Wheel graph homology classes via Lie graph homology

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Abstract. We give a new proof of the non-triviality of wheel graph homology classes using higher operations on Lie graph homology and a derived version of Koszul duality for modular operads.

1. Introduction

In the seminal paper [9], Willwacher constructed a family of non-trivial graph cohomology classes σ_{2j+1} by analyzing an isomorphism between the 0-th cohomology of a certain graph complex GC_2 with the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$. The complex GC_2 is (up to important details) the *commutative* variant of a construction called the Feynman transform [4].

In [2], the authors study a family of group extensions of the outer automorphism groups of free groups and use the Leray–Serre spectral sequence to compute the homology of a low genus portion of the *Lie* variant of this construction. They show in particular that in genus 1, the virtual cohomological dimension consists of a family of classes α_{2j+1}.

The purpose of this paper is to show that the families of classes σ_{2j+1} and α_{2j+1} correspond to each other under Koszul duality.

1.1. The correspondence: non-technical version

Let us explain the nature of this correspondence and the perspective from which such a result is expected. We omit several important technical details in this informal summary.

The statement that commutative and Lie structures are Koszul dual can be encoded in the language of operads. One first defines an operadic generalization of the bar construction of an algebra. One then proves that the homology of the bar construction of the commutative operad has the homotopy type of the Lie operad, and vice versa.

Operads have operations parameterized by graphs of genus 0 (trees), and may be generalized by considering structures with operations parameterized by all graphs. Such structures are called modular operads [4], and the analog of the bar construction is called the Feynman transform which we denote ft. The commutative and Lie operads may be considered modular operads by simply declaring operations corresponding to higher genus graphs to be zero.

When viewed as modular operads the analogous Koszul duality relationship between Com and Lie no longer holds. If it did, it would imply that ft(Com) had no homology in its higher genus summands which is patently false. Indeed, both commutative graph homology $H_*(\text{ft}(\text{Com}))$ and Lie graph homology $H_*(\text{ft}(\text{Lie}))$ have a rich and highly non-trivial structure in higher genera.

There is, however, a more subtle Koszul duality relationship between these modular operads. The modular operad $H_*(\text{ft}(\text{Lie}))$ encoding Lie graph homology contains a copy of the commutative modular operad Com but is Koszul dual (in a suitably derived sense) to the Lie modular operad Lie. Therefore in genus $g \geq 1$, the (derived) Feynman transform of Lie graph homology is acyclic on the one hand, and contains a subcomplex computing commutative graph homology on the other.

We thus conclude that *every commutative graph homology class of non-zero genus may be represented by a graph labeled by Lie graph homology classes*. This correspondence is realized via the boundary operator in a derived version of the Feynman transform. In particular, the differential in this acyclic complex depends not just on the modular operad structure of Lie graph homology, but also on its higher "Massey products" which we compute here-in. Under this correspondence, the homology class σ_{2j+1} corresponds to a graph whose lone non-trivial vertex label is built from α_{2j+1}.

1.2. The correspondence: technical version

The graph complex $(\text{GC}_2, d_{\text{GC}})$ of [9] which we consider here-in consists of graphs with no legs and no loops (aka tadpoles). The differential is a sum of all possible edge expansions, with signs encoded by a factor of the top exterior power of the set of edges of the graph. This complex splits over genus, and the complex GC_2^g is (after a shift in degree by Σ^{-2g}) a sub-complex of $\text{ft}(\text{Com})(g, 0)$. We will suppress the degree shift notation Σ^{-2g} in this introduction. Observe here the commutative operad is not twisted and the output $\text{ft}(\text{Com})$ is $\mathfrak{K}$-twisted in the parlance of [4].

By "Lie graph homology" we refer to the Feynman transform of the Lie operad, but with opposite parity conventions. Explicitly the input is $\mathfrak{K}$-twisted and output is untwisted (see Section 3.1); this matches the convention of [7]. The modular operad structure of Lie graph homology was studied in [2]. This modular operad is not formal and so, abstractly, higher operations exist which make Lie graph homology into a modular operad up-to-homotopy (called a weak modular operad) which is homotopy equivalent to the Feynman transform of an extension by zero [8]. The derived version of the Feynman transform ft mentioned above has a differential with additional terms taking these higher operations into account. Making these higher operations explicit in all genera is both unlikely to be possible and unnecessary for the task at hand. As we shall show, the wheel graph homology classes can be detected by higher operations on Lie graph homology landing in genus 1. With this in mind, we proceed as follows.

First we truncate Lie graph homology above genus 1 and endow the resulting semi-classical modular operad with explicit higher operations. The resulting weak modular

operad is denoted $\mathcal{H}_{\mathrm{Lie}}$. The projection $\mathcal{H}_{\mathrm{Lie}} \to \mathsf{Com}$ induces a map of $\mathcal{K}$-modular operads $\mathsf{ft}(\mathsf{Com}) \hookrightarrow \mathsf{ft}(\mathcal{H}_{\mathrm{Lie}})$. Declare a loop to be simple if its adjacent vertex is trivalent and of genus 0. The subspaces of the Feynman transform supported on graphs with simple loops form a dg submodule. Let $\bar{\mathsf{ft}}$ denote the passage to the quotient; we also write $\bar{\eta}$ for the projection of a vector η to this quotient. Lifting $\mathsf{ft}(\mathsf{Com}) \hookrightarrow \mathsf{ft}(\mathcal{H}_{\mathrm{Lie}})$ we have the following diagram of dg $\mathbb{S}$-modules:

$$\mathsf{GC}_2 \hookrightarrow \bar{\mathsf{ft}}(\mathsf{Com}) \hookrightarrow \bar{\mathsf{ft}}(\mathcal{H}_{\mathrm{Lie}}). \tag{1.1}$$

The left hand inclusion is split; we denote the projection $\pi \colon \bar{\mathsf{ft}}(\mathsf{Com}) \to \mathsf{GC}_2$.

We write $\omega_{2j+1} \in \mathsf{GC}_2^*$ for the wheel graph; a $2j + 1$-gon along with a central vertex connected to the other vertices, tensored with a fixed element in the top exterior power of the span of the set of edges of the graph (Figure 1). The main technical result of this paper is the following construction:

Theorem 1.1. *There exists an element* $\eta_{2j+1} \in \mathsf{ft}(\mathcal{H}_{\mathrm{Lie}})(2j + 1, 0)$ *such that* $d(\bar{\eta}_{2j+1}) \in \bar{\mathsf{ft}}(\mathsf{Com})$ *and* $\langle \omega_{2j+1}, \pi \circ d(\bar{\eta}_{2j+1}) \rangle \neq 0$.

The condition $\langle \omega_{2j+1}, \pi \circ d(\bar{\eta}_{2j+1}) \rangle \neq 0$ ensures that the elements $\pi \circ d(\bar{\eta}_{2j+1})$ are not boundaries in the submodule $\mathsf{GC}_2 \subset \bar{\mathsf{ft}}(\mathcal{H}_{\mathrm{Lie}})$. On the other hand, the elements $d(\bar{\eta}_{2j+1})$ are boundaries in the larger complex and so certainly must project to cycles in GC_2. Thus, each $\pi \circ d(\bar{\eta}_{2j+1})$ represents a non-trivial graph homology classes in GC_2. Equivalently we may state:

Corollary 1.2. *The homology class* $[\omega_{2j+1}] \in H_0(\mathsf{GC}_2^*)$ *is non-trivial.*

The proof of Theorem 1.1 is an explicit construction. The element η_{2j+1} has the underlying graph pictured in Figure 4, and carries a vertex label given by the VCD class α_{2j+1} composed with $2j - 2$ copies of the commutative product. In this way, we find η_{2j+1} not by contracting the exterior polygon (a logical first guess) but by contracting interior polygons (Figure 1). Contracting such interior polygons produces graphs having $n = 2j - 2$ adjacent loops and the construction of η_{2j+1} depends crucially on the representation theory of the Weyl group of $SO(2n + 1)$ to show that the class we construct is the unique class (up to scalar multiple) which does not vanish upon passage to coinvariants by the group of isomorphisms of such a graph.
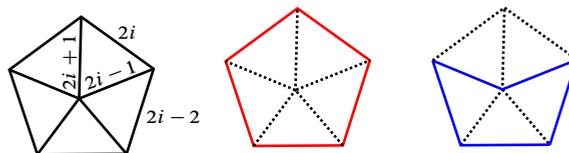
Figure 1. Left: The wheel graph ω_{2j+1} for $j = 2$ with edge order indicated. Center: The "exterior" $2j + 1$-gon is the pictured (red) subgraph. Right: The "interior" $2j + 1$-gon is the pictured (blue) subgraph.

Corollary 1.2 may also be derived from the results of [9], see [1, Theorem 2.6]. What is new here is the technique, as our proof completely avoids discussion of Drinfeld associators and $\mathfrak{grt}_1$. The interplay between Lie and commutative graph homology is subtle, and much interesting work remains. For example how to realize Morita and Eisenstein classes via Massey products on commutative graph homology, or how to relate open conjectures on either side of this correspondence. For now we simply offer the results of this paper as evidence that commutative and Lie graph homology may be effectively studied in tandem.

This paper is organized as follows. A brief overview of the prerequisites is given in Section 2. In Section 3, we compute relations in the modular operad for Lie graph homology and then construct higher operations for Lie graph homology landing in genus 1. The construction of the element η_{2j+1} and verification of its properties is given in Section 4. Throughout we mostly work with the "co-Feynman transform", and then linear dualize as the very last step (Section 4.4) to prove the results stated in this introduction.

1.3. Conventions

Throughout we work in the category of differential graded vector spaces over the field $k = \mathbb{Q}$ with homological grading conventions. We write A^* for the graded linear dual of A. We denote symmetric groups by S_n and denote irreducible representations of S_n by V_α for a partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n.

2. Background

2.1. Graphs

We briefly review the standard operadic definition of an abstract graph, see [8] for full details. A graph $\gamma = (V, F, a, \iota)$ consists of a finite non-empty set of vertices V, a finite set of flags (also called half-edges) F, a function $a \colon F \to V$ which indicates to which vertex a flag is adjacent, and an involution $\iota \colon F \to F$. The orbits of ι of order two are called the edges of γ and denoted $E(\gamma)$. The orbits of order one are called the legs γ and denoted $\mathrm{leg}(\gamma)$. A loop is an edge $\{f, \iota(f)\}$ for which $a(f) = a(\iota(f))$. The number $|a^{-1}(v)|$ is called the valence of the vertex v.

A graph determines a 1-dimensional CW complex and we say the graph is connected if this CW complex is connected. A genus labeling of a graph is a function $g \colon V \to \mathbb{Z}_{\geq 0}$. A connected, genus labeled graph is stable if $2g(v) + |a^{-1}(v)| \geq 3$ for every $v \in V$. A leg labeling of a graph is a bijection $\{1, \ldots, n\} \to \mathrm{leg}(\gamma)$, for the appropriate n. A modular graph is a stable graph along with a leg labeling. The total genus of a modular graph is $g(\gamma) := \beta_1(\gamma) + \sum_v g(v)$, where β_1 denotes the first Betti number of the associated CW complex. The type of a modular graph is the pair of non-negative integers $(g(\gamma), |\mathrm{leg}(\gamma)|)$. See Figure 2.

