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Geometric similarity invariants of Cowen–Douglas operators

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Abstract. In 1978, M. J. Cowen and R. G. Douglas introduced a class of operators $B_n(\Omega)$ (known as Cowen–Douglas class of operators) and associated a Hermitian holomorphic vector bundle to such an operator. They gave a complete set of unitary invariants in terms of the curvature and its covariant derivatives. At the same time, they asked whether one can use geometric ideas to find a complete set of similarity invariants of the Cowen–Douglas operators. We give a partial answer to this question. In this paper, we show that the curvature and the second fundamental form completely determine the similarity orbit of a norm dense class of the Cowen–Douglas operators. As an application we show that uncountably many (non-similar) strongly irreducible operators in $B_n(\mathbb{D})$ can be constructed from a given operator in $B_1(\mathbb{D})$. We also characterize a class of strongly irreducible weakly homogeneous operators in $B_n(\mathbb{D})$.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$ be a complex separable Hilbert space and the set of all bounded linear operators on $\mathcal{H}$, respectively. The Grassmannian $\mathrm{Gr}(n, \mathcal{H})$ is the set of all n-dimensional subspaces of the Hilbert space $\mathcal{H}$. For an open bounded connected subset Ω of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, a map $t : \Omega \to \mathrm{Gr}(n, \mathcal{H})$ is said to be a *holomorphic curve*, if there exist n (point-wise linearly independent) holomorphic functions $\gamma_1, \gamma_2, \dots, \gamma_n$ on Ω taking values in $\mathcal{H}$ such that $t(w) = \bigvee\{\gamma_1(w), \dots, \gamma_n(w)\}$, $w \in \Omega$. Each holomorphic curve $t : \Omega \to \mathrm{Gr}(n, \mathcal{H})$ gives rise to a rank n Hermitian holomorphic vector bundle E_t over Ω, namely, $E_t = \{(x, w) \in \mathcal{H} \times \Omega \mid x \in t(w)\}$ and $\pi : E_t \to \Omega$ is given by $\pi(x, w) = w$.

In a very influential paper [7], M. J. Cowen and R. G. Douglas initiated a systematic study of a class of bounded linear operators involving the intrinsic complex geometry. An operator T acting on $\mathcal{H}$ is said to be in the Cowen–Douglas class $B_n(\Omega)$ of rank n, associated with an open bounded subset Ω, if $T - w$ is surjective, $\dim \ker(T - w) = n$ for all $w \in \Omega$, and $\bigvee_{w \in \Omega} \ker(T - w) = \mathcal{H}$. M. J. Cowen and R. G. Douglas showed that each such operator T gives rise to a rank n Hermitian holomorphic vector bundle E_T over Ω, namely, $E_T = \{(x, w) \in \mathcal{H} \times \Omega \mid x \in \ker(T - w)\}$ and $\pi : E_T \to \Omega$ is given by $\pi(x, w) = w$.

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Two holomorphic curves $t, \widetilde{T} : \Omega \to \mathrm{Gr}(n, \mathcal{H})$ are said to be congruent if vector bundles E_t and $E_{\widetilde{T}}$ are locally equivalent as a Hermitian holomorphic vector bundle. Furthermore, t and $\widetilde{T}$ are said to be unitarily equivalent (denoted by $t \sim_u \widetilde{T}$), if there exists a unitary operator $U(\in \mathcal{B}(\mathcal{H}))$ such that $U(w)t(w) = \widetilde{T}(w)$, where $U(w) := U|_{E_t(w)}$ is the restriction of the unitary operator U to the fiber $E_t(w) = \pi^{-1}(w)$. It is easy to see, by using the Rigidity Theorem in [7], t and $\widetilde{T}$ are congruent if and only if t and $\widetilde{T}$ are unitarily equivalent. The holomorphic curves t and $\widetilde{T}$ are said to be similarity equivalent (denoted by $t \sim_s \widetilde{T}$) if there exists an invertible operator $X \in \mathcal{B}(\mathcal{H})$ such that $X(w)t(w) = \widetilde{T}(w)$, where $X(w) := X_{|E_t(w)}$ is the restriction of X to the fiber $E_t(w)$. In this case, we say that the vector bundles E_t and $E_{\widetilde{T}}$ are similarity equivalent.

For an open bounded connected subset Ω of $\mathbb{C}$, a Cowen–Douglas operator T with index n induces a non-constant holomorphic curve $t : \Omega \to \mathrm{Gr}(n, \mathcal{H})$, namely, $t(w) = \ker(T - w)$, $w \in \Omega$ and hence a Hermitian holomorphic vector bundle E_t (here, the vector bundle E_t is the same as E_T). Unitary and similarity invariants for the operator T are obtained from the vector bundle E_T. To describe these invariants, we need the curvature of the vector bundle E_T along with its covariant derivatives. We recall some of these notions from [7]. The metric of bundle E_T with respect to a holomorphic frame γ is given by $h_\gamma(w) = (\!(\langle \gamma_j(w), \gamma_i(w)\rangle)\!)_{i,j=1}^n$, $w \in \Omega$, where $\gamma(w) = \bigvee\{\gamma_1(w), \dots, \gamma_n(w)\}$, $w \in \Omega$. The connection compatible with the complex structure and the metric of the vector bundle E_T is canonically determined and locally it is given by the formula $h_\gamma^{-1}(\frac{\partial}{\partial w} h_\gamma)\, dw$ (see [35, Theorem 2.1]). The curvature of the Hermitian holomorphic vector bundle E_T is the $(1,1)$ form, namely,

$$\bar{\partial}\big(h_\gamma^{-1}(w)\partial h_\gamma(w)\big) = -\frac{\partial}{\partial \overline{w}}\Big(h_\gamma^{-1}(w)\frac{\partial}{\partial w} h_\gamma(w)\Big)\, dw \wedge d\overline{w}.$$

Let $\mathcal{K}_T(w)$ denote the coefficient of this $(1,1)$ form, i.e.,

$$\mathcal{K}_T(w) := -\frac{\partial}{\partial \overline{w}}\Big(h_\gamma^{-1}(w)\frac{\partial}{\partial w} h_\gamma(w)\Big)$$

and we call it the *curvature*. The curvature $\mathcal{K}_T$ can be thought of as a bundle map, following the definition of the covariant partial derivatives of bundle map, its covariant partial derivatives $\mathcal{K}_{T,w^i\overline{w}^j}$, $i, j \in \mathbb{N} \cup \{0\}$ are given by

(1) $\mathcal{K}_{T,w^i\overline{w}^{j+1}} = \frac{\partial}{\partial \overline{w}}(\mathcal{K}_{T,w^i\overline{w}^j})$;

(2) $\mathcal{K}_{T,w^{i+1}\overline{w}^j} = \frac{\partial}{\partial w}(\mathcal{K}_{T,w^i\overline{w}^j}) + [h_\gamma^{-1}\frac{\partial}{\partial w} h_\gamma, \mathcal{K}_{T,w^i\overline{w}^j}]$.

The curvature and its covariant partial derivatives are complete unitary invariants of an operator in the Cowen–Douglas class. M. J. Cowen and R. G. Douglas proved in [7] the following theorem.

Theorem 1.1. *Let T and $\widetilde{T}$ be two Cowen–Douglas operators with index n. Then T and $\widetilde{T}$ are unitarily equivalent if and only if there exists an isometric holomorphic bundle map $V : E_T \to E_{\widetilde{T}}$ such that*

$$V\big(\mathcal{K}_{T,w^i\overline{w}^j}\big) = \big(\mathcal{K}_{\widetilde{T},w^i\overline{w}^j}\big)V, \quad 0 \le i \le j \le i + j \le n,\ (i,j) \neq (0,n), (n,0).$$

In particular, if T and $\widetilde{T}$ are the Cowen–Douglas operators with index one, then $T \sim_u \widetilde{T}$ if and only if $\mathcal{K}_T = \mathcal{K}_{\widetilde{T}}$.

Theorem 1.1 says that the local complex geometric invariants are of global nature from the point of view of unitary equivalence. However, for similarity equivalence, the global invariants are not easily detectable by the local invariants, such as the curvature and its covariant derivatives. It is because the holomorphic bundle map determined by invertible operators is not rigid. In other words, one does not know when a bundle map with local isomorphism can be extended to an invertible operator in $\mathcal{B}(\mathcal{H})$. In the absence of characterization of equivalent classes under an invertible linear transformation, M. J. Cowen and R. G. Douglas proposed the following conjecture in [7].

Conjecture. Let $\mathbb{D}$ denote the open unit disc. Let $T, \widetilde{T} \in B_1(\mathbb{D})$ with the spectrums of T and $\widetilde{T}$ are closure of $\mathbb{D}$ (denoted by $\overline{\mathbb{D}}$). Then $T \sim_s \widetilde{T}$ if and only if

$$\lim_{w \to \partial\mathbb{D}} \frac{\mathcal{K}_T(w)}{\mathcal{K}_{\widetilde{T}}(w)} = 1.$$

This conjecture turned out to be false (cf. [5, 6]).

The class of Cowen–Douglas operators is very rich. In fact, the norm closure of the Cowen–Douglas operators contains the collection of all quasi-triangular operators with spectrum being connected. This follows from the famous similarity orbit theorem given by C. Apostol, L. A. Fialkow, D. A. Herrero, and D. Voiculescu (cf. [2]). Subsequently, the Cowen–Douglas operator has been one of the important ingredients in the research of operator theory (cf. [1, 3, 4, 6, 11, 12, 15, 18, 21, 22, 27, 29–32, 34, 36, 37]).

To find similarity invariants for the Cowen–Douglas operators in terms of geometric invariants, we need the following concepts and theorems.

Theorem 1.2 (Upper triangular representation theorem, [24]). *Let $T \in \mathcal{L}(\mathcal{H})$ be a Cowen–Douglas operator with index n, then there exists an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n$ and operators $T_{1,1}, T_{2,2}, \ldots, T_{n,n}$ in $B_1(\Omega)$ such that T takes the following form*

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} & \cdots & T_{1,n} \\ 0 & T_{2,2} & T_{2,3} & \cdots & T_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-1,n-1} & T_{n-1,n} \\ 0 & \cdots & \cdots & 0 & T_{n,n} \end{pmatrix}. \tag{1.1}$$

Let $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be a holomorphic frame of E_T with

$$\mathcal{H} = \bigvee\{\gamma_i(w), w \in \Omega, 1 \le i \le n\},$$

and $t_i : \Omega \to \mathrm{Gr}(1, \mathcal{H}_i)$ be a holomorphic frame of $E_{T_{i,i}}$, $1 \le i \le n$. Then we can find certain relations between $\{\gamma_i\}_{i=1}^n$ and $\{t_i\}_{i=1}^n$, prescribed in the following equations:

$$\begin{aligned} \gamma_1 &= t_1, \\ \gamma_2 &= \phi_{1,2}(t_2) + t_2, \\ \gamma_3 &= \phi_{1,3}(t_3) + \phi_{2,3}(t_3) + t_3, \\ &\vdots \\ \gamma_j &= \phi_{1,j}(t_j) + \cdots + \phi_{i,j}(t_j) + \cdots + t_j, \\ &\vdots \\ \gamma_n &= \phi_{1,n}(t_n) + \cdots + \phi_{i,n}(t_n) + \cdots + t_n, \end{aligned} \tag{1.2}$$

where $\phi_{i,j}$, $(i, j = 1, 2, \dots, n)$ are certain holomorphic bundle maps. We expect these bundle maps to reflect geometric similarity invariants of the operator T. However, it is far from enough on the coarse relation of the above equations. In particular, to use geometric terms such as curvature and second fundamental form for similarity invariants of the operator in (1.1), we have to give them more internal structures. For example, we may assume that $T_{i,i+1}$ are nonzero operators and intertwines $T_{i,i}$ and $T_{i+1,i+1}$, i.e., $T_{i,i}T_{i,i+1} = T_{i,i+1}T_{i+1,i+1}$, $1 \le i \le n-1$ (it is denoted by $\mathcal{F}\mathcal{B}_n(\Omega)$ (see [19])).

For a 2×2 block $\begin{pmatrix} T_{i,i} & T_{i,i+1} \\ 0 & T_{i+1,i+1} \end{pmatrix}$, if $T_{i,i}T_{i,i+1} = T_{i,i+1}T_{i+1,i+1}$, then the corresponding second fundamental form $\theta_{i,i+1}$, which is obtained by R. G. Douglas and G. Misra (see [10]), is

$$\theta_{i,i+1}(T)(w) = \frac{\mathcal{K}_{T_{i,i}}(w)\, d\overline{w}}{\left(\frac{\|T_{i,i+1}(t_{i+1}(w))\|^2}{\|t_{i+1}(w)\|^2} - \mathcal{K}_{T_{i,i}}(w)\right)^{1/2}}. \tag{1.3}$$

Let $T, \widetilde{T}$ have upper-triangular representation as in (1.1) and assume that $T_{i,i}, T_{i+1,i+1}$ and $\widetilde{T}_{i,i}, \widetilde{T}_{i+1,i+1}$ have intertwinings $T_{i,i+1}$ and $\widetilde{T}_{i,i+1}$, respectively. If $\mathcal{K}_{T_{i,i}} = \mathcal{K}_{\widetilde{T}_{i,i}}$, then from (1.3) it is easy to see that

$$\theta_{i,i+1}(T)(w) = \theta_{i,i+1}(\widetilde{T})(w) \Leftrightarrow \frac{\|T_{i,i+1}(t_{i+1}(w))\|^2}{\|t_{i+1}(w)\|^2} = \frac{\|\widetilde{T}_{i,i+1}(\widetilde{T}_{i+1}(w))\|^2}{\|\widetilde{T}_{i+1}(w)\|^2}.$$

If the upper-triangular representation in (1.1) has such a good internal structure, then a complete set of unitary (or similarity) invariants of T is obtained in terms of the curvature and the second fundamental form.

In this paper, we introduce a subset of the Cowen–Douglas operators denoted by $\mathcal{CF}\mathcal{B}_n(\Omega)$ (Definition 2.7). We prove that $\mathcal{CF}\mathcal{B}_n(\Omega)$ is norm dense in $B_n(\Omega)$ (Proposition 2.16). Hence it is meaningful to discuss the geometric similarity invariants for operators belonging to $\mathcal{CF}\mathcal{B}_n(\Omega)$. We would like to point out that similarity results for quasi-homogeneous Cowen–Douglas operators have been discussed in [23]. However, the class $\mathcal{CFB}_n(\Omega)$ is quite different from the class of quasi-homogeneous Cowen–Douglas operators. For example, if we take an operator T in $\mathcal{CFB}_n(\Omega)$ such that $T_{i,i} = T_{i+1,i+1}$, $1 \le i \le n-1$, or $T_{i,i}$ is not a homogeneous operator, then T is not a quasi-homogeneous Cowen–Douglas operator. Also, if we take a homogeneous Cowen–Douglas operator T (which is quasi-homogeneous by definition) in $B_n(\Omega)$ for $n \ge 3$, then T does not belong to $\mathcal{CFB}_n(\Omega)$.

