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Anick type automorphisms and new irreducible representations of Leavitt path algebras

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Abstract. In this article, we give a new class of automorphisms of Leavitt path algebras of arbitrary graphs. Consequently, we obtain Anick type automorphisms of these Leavitt path algebras and new irreducible representations of Leavitt algebras of type $(1, n)$.

1. Introduction

Given a row-finite directed graph E and any field K, Abrams and Aranda Pino in [3] and independently Ara, Moreno, and Pardo in [9] introduced the *Leavitt path algebra* $L_K(E)$. Abrams and Aranda Pino later extended the definition in [4] to all countable directed graphs. Goodearl in [17] extended the notion of Leavitt path algebras $L_K(E)$ to all (possibly uncountable) directed graphs E. Leavitt path algebras generalize the Leavitt algebras $L_K(1,n)$ of [20] and also contain many other interesting classes of algebras. In addition, Leavitt path algebras are intimately related to graph C^*-algebras (see [22]). During the past fifteen years, Leavitt path algebras have become a topic of intense investigation by mathematicians from across the mathematical spectrum. We refer the reader to [1, 2] for a detailed history and overview of Leavitt path algebras.

The study of the module theory over Leavitt path algebras was initiated in [8], in connection with some questions in algebraic K-theory. As an important step in the study of modules over a Leavitt path algebra $L_K(E)$, the simple $L_K(E)$-modules have been investigated in numerous articles; see, e.g., [5, 7, 10, 12, 24].

Although, in general, classification of simple $L_K(E)$-modules seems to be a quite difficult task, recently there have been obtained a number of interesting results describing special classes of simple modules for Leavitt path algebras among which we mention, for example, the following ones. Following the ideas of Smith [21], Chen [12] constructed two types of simple modules N_w and $V_{[p]}$ for the Leavitt path algebra $L_K(E)$ of an arbitrary graph E by using various sinks w in E and the equivalence class $[p]$ of infinite paths tail-equivalent to a fixed infinite path p in E, respectively. Ara and Rangaswamy [10] generalized Chen's result and constructed additional classes of non-isomorphic simple $L_K(E)$-modules $N_v^{B_{H(v)}}$, $N_v^{H(v)}$, and $V_{[c^\infty]}^f$ which associated respectively to both infinite emitters v and pairs (c, f) consisting of exclusive cycles c together with irreducible

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polynomials $f \in K[x] \setminus \{1-x\}$. They called all these simple modules over $L_K(E)$ *Chen modules*. Also, by using the structure of the primitive ideals over the Leavitt path algebra of an arbitrary graph described in [23], they showed that every primitive ideal of the Leavitt path algebra of an arbitrary graph can be realized as the annihilator of a Chen module. Rangaswamy [24] constructed an additional class of simple $L_K(E)$-modules, $N_{v\infty}$, by using infinite emitters v. By a different method from those presented in [10], Ánh and the second author [7] constructed simple $L_K(E)$-modules S_c^f associated to pairs (c, f) consisting of simple closed paths c together with irreducible polynomials f in $K[x]$. We also should mention that Ara and Rangaswamy [10] showed that all simple modules over the Leavitt path algebra of a finite graph in which every vertex is in at most one cycle are exactly N_w, $V_{[p]}$ and $V^f_{[c^\infty]} \cong S^f_c$, which are cited above. Koç and Özaydın [19] have classified all finite-dimensional modules for the Leavitt path algebra $L_K(E)$ of a row-finite graph E via an explicit Morita equivalence given by an effective combinatorial (reduction) algorithm on the graph. These obtained results induce our investigation to the study of simple modules for Leavitt path algebras of graphs having a vertex that is in at least two cycles. The most important case of this class is the Leavitt path algebra of a rose with $n \geq 2$ petals. It is exactly the Leavitt algebra $L_K(1, n)$ (see, e.g., Proposition 2.6).

As of the writing of this article, there are two known classes of non-isomorphic simple modules for Leavitt path algebras $L_K(R_n)$ of the rose R_n with $n \geq 2$ petals:

- simple modules $V_{[p]}$ associated to infinite irrational paths p;
- simple modules S_c^f associated to pairs (c, f) consisting of simple closed paths c together with irreducible polynomials f in $K[x]$.

The main goal of this article is to construct Anick type automorphisms of Leavitt path algebras of graphs having finitely many vertices and construct new classes of simple $L_K(R_n)$-modules by studying the twisted modules of the simple modules S_c^f under Anick type automorphisms of $L_K(R_n)$.

We should recall some history about investigations of automorphisms of graph C^*-algebras and Leavitt path algebras. In his beautiful paper [16], Cuntz initiated systematic investigations of the automorphism group of $\mathcal{O}_n$ ($n \geq 2$). In particular, he showed that there is a one-to-one correspondence between unitary elements of the Cuntz algebra $\mathcal{O}_n$ and endomorphisms of $\mathcal{O}_n$ via $u \mapsto \lambda_u$, where $\lambda_u(S_i) = uS_i$. The problem is that in general there is no easy way of verifying which unitaries u give rise to automorphisms λ_u. In [15], Conti and Szymański provided a remedy to this problem for a large class of endomorphisms related to unitary matrices in $M_{n^k}(\mathbb{C})$ contained in the UHF-subalgebra. In [14], motivated by Cuntz's idea [16], Conti, Hong, and Szymański initiated a systematic investigation of endomorphisms of graph C^*-algebras $C^*(E)$ of finite graphs E. They introduced a class of endomorphisms fixing all vertex projections λ_u of $C^*(E)$ corresponding to unitaries u in the multiplier algebra $M(C^*(E))$ which commute with all vertex projections. They studied localized endomorphisms of the graph algebra $C^*(E)$ of a finite graph E without sinks, that is, endomorphisms λ_u corresponding to unitaries u from the algebraic part of the core AF-subalgebra which commute with the vertex projections. Then, they obtained a criterion of invertibility of such localized endomorphisms, and provided a criterion of invertibility of the restriction of a localized endomorphism to the diagonal maximum abelian subalgebra (MASA), as well as gave combinatorial criteria for localized endomorphisms corresponding to permutation unitaries to be automorphisms. We should mention that all endomorphisms (hence also automorphisms) are studied which only point-wise fix the vertex projections. They may well move the diagonal MASA. In [11], Avery, Johansen, and Szymański studied permutative automorphisms of graph C^*-algebras $C^*(E)$ and Leavitt path algebras of finite graphs E without sinks or sources where every cycle has an exit, by introducing a notion of a permutation graph and use this concept to determine whether a given permutative endomorphism is an automorphism. In [18], motivated by [15], Johansen, Sørensen, and Szymański investigated polynomial endomorphisms of graph C^*-algebras $C^*(E)$ and Leavitt path algebras of finite graphs E without sinks or sources where every cycle has an exit, by introducing the coding graph corresponding to each such an endomorphism, and used this concept to give a criterion for the endomorphism to restrict to an automorphism of the diagonal MASA. We should note that Szymański et al. [11, 18] focused only on Leavitt path algebras over integral domains of characteristic 0.

In this article, motivated by the above works, we give a new class of automorphisms of Leavitt path algebras $L_K(E)$ of arbitrary graphs E over an arbitrary field K, by using special pairs (P, Q) consisting of matrices in $M_n(L_K(E))$ which commutes with all vertices in E, where n is an arbitrary positive integer. In particular, if E is a graph having finitely many vertices, then P is exactly an invertible matrix in $M_n(L_K(E))$ which commute with all vertices in E and Q is the inverse of P. These automorphisms fix all vertices, but do not globally preserve the diagonal MASA in general. It is interesting to note that our automorphisms include analogues of the Anick automorphism of the free associative algebra $K\langle x_1, x_2, x_3\rangle$ (cf. [13, p. 343]). The Anick automorphism is well known in the context of the tame generators problem which asks if the automorphism group of the K-algebra $K\langle x_1, \dots, x_n\rangle$ is generated by the so-called elementary automorphisms. It is notable that, in 2007, Umirbaev [25] solved this problem in the negative when $n = 3$ and K is of characteristic zero, by showing that the Anick automorphism cannot be obtained by composing elementary automorphisms.

The article is organized as follows. In Section 2, we provide a method to construct automorphisms of Leavitt path algebras of graphs (Theorem 2.2 and Corollary 2.3). Consequently, we obtain Anick type automorphisms of these Leavitt path algebras (Corollaries 2.5 and 2.8). In Section 3, based on Corollary 2.8 and the simple modules S_c^f mentioned above, we construct new classes of simple $L_K(R_n)$-modules (Theorems 3.6 and 3.8).