An isomorphism of abstract graphs is a pair of bijections between the respective vertices and flags which commutes with the adjacency and involution maps. An isomorphism
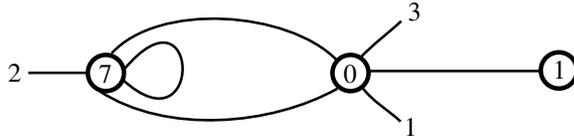
Figure 2. A modular graph of type $(10, 3)$ having 3 vertices, 11 flags, 4 edges, 1 loop and 3 legs.

of modular graphs is an isomorphism of abstract graphs which preserves the leg and genus labeling. If γ is a modular graph we write $\mathrm{Aut}(\gamma)$ for the group of automorphisms of γ viewed as a modular graph. For non-negative integers g and n with $2g + n \geq 3$ we fix once and for all a skeleton of the groupoid of graphs of type (g, n) and call it $\mathsf{Gr}(g, n)$.

A subgraph of a graph is a pair of subsets of V and F closed under a and ι. A nest N on a graph γ is a proper, connected subgraph of γ containing no legs. Given a nest N on a graph γ we define two auxiliary graphs. The modular graph γ/N is formed by contracting the edges of N and their adjacent vertices to a single vertex labeled to preserve the total genus of the graph. We call this new vertex N. The graph $\hat{N}$ is the graph formed by adding as legs all flags of $F(\gamma) \setminus F(N)$ which are adjacent to vertices in N. Note the legs of $\hat{N}$ are not numerically labeled, but are in bijective correspondence with the flags adjacent to the vertex N in γ/N.

Define $\mathcal{K}^*(\gamma)$ to be the top exterior power of the set $E(\gamma)$. Explicitly, $\mathcal{K}^*(\gamma)$ is a 1-dimensional vector space concentrated in degree $|E(\gamma)|$ which carries an alternating action of the group $S_{E(\gamma)}$. Define $\mathcal{K}(\gamma)$ to be the linear dual of $\mathcal{K}^*(\gamma)$. Observe $\mathcal{K}(\gamma)$ is naturally identified with $\mathcal{K}^*(\gamma)$, but is concentrated in degree $-|E(\gamma)|$. As we are using homological grading conventions, these definitions are opposite to the conventions in [4]. A mod 2 order on the edges of a graph is defined to be a choice of unit vector in $\mathcal{K}^*(\gamma)$ or equivalently $\mathcal{K}(\gamma)$.

2.2. Weak modular operads

A stable $\mathbb{S}$-module A is a family of differential graded S_n representations $A(g, n)$, indexed over pairs of non-negative integers (g, n) satisfying $2g + n \geq 3$. Given such an A we extend $A(g, -)$ to a functor valued in all finite sets via left Kan extension. In particular, if X is a finite set, $A(g, X)$ is non-canonically isomorphic to $A(g, |X|)$. Given a stable $\mathbb{S}$-module A and a modular graph γ with vertex v we define $A(v) := A(g(v), a^{-1}(v))$ and define $A(\gamma) := \bigotimes_{v \in V(\gamma)} A(v)$. Note $A(\gamma)$ inherits an action of the group $\mathrm{Aut}(\gamma)$.

Definition 2.1. Let A be a stable $\mathbb{S}$-module. A weak modular operad structure on A is a collection of degree -1 operations

$$\mathcal{K}(\gamma)^* \otimes A(\gamma) \xrightarrow{\mu_\gamma} A(g, n), \tag{2.1}$$

for each $\gamma \in \mathsf{Gr}(g, n)$ and for all (g, n) which are S_n-equivariant, $\mathrm{Aut}(\gamma)$-coinvariant and which satisfy the differential condition $\sum \mu_{\gamma/N} \circ_N \mu_{\hat{N}} = 0$.

Here $\mu_{\gamma/N} \circ_N \mu_{\widehat{N}}$ is the composition of operations given by plugging the output of $\mu_{\widehat{N}}$ into the vertex N of γ/N and the sum is over all nests N on γ. The factors of $\mathcal{K}$ are composed by pulling back the wedge product isomorphism

$$\mathcal{K}(\gamma/N) \otimes \mathcal{K}(\widehat{N}) \xrightarrow{\wedge} \mathcal{K}(\gamma),$$

which in turn encodes signs in the differential. We refer to [8, Proposition 3.21] for additional details. Classical modular operads are weak modular operads for which $\mu_\gamma = 0$ if $|E(\gamma)| > 1$. In this case, the differential condition collapses to the associativity of composition of these one-edged operations.

2.3. Co-Feynman transform

Let A be a weak modular operad. The co-Feynman transform of A, denoted $\mathsf{B}(A)$ is defined to be its bar construction viewed as an algebra over the Koszul resolution of the groupoid colored operad encoding modular operads [8]. Explicitly this means $\mathsf{B}(A)$ is the dg $\mathcal{K}$-modular co-operad given by

$$\mathsf{B}(A)(g,n) = \bigoplus_{\gamma \in \mathsf{Gr}(g,n)} \mathcal{K}^*(\gamma) \otimes_{\mathrm{Aut}(\gamma)} A(\gamma).$$

The $\mathcal{K}$-modular co-operad structure is co-free and specified by decomposition maps

$$\nu_\gamma \colon \mathsf{B}(A)(g,n) \to \mathcal{K}^*(\gamma) \otimes_{\mathrm{Aut}(\gamma)} \mathsf{B}(A)(\gamma)$$

for each $\gamma \in \mathsf{Gr}(g,n)$. These decomposition maps are defined on the summand of the source corresponding to a graph γ' by summing over all ways to add a single layer of nests to it such that $\gamma'/\bigsqcup N_i = \gamma$.

The weak $\mathcal{K}$-modular operad structure maps on A induce a degree -1 map of stable $\mathbb{S}$-modules $\mu \colon \mathsf{B}(A) \to A$. This map induces a differential $\partial \colon \mathsf{B}(A)(g,n) \to \mathsf{B}(A)(g,n)$. To describe ∂ it is sufficient to indicate its composite with projection π_γ to a summand indexed by a (g,n)-graph γ, which is defined by asserting that the following diagram commutes:

$$
\begin{array}{ccc}
\mathsf{B}(A)(g,n) & \xrightarrow{\ \ \partial\ \ } & \mathsf{B}(A)(g,n) \\
\downarrow{\scriptstyle \nu_\gamma} & & \downarrow{\scriptstyle \pi_\gamma} \\
\mathsf{B}(A)(\gamma) \otimes_{\mathrm{Aut}(\gamma)} \mathcal{K}^*(\gamma) & \xrightarrow{\ \ \overline{\mu}\ \ } & A(\gamma) \otimes_{\mathrm{Aut}(\gamma)} \mathcal{K}^*(\gamma).
\end{array}
\tag{2.2}
$$

Here $\overline{\mu}$ is the extension of μ by the Leibniz rule, composed with $- \otimes_{\mathrm{Aut}(\gamma)} \mathcal{K}^*(\gamma)$. In particular, $\overline{\mu}$ is supported on nested graphs in $\mathsf{B}(A)(\gamma)$ for which exactly one vertex of γ is not labeled by a corolla.

The co-Feynman transform of a weak modular operad has a bigrading

$$\mathsf{B}(A)(g,n) = \bigoplus_{r \geq 0, s \in \mathbb{Z}} \mathsf{B}(A)(g,n)^{r,s} \tag{2.3}$$

given by the number of edges r on the graph indexing a summand and s is the sum of the internal degrees of the vertex labels. With respect to this bigrading, the co-Feynman transform differential is supported on

$$\mathsf{B}(A)(g,n)^{r,s} \xrightarrow{\partial} \bigoplus_{0 \le e \le r} \mathsf{B}(A)(g,n)^{r-e,s+e-1}. \tag{2.4}$$

In particular, on each bigraded component the differential splits as $\bigoplus_e \partial_e$ corresponding to those terms which contract e edges, under the convention that contracting 0 edges means to apply the internal differential d_A.

Definition 2.2. When each $A(g,n)$ is finite dimensional in each graded component we define the (weak) Feynman transform ft to be the linear dual of the co-Feynman transform. In particular, the Feynman transform of a weak modular operad is a $\mathscr{K}$-modular operad.

This definition generalizes the usual Feynman transform [4], under the definition that a (strict) modular operad is a weak modular operad for which $\mu_\gamma = 0$ whenever γ has more than one edge.

Remark 2.3. Let (A, μ) be a weak modular operad with vanishing internal differentials $d_A = 0$. Then A along with only its one-edged contractions forms a (strict) modular operad. In this case, ∂_1 is itself square zero; it is the linear dual of the differential in the usual Feynman transform of this (strict) modular operad.

2.4. Comparing ft(Com) and GC₂

As above, Com is the operad encoding commutative algebras, viewed as a modular operad by extension to higher genus by 0. It follows that

$$\mathsf{ft}(\mathsf{Com})(g,n) \cong \bigoplus \mathscr{K}(\gamma)_{\mathrm{Aut}(\gamma)},$$

where the sum is taken over all $\gamma \in \mathrm{Gr}(g,n)$ for which $g(v) = 0$ for all $v \in V(\gamma)$.

Since all vertices have genus 0, no differential terms expand loops, and so the sum over those graphs with no loops is a subcomplex $\mathsf{NL}(g,n) \subset \mathsf{ft}(\mathsf{Com})(g,n)$. For each $g \ge 3$ we define the chain complex $\mathsf{GC}_2^g = (\Sigma^{2g} \mathsf{NL}(g,0))$. We then define $\mathsf{GC}_2 = \prod_g \mathsf{GC}_2^g$. Note that [9] uses cohomological conventions, so to recover exactly his GC_2, one must take the cochain complex associated to the chain complex which we have called GC_2 by negating the indices: $(V_i)^{\mathrm{op}} = V_{-i}$.

In particular, a homogeneous element in GC_2 is specified by a scalar multiple of an isomorphism class of a connected graph with no loops, all of whose vertices have valence 3 or greater, along with a mod 2 order on the set of edges. The degree of such a vector is $2g - E = E - 2V + 2$ and the differential is given by a sum of edge expansions. The alternating action on the edges of a representative has the effect that any isomorphism class of a graph with parallel edges vanishes.

2.5. Wheel graphs

As above, we define the wheel graph ω_{2j+1} by connecting a new vertex to all the edges in a $2j + 1$-gon. As a convention, the mod 2 edge order is chosen to coincide with [9, Proposition 9.1], see Figure 1.