Roughly speaking, for operators T in $\mathcal{CFB}_n(\Omega)$, the curvature and the second fundamental form give a complete set of similarity invariants. In a joint work with G. Misra (see [20]), the authors gave a complete set of unitary invariants in terms of the curvature and the second fundamental form for operators in $\mathcal{FB}_n(\Omega)$. In the general case, the first and the second authors obtained a complete set of similarity invariants by using ordered K_0-group. However, an ordered K_0-group is an algebraic invariant. We expect these algebraic invariants to supply more insight in the search of geometric invariants. Recently, R. G. Douglas, H. Kwon, S. Treil [9, 28] and K. Ji [17] use the curvature to describe similarity invariants for a subclass of the Cowen–Douglas operators. Here is one of the results from [9, 17].

Theorem 1.3. *Let $T \in B_n(\mathbb{D})$ be an m-hypercontraction and S_z is the multiplication on the weighted Bergman space. Then T is similar to $\bigoplus_{i=1}^n S_z^*$ if and only if there exists a bounded subharmonic function ψ defined on $\mathbb{D}$ such that*

$$\operatorname{trace}(\mathcal{K}_T) - \operatorname{trace}(\mathcal{K}_{S_z^*}) = \Delta\psi.$$

An operator T is said to be homogeneous if $\phi_\alpha(T)$ is unitarily equivalent to T for each Möbius transformation ϕ_α. G. Misra proved the following theorem.

Theorem 1.4 ([26]). *Let T_1 and T_2 be two homogeneous Cowen–Douglas operators with index one. Then T_1 is similar to T_2 if and only if T_1 is unitarily equivalent to T_2, i.e., $\mathcal{K}_{T_1} = \mathcal{K}_{T_2}$.*

The first and the second authors jointly with G. Misra extended the concepts of homogeneous operators to quasi-homogeneous operators as follows.

Definition 1.5 ([23]). Let $T \in \mathcal{FB}_n(\Omega)$ and T has an $n \times n$ upper-triangular matrix as in (1.1). Then the operator T is called a quasi-homogeneous operator, i.e., $T \in \mathcal{QB}_n(\Omega)$, if each $T_{i,i}$ is a homogeneous operator in $B_1(\Omega)$ and

$$T_{i,j}(t_j) \in \bigvee\{t_i^{(k)}, k \le j-i-1\}.$$

For the quasi-homogeneous operators, the curvature and the second fundamental form completely describe similarity invariants.

Theorem 1.6 ([23]). *Let $T, S \in \mathcal{QB}_n(\Omega)$, then*

$$\begin{cases} \mathcal{K}_{T_{i,i}} = \mathcal{K}_{\widetilde{T}_{i,i}}, \\ \theta_{i,i+1}(T) = \theta_{i,i+1}(\widetilde{T}) \end{cases} \Longrightarrow T \sim_s \widetilde{T} \text{ if and only if } T = \widetilde{T}.$$

We point out that even if T is a Cowen–Douglas operator with index one, its spectral picture (see [24, page 8]) is also very complicated. The following theorem, due to D. A. Herrero, shows its complexity.

Theorem 1.7 ([16]). *Let $T \in \mathcal{L}(\mathcal{H})$ be a quasi-triangular operator with connected spectral picture. If there exists a point w in the Fredholm domain of T such that* $\operatorname{ind}(T-w)=1$*, then for any $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K$ is a Cowen–Douglas operator with index one.*

It is due to the complexity of the structure of the Cowen–Douglas operators and the fact that the invertible operator is not an isometric bundle map. Therefore, for any two Cowen–Douglas operators T and $\tilde{T}$ with index one, to understand the similarity between T and $\tilde{T}$, we have to explore further the relation between $\mathcal{K}_T$ and $\mathcal{K}_{\tilde{T}}$.

We now summarize the content of this paper. In Section 2, we introduce a subclass of the Cowen–Douglas operators denoted by $\mathcal{CFB}_n(\Omega)$. We prove this class of operators is norm dense in the set of all of the Cowen–Douglas operators. In Section 3, we study an important property, named, Property (H). In Section 4, we show that the curvature and the second fundamental form completely characterize the similarity invariants for all the Cowen–Douglas operators in $\mathcal{CFB}_n(\Omega)$. In Section 5, we characterize a class of weakly homogeneous operators in $B_n(\mathbb{D})$. We also construct uncountably many strongly irreducible operators (non-similar) in $B_n(\mathbb{D})$ from a given operator in $B_1(\mathbb{D})$.

2. The operator class $\mathcal{CFB}_n(\Omega)$

In this section, we introduce a subclass of the class of the Cowen–Douglas operators which is denoted as $\mathcal{CFB}_n(\Omega)$. We show that $\mathcal{CFB}_n(\Omega)$ is norm dense in $B_n(\Omega)$. We first recall the definition of the subclass $\mathcal{FB}_n(\Omega)$ of $B_n(\Omega)$. This class has been studied in detail in [20].

Definition 2.1. $\mathcal{FB}_n(\Omega)$ is the set of all bounded linear operators T defined on some complex separable Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$, which are of the form

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ & T_{2,2} & \cdots & T_{2,n} \\ & & \ddots & \vdots \\ & & & T_{n,n} \end{pmatrix}$$

where the operator $T_{i,i} : \mathcal{H}_i \to \mathcal{H}_i$, defined on a complex separable Hilbert space $\mathcal{H}_i$, $1 \le i \le n$, is assumed to be in $B_1(\Omega)$ and $T_{i,i+1} : \mathcal{H}_{i+1} \to \mathcal{H}_i$, is assumed to be a nonzero intertwining bounded operator, namely, $T_{i,i}T_{i,i+1} = T_{i,i+1}T_{i+1,i+1}$, $1 \le i \le n-1$.

To define the class $\mathcal{CFB}_n(\Omega)$, we need following definitions.

Definition 2.2. Let T_1 and T_2 be bounded linear operators on $\mathcal{H}$. The Rosenblum operators τ_{T_1,T_2} and δ_{T_1} are maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$ defined as follows:

$$\tau_{T_1,T_2}(X) = T_1X - XT_2$$

and

$$\delta_{T_1}(X) = T_1X - XT_1,$$

where X is a bounded linear operator on $\mathcal{H}$.

Definition 2.3 (Property (H)). Let T_1 and T_2 be bounded linear operators on $\mathcal{H}$. We say that T_1, T_2 satisfy the *Property (H)* if the following condition holds: if X is a bounded linear operator defined on $\mathcal{H}$ such that $T_1X = XT_2$ and $X = T_1Z - ZT_2$, for some Z in $\mathcal{L}(\mathcal{H})$, then $X = 0$.

Remark 2.4. Let T be an operator in $B_1(\Omega)$. If we take $T_1 = T_2 = T$, then from [20] it follows that T_1, T_2 satisfy the Property (H).

We recall that $\{T\}'$ denotes the commutant, i.e., $\{T\}'$ is the set of all bounded linear operators that commute with T.

Definition 2.5. Let T be a bounded linear operator on $\mathcal{H}$. We say that T is strongly irreducible if there is no non-trivial idempotent in $\{T\}'$.

Lemma 2.6. *Let T_1 and T_2 be bounded linear operators on $\mathcal{H}$. Suppose that S_1 and S_2 are similar to T_1 and T_2, respectively. If T_1, T_2 satisfy the Property (H), then S_1, S_2 also satisfy the Property (H).*

Proof. Since T_i is similar to S_i, there exists an invertible operator X_i such that $X_iT_i = S_iX_i$, $1 \leq i \leq 2$. Let Y be a bounded linear operator such that $S_1Y = YS_2$ and $Y = S_1Z - ZS_2$ for some Z in $\mathcal{L}(\mathcal{H})$. It is easy see that

$$T_1X_1^{-1}YX_2 = X_1^{-1}YX_2T_2$$

and

$$\begin{aligned} X_1^{-1}YX_2 &= X_1^{-1}S_1X_1X_1^{-1}ZX_2 - X_1^{-1}ZX_2X_2^{-1}S_2X_2 \\ &= T_1X_1^{-1}ZX_2 - X_1^{-1}ZX_2T_2. \end{aligned}$$

Since T_1, T_2 satisfy the Property (H), $X_1^{-1}YX_2 = 0$ and hence $Y = 0$. Thus S_1, S_2 also satisfy the Property (H). This completes the proof. ∎

Definition 2.7. A Cowen–Douglas operator T with index n is said to be in $\mathcal{CFB}_n(\Omega)$[1], if T satisfies the following properties:

[1]Each operator $T \in \mathcal{CFB}_n(\Omega)$ possesses a flag structure and the entries of T satisfy the commuting relations. Hence we use the symbol "$\mathcal{CF}$" to specify the subclass $\mathcal{CFB}_n(\Omega)$.

(1) T can be written as an $n \times n$ upper-triangular matrix form under a topological direct decomposition of $\mathcal{H}$ and $\operatorname{diag}\{T\} := T_{1,1} \dot{+} T_{2,2} \dot{+} \cdots \dot{+} T_{n,n} \in \{T\}'$. Furthermore, each entry

$$T_{i,j} = \phi_{i,j} T_{i,i+1} T_{i+1,i+2} \cdots T_{j-1,j},$$

where $\phi_{i,j} \in \{T_{i,i}\}'$;

(2) $T_{i,i}, T_{i+1,i+1}$ satisfy the Property (H), i.e., $\ker \tau_{T_{i,i},T_{i+1,i+1}} \cap \operatorname{ran} \tau_{T_{i,i},T_{i+1,i+1}} = \{0\}$, $1 \leq i \leq n-1$.

Using the following concepts and lemmas, we prove that the class $\mathcal{CFB}_n(\Omega)$ is norm dense in $B_n(\Omega)$.

Definition 2.8 (Similarity invariant set). Let $\mathcal{F} = \{A_\alpha \in \mathcal{L}(\mathcal{H}), \alpha \in \Lambda\}$. We say $\mathcal{F}$ is a similarity invariant set, if for any invertible operator X,

$$X\mathcal{F}X^{-1} = \{XA_\alpha X^{-1} : A_\alpha \in \mathcal{F}\} = \mathcal{F}.$$

Definition 2.9 ([24]). If $\mathcal{K}(\mathcal{H})$ denotes the set of all compact operators acting on $\mathcal{H}$ and $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the projection of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra, then $\sigma_e(T)$, the essential spectrum of T, is the spectrum of $\pi(T)$ in $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and $\mathbb{C}\backslash\sigma_e(T)$ is called the Fredholm domain of T and is denoted by $\rho_F(T)$. Thus, $\sigma_e(T) = \sigma_{\text{le}}(T) \cup \sigma_{\text{re}}(T)$, where $\sigma_{\text{le}}(T) = \sigma_l(\pi(T))$ (left essential spectrum of T) and $\sigma_{\text{re}}(T) = \sigma_r(\pi(T))$ (right essential spectrum of T).

On the other hand, the intersection $\sigma_{\text{lre}}(T) := \sigma_{\text{le}}(T) \cap \sigma_{\text{re}}(T)$ is called Wolf spectrum and it includes the boundary $\partial\sigma_e(T)$ of $\sigma_e(T)$. Therefore, it is a non-empty compact subset of $\mathbb{C}$. Its complement $\mathbb{C}\backslash\sigma_{\text{lre}}(T)$ coincides with $\rho_{s-F}(T) := \{w \in \mathbb{C} : T - w$ is semi-Fredholm $\}$. Further, $\rho_{s-F}(T)$ is the disjoint union of the (possibly empty) open sets $\{\rho^n_{s-F}(T) : -\infty \leq n \leq +\infty\}$, where

$$\rho^n_{s-F}(T) = \{w \in \mathbb{C} : T - w \text{ is semi-Fredholm with } \operatorname{ind}(T - w) = n\}.$$

The spectrum picture of T, denoted by $\Lambda(T)$, is defined as the compact set $\sigma_{\text{lre}}(T)$, plus the data corresponding to the indices of $T - w$ for w in the bounded components of $\rho_{s-F}(T)$.

Lemma 2.10 ([16]). *Let $T \in B_n(\Omega)$. Then $\sigma_p(T^*) = \emptyset$, and $\sigma(T)$ is connected, where $\sigma_p(T^*)$ denotes the point spectrum of T^*.*

Lemma 2.11 ([16]). *Let $T \in \mathcal{L}(\mathcal{H})$, $\varepsilon > 0$, and let T be a quasi-triangular operator such that*

(i) *$\sigma(T)$ is connected;*

(ii) $\operatorname{ind}(T - w) > 0$, *$w \in \rho_F(T)$;*

(iii) *there exist a positive integer n and $w_0 \in \rho_F(T)$ such that* $\operatorname{ind}(T - w_0) = n$,

then there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K \in B_n(\Omega)$, where $w_0 \in \Omega \subset \rho_F(T)$.

Lemma 2.12 (Voiculescu Theorem, [33]). *Let $T \in \mathcal{L}(\mathcal{H})$ and ρ be a unital faithful $*$-representation of a separable C^*-subalgebra of the Calkin algebra containing the canonical image $\pi(T)$ and $\pi(I)$ on a separable space $\mathcal{H}_\rho$. Let $A = \rho(\pi(T))$ and k be a positive integer. Given ε, there exists $K \in \mathcal{K}(\mathcal{H})$, with $\|K\| < \varepsilon$, such that*

$$T - K \sim_u T \oplus A^{(\infty)} \sim_u T \oplus A^{(k)},$$

where $A^{(k)}$ denotes $\bigoplus_{i=1}^{k} A$, and $A^{(\infty)}$ denotes $\bigoplus_{i=1}^{\infty} A$.

Lemma 2.13 (Special case of the similarity orbit theorem, Apostle, Fialkow, Herrero, and Voiculescu [2]). *Let T and S be in $B_n(\Omega)$ satisfy the following conditions:*

(i) $\sigma_{\text{lre}}(T) = \sigma_{\text{lre}}(S)$ *and* $\sigma_{\text{lre}}(T)$ *is a perfect set;*

(ii) $\Lambda(T) = \Lambda(S)$,

where $\sigma_{\text{lre}}(T)$ and $\Lambda(T)$ denote the Wolf spectrum and spectral picture of T respectively (see [24, page 8]*). Then there exist two sequences of invertible operators $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} X_n T X_n^{-1} = S, \qquad \lim_{n\to\infty} Y_n S Y_n^{-1} = T.$$

Lemma 2.14. *Let $T \in B_n(\Omega)$, $\varepsilon > 0$ and Φ be an analytic Cauchy domain satisfying*

$$\sigma_{\text{lre}}(T) \subset \Phi \subset [\sigma_{\text{lre}}(T)]_\varepsilon := \{w \in \mathbb{C} : \text{dist}(w, \sigma_{\text{lre}}(T)) < \varepsilon\},$$

then there exists $T_\varepsilon \in B_n(\Omega_1)$ such that

(i) $\sigma_{\text{lre}}(T_\varepsilon) = \Phi$ *and* Ω_1 *is an open connected subset of* Ω*;*

(ii) $\|T - T_\varepsilon\| < \varepsilon$.

