$ for some $V = V^* \in M_2(\mathcal{A})$. For f and g as above, associated constants*

$$c_1^2 = \min_{|\lambda| \geq 1} \left(|f(\lambda)|^2 + |g(\lambda)|^2\right),$$

as well as $\widetilde{C}_3 = \widetilde{C}_3(f) + \widetilde{C}_3(g)$ in terms of the constants in Lemma 10, suppose that κ is such that

$$\tfrac{1}{2} c_1^2 + \mu^2 \kappa^2 - \kappa \widetilde{C}_3 \|H\| \|[D, H]\| > 0. \tag{17}$$

Then $(L^e_{\kappa,f,g})^2$ is a self-adjoint $\mathcal{T}$-Fredholm operator with spectrum $\text{mod}\mathcal{K}_\mathcal{T}$ *bounded from below by $\frac{1}{2} c_1^2 + \mu^2 \kappa^2 - \kappa \widetilde{C}_3 \|H\| \|[D, H]\|$.*

Proof. One computes

$$\left(L^e_{\kappa,f,g}\right)^2 = \begin{pmatrix} |f(H_+)|^2 + |g(H_+)|^2 + \kappa^2 D_0^* D_0 & \kappa B^* \\ \kappa B & |f(H_-)|^2 + |g(H_-)|^2 + \kappa^2 D_0 D_0^* \end{pmatrix}$$

with $B = D_0(f(H_+) + J_+ g(H_+)) - (f(H_-) + J_- g(H_-)) D_0$. Noting that

$$\begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} = \left([D, f(H)] + J[D, g(H)]\right)\gamma,$$

one deduces

$$\big(L^e_{\kappa,f,g}\big)^2 \geq c_1^2 + \kappa^2\mu^2 - \kappa\widetilde{C}_3\|H\|\|[D,H]\|,$$

and can conclude the proof by the same arguments as in the proof of Lemma 11. ■

Lemma 20. *The straight-line path*

$$t \in [0,1] \mapsto f_{t,0}(\lambda) + Jg_{t,0}(\lambda) = (1-t)(\lambda - Jm) + t(\lambda + Jm)$$

is homotopic to the straight-line path

$$t \in [0,1] \mapsto f_{t,1}(\lambda) + Jg_{t,1}(\lambda) = (1-t)(-J)e^{-\imath\frac{\pi}{2}G(\lambda)} + tJe^{\imath\frac{\pi}{2}G(\lambda)}$$

via

$$s \in [0,1] \mapsto f_{t,s}(\lambda) + Jg_{t,s}(\lambda) = (1-s)\big(f_{t,0}(\lambda) + Jg_{t,0}(\lambda)\big) + s\big(f_{t,1}(\lambda) + Jg_{t,1}(\lambda)\big)$$

in such a way that (17) *computed for* $f_{s,t}$ *and* $g_{s,t}$ *is uniformly bounded from below by a strictly positive number for any small enough* $\kappa > 0$*. Moreover, for* $t \in \{0,1\}$ *the two paths* $s \in [0,1] \mapsto |f_{t,s}(\lambda)|^2 + |g_{t,s}(\lambda)|^2$ *are uniformly bounded away from* 0.

Proof. In the statement, J is merely used as a symbol to join the two functions $f_{s,t}$ and $g_{s,t}$. One expands

$$|f_{s,t}(\lambda)|^2 + |g_{s,t}(\lambda)|^2 = \big((1-s)\lambda\big)^2 + \Big((1-2t)\big(m(1-s) + s\cos\big(\tfrac{\pi}{2}G(\lambda)\big)\big)\Big)^2 + \big(s\sin\big(\tfrac{\pi}{2}G(\lambda)\big)\big)^2,$$

which upon substituting the value of G for $|\lambda| \geq 1$ reduces to

$$\min_{|\lambda|\geq 1}\big(|f_{s,t}(\lambda)|^2 + |g_{s,t}(\lambda)|^2\big) = \min_{|\lambda|\geq 1}\big(((1-s)\lambda)^2 + ((1-2t)m(1-s))^2 + s^2\big) \geq \frac{1}{2}.$$

The constant $\widetilde{C}_3$ is obviously bounded by compactness. It remains to show that the term $|f_{s,t}(\lambda)|^2 + |g_{s,t}(\lambda)|^2$ is invertible for $t \in \{0,1\}$ and all λ. Invertibility can only fail at $\lambda = 0$ since that is the only point where the first and third summand of $|f_{s,t}(\lambda)|^2 + |g_{s,t}(\lambda)|^2$ have a common zero. But then

$$|f_{s,t}(0)|^2 + |g_{s,t}(0)|^2 = (1-2t)^2(m(1-s)+s)^2 = (m(1-s)+s)^2 \geq \min\{m^2, 1\},$$

which by continuity shows the last claim. Let us stress that it is the required invertibility that effectively fixes the signs of the coefficients of $g_{t,1}$. ■

Corollary 21. *Set* $W_\pm = e^{\imath\frac{\pi}{2}G(H_\pm)}$*. Then, for* κ *small enough,*

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,H}) = \mathrm{Sf}\left(\begin{pmatrix} -J_+W_+^* & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & J_-W_-^* \end{pmatrix}, \begin{pmatrix} J_+W_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -J_-W_- \end{pmatrix}\right).$$

Proof. Start out with (16), which, with the notations of Lemma 20, can be written out with diagonal entries $f_{0,0}(H_\pm) + J_\pm g_{0,0}(H_\pm)$ and $f_{1,0}(H_\pm) + J_\pm g_{1,0}(H_\pm)$. Now, the straight line path can be deformed due to Lemmata 19 and 20 combined with the homotopy invariance of the spectral flow under homotopies with invertible end points. Therefore, $\mathcal{T}\text{-Ind}(D^e_{\kappa,H})$ is equal to

$$\mathrm{Sf}\left(\begin{pmatrix} f_{0,s}(H_+) + J_+ g_{0,s}(H_+) & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -f_{0,s}(H_-) - J_- g_{0,s}(H_-) \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} f_{1,s}(H_+) + J_+ g_{1,s}(H_+) & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -f_{1,s}(H_-) - J_- g_{1,s}(H_-) \end{pmatrix}\right)$$

for all $s \in [0, 1]$. Use this for $s = 1$. As $f_{t,1}(\lambda) = 0$ and $g_{t,1}(\lambda) = -(1-t)e^{-\imath\frac{\pi}{2}G(\lambda)} + te^{\imath\frac{\pi}{2}G(\lambda)}$, replacing the definition of $W_\pm$, shows the claim. ■

Lemma 22. *For κ small enough,*

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,H}) = \mathrm{Sf}\left(\begin{pmatrix} 0 & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & 0 \end{pmatrix}, \begin{pmatrix} J_+ W_+^2 & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -J_- W_-^2 \end{pmatrix}\right).$$

Proof. Let us introduce the unitary

$$U = e^{\imath\frac{\pi}{4}G(H)} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \qquad U_\pm = e^{\imath\frac{\pi}{4}G(H_\pm)}.$$

Then $U_\pm^* J_\pm = J_\pm U_\pm$ again due to $J_\pm H_\pm J_\pm = -H_\pm$, and $U_\pm^2 = W_\pm$. Applying the adjoint action of U to the formula in Corollary 21 leads to

$$\begin{aligned}&\mathcal{T}\text{-Ind}(D^e_{\kappa,H}) \\ &= \mathrm{Sf}\left(\begin{pmatrix} -J_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & J_- \end{pmatrix}, \begin{pmatrix} J_+ W_+^2 & \kappa U_+^*(D_0^* \otimes \mathbf{1}_2)U_- \\ \kappa U_-^*(D_0 \otimes \mathbf{1}_2)U_+ & -J_- W_-^2 \end{pmatrix}\right).\end{aligned}$$

Now,

$$\begin{pmatrix} 0 & U_+^*(D_0^* \otimes \mathbf{1}_2)U_- \\ U_-^*(D_0 \otimes \mathbf{1}_2)U_+ & 0 \end{pmatrix} = U^*(D \otimes \mathbf{1}_2)U = D \otimes \mathbf{1}_2 + U^*[D \otimes \mathbf{1}_2, U].$$

As $[D \otimes \mathbf{1}_2, U]$ is bounded and then multiplied by κ, a homotopy as in the proof of Theorem 8 allows to remove the commutator so that

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,H}) = \mathrm{Sf}\left(\begin{pmatrix} -J_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & J_- \end{pmatrix}, \begin{pmatrix} J_+ W_+^2 & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -J_- W_-^2 \end{pmatrix}\right).$$

Finally, one can also homotopically remove the diagonal entries $\mp J_\pm$ of the left entry since these entries are required for neither the invertibility nor the Fredholm property. In fact, as $J(-1 + e^{\imath\pi G(H)}) \in M_2(\mathcal{A})$ is relatively compact to D, the Fredholm property of all involved operators can readily be checked. ■

Let us now complete the proof under the additional assumption that the spectral triple $(\mathcal{N}, D, \mathcal{A})$ is Lipschitz regular, which by definition means that $(\mathcal{N}, |D|, \mathcal{A})$ is also a spectral triple. Likewise, a self-adjoint $\mathcal{A}$-multiplier is said to be Lipschitz-differentiable if it is differentiable with respect to both D and $|D|$. Once the proof is achieved for Lipschitz regular spectral triples, it will be shown in Lemma 25 below that any spectral triple can be deformed into a Lipschitz regular one.

For an even spectral triple with grading γ consider $\pi_\pm = \frac{1}{2}(\gamma \pm \mathbf{1})$ and $\Sigma = \text{sgn}(D)$. If the triple is Lipschitz regular, then one can consider the representation $\rho : \mathcal{A} \to \mathcal{N}$ given by $\rho(a) = \pi_+ a \pi_+ + \pi_- \Sigma a \Sigma \pi_-$. Then $(\mathcal{N}, D, \rho(\mathcal{A}))$ is again an even spectral triple because

$$[D, \rho(A)] = \pi_+ \Sigma[|D|, A]\pi_+ + \pi_-[|D|, A]\Sigma\pi_-$$

is bounded if $[|D|, A]$ is bounded. The following lemma repackages a similar spectral flow argument from Section 6 of [43] that eventually connects to the index pairing:

Lemma 23. *Let $(\mathcal{N}, D, \mathcal{A})$ be a Lipschitz regular even spectral triple with invertible Dirac operator. Assume that $S = \mathbf{1} - 2P \in \mathcal{N}$ for a projection $P = P_0 + A$ where $A \in \mathcal{A}$ with $[D, A]$ and $[|D|, A]$ bounded and P_0 a projection with $[D, P_0] = 0 = [\gamma, P_0]$, i.e., P_0 is the scalar part of P. Setting $\rho(S) = \mathbf{1} - 2(P_0 + \rho(A))$, one then has*

$$\text{Sf}(S\gamma, \rho(S)\gamma) = \text{Sf}(\kappa D + S\gamma, \kappa D)$$

for κ small enough.

Proof. We use the family of approximate Dirac-operators $(D_R)_{R>0}$ of Lemma 48. For arbitrary $R > 0$ we consider the norm-continuous two-parameter family

$$(s, t) \in [0, 1] \times [0, 1] \mapsto T_{s,t} = s\kappa D_R + \big((1 - t)S + t\rho(S)\big)\gamma.$$

Since $(S\gamma)^2 = \mathbf{1} = (\rho(S))^2$ and D_R also anti-commutes with γ, one can write

$$T_{s,t}^2 = \kappa^2 |D_R|^2 + (1 - t)\kappa[D_R, S]\gamma + t\kappa[D_R, \rho(S)] + \mathbf{1} + 2t(1 - t)\{S, \rho(S) - S\}.$$

With the constant c from Lemma 48 a sufficient condition for the invertibility of the end-points of the homotopy at $t \in \{0, 1\}$ is therefore

$$\mathbf{1} - \kappa c(\|[D, S]\| + \|[D, \rho(S)]\|) > 0$$

and that is clearly the case for small enough κ. The differentiability of S implies $[\Sigma, S] \in \mathcal{K}_\mathcal{T}$ and thus $S - \rho(S) = \pi_- \Sigma[\Sigma, S]\pi_-$ is also a $\mathcal{T}$-compact, such that then $T_{s,t} = T_{s,0} \mod \mathcal{K}_\mathcal{T}$ is also Fredholm for all $s, t \in [0, 1]$.

In conclusion, we have shown

$$\text{Sf}(S\gamma, \rho(S)\gamma) = \text{Sf}(\kappa D_R + S\gamma, \kappa D_R + \rho(S)\gamma),$$

for arbitrary $R > 0$ and then, by concatenation,

$$\text{Sf}(\kappa D_R + S\gamma, \kappa D_R + \rho(S)\gamma) = \text{Sf}(\kappa D_R + S\gamma, \kappa D_R) + \text{Sf}(\kappa D_R, \kappa D_R + \rho(S)\gamma).$$

Finally, define the unitary $U = -\pi_- \Sigma \pi_+ + \pi_+ \Sigma \pi_-$ for which one checks that $U D_R U^* = -D_R$ and $U\rho(S)\gamma U^* = -\rho(S)\gamma$ and hence using invariance under unitary conjugation,

$$\begin{aligned}\mathrm{Sf}(\kappa D_R, \kappa D_R + \rho(S)\gamma) &= \mathrm{Sf}(-\kappa D_R, -(\kappa D_R + \rho(S)\gamma)) \\ &= -\,\mathrm{Sf}(\kappa D_R, \kappa D_R + \rho(S)\gamma) = 0.\end{aligned}$$

Lemma 49 concludes the proof since D_R converges to D in the gap metric.