2. Anick type automorphisms of Leavitt path algebras

The aim of this section is to describe automorphisms of Leavitt path algebras of arbitrary graphs (Theorem 2.2). Consequently, we provide a method to construct automorphisms of unital Leavitt path algebras in terms of invertible matrices (Corollary 2.3) and Anick type automorphisms of these Leavitt path algebras (Corollaries 2.5 and 2.8).

We begin this section by recalling some useful notions of graph theory. A (*directed*) *graph* is a quadruplet $E = (E^0, E^1, s, r)$ consisting of two disjoint sets E^0 and E^1, called *vertices* and *edges* respectively, together with two maps $s, r : E^1 \to E^0$. The vertices $s(e)$ and $r(e)$ are referred to as the *source* and the *range* of the edge e, respectively. A vertex v for which $s^{-1}(v)$ is empty is called a *sink*; a vertex v is *regular* if $0 < |s^{-1}(v)| < \infty$; a vertex v is an *infinite emitter* if $|s^{-1}(v)| = \infty$; and a vertex is *singular* if it is either a sink or an infinite emitter.

A *finite path of length* n in a graph E is a sequence $p = e_1 \cdots e_n$ of edges $e_1, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, we say that the path p starts at the vertex $s(p) := s(e_1)$ and ends at the vertex $r(p) := r(e_n)$, we write $|p| = n$ for the length of p. We consider the elements of E^0 to be paths of length 0. We denote by E^* the set of all finite paths in E. An edge f is an *exit* for a path $p = e_1 \cdots e_n$ if $s(f) = s(e_i)$ but $f \neq e_i$ for some $1 \le i \le n$. A finite path p of positive length is called a *closed path based at* v if $v = s(p) = r(p)$. A *cycle* is a closed path $p = e_1 \cdots e_n$, and for which the vertices $s(e_1), s(e_2), \dots, s(e_n)$ are distinct. A closed path c in E is called *simple* if $c \neq d^n$ for any closed path d and integer $n \geq 2$. We denoted by SCP(E) the set of all simple closed paths in E.

Definition 2.1. For an arbitrary graph $E = (E^0, E^1, s, r)$ and any field K, the *Leavitt path algebra* $L_K(E)$ *of the graph* E *with coefficients in* K is the K-algebra generated by the union of the set E^0 and two disjoint copies of E^1, say E^1 and $\{e^* \mid e \in E^1\}$, satisfying the following relations for all $v, w \in E^0$ and $e, f \in E^1$:

(1) $vw = \delta_{v,w} w$,

(2) $s(e)e = e = er(e)$ and $e^*s(e) = e^* = r(e)e^*$,

(3) $e^* f = \delta_{e,f} r(e)$,

(4) $v = \sum_{e \in s^{-1}(v)} ee^*$ for any regular vertex v,

where δ is the Kronecker delta.

If E^0 is finite, then $L_K(E)$ is a unital ring having identity $1 = \sum_{v \in E^0} v$ (see, e.g., [3, Lemma 1.6]). It is easy to see that the mapping, given by $v \mapsto v$ for all $v \in E^0$, and $e \mapsto e^*$, $e^* \mapsto e$ for all $e \in E^1$, produces an involution on the algebra $L_K(E)$, and for any path $p = e_1 e_2 \cdots e_n$, the element $e_n^* \cdots e_2^* e_1^*$ of $L_K(E)$ is denoted by p^*. It can be shown [3, Lemma 1.7] that $L_K(E)$ is spanned as a K-vector space by

$$\{pq^* \mid p, q \in F(E),\ r(p) = r(q)\}.$$

Indeed, $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra: $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$, where for each $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is the set

$$\text{span}_K \{pq^* \mid p, q \in E^*,\ r(p) = r(q),\ |p| - |q| = n\}.$$

Also, $L_K(E)$ has the following property: if $\mathcal{A}$ is a K-algebra generated by a family of elements $\{a_v, b_e, c_{e^*} \mid v \in E^0,\ e \in E^1\}$ satisfying the relations analogous to (1)–(4) in Definition 2.1, then there always exists a K-algebra homomorphism $\varphi : L_K(E) \to \mathcal{A}$ given by $\varphi(v) = a_v$, $\varphi(e) = b_e$, and $\varphi(e^*) = c_{e^*}$. We will refer to this property as the universal property of $L_K(E)$.

In [16], Cuntz showed that there is a one-to-one correspondence between unitary elements of the Cuntz algebra $\mathcal{O}_n$ and endomorphisms of $\mathcal{O}_n$ via $u \mapsto \lambda_u$, where $\lambda_u(S_i) = uS_i$, and provided criteria for these endomorphisms to be automorphisms. In [14], motivated by Cuntz's results, Conti, Hong, and Szymański introduced a class of endomorphisms fixing all vertex projections λ_u of $C^*(E)$ corresponding to unitaries in the multiplier algebra $M(C^*(E))$ which commute with all vertex projections. Then, they studied localized endomorphisms of the graph algebra $C^*(E)$ of a finite graph without sinks, that is, endomorphisms λ_u corresponding to unitaries u from the algebraic part of the core AF-subalgebra which commute with the vertex projections, and obtained a criterion of invertibility of such localized endomorphisms, as well as gave combinatorial criteria for localized endomorphisms corresponding to permutation unitaries to be automorphisms.

Szymański et al. [11, 18] studied permutative automorphisms and polynomial endomorphisms of graph C^*-algebras $C^*(E)$ and Leavitt path algebras $L_K(E)$, where E is a finite graph without sinks or sources in which every cycle has an exit, and K is an integral domain of characteristic 0.

The following theorem provides us with a method to construct automorphisms of Leavitt path algebras of arbitrary graphs over an arbitrary field.

Theorem 2.2. *Let K be a field, n a positive integer, E a graph, and v and w vertices in E (they may be the same). Let $e_1, e_2, \dots, e_n$ be distinct edges in E with $s(e_i) = v$ and $r(e_i) = w$ for all $1 \le i \le n$. Let $P = (p_{i,j})$ and $Q = (q_{i,j})$ be elements of $M_n(L_K(E))$ such that $wP = Pw$, $wQ = Qw$, and $wPQ = wQP = wI_n$. Then the following statements hold.*

(i) *There exists a unique homomorphism $\varphi_{P,Q} : L_K(E) \to L_K(E)$ of K-algebras satisfying*

$$\varphi_{P,Q}(u) = u, \quad \varphi_{P,Q}(e) = e, \quad \textit{and} \quad \varphi_{P,Q}(e^*) = e^*$$

for all $u \in E^0$ and $e \in E^1 \setminus \{e_1, \dots, e_n\}$, and

$$\varphi_{P,Q}(e_i) = \sum_{k=1}^{n} e_k p_{k,i} \quad \textit{and} \quad \varphi_{P,Q}(e_i^*) = \sum_{k=1}^{n} q_{i,k} e_k^*$$

for all $1 \le i \le n$.

(ii) *If $w\varphi_{P,Q}(p_{i,j}) = wp_{i,j}$ for all $1 \le i, j \le n$, or $w\varphi_{P,Q}(q_{i,j}) = wq_{i,j}$ for all $1 \le i, j \le n$, then $\varphi_{P,Q}$ is an isomorphism and $\varphi_{P,Q}^{-1} = \varphi_{Q,P}$.*

Proof. We first note that $wp_{k,i} = p_{k,i}w$ and $wq_{i,k} = q_{i,k}w$ for all k, i (since $wP = Pw$ and $wQ = Qw$), and $\sum_{k=1}^{n} wp_{i,k}q_{k,j} = \delta_{i,j}w = \sum_{k=1}^{n} wq_{i,k}p_{k,j}$ for all i, j, where δ is the Kronecker delta.

(i) We define the elements $\{Q_u \mid u \in E^0\}$ and $\{T_e, T_{e^*} \mid e \in E^1\}$ of $L_K(E)$ by setting $Q_u = u$,

$$T_e = \begin{cases} \sum_{k=1}^{n} e_k p_{k,i} & \text{if } e = e_i \text{ for some } 1 \leq i \leq n, \\ e & \text{otherwise,} \end{cases}$$

and

$$T_{e^*} = \begin{cases} \sum_{k=1}^{n} q_{i,k} e_k^* & \text{if } e = e_i \text{ for some } 1 \leq i \leq n, \\ e^* & \text{otherwise.} \end{cases}$$

We claim that $\{Q_u, T_e, T_{e^*} \mid u \in E^0, e \in E^1\}$ is a family in $L_K(E)$ satisfying the relations analogous to (1)–(4) in Definition 2.1. Indeed, we have $Q_u Q_{u'} = uu' = \delta_{u,u'} u = \delta_{u,u'} Q_u$ for all $u, u' \in E^0$, showing relation (1).