With this convention the wheel graph ω_{2j+1} may be viewed as a degree $4j + 2$ element of $\mathsf{B}(\mathsf{Com})(2j + 1, 0)$, having each vertex labeled by the commutative product of suitable valence. Note that contracting any edge in a wheel graph produces parallel edges with commutative labels and hence:

Lemma 2.4. *The wheel graph ω_{2j+1} is a cycle in the complex* $\mathsf{B}(\mathsf{Com})(2j + 1, 0)$.

We remark that the vector spaces $\mathsf{B}(\mathsf{Com})(2j + 1, 0)$ and $\mathsf{ft}(\mathsf{Com})(2j + 1, 0)$ are canonically isomorphic, via $\mathsf{Com}(g, n) \cong \mathsf{Com}(g, n)^*$, and so the wheel graph with the above conventions also specifies an element $\omega^*_{2j+1} \in \mathsf{ft}(\mathsf{Com})(2j + 1, 0)$. Lemma 2.4 shows that a vector for which ω^*_{2j+1} appears with non-zero coefficient is not a boundary in $\mathsf{ft}(\mathsf{Com})(2j + 1, 0)$.

3. Higher operations on Lie graph homology

3.1. Lie graph homology

Following [4], after [7], we now define Lie graph homology to be the modular operad $H_*(\mathsf{ft}(\Sigma s^{-1}\mathsf{Lie}))$, where s denotes the (cyclic) operadic suspension and Σ denotes a shift up in degree. In particular, extension by zero makes $\Sigma s^{-1}\mathsf{Lie}$ a $\mathfrak{K}$-modular operad, and its Feynman transform is a modular operad. Following [2], we will denote this modular operad by $H_*(\Gamma)$ and use the notation $H_*(\Gamma_{g,n}) = H_*(\Gamma)(g, n)$.

For our purposes, however, we will only require the following partial characterization of this modular operad in genus ≤ 1.

Lemma 3.1 ([2]). *The graded vector spaces $H_*(\Gamma_{g,n})$ form a modular operad with the following properties:*

(1) *The underlying cyclic operad $H_*(\Gamma_{0,-})$ is canonically isomorphic to the commutative (cyclic) operad.*

(2) *As an S_n module,*

$$H_i(\Gamma_{1,n}) \cong \begin{cases} V_{n-i,1^i} & \text{if } i \text{ is even and } 0 \leq i \leq n - 1, \\ 0 & \text{else.} \end{cases}$$

(3) *The modular operadic composition map*

$$H_0(\Gamma_{0,3}) \otimes H_d(\Gamma_{1,n}) \to H_d(\Gamma_{1,n+1})$$

is injective.

Notice that $H_{2j}(\Gamma_{1,2j+1})$ is the alternating representation. We fix generators $\alpha_{2j+1} \in H_{2j}(\Gamma_{1,2j+1})$ once and for all. We will also write $m_t \in \mathsf{Com}(t) = H_0(\Gamma_{0,t+1})$ for the commutative product.

We may iterate the modular operadic composition map above to form

$$H_0(\Gamma_{0,t+1}) \otimes H_d(\Gamma_{1,n}) \xrightarrow{\circ_e} H_d(\Gamma_{1,n+t-1}). \tag{3.1}$$

Here, $\circ_e$ is the modular operadic composition which corresponds to gluing along a tree with one edge adjacent to vertices of type $(0, t+1)$ and $(1, n)$. To be precise, this composition is only well defined after a choice of labeling of the $t + n - 1$ legs of the tree by the set $\{1, \ldots, t + n - 1\}$. We fix the convention that $\circ_e$ corresponds to labeling the genus 0 vertex by $\{1, \ldots, t\}$ and the genus 1 vertex by $\{t + 1, \ldots, t + n - 1\}$. Observe that repeated application of Lemma 3.1 (3) implies that $m_t \circ_e \alpha_{2j+1} \neq 0$.

We now calculate relations between compositions in the modular operad $H_*(\Gamma)$.

Lemma 3.2. *The non-zero homology class $m_t \circ_e \alpha_{2j+1} \in H_{2j}(\Gamma_{1,t+2j})$ satisfies*

$$\sum_{i=1}^{t+1}(i, t+1)(m_t \circ_e \alpha_{2j+1}) = 0,$$

where $(i, t+1) \in S_{t+2j}$ denotes a transposition.

Proof. The map $\circ_e$ of equation (3.1) is $S_t \times S_{2j}$ equivariant, where the target carries the restricted action along the standard inclusion $S_t \times S_{2j} \hookrightarrow S_{t+2j}$. By abuse of notation we also write $\circ_e$ for the adjoint,

$$\mathrm{Ind}_{S_t \times S_{2j}}^{S_{t+2j}}(H_0(\Gamma_{0,t+1}) \otimes H_{2j}(\Gamma_{1,2j+1})) \xrightarrow{\circ_e} H_{2j}(\Gamma_{1,t+2j}). \tag{3.2}$$

Using the Littlewood–Richardson rule ([3, p. 456]) we compute the irreducible decomposition of the source of equation (3.2) to be $V_{t+1,1^{2j-1}} \oplus V_{t,1^{2j}}$.

By Lemma 3.1, the target of $\circ_e$ is an irreducible S_{t+2j}-representation of type $V_{t,1^{2j}}$. Thus, any vector z for which $\mathbb{Q}[S_{t+2j}] \cdot z \cong V_{t+1,1^{2j-1}}$ must be in the kernel of $\circ_e$. To produce such a vector z we embed the problem in the group ring. That is, consider the $S_t \times S_{2j}$ equivariant map $H_0(\Gamma_{0,t+1}) \otimes H_{2j}(\Gamma_{1,2j+1}) \to \mathbb{Q}[S_{t+2j}]$ defined by

$$m_t \otimes \alpha_{2j+1} \mapsto y := \left(\sum_{\sigma \in S_t} \sigma\right)\left(\sum_{\sigma \in S_{\{t+1,\ldots,t+2j\}}} \mathrm{sgn}(\sigma)\sigma\right). \tag{3.3}$$

Form the Young diagram of shape $t + 1, 1^{2j-1}$ labeled numerically right to left, then down. So $t + 1$ is in the pivot position. Call this tableau ζ. Its associated Young symmetrizer c_ζ is

$$c_\zeta = (\mathrm{id} + (1, t+1) + (2, t+1) + \cdots + (t, t+1))y.$$

Since y is in the image of the map defined in equation (3.3), c_ζ is in the image of the adjoint morphism from the induced representation. By construction c_ζ generates a copy of $V_{t+1,1^{2j-1}}$ under left multiplication by $\mathbb{Q}[S_{t+2j}]$, hence so does

$$z := (\mathrm{id} + (1, t+1) + (2, t+1) + \cdots + (t, t+1))(m_t \otimes \alpha_{2j+1}).$$

Thus, this z is in the kernel of the $\circ_e$ in equation (3.2) as desired. Moreover, the kernel is spanned by the S_{t+2j} orbit of this z. ∎

3.2. The weak semi-classical modular operad $\mathcal{H}_{\mathrm{Lie}}$

In this section, we endow the genus ≤ 1 spaces of $H_*(\Gamma_{g,n})$ with higher operations. The result will be a weak modular operad which we denote $(\mathcal{H}_{\mathrm{Lie}}, \mu)$.

As stable $\mathbb{S}$-modules we define

$$\mathcal{H}_{\mathrm{Lie}}(g,n) = \begin{cases} H_*(\Gamma_{g,n}) & \text{if } g < 2, \\ 0 & \text{if } g \geq 2. \end{cases}$$

The operations μ_γ are defined as follows. We define $\mu_\gamma = 0$ unless one of the following two mutually exclusive conditions is met:

- γ has genus < 2 and only one edge, or
- γ has genus 1, and the underlying leg free graph of γ is a $2j+1$-gon, for some $j \geq 1$.

In the first case, we define μ_γ to be the operation induced by the modular operad structure on $H_*(\Gamma)$. In the second case, we proceed as follows.

Let p_{2j+1} be the standard trivalent $2j+1$-gon. By this we mean the modular graph formed by attaching a leg to each vertex of a $2j+1$-gon. The vertices have genus label 0. The leg labels are in the dihedral order, and we give this modular graph an edge ordering $e_1 < \cdots < e_{2j+1}$ such that the i-th edge connects the vertices adjacent to flags i and $i+1$ (mod $2j+1$).

Observe that $\mathcal{H}_{\mathrm{Lie}}(\mathsf{p}_{2j+1}) \cong k$. We define

$$\mu_{\mathsf{p}_{2j+1}} \colon \mathcal{H}_{\mathrm{Lie}}(\mathsf{p}_{2j+1}) \otimes \mathcal{K}(\mathsf{p}_{2j+1}) \to \mathcal{H}_{\mathrm{Lie}}(1, 2j+1)$$

by $\mu_{\mathsf{p}_{2j+1}}(1 \otimes e_1 \wedge \cdots \wedge e_{2j+1}) = \alpha_{2j+1}$.

Lemma 3.3. *The above operations extend to a unique weak modular operad structure on $\mathcal{H}_{\mathrm{Lie}}$.*

Proof. We refer to Definition 2.1. The S_n equivariance defines $\mu_{\hat{\mathsf{p}}}$ for any other edge ordered trivalent polygon $\hat{\mathsf{p}}$. One easily checks that this definition is not over-prescribed, since symmetries of a $2j+1$-gon induce permutations of the edges and the legs which have matching parity.

We then want to show that if P is a non-trivalent graph whose underlying leg free graph is a $2j+1$-gon, that μ_{P} is determined by the above operations. For this we induct on the number of non-trivalent vertices. First suppose that this number is 1, at a vertex v of $\mathsf{P}_1 := \mathsf{P}$.
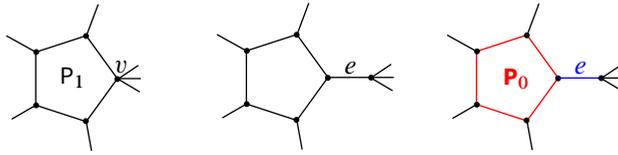
Figure 3. The Massey project associated to $\mathsf{P} = \mathsf{P}_1$ is defined by expanding the edge e, applying the Massey product μ_{P_0} and then the modular operadic composition $\circ_e$. Leg labels are suppressed in the figure.

Let γ be the graph formed by blowing up v to separate the two flags which belong to edges of the polygonal subgraph from the rest of the flags at v; see Figure 3. In particular, γ has $2j + 2$ edges, $2j + 1$ of which form a polygon P_0 with trivalent vertices. Let N be a nest on γ. By the above definition, the composition $\mu_{\gamma/N} \circ_N \mu_{\widehat{N}}$ will be zero unless $N = \{e\}$ or $\widehat{N} = \mathsf{P}_0$ (pictured blue and red in Figure 3).