Let us also note that formally the argument still makes sense if one directly substitutes D for D_R, except that the homotopy above could then not be Riesz- or gap-continuous in general, as that would imply that the family at $s = 0$ also has compact resolvents if D has a compact resolvent (hence the approximation argument fixes a technical error in the proof of [43, Lemma 16] where it was tacitly assumed that Riesz-continuity holds). ∎

Proof of Theorem 17. Due to Lemma 25 below, one may assume without loss of generality that T is a Lipschitz differentiable $\mathcal{A}$-multiplier and the Dirac operator D is Lipschitz regular. Then Lemma 23 can be applied to the expression in Lemma 22, by choosing $S = Je^{\imath\pi G(H)} = \mathrm{diag}(J_+W_+^2, J_-W_-^2)$ and $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus,

$$\begin{aligned}\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) &= \mathrm{Sf}\left(\begin{pmatrix} 0 & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & 0 \end{pmatrix}, \begin{pmatrix} S_+ & \kappa D_0^* \otimes \mathbf{1}_2 \\ \kappa D_0 \otimes \mathbf{1}_2 & -S_- \end{pmatrix}\right) \\ &= -\,\mathrm{Sf}\left(\begin{pmatrix} S_+ & 0 \\ 0 & -S_- \end{pmatrix}, \begin{pmatrix} S_+ & 0 \\ 0 & -FS_+F^* \end{pmatrix}\right) \\ &= \mathrm{Sf}(S_-, FS_+F^*)\end{aligned}$$

where F is the phase of D_0. Recalling Proposition 15, Lemma 24 below now implies

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) = \mathcal{T}\text{-Ind}_{P_+ \cdot P_-}(P_+F^*P_-),$$

which according to (13) and Definition 16 concludes the proof. ∎

Lemma 24. *The skew-corner index can also be computed as a spectral flow via*

$$\mathcal{T}\text{-Ind}_{P_+ \cdot P_-}(P_+F^*P_-) = \mathrm{Sf}(\mathbf{1} - 2P_-, F(\mathbf{1} - 2P_+)F^*).$$

Proof. By definition (22) of the skew-corner index,

$$\mathcal{T}\text{-Ind}_{P_+ \cdot P_-}(P_+F^*P_-) = \mathcal{T}(\mathrm{Ker}(P_+F^*P_-) \cap P_-) - \mathcal{T}(\mathrm{Ker}(P_-FP_+) \cap P_+),$$

while the spectral flow on the right-hand side can be computed from Definition 42 using only the endpoints

$$\begin{aligned}&\mathrm{Sf}(\mathbf{1} - 2P_-, F(\mathbf{1} - 2P_+)F^*) \\ &\quad = \mathcal{T}\big(P_- \cap (F(\mathbf{1} - P_+)F^*)\big) - \mathcal{T}\big((\mathbf{1} - P_-) \cap (FP_+F^*)\big),\end{aligned}$$

because $P_- - FP_+F^* \in \mathcal{K}_{\mathcal{T}}$ follows from $[P, \Sigma] \in \mathcal{K}_{\mathcal{T}}$ which holds in any spectral triple. Since F is unitary here, one can check that

$$\mathrm{Ker}(P_+F^*P_-) \cap P_- = (F(\mathbf{1} - P_+)F^*) \cap P_-$$

and

$$\mathrm{Ker}(P_-FP_+) \cap P_+ = (F^*(\mathbf{1} - P_-)F) \cap P_+ = F^*\big((\mathbf{1} - P_-) \cap (FP_+F^*)\big)F,$$

such that taking traces gives the desired equality. ∎

To complete the proof of Theorem 17, it remains to show that the Lipschitz regularity can be assumed without loss of generality. The gist of the argument, going back to a trick by Kaad [26, Proposition 5.1], is that by replacing the Dirac operator D with $D(1 + D^2)^{-\frac{r}{2}}$ for some $0 < r < 1$, one obtains an equivalent spectral triple which is Lipschitz regular. It needs to be verified that this construction is compatible with differentiability to ensure that the Callias-type operators stay Fredholm even for the regularized triple:

Lemma 25. *If H is a bounded self-adjoint differentiable multiplier and $f \in \mathcal{S}^{\rho}(\mathbb{R})$ for $\rho < 1$, then H is differentiable with respect to $f(D)$. Assuming also $(1 + f^2)^{-\frac{1}{2}} \in \mathcal{S}^{\beta}(\mathbb{R})$ for some $\beta < 0$, one has $(f(D) + \imath)^{-1}A \in \mathcal{K}_{\mathcal{T}}$ for each $A \in \mathcal{A}$ and so $(f(D), \mathcal{N}, \mathcal{A})$ is again a spectral triple.*

Proof. For the differentiability of H it is enough to verify that there is a core $\mathcal{E} \subset \mathrm{Dom}(f(D))$ for which $H\mathcal{E} \subset \mathrm{Dom}(f(D))$ and that $[f(D), H]$ extends from $\mathcal{E}$ to a bounded operator. One can use $\mathcal{E} = \mathrm{Dom}(D) \subset \mathrm{Dom}(f(D))$ which is preserved by H by assumption.

Next, let us choose a smooth switch function χ equal to 1 on $[-1, 1]$ and vanishing outside $(-2, 2)$ and regularize $f_R(\lambda) = f(\lambda)\chi(\lambda R^{-1})$. There is for any $k \in \mathbb{N}$ a constant c_k such that $|\partial^k \chi(\lambda)| \leq c_k(1 + \lambda^2)^{-\frac{k}{2}}$ and then by scaling,

$$|\partial^k f_R| \leq \sum_{m=0}^{k} C_m(1 + x^2)^{\frac{\rho-k}{2}} \frac{1}{R^{k-m}} c_{k-m} \leq \widehat{C}_k(1 + x^2)^{\frac{\rho-k}{2}}$$

with constants uniformly in $R \geq 1$. With an almost analytic continuation $\tilde{f}_{R,N}$ and $\psi \in \mathcal{E}$, one can then write

$$\begin{aligned}[f_R(D), H]\psi &= \int_G (\partial_{\overline{z}} \tilde{f}_{R,N}(z))[(D - z)^{-1}, H]\psi \,\mathrm{d}z \wedge \mathrm{d}\overline{z} \\ &= -\int_G (\partial_{\overline{z}} \tilde{f}_{R,N}(z))(D - z)^{-1}[D, H](D - z)^{-1}\psi \,\mathrm{d}z \wedge \mathrm{d}\overline{z}\end{aligned}$$

and since $f_R(D)$ and H are bounded, the latter expression also holds for all $\psi \in \mathcal{H}$. Since

$$\begin{aligned}&\|(\partial_{\overline{z}} \tilde{f}_{R,N}(z))(D - z)^{-1}[D, H](D - z)^{-1}\| \\ &\quad\leq c_N \widetilde{C}_{R,N+1}(1 + x^2)^{\frac{\rho-1-N}{2}} |\Im m(z)|^{-2+N} \|[D, H]\|\end{aligned}$$

with $\widetilde{C}_{R,N+1}$ bounded uniformly in R, the integral is also bounded uniformly in R when substituting $N \geq 2$ and hence $\sup_{R\geq 1} \|[f_R(D), H]\| < \infty$. For $\psi \in \mathcal{E}$ one also has $H\psi \in \mathcal{E}$ and the spectral representation shows

$$Hf(D) = H \lim_{R\to\infty} f_R(D)\psi = \lim_{R\to\infty} Hf_R(D)\psi$$

and

$$f(D)H = \lim_{R\to\infty} f_R(D)H\psi.$$

Hence $[f(D), H]\psi = \lim_{R\to\infty}[f_R(D), H]\psi$ for all $\psi \in \mathcal{E}$ which implies that the commutator extends to a bounded operator. Finally, $(f(D) + \imath)^{-1}A \in \mathcal{K}_{\mathcal{T}}$ for $H \in \mathcal{A}$ again follows from the functional calculus since $(f(D) + \imath)^{-1}$ can be expressed as a norm-convergent integral of terms $(D + z)^{-1}A \in \mathcal{K}_{\mathcal{T}}$. ∎

Assuming that the Dirac operator D is invertible and let $0 < \rho < 1$, this lemma implies that for $D^{(\rho)} = D(1 + D^2)^{-\frac{\rho}{2}}$ one has spectral triples $(\mathcal{N}, D^{(\rho)}, \mathcal{A})$ and $(\mathcal{N}, |D^{(\rho)}|, \mathcal{A})$ (for the latter, note that $\lambda \mapsto |\lambda|(1 + \lambda^2)^{-\frac{\rho}{2}}$ may be replaced by a smooth function as $0 \notin \sigma(D)$). Moreover, any bounded D-differentiable multiplier is $D^{(\rho)}$- and $|D^{(\rho)}|$-differentiable. Hence, one can replace the spectral triple with a Lipschitz regular one for which H is Lipschitz differentiable.

Lemma 26. *If T is a bounded Callias potential and $(D^{(\rho)})^e_{\kappa,T}$ the even Callias-type operator obtained from pairing with the Dirac operator $D^{(\rho)}$, then $\mathcal{T}$-*Ind$((D^{(\rho)})^e_{\kappa,T})$ *does not depend on $\rho \in [\frac{1}{2}, 1]$ for small enough κ.*

Proof. Due to the inequality $\|[D^{(\rho)}, T]\| \leq \|D^{-\rho}\| \|[D, T]\|$ one can choose κ so small that $(D^{(\rho)})^e_{\kappa,T}$ is $\mathcal{T}$-Fredholm for all $\rho \in [\frac{1}{2}, 1]$ and then the result follows from homotopy invariance since the path $r \in [\frac{1}{2}, 1] \mapsto D^{(r)}$ is gap-continuous and T a bounded perturbation. ∎

6. Callias-type operators with unbounded potentials

This section introduces a class of unbounded Callias potentials for which it is possible to reduce the computation of the index to the bounded case. This then allows to state and prove unbounded versions of Theorems 8 and 17.

Definition 27. An unbounded $\mathcal{A}$-multiplier T is a closed operator affiliated to $\mathcal{N}$ in such a way that the bounded transform

$$F(T) = T(1 + T^*T)^{-\frac{1}{2}}$$

is a multiplier in $M(\mathcal{A}, \mathcal{N})$ and $(1 + T^*T)^{-\frac{1}{2}}\mathcal{A}$ is a dense subset of $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{N}$ act non-degenerately on $\mathcal{H}$, then $M(\mathcal{A}, \mathcal{N})$ is the usual multiplier algebra and T is affiliated to $\mathcal{A}$ in the C^*-algebraic sense of Woronowicz [49], however, we do not ask for that since one may want to pass to proper non-dense subalgebras of $\mathcal{A}$ later. Left multiplication by an unbounded multiplier $A \mapsto TA$ gives a closed operator from a dense subset of $\mathcal{A}$ to $\mathcal{A}$. Since functional calculus factors through the bounded transform each C_0-function of a self-adjoint multiplier lies in $M(\mathcal{A}, \mathcal{N})$.

Definition 28. A self-adjoint operator H affiliated to $\mathcal{N}$ is said to be D-differentiable (with respect to the self-adjoint operator D) if there is a core $\mathcal{E}$ for D such that the following holds for each $\mu \in \mathbb{R} \setminus \{0\}$:

(i) $(H - \imath\mu)^{-1}\mathcal{E} \subset \mathrm{Dom}(D) \cap \mathrm{Dom}(H)$ and $D(H - \imath\mu)^{-1}\mathcal{E} \subset \mathrm{Dom}(H)$.

(ii) The operator $[D, H](H - \imath\mu)^{-1}$ extends from $\mathcal{E}$ to a bounded operator in $\mathcal{N}$.

If the two conditions hold for some core $\mathcal{E}$, then they also hold for $\mathcal{E} = \mathrm{Dom}(D)$ [27, Proposition 7.3]. The dense subspace $\mathcal{D}_H = (H + \imath)^{-1}\mathrm{Dom}(D)$ is dense in $\mathrm{Dom}(D) \cap \mathrm{Dom}(H)$ with respect to the graph norm $\|\psi\|_{D,H} = \|\psi\| + \|D\psi\| + \|H\psi\|$. In particular, the commutator $[D, H]$ is also densely defined and symmetric on $\mathcal{D}_H$. A bounded operator H is differentiable if and only if it preserves $\mathrm{Dom}(D)$ and $[D, H]$ extends to a bounded operator.

The above notion of differentiability is chosen precisely such that the self-adjointness criteria from [27] imply well-definedness of the following:

Proposition 29. *For a differentiable $\mathcal{A}$-multiplier H introduce the Callias-type operators*

$$D_{\kappa,H} = \kappa D + \imath H, \qquad D^*_{\kappa,H} = \kappa D - \imath H,$$

on the domain $\mathrm{Dom}(D) \cap \mathrm{Dom}(H)$ *as well as*

$$L_{\kappa,H} = \begin{pmatrix} 0 & D^*_{\kappa,H} \\ D_{\kappa,H} & 0 \end{pmatrix},$$

on the domain $(\mathrm{Dom}(D) \cap \mathrm{Dom}(H))^{\times 2}$. *Then $L_{\kappa,H}$ is self-adjoint and therefore $D_{\kappa,H}$ and $D^*_{\kappa,H}$ are adjoints to each other.*

Since D and H are affiliated to $\mathcal{N}$ one can check from the domains that the Callias-type operators and $L_{\kappa,H}$ are affiliated to $\mathcal{N}$ and $M_2(\mathcal{N})$ respectively (again, a closed operator T is affiliated if it commutes with each unitary $U \in \mathcal{N}'$).