For (2), we always have $Q_{s(e)} T_e = T_e = T_e T_{r(e)}$ and $T_{e^*} Q_{s(e)} = T_{e^*} = Q_{r(e)} T_{e^*}$ for all $e \in E^1 \setminus \{e_1, \ldots, e_n\}$. For each $1 \leq i \leq n$, since

$$ve_k = e_k w = e_k, \quad we_k^* = e_k^* v = e_k^*, \quad wp_{k,i} = p_{k,i} w, \quad \text{and} \quad wq_{i,k} = q_{i,k} w$$

for all k, we have

$$\begin{aligned} Q_v Q_{e_i} &= v \sum_{k=1}^{n} e_k p_{k,i} = \sum_{k=1}^{n} e_k p_{k,i} = Q_{e_i}, \\ Q_{e_i} Q_w &= \sum_{k=1}^{n} e_k p_{k,i} w = \sum_{k=1}^{n} e_k w p_{k,i} = \sum_{k=1}^{n} e_k p_{k,i} = Q_{e_i}, \\ Q_w T_{e_i^*} &= w \sum_{k=1}^{n} q_{i,k} e_k^* = \sum_{k=1}^{n} q_{i,k} w e_k^* = \sum_{k=1}^{n} q_{i,k} e_k^* = T_{e_i^*}, \\ T_{e_i^*} Q_v &= \sum_{k=1}^{n} q_{i,k} e_k^* v = \sum_{k=1}^{n} q_{i,k} e_k^* = T_{e_i^*}. \end{aligned}$$

For (3), we obtain that $T_{e^*} T_f = e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1 \setminus \{e_1, \ldots, e_n\}$. For each $f \in E^1 \setminus \{e_1, \ldots, e_n\}$ and $1 \leq i \leq n$, we have

$$T_{e_i^*} T_f = \sum_{k=1}^{n} q_{i,k} e_k^* f = 0 \quad \text{and} \quad T_{f^*} T_e = \sum_{k=1}^{n} f^* e_k p_{k,i} = 0,$$

since $e_k^* f = f^* e_k = 0$. For $i, j \in \{1, \ldots, n\}$, we have

$$\begin{aligned} T_{e_i^*} T_{e_j} &= \sum_{k=1}^{n} \sum_{l=1}^{n} q_{i,k} e_k^* e_l p_{l,j} = \sum_{k=1}^{n} \sum_{l=1}^{n} q_{i,k} \delta_{k,l} w p_{l,j} \\ &= \sum_{k=1}^{n} w q_{i,k} p_{k,j} = \delta_{i,j} w = \delta_{i,j} Q_w, \end{aligned}$$

since $e_k^* e_l = \delta_{k,l} w$ and $wp_{l,j} = p_{l,j} w$.

For (4), let u be a regular vertex in E. If $u \neq v$, then

$$\sum_{e\in s^{-1}(u)} T_eT_{e^*} = \sum_{e\in s^{-1}(u)} ee^* = u = Q_u.$$

Consider the case when $u = v$, that is, v is a regular vertex. Write

$$s^{-1}(v) = \{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$$

for some distinct $e_{n+1}, \dots, e_m \in E^1$ with $n \leq m < \infty$. We note that

$$\begin{aligned}\sum_{i=1}^{n} T_{e_i}T_{e_i^*} &= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} e_k p_{k,i}q_{i,l}e_l^* = \sum_{k=1}^{n}\sum_{l=1}^{n} e_k w\Big(\sum_{i=1}^{n} p_{k,i}q_{i,l}\Big)e_l^* \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} e_k(\delta_{k,l}w)e_l^* = \sum_{k=1}^{n} e_ke_k^*,\end{aligned}$$

and so, we have

$$\sum_{e\in s^{-1}(v)} T_eT_{e^*} = \sum_{i=1}^{m} T_{e_i}T_{e_i^*} = \sum_{i=1}^{m} e_ie_i^* = v = Q_v,$$

thus showing the claim. Then, by the universal property of $L_K(E)$, there exists a K-algebra homomorphism $\varphi_{P,Q} : L_K(E) \to L_K(E)$, which maps $u \mapsto Q_u$, $e \mapsto T_e$, and $e^* \mapsto T_{e^*}$, as desired.

(ii) Let P' and Q' be elements of $M_n(L_K(E))$ obtained from P and Q by applying the homomorphism $\varphi_{P,Q}$, respectively. Assume that $w\varphi_{P,Q}(p_{i,j}) = wp_{i,j}$ for all $1 \leq i, j \leq n$. Then, since $\varphi_{P,Q}$ is a K-algebra homomorphism and $wPQ = wI_n$, we have $wP'Q' = wI_n$ and $wPQ' = wI_n$. This implies that

$$wQ' = wQPQ' = QwPQ' = QwI_n = wQI_n = wQ,$$

that means, $wq_{i,j} = w\varphi_{P,Q}(q_{i,j})$ for all i, j. Similarly, we receive the fact that if $wQ' = wQ$, then $wP' = wP$. Therefore, in any case, we have both $wP' = wP$ and $wQ' = wQ$.

We claim that $\varphi_{P,Q}\varphi_{Q,P} = id_{L_K(E)}$. Indeed, it suffices to check that

$$\varphi_{P,Q}\varphi_{Q,P}(e_i) = e_i \quad \text{and} \quad \varphi_{P,Q}\varphi_{Q,P}(e_i^*) = e_i^* \quad \text{for all } 1 \leq i \leq n.$$

For each $1 \leq i \leq n$, by definition of $\varphi_{Q,P}$, $\varphi_{Q,P}(e_i) = \sum_{k=1}^{n} e_kq_{k,i} = \sum_{k=1}^{n} e_kwq_{k,i}$ and $\varphi_{Q,P}(e_i^*) = \sum_{k=1}^{n} p_{i,k}e_k^* = \sum_{k=1}^{n} p_{i,k}we_k^* = \sum_{k=1}^{n} wp_{i,k}e_k^*$, so

$$\begin{aligned}\varphi_{P,Q}\varphi_{Q,P}(e_i) &= \varphi_{P,Q}\Big(\sum_{k=1}^{n} e_kwq_{k,i}\Big) = \sum_{k=1}^{n} \varphi_{P,Q}(e_k)w\varphi_{P,Q}(q_{k,i}) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} e_lp_{l,k}wq_{k,i} = \sum_{l=1}^{n} e_l\Big(\sum_{k=1}^{n} wp_{l,k}q_{k,i}\Big) \\ &= \sum_{l=1}^{n} e_l\delta_{l,i}w = e_iw = e_i\end{aligned}$$

and

$$\begin{aligned}\varphi_{P,Q}\varphi_{Q,P}(e_i^*) &= \varphi_{P,Q}\left(\sum_{k=1}^{n} wp_{i,k}e_k^*\right) = \sum_{k=1}^{n} w\varphi_{P,Q}(p_{i,k})\varphi_{P,Q}(e_k^*) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} wp_{i,k}q_{k,l}e_l^* = \sum_{l=1}^{n}\left(\sum_{k=1}^{n} wp_{i,k}q_{k,l}\right)e_l^* \\ &= \sum_{l=1}^{n}\delta_{i,l}we_l^* = we_i^* = e_i^*,\end{aligned}$$

proving the claim. This implies that $\varphi_{P,Q}$ is surjective.

We next prove that $\varphi_{P,Q}$ is injective. To the contrary, suppose there exists a nonzero element $x \in \ker(\varphi_{P,Q})$. Then, by the reduction theorem (see, e.g., [2, Theorem 2.2.11]), there exist $a, b \in L_K(E)$ such that either $axb = u \neq 0$, for some $u \in E^0$, or $axb = p(c) \neq 0$, where c is a cycle in E without exits and $p(x)$ is a nonzero polynomial in $K[x, x^{-1}]$.

In the first case, since $axb \in \ker(\varphi_{P,Q})$, this would imply that $u = \varphi_{P,Q}(u) = 0$ in $L_K(E)$; but each vertex is well known to be a nonzero element inside the Leavitt path algebra, which is a contradiction.