Thus, applying the differential condition of Definition 2.1 with internal differential $d = 0$, it must be the case that

$$0 = \mu_{\mathsf{P}_1} \circ \mu_e + \mu_{\gamma/\mathsf{P}_0} \circ \mu_{\mathsf{P}_0}. \tag{3.4}$$

Here we write $\circ_e = \mu_e$ for the modular operadic composition map which contracts the edge e. But this map $\mu_e \colon \mathcal{H}_{\mathrm{Lie}}(\gamma) \to \mathcal{H}_{\mathrm{Lie}}(\mathsf{P})$ is simply a composition in the commutative operad and so is an isomorphism, $\mathcal{H}_{\mathrm{Lie}}(\gamma) \xrightarrow{\cong} \mathcal{H}_{\mathrm{Lie}}(\mathsf{P})$. Thus, equation (3.4) uniquely determines μ_{P_1}.

For the induction step we repeat the above argument, reducing the number of non-trivalent vertices by one at each step. The fact that the operation is independent of the choice of order of the non-trivalent vertex follows from the fact that $H_*(\Gamma)$ is a (strong) modular operad.

Thus, there is at most one weak modular structure on $\mathcal{H}_{\mathrm{Lie}}$ extending the above operations. Conversely the maps defined above have the requisite degree and equivariance, so it remains to show that the differential condition

$$\sum_{N \text{ on } \gamma} \mu_{\gamma/N} \circ_N \mu_{\widehat{N}} = 0 \tag{3.5}$$

is satisfied for every modular graph γ.

If γ has total genus ≥ 2 or has fewer than 2 edges, all terms in equation (3.5) are 0 by definition so there is nothing to check. If γ has exactly two edges, then equation (3.5) is merely the associativity axiom for the (strict) modular operad structure on $H_*(\Gamma)$.

So we now assume γ has more than two edges and has total genus ≤ 1. Let N be a nest on γ. Either N or γ/N has two or more edges, thus for $\mu(\gamma/N) \circ_N \mu(\widehat{N})$ to be non-zero requires that one of γ/N and N is a polygon with an odd number of sides and the other must be a lone edge. In particular, γ must have an even number of edges, have first Betti number 1, and hence only genus 0 vertices. There are two cases for such a γ. Either its

lone cycle has an odd number of edges, in which case there is an additional edge pointing outward, or its lone cycle has an even number of edges, in which case the leg free graph underlying γ must be a $2n$-gon.

The first case follows as above. In particular, suppose γ has $2n$ edges and its lone cycle is of length $2n - 1$. Let e be the unique edge of γ which is not in the cycle. This edge is connected to vertices v and w, with v belonging to the polygon and w not. If v is a trivalent vertex in γ, then the differential condition was verified above. If v is not trivalent, the differential condition is verified by a double iteration of equation (3.4).

So we now consider the second case. Suppose $\gamma = \mathsf{P}_t$ is a polygon with $2n \geq 4$ sides whose vertices have genus 0, and which has $t \geq 2n$ legs labeled $\{1, \ldots, t\}$, with at least one leg at each vertex. The only non-zero terms in the differential condition are given by first choosing N to be a single edge. Thus the differential condition in this case is $0 = \sum_{e \in \mathsf{P}_t} \mu_{\mathsf{P}_t/e} \circ \mu_e$. This condition can be rephrased as saying the following composite vanishes:

$$
\begin{array}{c}
\mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(1, t)^{2n,0} \supset \mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_t) \otimes \mathcal{K}^*(\mathsf{P}_t) \\
\downarrow {\scriptstyle \partial_1} \\
\mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(1, t)^{2n-1,0} \supset \bigoplus_{e \in \mathsf{P}_t} \mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_t/e) \otimes \mathcal{K}^*(\mathsf{P}_t/e) \xrightarrow{\oplus_e \mu_{\mathsf{P}_t/e}} H_{2n-2}(\Gamma_{1,t}).
\end{array}
\tag{3.6}
$$

Let us first consider the case $t = 2n$. Since the maps in diagram (3.6) are S_{2n}-equivariant it is sufficient to consider the case that the legs of P_{2n} are labeled in the dihedral order. The source of this diagram is closed under the $\mathbb{Z}_{2n}$ action restricted along the standard inclusion $\mathbb{Z}_{2n} \cong \langle \sigma := (12 \ldots 2n) \rangle \subset S_{2n}$. Denoting this 1-dimensional $\mathbb{Z}_{2n}$ representation $\mathcal{E} := \mathrm{Res}^{S_{2n}}_{\mathbb{Z}_{2n}}(\mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_{2n}) \otimes \mathcal{K}^*(\mathsf{P}_{2n}))$, we compute its character $\chi_{\mathcal{E}}(\sigma) = \mathrm{sgn}(\sigma) = -1$. This in turn determines the isomorphism type of the irreducible $\mathbb{Z}_{2n}$-representation $\mathcal{E}$; namely the $2n$ cycle σ acts by -1.

To show the composite in diagram (3.6) is 0, it thus suffices to show that there are no copies of this irreducible representation appearing in $\mathrm{Res}^{S_{2n}}_{\mathbb{Z}_{2n}}(H_{2n-2}(\Gamma_{1,2n}))$. The irreducible representations of $\mathbb{Z}_{2n}$ over the algebraic closure $\overline{\mathbb{Q}}$ are all 1-dimensional and are given by letting σ act by multiplication of a root of $x^{2n} - 1$. Let $\omega = e^{i\pi/n}$ and write W_i for the irreducible representation corresponding to multiplication by ω^i. Let $V_{2n} \oplus V_{2n-1,1}$ be the permutation representation of S_{2n}. One easily calculates its restriction

$$
\mathrm{Res}^{S_{2n}}_{\mathbb{Z}_{2n}}(V_{2n} \oplus V_{2n-1,1}) = \bigoplus_{i=0}^{2n-1} W_i.
$$

Since $V_{2n-1,1} \otimes V_{1^{2n}} \cong V_{2,1^{2n-2}}$, the number of copies of $\mathcal{E} \cong W_n$ appearing in $\mathrm{Res}^{S_{2n}}_{\mathbb{Z}_{2n}}(V_{2,1^{2n-2}})$ is the number of copies of W_0 appearing in $\mathrm{Res}^{S_{2n}}_{\mathbb{Z}_{2n}}(V_{2n-1,1})$ which is $1 - 1 = 0$, as desired.

Whence the case $t = 2n$. Now suppose $t > 2n$. Choose an ordering of the vertices of P compatible with the dihedral ordering and let T_i be the set of flags adjacent to the

i-th vertex. In particular, $\{1, \ldots, t\} = \bigsqcup T_i$. Let $\widehat{T}_i$ be the set T_i along with an added basepoint, called the root.

Consider the following diagram:

$$
\begin{array}{ccc}
\left(\mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_{2n}) \otimes \mathcal{K}^*(\mathsf{P}_{2n}) \right) \otimes \left(\bigotimes_{i=1}^{2n} \mathcal{H}_{\mathrm{Lie}}(0, \widehat{T}_i) \right) & \xrightarrow{\ \cong\ } & \mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_t) \otimes \mathcal{K}^*(\mathsf{P}_t) \\[1mm]
\Big\downarrow{\scriptstyle \partial_1 \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \partial_1} \\[1mm]
\bigoplus_{e \in \mathsf{P}_{2n}} \mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_{2n}/e) \otimes \mathcal{K}^*(\mathsf{P}_{2n}/e) \otimes \left(\bigotimes_{i=1}^{2n} \mathcal{H}_{\mathrm{Lie}}(0, \widehat{T}_i) \right) & \xrightarrow{\ \cong\ } & \bigoplus_{e \in \mathsf{P}_t} \mathcal{H}_{\mathrm{Lie}}(\mathsf{P}_t/e) \otimes \mathcal{K}^*(\mathsf{P}_t/e) \\[1mm]
\Big\downarrow{\scriptstyle \oplus_e \, \mu_{\mathsf{P}_{2n}/e} \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \oplus_e \, \mu_{\mathsf{P}_t/e}} \\[1mm]
H_{2n-2}(\Gamma_{1,2n}) \otimes \left(\bigotimes_i H_0(\Gamma_{0, \widehat{T}_i}) \right) & \xrightarrow{\ \prod \circ_i\ } & H_{2n-2}(\Gamma_{1,t}).
\end{array}
$$

The right hand side of this diagram is exactly diagram (3.6). The left hand side of this diagram is diagram (3.6) in the prior case $t = 2n$, then tensored with the 1-dimensional, trivial $\mathbb{Z}_{2n}$-representation $\bigotimes_{i=1}^{2n} \mathcal{H}_{\mathrm{Lie}}(0, \widehat{T}_i)$. The horizontal arrows are contractions using the modular operad structure along the graph identifying the root of $\widehat{T}_i$ with leg i of P_{2n}.

The commutativity of the top square can be seen just by looking at each summand – both routes give the same graph with commutative labels. The commutativity of the bottom square follows immediately from the definition of the Massey product associated to each P_t/e.

Since the left hand side of the diagram vanishes, by the $t = 2n$ case considered above, and since the top horizontal arrow is an isomorphism, the right hand side of the diagram also vanishes, as desired. ∎

Viewing Com as a modular operad, as in Section 2.4, we see immediately that there is a level-wise surjective morphism of weak modular operads $\mathcal{H}_{\mathrm{Lie}} \to \mathsf{Com}$. Taking the weak Feynman transform, we have a level-wise injective morphism of $\mathcal{K}$-twisted modular operads $\mathsf{ft}(\mathsf{Com}) \hookrightarrow \mathsf{ft}(\mathcal{H}_{\mathrm{Lie}})$.

4. Proof of the main results

Let us regard the wreath product $S_2 \wr S_n$ as follows. Its underlying set is $S_2^n \times S_n$. To an element $((\alpha_1, \ldots, \alpha_n), \tau)$ in this set we associate a permutation in S_{2n} by first acting by τ on the ordered set $\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}$ of size n and then acting by α_i on the ordered set $\{2\tau(i) - 1, 2\tau(i)\}$ for each i. This defines an injective map of sets

$$ S_2^{\times n} \times S_n \hookrightarrow S_{2n}, $$

and $S_2 \wr S_n$ carries the unique group structure for which this map is a homomorphism. The wreath product $S_2 \wr S_n$ has a 1-dimensional representation given by letting an element $((\alpha_1, \ldots, \alpha_n), \tau)$ act by multiplication by $\mathrm{sgn}(\tau)$. We call this representation L_n (L stands for loops).

In what follows, we abuse notation by regarding the sequence of injections

$$(S_2)^{r-1} \hookrightarrow S_2 \wr S_{r-1} \hookrightarrow S_2 \wr S_{r-1} \times S_{n-r+2} \hookrightarrow S_{2r-2} \times S_{n-r+2} \hookrightarrow S_{n+r}$$

as a sequence of subgroups. We write Res_H^G for the restriction of a representation of a group G to a representation of a subgroup H.