Definition 30. An unbounded self-adjoint $\mathcal{A}$-multiplier H is asymptotically invertible if there is a positive self-adjoint element $V \in \mathcal{A}$ such that $H^2 + V$ has a bounded inverse (which then lies in $M(\mathcal{A}, \mathcal{N})$). A self-adjoint D-differentiable $\mathcal{A}$-multiplier H that is asymptotically invertible will be called an (unbounded) Callias potential.

The main result is that the index theorem as stated in Theorem 8 extends to unbounded Callias potentials. For the formulation, let us note that Proposition 6 remains valid if $[\chi(H + \mathcal{A} < 0)]_0 \in K_0(M(\mathcal{A}, \mathcal{N})/\mathcal{A})$ is replaced by $[\chi(F(H) + \mathcal{A} < 0)]_0$. In particular, the index pairing $\langle [e^{\imath\pi(G(H)+1)}]_1, [D]\rangle \in \mathbb{R}$ is well defined.

Theorem 31. *Let H be a (possibly unbounded) Callias potential for the semifinite spectral triple $(\mathcal{N}, D, \mathcal{A})$. Then there is a $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0]$,*

$$\mathcal{T}\text{-Ind}(D_{\kappa,H}) = \langle [e^{\imath\pi(G(H)+\mathbf{1})}]_1, [D] \rangle$$

and G any switch function chosen as in Proposition 6. More precisely, if there exists some $g > 0$ and positive self-adjoint $V \in \mathcal{A}$ such that $H^2 + V > g^2\mathbf{1}$, one can choose

$$\kappa_0 = \| [D, H](H + \imath)^{-1} \|^{-1} \frac{g^2}{\sqrt{1 + g^2}}.$$

Such a V exists for all $g > 0$ if and only if the resolvent of H is $\mathcal{A}$-compact, i.e., $(H + \mu\imath)^{-1} \in \mathcal{A}$, in which case all $\kappa \in (0, \infty)$ are allowed and the resolvent of L_κ is $\mathcal{T}$-compact.

Let us briefly discuss the last mentioned situation for the classical case of a Riemannian manifold X. Then $\mathcal{A} = C_0(X, \mathcal{K}(\mathcal{H}))$ and H is given by an operator-valued bounded function $x \in X \mapsto H_x \in \mathcal{B}(\mathcal{H})$. If now $x \mapsto H_x$ grows at infinity, one can indeed choose g arbitrarily large and still find V such that $H^2 + V > g^2\mathbf{1}$. This is the situation considered in the work of Robbin and Salamon [42].

Checking the Fredholm property is more difficult in the unbounded case, since there are domain issues and also the commutator $[D, H]$ is not bounded, but only relatively H-bounded:

Proposition 32. *For a Callias potential H there exists some $\kappa_0 > 0$ such that $D_{\kappa,H}$ is $\mathcal{T}$-Fredholm for all $0 < \kappa \leq \kappa_0$. In particular, κ_0 can be chosen as in Theorem 31.*

Proof. As recalled above, $\mathcal{D}_H = (H + \imath)^{-1}\text{Dom}(D)$ is a core for both $D_{\kappa,H}$ and $D^*_{\kappa,H}$ and contained in the domain of $[D, H]$. For $\psi \in \mathcal{D}_H$ let us consider the quadratic form

$$\langle (\kappa D + \imath H)\psi, (\kappa D + \imath H)\psi \rangle \geq \langle H\psi, H\psi \rangle + \kappa \langle \psi, \imath [H, D]\psi \rangle,$$

which is estimated as in the proof of [27, Lemma 7.5]

$$\pm \langle \psi, \imath\kappa [H, D]\psi \rangle \leq \Big(s + \frac{\kappa^2 b^2}{4s} \Big) \langle \psi, \psi \rangle + s \langle H\psi, H\psi \rangle$$

for any $0 < s < 1$. Assume now that $H^2 + V > g^2\mathbf{1}$ for some positive self-adjoint $V \in \mathcal{A}$. Fixing $s = \frac{g^2}{2(1+g^2)}$ and setting $b = \| [D, H](H + \imath)^{-1} \|$ one checks that

$$H^2 + V > g^2\mathbf{1} \geq \Big(\frac{s}{1 - s} + \frac{\kappa^2}{4s(1 - s)} b^2 \Big) \mathbf{1}$$

for all $0 \leq \kappa \leq \kappa_0 = \frac{g^2}{b\sqrt{1+g^2}}$ (which was obtained by maximizing the right-hand side over all $0 < s < 1$).

Substituting that particular choice of s, one has

$$\langle D_{\kappa,H}\psi, D_{\kappa,H}\psi \rangle + (1 - s)\langle \psi, V\psi \rangle \geq (1 - s)(\langle H\psi, H\psi \rangle + \langle \psi, V\psi \rangle - g^2 \langle \psi, \psi \rangle) > 0,$$

and hence the strict positivity of $H^2 + V - g\mathbf{1}$ implies that $D^*_{\kappa,H} D_{\kappa,H} + (1-s)V \otimes \mathbf{1}_2$ is invertible. A similar argument also yields $D_{\kappa,H} D^*_{\kappa,H} + (1-s)V > 0$.

Also V is $\mathcal{T}$-compact relative to $L_{\kappa,H}$ and thus $L^2_{\kappa,H}$, since

$$\widetilde{V}(L_{\kappa,H} + \imath)^{-1} = \widetilde{V}(D \otimes \sigma_1 + \imath)^{-1} + \widetilde{V}(D \otimes \sigma_1 + \imath)^{-1}(H \otimes \sigma_1)(L_{\kappa,H} + \imath)^{-1},$$

where $\widetilde{V}(D \otimes \sigma_1 + \imath)^{-1} \in M_2(\mathcal{K}_{\mathcal{T}})$ and $(H \otimes \sigma_1)(L_{\kappa,H} + \imath)^{-1}$ since $(L_{\kappa,H} + \imath)^{-1}$ is bounded as an operator from $\mathcal{H}$ to $\mathrm{Dom}(H)$. Since $\mathcal{K}_{\mathcal{T}}$ is an ideal, this completes the proof. ∎

The last statement in Theorem 31 follows immediately from the above, since one can take for V a spectral function of H^2 as in Proposition 5, respectively for each function $f \in C_c(\mathbb{R})$ one can find g so large that the proof implies $f(H) \in \mathcal{A}$ and $f(L_\kappa) \in M_2(\mathcal{K}_{\mathcal{T}})$. Hence the same holds for all C_0-functions.

The dependence of $D_{\kappa,H}$ on κ is still gap-continuous and so the index again does not depend on κ as long as it is small enough. Let us now proceed to prove that the bounded transform maps unbounded Callias potentials to bounded ones with the same index. This fact immediately concludes the proof of Theorem 31 since one can obviously write $[e^{\imath\pi(G(H)+\mathbf{1})}]_1 = [e^{\imath\pi((\widetilde{G}\circ F)(H)+\mathbf{1})}]_1$ for another switch function $\widetilde{G}$ and then apply Theorem 8. The technically most difficult part of the proof is to verify the differentiability of $F(H)$. For decaying functions the differentiability of spectral functions of H again follows from the Helffer–Sjöstrand calculus:

Lemma 33. *If H is a self-adjoint differentiable multiplier and $f \in \mathcal{S}^\rho(\mathbb{R})$ for some $\rho < 0$, then $f(H)$ preserves the domain of D and $[D, f(H)]$ extends from* Dom(D) *to a bounded operator.*

Proof. Since H is D-differentiable one has $(H - z)^{-1}\psi \in \mathrm{Dom}(D)$ for any $\psi \in \mathcal{E}$. Let us estimate

$$\begin{aligned}\|D(H-z)^{-1}\psi\| &\le \|(H-z)^{-1}D\psi\| + \|(H-z)^{-1}[H,D](H-z)^{-1}\psi\| \\ &\le |\Im m z|^{-1}\|D\psi\| + |\Im m(z)|^{-1}(1 + |\imath + z||\Im m(z)|^{-1}) \\ &\quad \cdot \|[H,D](H-\imath)^{-1}\|\|\psi\|,\end{aligned}$$

due to

$$[H,D](H-z)^{-1} = [H,D](H+\imath)^{-1} + [H,D](H+\imath)^{-1}(\imath + z)(H-z)^{-1}.$$

Using an extension $\tilde{f}$ of f satisfying (2) with $N \ge 2$ and $\rho < 0$, the integral representation (3) therefore also converges in the graph norm of D and defines a bounded operator $f(H) : \mathrm{Dom}(D) \to \mathrm{Dom}(D)$. Using

$$[D, f(H)] = -\int_G (\partial_{\overline{z}}\tilde{f}(z))(H-z)^{-1}[D,H](H-z)^{-1}\,\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

the boundedness of the commutator follows similarly. ∎

The following result is morally similar to bounds obtained in [13], but the proof presented here avoids the use of double operator integrals.

Lemma 34. *If H is an unbounded D-differentiable $\mathcal{A}$-multiplier, the bounded transform $F(H)$ is also D-differentiable with $\|[D, F(H)]\| \leq \|[D, H](H + \imath)^{-1}\|$.*

Proof. One needs to check that $F(H)$ maps a core of D into $\mathrm{Dom}(D)$ and extends from there to a bounded operator. As recalled below Definition 28, H is also differentiable with the core $\mathcal{E} = \mathrm{Dom}(D)$ and $\mathcal{D}_H = (H + \imath)^{-1}\mathrm{Dom}(D)$ is a core of D.

Applying Lemma 33 (with $\rho = -1$) to the smooth function $F(H)(H + \imath)^{-1}$, implies $F(H)\mathcal{D}_H \subset \mathrm{Dom}(D)$. It remains to show that the commutator $[D, F(H)]$ extends from $\mathcal{D}_H$ to a bounded operator. For the commutator one has from the integral representation of fractional powers (compare [12, Proposition 2.10]) an integral formula

$$[D, F(H)]\psi = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{\lambda}}\Big((1 + H^2 + \lambda)^{-1}(1 + \lambda)[D, H](1 + H^2 + \lambda)^{-1} - H(1 + H^2 + \lambda)^{-1}[D, H]H(1 + H^2 + \lambda)^{-1}\Big)\psi \,\mathrm{d}\lambda.$$

It converges absolutely in the norm of $\mathcal{H}$ for each $\psi \in \mathrm{Dom}(D)$. For the bounded self-adjoint operator $T = \imath(H - \imath)^{-1}[D, H](H + \imath)^{-1}$, one has

$$T^2 \leq \|[D, H](H + \imath)^{-1}\|^2(1 + H^2)^{-1},$$

and hence

$$T \leq |T| \leq \|[D, H](H + \imath)^{-1}\|(1 + H^2)^{-\frac{1}{2}}$$

by operator monotonicity of the square root. Therefore

$$\langle\psi, \imath[D, H]\psi\rangle = \langle(H + \imath)\psi, T(H + \imath)\psi\rangle \leq \|[D, H](H + \imath)^{-1}\|\langle\psi, (1 + H^2)^{\frac{1}{2}}\psi\rangle$$

holds for all $\psi \in \mathcal{D}_H$. On the same domain, one then has

$$\langle\psi, \imath[D, F(H)]\psi\rangle \leq \frac{\|[D, H](H + \imath)^{-1}\|}{\pi}\int_0^\infty \langle\psi, f_\lambda(H)\psi\rangle \,\mathrm{d}\lambda$$

with the positive continuous function

$$\begin{aligned} f_\lambda(H) &= \frac{1}{\sqrt{\lambda}}\big((1 + H^2 + \lambda)^{-1}(1 + \lambda)(1 + H^2)^{\frac{1}{2}}(1 + H^2 + \lambda)^{-1} \\ &\quad + H(1 + H^2 + \lambda)^{-1}(1 + H^2)^{\frac{1}{2}}H(1 + H^2 + \lambda)^{-1}\big) \\ &= \frac{\sqrt{1 + H^2}}{\sqrt{\lambda}(1 + H^2 + \lambda)}. \end{aligned}$$

Using the spectral measure μ_ψ with respect to H, the integral becomes

$$\int_0^\infty \langle\psi, f_\lambda(H)\psi\rangle \,\mathrm{d}\lambda = \int_{\mathbb{R}}\int_0^\infty \frac{\sqrt{1 + x^2}}{\sqrt{\lambda}(1 + x^2 + \lambda)}\,\mathrm{d}\lambda \,\mathrm{d}\mu_\psi(x) = \pi\int_{\mathbb{R}} \mathrm{d}\mu_\psi(x) = \pi,$$

which shows that the commutator defines a bounded quadratic form and hence has a bounded extension with $\|[D, F(H)]\| \leq \|[D, H](H + \imath)^{-1}\|$. ∎

Proposition 35. *If H is an unbounded Callias potential, then the bounded transform $F(H)$ is also a Callias potential. Furthermore, there exists a constant $\kappa_0 > 0$ such that for all $0 < \kappa \le \kappa_0$*

$$\mathcal{T}\text{-Ind}(D_{\kappa,F(H)}) = \mathcal{T}\text{-Ind}(D_{\kappa,H}).$$

Proof. By assumption there is some self-adjoint $V \in \mathcal{A}$ and $g > 0$ such that one has that $H^2 + V > g^2\mathbf{1} > 0$ holds. Let χ be a smooth non-increasing function which is equal to 1 on $[0, \frac{g^2}{4})$ and vanishes outside $[0, g^2)$. We note that the proof of Proposition 5 can be adapted for unbounded $\mathcal{A}$-multipliers since the resolvents $(H^2 + V - z)^{-1}$ and $(H^2 - z)^{-1}$ are readily checked to lie in the norm-closed algebra $M(\mathcal{A}, \mathcal{N})$. Hence one concludes $\widetilde{V} = \chi(H^2)$ lies in $\mathcal{A}$.