So we are in the second case: there exists a cycle c in E without exits such that $axb = \sum_{i=-l}^{m} k_i c^i \neq 0$, where $k_i \in K$, l, and m are nonnegative integers, and we interpret c^i as $(c^*)^{-i}$ for negative i and interpret c^0 as $u := s(c)$. Write $c = g_1 q_2 \cdots g_t$, where $g_i \in E^1$ and t is a positive integer. If $g_i \in E^1 \setminus \{e_1, \ldots, e_n\}$ for all $1 \leq i \leq t$, then $\varphi_{P,Q}(c) = c$ and $\varphi_{P,Q}(c^*) = c^*$, so $0 \neq \sum_{i=-l}^{m} k_i c^i = \sum_{i=-l}^{m} k_i \varphi_{P,Q}(c^i) = \varphi_{P,Q}(axb) = 0$ in $L_K(E)$, a contradiction. Consider the case that there exists a $1 \leq k \leq t$ such that $g_k = e_i$ for some i. Then, since c is a cycle without exits, we must have $n = 1$ and k is a unique element such that $g_k = e_1$. Let $\alpha := g_{k+1} \cdots g_t g_1 \cdots g_{k-1} e_1$. We have that α is a cycle in E without exits and $s(\alpha) = w$. Since $n = 1$, $P = p_{1,1}$, and $Q = q_{1,1}$ are two elements of $L_K(E)$ with $wp_{1,1}q_{1,1} = w = wq_{1,1}p_{1,1}$, so $wp_{1,1}w$ is also a unit of $wL_K(E)w$ with $(wp_{1,1}w)^{-1} = wq_{1,1}w$. Moreover, $\varphi_{P,Q}(wp_{1,1}w) = \varphi_{P,Q}(w)\varphi_{P,Q}(p_{1,1})\varphi_{P,Q}(w) = w\varphi_{P,Q}(p_{1,1})w = wp_{1,1}w$ and $\varphi_{P,Q}(wq_{1,1}w) = \varphi_{P,Q}(w)\varphi_{P,Q}(q_{1,1})\varphi_{P,Q}(w) = w\varphi_{P,Q}(q_{1,1})w = wq_{1,1}w$. By [2, Lemma 2.2.7], we have

$$wL_K(E)w = \left\{\sum_{i=l}^{h} k_i\alpha^i \mid k_i \in K,\ l \leq h,\ h, l \in \mathbb{Z}\right\} \cong K[x, x^{-1}]$$

via an isomorphism that sends v to 1, α to x, and α^* to x^{-1}; and so $wp_{1,1}w = a\alpha^s$ and $wq_{1,1}w = a^{-1}\alpha^{-s}$ for some $a \in K \setminus \{0\}$ and $s \in \mathbb{Z}$. If $s \geq 0$, then

$$\begin{aligned}a\alpha^s &= wp_{1,1}w = \varphi_{P,Q}(wp_{1,1}w) = \varphi_{P,Q}(a\alpha^s) = a\varphi_{P,Q}(\alpha)^s \\ &= a\big(\varphi_{P,Q}(g_{k+1} \cdots g_t g_1 \cdots g_{k-1} e_1)\big)^s = a(g_{k+1} \cdots g_t g_1 \cdots g_{k-1} e_1 p_{1,1})^s \\ &= a(g_{k+1} \cdots g_t g_1 \cdots g_{k-1} e_1 wp_{1,1}w)^s = a^{s+1}\alpha^{s(s+1)}\end{aligned}$$

in $wL_K(E)w$, so $s = 0$, that is, $wp_{1,1}w = aw$ and $wq_{1,1}w = a^{-1}w$. If $s \leq 0$, then since $\varphi_{P,Q}(wq_{1,1}w) = wq_{1,1}w$, and by repeating the argument described in the first case, we obtain that $s = 0$, $wp_{1,1}w = aw$, and $wq_{1,1}w = a^{-1}w$. This implies that

$$\begin{aligned}\varphi_{P,Q}(c) &= \varphi_{P,Q}(g_1 \cdots g_{k-1}e_1 g_{k+1} \cdots g_t) = (g_1 \cdots g_{k-1})e_1 p_{1,1}(g_{k+1} \cdots g_t) \\ &= (g_1 \cdots g_{k-1}e_1)wp_{1,1}w(g_{k+1} \cdots g_t) = (g_1 \cdots g_{k-1}e_1)aw(g_{k+1} \cdots g_t) = ac\end{aligned}$$

and

$$\begin{aligned}\varphi_{P,Q}(c^*) &= \varphi_{P,Q}(g_t^* \cdots g_{k+1}^* e_1^* g_{k-1}^* \cdots g_1^*) = (g_t^* \cdots g_{k+1}^*)q_{1,1}e_1^*(g_{k-1}^* \cdots g_1^*) \\ &= (g_t^* \cdots g_{k+1}^*)wq_{1,1}w(e_1^* g_{k-1}^* \cdots g_1^*) = (g_t^* \cdots g_{k+1}^*)a^{-1}w(e_1^* g_{k-1}^* \cdots g_1^*) \\ &= a^{-1}c^*,\end{aligned}$$

so $\varphi_{P,Q}(c^l) = a^l c^l$ for all $l \in \mathbb{Z}$. We then have

$$0 \neq \sum_{i=-l}^{m} k_i a^i c^i = \sum_{i=-l}^{m} k_i \varphi_{P,Q}(c^i) = \varphi_{P,Q}(axb) = 0$$

in $L_K(E)$, which is a contradiction.

In any case, we arrive at a contradiction, and so we infer that $\varphi_{P,Q}$ is injective, thus $\varphi_{P,Q}$ is an isomorphism with $\varphi_{P,Q}^{-1} = \varphi_{Q,P}$, finishing the proof. ∎

Consequently, we obtain a method to construct automorphisms of unital Leavitt path algebras in terms of invertible matrices.

Corollary 2.3. *Let K be a field, n a positive integer, E a graph with finitely many vertices, and v and w vertices in E (they may be the same). Let $e_1, e_2, \ldots, e_n$ be distinct edges in E with $s(e_i) = v$ and $r(e_i) = w$ for all $1 \leq i \leq n$. Let $P = (p_{i,j})$ be a unit of $M_n(L_K(E))$ with $wP = Pw$ and $P^{-1} = (q_{i,j})$. Then the following statements hold.*

(i) *There exists a unique homomorphism $\varphi_P : L_K(E) \to L_K(E)$ of K-algebras satisfying*

$$\varphi_P(u) = u, \quad \varphi_P(e) = e, \quad \text{and} \quad \varphi_P(e^*) = e^*$$

for all $u \in E^0$ and $e \in E^1 \setminus \{e_1, \ldots, e_n\}$, and

$$\varphi_P(e_i) = \sum_{k=1}^{n} e_k p_{k,i} \quad \text{and} \quad \varphi_P(e_i^*) = \sum_{k=1}^{n} q_{i,k} e_k^*$$

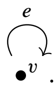
for all $1 \leq i \leq n$.

(ii) *If $\varphi_P(p_{i,j}) = p_{i,j}$ for all $1 \leq i, j \leq n$, then φ_P is an isomorphism and $\varphi_P^{-1} = \varphi_{P^{-1}}$.*

Proof. Since $wP = Pw$, we have $P^{-1}wPP^{-1} = P^{-1}PwP^{-1}$, so $wP^{-1} = P^{-1}w$. Since $PP^{-1} = I_n = P^{-1}P$, it is obvious that $wPP^{-1} = wI_n = wP^{-1}P$. Therefore, the pair of the matrices P and P^{-1} satisfies the conditions analogous to the one of the matrices P and Q in Theorem 2.2. Then, by Theorem 2.2, we immediately obtain the statements, thus finishing the proof. ■

For clarification, we illustrate Theorem 2.2 and Corollary 2.3 by presenting the following example.

Examples 2.4. Let K be a field and R_1 the following graph:

$R_1 =$  .

Then $L_K(R_1) \cong K[x, x^{-1}]$ via an isomorphism that sends v to 1, e to x, and e^* to x^{-1}. Let $P = e^*$. We have that P is a unit of $L_K(R_1)$ with $P^{-1} = e$. Then, by Corollary 2.3, we obtain the endomorphism φ_P defined by $v \mapsto v$, $e \mapsto eP = ee^* = v$, and $e^* \mapsto P^{-1}e^* = ee^* = v$. We have that φ_P is not isomorphic and $\varphi_P(P) = \varphi_P(e^*) = v \neq e^* = P$ in $L_K(R_1)$. This implies that the hypothesis "$\varphi_P(p_{i,j}) = p_{i,j}$ for all $1 \leq i, j \leq n$" in part (ii) of Corollary 2.3 cannot be removed.