Lemma 4.1. *Let $2 \leq r < n$ be integers.*

- *The irreducible decomposition of $\mathrm{Res}_{(S_2 \wr S_{r-1}) \times S_{n-r+2}}^{S_{n+r}}(V_{r,1^n})$ contains a unique summand of the form $L_{r-1} \boxtimes V_\beta$. It is of the form $L_{r-1} \boxtimes V_{1^{n-r+2}}$.*

- *The irreducible decomposition of $\mathrm{Res}_{(S_2 \wr S_q) \times S_{n-2q+r}}^{S_{n+r}}(V_{r,1^n})$ has no summand of the form $L_q \boxtimes -$ for $q \geq r$.*

Proof. Fix $q \geq r-1$. The number of copies of a summand $V_\alpha \boxtimes V_\beta$ appearing in

$$\mathrm{Res}_{S_{2q} \times S_{n+r-2q}}^{S_{n+r}}(V_{r,1^n}) \tag{4.1}$$

is computed via the Littlewood–Richardson rule [3]. In this case, because $V_{r,1^n}$ is a hook, each α and β appearing with non-zero coefficient must also be hooks. If $q > r-1$, the number of summands of the form $V_{q+1,1^{q-1}} \boxtimes V_\beta$ appearing in the decomposition of the representation in equation (4.1) is zero. If $q = r-1$, there is a unique summand of the form $V_{q+1,1^{q-1}} \boxtimes V_\beta$ appearing in the decomposition of the representation in equation (4.1) is zero. It is of the form $V_{r,1^{r-2}} \boxtimes V_{1^{n-r+2}}$.

It now remains to analyze the irreducible decomposition of $\mathrm{Res}_{S_2 \wr S_q}^{S_{2q}}$ of a hook. The needed calculation, modulo Frobenius reciprocity, is explicitly presented in [5, Proposition 2.3′ (iv)] (see also [6]) which says that the number of copies of L_q appearing in the restriction of a hook $\mathrm{Res}_{S_2 \wr S_q}^{S_{2q}}(V_{x,1^y})$ is 0 unless $x = q+1$ and $y = q-1$, in which case it is 1. This completes the proof.

While it was convenient that the calculation we needed was available in the literature, there is an argument internal to this article which may also be used to prove this. One first uses the Pieri rule to see that a summand $V_\alpha \boxtimes V_\beta$ appearing in equation (4.1) has an $(S_2)^q \subset S_2 \wr S_q \subset S_{2q}$ invariant subspace if and only if $q = r-1$, $\alpha = (r, 2q-r)$ and $\beta = (1^{n+r-2q})$. Any summand of type $L_q \boxtimes -$ would restrict to an $(S_2)^q$ invariant subspace, which establishes the second statement. On the other hand, when $q = r-1$ there is a unique $(S_2)^{r-1}$ invariant subspace in $V_{r,1^{r-2}}$. Since it is unique, it must contain the image of the gluing operation which grafts $H_0(\Gamma_{0,3})$ onto each input of $H_{r-2}(\Gamma_{1,r-1}) \cong V_{1^{r-1}}$, landing in $H_{r-2}(\Gamma_{1,2r-2}) \cong V_{r,1^{r-2}}$. This image is non-zero by Lemma 3.1. Since we are gluing on to the alternating representation $H_{r-2}(\Gamma_{1,r-1})$, it must be the case that this $(S_2)^{r-1}$ invariant subspace lifts to a $S_2 \wr S_{r-1}$ representation which is alternating with respect to the S_{r-1} factor, i.e., to a copy of L_{r-1}, which establishes the first statement. ∎

Corollary 4.2. *Let* $\gamma \in \mathsf{Gr}(g, n)$ *with a vertex* v *of genus* $g(v) = 1$ *and with valence* $|a^{-1}(v)| = m$. *Consider a homogeneous element*

$$[x] \in \mathcal{H}_{\mathrm{Lie}}(\gamma) \otimes_{\mathrm{Aut}(\gamma)} \mathcal{R}^*(\gamma) \subset \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(g, n)$$

whose vertex v *carries a label in* $\mathcal{H}_{\mathrm{Lie}}(v) \cong H_*(\Gamma_{1,m})$ *of degree* i. *If* v *is adjacent to* $m - i$ *or more loops then* $[x] = 0$.

Proof. Let q be the number of loops adjacent to v. Then $\mathrm{Aut}(\gamma)$ contains a subgroup isomorphic to $S_2 \wr S_q$ generated by transposing the pair of flags in a loop and permuting the set of loops. This subgroup acts on $H_i(\Gamma_{1,a^{-1}(v)}) \otimes \mathcal{R}^*(\gamma)$ and since

$$H_i(\Gamma_{1,a^{-1}(v)}) \cong V_{m-i,1^i},$$

the invariants of this action correspond to the copies of L_q appearing in $\mathrm{Res}^{S_{2q}}_{S_2 \wr S_q}(V_{m-i,1^i})$. From the proof of Lemma 4.1 we see that the number of such copies is 0 when $q \geq m - i$. Any such $\mathrm{Aut}(\gamma)$-invariant element $[x]$ would require an $S_2 \wr S_q$ invariant element of $H_i(\Gamma_{1,a^{-1}(v)}) \otimes \mathcal{R}^*(\gamma)$ labeling v. Since there are no such elements when $q \geq m - i$, the $\mathrm{Aut}(\gamma)$-coinvariants vanish. ∎

Definition 4.3. For a non-negative integer j we define $\theta_{2j+1} \in \mathsf{Gr}(2j + 1, 0)$ as follows. It has two vertices; one of genus 0, call it v_0, and one of genus 1, call it v_1. It has $2j + 1$ edges, 3 of which connect the two vertices and the remaining $2j - 2$ of which are loops connected to the vertex of genus 1. See Figure 4.

Recall that $\mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(g, n)^{r,s}$ denotes the bigraded component of the chain complex $\mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(g, n)$ having r edges and internal degree s.

Lemma 4.4. *The subspace*

$$(\mathcal{R}^*(\theta_{2j+1}) \otimes_{\mathrm{Aut}(\theta_{2j+1})} \mathcal{H}_{\mathrm{Lie}}(\theta_{2j+1}))^{2j+1,2j} \subset \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(2j + 1, 0)^{2j+1,2j}$$

is 1-*dimensional.*
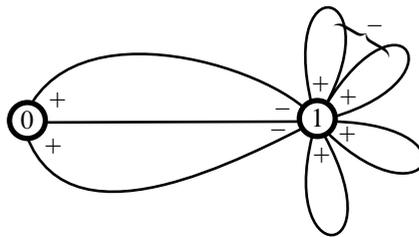
Figure 4. The case $j = 3$. The graph θ_7 (pictured) underlying the element $\beta_7 \in \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(7, 0)$. The signs indicate the effect of transposing at the vertices. The bracket labeled by a minus sign indicates that the block permutation exchanging pairs of flags on two loops gives a minus sign. The label of the genus 1 vertex is (up to scalar multiple) the unique class for which $\mathrm{Aut}(\theta_7)$ acts by such signed multiples.

Proof. Observe that $\mathrm{Aut}(\theta_{2j+1}) \cong S_3 \times (S_2 \wr S_{2j-2})$; the S_3 permutes the non-loop edges, the factors of S_2 transpose the flags in a loop and the S_{2j-2} permutes the loop edges. As an $\mathrm{Aut}(\theta_{2j+1})$-module, the one-dimensional vector space $\mathcal{K}^*(\theta_{2j+1})$ has representation type isomorphic to $V_{1,1,1} \boxtimes L_{2j-2}$. Thus, the $\mathrm{Aut}(\theta_{2j+1})$ fixed points of $\mathcal{K}^*(\theta_{2j+1}) \otimes \mathcal{H}_{\mathrm{Lie}}(\theta_{2j+1})$ are given by the number of copies of $V_{1,1,1} \boxtimes L_{2j-2}$ in the irreducible decomposition of $V_{2j-1,1^{2j}} \cong \mathcal{H}_{\mathrm{Lie}}(1, a^{-1}(v_1))$. Applying Lemma 4.1 with $r = 2j - 1$ and $n = 2j$, we see there is exactly one such copy. ∎

4.1. Definition of β

We now construct a canonical basis vector spanning

$$(\mathcal{K}^*(\theta_{2j+1}) \otimes_{\mathrm{Aut}(\theta_{2j+1})} \mathcal{H}_{\mathrm{Lie}}(\theta_{2j+1}))^{2j+1,2j}.$$

Define the set $X := a^{-1}(v_1)$. This set is partitioned by the edges of θ_{2j+1} into three blocks of size 1 and $2j - 2$ blocks of size 2. Choose an auxiliary order on the set of edges of θ such that the three non-loop edges are in the first three positions, and choose an order on the flags within each block. This fixes a total order on the set X and hence an isomorphism:

$$H_{2j}(\Gamma_{1,4j-1}) \cong H_{2j}(\Gamma_{1,X}). \tag{4.2}$$

Define $x_{2j+1} \in H_{2j}(\Gamma_{1,X})$ to be the composition

$$(\cdots((\alpha_{2j+1} \circ_{2j+1} m_2) \circ_{2j} m_2) \cdots) \circ_4 m_2 \in H_{2j}(\Gamma_{1,4j-1})$$

composed with this isomorphism. Here, m_2 is the generator of $H_0(\Gamma_{0,3}) = \mathrm{Com}(2)$, and $x_{2j+1} \neq 0$ by Lemma 3.1. Observe that permuting the two flags on m_2 acts by $+$, where-as permuting blocks of the permutation on X by σ is the same as composing with $\sigma \alpha_{2j+1} = \mathrm{sgn}(\sigma)\alpha_{2j+1}$ (by equivariance of the operadic compositions). Therefore, the element x_{2j+1} spans the unique copy of $V_{1,1,1} \boxtimes L_{2j-2}$ in $\mathrm{Res}_{S_3 \times (S_2 \wr S_{2j-2})}^{S_{4j-1}} H_{2j}(\Gamma_{1,X})$, where the S_{4j-1} action is inherited from the isomorphism in equation (4.2). The class $x_{2j+1} \in H_{2j}(\Gamma_{1,X})$ depends on the choice of isomorphism in equation (4.2), but only up to sign.

Define

$$\beta_{2j+1} \in (\mathcal{K}^*(\theta_{2j+1}) \otimes_{\mathrm{Aut}(\theta_{2j+1})} \mathcal{H}_{\mathrm{Lie}}(\theta_{2j+1}))^{2j+1,2j} \tag{4.3}$$

to be the element formed by labeling v_1 with x_{2j+1} and v_0 by m_2 and using the mod 2 edge order induced by the choice above. Observe that β_{2j+1} is independent of the choices made. If we had picked a different edge order the result would differ by two factors of the sign of the corresponding permutation; if we had picked a different loop orientation the result would differ by a transposition of the commutative product.

Here are some features of β_{2j+1} and its underlying graph θ_{2j+1} for particular values of j:

j	total genus	int. deg.	loops	valence of v_1	Rep. type at v_1
1	3	2	0	3	$V_{1,1,1}$
2	5	4	2	7	$V_{3,1^4}$
3	7	6	4	11	$V_{5,1^6}$
j	$2j+1$	$2j$	$2j-2$	$4j-1$	$V_{2j-1,1^{2j}}$

When j is fixed we may abbreviate the notation $\theta := \theta_{2j+1}$; $\beta := \beta_{2j+1}$; $\omega := \omega_{2j+1}$ and so on.