The spectral mapping property implies $F(H)^2 + \widetilde{V} > \tilde{g}^2$ for some positive constant $\tilde{g} > 0$ and therefore $F(H)$ is asymptotically invertible. Lemma 34 also shows that $F(H)$ is differentiable.

To show the equality of indices it is enough to prove that the joint potential given by $H \oplus (-F(H))$ has index 0, due to the additivity of the index. Consider the modified potential

$$\widehat{H}_m = \begin{pmatrix} H & m\widetilde{V} \\ m\widetilde{V} & -F(H) \end{pmatrix},$$

which we check to be a differentiable self-adjoint multiplier with respect to $D \otimes \mathbf{1}_2$. That statement is clear for $m = 0$ and that $\widehat{H}_m$ is again a multiplier follows easily from perturbation formulas for the bounded transform (such as [12, Lemma 2.7]). Let $\mathcal{E} = \mathrm{Dom}(D \otimes \mathbf{1}_2)$, then due to the domain of self-adjointness one has

$$(\widehat{H}_m - \imath)^{-1}\mathcal{E} \subset (\mathrm{Dom}(D) \cap \mathrm{Dom}(H)) \oplus \mathrm{Dom}(D) = \mathrm{Dom}(D \otimes \mathbf{1}_2) \cap \mathrm{Dom}(\widehat{H}_m),$$

and using the resolvent identity to compare with $\widehat{H}_0$, one also has

$$\begin{aligned} &D(\widehat{H}_m - \imath\mu)^{-1}\mathcal{E} \\ &\quad \subset D(\widehat{H}_0 - \imath\mu)^{-1}\mathcal{E} + D(\widehat{H}_0 - \imath\mu)^{-1} \begin{pmatrix} 0 & m\widetilde{V} \\ m\widetilde{V} & 0 \end{pmatrix} (\widehat{H}_m - \imath\mu)^{-1}\mathcal{E} \subset \mathcal{E} \end{aligned}$$

since $\widetilde{V}$ preserves $\mathrm{Dom}(D)$ and $(\widehat{H}_m - \imath\mu)^{-1}$ preserves $\mathcal{E}$. Finally, since $[D, \widetilde{V}]$ and $[D \otimes \mathbf{1}_2, \widehat{H}_0](\widehat{H}_0 - \imath\mu)^{-1}$ extend to bounded operators, another application of the resolvent identity implies that $[D \otimes \mathbf{1}_2, \widehat{H}_m](\widehat{H}_m - \imath\mu)^{-1}$ extends to a bounded operator as well.

Since $\widehat{H}_0$ is asymptotically invertible and $\widehat{H}_m - \widehat{H}_0 \in M_2(\mathcal{A})$, we have shown that $\widehat{H}_m$ is a Callias potential for any $m \ge 0$. From the above one also sees

$$\max_{m\in[0,1]} \|[D, \widehat{H}_m](\widehat{H}_m + \imath)^{-1}\| < \infty$$

such that the proof of Proposition 32 implies that there is some κ_0 such that $D_{\kappa,\widehat{H}_m}$ is $\mathcal{T}$-Fredholm for all $0 < \kappa \le \kappa_0$ and all $0 \le m \le 1$.

For any $m \geq 0$ one can check that $\widehat{H}_m$ is invertible from the square

$$(\widehat{H}_m)^2 = \begin{pmatrix} H^2 + m^2\widetilde{V}^2 & m(H - F(H))\widetilde{V} \\ m(H - F(H))\widetilde{V} & F(H)^2 + m^2\widetilde{V}^2 \end{pmatrix},$$

which can be diagonalized in the spectral representation. The off-diagonal part $m\widetilde{V}$ does not affect the index since it is relatively $\mathcal{T}$-compact with respect to D. Fixing any $m > 0$, one has $\mathcal{T}\text{-Ind}(D_{\kappa,\widehat{H}_m}) = 0$ for small enough κ since $D^*_{\kappa,\widehat{H}_m} D_{\kappa,\widehat{H}_m}$ and similarly $D_{\kappa,\widehat{H}_m} D^*_{\kappa,\widehat{H}_m}$ become invertible. More precisely, this follows from the proof of Proposition 32, since $\widehat{H}_m^2 > c_m^2 \mathbf{1}$ allows one to choose $V = 0$ there. Since the index does not depend on m and κ,

$$0 = \mathcal{T}\text{-Ind}(D_{\kappa,\widehat{H}_m}) = \mathcal{T}\text{-Ind}(D_{\kappa,H\oplus(-F(H))}) = \mathcal{T}\text{-Ind}(D_{\kappa,H}) - \mathcal{T}\text{-Ind}(D_{\kappa,F(H)}),$$

concluding the proof. ∎

Finally, let us turn to the even unbounded case.

Definition 36. An unbounded $\mathcal{A}$-multiplier T is an even Callias potential if $H = \left(\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right)$ is an unbounded Callias potential for the spectral triple $(M_2(\mathcal{N}), D \otimes \mathbf{1}_2, \mathcal{A} \otimes \mathbf{1}_2)$. The associated even Callias-type operator is defined as

$$D^e_{\kappa,T} = \begin{pmatrix} T_+ & \kappa D_0^* \\ \kappa D_0 & -T_-^* \end{pmatrix}$$

on the domain $(\mathrm{Dom}(D_0) \cap \mathrm{Dom}(T_+)) \oplus (\mathrm{Dom}(D_0^*) \cap \mathrm{Dom}(T_-^*))$.

The unitary equivalence (14) and the self-adjointness of $L_{\kappa,H}$ again implies that $D^e_{\kappa,T}$ is a closed affiliated operator on the stated domain and $(D^e_{\kappa,T})^* = D^e_{\kappa,T^*}$. Now, the generalization of Theorem 17 to unbounded potentials reads as follows:

Theorem 37. *Let T be an even Callias potential with $H = \left(\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right)$. Then*

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) = \left\langle [\tfrac{1}{2}(\mathbf{1} - Je^{\imath\pi G(H)}), [D_0]]_0 \right\rangle$$

for each κ and G as in Theorem 31.

The proof is immediate from the bounded version and the next result.

Proposition 38. *If T is an unbounded even Callias potential, there exists κ_0 such that $D^e_{\kappa,T}$ is $\mathcal{T}$-Fredholm for all $0 < \kappa \leq \kappa_0$ and $\mathcal{T}\text{-Ind}(D^e_{\kappa,T})$ does not depend on $0 < \kappa \leq \kappa_0$. Also, the bounded transform $F(T) = T(1 + T^*T)^{-\frac{1}{2}}$ is a Callias-admissible potential with the same index*

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) = \mathcal{T}\text{-Ind}(D^e_{\kappa,F(T)}),$$

for small enough κ.

Proof. Due to the block decomposition (14), $D^e_{\kappa,T}$ is $\mathcal{T}$-Fredholm if $L_{\kappa,H}$ is $\mathcal{T}$-Fredholm. Hence the existence of κ_0 follows from the odd case (Proposition 32).

That $F(T)$ is again a Callias potential is also clear from considerations of the odd case and the fact that we defined differentiability using a doubling construction. The doubled Hamiltonian $\widehat{H}_m$ from the proof of Proposition 35 is unitarily equivalent to an off-diagonal matrix

$$\widehat{H}_m \sim \begin{pmatrix} 0 & \widehat{T}_m^* \\ \widehat{T}_m & 0 \end{pmatrix},$$

with the operator

$$\widehat{T}_m = \begin{pmatrix} T & -m\chi(T^*T) \\ m\chi(TT^*) & F(T^*) \end{pmatrix},$$

which is therefore an even Callias potential and invertible for any $m > 0$. One argues as in the odd case that

$$0 = \mathcal{T}\text{-Ind}(D^e_{\kappa,\widehat{T}_m}) = \mathcal{T}\text{-Ind}(D^e_{\kappa,T\oplus(-F(T^*))}) = \mathcal{T}\text{-Ind}(D^e_{\kappa,T}) + \mathcal{T}\text{-Ind}(D^e_{\kappa,-F(T^*)})$$

for small enough κ.

Replacing a Callias potential T by λT with $|\lambda| = 1$, again gives a Callias potential and since $\mathbb{S}^1$ is connected, $\mathcal{T}\text{-Ind}(D_{\kappa,T}) = \mathcal{T}\text{-Ind}(D_{\kappa,-T})$ by homotopy. Conjugating the potential gives a factor of -1 and thus

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) = (-1)\mathcal{T}\text{-Ind}(D^e_{\kappa,-T^*}) = \mathcal{T}\text{-Ind}(D^e_{\kappa,F(T)}),$$

concluding the proof. ∎

7. Comparison with the unbounded Kasparov product

This section outlines how the Callias index arises as an unbounded representative of a KK-group. For simplicity, it will be assumed that the Dirac operator of the semifinite spectral triple $(\mathcal{A}, \mathcal{N}, D)$ is invertible (see Section 4 on how to achieve this).

Definition 39. Let $\mathscr{A}$ and $\mathscr{B}$ be separable C^*-algebras. An unbounded Kasparov cycle $(\mathcal{A}, E, D)$ is a tuple consisting of a countably generated $\mathscr{A}$-$\mathscr{B}$-Hilbert-C^*-module E together with an odd regular self-adjoint unbounded operator $D : \mathrm{Dom}(D) \subset E \to E$ such that

- (i) $\mathcal{A} \subset \mathscr{A}$ is a dense $*$-subalgebra such that each $a \in \mathcal{A}$ preserves $\mathrm{Dom}(D)$ and the graded commutator $[D, a]$ extends to a bounded operator on E.
- (ii) The products $a(D - \imath)^{-1}$ are $\mathscr{B}$-compact for all $a \in \mathscr{A}$.

The remainder of the section considers a semifinite spectral triple $(\mathcal{A}, \mathcal{N}, D)$ with separable C^*-algebra $\mathscr{A} = \overline{\mathcal{A}}$. Such a spectral triple naturally defines an unbounded or bounded Kasparov cycle if and only if $\mathcal{K}_{\mathcal{T}}$ is σ-unital. Since that condition does not generally hold in the semifinite setting one passes [15, 29] to the norm-closed subalgebra $\mathcal{C} \subset \mathcal{K}_{\mathcal{T}}$ generated by all $*$-algebraic combinations of elements

$$A[F(D), B], \qquad [F(D), B], \qquad F(D)A[F(D), B], \qquad \varphi(D)A \tag{18}$$

with $A, B \in \mathscr{A}$, $F(D) = D(1 + D^2)^{-\frac{1}{2}}$ and $\varphi \in C_0(\mathbb{R})$. Then $(\mathscr{A}, \mathcal{C}, F(D)) \in KK(\mathscr{A}, \mathcal{C})$ defines a bounded ungraded Kasparov cycle and $\mathcal{C}$ is the smallest C^*-algebra for which this is the case. An important technical point is that the unbounded Kasparov cycle is constructed precisely from D and $\mathcal{A}$ since only then the Callias operators can be interpreted as unbounded Kasparov products. In most cases, $(\mathcal{A}, \mathcal{C}, F(D))$ is already the bounded transform of a well-defined Kasparov cycle:

Proposition 40. *Set $\alpha_t(A) = e^{\imath Dt} A e^{-\imath Dt}$ and let $\mathscr{A}_\alpha \subset \mathcal{N}$ be the smallest α-invariant C^*-algebra containing $\mathscr{A}$. If $\mathscr{A}_\alpha$ acts non-degenerately on $\mathcal{H}$, then $(\mathcal{A} \otimes \mathbb{C}_1, \mathcal{C} \otimes \mathbb{C}_2, D\sigma_2)$ is an unbounded Kasparov cycle.*

Proof. The only non-trivial point in Definition 39 is the regular self-adjointness of D on $\mathcal{C}$ or, equivalently, that D is affiliated to $\mathcal{C}$. To prove the latter we derive a better characterization of the algebra $\mathcal{C}$.

The affiliation of D to $\mathcal{N}$ implies that α defines a weak-$*$-continuous action on $\mathcal{N}$. Since any $A \in \mathcal{A}$ is differentiable, the identity $\alpha_t(A) - A = \int_0^t \imath\alpha_s([D, A])\, \mathrm{d}s$ (with convergence in the weak-$*$-topology) implies that the orbit under α is norm-continuous, hence $\mathscr{A}_\alpha$ is still separable and α extends to a strongly continuous $\mathbb{R}$-action on $\mathscr{A}_\alpha$. Now define $\mathcal{C}_\alpha$ to be the separable C^*-subalgebra of $\mathcal{K}_{\mathcal{T}}$ spanned by the elements (18), but with $\mathscr{A}$ replaced by $\mathscr{A}_\alpha$.