In light of the well-known Anick automorphism (see [13, p. 343]) of the free associative algebra $K\langle x, y, z\rangle$, we construct Anick type automorphisms of unital Leavitt path algebras.

Corollary 2.5 (Anick type automorphism). *Let K be a field, E a graph with finitely many vertices, and v and w vertices in E (they may be the same). Let e_1 and e_2 be two distinct edges in E with $s(e_i) = v$ and $r(e_i) = w$ for all i. Let $A_E(e_1, e_2)$ be the K-subalgebra of $L_K(E)$ generated by the sets E^0, $E^1 \setminus \{e_2\}$ and $\{e^* \mid e \in E^1 \setminus \{e_1\}\}$. Then, for any $p \in A_E(e_1, e_2)$ with $wp = pw$, there exists a unique automorphism σ_p of the K-algebra $L_K(E)$ satisfying*

$$\sigma_p(e_2) = e_2 + e_1 p, \quad \sigma_p(e_1^*) = e_1^* - pe_2^*,$$
$$\sigma_p^{-1}(e_2) = e_2 - e_1 p, \quad \sigma_p^{-1}(e_1^*) = e_1^* + pe_2^*$$

and $\sigma_p(q) = q$ for all $q \in A_E(e_1, e_2)$.

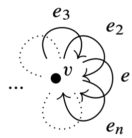
Proof. Let $P = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in M_2(L_K(E))$. We then have that P is a unit of $M_2(L_K(E))$ with $P^{-1} = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}$. It is clear that $\sigma_p = \varphi_P$, which is described in Corollary 2.3 (i), and $\varphi_P(q) = q$ for all $q \in A_E(e_1, e_2)$. Then, using Corollary 2.3, we immediately receive the corollary, thus finishing the proof. ■

Let K be a field and $n \geq 2$ any integer. Then the *Leavitt K-algebra of type* $(1; n)$, denoted by $L_K(1, n)$, is the K-algebra

$$K\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle / \left\langle \sum_{i=1}^{n} x_i y_i - 1, y_i x_j - \delta_{i,j} 1 \mid 1 \leq i, j \leq n \right\rangle.$$

Notationally, it is often more convenient to view $L_K(1,n)$ as the free associative K-algebra on the $2n$ variables $x_1,\dots,x_n,y_1,\dots,y_n$ subject to the relations $\sum_{i=1}^n x_i y_i = 1$ and $y_i x_j = \delta_{i,j}1$ $(1 \le i,\ j \le n)$; see [20] for more details.

For any integer $n \ge 2$, we let R_n denote the *rose with n petals* graph having one vertex and n loops:

$R_n =$  .

Then $L_K(R_n)$ is defined to be the K-algebra generated by v, $e_1,\dots,e_n$, $e_1^*,\dots,e_n^*$, satisfying the relations

$$v^2 = v, \quad ve_i = e_i = e_i v, \quad ve_i^* = e_i^* = e_i^* v, \quad e_i^* e_j = \delta_{i,j} v, \quad \text{and} \quad \sum_{i=1}^n e_i e_i^* = v$$

for all $1 \le i, j \le n$. In particular, $v = 1_{L_K(R_n)}$.

Proposition 2.6 ([2, Proposition 1.3.2]). *Let $n \ge 2$ be any positive integer, K a field, and R_n the rose with petals. Then $L_K(1,n) \cong L_K(R_n)$ as K-algebras.*

Proof. We can show that the map $\varphi : L_K(1,n) \to L_K(R_n)$, given by the extension of $\varphi(1) = v$, $\varphi(x_i) = e_i$, and $\varphi(y_i) = e_i^*$, is a K-algebra isomorphism. ∎

With Proposition 2.6 in mind, for the remainder of this article we investigate the structure of the Leavitt algebra $L_K(1,n)$ by equivalently investigating the structure of the Leavitt path algebra $L_K(R_n)$.

Notation 2.7. For any integer $n \ge 2$ and any field K, we denote by $A_{R_n}(e_1,e_2)$ the K-subalgebra of $L_K(R_n)$ generated by

$$v, e_1, e_3, \dots, e_n, e_2^*, \dots, e_n^*.$$

We should mention that by [6, Theorem 1], the following elements form a basis of the K-algebra $A_{R_n}(e_1,e_2)$: (1) v, (2) $p = e_{k_1}\cdots e_{k_m}$, where $k_i \in \{1,3,\dots,n\}$, (3) $q^* = e_{t_1}^*\cdots e_{t_h}^*$, where $t_i \in \{2,3,\dots,n\}$, (4) pq^*, where p and q^* are defined as in items (2) and (3), respectively.

The following result provides us with Anick type automorphisms of Leavitt algebras of type $(1,n)$.

Corollary 2.8. *Let $n \ge 2$ be a positive integer, K a field, and R_n the rose with n petals. Then, for any $p \in A_{R_n}(e_1,e_2)$, there exists a unique automorphism σ_p of the K-algebra $L_K(R_n)$ satisfying $\sigma_p(e_2) = e_2 + e_1 p$, $\sigma_p(e_1^*) = e_1^* - pe_2^*$, $\sigma_p^{-1}(e_2) = e_2 - e_1 p$, $\sigma_p^{-1}(e_1^*) = e_1^* + pe_2^*$, and $\sigma_p(q) = q$ for all $q \in A_{R_n}(e_1,e_2)$.*

Proof. Since $vp = p = pv$, the corollary immediately follows from Corollary 2.5. ∎

3. New irreducible representations of $L_K(R_n)$

In this section, we study the twisted modules of the simple $L_K(R_n)$-modules S_c^f mentioned in Section 1 under Anick type automorphisms of $L_K(R_n)$ introduced in Corollary 2.8. In particular, we obtain new classes of simple $L_K(R_n)$-modules (Theorems 3.6 and 3.8).

Let E be an arbitrary graph. An *infinite path* $p := e_1 \cdots e_n \cdots$ in a graph E is a sequence of edges $e_1, \dots, e_n, \dots$ such that $r(e_i) = s(e_{i+1})$ for all i. We denote by E^∞ the set of all infinite paths in E. For $p := e_1 \cdots e_n \cdots \in E^\infty$ and $n \geq 1$, Chen [12] defines $\tau_{>n}(p) = e_{n+1}e_{n+2}\cdots$, and $\tau_{\leq n}(p) = e_1 e_2 \cdots e_n$. Two infinite paths p, q are said to be *tail-equivalent* (written $p \sim q$) if there exist positive integers m, n such that $\tau_{>n}(p) = \tau_{>m}(q)$. Clearly $\sim$ is an equivalence relation on E^∞, and we let $[p]$ denote the $\sim$ equivalence class of the infinite path p.

Let c be a closed path in E. Then the path $ccc\cdots$ is an infinite path in E, which we denote by c^∞. Note that if c and d are closed paths in E such that $c = d^n$, then $c^\infty = d^\infty$ as elements of E^∞. The infinite path p is called *rational* in case $p \sim c^\infty$ for some closed path c. If $p \in E^\infty$ is not rational, we say p is *irrational*. We denote by E^∞_{rat} and E^∞_{irr} the sets of rational and irrational paths in E, respectively.

Given a field K and an infinite path p, Chen [12] defines $V_{[p]}$ to be the K-vector space having $\{q \in E^\infty \mid q \in [p]\}$ as a basis, that is, having basis consisting of distinct elements of E^∞ which are tail-equivalent to p. $V_{[p]}$ is made a left $L_K(E)$-module by defining, for all $q \in [p]$ and all $v \in E^0$, $e \in E^1$,

$$v \cdot q = q \text{ or } 0 \text{ according as } v = s(q) \text{ or not;}$$
$$e \cdot q = eq \text{ or } 0 \text{ according as } r(e) = s(q) \text{ or not;}$$
$$e^* \cdot q = \tau_1(q) \text{ or } 0 \text{ according as } q = e\tau_1(q) \text{ or not.}$$

In [12, Theorem 3.2], Chen showed the following result.