4.2. Analysis of $\partial^{-1}(\beta)$

We define a loop in a stable graph γ to be simple if the vertex v to which the loop is adjacent satisfies $|a^{-1}(v)| = 3$ and $g(v) = 0$.

Definition 4.5. Define $\widehat{\mathsf{Gr}}(g,n) \subset \mathsf{Gr}(g,n)$ to be the subset of graphs which do not contain simple loops. For a weak modular operad A we then define

$$\widehat{\mathsf{B}}(A)(g,n) = \bigoplus_{\gamma \in \widehat{\mathsf{Gr}}(g,n)} A(\gamma) \otimes_{\mathrm{Aut}(\gamma)} \mathcal{K}^*(\gamma) \subset \mathsf{B}(A)(g,n).$$

Lemma 4.6. *The submodule $\widehat{\mathsf{B}}(A) \subset \mathsf{B}(A)$ is closed under the co-Feynman transform differential ∂.*

Proof. Given a modular graph γ with no simple loops, terms in the weak co-Feynman transform differential are indexed by contractions of subgraphs of γ. If such a differential term has a simple loop it must have been created by contracting a subgraph of type $(0, 3)$. However, $\mathsf{Gr}(0, 3)$ consists only of the $(0, 3)$-corolla, so no such contraction is possible. ∎

We remark that since the weak modular operad $\mathcal{H}_{\mathrm{Lie}}$ has internal differential 0, the summand of ∂ which contracts just 1 edge, call it ∂_1, is itself a differential (Remark 2.3). In this case Lemma 4.6 also shows $(\widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}})(g,n), \partial_1)$ is a subcomplex of $(\mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(g,n), \partial_1)$.

Proposition 4.7. *The composition of ∂_1 with projection to the θ_{2j+1} summand of $\widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}})(2j + 1, 0)^{2j+1,2j}$ is zero. In particular, β_{2j+1} does not appear with non-zero coefficient in any ∂_1 boundary.*

Proof. As above, let $X := a^{-1}(v_1)$. The set X is partitioned by the edges of θ_{2j+1} into 3 blocks of size 1 and $2j - 2$ blocks of size 2. Label the blocks of size 1 by a_1, a_2, a_3 and label the elements of each block of size two by l_1^i, l_2^i label, where i indexes the $2j - 2$ loops of θ_{2j-2}. As above, we write $x_{2j+1} \in H_{2j}(\Gamma_{1,X})$ for a basis vector spanning the unique invariant subspace of $\mathrm{Res}^{S_X}_{\mathrm{Aut}(\theta_{2j+1})} H_{2j}(\Gamma_{1,X})$ isomorphic to $V_{1,1,1} \boxtimes L_{2j-2}$. The $S_{\{a_1,a_2,a_3\}}$ action on x_{2j+1} is alternating while each $S_{\{l_1^i,l_2^i\}}$ action on x_{2j+1} is the identity.

Further, write ∂_1^θ for the composition of ∂_1 with projection to the θ_{2j+1} summand of $\widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}})(2j+1,0)$. Define Θ to be the set of graphs γ for which the following composite is non-zero:

$$(\mathcal{R}^*(\gamma) \otimes_{\mathrm{Aut}(\gamma)} \mathcal{H}_{\mathrm{Lie}}(\gamma))^{2j+2,2j} \hookrightarrow \widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}})(2j+1,0)^{2j+2,2j}$$

$$\downarrow \partial_1^\theta$$

$$(\mathcal{R}^*(\theta_{2j+1}) \otimes_{\mathrm{Aut}(\theta_{2j+1})} \mathcal{H}_{\mathrm{Lie}}(\theta_{2j+1}))^{2j+1,2j}.$$

To prove the claim it is sufficient to show that the set Θ is empty. By way of contradiction, suppose $\gamma \in \Theta$. Then γ has an edge $e = \{r, s\}$ such that $\gamma/e \cong \theta$.

Note that the vertex of γ/e corresponding to e must be sent to the vertex v_1 of θ_{2j+1} since the vertex v_0 is of type $(0,3)$ and hence indecomposable. Note also that the edge e of γ cannot be a loop for degree reasons – such a γ would have only genus 0 vertices, and so internal degree $0 \neq 2j + 1$.

Therefore the edge e of γ is adjacent to two vertices whose genera add to 1. Let $Y \cup \{r\}$ be the flags of γ adjacent to the genus 1 vertex and $Z \cup \{s\}$ be the flags of γ adjacent to the genus 0 vertex. The isomorphism $\gamma/E \cong \theta$ specifies a partition

$$X = Y \sqcup Z,$$

along with a linear map

$$H_{2j}(\Gamma_{1,Y\cup\{r\}}) \otimes H_0(\Gamma_{0,Z\cup\{s\}}) \xrightarrow{\circ_e} H_{2j}(\Gamma_{1,X}).$$

This linear map is $S_Y \times S_Z$ equivariant. In particular, $|X| = |Y| + |Z| = 4j - 1$. We say a loop l^i of θ is split (by e) if both $\{l_1^i, l_2^i\} \cap Y$ and $\{l_1^i, l_2^i\} \cap Z$ are non-empty.

Such a differential term being nonvanishing implies the following:

- $|Y \cup r| \geq 2j + 1$ and hence $|Y| \geq 2j$. Thus $3 \leq |Z \cup s| \leq 2j$.

- $|\{a_1, a_2, a_3\} \cap Z| < 2$, since the representation type of $H_0(\Gamma_{0,Z\cup\{s\}})$ is trivial. So we suppose without loss of generality that $a_2, a_3 \in Y$.

- At most one loop is split, since otherwise we would create parallel edges with an alternating action of $\{l_1^i, l_1^h\}$ at one vertex an identity action of $\{l_2^i, l_2^h\}$ at the other vertex which pass equivariantly to the identity action of both $\{l_1^i, l_2^i\}$ and $\{l_1^h, l_2^h\}$ on X.

Let ℓ be the number of loops adjacent to Y. By Corollary 4.2 we know

$$\ell \leq |Y| + 1 - 2j - 1 = |Y| - 2j,$$

and hence

$$2j \leq |Y| - \ell.$$

Consider the possible cases for such a $\gamma \in \Theta$.

Case $a_1 \in Z$ and no loop is split: Then $|Y| = 2\ell + 2$ which implies $2j - 2 \leq \ell$ and hence $2j - 2 = \ell$, since $2j - 2$ is the total number of loops. But then the arity of Z would be 2, contradiction.

Case $a_1 \in Z$ and one loop is split: Then $|Y| = 2\ell + 3$ which implies $2j - 3 \leq \ell$ and hence $2j - 3 = \ell$, since $2j - 2$ is the total number of loops and one was split so cannot be adjacent to Y. This means that Y carries the maximum number of loops for its given valence, so is alternating off the loops (Lemma 4.1). Let l_1, l_2 be the split loop with $l_1 \in Y$ and $l_2 \in Z$. Thus the action of l_1, a_2 is alternating on Y and hence on X. But the action of a_1, a_2 is alternating whilst a_1, l_2 is id on Z and hence X, which is a contradiction of the fact that the transpositions $(a_1 a_2)$, $(a_1 l_2)$ and $(l_1 l_2)$ generate the symmetric group of a_1, a_2, l_1, l_2.

So we conclude $a_1 \in Y$ and proceed to:

Case: Suppose one loop is split. Then $|Y| = 2\ell + 4$ and so $2j - 4 \leq \ell < 2j - 2$, but only one loop was split so by stability considerations, one must go on Z, hence $\ell = 2j - 4$. Let l^i be the split loop and l^h be the loop on Z. Equivariance will imply that all permutations of $\{l_1^i, l_2^i, l_1^h, l_2^h\}$ must act by the identity on X, (since we can switch l_2^i, l_2^h adjacent to Z). This contradicts the definition of x_{2j+1} which says that $(l_1^i l_1^h)(l_2^i l_2^h)$ must act by -1.

We thus conclude no loop is split, hence $|Y| = 2\ell + 3$ and so $2j - 3 \leq \ell \leq 2j - 2$. But if $\ell = 2j - 2$, then the vertex adjacent to Z would be unstable. So the only remaining possibility is that $\ell = 2j - 3$, which in turn implies that Z is a vertex of valence 3, genus 0 and adjacent to 1 loop. But such graphs are excluded from $\widehat{\mathsf{B}}(\mathcal{H}_{\text{Lie}})$ by definition. We thus conclude Θ is empty, as desired. ∎

We remark that the subcomplex

$$\widehat{\mathsf{B}}(\mathcal{H}_{\text{Lie}})(2j + 1, 0) \subset \mathsf{B}(\mathcal{H}_{\text{Lie}})(2j + 1, 0)$$

does not split, and we do not assert that β_{2j+1} is a non-boundary when viewed in $\mathsf{B}(\mathcal{H}_{\text{Lie}})(2j + 1, 0)$, although the above proof shows that its inverse image is supported on a 1-dimensional subspace of $\mathsf{B}(\mathcal{H}_{\text{Lie}})(2j + 1, 0)$.

Corollary 4.8. *Let $\xi \in \widehat{\mathsf{B}}(\mathcal{H}_{\text{Lie}})(2j + 1, 0)^{4j+2-s,s}$ be a vector of positive internal degree $s > 0$. Then the projection of $\partial(\xi)$ to the θ_{2j+1} summand is zero.*

Proof. The Lemma establishes the case $s = 2j$. Suppose $0 < s < 2j$. Without loss of generality we may assume ξ is a homogeneous element supported on a summand index by a modular graph γ. A term in $\partial(\xi)$ is non-zero upon projection to the θ_{2j+1} summand only if it is possible to contract a subgraph γ' such that $\gamma/\gamma' \cong \theta$ where γ' has $2j + 1 - s > 1$ edges. As above, such an isomorphism must send the vertex corresponding to γ' to v_1, so the total genus of γ' must be 1. By definition of $\mathcal{H}_{\text{Lie}}$, such an operation is non-trivial only if γ' has first Betti number 1. These two conditions are true simultaneously only if each vertex of γ' has genus 0, which in turn implies that each vertex carries a label in some $H_*(\Gamma_{0,m})$, which is concentrated in internal degree 0. The only other vertex of γ has genus 0 as well, hence such an element must be supported on internal degree 0. ∎

4.3. Co-operadic non-zero coefficient lemma

Lemma 4.9. *The differential of the wheel graph $\partial(\omega_{2j+1})$ contains β_{2j+1} with non-zero coefficient.*