One can next form the crossed product algebra $\mathscr{A}_\alpha \rtimes_\alpha \mathbb{R}$ for which we now recall some universal properties [41]: There are embeddings $\imath_{\mathscr{A}_\alpha} : \mathscr{A}_\alpha \to M(\mathscr{A}_\alpha \rtimes_\alpha \mathbb{R})$ and $\imath_{\mathbb{R}} : C_c(\mathbb{R}) \to M(\mathscr{A}_\alpha \rtimes_\alpha \mathbb{R})$ such that $\mathscr{A}_\alpha \rtimes_\alpha \mathbb{R}$ is generated by the linear span of $\imath_{\mathscr{A}_\alpha}(\mathscr{A}_\alpha)\imath_{\mathbb{R}}(C_0(\mathbb{R}))$. Furthermore, given a non-degenerate covariant representation (π, U) on a Hilbert space $\mathcal{H}$, there is a non-degenerate representation $(\pi \times U) : \mathscr{A}_\alpha \rtimes_\alpha \mathbb{R} \to \mathcal{B}(\mathcal{H})$ for which $(\pi \times U) \circ \imath_{\mathscr{A}_\alpha} = \pi$ and $((\pi \times U) \circ \imath_{\mathbb{R}})(\varphi) = \varphi(D)$ for D the self-adjoint generator of U. Also $\imath_{\mathbb{R}}$ extends uniquely to $C_b(\mathbb{R})$ with the same property.

The identical map $\pi : \mathscr{A}_\alpha \to \mathcal{B}(\mathcal{H})$ and $U : \mathbb{R} \to \exp(\imath D\cdot)$ trivially form a covariant representation (π, U). By definition one has $(\pi \times U)(\mathscr{A}_\alpha \rtimes_\alpha \mathbb{R}) \subset \mathcal{C}_\alpha$ since the latter contains the generators $\pi(a)\varphi(D)$. Moreover, one has equality $(\pi \times U)(\mathscr{A}_\alpha \rtimes_\alpha \mathbb{R}) = \mathcal{C}_\alpha$, since $[\imath_{\mathbb{R}}(F), \imath_{\mathscr{A}_\alpha}(a)] \in \mathscr{A}_\alpha \rtimes_\alpha \mathbb{R}$ holds for any smooth switch function like F (see, e.g., [33]).

From the above one concludes that a dense subset of $\mathcal{C}_\alpha$ is given by all elements of the form

$$\sum_{k=1}^{N} \varphi_k(D)\alpha_{t_k}(A_k) = \sum_{k=1}^{N} \varphi_k(D) e^{\imath Dt_k} A_k e^{-\imath Dt_k}$$

with $\varphi_1, \dots, \varphi_N \in C_c(\mathbb{R})$ and $A_1, \dots, A_N \in \mathscr{A}$. Therefore D is densely defined on $\mathcal{C}_\alpha$ and clearly $(D + \imath)^{-1}\mathcal{C}_\alpha \subset \mathcal{C}_\alpha$ is also a norm-dense subset. Hence, D is affiliated to $\mathcal{C}_\alpha$ and regular self-adjoint in the Hilbert-module sense.

To complete the proof let us show that $\mathcal{C}_\alpha = \mathcal{C}$, for which it is only necessary to verify $\varphi(D)Ae^{-\imath Dt} \in \mathcal{C}$ for all $\varphi \in C_c(\mathbb{R})$, $t \in \mathbb{R}$, $A \in \mathcal{A}$. Choose approximate units $(B_n)_{n\in\mathbb{N}}$ for $\mathcal{A}$ and $(\Psi_m)_{m\in\mathbb{N}}$ for $C_0(\mathbb{R})$. Then

$$\varphi(D)Ae^{-\imath Dt} = \lim_{n\to\infty} \varphi(D)AB_n e^{-\imath Dt} = \lim_{n\to\infty}\lim_{m\to\infty} \varphi(D)AB_n e^{-\imath Dt}\Psi_m(D),$$

converges in norm, since the B_n and $\Psi_m(D)$ are approximate units for $(\pi \times U)(\mathcal{A}_\alpha \rtimes_\alpha \mathbb{R})$ as well. That shows that $\mathcal{C}$ is norm-dense in $\mathcal{C}_\alpha$. ■

Let us now compare our main result for unbounded Callias operators to approaches using the unbounded Kasparov product (specifically [28, 46] which treat the classical case). Semifinite spectral triples often arise in non-commutative geometry from an unbounded Kasparov cycle $(\mathcal{A} \otimes \mathbb{C}_1, E_{\mathcal{B}} \otimes \mathbb{C}_2, D\sigma_2) \in KK^{-1}(\mathcal{A}, \mathcal{B})$ where $\mathcal{B}$ is a separable C^*-algebra that carries a densely defined faithful lower semicontinuous trace $\mathcal{T}$, $E_{\mathcal{B}}$ a countably generated right $\mathcal{B}$-module and D a regular self-adjoint unbounded operator on $E_{\mathcal{B}}$. Both $\mathcal{A}$ and $\mathcal{B}$ act naturally on the Hilbert space $\mathcal{H}$ obtained by completing the submodule of $E_{\mathcal{B}}$ for which $\mathcal{T}_{\mathcal{B}}(\langle e, e\rangle_{E_{\mathcal{B}}}) < \infty$ in the obvious norm. In that situation, one obtains a semifinite spectral triple $(\mathcal{N}, D, \mathcal{A})$ with $\mathcal{N} = \mathcal{B}''$ to which $\mathcal{T}_{\mathcal{B}}$ extends as a normal semifinite faithful trace $\mathcal{T}$. All examples described in Section 8 below can be written in that form for some natural algebra $\mathcal{B}$. If $\mathcal{A}$ acts non-degenerately, it is completely general since, as shown above, for any semifinite spectral triple the minimal choice $\mathcal{B} = \mathcal{C} = E_{\mathcal{B}}$ is available.

If the potential H is a self-adjoint unbounded $\mathcal{A}$-multiplier with resolvent in $\mathcal{A}$, then it defines an odd unbounded Kasparov cycle $(\mathbb{C}, \mathcal{A} \otimes \mathbb{C}_1, H\sigma_2) \in KK^1(\mathbb{C}, \mathcal{A})$. The Callias operator $H\sigma_1 + D\sigma_2$ on $E_{\mathcal{B}}$ should then represent the product class

$$[H\sigma_2] \otimes_{\mathcal{A}\otimes\mathbb{C}_1} [D\sigma_1] \in KK(\mathbb{C}, \mathcal{B}) \simeq K_0(\mathcal{B}),$$

given some compatibility conditions between H and D which are similar to the differentiability that we impose here (a possible set of conditions may be derived from, e.g., [35, Theorem 7.4]). Composing the product class with the homomorphism $\mathcal{T}_* : K_0(\mathcal{B}) \to \mathbb{C}$ computes the $\mathcal{T}$-index, while applying $\mathcal{T}_*$ to the product of the bounded transforms $[F(H\sigma_1)] \otimes_{\mathcal{A}\otimes\mathbb{C}_1} [F(D\sigma_2)]$ recovers the index pairing $\langle [e^{\imath\pi F(H)}]_1, [D]\rangle$ (see [15]). Since the bounded and unbounded picture of KK-theory are isomorphic [26,47], one concludes that Theorem 31 holds in that special case. For the case of a spectral triple over $\mathcal{B}(\mathcal{H})$, a more detailed proof can also be found in [7], though compared to our notations the regularity assumptions are formulated in terms of the Cayley transform of H, which is another unitary representing the class $[e^{\imath\pi F(H)}]_1$. The even case can be handled similarly with the product represented by the self-adjoint operator $D + \gamma H$ (see [10]).

The KK-theoretic approach has certain advantages, in particular, the class of the Callias operator may carry finer topological invariants besides the numerical index and also the associativity of the Kasparov product can then be further applied to prove more specialized formulas for the index. A severe limitation is that apparently the potential H

must always be unbounded with $\mathcal{A}$-compact resolvent, i.e., $(H+\imath)^{-1} \in \mathcal{A}$ which is a stronger condition than our asymptotic invertibility. That is necessary for the obvious cycle to define a class $KK^1(\mathbb{C}, \mathcal{A})$ in unbounded KK-theory, though there might be more complicated constructions to handle a bounded potential that is invertible up to $\mathcal{A}$. In the classical commutative case of potentials on a manifold, one can already treat the larger class of asymptotically invertible potentials with pointwise compact resolvents by amplifying them with a growing function on the underlying manifold (as it is done in [28, 46]).

8. Examples

8.1. The classical Callias index theorem

The first example, which was already discussed briefly above is the classical geometric situation with Callias-type operators on a Riemannian manifold X. We will consider a possibly infinite-dimensional vector bundle E over X with typical fiber isomorphic to a Hilbert space $\mathcal{H}_0$. For D a weakly elliptic first-order differential operator we have a spectral triple $(\mathcal{B}(L^2(X, \mathcal{H}_0)), D, C_c^\infty(X) \otimes \mathbb{K}(\mathcal{H}_0))$. If $H = (H_x)_{x \in X}$ is a self-adjoint fibered operator that is differentiable (with derivative $(\nabla H_x)_{x \in X}$ bounded relative to H) and invertible modulo $C_0(X) \otimes \mathbb{K}(\mathcal{H}_0)$, then the technical conditions of our index theorem can all be verified and it reproduces the result of Kaad and Lesch [28] which used the Kasparov product. If $\mathcal{H}_0$ is finite-dimensional and H invertible outside a compact set K, there are other ways to compute the index, for example, the original index theorem by Callias [11] on $\mathbb{R}^n$ expresses the index as an integral over the boundary of a large enough sphere and more generally the index theorem by Anghel [2] gives a similar generalization to manifolds with warped ends. In both of these situations, one only needs to know the potential H on a lower-dimensional submanifold that envelops all singular points of the potential. Explicitly, in the case $X = \mathbb{R}^n$, n odd, with D the Euclidean Dirac operator one has

$$\mathrm{Ind}(\kappa D + \imath H) \sim \int_{\partial B_R(0)} \mathrm{Tr}((Q\, \mathrm{d}Q)^{\wedge(n-1)}) \tag{19}$$

for $Q = H|H|^{-1}$ invertible outside $B_R(0)$. In contrast, our index formula with standard index computations would give a volume integral

$$\mathrm{Ind}(\kappa D + \imath H) \sim \int_{\mathbb{R}^n} \mathrm{Tr}((U^*\, \mathrm{d}U)^{\wedge n}), \tag{20}$$

with $U = e^{\imath\pi(G(H)+1)}$ or some other representative with sufficiently fast decaying derivatives.

There is a simple K-theoretical relation between those formulations. Let us assume that H is not only invertible up to $\mathcal{A} = C_0(X)$, but already up to $\mathcal{A}' = C_0(K)$ for $K \subset X$ a compact set. Since $\mathcal{A}'$ is an ideal in $\mathcal{A}$, one then also has $H \in M(\mathcal{A}', \mathcal{N})$ and so the exponential map in K-theory gives an element of $K_1(C_0(K))$. One can do even better since there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(K) & \longrightarrow & C_b(X) & \overset{q}{\longrightarrow} & C_b(X)/C_0(K) & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \rho & & \downarrow r & & \\ 0 & \longrightarrow & C_0(K^\circ) & \longrightarrow & C(K) & \overset{q}{\longrightarrow} & C(\partial K) & \longrightarrow & 0 \end{array}$$

where r is the restriction and ρ acts as the identity on $C_0(K)$. Hence the naturality of the exponential map implies

$$\partial_0[\chi(H + \mathcal{A} < 0)]_0 = \partial_0[\chi(r(H) < 0)]_0.$$

This shows that, as expected, the spectral flow $\mathrm{Sf}_D(H)$ naturally depends only on the class that $r(H) = H|_{\partial K}$ defines in $K_0(C(\partial K))$. For a more detailed examination of those K-theoretical aspects of Callias-type operators we refer to the work of Bunke [10]. The equality between the expressions (20) and (19) can be derived from the boundary maps of K-homology [3], which are dual to the K-theoretic connecting maps, however, in a more general setting constructing an explicit representative for the odd K-homology class on $C(\partial K)$ that computes the same pairing as the spectral flow can involve subtle geometric and analytic issues.

In that sense, our main index theorem on Callias-type operators contains only part of the information of the original theorem by Callias. It is an interesting problem to find additional analytic and algebraic data from which one can canonically construct a spectral triple for the boundary class also in the non-commutative case. Partial solutions may be provided by the construction of relative spectral triples [21].

Another natural question is whether it is possible to reduce the index computation to a compact hypersurface also in the case of an infinite-dimensional vector bundle. In general, the answer is negative, though, since due to Kuipers' theorem any potential H that is invertible on ∂K may be homotopic to another potential $\widetilde{H}$ that has the same index but is flat in the sense that $\widetilde{H}|_{\partial K} = \mathbf{1} - 2P_0$ with $P_0 \in \mathcal{B}(\mathcal{H}_0)$ some fixed projection. Hence the non-trivial index is invisible on ∂K. One way to evade such counterexamples is to consider a more restricted class of Callias-type operators, for example, those of the form $H = H_0 + V$ with H_0 a fixed self-adjoint operator with compact resolvent and $V : X \to \mathcal{B}(\mathcal{H}_0)$ a norm-continuous family of self-adjoint operators. In that case, again one can compute the index from a K_1-class over $C^*(H_0) + C_0(K) \otimes \mathbb{K}$ obtained through the exponential map from a K_0-class over $C^*(H_0)|_{\partial K} + C(\partial K) \otimes \mathbb{K}$. It must therefore be possible to compute the index using only the potential on the boundary, though we are not aware of any known formulas.