Theorem 3.1 ([12, Theorem 3.2]). *Let K be a field, E an arbitrary graph, and $p, q \in E^\infty$. Then the following holds:*

(1) *$V_{[p]}$ is a simple left $L_K(E)$-module;*

(2) $\text{End}_K(V_{[p]}) \cong K$;

(3) *$V_{[p]} \cong V_{[q]}$ if and only if $p \sim q$, which happens precisely when $V_{[p]} = V_{[q]}$.*

Theorem 3.1 provides us with the following two classes of simple modules for the Leavitt path algebra $L_K(E)$ of an arbitrary graph E:

- $V_{[\alpha]}$, where $\alpha \in E^\infty_{\text{irr}}$;
- $V_{[\beta]}$, where $\beta \in E^\infty_{\text{rat}}$.

We note that for any $\beta \in E^\infty_{\text{rat}}$, $V_{[\beta]} = V_{[c^\infty]}$ for some $c \in \text{SCP}(E)$. By [5, Theorem 2.8], we have $V_{[\beta]} = V_{[c^\infty]} \cong L_K(E)v/L_K(E)(c - v)$ as left $L_K(E)$-modules; i.e., it is finitely presented; while $V_{[\alpha]}$ ($\alpha \in E^\infty_{\text{irr}}$) is, in general, not finitely presented by [7, Corollary 3.5].

In [7], Ánh and the second author constructed simple $L_K(E)$-modules S_c^f associated to pairs (f, c) consisting of simple closed paths c together with irreducible polynomials f in $K[x]$. We will represent again this result in Theorem 3.2 below. To do so, we need some notions.

Let K be a field, E a graph, and c a closed path in E based at v. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial in $K[x]$. We denote by $f(c)$ the element

$$f(c) := a_0v + a_1c + \cdots + a_nc^n \in L_K(E).$$

We denote by $K[c]$ the subalgebra of $L_K(E)$ generated by v and c. By the $\mathbb{Z}$-grading on $L_K(E)$, $K[c]$ is isomorphic to the polynomial algebra $K[x]$ by the map: $v \mapsto 1$ and $c \mapsto x$. We denote by $\mathrm{Irr}(K[x])$ the set of all irreducible polynomials in $K[x]$ written in the form $1 - a_1x - \cdots - a_nx^n$.

Theorem 3.2 (cf. [7, Theorems 4.3 and 4.7]). *Let K be a field, E an arbitrary graph, c a simple closed path in E based at v, and $f(x) = 1 - a_1x - \cdots - a_nx^n$ an irreducible polynomial in $K[x]$. Then the following holds:*

(1) *the cyclic left $L_K(E)$-module S_c^f generated by z subject to $z = (a_1c + \cdots + a_nc^n)z$ is simple, and its endomorphism ring is isomorphic to $K[x]/K[x]f(x)$. Moreover,*

$$S_c^f \cong L_K(E)v/L_K(E)f(c),$$

as left $L_K(E)$modules, via the map $z \mapsto v + L_K(E)f(c)$;

(2) *for any $g \in \mathrm{Irr}(K[x])$ and any simple closed path d in E, $S_c^f \cong S_d^g$ as left $L_K(E)$-modules if and only if $f = g$ and $c^\infty \sim d^\infty$.*

Proof. (1) We note that $vz = z$ and $z = c^n f_1(c)^n z$ for all $n \geq 1$, where $f_1(c) = a_1v + a_2c + \cdots + a_nc^{n-1}$. By the $\mathbb{Z}$-grading on $L_K(E)$, $L_K(E)v/L_K(E)f(c) \neq 0$. Since $f(c)(v + L_K(E)f(c)) = f(c) + L_K(E)f(c) = 0$ in $L_K(E)v/L_K(E)f(c)$, there exists a surjective $L_K(E)$-homomorphism $\theta : S_c^f \to L_K(E)v/L_K(E)f(c)$ such that $\theta(z) = v + L_K(E)f(c)$, and so $S_c^f \neq 0$.

We claim that S_c^f is a simple left $L_K(E)$-module. Indeed, let y be a nonzero element in S_c^f. Since $S_c^f = L_K(E)z$, y may be written in the form $y = rz$ and $0 \neq r = \sum_{i=1}^m k_i\mu_i\nu_i^* \in L_K(E)$, where m is minimal such that $k_i \in K \setminus \{0\}$ and $\mu_i, \nu_i \in E^*$ with $r(\mu_i) = r(\nu_i)$ for all $1 \leq i \leq m$. Let n be a positive integer such that $|\nu_i \leq n|c|$ for all $1 \leq i \leq m$. We then have

$$y = \left(\sum_{i=1}^m k_i\mu_i\nu_i^*\right)z = \left(\sum_{i=1}^m k_i\mu_i\nu_i^*\right)c^n f_1(c)^n z = \left(\sum_{i=1}^m k_i\mu_i\nu_i^* c^n f_1(c)^n\right)z.$$

By the minimality of m, $\nu_i^*c^n \neq 0$ for all $1 \leq i \leq m$. Then, for each i, there exists $\delta_i \in E^*$ such that $c^n = \nu_i\delta_i$ and $r(\delta_i) = v := r(c) = s(c)$. This implies that

$$y = \left(\sum_{i=1}^m k_i\mu_i\nu_i^* c^n f_1(c)^n\right)z = \left(\sum_{i=1}^m k_i\mu_i\delta_i f_1(c)^n\right)z = \sum_{i=1}^m k_i\alpha_i f_1(c)^n z,$$

where $\alpha_i = \mu_i \delta_i$ $(i = 1, \dots, m)$. We note that $\alpha_i c^\infty = \alpha_j c^\infty$ in E^∞ if and only if $\alpha_i = \alpha_j c^{n_i}$ for some $n_i \in \mathbb{Z}^+$, or $\alpha_j = \alpha_i c^{n_j}$ for some $n_j \in \mathbb{Z}^+$, and $f(c)z = 0$ in S_c^f. Consequently, y may be written in the form $y = \sum_{i=1}^d \beta_i p_i(c)z$, where $p_i(c)$'s are nonzero elements in $K[c]/K[c]f(c)$ and β_i's are paths in E^* such that $\beta_i c^\infty$'s are distinct infinite paths in E^∞, and so there exists a positive integer t such that $\tau_{\le t}(\beta_i c^\infty)$'s are distinct paths in E^*. This implies that $\tau_{\le t+j}(\beta_i c^\infty)$'s are distinct paths in E^* for all $j \ge 0$. Therefore, without loss of generality, we may assume that $\tau_{\le t}(\beta_1 c^\infty) = \beta_1 c^l$ for some $l \ge 1$. We then have

$$\begin{aligned}\tau_{\le t}(\beta_1 c^\infty)^* y &= \tau_{\le t}(\beta_1 c^\infty)^* \Big(\sum_{i=1}^d \beta_i p_i(c) z \Big) \\ &= \tau_{\le t}(\beta_1 c^\infty)^* \Big(\sum_{i=1}^d \beta_i c^l p_i(c) f_1(c)^l z \Big) \\ &= p_1(c) f_1(c)^l z.\end{aligned}$$

Since $p_1(c) f_1(c)^l$ is a nonzero element in $K[c]/K[c]f(c)$ and

$$K[c]/K[c]f(c) \cong K[x]/K[x]f(x)$$

is a field, there exist q and $h \in K[c]$ such that $qp_1(c)f_1(c)^l = v + hf(c)$, and so

$$q\tau_{\le t}(\beta_1 c^\infty)^* y = qp_1(c) f_1(c)^l z = vz + hf(c)z = z.$$

This implies that $z \in L_K(E)y$, and hence $S_c^f = L_K(E)z = L_K(E)y$. Consequently, S_c^f is a simple left $L_K(E)$-module, showing the claim. This implies that θ is an isomorphism.