Proof. By Lemma 4.4, it suffices to show that the projection of $\partial(\omega_{2j+1})$ to the θ_{2j+1} summand of $\mathsf{B}(\mathscr{H}_{\mathrm{Lie}})(2j+1,0)^{2j+1,2j}$ is non-zero. After diagram (2.2), it is sufficient to show that composition in the following diagram is not zero:

$$
\begin{array}{ccc}
\omega_{2j+1} \in \mathsf{B}(\mathscr{H}_{\mathrm{Lie}})(2j+1,0)^{4j+2,0} & \xrightarrow{\quad\partial\quad} & \mathsf{B}(\mathscr{H}_{\mathrm{Lie}})(2j+1,0)^{2j+1,2j} \\
\Big\downarrow{\nu_\theta} & & \Big\downarrow{\pi_\theta} \\
\mathsf{B}(\mathscr{H}_{\mathrm{Lie}})(\theta_{2j+1})^{2j+1} \otimes_{\mathrm{Aut}(\theta)} \mathscr{R}^*(\theta_{2j+1}) & \xrightarrow{\quad\bar{\mu}\quad} & \mathscr{H}_{\mathrm{Lie}}(\theta_{2j+1})^{2j} \otimes_{\mathrm{Aut}(\theta)} \mathscr{R}^*(\theta_{2j+1}).
\end{array}
$$
$$(4.4)$$

By definition, $\nu_\theta(\omega_{2j+1})$ is determined by summing over ways to nest the graph ω_{2j+1} such that collapsing nests gives the graph θ_{2j+1}. Such a nesting specifies two induced graphs of ω_{2j+1}, $\widehat{N}_0$ and $\widehat{N}_1$ which collapse to the two vertices v_0 and v_1 of θ_{2j+1}. Since the vertex v_0 is of type $(0,3)$, the induced stable graph $\widehat{N}_0$ must be a corolla and N_0 must consist of a lone vertex. Given such an N_0, there is a unique such N_1 for which the composite with μ_X is non-zero. It is given by the unique $2j+1$-gon N_1 which is a subgraph of ω_{2j+1} and which misses a distinguished vertex N_0 (right hand side of Figure 1).

When $j=1$, a direct computation shows that the four choices for a lone vertex N_0 are sent by ν_{θ_3} to the same element, hence $\nu_{\theta_3} \neq 0$. Indeed, with the convention that the edges of the nested triangle appear last in the decomposition, one must apply the permutations (14), (25), (36) and $(14)(25)(36)$ to the conventional edge ordering of ω_3 (Figure 1) to decompose, and these permutations are all odd. Then since $\alpha_3 = x_3$, μ_X is simply the Massey product which contracts the triangle, which is not zero. Whence the case $j=1$.

So we now assume $j>1$. In this case there are $2j+1$ choices for such an N_0, corresponding to the outer vertices of ω. Therefore, $\nu_\theta(\omega)$ is a sum of $2j+1$ terms, corresponding to the choice of an outer vertex and the complementary $2j+1$-gon. Since these terms are related by an automorphism of ω, it is enough to show that any one of them is non-zero when composed with μ_X.

So let us fix such an N_0 and N_1. The term in the sum $\nu_\theta(\omega)$ corresponding to this choice of nesting is given by choosing an isomorphism $\omega/N_1 \cong \theta$, which in turn specifies a labeling of the flags of $\widehat{N}_1$ by the set $X := a^{-1}(v_1)$. We import the notation

$$
X = a^{-1}(v_1) = \{a_1, a_2, a_3, \ell_1^i, \ell_2^i \mid 1 \leq i \leq 2j-2\}
$$

from the proof of Proposition 4.7. Since ω has a unique non-trivalent vertex, so does $\widehat{N}_1$. The conditions on this X-labeling of the flags of $\widehat{N}_1$ coming from the isomorphism $\omega/N_1 \cong \theta$ are that the non-trivalent vertex of $\widehat{N}_1$ must be adjacent to flags labeled by exactly one of the a_i, and one flag from each loop. The two vertices adjacent to the non-trivalent vertex must have the other a labels (Figure 5). Let $P \subset \mathsf{B}(\mathscr{H}_{\mathrm{Lie}})(1, X)$ be the
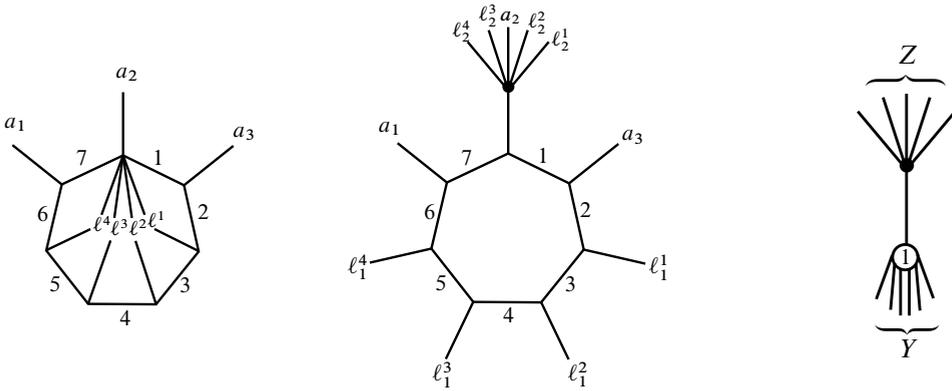
Figure 5. At left, the vector ρ in the case $j = 3$ is a graph with 11 legs whose underlying leg free graph is a 7-gon. The image of ρ under the heptagonal Massey product is determined by first contracting the heptagon in the center to the VCD class $\alpha_7 \in H_6(\Gamma_{1,Y})$ and then contracting the lone edge in the graph on the right.

span of such X-labeled, $2j + 1$-gons. In particular, $\dim(P) = 3!(2j - 2)!2^{2j-2}/2$. We conclude that the bottom row of the diagram is supported on the restriction

$$v_\theta(\omega_{2j+1})_{|P} \in (\mathsf{Com}(2) \otimes P) \otimes_{\mathrm{Aut}(\theta)} \mathcal{R}^*(\theta_{2j+1}) \subset \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(\theta_{2j+1}) \otimes_{\mathrm{Aut}(\theta)} \mathcal{R}^*(\theta_{2j+1})$$

where $\mathsf{Com}(2)$ labels v_0 and P labels v_1.

The bottom row in diagram (4.4) is given by contracting the X-labeled graph at the vertex v_1 of θ_{2j+1}, via the associated Massey product. It thus remains to show that contraction of such X-labeled polygons

$$(\mathsf{Com}(2) \otimes P) \otimes \mathcal{R}^*(\theta_{2j+1}) \xrightarrow{\mu_X} \mathcal{H}_{\mathrm{Lie}}(\theta_{2j+1})^{2j} \otimes \mathcal{R}^*(\theta_{2j+1})$$

is non-zero upon passage to $\mathrm{Aut}(\theta)$-coinvariants. Since this map is $\mathrm{Aut}(\theta)$ invariant, it is enough to know that the class $x_{2j+1} \in \mathcal{H}_{\mathrm{Lie}}(1, X)$ is in the image of the contraction when restricted to $P \subset \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(1, X) \to \mathcal{H}_{\mathrm{Lie}}(1, X)$. The remainder of the proof is dedicated to this calculation.

Recall that the class x_{2j+1} is defined by choosing a total order on the set X, which fixes an isomorphism $H_{2j}(\Gamma_{1,4j-1}) \cong H_{2j}(\Gamma_{1,X})$ (see equation (4.2)). By abuse of notation we write $x_{2j+1} \in H_{2j}(\Gamma_{1,4j-1})$ for the image of x_{2j+1} under this isomorphism. Note that since x_{2j+1} depends on this isomorphism only up to sign, the span is independent of choice. Thus, using the bijection $X \cong \{1, \ldots, 4j - 1\}$, we may view P as carrying numerical labels, and it suffices to show the contraction

$$P \subset \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(1, 4j - 1) \to H_{2j}(\Gamma_{1,4j-1})$$

surjects onto the span of x_{2j+1}. To be pedantic, the bijection $X \cong \{1, \ldots, 4j - 1\}$ sends $a_i \mapsto i$ for $1 \leq i \leq 3$ and $\ell_r^i \mapsto 2(i + 1) + (r - 1)$. As such we call a pair of numbers

$2(i + 1), 2(i + 1) + 1$ with $1 \leq i \leq 2j - 2$, a loop, called loop i or the i-th loop. We say a representative of a loop is one of its entries.

The contraction map $P \to H_{2j}(\Gamma_{1,2j-1})$ is given by the Massey product which contracts each such $2j + 1$-gon. The image of this contraction is determined by expanding the distinguished (non-trivalent) vertex and its legs to a new edge, contracting the $2j + 1$-gon, and then contracting the expanded edge (Figure 5). In other words, this map factors as

$$P \to \text{Ind}_{S_{2j-1} \times S_{2j}}^{S_{4j-1}} (H_0(\Gamma_{0,2j}) \otimes H_{2j}(\Gamma_{1,2j+1})) \xrightarrow{\circ_e} H_{2j}(\Gamma_{1,4j-1}). \qquad (4.5)$$

Notice the map $\circ_e$ is surjective, since it is non-zero and the target is irreducible.

We now analyze $\circ_e^{-1}(x_{2j+1})$. Recall x_{2j+1} spans a representation of $\text{Aut}(\theta)$ for which $S_{\{1,2,3\}}$ acts by the alternating representation, transpositions of representatives of a loop act by the identity and where each $(45)(2i, 2i + 1)$ with $1 < i - 1 \leq 2j - 2$ acts by -1.

The space $\text{Ind}_{S_{2j-1} \times S_{2j}}^{S_{4j-1}} (H_0(\Gamma_{0,2j}) \otimes H_{2j}(\Gamma_{1,2j+1}))$ has a basis given by

$$i_1 \wedge i_2 \wedge \cdots \wedge i_{2j} \quad \text{where } 1 \leq i_1 < i_2 < \cdots < i_{2j} \leq 4j - 1.$$

The S_{4j-1} action is by permutation, with permutation of wedge products acting by the sign of the permutation. Write a preimage of x_{2j+1} in this basis,

$$\sum_{1 \leq i_1 < \cdots < i_{2j} \leq 4j-1} c(i_1, i_2, \ldots, i_{2j}) i_1 \wedge i_2 \wedge \cdots \wedge i_{2j} \in \circ_e^{-1}(x). \qquad (4.6)$$

Since $S_{\{1,2,3\}}$ acts by the sign representation on x_{2j+1}, and $\circ_e$ is equivariant, $c(i_1, \ldots, i_{2j}) = 0$ unless $\{i_1, \ldots, i_{2j}\} \cap \{1, 2, 3\} \geq 2$. Likewise, $c(i_1, \ldots, i_{2j}) = 0$ if it lists both representatives of any loop since transposition of loop representatives acts by the identity.