Proof. Let $\delta = \text{dist}(\Phi, \partial[\sigma_{\text{lre}}(T)]_\varepsilon)$. By Lemma 2.12, there exist an operator A and an operator K_1 with $\|K_1\| < \frac{\delta}{3}$, and a unitary operator U such that

$$U(T + K_1)U^* = A \oplus T, \quad \text{where } \sigma_{\text{lre}}(A) = \sigma_{\text{lre}}(T).$$

By [24, Theorem 1.25] and [24, Proposition 1.22], there exists a compact operator K_2 with $\|K_2\| < \frac{\delta}{3}$ such that

$$U(T + K_1 + K_2)U^* = \begin{pmatrix} N & A_{1,2} \\ 0 & A_\infty \end{pmatrix} \oplus T = B \oplus T,$$

and $\sigma_e(B) = \sigma(B) = \sigma_e(T)$, $\sigma_{\text{lre}}(B) = \sigma_{\text{lre}}(T)$, $\sigma(A_\infty) = \sigma_e(A_\infty) = \sigma_e(T)$, where $B = \begin{pmatrix} N & A_{1,2} \\ 0 & A_\infty \end{pmatrix}$ and N is a diagonal normal operator of uniform infinite multiplicity and $\sigma(N) = \sigma_{\text{lre}}(T)$.

Let $L = \begin{pmatrix} M & A_{1,2} \\ 0 & A_\infty \end{pmatrix} \oplus T$, where M is a diagonal normal operator of a uniform infinite multiplication operator with $\sigma(M) = \sigma_e(M) = \Phi$. By direct calculation, we can see that

$$\|L - U(T + K_1 + K_2)U^*\| < \varepsilon - \delta.$$

Let T'_ε denote U^*LU. By Lemma 2.10, we have that $\sigma_p(T^*) = \emptyset$, then it follows that $\sigma(T'_\varepsilon)$ is connected and $\sigma_p(T'^*_\varepsilon) = \emptyset$. When ε is small enough, we can find $w \in \Omega$ and $\delta_1 > 0$ such that $\Omega_1 = O_{w,\delta_1}$, the neighborhood of w such that $\Omega_1 \subset \Omega$.

Applying Lemma 2.11 to T'_ε, there is a compact operator K_3 with $\|K_3\| < \varepsilon$ such that $T_\varepsilon = T'_\varepsilon + K_3 \in B_n(\Omega_1)$, then T_ε satisfies all the requirements of the lemma. ■

Lemma 2.15 ([7]). *Let $T \in B_n(\Omega)$ and $\Omega_1 \subseteq \Omega$ be an open connected set. Then*

$$B_n(\Omega) \subseteq B_n(\Omega_1).$$

Proposition 2.16. *$\mathcal{CFB}_n(\Omega)$ is norm dense in $B_n(\Omega)$.*

Proof. First, we shall prove that $\mathcal{CFB}_n(\Omega)$ is a similarity invariant set. For any $T \in \mathcal{CFB}_n(\Omega)$, by Definition 2.7, there exist n idempotents $\{P_i\}_{i=1}^n$ such that

(1) $\sum_{i=1}^n P_i = I$, $P_iP_j = 0, i \neq j$;

(2) $T = ((T_{i,j}))_{n\times n}$, $T_{i,j} = P_iTP_j = 0$, if $i > j$;

(3) $T_{i,i}T_{i,j} = T_{i,j}T_{j,j}$, $1 \leq i, j \leq n$.

Let X be an invertible operator, set $Q_i := XP_iX^{-1}$, $1 \leq i \leq n$. We have

$$\begin{aligned}
\sum_{i=1}^n Q_i &= X\Big(\sum_{i=1}^n P_i\Big)X^{-1} = I,\\
Q_iQ_j &= X(P_iP_j)X^{-1} = 0, \quad i \neq j,\\
Q_iXTX^{-1}Q_j &= XP_iX^{-1}XTX^{-1}XP_jX^{-1} = XT_{i,j}X^{-1} = 0 \quad \text{for } i > j,
\end{aligned}$$

and

$$\begin{aligned}
Q_iXTX^{-1}Q_iQ_iXTX^{-1}Q_j &= Q_iXTX^{-1}Q_iXTX^{-1}Q_j\\
&= XP_iX^{-1}XTX^{-1}XP_iX^{-1}XTX^{-1}XP_jX^{-1}\\
&= XP_iTP_iP_iTP_jX^{-1} = XT_{i,i}T_{i,j}X^{-1}\\
&= XT_{i,j}T_{j,j}X^{-1} = XP_iTP_jP_jTP_jX^{-1}\\
&= Q_iXTX^{-1}Q_jQ_jXTX^{-1}Q_j.
\end{aligned}$$

Thus, under the decomposition $\mathcal{H} = \operatorname{ran} Q_1 \dotplus \operatorname{ran} Q_2 \dotplus \cdots \dotplus \operatorname{ran} Q_n$, the operator T admits the upper-triangular matrix representation, i.e.,

$$XTX^{-1} = \begin{pmatrix}
Q_1XTX^{-1}Q_1 & Q_1XTX^{-1}Q_2 & Q_1XTX^{-1}Q_3 & \cdots & Q_1XTX^{-1}Q_n\\
0 & Q_2XTX^{-1}Q_2 & Q_2XTX^{-1}Q_3 & \cdots & Q_2XTX^{-1}Q_n\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
0 & \cdots & 0 & Q_{n-1}XTX^{-1}Q_{n-1} & Q_{n-1}XTX^{-1}Q_n\\
0 & \cdots & \cdots & 0 & Q_nXTX^{-1}Q_n
\end{pmatrix}. \tag{2.1}$$

Note that

$$\begin{aligned}
Q_iXTX^{-1}Q_j &= XP_iX^{-1}XTX^{-1}XP_jX^{-1}\\
&= XP_iTP_jX^{-1}\\
&= XT_{i,j}X^{-1}\\
&= X\phi_{i,j}T_{i,i+1}T_{i+1,i+2},\dots,T_{j-1,j}X^{-1}\\
&= X\phi_{i,j}X^{-1}XT_{i,i+1}X^{-1}XT_{i+1,i+2}X^{-1},\dots,XT_{j-1,j}X^{-1}\\
&= X\phi_{i,j}X^{-1}XP_iTP_{i+1}X^{-1}XP_{i+1}TP_{i+2}X^{-1},\dots,XP_{j-1}TP_jX^{-1}\\
&= X\phi_{i,j}X^{-1}Q_iXTX^{-1}Q_{i+1}Q_{i+1}XTX^{-1}Q_{i+2},\dots,Q_{j-1}XTX^{-1}Q_j
\end{aligned}$$

and

$$\begin{aligned}
X\phi_{i,j}X^{-1}Q_iXTX^{-1}Q_i &= X\phi_{i,j}X^{-1}XP_iX^{-1}XTX^{-1}XP_iX^{-1}\\
&= Q_iXTX^{-1}Q_iX\phi_{i,j}X^{-1}.
\end{aligned}$$

Finally, we will show that the Property (H) remains intact under the similarity transformation for operators in $\mathcal{CFB}_n(\Omega)$. Since $Q_iXTX^{-1}Q_i = XP_iTP_iX^{-1}$ for $1 \le i \le n$, by Lemma 2.6, $Q_iXTX^{-1}Q_i$, $Q_{i+1}XTX^{-1}Q_{i+1}$ satisfy the Property (H). Hence XTX^{-1} also satisfies the Property (H). Thus XTX^{-1} belongs to $\mathcal{CFB}_n(\Omega)$. Hence $\mathcal{CFB}_n(\Omega)$ is a similarity invariant set.

Now, by using the similarity orbit theorem, we prove that $\mathcal{CFB}_n(\Omega)$ is norm dense in $B_n(\Omega)$.

By Lemma 2.14, we only need to prove that for any $T \in B_n(\Omega)$ with $\sigma_{\mathrm{lre}}(T) = \overline{\Phi}$, Φ is an analytic Cauchy domain, we can find $T_\varepsilon \in \mathcal{CFB}_n(\Omega)$ such that $\|T_\varepsilon - T\| < \varepsilon$.

Since $\sigma_{\mathrm{lre}}(T) = \Phi$, $\rho_{s-F}(T)$ only has finite many components denoted by $\{\Omega_i, n_i\}_{i=1}^n$, where $n_i = \dim\ker(T - w_i)$, for any $w_i \in \Omega_i$, $i = 1, 2, \dots, n$. By Lemma 2.15, without loss of generality, we can assume that $\Omega_1 = \Omega$. Since Φ is an analytic Cauchy domain, Ω is an analytic connected Cauchy domain. Let $H_{z_i}(\Omega_i)$ be the multiplication operator on $\mathcal{H}^2(\Omega_i, d\mu_i)$ and

$$B = H_{z_1}(\Omega_1) \oplus \bigoplus_{i=2}^{n} H_{z_i}^{(n_i)}(\Omega_i) \oplus M,$$

where μ_i is a Lebesgue measure and M is a diagonal normal operator such that $\sigma(M) = \sigma_{\mathrm{lre}}(M) = \overline{\Phi}$. Applying Lemma 2.11 to the operator B, there exists a compact operator K with $\|K\| < \varepsilon$ such that $B + K \in B_1(\Omega)$. Let

$$T_\varepsilon = \begin{pmatrix}
H_{z_1}(\Omega_1) & I & 0 & \cdots & 0 & 0\\
0 & H_{z_1}(\Omega_1) & I & \cdots & 0 & 0\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\
0 & \cdots & 0 & H_{z_1}(\Omega_1) & I & 0\\
0 & \cdots & 0 & 0 & H_{z_1}(\Omega_1) & 0\\
0 & \cdots & \cdots & 0 & 0 & B+K
\end{pmatrix}. \tag{2.2}$$

The spectrum pictures of T_ε and T are the same. Thus, by Lemma 2.13, there exists invertible operators $\{X_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} X_n T_\varepsilon X_n^{-1} = T$. Since T_ε is in $\mathcal{CFB}_n(\Omega)$ and $\mathcal{CFB}_n(\Omega)$ is a similarity invariant set, $X_n T_\varepsilon X_n^{-1}$ in $\mathcal{CFB}_n(\Omega)$ for all n. This finishes the proof of this theorem. ∎

Remark 2.17. Let $T \in \mathcal{CFB}_n(\Omega)$ and $T = ((T_{i,j}))_{n\times n}$ be the $n \times n$ upper-triangular matrix form under a topological direct decomposition of $\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{H}_2 \dotplus \cdots \dotplus \mathcal{H}_n$. Let t_n be a nonzero section of $E_{T_{n,n}}$. Set $t_i := T_{i,i+1}(t_{i+1})$, $1 \le i \le n-1$. It is easy to see that t_i is a section of the vector bundle $E_{T_{i,i}}$. We define $\theta_{i,i+1}(T) = \frac{\|T_{i,i+1}(t_{i+1})\|^2}{\|t_{i+1}\|^2}$ and call it generalized second fundamental form.

Remark 2.18. For any topological direct decomposition of $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_1 \dotplus \mathcal{H}_2 \dotplus \cdots \dotplus \mathcal{H}_n,$$

there exist n idempotents $P_1, P_2, \ldots, P_n$ such that $\sum_{i=1}^{n} P_i = I$, $P_i P_j = 0, i \neq j$ and $\operatorname{ran} P_i = \mathcal{H}_i$. Then we can find an invertible operator X such that $\{Q_i\}_{i=1}^{n} = \{X P_i X^{-1}\}_{i=1}^{n}$ is a set of orthogonal projections with $Q_i Q_j = 0, i \neq j$. Furthermore,

$$\mathcal{H} = X\mathcal{H}_1 \oplus X\mathcal{H}_2 \oplus \cdots \oplus X\mathcal{H}_n,$$

where $X\mathcal{H}_i = \operatorname{ran} Q_i$. Suppose $T \in \mathcal{CFB}_n(\Omega)$ has the upper-triangular matrix representation according to a topological direct decomposition of $\mathcal{H}$. By the proof of Proposition 2.16, we see that $XTX^{-1} \in \mathcal{CFB}_n(\Omega)$ according to an orthogonal direct decomposition of $\mathcal{H}$ induced by X above.

From now on, we assume that the operators in $\mathcal{CFB}_n(\Omega)$ have an upper-triangular matrix representation with respect to an orthogonal direct sum decomposition of $\mathcal{H}$.

3. Sufficient conditions for the Property (H)

In this section, we study the "Property (H)". This property plays a vital role in our study on the similarity problem for the operators in the class $\mathcal{CFB}_n(\Omega)$. We would like to know under what conditions two bounded linear operators in $B_1(\Omega)$ satisfy the Property (H).

Let T_1, T_2 be bounded linear operators in $B_1(\Omega)$ and X be a bounded operator such that $T_1 X = X T_2$ and $X = T_1 Y - Y T_2$ for some bounded linear operator Y. We would like to find a sufficient condition, so that X becomes zero. It is well known that $T_i \sim_u (M_z^*, \mathcal{H}_{K_i})$, $1 \le i \le 2$.

First, we discuss a condition that ensures the intertwining operator between T_1 and T_2 will be the zero operator. More precisely, if $T_1 X = X T_2$, then $X = 0$. A sufficient condition for this is

$$\lim_{w\to\partial\Omega} \frac{K_1(w,w)}{K_2(w,w)} = \infty. \tag{3.1}$$

Indeed, when $T_1X = XT_2$, there exists a holomorphic function ϕ defined on Ω such that $X(K_2(\cdot,w)) = \overline{\phi(w)}K_1(\cdot,w))$. By condition (3.1) and the maximum modulus principle, it follows that $\phi = 0$ and hence $X = 0$. For example, we consider S_1^* and S_2^*, the adjoints of Hardy shift and Bergman shift, respectively. It is well known that there is no nonzero bounded linear operator X such that $S_2^*X = XS_1^*$ (since $\lim_{w\to\partial\mathbb{D}} \frac{(1-|w|^2)^{-2}}{(1-|w|^2)^{-1}} = \lim_{w\to\partial\mathbb{D}}(1-|w|^2)^{-1} = \infty$).