Finally, let us note an interpretation as spectral flow under the additional assumption H_x is invertible for all but finitely many isolated points $x_0, \ldots, x_N \in X$. In that case, (19) decomposes into a sum of contributions of small spheres around each x_i, each of which is individually integer-valued. This is analogous to the way that usual spectral flow counts the number of eigenvalue crossings. In physics, one uses such expressions to assign topological charges to stable band-touching points and the spectral flow is therefore the total charge. A particular set-up of this type is considered in [32] where the potential H is a linear combination of Clifford algebra generators with essentially commuting coefficients. If the critical points of H are not isolated or $\mathcal{H}_0$ is infinite-dimensional the spectral flow is more difficult to interpret. For even dimensions our analogue of Section 5 can be used to define topological charges for potentials satisfying a chiral symmetry of the type $JHJ = -H$.

8.2. The Boutet de Monvel index theorem

The framework of Section 8.1 can be transposed to an even $2n$-dimensional complete Riemannian manifold X. Let then E be a possibly infinite-dimensional vector bundle over X with typical fiber Hilbert space $\mathcal{H}_0$. Furthermore, there is supposed to exist an involution $\gamma = (\gamma_x)_{x \in X} : E \to E$ and a weakly elliptic first-order differential operator D satisfying $\gamma D \gamma = -D$ and which results in an even spectral triple

$$\big(\mathcal{B}(L^2(X, \mathcal{H}_0)), D, C_c^\infty(X) \otimes \mathbb{K}(\mathcal{H}_0)\big).$$

The even Callias potential is then given by a self-adjoint multiplication operator $H = (H_x)_{x \in X}$ on $E \oplus E$ satisfying $\gamma H \gamma = H$ and $JHJ = -H$ with $J = \operatorname{diag}(\mathbf{1}_\mathcal{E}, -\mathbf{1}_\mathcal{E})$ and being invertible modulo $C_0(X) \otimes \mathbb{K}(\mathcal{H}_0)$. In the grading of γ one then has $H = H_+ \oplus H_-$ and both $H_\pm$ are off-diagonal in the grading of J with lower left entry $T_\pm$.

The set-up of the index theorem of Boutet de Monvel [8, 25] assumes that X is a strongly pseudoconvex domain in $\mathbb{C}^n$ equipped with the Bergmann metric, E is finite-dimensional and given by the differential forms on X of type (n, p) graded by the parity of p and then D is the Dolbeaut operator. Furthermore, $H_+ = H_-$ is supposed to extend smoothly to the boundary ∂X. A related setting is obtained by simply choosing $X = \mathbb{R}^{2n}$ with the Euclidean metric and D the associated Dirac operator on $E = \mathbb{R}^{2n} \otimes \mathbb{C}^N$ where $\mathbb{C}^N$ is the representation space of the Clifford algebra with N generators. Then the index theorem states [8, 10, 25]

$$\mathcal{T}\text{-Ind}(D^e_{\kappa,T}) \sim \int_{\partial X} \operatorname{Tr}(J(Q\,\mathrm{d}Q)^{\wedge(2n-1)}),$$

where again $Q = H|H|^{-1}$ and in the case $X = \mathbb{R}^{2n}$ one replaces ∂X by $\partial B_R(0)$ with R sufficiently large. The right-hand side is an odd Chern number in the representation given, e.g., [18]. Theorems 17 and 37 connect $\mathcal{T}\text{-Ind}(D^e_{\kappa,T})$ to an integral over X rather than its boundary, and hence do not directly provide the right side. However, in the case of finite-dimensional fibers one can again argue as in Section 8.1. To determine conditions under which such a formula holds also with infinite-dimensional fibers remains an open problem.

8.3. A generalized Robbin–Salamon theorem

As a first non-commutative example we consider a generalized Robbin–Salamon theorem. Let $\mathcal{C}$ be a separable C^*-algebra with a densely defined faithful lower semicontinuous trace τ and a strongly continuous automorphic $\mathbb{R}$-action α that leaves τ invariant (everything below directly transposes to $\mathbb{R}^n$-actions). Then the crossed product algebra $\mathcal{A} = \mathcal{C} \rtimes_\alpha \mathbb{R}$ has an induced dual trace $\mathcal{T}$ and a dual $\mathbb{R}$-action $\hat{\alpha}$. Let further $\mathcal{M}$ be the von Neumann algebra generated by $\mathcal{C}$ in the semicyclic GNS representation for $\mathcal{T}$ and $\mathcal{H}$ the corresponding representation space. The regular representation π of $\mathcal{A}$ acts on the Hilbert space $L^2(\mathbb{R}, \mathcal{H})$ such that α is generated by right translation, i.e., $\pi \circ \alpha_t = \mathrm{Ad}(U_t) \circ \pi$ with $U_t = e^{t\partial}$. Then $\mathcal{T}$ extends to a semifinite normal faithful trace on the von Neumann algebra $\mathcal{N} = \pi(\mathcal{A})''$. Let $\mathcal{A}$ be the dense $*$-subalgebra of elements $A \in \mathcal{A}$ that can be written in the form $\pi(A) = \int_{\mathbb{R}} \pi(f(t))U_t \,\mathrm{d}t$ with a function $f : \mathbb{R} \to \mathrm{Dom}(\tau) \subset \mathcal{A}$ that is smooth and rapidly decaying in the norm $\|A\| + \tau(|A|)$ and $D = -\imath\partial$ then $(\mathcal{N}, D, \mathcal{A})$ is a semifinite spectral triple (compare, e.g., [15]). If H is a differentiable self-adjoint multiplier invertible modulo $\mathcal{A}$, then the index theorem implies that

$$\widehat{\mathcal{T}}\text{-Ind}(\kappa D + H) = \mathrm{Sf}_D(H),$$

and by choosing a representative $U \in \mathbf{1} + \mathcal{A}$ of the class $[e^{\imath\pi(G(H)+\mathbf{1})}]_1$, one can compute the spectral flow [38]

$$\mathrm{Sf}_D(H) = \langle [U]_1, [D] \rangle = \mathcal{T}((\mathbf{1} - U^*)[\partial, U]). \tag{21}$$

The right-hand side is the non-commutative winding number as is expected for an analytic formula for spectral flow. The appropriate setting for the theorem in [42] is a trivial action α in which case $\mathcal{A} = C_0(\mathbb{R}, \mathcal{C})$ consists of paths in the von Neumann algebra $\mathcal{M} = \pi(\mathcal{C})''$. Since τ-traceclass elements are dense in $\mathcal{C}$ one has $\mathcal{C} \subset \mathcal{K}_\tau$ and hence invertibility of $H \in M(\mathcal{A}) \subset C_b(\mathbb{R}, \mathcal{M})$ modulo $\mathcal{A}$ means that H is a continuous path of τ-Fredholm operators. In that case, the right-hand side of (21) computes the usual semifinite spectral flow, in fact, it is almost exactly the definition of spectral flow for gap-continuous paths (see Appendix B, note, however, that for an unbounded H to be a multiplier here, it must describe a Riesz-continuous path).

8.4. Index theorems for topological insulators

Let us now discuss a more complicated non-commutative example coming from the theory of topological insulators [40]. To keep the discussion simple we consider a two-dimensional example with magnetic field, but no disordered potential. Thus, the observable algebra is the two-dimensional non-commutative torus $\mathfrak{a}_\theta$ with twisting angle θ generated by two unitaries with the commutation relation $v_1 v_2 = e^{\imath\theta} v_2 v_1$. Let $c_*(\mathbb{Z})$ be the algebra of sequences which admit limits for $\pm\infty$ and $c_0(\mathbb{Z})$ the subalgebra for which those limits vanish. Let then $\widehat{\mathcal{A}}$ be the C^*-algebra generated by $c_*(\mathbb{Z})$ and the unitaries v_1,v_2 with the additional commutation relations $f v_1 = (f \circ T_1) v_1$ and $f v_2 = v_2 f$ with $T_1 : c_*(\mathbb{Z}) \to c_*(\mathbb{Z})$ left translation. Each element $a \in \widehat{\mathcal{A}}$ has a representation as a formal sum $a = \sum_{x,y\in\mathbb{Z}} f_{x,y} v_1^x v_2^y$. Consider the ideal $\mathcal{A} \subset \widehat{\mathcal{A}}$ of those elements for which the coefficient functions are in $c_0(\mathbb{Z})$. Then one has an exact sequence

$$0 \to \mathcal{A} \to \widehat{\mathcal{A}} \to \mathfrak{a}_\theta \oplus \mathfrak{a}_\theta \to 0$$

obtained by evaluation at $\pm\infty$. On $\mathfrak{a}_\theta$ one can introduce a finite trace τ and on $\mathcal{A}$ a densely defined lower semicontinuous trace $\hat{\tau}$ such that

$$\tau(v_1^x v_2^y) = \delta_{x,0}\delta_{y,0}, \qquad \hat{\tau}(f v_1^x v_2^y) = \delta_{x,0}\delta_{y,0}\sum_{k\in\mathbb{Z}} f(k),$$

holds for all $x, y \in \mathbb{Z}$ and $f \in \ell^1(\mathbb{Z})$. Let $\mathcal{A}$ be the subalgebra of all $\sum_{x,y\in\mathbb{Z}} f_{x,y} v_1^x v_2^y \in \mathcal{A}$ for which $|f_{x,y}(k)|$ decays faster than any inverse polynomial in x, y, k. One can represent $\widehat{\mathcal{A}}$ on the Hilbert space $\ell^2(\mathbb{Z}^2)$ in such a way that $c_*(\mathbb{Z})$ acts by multiplication and v_1,v_2 as magnetic shifts. Furthermore, there are the two position operators X_1,X_2 acting on the standard basis of $\ell^2(\mathbb{Z}^2)$ by $X_i e_x = x_i e_x$. The commutators $[X_i, \cdot]$ produce densely defined derivations on $\mathcal{A}$ and $\mathfrak{a}_\theta$. As the Dirac operator, let us use $D = X_2$ which results in a spectral triple $(\mathcal{B}(\ell^2(\mathbb{Z}, \mathbb{C}^2)), D, \mathcal{A})$. One can therefore consider Callias-type operator with potentials H in the multiplier algebra $M(\mathcal{A})$. Any such multiplier has a representation $H = \sum_{x,y\in\mathbb{Z}} h_{x,y} v_1^x v_2^y$ with coefficient functions $h_{x,y} \in \ell^\infty(\mathbb{Z})$. Assume that $\|h_{x,y}\|_\infty$ decays faster than any inverse polynomial in x, y from which one can check that an H is bounded and differentiable in our sense. If H is invertible modulo $\mathcal{A}$, then by Theorem 8 its index is given by

$$\mathrm{Ind}(\kappa D + \imath H) = \mathrm{Sf}_D(H) = \langle [U]_1, [X_2]\rangle,$$

with $U = e^{\imath\pi(G(H)+\mathbf{1})} \in \mathbf{1} + \mathcal{A}$. In fact, one has $U \in \mathbf{1} + \mathcal{A}$ and this index can be computed explicitly [39]

$$\langle [U]_1, [X_2]\rangle = \hat{\tau}((\mathbf{1} - U^*)[X_2, U]).$$

In physics, the algebra $\mathfrak{a}_\theta$ describes an observable algebra for two-dimensional tight-binding models which are invariant under magnetic translations, while $M(\mathcal{A})$ more generally allows modulations with respect to the x_1-direction. In particular, a self-adjoint multiplier $H \in \widehat{\mathcal{A}} \subset M(\mathcal{A})$ represents the Hamiltonian for a system with an interface at the line $x_1 = 0$ between two asymptotic "bulk" Hamiltonians $H_\pm \in \mathfrak{a}_\theta$ which describe the local Hamiltonian far away from the interface for $x_1 \to \pm\infty$. The number $\mathrm{Sf}_D(H)$ again makes sense as a non-commutative spectral flow: if the flow along the "path" H connecting the invertible Hamiltonians H_+, H_- is non-trivial, then H itself cannot be invertible ("the spectral gap closes") and this fact only depends on H up to homotopy.

The known results on the bulk-boundary correspondence of such operators form a close analogue of the Callias index formula. Indeed, for $H \in \widehat{\mathcal{A}}$ the class $[U]_1 \in K_1(\mathcal{A})$ of the spectral flow is the image under the exponential map of the class in $[P_+ \oplus P_-] \in K_0(\mathfrak{a}_\theta \oplus \mathfrak{a}_\theta)$ of the Fermi projections $P_\pm = \chi(H_+ < 0)$. It is then known (e.g., [31]) that

$$\begin{aligned}\langle \partial_0[P]_0, [X_2]\rangle &= \langle [P]_0, [X_1 \otimes \sigma_1 + X_2 \otimes \sigma_2]\rangle \\ &= \imath\tau(P_+[[X_1, P_+][X_2, P_+]]) - \imath\tau(P_-[[X_1, P_-][X, P_-]])\end{aligned}$$

which shows that the index can be computed from the boundaries at $\pm\infty$. Compared to the Callias index formula the situation seems inverted since the boundary now actually represents a higher-dimensional space. This is not too unusual since cyclic cohomology is 2-periodic and so dualities can affect the apparent dimensions of cocycles and algebras.

This example can be generalized in different ways. The non-commutative torus does not really play a role in the arguments. One can construct analogous spectral triples and bulk-boundary sequences for any twisted crossed product $\mathcal{C} \rtimes \mathbb{R}^d$ or $\mathcal{C} \rtimes \mathbb{Z}^d$ where the base algebra $\mathcal{C}$ admits a densely defined faithful lower semicontinuous trace. When one generalizes, this naturally leads to semifinite spectral triples, e.g., if one chooses $\mathcal{C} = C(K)$ for some compact metric space K with measure μ, the spectral triple will be based on the type-I-von Neumann algebra $L^\infty(K,\mu) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^2))$ with trace $\int_K \mathrm{d}\mu \otimes \mathcal{T}$. For higher dimension $d > 2$ one can also construct spectral triples with Dirac operators that involve less than $d - 1$ spatial directions, so that the spectral triple is then naturally based on a type-II_∞-von Neumann algebra. Examples for multipliers H with non-trivial indices can then be given in terms of Hamiltonians of so-called weak topological insulators, see [43].