Let $\varphi : S_c^f \to S_c^f$ be a nonzero $L_K(E)$-homomorphism. By the same approach as above, $\varphi(z)$ may be written in the form $0 \ne \varphi(z) = \sum_{i=1}^m \beta_i p_i(c)z$, where $m \ge 1$, $p_i(c)$'s are nonzero elements in $K[c]/K[c]f(c)$, and β_i's are paths in E^* such that $\beta_i c^\infty$'s are distinct infinite paths in E^∞. If $c^\infty \ne \beta_i c^\infty$ in E^∞ for all $1 \le i \le m$, then there exists a positive integer d such that $\tau_{\le d}(c^\infty)$ and $\tau_{\le d}(\beta_i c^\infty)$'s are distinct paths in E^*, and so $(c^*)^d \beta_i c^d = 0$ for all i. This implies that

$$\begin{aligned}\varphi\big((c^*)^d z\big) = (c^*)^d \varphi(z) &= (c^*)^d \Big(\sum_{i=1}^m \beta_i p_i(c) z \Big) \\ &= (c^*)^d \Big(\sum_{i=1}^m \beta_i c^d p_i(c) f_1(c)^d z \Big) = 0.\end{aligned}$$

On the other hand, we have that φ is an automorphism (since φ is nonzero and S_c^f is simple) and $(c^*)^d z = f_1(c)^d z \ne 0$, and so $\varphi((c^*)^d z) \ne 0$, a contradiction. This implies that there exists an i such that $\beta_i c^\infty = c^\infty$ in E^∞. Without loss of generality, we may assume that $\beta_1 c^\infty = c^\infty$. We then have that $\beta_1 = c^t$ for some $t \ge 0$, and $\tau_{\le d}(c^\infty) = \tau_{\le d}(\beta_1 c^\infty)$ and $\tau_{\le d}(\beta_i c^\infty)$ $(i = 2, \dots, m)$ are distinct paths in E^*, so $\tau_{\le d+l}(c^\infty) = \tau_{\le d+l}(\beta_1 c^\infty)$ and $\tau_{\le d+l}(\beta_i c^\infty)$ $(i = 2, \dots, m)$ are distinct paths in E^* for all $l \ge 0$.

Therefore, without loss of generality, we may assume that $\tau_{\leq d}(c^\infty) = \tau_{\leq d}(\beta_1 c^\infty) = c^l$ for some $l \geq t$, and so

$$\begin{aligned}\varphi((c^*)^l z) &= (c^*)^l \varphi(z) = (c^*)^l (c^t p_1(c) z + \sum_{i=2}^{m} \beta_i p_i(c) z)\\ &= (c^*)^l (c^l p_1(c) f_1(c)^{l-t} z + \sum_{i=2}^{m} \beta_i c^l p_i(c) f_1(c)^l z)\\ &= p_1(c) f_1(c)^{l-t} z + \sum_{i=2}^{m} (c^*)^l \beta_i c^l p_i(c) f_1(c)^l z = p_1(c) f_1(c)^{l-t} z.\end{aligned}$$

This implies that

$$\varphi(z) = \varphi\big(c^l (c^*)^l z\big) = c^l \varphi\big((c^*)^l z\big) = c^l p_1(c) f_1(c)^{l-t} z,$$

so $\varphi(z)$ can be written in the form $\varphi(z) = p(c)z$, where

$$p(c) \in K[c]/K[c]f(c) \cong K[x]/K[x]f(x).$$

Conversely, let $p(c)$ be a nonzero element in $K[c]/K[c]f(c)$. We then have $p(c)z \neq 0$ in S_c^f (since $vz = z \neq 0$ and $p(c)$ is a unit in $K[c]/K[c]f(c)$) and $f(c)(p(c)z) = (f(c)p(c))z = (p(c)f(c))z = p(c)(f(c)z) = 0$, and so there exists a nonzero $L_K(E)$-homomorphism $\pi : S_c^f \to S_c^f$ such that $\pi(z) = p(c)z$. Therefore, we have

$$\mathrm{End}_{L_K(E)}(S_c^f) \cong K[c]/K[c]f(c) \cong K[x]/K[x]f(x).$$

(2) Write $g(x) = 1 - b_1 x - \cdots - b_m x^m \in K[x]$ and $c = e_1 \cdots e_t$. Assume that S_d^g is the left $L_K(E)$-module generated by z' subject to

$$z' = (b_1 d + \cdots + b_m d^m) z' = d g_1(d) z'.$$

($\Rightarrow$) Assume that $\varphi : S_c^f \to S_d^g$ is an $L_K(E)$-isomorphism. Then, by the same approach as above, $\varphi(z) = \sum_{i=1}^{s} \alpha_i a_i z'$, where a_i's are nonzero elements in $K[d]/K[d]g(d)$ and α_i's are paths in E^* such that $\alpha_i d^\infty$'s are distinct infinite paths in E^∞. If $c^\infty \neq \alpha_i d^\infty$ in E^∞ for all $1 \leq i \leq s$, then there exist a positive integer t such that $(c^*)^t \alpha_i d^t = 0$ for all $1 \leq i \leq s$. Then, since $z' = d g_1(d) z'$, so

$$\varphi(z) = \sum_{i=1}^{s} \alpha_i a_i z' = \sum_{i=1}^{s} \alpha_i a_i d^t g_1(d)^t z' = \sum_{i=1}^{s} \alpha_i d^t a_i g_1(d)^t z'$$

and

$$\varphi\big((c^*)^t z\big) = (c^*)^t \varphi(z) = \sum_{i=1}^{s} (c^*)^t \alpha_i d^t a_i g_1(d)^t z' = 0.$$

On the other hand, we note that $(c^*)^t z = f_1(c)^t z \neq 0$ in S_c^f, so $\varphi((c^*)^t z) \neq 0$, since φ is an $L_K(E)$-isomorphism, a contradiction. This implies that $c^\infty = \alpha_i d^\infty$ for some

$1 \le i \le s$, so $c^\infty \sim d^\infty$. Since c and d are simple closed paths in E, we must have $d = c_j := e_j \cdots e_t e_1 \cdots e_{j-1}$ for some $1 \le j \le t$.

Let $z'' := e_1 \cdots e_{j-1} z'$ if $j \ne 1$ and $z'' := z'$ if $j = 1$. Since $z' = (e_1 \cdots e_{j-1})^* z'' \ne 0$, $z'' \ne 0$ in $S^g_d = S^g_{c_j}$. We then have $e_j \cdots e_t g(c) z'' = c_j g(c_j) z' = d g(d) z' = 0$ in S^g_d, and so $g(c) z'' = (e_j \cdots e_t)^* e_j \cdots e_t g(c) z'' = 0$ in S^g_d. By item (1), S^g_d can be also generated by z'' subject to $g(c) z'' = 0$.

By repeating approach described in the proof of item (1), we obtain that $\varphi(z) = p(c) z''$, where $p(c)$ is a nonzero element of $K[c]/K[c]g(c)$. We then have

$$0 = \varphi(0) = \varphi\big(f(c)z\big) = f(c)\varphi(z) = f(c)\big(p(c)z''\big) = \big(p(c)f(c)\big)z'',$$

and so $p(c)f(c) = 0$ in $K[c]/K[c]g(c)$, by item (1). Since $p(c)$ is a unit in $K[c]/K[c]g(c)$, $f(c) \in K[c]g(c)$. Then, since f is an irreducible polynomial in $K[x]$, we must have $f = g$.

($\Leftarrow$) Assume that $f = g$ and $c^\infty \sim d^\infty$. Since c and d are simple closed paths, $d = c_j := e_j \cdots e_t e_1 \cdots e_{j-1}$ for some $1 \le j \le t$. Then, by repeating method described as in the direction ($\Rightarrow$), we obtain that $S^f_c \cong S^f_d$, thus finishing the proof. ∎

We should mention the following useful remark.

Remark 3.3. Let K be a field, E a graph, and $c = e_1 \cdots e_t$ a simple closed path in E based at v. Let $f(x) \in \mathrm{Irr}(K[x])$.

(1) We denote by Π_c the set of all the following closed paths

$$c_1 := c, \quad c_2 := e_2 \cdots e_t e_1, \ldots, c_n := e_n e_1 \cdots e_{n-1}.$$

By Theorem 3.2, all modules $S^f_{c_i}$ are isomorphic to each other, and for a simple closed path d, $S^f_d \cong S^f_c$ if and only if $d \in \Pi_c$. Consequently, if one can represent their isomorphism class by a simple module $S^f_{\Pi_c}$ isomorphic to some $S^f_{c_i}$, then $S^f_{\Pi_c}$ is well defined and depends only on the (f, Π_c).

(2) If $f(x) = 1 - x \in K[x]$, then by Theorem 3.2 (1) and [5, Theorem 2.8], we have $S^f_c \cong L_K(E)v/L_K(E)(c - v) \cong V_{[c^\infty]}$ as left $L_K(E)$-modules.

(3) It was shown in the proof of Theorem 3.2 that every element y of S^f_c may be written in the form $y = \sum_{i=1}^n \alpha_i p_i(c) z$, where α_i's are paths in E such that $\alpha_i c^\infty$'s are distinct infinite paths in E^∞ and $p_i(c)$'s are nonzero elements of $K[c]/K[c]f(c)$.