However, it is not clear what would be a sufficient condition for the Property (H) in terms of reproducing kernels as above. Now we will discuss some criteria to decide when given operators T_1, T_2 satisfy the Property (H).

Lemma 3.1 ([14]). *Let X, T be bounded linear operators defined on $\mathcal{H}$. If $X \in \ker\delta_T \cap \operatorname{ran}\delta_T$, then $\sigma(X) = \{0\}$.*

Lemma 3.2 ([20]). *Suppose T_1 and T_2 are two Cowen–Douglas operators in $B_1(\Omega)$, and S is a bounded operator that intertwines T_1 and T_2, i.e., $T_1S = ST_2$. Then S is nonzero if and only if the range of S is dense.*

Proposition 3.3. *Let T_1, T_2 be bounded linear operators defined on $\mathcal{H}$. If $\ker\tau_{T_2,T_1} \neq \{0\}$ and $\{T_2\}'$ is semi-simple, then T_1, T_2 satisfy the Property (H).*

Proof. We want to show that T_1, T_2 satisfy the Property (H), i.e., $\ker\tau_{T_1,T_2}\cap\operatorname{ran}\tau_{T_1,T_2}=0$. Suppose on contrary $\ker\tau_{T_1,T_2}\cap\operatorname{ran}\tau_{T_1,T_2}\neq 0$. Let $X\in\ker\tau_{T_1,T_2}\cap\operatorname{ran}\tau_{T_1,T_2}$ and X is nonzero. There exists a bounded operator Z such that $X = T_1Z - ZT_2$ and $T_1X = XT_2$. Since $\ker\tau_{T_2,T_1}\neq\{0\}$, there exists a nonzero bounded linear operator Y such that $YT_1 = T_2Y$. We have

$$YX = YT_1Z - YZT_2 = T_2YZ - YZT_2$$

and

$$YXT_2 = YT_1X = T_2YX.$$

Thus, $YX \in \ker\tau_{T_2}\cap\operatorname{ran}\tau_{T_2}$. By Lemma 3.1, it follows that $\sigma(YX) = 0$. Since X is nonzero, by Lemma 3.2, the range of X is dense. Since $\{T_2\}'$ is semi-simple and $X \neq 0$, we have $Y = 0$. This is a contradiction. This completes the proof. ∎

Proposition 3.4. *Let T_1, T_2 be bounded linear operators on $\mathcal{H}$ and S_2 be the right inverse of T_2. If $\lim_{n\to\infty}\frac{\|T_1^n\|\|S_2^n\|}{n} = 0$, then the Property (H) holds.*

Proof. Let X, Y be linear bounded operators on $\mathcal{H}$ such that $T_1X = XT_2$ and $X = T_1Y - YT_2$. We claim that $T_1^nY - YT_2^n = nT_1^{n-1}X$ for $n\in\mathbb{N}$. In fact, for $n = 1$, the conclusion follows from the assumption. For $n > 1$, we have

$$\begin{aligned} T_1^nY - YT_2^n &= T_1^nY - T_1^{n-1}YT_2 + T_1^{n-1}YT_2 - T_1^{n-2}YT_2^2 + T_1^{n-2}YT_2^2 - \cdots \\ &\quad + T_1YT_2^{n-1} - YT_2^n \\ &= T_1^{n-1}(T_1Y - YT_2) + T_1^{n-2}(T_1Y - YT_2)T_2 + \cdots \\ &\quad + (T_1Y - YT_2)T_2^{n-1} \end{aligned}$$

$$
\begin{aligned}
&= T_1^{n-1}X + T_1^{n-2}XT_2 + \cdots + XT_2^{n-1} \\
&= nT_1^{n-1}X \qquad (\text{or } nXT_2^{n-1}).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
T_1^n Y S_2^n - Y T_2^n S_2^n &= nT_1^{n-1}XS_2^n \\
&= nXT_2^{n-1}S_2^n \\
&= nXS_2.
\end{aligned}
$$

Since $T_2^n S_2^n = I$, we have

$$
Y = T_1^n Y S_2^n - nXS_2, \quad n \in \mathbb{N}.
$$

Therefore, for $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\|Y\| &= \|nXS_2 - T_1^n Y S_2^n\| \\
&\geq n\|XS_2\| - \|T_1^n Y S_2^n\| \\
&= n\Big(\|XS_2\| - \frac{\|T_1^n Y S_2^n\|}{n}\Big).
\end{aligned} \tag{3.2}
$$

If $X = 0$, we are done. Suppose X is nonzero. Since Y is a bounded linear operator, from equation (3.2), it follows that $Y = 0$ and hence $X = 0$. This is a contraction. This completes the proof. ■

Proposition 3.5. *Let $A, B \in B_1(\mathbb{D})$ be backward shift operators with weighted sequences $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$, respectively. If $\lim_{n\to\infty} n\frac{\prod_{k=1}^n b_k}{\prod_{k=1}^n a_k} = \infty$, then the following statements hold:*

(i) *If X intertwines A and B, i.e., $AX = XB$, then there exists an ONB $\{e_i\}_{i=1}^\infty$ of $\mathcal{H}$ such that the matrix form of X with respect to $\{e_i\}_{i=1}^\infty$ has the form*

$$
X = \begin{pmatrix}
x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} & \cdots \\
 & x_{2,2} & x_{2,3} & \cdots & x_{2,n} & \cdots \\
 & & \ddots & \ddots & \vdots & \cdots \\
 & & & x_{n-1,n-1} & x_{n-1,n} & \cdots \\
 & & & & x_{n,n} & \ddots \\
 & & & & & \ddots
\end{pmatrix},
$$

where $x_{n,n+j} = \frac{\prod_{k=1}^{n-1} b_{k+j}}{\prod_{k=1}^{n-1} a_k} x_{1,1+j}$, $j = 0, 1, 2, \ldots, n = 1, 2, \ldots$

(ii) *$X \in \ker \tau_{A,B} \cap \operatorname{ran} \tau_{A,B}$ if and only if $X = 0$.*

Furthermore, if we replace A and B by $\phi(A)$ and $\phi(B)$, respectively, where ϕ is a univalent analytic function defined on $\overline{\mathbb{D}}$, then above conclusions continue to hold.

Proof. Commuting relation $AX = XB$ forces X to be in upper triangular form. We consider the following equation

$$\begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & \cdots \\ & 0 & a_2 & \cdots & 0 & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & a_{n-1} & \cdots \\ & & & & 0 & a_n \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} & \cdots \\ & x_{2,2} & x_{2,3} & \cdots & x_{2,n} & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & x_{n-1,n-1} & x_{n-1,n} & \cdots \\ & & & & x_{n,n} & \ddots \\ & & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} & \cdots \\ & x_{2,2} & x_{2,3} & \cdots & x_{2,n} & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & x_{n-1,n-1} & x_{n-1,n} & \cdots \\ & & & & x_{n,n} & \ddots \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 & \cdots \\ & 0 & b_2 & \cdots & 0 & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & b_{n-1} & \cdots \\ & & & & 0 & b_n \\ & & & & & \ddots \end{pmatrix},$$

by comparing the elements in (i, j) position and after a simple calculation, we will get the statement (i), i.e.,

$$x_{n,n+j} = \frac{\prod_{k=1}^{n-1} b_{k+j}}{\prod_{k=1}^{n-1} a_k} x_{1,1+j}, \quad j = 0, 1, 2, \ldots, \ n = 1, 2, \ldots \tag{3.3}$$

Thus we only need to prove statement (ii). Notice that $Ae_1 = Be_1 = 0$, $Xe_1 = x_{1,1}e_1$ and

$$AYe_1 - YBe_1 = Xe_1 = x_{1,1}e_1.$$

Since $AYe_1 = x_{1,1}e_1$, there exist $\alpha_1^1, \alpha_2^1 \in \mathbb{C}$ such that $Ye_1 = \alpha_1^1 e_1 + \alpha_2^1 e_2$. From

$$AYe_2 - YBe_2 = Xe_2 = x_{1,2}e_1 + x_{2,2}e_2,$$

it follows that

$$\begin{aligned} AYe_2 &= b_1\alpha_1^1 e_1 + b_1\alpha_2^1 e_2 + x_{1,2}e_1 + x_{2,2}e_2 \\ &= (b_1\alpha_1^1 + x_{1,2})e_1 + (b_1\alpha_2^1 + x_{2,2})e_2. \end{aligned}$$

Similarly, we can find $\alpha_1^2, \alpha_2^2, \alpha_3^2 \in \mathbb{C}$ such that

$$Ye_2 = \alpha_1^2 e_1 + \alpha_2^2 e_2 + \alpha_3^2 e_3.$$

Inductively we see that for any $n > 0$,

$$Ye_n \in \bigvee\{e_1, e_2, \ldots, e_{n+1}\}.$$

It follows that the matrix form of Y according to $\{e_i\}_{i=1}^{\infty}$ is as follows:

$$Y = \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,n} & \cdots & \cdots \\ y_{2,1} & y_{2,2} & y_{2,3} & \cdots & y_{2,n} & \cdots & \cdots \\ & y_{3,2} & y_{3,3} & \cdots & y_{3,n} & \cdots & \cdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & y_{n-1,n-2} & y_{n-1,n-1} & y_{n-1,n} & \cdots \\ & & & & y_{n,n-1} & y_{n,n} & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}.$$

From the equation $AY - YB = X$, it is easy to see that

$$y_{n,n-1} = n\frac{\prod_{k=1}^{n} b_k}{a_{n+1}\prod_{k=1}^{n} a_k} x_{1,1}.$$

By the assumption of the lemma,

$$\lim_{n\to\infty} n\frac{\prod_{k=1}^{n} b_k}{\prod_{k=1}^{n} a_k} = \infty.$$

So we have $x_{1,1} = 0$ and $y_{n,n-1} = 0$. By equation (3.3), we get $x_{n,n} = 0$, $n = 1, 2, \ldots$

Now assume that $x_{1,2} \neq 0$, then it follows that

$$y_{n,n} = n\frac{\prod_{k=1}^{n} b_{k+1}}{a_{n+1}\prod_{k=1}^{n} a_k} x_{1,2} + \frac{\prod_{k=1}^{n} b_k}{a_{n+1}\prod_{k=1}^{n} a_k} y_{1,1}.$$

Since $A, B \in B_1(\mathbb{D})$, there exist $d, M_0 \in \mathbb{R}^+$ such that

$$\min_k\{|a_k|, |b_k|\} > d, \qquad \max_k\{|a_k|, |b_k|\} < M_0.$$

We have

$$n\frac{\prod_{k=1}^{n} b_{k+1}}{a_{n+1}\prod_{k=1}^{n} a_k} \geq n\frac{b_{n+1}}{a_{n+1}b_1}\frac{\prod_{k=1}^{n} b_k}{\prod_{k=1}^{n} a_k} \geq \frac{d}{M_0 b_1} n \frac{\prod_{k=1}^{n} b_k}{\prod_{k=1}^{n} a_k} \to \infty.$$

Notice that

$$\frac{\prod_{k=1}^{n} b_k}{a_{n+1}\prod_{k=1}^{n} a_k} y_{1,1}$$

is bounded. If $x_{1,2} \neq 0$, then $y_{n,n} \to \infty$ as $n \to \infty$. Thus we have $x_{1,2} = 0$. Using equation (3.3) again, we get $x_{n,n+1} = 0$ for any $n > 1$.

We set $E_0 := \operatorname{diag}\{y_{1,1}, y_{2,2}, \ldots, y_{n,n}, \ldots\}$, by direct calculation, it is easy to see that $AE_0 = E_0 B$ and hence

$$A(Y - E_0) - (Y - E_0)B = X.$$

So for the sake of simplicity, we continue to denote $Y - E_0$ by Y. In this case, we also have
$$y_{n,n+1} = n\frac{\prod_{k=1}^{n} b_{k+2}}{\prod_{k=1}^{n} a_k} x_{1,3} + \frac{\prod_{k=1}^{n} b_{k+1}}{a_{n+1}\prod_{k=1}^{n} a_k} y_{1,2}.$$
By a similar argument as above, we have $x_{1,3} = 0$. We set

$$E_1 = \begin{pmatrix} 0 & y_{1,2} & 0 & \cdots & 0 & \cdots \\ & 0 & y_{2,3} & \cdots & 0 & \cdots \\ & & \ddots & \ddots & \vdots & \cdots \\ & & & 0 & y_{n-1,n} & \cdots \\ & & & & 0 & y_{n,n+1} \\ & & & & & \ddots \end{pmatrix}.$$

Thus we have $AE_1 = E_1 B$ and $A(Y - E_1) + (Y - E_1)B = X$. We again continue to denote $Y - E_1$ by Y. By repeating the above process, we see that for any j, we have $x_{n,n+j} = 0$. Thus $X = 0$. This finishes the proof of statement (ii).

At last, we will show the conclusion above continues to hold for $\phi(A)$ and $\phi(B)$. Without of loss generality, we can also assume that $\phi(z) = \sum_{n=1}^{\infty} k_n z^n$. Suppose that there exists a bounded operator X such that $\phi(A)X = X\phi(B)$. By a similar argument, we have
$$x_{n,n} = \frac{\prod_{k=1}^{n-1} b_k}{\prod_{k=1}^{n-1} a_k} x_{1,1}, \quad n = 1, 2, \ldots$$
Suppose that $\phi(A)Y - Y\phi(B) = X$. By direct calculation, we see that
$$y_{n+1,n} = n\frac{\prod_{k=1}^{n-1} b_k}{\prod_{k=1}^{n-1} a_k} x_{1,1}, \quad n = 1, 2, \ldots$$
Thus we have $x_{1,1} = 0$ and $y_{n,n-1} = 0$. By equation (3.3), we have $x_{n,n} = 0, n = 1, 2, \ldots$ We set $E_0 = \operatorname{diag}\{y_{1,1}, y_{2,2}, \ldots, y_{n,n}, \ldots\}$, by direct calculation, we have $AE_0 = E_0 B$ and hence $\phi(A)E_0 = E_0\phi(B)$ and
$$\phi(A)(Y - E_0) - (Y - E_0)\phi(B) = X.$$
So for sake of simplicity, we still use Y to denote $Y - E_0$. Now, repeating the proof of statement (ii), it can also be shown that X is equal to the zero operator. ∎

Corollary 3.6. *Let $M_{i,z}$ be the multiplication operator on the reproducing kernel Hilbert space H_{K_i}, where $K_i(z,w) = \frac{1}{(1-z\overline{w})^{\lambda_i}}$, $z, w \in \mathbb{D}$, $1 \le i \le 2$. If $\lambda_2 - \lambda_1 < 2$, then $M_{1,z}^*$ and $M_{2,z}^*$ satisfy the Property (H).*

Proof. Let $a_n(\lambda_i)$ denote the coefficient of $\overline{w}^n z^n$ in the power series expansion for $K_i(z,w)$, $i = 1, 2$. Then $M_{i,z}^*$ is a backward weight shift with $w_n^{(\lambda_i)} = \frac{\sqrt{a_n(\lambda_i)}}{\sqrt{a_{n+1}(\lambda_i)}}$, $i = 1, 2$.