To convey additional conditions that the coefficients in equation (4.6) must satisfy we fix some new notation. For a subset $I \subset \{1, \ldots, 2j - 2\}$ we define $\mathcal{S}_I$ to be the set of lists of representatives of those loops not appearing in I, such that no list has two representatives of the same loop. In particular, each list has $2j - 2 - |I|$ entries and there are $2^{(2j-2-|I|)}$ such lists. Let S_I be the formal sum of wedge products appearing in $\mathcal{S}_I$. For brevity we write $\mathcal{S} := \mathcal{S}_\emptyset$, $S := S_\emptyset$, $\mathcal{S}_i := \mathcal{S}_{\{i\}}$ and $S_i := S_{\{i\}}$. We will denote lists of loop representatives by $\vec{s} \in \mathcal{S}_I$. By abuse of notation we also write $\vec{s}$ for the associated wedge product. We write $\vec{s}^\perp$ for the list of complementary representatives. For example if $j = 2$ then $\mathcal{S} = \{(4, 6), (5, 6), (4, 7), (5, 7)\}$, $\mathcal{S}_2 = \{(4), (5)\}$, $S = 4 \wedge 6 + 5 \wedge 6 + 4 \wedge 7 + 5 \wedge 7$ and $S_2 = 4 \wedge 5$. If $\vec{s} = (4, 6)$ then $\vec{s}^\perp = (5, 7)$, and by abuse of notation me may also write $\vec{s} = 4 \wedge 6$.

With this notation we describe conditions on the coefficients in equation (4.6). First,

$$c(1, 2, 3, \vec{a}) = (-1)^{i-1} c(1, 2, 3, \vec{b}) \quad \text{if } \vec{a} \in \mathcal{S}_1 \text{ and } \vec{b} \in \mathcal{S}_i.$$

This condition is forced by the equivariance of $\circ_e$. Transposing representatives of a loop acts by the identity, so the coefficient is independent of choice of list in $\mathcal{S}_i$ (resp. $\mathcal{S}_1$). On the other hand, a list in $\mathcal{S}_1$ is missing one loop, namely $\{4, 5\}$, and the permutation

(45)$(2(i + 1), 2(i + 1) + 1)$ acts by -1 for any $i > 1$. The result may be compared with a list in $\mathcal{S}_i$ by applying an $i - 1$ cycle, acting by $(-1)^i$, hence the claim.

Second, consider coefficients whose index has exactly two of $1, 2, 3$. The condition that both representatives of a loop cannot appear in a wedge, means that wedges must have one representative from each loop. This gives conditions

$$c(1, 2, \vec{a}) = -c(1, 3, \vec{a}) = c(2, 3, \vec{a}) \quad \text{and} \quad c(1, 2, \vec{a}) = c(1, 2, \vec{b}) \quad \text{for each } \vec{a}, \vec{b} \in \mathcal{S}.$$

Define $w, v \in \operatorname{Ind}_{S_{2j-1} \times S_{2j}}^{S_{4j-1}}(H_0(\Gamma_{0,2j}) \otimes H_{2j}(\Gamma_{1,2j+1}))$ by

$$w = \sum_{1 \leq i \leq 2j-2} (-1)^i \, 1 \wedge 2 \wedge 3 \wedge S_i \quad \text{and} \quad v = 1 \wedge 2 \wedge S - 1 \wedge 3 \wedge S + 2 \wedge 3 \wedge S. \quad (4.7)$$

The above conditions on coefficients show that a vector in $\circ_e^{-1}(x_{2j+1})$ must be a linear combination of w and v. To find this linear combination, we invoke Lemma 3.2 which, in the present notation states

$$(\operatorname{id} + (1, 2j) + (2, 2j) + \cdots + (2j - 1, 2j)) 2j \wedge 2j + 1 \wedge \cdots \wedge 4j - 1 \quad (4.8)$$

is in the kernel of $\circ_e$. Consequently, any permutation of this relation is in $\ker(\circ_e)$ as well. Using this relation, we now calculate the linear combination of v and w which is in $\ker(\circ_e)$. Writing $\sim$ for the induced equivalence relation, the calculation follows from the following two claims:

Claim 1: $1 \wedge 2 \wedge S \sim -1 \wedge 3 \wedge S \sim 2 \wedge 3 \wedge S \sim -1 \wedge 2 \wedge 3 \wedge S_1$.

Proof. Let us prove that $2 \wedge 3 \wedge S \sim -1 \wedge 2 \wedge 3 \wedge S_1$, with the other two following similarly. $2 \wedge 3 \wedge S$ is a sum of $(2j - 2)^2$ wedge products of length $2j$. Each of the terms in the sum have 4 or 5 appearing once in the third position, and 1 does not appear at all. Hence terms appearing in this sum can be paired to write

$$2 \wedge 3 \wedge S = \sum_{\vec{b} \in S_1} 2 \wedge 3 \wedge (4 \oplus 5) \wedge \vec{b}.$$

To each such $\vec{b}$ we apply a permutation of the relation in equation (4.8) to $2 \wedge 3 \wedge 1 \wedge \vec{b}$ to find

$$2 \wedge 3 \wedge 4 \oplus 5 \wedge \vec{b} = ((14) \oplus (15)) 2 \wedge 3 \wedge 1 \wedge \vec{b} \sim -2 \wedge 3 \wedge 1 \wedge \vec{b} - 2 \wedge 3 \wedge \left(\bigoplus_{b \in \vec{b}^\perp} b \right) \wedge \vec{b}.$$

The sum over all $\vec{b}$ of $-2 \wedge 3 \wedge (\sum_{b \in \vec{b}^\perp} b) \wedge \vec{b}$ vanishes since each term has two representatives of one loop, and hence pairs with the term in which the representatives appear in the transposed order. Combining the previous two equations with this observation proves the claim. ∎

Claim 2: $(-1)^{i-1} 1 \wedge 2 \wedge 3 \wedge S_i \sim 1 \wedge 2 \wedge 3 \wedge S_1$.

Proof. The claim is vacuous for $i = 1$, so fix $1 < i \leq 2j - 2$. Then each term in $1 \wedge 2 \wedge 3 \wedge S_i$ has 4 or 5 appearing in it. Apply the relations to each term with a 4 appearing,

$$1 \wedge 2 \wedge 3 \wedge 4 \wedge \vec{b} \sim -1 \wedge 2 \wedge 3 \wedge \left(5 \oplus 2(i+1) \oplus (2(i+1)+1) \oplus \left(\bigoplus_{b \in \vec{b}^\perp} b \right) \right) \wedge \vec{b}$$

for each $\vec{b} \in \mathcal{S}_{\{1,i\}}$. The terms replacing 4 with 5 cancel with the terms in which 5 originally appeared. The terms where both representatives of a loop appear cancel in pairs. The remaining terms replace 4 with a representative of loop i. To compare these terms with $1 \wedge 2 \wedge 3 \wedge S_1$, it suffices to permute the order of the representatives of the loops into numerical order. This is done via a cycle of length $i - 1$, so produces a factor of $(-1)^{i-2}$. Combining this with the factor of -1 already appearing yields the result. ∎

These two claims together imply $(2j - 2)v - 3w \in \ker(\circ_e)$. From this calculation we conclude that $\circ_e(v) \neq 0$, since if it were zero, it would imply $\circ_e^{-1}(x_{2j+1}) = 0$, but iteration of Lemma 3.1 shows this not to be the case.

Finally, it remains to observe that the vector v is in the image of P in equation (4.5). Indeed, the map $P \to \mathrm{Ind}_{S_{2j-1} \times S_{2j}}^{S_{4j-1}}(H_0(\Gamma_{0,2j}) \otimes H_{2j}(\Gamma_{1,2j+1}))$ sends each X-labeled $2j + 1$-gon to a wedge containing two of $\{1, 2, 3\}$ and exactly one representative of each loop. The vector v is defined as a linear combination of such wedge products, so it is in the image. In particular, we have shown the contraction map $P \to \mathcal{H}_{\mathrm{Lie}}(1, X)$ surjects onto the span of x_{2j+1}, from which the statement follows. ∎

4.4. Proof of Main Results

To conclude, we observe how our main results stated in the introduction follow by linear dualizing the results of this section.

Proof of Theorem 1.1. As above, $\widehat{\mathsf{B}}$ denotes the subcomplex of the co-Feynman transform with no simple loops. Dualizing the inclusion $\widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}}) \hookrightarrow \mathsf{B}(\mathcal{H}_{\mathrm{Lie}})$, we have a projection $\mathsf{ft}(\mathcal{H}_{\mathrm{Lie}}) \twoheadrightarrow \bar{\mathsf{ft}}(\mathcal{H}_{\mathrm{Lie}})$ which quotients by simple loops. Since β_{2j+1} spans the θ_{2j+1} summand of $\mathsf{B}(\mathcal{H}_{\mathrm{Lie}})(2j + 1, 0)^{2j+1, 2j}$, we may define its characteristic functional

$$\eta_{2j+1} := \beta_{2j+1}^* \in \mathsf{ft}(\mathcal{H}_{\mathrm{Lie}})(2j + 1, 0),$$

extending by 0 off this summand. We write $\bar{\eta}_{2j+1} \in \bar{\mathsf{ft}}(\mathcal{H}_{\mathrm{Lie}})(2j + 1, 0)$ for its image under projection. Corollary 4.8 shows that no vector in $\partial(\ker(\widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}}) \twoheadrightarrow \widehat{\mathsf{B}}(\mathsf{Com})))$ has a non-zero coefficient of β_{2j+1}. Therefore

$$d(\bar{\eta}_{2j+1})(\ker(\widehat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}}) \twoheadrightarrow \widehat{\mathsf{B}}(\mathsf{Com}))) = 0,$$

from which we conclude $d(\bar{\eta}_{2j+1}) \in \bar{\mathsf{ft}}(\mathsf{Com})$. Finally, projecting $d(\bar{\eta}_{2j+1})$ along

$$\bar{\mathsf{ft}}(\mathsf{Com})(2j + 1, 0) \twoheadrightarrow \Sigma^{-4j-2} \mathsf{GC}_2^{2j+1},$$

we find $d(\bar{\eta}_{2j+1})(\omega_{2j+1}) \neq 0$ from Lemma 4.9. ∎

Proof of Corollary 1.2. The wheel graph $\omega_{2j+1} \in \Sigma^{4j+2} \mathsf{GC}_2^*$ is a cycle, it remains to see that it cannot be a boundary. Suppose that it were a boundary, so that $d_{\mathsf{GC}_2^*}(\xi) = \omega_{2j+1}$. The vector ξ has a canonical preimage $\hat{\xi} \in \hat{\mathsf{B}}(\mathcal{H}_{\mathrm{Lie}})$ for which

$$\partial(\hat{\xi}) = \partial_1(\hat{\xi}) + \partial_{>1}(\hat{\xi}) = \omega_{2j+1} + \text{terms of higher internal degree}.$$

Since $\partial^2(\hat{\xi}) = 0$, it must be the case that the non-zero scalar multiple of β_{2j+1} appearing in $\partial(\omega_{2j+1})$ (after Lemma 4.9) is canceled out by another term in $\partial(\hat{\xi})$. However, Corollary 4.8 shows that this cannot happen. We conclude ω_{2j+1} is not a boundary. ∎

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