A. Semifinite index

Let $\mathcal{N}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace $\mathcal{T}$. This appendix briefly reviews the theory of semifinite index and its continuity properties [4, 34, 45]. The domain of $\mathcal{T}$ is by definition given by $\mathcal{N}_\mathcal{T} = \{A \in \mathcal{N} : \mathcal{T}(|A|) < \infty\}$. It is a $*$-algebra and an ideal in $\mathcal{N}$. It becomes a Banach-$*$-algebra when supplied with the submultiplicative norm $\|A\|_\mathcal{T} = \|A\| + \mathcal{T}(|A|)$ and its C^*-completion $\mathcal{K}_\mathcal{T} \subset \mathcal{N}$ is the algebra of $\mathcal{T}$-compact operators. The quotient $\mathcal{N}/\mathcal{K}_\mathcal{T}$ is called the Calkin algebra and the quotient map will be denoted by π. The $\mathcal{T}$-essential spectrum of $A \in \mathcal{N}$ is then defined by $\sigma_{\mathrm{ess}}(A) = \sigma(\pi(A)) = \sigma(A + \mathcal{K}_\mathcal{T})$.

A possibly unbounded operator T affiliated to $\mathcal{N}$ is called $\mathcal{T}$-Fredholm if there is a continuous function $\chi : [0,\infty) \to [0,1]$ with $\chi(0) = 1$ such that $\chi(T^*T) \in \mathcal{K}_\mathcal{T}$ and $\chi(TT^*) \in \mathcal{K}_\mathcal{T}$. This is equivalent to the existence of a pair of operators $K, K' \in \mathcal{K}_\mathcal{T}$ such that $T^*T + K$ and $TT^* + K'$ are invertible. The set of $\mathcal{T}$-Fredholm operators will be denoted by $\mathcal{F}(\mathcal{N})$ and the intersection with the self-adjoint operators by $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$.

If $T \in \mathcal{F}(\mathcal{N})$, then also $T^* \in \mathcal{F}(\mathcal{N})$ and furthermore the kernel projection $\mathrm{Ker}(T)$ lies in $\mathcal{K}_\mathcal{T}$. It is important to note that any $\mathcal{T}$-compact projection is automatically $\mathcal{T}$-finite and therefore one has a well-defined index

$$\mathcal{T}\text{-Ind}(T) = \mathcal{T}(\mathrm{Ker}(T)) - \mathcal{T}(\mathrm{Ker}(T^*)) \in \mathbb{R}.$$

The index is invariant under addition of $\mathcal{K}_\mathcal{T}$ perturbations and constant on norm-connected components of $\mathcal{F}(\mathcal{N}) \cap \mathcal{N}$.

An element $T \in P\mathcal{N}Q$ for two projections $P, Q \in \mathcal{N}$ is called $P \cdot Q$-Fredholm if T^*T and TT^* are $\mathcal{T}$-Fredholm in the corner algebras $Q\mathcal{N}Q$ and $P\mathcal{N}P$ respectively. One then defines more generally the skew-corner index [16] by

$$\mathcal{T}\text{-Ind}_{P \cdot Q}(T) = \mathcal{T}(\mathrm{Ker}(T) \cap Q) - \mathcal{T}(\mathrm{Ker}(T^*) \cap P). \tag{22}$$

For the study of the spectral flow of unbounded Fredholm operators in Appendix B, one needs to introduce topologies on $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$, of which there are several distinct ones [34, 48]. The most important ones are the Riesz topology and the gap topology, induced respectively by the metrics

$$d_R : \mathcal{F}_{\mathrm{sa}} \times \mathcal{F}_{\mathrm{sa}} \to \mathbb{R}_{\geq}, \qquad d_R(T_1, T_2) = \|F(T_1) - F(T_2)\|,$$

where F is the bounded transform, and

$$\begin{aligned} d_G : \mathcal{F}_{\mathrm{sa}} \times \mathcal{F}_{\mathrm{sa}} \to \mathbb{R}_{\geq}, \qquad d_G(T_1, T_2) &= \|\mathcal{C}(T_1) - \mathcal{C}(T_2)\| \\ &= \|(T_1 + \imath)^{-1} - (T_2 + \imath)^{-1}\|, \end{aligned}$$

with $\mathcal{C}(T) = (T - \imath)(T + \imath)^{-1}$ the Cayley transform. A sequence of self-adjoint operators converges in the gap topology if and only if it converges in the norm-resolvent sense. Hence, if $(T_t)_{t\in[0,1]}$ is a path in $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$ that is gap-continuous, then $(f(T_t))_{t\in[0,1]}$ is therefore a norm-continuous path in $\mathcal{N}$ for any $f \in C_0(\mathbb{R})$. The gap topology is weaker than the Riesz topology since it does not imply continuity under the bounded transform.

By the theorem of Cordes–Labrousse, all those topologies are equivalent to the norm-topology when restricting to bounded self-adjoint operators. The gap topology can be extended to non-self-adjoint Fredholm operators by setting

$$\widetilde{D}_G(T_1, T_2) = d_G\left(\begin{pmatrix} 0 & T_1^* \\ T_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & T_2^* \\ T_2 & 0 \end{pmatrix}\right),$$

and the $\mathcal{T}$-index of unbounded $\mathcal{T}$-Fredholm operators stays constant under gap-continuous homotopies [48, Proposition 5.2].

B. Semifinite spectral flow

This appendix recalls the definition and properties of the spectral flow in a semifinite von Neumann algebra. For paths of bounded self-adjoint operators this is reviewed in [4]. For paths of unbounded self-adjoint operators, depending on the notion of continuity, there are different possible ways to define spectral flow which we now describe in some detail since it is relevant for the main part of this article. Spectral flow for gap-continuous paths using the notion of a non-commutative winding number has been introduced in the Hilbert space setting by [5] and extended to the semifinite setting by [48].

Consider the Banach $*$-algebra of differentiable paths $C_0^1([0,1], \mathcal{N}_{\mathcal{T}})$ with norm

$$\|f\|_C = \sup_{t\in[0,1]} \|f(t)\|_{\mathcal{T}} + \|f'(t)\|_{\mathcal{T}},$$

which is dense in the C^*-algebra $C_0([0,1], \mathcal{K}_{\mathcal{T}})$ with spectrally invariant inclusion (the latter follows from the inequality $\|fg\|_C \leq \|f\|\|g\|_C + \|f\|_C\|g\|$ via a standard argument using geometric series, see also [44]). The non-commutative winding number defined by

$$\mathrm{wind}_{\mathcal{T}} : C_0^1([0,1],\mathcal{N}_{\mathcal{T}}) \times C_0^1([0,1],\mathcal{N}_{\mathcal{T}}) \to \mathbb{C}, \qquad \mathrm{wind}(f_1,f_2) = \imath \int_0^1 \mathcal{T}(f_1(t) f_2'(t)),$$

is a cyclic 1-cocycle and therefore pairs with odd K-theory groups. Due to spectral invariance, one has $K_1(C_0^1([0,1],\mathcal{N}_{\mathcal{T}})) \simeq K_1(C_0([0,1],\mathcal{K}_{\mathcal{T}}))$ and more strongly any class $[f]_1 \in K_1(C_0([0,1],\mathcal{K}_{\mathcal{T}}))$ defined by a unitary path $f \in \mathbf{1} + C_0([0,1],\mathcal{K}_{\mathcal{T}})$ can be represented by a unitary path $\tilde{f} \in \mathbf{1} + C_0^1([0,1],\mathcal{N}_{\mathcal{T}})$ such that the real-valued pairing

$$\langle [f]_1, \mathrm{wind}_{\mathcal{T}} \rangle = \mathrm{wind}_{\mathcal{T}}(\tilde{f}^* - \mathbf{1}, \tilde{f} - \mathbf{1})$$

is well defined and does not depend on the choice of representative.

Definition 41 ([48]). Let $t \in [0,1] \mapsto T_t$ be a gap-continuous path in $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$ with invertible endpoints. One can always choose a so-called switch function $G : \mathbb{R} \to \mathbb{R}$ for the path, which is a smooth function with $\mathrm{supp}(G') \subset (-1,1)$, $G(\pm 1) = \pm 1$ and $\mathbf{1} - G(T_t)^2 \in \mathcal{K}_{\mathcal{T}}$ for all t. Then the norm-continuous unitary path $t \in [0,1] \mapsto e^{\imath\pi(G(T_t)+\mathbf{1})}$ lies in $\mathbf{1} + \mathcal{K}_{\mathcal{T}}$ and the spectral flow

$$\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) = \langle [e^{\imath\pi(G(T)+\mathbf{1})}]_1, \mathrm{wind}_{\mathcal{T}} \rangle$$

is well defined and does not depend on the choice of G.

The important technical point here is that $e^{\imath\pi G} - 1$ is a continuous compactly supported $C_0(\mathbb{R})$-function which vanishes on the $\mathcal{T}$-essential spectrum. Therefore gap-continuity is sufficient, but in exchange the endpoints have to be invertible.

For Riesz-continuous paths the spectral flow can also be defined more directly as the flow of spectrum from the negative to the positive:

Definition 42 ([4]). For projections $P, Q \in \mathcal{N}$ with $\|\pi(P-Q)\| < 1$ define the essential codimension

$$\mathrm{ec}(P,Q) = \mathcal{T}\big((1-P)\cap Q\big) - \mathcal{T}\big((1-Q)\cap P\big).$$

For a Riesz-continuous path $t \in [0,1] \mapsto T_t$ in $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$ one can always choose a partition $0 = t_0 < t_1 < \cdots < t_{K+1} = 1$ such that

$$\|\pi(\chi(T_s \geq 0) - \chi(T_t \geq 0))\| \leq \frac{1}{2}, \qquad \forall s,t \in [t_k, t_{k+1}],$$

holds for all $k = 0,\ldots,K$. In that case, the spectral flow is given by

$$\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) = \sum_{k=0}^{K} \mathrm{ec}\big(\chi(T_{t_k} \geq 0), \chi(T_{t_{k+1}} \geq 0)\big).$$

If the endpoints of the Riesz-continuous path are invertible, both notions coincide. Therefore, the spectral flow for gap-continuous paths is well defined by choosing for each endpoint a $\mathcal{T}$-compact perturbation Q_0, Q_1 such that $T_i + Q_i$ is invertible, and then setting

$$\begin{aligned}\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) = \mathrm{Sf}\big(\{T_t + (1-t)Q_0 + tQ_1\}_{t\in[0,1]}\big) + \mathrm{Sf}\big(\{T_0 + tQ_0\}_{t\in[0,1]}\big)\\ + \mathrm{Sf}\big(\{T_1 + (1-t)Q_1\}_{t\in[0,1]}\big),\end{aligned}$$

where the second two terms use the definition for Riesz-continuous paths and only the former the one for gap-continuous ones. One has the following properties:

Proposition 43. *Let $t \in [0,1] \mapsto T_t$ and $t \in [0,1] \mapsto T'_t$ be gap-continuous paths in $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$.*

(i) (Triviality) *If T_t has a bounded inverse for each $t \in [0,1]$ then* $\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) = 0$.

(ii) (Homotopy invariance) *If the two paths are connected by a gap-continuous (respectively Riesz-continuous) homotopy $(t,s) \in [0,1] \times [0,1] \mapsto T_{s,t}$ within $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$ with $T_{0,t} = T_t$, $T_{1,t} = T'_t$ and such that the endpoints $T_{s,0}$ and $T_{s,1}$ are invertible for each $s \in [0,1]$, then*

$$\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) = \mathrm{Sf}(\{T'_t\}_{t\in[0,1]}).$$

(iii) (Concatenation) *If $T_1 = T'_0$, then*

$$\mathrm{Sf}(\{T_t\}_{t\in[0,1]} * \{T'_t\}_{t\in[0,1]}) = \mathrm{Sf}(\{T_t\}_{t\in[0,1]}) + \mathrm{Sf}(\{T'_t\}_{t\in[0,1]}),$$

with $*$ denoting concatenation of paths.

(iv) (Homomorphism)

$$\mathrm{Sf}(\{T_t \oplus T'_t\}_{t\in[0,1]}) = \mathrm{Sf}(\{T_t\}_{t\in[0,1]}) + \mathrm{Sf}(\{T'_t\}_{t\in[0,1]}).$$

For straight-line paths the spectral flow will be abbreviated by

$$\mathrm{Sf}(T_0, T_1) = \mathrm{Sf}(\{(1-t)T_0 + tT_1\}_{t\in[0,1]}).$$

Let us now recall some relations between spectral flow and the $\mathcal{T}$-Fredholm index:

Proposition 44 ([48, Proposition 5.1]). *For T a possibly unbounded $\mathcal{T}$-Fredholm operator and any $m > 0$,*

$$\mathcal{T}\text{-}\mathrm{Ind}(T) = \mathrm{Sf}\left(\begin{pmatrix} -m & T^* \\ T & m \end{pmatrix}, \begin{pmatrix} m & T^* \\ T & -m \end{pmatrix}\right).$$

Then there are more specific spectral flow formulas for unitary conjugates [14, Theorem 4.2].