By Stirling's formula (see [23, page 2879]), we have that

$$\prod_{k=0}^{n} w_k^{(\lambda_i)} = \frac{\sqrt{a_0(\lambda_i)}}{\sqrt{a_{n+1}(\lambda_i)}} \sim (n+1)^{\frac{1-\lambda_i}{2}}, \quad i = 1, 2.$$

Then we have that

$$\frac{\prod_{k=0}^{n} w_k^{(\lambda_2)}}{\prod_{k=0}^{n} w_k^{(\lambda_1)}} \sim (n+1)^{-\frac{\lambda_2-\lambda_1}{2}}.$$

If $\lambda_2 - \lambda_1 < 2$, then $\lim_{n\to\infty} n \frac{\prod_{k=0}^{n} w_k^{(\lambda_2)}}{\prod_{k=0}^{n} w_k^{(\lambda_1)}} = \infty$. Thus, by Proposition 3.5, $M_{1,z}^*$ and $M_{2,z}^*$ satisfy the Property (H). ∎

4. Similarity of Operators in $\mathcal{CFB}_n(\Omega)$

In this section, we give complete similarity invariants for operators in $\mathcal{CFB}_n(\Omega)$, which involve the curvature and the second fundamental form. This is quite different from the case of a quasi-homogeneous operator class (see Theorem 4.12). To prove the main theorem of this section, we need the following concepts and lemmas.

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be strongly irreducible if there is no non-trivial idempotent operator in $\{T\}'$, where $\{T\}'$ denotes the commutant of T, i.e., $\{T\}' = \{B \in \mathcal{L}(\mathcal{H}) : TB = BT\}$. It can be proved that for any $T \in B_1(\Omega)$, T is strongly irreducible. We denote the set of all the strongly irreducible operators by the symbol "(SI)".

An operator T in $\mathcal{L}(\mathcal{H})$ is said to have finite strongly irreducible decomposition, if there exist idempotents $P_1, P_2, \ldots, P_n$ in $\{T\}'$ such that

1. $P_i P_j = \delta_{ij} P_i$ for $1 \le i, j \le n < +\infty$, where $\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j; \end{cases}$
2. $\sum_{i=1}^{n} P_i = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$;
3. $T|_{P_i\mathcal{H}}$ is strongly irreducible for $i = 1, 2, \ldots, n$.

Every Cowen–Douglas operator can be written as the direct sum of finitely many strongly irreducible Cowen–Douglas operators (see [24, Chapter 3]). We call $P = (P_1, P_2, \ldots, P_n)$ a unit finite strongly irreducible decomposition of T. Let T have a finite (SI) decomposition and $P = \{P_i\}_{i=1}^{n}$ and $Q = \{Q_i\}_{i=1}^{m}$ be two unit finite (SI) decompositions of T. We say that T has a unique, strongly irreducible decomposition up to similarity if the following conditions are satisfied:

1. $m = n$; and
2. there exists an invertible operator X in $\{T\}'$ and a permutation Π of the set $\{1, 2, \ldots, n\}$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for $1 \le i \le n$.

Lemma 4.1 ([25, Theorem 5.5.12]). *Let T be a Cowen–Douglas operator in $B_n(\Omega)$. The operator T has a unique* (SI) *decomposition.*

Lemma 4.2 ([25, Theorem 5.5.13]). *Let $T = \bigoplus_{i=1}^{k} T_i^{(n_i)}$ and $\tilde{T} = \bigoplus_{j=1}^{s} \tilde{T}_j^{(m_j)}$ be two Cowen–Douglas operators, where $T_i, \tilde{T}_j \in$ (SI) for any i, j and $T_i \not\sim_s T_{i'}$, $\tilde{T}_j \not\sim_s \tilde{T}_{j'}$. Then $T \sim_s \tilde{T}$ if and only if $k = s$ and there exists a permutation Π such that $T_i \sim_s \tilde{T}_{\Pi(i)}$ and $n_i = m_{\Pi(i)}$, $i = 1, 2, \ldots, k$.*

By Lemma 4.1 and Lemma 4.2, we only need to consider when two strongly irreducible operators in $\mathcal{CFB}_n(\Omega)$ are similar equivalent. The similarity classification for the general case will follow by Lemma 4.2. Thus, in the following, we will assume $T \in \mathcal{CFB}_n(\Omega)$ is a strongly irreducible operator.

Lemma 4.3. *Let $T \in \mathcal{CFB}_n(\Omega)$. Then T is strongly irreducible if and only if $T_{i,i+1} \neq 0$ for any $i = 1, 2, \ldots, n-1$.*

Proof. Let T be a strongly irreducible operator in $\mathcal{CFB}_n(\Omega)$. Suppose on contrary that $T_{k-1,k} = 0$ for some k. For i, j with $i + 1 \leq k \leq j$, we have

$$T_{i,j} = \phi_{i,j} T_{i,i+1} T_{i+1,i+2} \cdots T_{k-1,k} \cdots T_{j-1,j} = 0.$$

Thus T has the following matrix form:

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,k-1} & 0 & 0 & \cdots & 0 \\ 0 & T_{2,2} & \cdots & T_{2,k-1} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{k-1,k-1} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & T_{k,k} & T_{k,k+1} & \cdots & T_{k,n} \\ 0 & \cdots & \cdots & 0 & 0 & T_{k+1,k+1} & \cdots & T_{k+1,n} \\ 0 & 0 & \cdots & \cdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & T_{n,n} \end{pmatrix}. \tag{4.1}$$

So T is strongly reducible. This is a contradiction to the fact that T is strongly irreducible. This finishes the proof of the necessary part.

For the sufficient part, suppose that each $T_{i,i+1}$ is a nonzero operator. By Definition 2.7, $T_{i,i}$ and $T_{i+1,i+1}$ satisfy the Property (H). Since $T_{i,i}T_{i,i+1} = T_{i,i+1}T_{i+1,i+1}$, it follows that $T_{i,i+1} \notin \operatorname{ran} \tau_{T_{i,i}T_{i+1,i+1}}$. By a same proof of [20, Proposition 2.22], it is easy to see that T is strongly irreducible. ∎

Lemma 4.4. *Let $T = ((T_{i,j}))_{n\times n}$, $\tilde{T} = ((\tilde{T}_{i,j}))_{n\times n}$ be operators in $\mathcal{CFB}_n(\Omega)$. If $T_{i,i} = \tilde{T}_{i,i}$ and $T_{i,i+1} = \tilde{T}_{i,i+1}$, then there exists a bounded operator K such that $X = I + K$ is invertible and $XT = \tilde{T}X$.*

Proof. To find K, we need to solve the equation

$$(I + K)T = \tilde{T}(I + K). \tag{4.2}$$

We set $X := I + K$, where

$$K = \begin{pmatrix} 0 & K_{1,2} & K_{1,3} & \cdots & K_{1,n} \\ 0 & 0 & K_{2,3} & \cdots & K_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & K_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

From equation (4.2), we have

$$\begin{pmatrix} I & K_{1,2} & K_{1,3} & \cdots & K_{1,n} \\ 0 & I & K_{2,3} & \cdots & K_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & K_{n-1,n} \\ 0 & \cdots & \cdots & 0 & I \end{pmatrix} \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} & \cdots & T_{1,n} \\ 0 & T_{2,2} & T_{2,3} & \cdots & T_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-1,n-1} & T_{n-1,n} \\ 0 & \cdots & \cdots & 0 & T_{n,n} \end{pmatrix}$$
$$= \begin{pmatrix} T_{1,1} & T_{1,2} & \widetilde{T}_{1,3} & \cdots & \widetilde{T}_{1,n} \\ 0 & T_{2,2} & T_{2,3} & \cdots & \widetilde{T}_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-1,n-1} & T_{n-1,n} \\ 0 & \cdots & \cdots & 0 & T_{n,n} \end{pmatrix} \begin{pmatrix} I & K_{1,2} & K_{1,3} & \cdots & K_{1,n} \\ 0 & I & K_{2,3} & \cdots & K_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & K_{n-1,n} \\ 0 & \cdots & \cdots & 0 & I \end{pmatrix}. \tag{4.3}$$

To find $K_{i,j}$, we take the following steps.

Step 1: For $1 \le i \le n-1$, by equating the $(i, i+1)$th entry of equation (4.3), we have $T_{i,i+1} + K_{i,i+1}T_{i+1,i+1} = T_{i,i}K_{i,i+1} + T_{i,i+1}$, i.e., $K_{i,i+1}T_{i+1,i+1} = T_{i,i}K_{i,i+1}$. For $1 \le i \le n-2$, by comparing the $(i, i+2)$th entry of equation (4.3), we have

$$\begin{aligned} &T_{i,i+2} + K_{i,i+1}T_{i+1,i+2} + K_{i,i+2}T_{i+2,i+2} \\ &\qquad = T_{i,i}K_{i,i+2} + T_{i,i+1}K_{i+1,i+2} + \widetilde{T}_{i,i+2}. \end{aligned} \tag{4.4}$$

If $T_{i,i}K_{i,i+2} = K_{i,i+2}T_{i+2,i+2}$, $1 \le i \le n-2$, then from equation (4.4) we get

$$T_{i,i+2} + K_{i,i+1}T_{i+1,i+2} = T_{i,i+1}K_{i+1,i+2} + \widetilde{T}_{i,i+2}. \tag{4.5}$$

Choose $K_{n-1,n}$ such that $K_{n-1,n}T_{n,n} = T_{n-1,n-1}K_{n-1,n}$. For $1 \le i \le n-2$, from equation (4.5), we get $K_{i,i+1}$ which satisfies $K_{i,i+1}T_{i+1,i+1} = T_{i,i}K_{i,i+1}$.

Step 2: We compare the $(i, i+3)$th entry of equation (4.3), we get

$$\begin{aligned} &T_{i,i+3} + K_{i,i+1}T_{i+1,i+3} + K_{i,i+2}T_{i+2,i+3} + K_{i,i+3}T_{i+3,i+3} \\ &\qquad = T_{i,i}K_{i,i+3} + T_{i,i+1}K_{i+1,i+3} + \widetilde{T}_{i,i+2}K_{i+2,i+3} + \widetilde{T}_{i,i+3}. \end{aligned} \tag{4.6}$$

If $T_{i,i}K_{i,i+3} = K_{i,i+3}T_{i+3,i+3}$, $1 \leq i \leq n-3$, then from equation (4.6) we have

$$\begin{aligned} T_{i,i+3} &+ K_{i,i+1}T_{i+1,i+3} + K_{i,i+2}T_{i+2,i+3} \\ &= T_{i,i+1}K_{i+1,i+3} + \widetilde{T}_{i,i+2}K_{i+2,i+3} + \widetilde{T}_{i,i+3}. \end{aligned} \tag{4.7}$$

Choose $K_{n-2,n}$ such that $K_{n-2,n}T_{n,n} = T_{n-2,n-2}K_{n-2,n}$. For $1 \leq i \leq n-3$, from equation (4.7), we get $K_{i,i+2}$ which satisfies $T_{i,i}K_{i,i+2} = K_{i,i+2}T_{i+2,i+2}$.

Step 3: By following the previous steps, suppose we have solved $K_{i,i+l}$ for $1 \leq i \leq n-l$, $1 \leq l \leq j-2$.

By comparing the $(i, i+j)$th entry of equation (4.3), we have

$$\begin{aligned} T_{i,i+j} &+ K_{i,i+1}T_{i+1,i+j} + K_{i,i+2}T_{i+2,i+j} + \cdots + K_{i,i+j}T_{i+j,i+j} \\ &= T_{i,i}K_{i,i+j} + T_{i,i+1}K_{i+1,i+j} + \widetilde{T}_{i,i+2}K_{i+2,i+j} + \cdots \\ &\quad + \widetilde{T}_{i,i+j-1}K_{i+j-1,i+j} + \widetilde{T}_{i,i+j}. \end{aligned} \tag{4.8}$$

If $T_{i,i+j}K_{i,i+j} = K_{i,i+j}T_{i+j,i+j}$, $1 \leq i \leq n-j$, then from equation (4.8) we get

$$\begin{aligned} T_{i,i+j} &+ K_{i,i+1}T_{i+1,i+j} + K_{i,i+2}T_{i+2,i+j} + \cdots + K_{i,i+j-1}T_{i+j-1,i+j} \\ &= T_{i,i+1}K_{i+1,i+j} + \widetilde{T}_{i,i+2}K_{i+2,i+j} + \cdots + \widetilde{T}_{i,i+j-1}K_{i+j-1,i+j} + \widetilde{T}_{i,i+j}. \end{aligned} \tag{4.9}$$

Choose $K_{n-j+1,n}$ such that $K_{n-j+1,n}T_{n,n} = T_{n-j+1,n-j+1}K_{n-j+1,n}$. For $1 \leq i \leq n-j$, from equation (4.9), we get $K_{i,i+j-1}$ which satisfies

$$T_{i,i}K_{i,i+j-1} = K_{i,i+j-1}T_{i+j-1,i+j-1}. \qquad \blacksquare$$

We recall a result from [20], which describes an invertible operator intertwining any two operators in $\mathcal{F}\mathcal{B}_n(\Omega)$.

Proposition 4.5. *If X is an invertible operator intertwining two operators T and $\widetilde{T}$ from $\mathcal{F}\mathcal{B}_n(\Omega)$, then X and X^{-1} are upper triangular.*

Lemma 4.6. *Let T and $\widetilde{T}$ be operators in $\mathcal{C}\mathcal{F}\mathcal{B}_n(\Omega)$. Let X be a bounded linear operator of the form*

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & \cdots & X_{1,n} \\ 0 & X_{2,2} & X_{2,3} & \cdots & X_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{n-1,n-1} & X_{n-1,n} \\ 0 & \cdots & \cdots & 0 & X_{n,n} \end{pmatrix}.$$

If $X\widetilde{T} = TX$ and X is invertible, then

$$\begin{pmatrix} X_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & X_{2,2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & 0 & X_{n,n} \end{pmatrix} \begin{pmatrix} \widetilde{T}_{1,1} & \widetilde{T}_{1,2} & 0 & \cdots & 0 \\ 0 & \widetilde{T}_{2,2} & \widetilde{T}_{2,3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \widetilde{T}_{n-1,n-1} & T_{n-1,n} \\ 0 & \cdots & \cdots & 0 & \widetilde{T}_{n,n} \end{pmatrix}$$
$$= \begin{pmatrix} T_{1,1} & T_{1,2} & 0 & \cdots & 0 \\ 0 & T_{2,2} & T_{2,3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-1,n-1} & T_{n-1,n} \\ 0 & \cdots & \cdots & 0 & T_{n,n} \end{pmatrix} \begin{pmatrix} X_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & X_{2,2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & 0 & X_{n,n} \end{pmatrix}.$$

Proof. By equating the entries of $X\widetilde{T} = TX$, we get

$$X_{i,i}\widetilde{T}_{i,i} = T_{i,i}X_{i,i}, \quad 1 \le i \le n.$$

We set $Y := X \operatorname{diag}\{X_{1,1}^{-1}, X_{2,2}^{-1}, \ldots, X_{n,n}^{-1}\}$, and it is easy to see that

$$Y = \begin{pmatrix} I & X_{1,2}X_{2,2}^{-1} & X_{1,3}X_{3,3}^{-1} & \cdots & X_{1,n}X_{n,n}^{-1} \\ 0 & I & X_{2,3}X_{3,3}^{-1} & \cdots & X_{2,n}X_{n,n}^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & X_{n-1,n}X_{n,n}^{-1} \\ 0 & \cdots & \cdots & 0 & I \end{pmatrix}.$$

From $T = X\widetilde{T}X^{-1}$, we get $TY = Y \operatorname{diag}\{X_{1,1} \cdots X_{n,n}\}\widetilde{T} \operatorname{diag}\{X_{1,1} \cdots X_{n,n}\}^{-1}$ which is equivalent to

$$\begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} & \cdots & T_{1,n} \\ 0 & T_{2,2} & T_{2,3} & \cdots & T_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-1,n-1} & T_{n-1,n} \\ 0 & \cdots & \cdots & 0 & T_{n,n} \end{pmatrix} \begin{pmatrix} I & X_{1,2}X_{2,2}^{-1} & X_{1,3}X_{3,3}^{-1} & \cdots & X_{1,n}X_{n,n}^{-1} \\ 0 & I & X_{2,3}X_{3,3}^{-1} & \cdots & X_{2,n}X_{n,n}^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & X_{n-1,n}X_{n,n}^{-1} \\ 0 & \cdots & \cdots & 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} I & X_{1,2}X_{2,2}^{-1} & X_{1,3}X_{3,3}^{-1} & \cdots & X_{1,n}X_{n,n}^{-1} \\ 0 & I & X_{2,3}X_{3,3}^{-1} & \cdots & X_{2,n}X_{n,n}^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & X_{n-1,n}X_{n,n}^{-1} \\ 0 & \cdots & \cdots & 0 & I \end{pmatrix} \begin{pmatrix} T_{1,1} & X_{1,1}\widetilde{T}_{1,2}X_{2,2}^{-1} & \cdots & \cdots & X_{1,1}\widetilde{T}_{1,n}X_{n,n}^{-1} \\ 0 & T_{2,2} & X_{2,2}\widetilde{T}_{2,3}X_{3,3}^{-1} & \cdots & X_{2,2}\widetilde{T}_{2,n}X_{n,n}^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & T_{n,n} \end{pmatrix}. \tag{4.10}$$

For $1 \le i \le n-1$, from equation (4.10), we get

$$\begin{pmatrix} T_{i,i} & T_{i,i+1} \\ 0 & T_{i+1,i+1} \end{pmatrix} \begin{pmatrix} I & X_{i,i+1}X_{i+1,i+1}^{-1} \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} I & X_{i,i+1}X_{i+1,i+1}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} T_{i,i} & X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1} \\ 0 & T_{i+1,i+1} \end{pmatrix},$$

which is equivalent to

$$T_{i,i+1} - X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1} = X_{i,i+1}X_{i+1,i+1}^{-1}T_{i+1,i+1} - T_{i,i}X_{i,i+1}X_{i+1,i+1}^{-1}.$$

Consider

$$\begin{aligned} T_{i,i}(T_{i,i+1} - X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1}) &= T_{i,i+1}T_{i+1,i+1} - X_{i,i}\widetilde{T}_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1} \\ &= T_{i,i+1}T_{i+1,i+1} - X_{i,i}\widetilde{T}_{i,i+1}\widetilde{T}_{i+1,i+1}X_{i+1,i+1}^{-1} \\ &= T_{i,i+1}T_{i+1,i+1} - X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1}T_{i+1,i+1} \\ &= (T_{i,i+1} - X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1})T_{i+1,i+1}. \end{aligned}$$

In other words, $(T_{i,i+1} - X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1})$ belongs to

$$\ker(\tau_{T_{i,i},T_{i+1,i+1}}) \cap \operatorname{ran}(\tau_{T_{i,i},T_{i+1,i+1}}).$$

Since T satisfies the Property (H), we have

$$T_{i,i+1} = X_{i,i}\widetilde{T}_{i,i+1}X_{i+1,i+1}^{-1}, \quad 1 \le i \le n-1.$$

Hence

$$\begin{pmatrix} X_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & X_{2,2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & 0 & X_{n,n} \end{pmatrix} \begin{pmatrix} \widetilde{T}_{1,1} & \widetilde{T}_{1,2} & 0 & \cdots & 0 \\ 0 & \widetilde{T}_{2,2} & \widetilde{T}_{2,3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \widetilde{T}_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & 0 & \widetilde{T}_{n,n} \end{pmatrix}$$
$$= \begin{pmatrix} T_{1,1} & T_{1,2} & 0 & \cdots & 0 \\ 0 & T_{2,2} & T_{2,3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & 0 & T_{n,n} \end{pmatrix} \begin{pmatrix} X_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & X_{2,2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & 0 & X_{n,n} \end{pmatrix}.$$ ∎

Corollary 4.7. *Let $T = ((T_{i,j}))_{n\times n}$, $\widetilde{T} = ((\widetilde{T}_{i,j}))_{n\times n}$ be any two operators in $\mathcal{CFB}_n(\Omega)$. Suppose that $T_{i,j} = T_{i,i+1}T_{i+1,i+2}\cdots T_{j-1,j}$ and $\widetilde{T}_{i,j} = \widetilde{T}_{i,i+1}\widetilde{T}_{i+1,i+2}\cdots\widetilde{T}_{j-1,j}$ for $1 \le i < j \le n$. T is similar to $\widetilde{T}$ if and only if $X_{i,i}T_{i,i} = \widetilde{T}_{i,i}X_{i,i}$ and $X_{i,i}T_{i,j} = \widetilde{T}_{i,j}X_{j,j}$, where $X_{i,i} \in \mathcal{L}(\mathcal{H}_i, \widetilde{\mathcal{H}}_i)$ is an invertible linear operator for $1 \le i \le n$.*

Proof. Proof of the sufficient part follows easily. We will sketch here the proof of the necessary part. By Lemma 4.6, there exist invertible operators $X_{1,1}, X_{2,2}, \ldots, X_{n,n}$ such that

$$X_{i,i}T_{i,i} = \widetilde{T}_{i,i}X_{i,i}, \quad 1 \le i \le n$$

and

$$X_{i,i}T_{i,i+1} = \widetilde{T}_{i,i+1}X_{i+1,i+1}, \quad 1 \le i \le n-1.$$

For $1 \le i < j \le n$, it is easy to see that

$$X_{i,i}T_{i,j} = \widetilde{T}_{i,j}X_{j,j}.$$ ∎

We state and prove a result that shows the problem of finding invertible intertwining and $U + K$ intertwining between any two operators in $B_1(\Omega)$ is the same as finding a bounded linear operator with a relation in terms of the curvature of the given operators. Let $\pi : E \to \Omega$ and $\tilde{\pi} : \tilde{E} \to \Omega$ be vector bundles. We set $\mathcal{H} := \overline{\text{span}}\{\pi^{-1}(w) : w \in \Omega\}$ and $\tilde{\mathcal{H}} := \overline{\text{span}}\{\tilde{\pi}^{-1}(w) : w \in \Omega\}$. We say that a bundle map $\Phi : E \to \tilde{E}$ is a *bounded bundle map* if Φ induces a bounded linear map from $\mathcal{H}$ to $\tilde{\mathcal{H}}$, where $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are Hilbert spaces.

Proposition 4.8. *Let $T, \tilde{T} \in B_1(\Omega)$. Let $\mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the Calkin algebra, $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$. Suppose that $\mathcal{K}_T(w) - \mathcal{K}_{\tilde{T}}(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \ln \Psi(w)$, $w \in \Omega$. Then we have the following statements:*

(1) *T is unitarily equivalent to $\tilde{T}$ if and only if $\Psi(w) = |\phi(w)|^2$, for some holomorphic function ϕ on Ω.*

(2) *T is similar to $\tilde{T}$ if and only if*

$$\Psi(w) = \frac{\|\Phi(t(w))\|^2}{\|t(w)\|^2} + 1,$$

where E is a Hermitian holomorphic line bundle, $\Phi : E_T \to E$ is a bounded bundle map and t is a nonzero section of the bundle E_T.

(3) *$T \sim_{U+K} \tilde{T}$ if and only if there exists a bounded linear operator X such that $\pi(X) = \alpha[I]$, $1 > \alpha > 0$ and*

$$\Psi(w) = \ln\left(\frac{\|X(t(w))\|^2}{\|t(w)\|^2} + (1 - \alpha^2)\right),$$

where t is a nonzero section of E_T.

Proof. First, the statement (1) is well known ([8, page 326]). Then we only need to prove statement (2) and (3). For statement (2), assume that T is similar to $\tilde{T}$, i.e., there exists a bounded invertible operator Y such that $TY = Y\tilde{T}$. Without loss of generality, we can assume that $Y^*Y - I \geq 0$. Otherwise, we can choose some kY instead of Y for some $k > 0$. Thus there exists a bounded linear operator X such that $Y^*Y = I + X^*X$. Since $TY = Y\tilde{T}$ and Y is invertible, $Y(t(\cdot))$ is a nonzero section of $E_{\tilde{T}}$. For w in Ω, we have

$$\begin{aligned}\mathcal{K}_{\tilde{T}}(w) &= -\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|Y(t(w)\|^2) \\ &= -\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|X(t(w))\|^2 + \|t(w)\|^2).\end{aligned}$$

Thus we have

$$\mathcal{K}_T(w) - \mathcal{K}_{\tilde{T}}(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \ln\left(\frac{\|X(t(w))\|^2}{\|t(w)\|^2} + 1\right).$$

Let $\Phi : E \to \tilde{E}$ be the bounded bundle map which induces a bounded operator X, then this finishes the proof of necessary part.

From the given condition, there exists a bounded operator X such that

$$\begin{aligned}\mathcal{K}_{\widetilde{T}}(w) &= -\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|X(t(w))\|^2 + \|t(w)\|^2)\\ &= -\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\langle (I + X^*X)(t(w)), t(w)\rangle)\\ &= -\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|(I + X^*X)^{1/2}(t(w))\|^2).\end{aligned}$$

We set $Y := (I + X^*X)^{1/2}$. Clearly, Y is an invertible operator and $\mathcal{K}_{\widetilde{T}}(w) = \mathcal{K}_{YTY^{-1}}(w)$, $w \in \Omega$. Thus, $\widetilde{T}$ is unitarily equivalent to YTY^{-1} and hence $\widetilde{T}$ is similar to T.

Now we give the proof of statement (3). Suppose that $Y = U + K$ is an invertible operator and $YT = \widetilde{T}Y$, where U is a unitary operator and K is a compact operator. We set $\widetilde{K} := U^*K$, it is easy to see that $Y^*Y = I + \widetilde{K} + \widetilde{K}^* + \widetilde{K}^*\widetilde{K}$. We set $G := \widetilde{K} + \widetilde{K}^* + \widetilde{K}^*\widetilde{K}$. Since G is a self-adjoint compact operator, there exists $\{\lambda_k\}_{k=1}^{\infty}$ such that

$$G = \lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} \oplus \cdots \oplus \lambda_k I_{n_k} \oplus \cdots ,$$

where $\dim(\ker(G - \lambda_k)) = n_k > 0$ and $\lim_{k\to\infty} \lambda_k = 0$. Since $I + G$ is a positive operator and $\lim_{k\to\infty} \lambda_k = 0$, there exists $1 > \alpha > 0$ such that $\alpha I + G$ is positive operator. Now, set

$$K_1 := \bigoplus_{k=1}^{\infty} \big((\alpha - \lambda_k)^{\frac{1}{2}} - \alpha^{\frac{1}{2}}\big) I_{n_k}.$$

Then K_1 is a compact operator. By direct calculation, we have that

$$\big(\alpha^{\frac{1}{2}} I + K_1\big)^*\big(\alpha^{\frac{1}{2}} I + K_1\big) = \alpha I + G.$$

We set $X := \alpha^{\frac{1}{2}} I + K_1$, then we have that $\pi(X) = \alpha^{\frac{1}{2}}[I]$ and

$$X^*X + (1 - \alpha)I = (I + \widetilde{K})^*(I + \widetilde{K}).$$

It follows that

$$\mathcal{K}_T(w) - \mathcal{K}_{\widetilde{T}}(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \ln\left(\frac{\|X(t(w))\|^2}{\|t(w)\|^2} + (1 - \alpha)\right).$$

This finishes the proof of the necessary part. The sufficient part will follows from the same argument as above. ∎

Remark 4.9. R. G. Douglas, H. Kwon and S. Treil proved that for any n-hypercontraction $T \in B_1(\mathbb{D})$, T is similar to S_z^* if and only if there exists a bounded subharmonic function Ψ defined on $\mathbb{D}$ such that $\mathcal{K}_T - \mathcal{K}_{S_z^*} = \frac{\partial^2}{\partial w \partial \overline{w}} \Psi$. In Proposition 4.8, we gave a concrete description of such function Ψ. In the following, we point out that Ψ is also a bounded subharmonic function.