Theorem 45. *Let D be a self-adjoint invertible $\mathcal{T}$-Fredholm operator affiliated to $\mathcal{N}$. If $U \in \mathcal{N}$ is a unitary that preserves* $\mathrm{Dom}(D)$*, $[D, U]$ extends to a bounded operator in $\mathcal{N}$ and such that $(D + \imath)^{-1}(U - \mathbf{1}) \in \mathcal{K}_{\mathcal{T}}$ and $(D + \imath)^{-1}[D, U] \in \mathcal{K}_{\mathcal{T}}$, then*

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}(U^* DU, D)$$

where $P = \chi(D > 0)$.

The bounded version of this formula is [4, Section 5].

Proposition 46. *If T is a self-adjoint involution and $U \in \mathcal{N}$ a unitary with $[T, U] \in \mathcal{K}_{\mathcal{T}}$, then*

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}(T, U^* T U)$$

where $P = \chi(T < 0)$.

Theorem 45 requires that $[D, U]$ must be relatively D-compact for the path to be Fredholm. While such a condition often is satisfied in applications, it is sometimes inconvenient, e.g., in the setting of spectral triples without smoothness assumptions. We therefore provide an alternative:

Proposition 47. *Let D be a self-adjoint invertible $\mathcal{T}$-Fredholm operator affiliated to $\mathcal{N}$ and $U \in \mathcal{N}$ a unitary that preserves* $\mathrm{Dom}(D)$*, $[D, U]$ extends to a bounded operator in $\mathcal{N}$ and $(U - \mathbf{1})(D + \imath)^{-1}$, $(U^* - \mathbf{1})(D + \imath)^{-1} \in \mathcal{K}_{\mathcal{T}}$. Set $P = \chi(D < 0)$, then*

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\left(\begin{pmatrix} \kappa D & \mathbf{1} \\ \mathbf{1} & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & U^* \\ U & -\kappa D \end{pmatrix}\right)$$

holds for all $\kappa > 0$ so small that $\kappa \|[D, U]\| < 1$.

The proof starts out with a technical lemma:

Lemma 48. *Let D be an unbounded self-adjoint invertible operator and H a bounded self-adjoint operator which preserves* $\mathrm{Dom}(D)$ *and for which $[D, H]$ extends to a bounded operator. Choose an even smooth function $g : \mathbb{R} \to [0, 1]$ supported in $[-2, 2]$ and equal to 1 on $[-1, 1]$. Set $\chi_R = g(R^{-1} D)$ and define a net $(D_R)_{R>0}$ of bounded self-adjoint operators by*

$$D_R = D\chi_R + R(1 - \chi_R)\mathrm{sgn}(D).$$

Then D_R converges to D with respect to the gap metric for $R \to \infty$ and there exists a universal constant $c > 0$ such that

$$\|[D_R, H]\| \leq c\|[D, H]\| \tag{23}$$

independent of D, R and H.

Proof. The convergence is readily seen in the spectral representation. For (23) let us first recall the bound [24, Lemma 10.15]

$$\|[f(D), H]\| \leq (2\pi)^{-1}\left(\int_{\mathbb{R}} |t|\,|\widehat{f}(t)|\,\mathrm{d}t\right)\|[D, H]\| = (2\pi)^{-1}\|\widehat{f'}\|_{L^1(\mathbb{R})}\|[D, H]\|$$

applicable to smooth functions $f \in C_c^\infty(\mathbb{R})$ where $\widehat{f}$ denotes the Fourier transform. By multiplying with a smooth approximate unit of the Fourier algebra $\mathcal{F}^{-1}L^1(\mathbb{R})$ the bound generalizes to functions f without compact support, but for which $f' \in C_c^\infty(\mathbb{R})$. Since one can write $D_R = f(R^{-1}D)$ for such a function $f \in C^\infty(\mathbb{R})$ a scaling argument therefore shows that (23) holds with $c = (2\pi)^{-1}\|\widehat{f'}\|_{L^1(\mathbb{R})}$. ■

Proof (of Proposition 47). It is sufficient to prove the result for some $\kappa > 0$ that is as small as necessary, since one can then increase κ up to the stated value using homotopy invariance.

By a standard argument $[P, U] \in \mathcal{K}_{\mathcal{T}}$ [24] and thus Proposition 46 and additivity imply

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\left(\begin{pmatrix}\kappa(\mathbf{1}-2P) & 0\\ 0 & -\kappa(\mathbf{1}-2P)\end{pmatrix}, \begin{pmatrix}\kappa U^*(\mathbf{1}-2P)U & 0\\ 0 & -\kappa(\mathbf{1}-2P)\end{pmatrix}\right).$$

The endpoints of the path are invertible and introducing an off-diagonal constant term only increases the spectral gap, thus

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\left(\begin{pmatrix}\kappa(\mathbf{1}-2P) & \mathbf{1}\\ \mathbf{1} & -\kappa(\mathbf{1}-2P)\end{pmatrix}, \begin{pmatrix}\kappa U^*(\mathbf{1}-2P)U & U^*\\ U & -\kappa(\mathbf{1}-2P)\end{pmatrix}\right).$$

To check the Fredholm property along that straight-line homotopy one notes

$$\begin{pmatrix}\kappa(\mathbf{1}-2P) - \kappa t U[P, U^*] & s(t + (1-t)U^*)\\ s(t+(1-t)U) & -\kappa(\mathbf{1}-2P)\end{pmatrix}^2 \geq \kappa^2 + s^2(1-2t)^2 \mod \mathcal{K}_{\mathcal{T}}.$$

The right endpoint at $(s,t) = (1,1)$ has a spectral gap in the interval $[-1,1]$ and assuming $\kappa < \frac{1}{4}$, the gap is not closed if one replaces $U^*(\mathbf{1}-2P)U = (\mathbf{1}-2P) - 2U^*[P,U]$ by $\mathbf{1} - 2P$ using an additive perturbation with norm $\|[P,U]\| \leq 2$. That compact perturbation also does not affect the Fredholm properties, therefore

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\left(\begin{pmatrix}\kappa(\mathbf{1}-2P) & \mathbf{1}\\ \mathbf{1} & -\kappa(\mathbf{1}-2P)\end{pmatrix}, \begin{pmatrix}\kappa(\mathbf{1}-2P) & U^*\\ U & -\kappa(\mathbf{1}-2P)\end{pmatrix}\right).$$

For arbitrary $R > 0$ we use the approximation D_R of Lemma 48 and consider the norm-continuous homotopy

$$(\gamma, t) \in [0,1]\times[0,1] \mapsto T_{\gamma,t} = \begin{pmatrix}\kappa D_R|D_R|^{\gamma-1} & t\mathbf{1} + (1-t)U^*\\ t\mathbf{1} + (1-t)U & -\kappa D_R|D_R|^{\gamma-1}\end{pmatrix}.$$

We must show that all $T_{\gamma,t}$ are Fredholm with invertible endpoints at $t \in \{0,1\}$ for some small enough κ. At the left endpoint

$$T_{\gamma,0}^2 = \begin{pmatrix}\kappa D_R|D_R|^{\gamma-1} & \mathbf{1}\\ \mathbf{1} & -\kappa D_R|D_R|^{\gamma-1}\end{pmatrix}^2 = (\kappa^2|D_R|^{2\gamma} + \mathbf{1}) \otimes \mathbf{1}_2$$

and at the right

$$T_{\gamma,1}^2 = \begin{pmatrix} \kappa D_R|D_R|^{\gamma-1} & U^* \\ U & -\kappa D_R|D_R|^{\gamma-1} \end{pmatrix}^2$$
$$= (\kappa^2|D_R|^{2\gamma} + \mathbf{1} - \kappa c\|D^{-1}\|^{\gamma-1}\|[D,U]\|)\mathbf{1}_2,$$

with the constant c from (23) and where we used a known estimate for the commutator with fractional powers [24, (10.58)],

$$\big\|\big[D_R|D_R|^{\gamma-1}, U\big]\big\| \le \big\||D_R|^{\gamma-1}\big\|\big\|[D_R, U]\big\| \le c\|D^{-1}\|^{\gamma-1}\|[D,U]\|.$$

Hence invertibility holds if we assume that κ is small enough. The relative compactness further implies

$$[f(D), U] = f(D)(U - \mathbf{1}) - (U - \mathbf{1})f(D) \in \mathcal{K}_{\mathcal{T}}$$

for any function $f \in C_c(\mathbb{R})$ and since also $[P, U] \in \mathcal{K}_{\mathcal{T}}$ one concludes $[D_R|D_R|^{\gamma-1}, U] \in \mathcal{K}_{\mathcal{T}}$ for all $\gamma \in [0, 1]$. Computing $T_{\gamma,t}^2$, one therefore finds

$$\begin{pmatrix} \kappa D_R|D_R|^{\gamma-1} & t\mathbf{1} + (1-t)U^* \\ t\mathbf{1} + (1-t)U & -\kappa D_R|D_R|^{\gamma-1} \end{pmatrix}^2$$
$$= \big(\kappa^2|D_R|^{2\gamma} + 1 + t(1-t)(U + U^* - 2)\big)\mathbf{1}_2 \mod \mathcal{K}_{\mathcal{T}}$$
$$\ge \kappa^2 \min(\|D^{-1}\|^{-2\gamma}, R^{2\gamma})\mathbf{1}_2 \mod \mathcal{K}_{\mathcal{T}}$$

where we used $\|t(1-t)(U + U^* - 2)\| \le 1$. Thus $T_{\gamma,t}$ is Fredholm for all $\gamma, t \in [0, 1]$ and homotopy invariance implies

$$\mathcal{T}\text{-Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\left(\begin{pmatrix} \kappa D_R & \mathbf{1} \\ \mathbf{1} & -\kappa D_R \end{pmatrix}, \begin{pmatrix} \kappa D_R & U^* \\ U & -\kappa D_R \end{pmatrix}\right)$$

for all $R > 0$ and some fixed $\kappa > 0$. The proof is then completed by taking the limit $R \to \infty$ as the following Lemma shows. ∎

Lemma 49. (i) *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of gap-continuous paths $T_n = (T_{n,t})_{t\in[0,1]}$ in $\mathcal{F}_{\mathrm{sa}}(\mathcal{N})$ converging uniformly in the gap metric to some path T, i.e.,*

$$\lim_{n\to\infty} \sup_{t\in[0,1]} d_G(T_{n,t}, T_t) = 0. \tag{24}$$

If the endpoints of T_n, T are invertible and $|T_{n,t}| > g\mathbf{1} \mod \mathcal{K}_{\mathcal{T}}$ holds for all $t \in [0, 1]$, $n \in \mathbb{N}$ with some fixed constant $g > 0$ then the spectral flow is continuous

$$\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) = \lim_{n\to\infty} \mathrm{Sf}(\{T_{n,t}\}_{t\in[0,1]}).$$

(ii) *The convergence condition (24) holds in particular for paths of the form $T_{n,t} = D_n + H_t$ with $(D_n)_{n\in\mathbb{N}}$ the sequence of self-adjoint operators affiliated to $\mathcal{N}$ that converges to D with respect to the gap metric and H a norm-continuous path in $\mathcal{N}_{\mathrm{sa}}$.*

Proof. For (ii) we note the resolvent identity

$$\begin{aligned}(D + H_t + \imath)^{-1} - (D_n + H_t + \imath)^{-1} \\ = \big(1 - (D + H_t + \imath)^{-1} H_t\big)\big((D + \imath)^{-1} - (D_n + \imath)^{-1}\big) \\ \cdot \big(1 - H_t (D_n + H_t + \imath)^{-1}\big)\end{aligned}$$

which implies $d_G(T_n, T) \leq (1 + \|H\|)^2 d_G(D_n, D)$. Similarly, one estimates

$$\|(T + z)^{-1} - (S + z)^{-1}\| \leq \left(1 + \frac{|\imath - z|}{|\Im m(z)|}\right)^2 d_G(T, S)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ and hence the Helffer–Sjöstrand calculus may be used to show that the map $T \in \mathcal{F}_{\mathrm{sa}}(\mathcal{N}) \mapsto f(T) \in \mathcal{N}$ is uniformly continuous for each fixed $f \in C_c^\infty(\mathbb{R})$ in the sense that

$$\|f(T) - f(S)\| \leq C_f d_G(T, S). \tag{25}$$

By assumption on T_n there is a gap in the $\mathcal{T}$-essential spectrum which is independent of n and t. We may therefore also choose the normalizing function G in Definition 41 to be independent of those parameters. Combining (24) and (25) shows

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \|e^{\imath\pi(G(T_{n,t})+1)} - e^{\imath\pi(G(T_t)+1)}\| = 0,$$

i.e., the unitary path determining the spectral flow is norm-convergent. Consequently, $[e^{\imath\pi(G(T_n)+1)}]_1$ is eventually constant with limit $[e^{\imath\pi(G(T)+1)}]_1$, which implies

$$\begin{aligned}\mathrm{Sf}(\{T_t\}_{t\in[0,1]}) &= \langle [e^{\imath\pi(G(T)+1)}]_1, \mathrm{wind}_{\mathcal{T}}\rangle = \lim_{n\to\infty} \langle [e^{\imath\pi(G(T_n)+1)}]_1, \mathrm{wind}_{\mathcal{T}}\rangle \\ &= \lim_{n\to\infty} \mathrm{Sf}(\{T_{n,t}\}_{t\in[0,1]}),\end{aligned}$$

so that the proof is concluded. ∎

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