Since Φ is bounded, it is easy to see that $\Psi(w) = \ln(\frac{\|\Phi(t(w))\|^2}{\|t(w)\|^2} + 1)$ is a bounded function. When T is an n-hypercontraction, by the operator model theorem, there exists a holomorphic bundle $\mathcal{E}$ such that

$$E_T = E_{S_z^*} \otimes \mathcal{E},$$

where $\mathcal{E}(w) = \bigvee\{D_T(t(w))\}$, $D_T := \sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k$, and t is a nonzero section of E_T. Thus, we have that

$$\mathcal{K}_T(w) - \mathcal{K}_{S_z^*}(w) = \mathcal{K}_{\mathcal{E}}(w) = -\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|D_T(t(w))\|^2).$$

Notice that $\langle D(T)(t(z)), D(T)(t(w))\rangle$ is a positive semidefinite reproducing kernel (see [10]), so we have that

$$-\frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|D_T(t(w))\|^2) < 0,$$

then it follows that

$$\frac{\partial^2}{\partial w \partial \overline{w}} \ln\left(\frac{\|\Phi(t(w))\|^2}{\|t(w)\|^2} + 1\right) = \mathcal{K}_{S_z^*}(w) - \mathcal{K}_T(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \ln(\|D_T(t(w))\|^2) > 0.$$

Corollary 4.10. *Let T and $\widetilde{T}$ be operators in $B_1(\Omega)$. Suppose that $\{T\}' \cong \mathcal{H}^\infty(\Omega)$. If there exists $\phi_i \in \mathcal{H}^\infty(\Omega)$, $i = 1, 2$ such that*

$$\mathcal{K}_T(w) - \mathcal{K}_{\widetilde{T}}(w) = \frac{\partial^2}{\partial w \partial \overline{w}} \ln(|\phi_1(w)|^2 + |\phi_2(w)|^2), \quad w \in \Omega,$$

then T is similar to $\widetilde{T}$.

Remark 4.11. It is challenging to decide when an intertwining operator (or a holomorphic bundle map) between two Cowen–Douglas operators is invertible. Thus, it is natural to find the bounded bundle map first before getting such an invertible bundle map. Thus, Proposition 4.8 gives a way to describe the similarity of two operators in $B_1(\Omega)$ by searching for the bounded bundle map to match with the difference of curvatures. For the $U + K$ similarity case, by using Proposition 4.8, we see that the bounded operator appear in the difference of curvature can also be in the form of a unitary operator plus a compact operator but may not be invertible.

Now we state and prove one of the main theorems of the paper.

Theorem 4.12. *Let T and $\widetilde{T}$ be operators in $\mathcal{CFB}_n(\Omega)$. T is similar to $\widetilde{T}$ if and only if the following statements hold:*

(1) $\mathcal{K}_{T_{i,i}} - \mathcal{K}_{\widetilde{T}_{i,i}} = \frac{\partial^2}{\partial w \partial \overline{w}} \ln(\phi_i)$, $1 \leq i \leq n$;

(2) $\frac{\phi_i}{\phi_{i+1}} \theta_{i,i+1}(T) = \theta_{i,i+1}(\widetilde{T})$, $1 \leq i \leq n-1$,

where $\phi_i = \frac{\|\Phi_i(t_i)\|^2}{\|t_i\|^2} + 1$, $\Phi_i : E_{T_{i,i}} \to E_i$ is a bounded bundle map, t_i is a nonzero section of bundle $E_{T_{i,i}}$, E_i is a Hermitian holomorphic line bundle for $1 \leq i \leq n$.

Proof. Suppose conditions (1) and (2) are satisfied. By Proposition 4.8, there exist invertible operators $X_1, X_2, \dots, X_n$ such that

$$T_{i,i} = X_i \widetilde{T}_{i,i} X_i^{-1}, \quad i = 1, 2, \dots, n-1.$$

Let $\overline{T}$ denote the following operator:

$$\overline{T} = \begin{pmatrix} \widetilde{T}_{1,1} & X_1 T_{1,2} X_2^{-1} & \cdots & \cdots & X_1 T_{1,n} X_n^{-1} \\ 0 & \widetilde{T}_{2,2} & X_2 T_{2,3} X_3^{-1} & \cdots & X_2 T_{2,n} X_n^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{T}_{n-1,n-1} & X_{n-1} T_{n-1,n} X_n^{-1} \\ 0 & 0 & \cdots & 0 & \widetilde{T}_{n,n} \end{pmatrix}.$$

By the definition of $\overline{T}$, it follows that $\overline{T}$ is similar to T.

We set

$$A := \begin{pmatrix} \widetilde{T}_{1,1} & X_1 T_{1,2} X_2^{-1} & 0 & \cdots & 0 \\ 0 & \widetilde{T}_{2,2} & X_2 T_{2,3} X_3^{-1} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{T}_{n-1,n-1} & X_{n-1} T_{n-1,n} X_n^{-1} \\ 0 & 0 & \cdots & 0 & \widetilde{T}_{n,n} \end{pmatrix}$$

and

$$B := \begin{pmatrix} \widetilde{T}_{1,1} & \widetilde{T}_{1,2} & 0 & \cdots & 0 \\ 0 & \widetilde{T}_{2,2} & \widetilde{T}_{2,3} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{T}_{n-1,n-1} & \widetilde{T}_{n-1,n} \\ 0 & 0 & \cdots & 0 & \widetilde{T}_{n,n} \end{pmatrix}.$$

It is easy to see that

$$\theta_{i,i+1}(\overline{T}) = \theta_{i,i+1}(A), \quad \theta_{i,i+1}(\widetilde{T}) = \theta_{i,i+1}(B), \quad 1 \le i \le n-1.$$

We claim that A is unitarily equivalent to B. In fact, by [20, Theorem 3.6], we only need to prove the second fundamental form of A and B are same. Clearly, $X_{i+1}(t_{i+1}(\cdot))$ is a nonzero section of $E_{\widetilde{T}_{i+1,i+1}}$,

$$\begin{aligned} \theta_{i,i+1}(A)(w) &= \frac{\|X_i T_{i,i+1} X_{i+1}^{-1} X_{i+1}(t_{i+1}(w))\|^2}{\|X_{i+1}(t_{i+1}(w))\|^2} \\ &= \frac{\|X_i T_{i,i+1}(t_{i+1}(w))\|^2}{\|X_{i+1}(t_{i+1}(w))\|^2} \end{aligned}$$

and

$$\theta_{i,i+1}(B)(w) = \frac{\|\widetilde{T}_{i,i+1} X_{i+1}(t_{i+1}(w))\|^2}{\|X_{i+1}(t_{i+1}(w))\|^2}.$$

Since

$$\phi_i(w) = \frac{\|X_i(t_i(w))\|^2}{\|t_i(w)\|^2}, \quad 1 \le i \le n,$$

we have

$$\begin{aligned}
\theta_{i,i+1}(B) &= \theta_{i,i+1}(\widetilde{T}) \\
&= \frac{\phi_i(w)}{\phi_{i+1}(w)} \theta_{i,i+1}(T) \\
&= \frac{\|X_i(T_{i,i+1}(t_{i+1}(w)))\|^2}{\|T_{i,i+1}(t_{i+1}(w))\|^2} \frac{\|t_{i+1}(w)\|^2}{\|X_{i+1}(t_{i+1}(w))\|^2} \frac{\|T_{i,i+1}(t_{i+1}(w))\|^2}{\|t_{i+1}(w)\|^2} \\
&= \frac{\|X_i T_{i,i+1}(t_{i+1}(w))\|^2}{\|X_{i+1}(t_{i+1}(w))\|^2} \\
&= \theta_{i,i+1}(A).
\end{aligned}$$

Thus, there exists a diagonal unitary operator $V = \operatorname{diag}\{V_1, V_2, \ldots, V_n\}$ such that $V^* B V = A$. We set $X := \operatorname{diag}\{X_1, X_2, \ldots, X_n\}$, consider

$$\begin{aligned}
VXTX^{-1}V^* &= V\bar{T}V^* \\
&= \begin{pmatrix}
\widetilde{T}_{1,1} & \widetilde{T}_{1,2} & V_1X_1T_{1,3}X_3^{-1}V_3^* & V_1X_1T_{1,4}X_4^{-1}V_4^* & \cdots & V_1X_1T_{1,n}X_n^{-1}V_n^* \\
0 & \widetilde{T}_{2,2} & \widetilde{T}_{2,3} & V_2X_2T_{2,4}X_4V_4^* & \cdots & V_2X_2T_{2,n}X_n^{-1}V_n^* \\
0 & 0 & \widetilde{T}_{3,3} & \widetilde{T}_{3,4} & \cdots & V_3X_3T_{3,n}X_n^{-1}V_n^* \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \widetilde{T}_{n-2,n-2} & \widetilde{T}_{n-2,n-1} & V_{n-2}X_{n-2}T_{n-2,n}X_n^{-1}V_n^* \\
0 & 0 & 0 & \cdots & \widetilde{T}_{n-1,n-1} & \widetilde{T}_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & \widetilde{T}_{n,n}
\end{pmatrix}.
\end{aligned}$$

By Lemma 4.4, there exist a bounded operator K such that $I + K$ is invertible and

$$(I + K)VXTX^{-1}V^*(I + K)^{-1} = \widetilde{T}.$$

Thus T is similar to $\widetilde{T}$.

On the other hand, suppose that T is similar to $\widetilde{T}$, i.e., there is an invertible linear operator X such that $TX = X\widetilde{T}$. By Proposition 4.5, X is upper triangular. By Lemma 4.6, we have

$$\begin{aligned}
&\begin{pmatrix}
X_{1,1} & 0 & 0 & \cdots & 0 \\
0 & X_{2,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & X_{n-1,n-1} & 0 \\
0 & \cdots & \cdots & 0 & X_{n,n}
\end{pmatrix}
\begin{pmatrix}
T_{1,1} & T_{1,2} & 0 & \cdots & 0 \\
0 & T_{2,2} & T_{2,3} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & T_{n-1,n-1} & 0 \\
0 & \cdots & \cdots & 0 & T_{n,n}
\end{pmatrix} \\
&= \begin{pmatrix}
\widetilde{T}_{1,1} & \widetilde{T}_{1,2} & 0 & \cdots & 0 \\
0 & \widetilde{T}_{2,2} & \widetilde{T}_{2,3} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \widetilde{T}_{n-1,n-1} & 0 \\
0 & \cdots & \cdots & 0 & \widetilde{T}_{n,n}
\end{pmatrix}
\begin{pmatrix}
X_{1,1} & 0 & 0 & \cdots & 0 \\
0 & X_{2,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & X_{n-1,n-1} & 0 \\
0 & \cdots & \cdots & 0 & X_{n,n}
\end{pmatrix}.
\end{aligned}$$

It follows that

$$X_{i,i}T_{i,i} = \widetilde{T}_{i,i}X_{i,i}, \quad X_{i,i}T_{i,i+1} = \widetilde{T}_{i,i+1}X_{i+1,i+1}.$$

We set

$$\phi_i(w) := \frac{\|X_i(t_i(w))\|^2}{\|t_i(w)\|^2}, \quad 1 \le i \le n.$$

By following the same argument as in the sufficient part and from Proposition 4.8, we can conclude that

$$\mathcal{K}_{T_{i,i}} - \mathcal{K}_{\widetilde{T}_{i,i}} = \frac{\partial^2}{\partial w \partial \overline{w}} \ln(\phi_i)$$

and

$$\frac{\phi_i}{\phi_{i+1}}\theta_{i,i+1}(T) = \frac{\|X_i T_{i,i+1}(t_{i+1})\|^2}{\|X_{i+1}(t_{i+1})\|^2} = \frac{\|\widetilde{T}_{i,i+1}X_{i+1}(t_{i+1})\|^2}{\|X_{i+1}(t_{i+1})\|^2} = \theta_{i,i+1}(\widetilde{T}).$$

This finishes the proof of the necessary part. ∎

5. Applications

In this section, we construct uncountably many (non-similar) strongly irreducible operators in $B_n(\mathbb{D})$, where $\mathbb{D}$ $(:= \{z \in \mathbb{C} : |z| < 1\})$ is the unit disc. We also characterize a class of strongly irreducible weakly homogeneous operators in $B_n(\mathbb{D})$. Let $M_{i,z}$ be the multiplication operator on a reproducing kernel Hilbert space $\mathcal{H}_{K_i}$ of holomorphic functions defined on $\mathbb{D}$, $1 \le i \le n$. Suppose $\{M_{i,z}\}' = H^\infty(\mathbb{D})$ and $M^*_{i,z} \in B_1(\mathbb{D})$, $1 \le i \le n$. We set

$$T := \begin{pmatrix} M^*_{1,z} & M^*_{\phi_{1,2}} & M^*_{\phi_{1,3}} & \cdots & M^*_{\phi_{1,n}} \\ & M^*_{2,z} & M^*_{\phi_{2,3}} & \cdots & M^*_{\phi_{2,n}} \\ & & \ddots & \ddots & \vdots \\ & & & M^*_{n-1,z} & M^*_{\phi_{n-1,n}} \\ & & & & M^*_{n,z} \end{pmatrix},$$

$$\widetilde{T} := \begin{pmatrix} M^*_{1,z} & M^*_{\widetilde{\phi}_{1,2}} & M^*_{\widetilde{\phi}_{1,3}} & \cdots & M^*_{\widetilde{\phi}_{1,n}} \\ & M^*_{2,z} & M^*_{\widetilde{\phi}_{2,3}} & \cdots & M^*_{\widetilde{\phi}_{2,n}} \\ & & \ddots & \ddots & \vdots \\ & & & M^*_{n-1,z} & M^*_{\widetilde{\phi}_{n-1,n}} \\ & & & & M^*_{n,z} \end{pmatrix}$$

and

$$T_1 := \begin{pmatrix} M^*_{1,z} & M^*_{\phi_{1,2}} & 0 & \cdots & \cdots & 0 \\ & M^*_{2,z} & M^*_{\phi_{2,3}} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & M^*_{n-2,z} & M^*_{\phi_{n-2,n-1}} & 0 \\ & & & & M^*_{n-1,z} & M^*_{\phi_{n-1,n}} \\ & & & & & M^*_{n,z} \end{pmatrix},$$

$$\widetilde{T}_1 := \begin{pmatrix} M^*_{1,z} & M^*_{\widetilde{\phi}_{1,2}} & 0 & \cdots & \cdots & 0 \\ & M^*_{2,z} & M^*_{\widetilde{\phi}_{2,3}} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & M^*_{n-2,z} & M^*_{\widetilde{\phi}_{n-2,n-1}} & 0 \\ & & & & M^*_{n-1,z} & M^*_{\widetilde{\phi}_{n-1,n}} \\ & & & & & M^*_{n,z} \end{pmatrix}.$$

Assume that $\phi_{i,i+1}, \widetilde{\phi}_{i,i+1}$, $1 \le i \le n-1$, are all nonzero bounded holomorphic functions. Then by Lemma 4.3, it follows that T, $\widetilde{T}$, T_1 and $\widetilde{T}_1$ are all strongly irreducible operators.

Proposition 5.1. *Let T and $\widetilde{T}$ be elements of $\mathcal{CFB}_n(\mathbb{D})$. Then T is similar to $\widetilde{T}$ if and only if the zeros, along with its multiplicity, of $\phi_{i,i+1}$ and $\widetilde{\phi}_{i,i+1}$ are the same, and both*

$$\frac{\phi_{i,i+1}}{\widetilde{\phi}_{i,i+1}} \quad \text{and} \quad \frac{\widetilde{\phi}_{i,i+1}}{\phi_{i,i+1}}$$

are elements of $H^\infty(\mathbb{D})$, $1 \le i \le n-1$.

Proof. By Lemma 4.4 and Lemma 4.6, T is similar to $\widetilde{T}$ if and only if T_1 is similar to $\widetilde{T}_1$. First, assume that the given conditions are satisfied. We need to show that T_1 is similar to $\widetilde{T}_1$. We set $\psi_{i,i+1} := \frac{\phi_{i,i+1}}{\widetilde{\phi}_{i,i+1}}$, $1 \le i \le n-1$, $X_1 := M^*_{\psi_{1,2}} M^*_{\psi_{2,3}} \cdots M^*_{\psi_{n-1,n}}$, $X_2 := M^*_{\psi_{2,3}} M^*_{\psi_{3,4}} \cdots M^*_{\psi_{n-1,n}}, \ldots, X_{n-1} := M^*_{\psi_{n,n-1}}$, $X_n := I$. It is easy to see that

$$X = \begin{pmatrix} X_1 & 0 & 0 & \ldots & 0 \\ 0 & X_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & X_n \end{pmatrix}$$

is invertible and $T_1 X = X \widetilde{T}_1$.

Conversely, suppose that T_1 is similar to $\widetilde{T}_1$. So, by Lemma 4.6, there exist invertible operators $X_1, X_2, \ldots, X_n$ such that $X_i M_{i,z}{}^* = M_{i,z}{}^* X_i$, $1 \le i \le n$, and $X_i M^*_{\phi_{i,i+1}} = M^*_{\widetilde{\phi}_{i,i+1}} X_{i+1}$, $1 \le i \le n-1$. From the commuting relation $X_i M_{i,z}{}^* = M_{i,z}{}^* X_i$ and

$\{M_{i,z}\}' = H^\infty(\mathbb{D})$, there exists a ψ_i in $H^\infty(\mathbb{D})$ such that $X_i = M^*_{\psi_i}$ for $1 \le i \le n$. Since X_i is invertible, ψ_i is nonzero, and $\frac{1}{\psi_i}$ is a bounded holomorphic function. From $X_i M^*_{\phi_{i,i+1}} = M^*_{\widetilde{\phi}_{i,i+1}} X_{i+1}$, we get $\psi_i(z)\phi_{i,i+1}(z) = \widetilde{\phi}_{i,i+1}(z)\psi_{i+1}(z)$ for all z in $\mathbb{D}$. Thus it follows that the zeros, along with its multiplicity, of the functions $\phi_{i,i+1}$ and $\widetilde{\phi}_{i,i+1}$ are the same and both $\frac{\phi_{i,i+1}}{\widetilde{\phi}_{i,i+1}}$ and $\frac{\widetilde{\phi}_{i,i+1}}{\phi_{i,i+1}}$ are bounded holomorphic functions on $\mathbb{D}$. ∎

Corollary 5.2. *Let $M_{i,z}$ be the multiplication operator on the reproducing kernel Hilbert space $\mathcal{H}_{K_i}$, where $K_i(z,w) = \frac{1}{(1-z\overline{w})^{\lambda_i}}$, $z, w \in \mathbb{D}$, $1 \le i \le n$. Suppose that $1 \le \lambda_i \le \lambda_{i+1} < \lambda_i + 2$ for $1 \le i \le n-1$. We set*

$$T_{(a_1,a_2,\ldots,a_{n-1},m_1,m_2,\ldots,m_{n-1})} := \begin{pmatrix} M^*_{1,z} & M^*_{(z-a_1)^{m_1}} & M^*_{\phi_{1,3}} & \cdots & M^*_{\phi_{1,n-1}} & M^*_{\phi_{1,n}} \\ & M^*_{2,z} & M^*_{(z-a_2)^{m_2}} & M^*_{\phi_{2,4}} & \cdots & M^*_{\phi_{2,n}} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & M^*_{n-2,z} & M^*_{(z-a_{n-2})^{m_{n-2}}} & M^*_{\phi_{n-2,n}} \\ & & & & M^*_{n-1,z} & M^*_{(z-a_{n-1})^{m_{n-1}}} \\ & & & & & M^*_{n,z} \end{pmatrix}$$

and

$$\widetilde{T}_{(b_1,b_2,\ldots,b_{n-1},l_1,l_2,\ldots,l_{n-1})} := \begin{pmatrix} M^*_{1,z} & M^*_{(z-b_1)^{l_1}} & M^*_{\widetilde{\phi}_{1,3}} & \cdots & M^*_{\widetilde{\phi}_{1,n-1}} & M^*_{\widetilde{\phi}_{1,n}} \\ & M^*_{2,z} & M^*_{(z-b_2)^{l_2}} & M^*_{\widetilde{\phi}_{2,4}} & \cdots & M^*_{\widetilde{\phi}_{2,n}} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & M^*_{n-2,z} & M^*_{(z-b_{n-2})^{l_{n-2}}} & M^*_{\widetilde{\phi}_{n-2,n}} \\ & & & & M^*_{n-1,z} & M^*_{(z-b_{n-1})^{l_{n-1}}} \\ & & & & & M^*_{n,z} \end{pmatrix},$$

where $a_i, b_i \in \mathbb{D}$ and $m_i, l_i \in \mathbb{N}$ for $1 \le i \le n-1$ and

$$\frac{\phi_{i,j}}{(z-a_i)^{m_i}(z-a_{i+1})^{m_{i+1}}\ldots(z-a_{j-1})^{m_{j-1}}},$$
$$\frac{\widetilde{\phi}_{i,j}}{(z-b_i)^{l_i}(z-b_{i+1})^{l_{i+1}}\ldots(z-b_{j-1})^{l_{j-1}}}$$

are bounded holomorphic functions on $\mathbb{D}$, with $2 \le i < j \le n$, $j-i \ge 2$. Therefore $T_{(a_1,a_2,\ldots,a_{n-1},m_1,m_2,\ldots,m_{n-1})}$ is similar to $\widetilde{T}_{(b_1,b_2,\ldots,b_{n-1},l_1,l_2,\ldots,l_{n-1})}$ if and only if $a_i = b_i$ and $m_i = l_i$ for $1 \le i \le n-1$.

Let Möb denote the Möbius group of all biholomorphic automorphisms of the unit disc $\mathbb{D}$.

Definition 5.3. A bounded linear operator T, defined on a Hilbert space $\mathcal{H}$, is said to be weakly homogeneous if $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $\phi(T)$ is similar to T for all ϕ in Möb.

Now we state and prove a result which characterizes a class of weakly homogeneous operators in $B_n(\mathbb{D})$. This generalizes [13, Theorem 3.6].

Proposition 5.4. *Let $M_{i,z}$ be the multiplication operator on the reproducing kernel Hilbert space $\mathcal{H}_{K_i}$, where $K_i(z,w) = \frac{1}{(1-z\overline{w})^{\lambda_i}}$, $z, w \in \mathbb{D}$, $1 \leq i \leq n$. Suppose that $1 \leq \lambda_i \leq \lambda_{i+1} < \lambda_i + 2$ for $1 \leq i \leq n-1$. We set*

$$T := \begin{pmatrix} M_{1,z}^* & M_{\psi_{1,2}}^* & M_{\phi_{1,3}}^* & \cdots & M_{\phi_{1,n-1}}^* & M_{\phi_{1,n}}^* \\ & M_{2,z}^* & M_{\psi_{2,3}}^* & M_{\phi_{2,4}}^* & \cdots & M_{\phi_{2,n}}^* \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & M_{n-2,z}^* & M_{\psi_{n-2,n-1}}^* & M_{\phi_{n-2,n}}^* \\ & & & & M_{n-1,z}^* & M_{\psi_{n-1,n}}^* \\ & & & & & M_{n,z}^* \end{pmatrix},$$

where $\psi_{i,i+1} \in C(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$, $1 \leq i \leq n-1$, is nonzero and $\phi_{i,j} \in H^\infty(\mathbb{D})$ for $1 \leq i < j \leq n$ and $j - i \geq 2$. The operator T is weakly homogeneous if and only if each $\psi_{i,i+1}$, $1 \leq i \leq n-1$, is non-vanishing.

Proof. We set

$$T_1 := \begin{pmatrix} M_{1,z}^* & M_{\psi_{1,2}}^* & 0 & \cdots & \cdots & 0 \\ & M_{2,z}^* & M_{\psi_{2,3}}^* & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & M_{n-2,z}^* & M_{\psi_{n-2,n-1}}^* & 0 \\ & & & & M_{n-1,z}^* & M_{\psi_{n-1,n}}^* \\ & & & & & M_{n,z}^* \end{pmatrix}.$$

By Corollary 3.6, Lemma 4.4 and Lemma 4.6, it follows that T is weakly homogeneous if and only if T_1 is weakly homogeneous. First, we show the necessary part, i.e., if the given conditions are satisfied, then T_1 is weakly homogeneous. It suffices to show that T_1^* is weakly homogeneous. To show this, we consider

$$\begin{aligned} X_1 &= M_{\frac{(\psi_{1,2}\circ\phi^{-1})((\phi^{-1})'\circ\phi)}{\psi_{1,2}}} M_{\frac{(\psi_{2,3}\circ\phi^{-1})((\phi^{-1})'\circ\phi)}{\psi_{2,3}}} \cdots M_{\frac{(\psi_{n-1,n}\circ\phi^{-1})((\phi^{-1})'\circ\phi)}{\psi_{n-1,n}}} C_{\phi^{-1}}, \\ X_2 &= M_{\frac{(\psi_{2,3}\circ\phi^{-1})((\phi^{-1})'\circ\phi)}{\psi_{2,3}}} \cdots M_{\frac{(\psi_{n-1,n}\circ\phi^{-1})((\phi^{-1})'\circ\phi)}{\psi_{n-1,n}}} C_{\phi^{-1}}, \\ &\vdots \\ X_{n-1} &= M_{\frac{(\psi_{n-1,n}\circ\phi^{-1})((\phi^{-1})'\circ\phi)}{\psi_{n-1,n}}} C_{\phi^{-1}}, \\ X_n &= C_{\phi^{-1}}, \end{aligned}$$

where $C_{\phi^{-1}}(f) := f \circ \phi^{-1}$, $f \in \mathrm{Hol}(\mathbb{D})$, $\phi \in$ Möb. As each X_i, $1 \le i \le n$, is invertible, so

$$X_\phi = \begin{pmatrix} X_1 & 0 & 0 & \dots & 0 \\ 0 & X_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & X_n \end{pmatrix}$$

is invertible and note that $\phi(T_1^*)X_\phi = X_\phi T_1^*$.

Conversely, assume that T_1 is weakly homogeneous. So, by Lemma 4.6, there exist invertible operators $X_1, X_2, \dots, X_n$ such that $X_i M_{i,z}^* = \phi(M_{i,z}^*)X_i$, $1 \le i \le n$, and $X_i M_{\psi_{i,i+1}}^* = \phi'(M_{i,z}^*)M_{\psi_{i,i+1}}^* X_{i+1}$, $1 \le i \le n-1$. We set $t_i(w) := K_{i,\overline{w}}(\cdot)$ and $t_{i,\phi} := t_i \circ \phi^{-1}$, where $K_{i,\overline{w}}(z) = K_i(z, \overline{w})$ for $z, w \in \mathbb{D}$. Since $X_i M_{i,z}^* = \phi(M_{i,z}^*)X_i$, there is a ψ_i in $H^\infty(\mathbb{D})$ such that $X_i(t_i(w)) = \psi_i(w)t_{i,\phi}(w)$ for all w in $\mathbb{D}$, $1 \le i \le n$. For $1 \le i \le n$, X_i is invertible, so $\psi_i(w) \neq 0$ for all w in $\mathbb{D}$, and $\frac{1}{\psi_i}$ is a bounded holomorphic function. From $X_i M_{\psi_{i,i+1}}^* = \phi'(M_{i,z}^*)M_{\psi_{i,i+1}}^* X_{i+1}$, $1 \le i \le n-1$, we get

$$\psi_i(w)\overline{\psi_{i,i+1}(\overline{w})} = \psi_{i+1}(w)\overline{\psi_{i,i+1}(\overline{\phi^{-1}(w)})}\phi'(\phi^{-1}(w)), \quad w \in \mathbb{D}. \tag{5.1}$$

We claim that $\psi_{i,i+1}$, $1 \le i \le n-1$, is non-vanishing. Suppose on contrary there exists a point $w_0 \in \mathbb{D}$ such that $\psi_{i,i+1}(w_0) = 0$. Since Möb acts transitively on $\mathbb{D}$, by equation (5.1), it follows that $\psi_{i,i+1}(w) = 0$ for all $w \in \mathbb{D}$ and hence $\psi_{i,i+1}(w) = 0$ for all w in $\overline{\mathbb{D}}$. This contradicts that $\psi_{i,i+1}$ is a nonzero function. Thus $\psi_{i,i+1}(w) \neq 0$ for all $w \in \mathbb{D}$.

Now we show that $\psi_{i,i+1}$, $1 \le i \le n-1$, is non-vanishing on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Replacing ϕ by a biholomorphic map $z \mapsto e^{i\theta}z$ in equation (5.1), we obtain

$$\psi_i(w)\overline{\psi_{i,i+1}(\overline{w})} = e^{i\theta}\psi_{i+1}(w)\overline{\psi_{i,i+1}(e^{i\theta}\overline{w})}, \quad w \in \mathbb{D}, \theta \in \mathbb{R}. \tag{5.2}$$

Suppose there exists a point $e^{i\theta_0}$ such that $\psi_{i,i+1}(e^{i\theta_0}) = 0$. Choose a sequence $\{w_n\}$ in $\mathbb{D}$ such that $w_n \to e^{-i\theta_0}$ as $n \to \infty$. From equation (5.2), we get

$$\psi_i(w_n)\overline{\psi_{i,i+1}(\overline{w}_n)} = e^{i\theta}\psi_{i+1}(w_n)\overline{\psi_{i,i+1}(e^{i\theta}\overline{w}_n)}. \tag{5.3}$$

Since $\psi_{i,i+1} \in C(\overline{\mathbb{D}})$ and ψ_i, ψ_{i+1} are bounded above and below on $\mathbb{D}$, from equation (5.3), as $n \to \infty$, we get $\psi_{i,i+1}(e^{i(\theta+\theta_0)}) = 0$ for all $\theta \in \mathbb{R}$. Thus $\psi_{i,i+1}$ is zero on at every point of $\mathbb{T}$, and hence $\psi_{i,i+1}$ vanishes identically on $\overline{\mathbb{D}}$. This again contradicts the hypothesis that $\psi_{i,i+1}$ is a nonzero function. This completes the proof. ■

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