We next construct new classes of simple modules for the Leavitt path algebra $L_K(R_n)$ by using Theorem 3.2, Corollary 2.8, and special closed paths in R_n.

Notation 3.4. For any integer $n \ge 2$, we denote by $C_s(R_n)$ the set of simple closed paths of the form $c = e_{k_1} e_{k_2} \cdots e_{k_m}$, where $k_i \in \{1, 3, \ldots, n\}$ for all $1 \le i \le m - 1$ and $k_m = 2$, in R_n.

Let $c = e_{k_1} e_{k_2} \cdots e_{k_m} \in C_s(R_n)$, $p \in A_{R_n}(e_1, e_2)$, and $f = 1 - a_1 x_1 - \cdots - a_n x^n = 1 - x f_1(x) \in \mathrm{Irr}(K[x])$. We have a left $L_K(R_n)$-module $S^{f,p}_c$, which is the twisted module $(S^f_c)^{\sigma_p}$, where σ_p is the automorphism of $L_K(R_n)$ defined in Corollary 2.8. Denoting by $*$ the module operation in $S^{f,p}_c$, we have the following useful fact.

Lemma 3.5. *Let K be a field, $n \geq 2$ a positive integer, and R_n the rose with n petals. Let $p \in A_{R_n}(e_1, e_2)$ be an arbitrary element, $c \in C_s(R_n)$, and $f \in \mathrm{Irr}(K[x])$. Then the following statements hold:*

(1) $c * y = cy + e_{k_1} \cdots e_{k_{m-1}} e_1 py$ *for all* $y \in S_c^{f,p}$*;*

(2) $(c^*)^m * z = (c^*)^m z$ *for all* $m \geq 1$*, where* z *is a generator of the left* $L_K(E)$*-module* S_c^f *which is described in Theorem 3.2.*

Proof. (1) By Corollary 2.8, we have $\sigma_p(c) = c + e_{k_1} \cdots e_{k_{m-1}} e_1 p$, so $c * y = \sigma_p(c)y = cy + e_{k_1} \cdots e_{k_{m-1}} e_1 py$ for all $y \in S_c^{f,p}$, as desired.

(2) If $e_{k_i} \neq e_1$ for all $1 \leq i \leq m-1$, then $\sigma_p(c^*) = c^*$, and so $c^* * z = c^* z$, as desired. Consider the case that $e_{k_i} = e_1$ for some $1 \leq i \leq m-1$. Let ℓ be the number of all elements $1 \leq i \leq m-1$ such that $e_{k_i} = e_1$. We use induction on ℓ to establish the claim that $\sigma_p(c^*)c = 1$ in $L_K(R_n)$. If $\ell = 1$, there is a unique element $1 \leq i \leq m-1$ such that $e_{k_i} = e_1$. We then have

$$\sigma_p(c^*) = e_{k_m}^* \cdots e_{k_{i+1}}^* (e_1^* - pe_2^*) e_{k_{i-1}}^* \cdots e_{k_1}^* = c^* - e_{k_m}^* \cdots e_{k_{i+1}}^* pe_2^* e_{k_{i-1}}^* \cdots e_{k_1}^*,$$

and so

$$\sigma_p(c^*)c = (c^* - e_{k_m}^* \cdots e_{k_{i+1}}^* pe_2^* e_{k_{i-1}}^* \cdots e_{k_1}^*)c = 1,$$

since $e_2^* e_{k_i} = e_2^* e_1 = 0$, as desired. Now we proceed inductively. For $\ell > 1$, let $j := \min\{i \mid 1 \leq i \leq m-1 \text{ and } e_{k_i} = e_1\}$. We have

$$\sigma_p(c^*) = \sigma_p(e_{k_m}^* \cdots e_{k_{j+1}}^*)(e_1^* - pe_2^*) e_{k_{j-1}}^* \cdots e_{k_1}^*.$$

It is clear that $e_{k_{j+1}} \cdots e_{k_m} \in C_s(R_n)$. Then, by the induction hypothesis, we obtain that $\sigma_p(e_{k_m}^* \cdots e_{k_{j+1}}^*) e_{k_{j+1}} \cdots e_{k_m} = 1$. This implies that

$$\begin{aligned}\sigma_p(c^*)c &= \sigma_p(e_{k_m}^* \cdots e_{k_{j+1}}^*)(e_1^* - pe_2^*) e_{k_{j-1}}^* \cdots e_{k_1}^* c \\ &= \sigma_p(e_{k_m}^* \cdots e_{k_{j+1}}^*) e_{k_{j+1}} \cdots e_{k_m} = 1,\end{aligned}$$

since $e_2^* e_{k_j} = e_2^* e_1 = 0$, thus showing the claim. By induction we get that $\sigma_p((c^*)^m)c^m = \sigma_p(c^*)^m c^m = 1$ in $L_K(R_n)$ for all $m \geq 1$.

By Theorem 3.2, $z = cf_1(c)z$, so $z = c^m f_1(c)^m z$ and $(c^*)^m z = f_1(c)^m z$ for all $m \geq 1$. We then have

$$\begin{aligned}(c^*)^m * z &= \sigma_p\big((c^*)^m\big)z = \sigma_p\big((c^*)^m\big)\big(c^m f_1(c)^m z\big) \\ &= \big(\sigma_p\big((c^*)^m\big)c^m\big) f_1(c)^m z = f_1(c)^m z = (c^*)^m z\end{aligned}$$

for all $m \geq 1$, thus finishing the proof. ■

We are now in position to provide the first main result of this section.

Theorem 3.6. *Let K be a field, $n \geq 2$ a positive integer, and R_n the rose with n petals. Let p and $q \in A_{R_n}(e_1, e_2)$ be two arbitrary elements, let $c, d \in C_s(R_n)$, and let $f, g \in$* $\mathrm{Irr}(K[x])$. *Then the following holds:*

(1) $S_c^{f,p}$ *is a simple left* $L_K(R_n)$*-module;*

(2) $S_c^{f,p} \cong S_d^{g,q}$ *as left* $L_K(R_n)$*-modules if and only if* $f = g$, $c = d$, *and* $p - q = rf(c)$ *for some* $r \in L_K(R_n)$*;*

(3) *for any simple closed path* α *in* R_n, $S_c^{f,p} \cong S_{\Pi_\alpha}^f$ *as left* $L_K(R_n)$*-modules if and only if* $\alpha \in \Pi_c$ *and* $p = rf(c)$ *for some* $r \in L_K(R_n)$*;*

(4) $\mathrm{End}_{L_K(R_n)}(S_c^{f,p}) \cong K[x]/K[x]f(x)$*;*

(5) $S_c^{f,p} \cong L_K(R_n)/L_K(R_n)f(\sigma_p^{-1}(c))$.

Proof. (1) It follows from the fact that S_c^f is a simple left $L_K(R_n)$-module (by Theorem 3.2) and σ_p is an automorphism of $L_K(R_n)$ (by Corollary 2.8).

(2) Assume that $\varphi : S_c^{f,p} \to S_d^{g,q}$ is an $L_K(R_n)$-isomorphism. Let z and z' be generators of the left $L_K(E)$-modules $S_c^{f,p}$ and $S_d^{g,q}$ which are described in Theorem 3.2, respectively. We then have $0 \neq \varphi(z) = \sum_{i=1}^t \alpha_i a_i z'$ in $S_d^{g,q}$, where α_i's are nonzero elements of $K[d]/K[d]g(d)$ and α_i's are paths in $(R_n)^*$ such that $\alpha_i d^\infty$'s are distinct infinite paths in $(R_n)^\infty$. By Theorem 3.2 and Lemma 3.5, we note that $z = c^k f_1(c)^k z$ and $(c^*)^k * z = (c^*)^k z$ in $S_c^{f,p}$ for all $k \geq 1$, and $z' = d^l g_1(d)^l z'$ and $(d^*)^l * z' = (d^*)^l z'$ in $S_d^{g,q}$ for all $l \geq 1$. Therefore, by repeating approach described in the proof of the direction $(\Rightarrow)$ of Theorem 3.2 (2), we obtain that $c^\infty \sim d^\infty$ and $\varphi((c^*)^k z) = az'$ for some $k \geq 1$ and a nonzero element $a \in K[d]/K[d]g(d)$. Then, since $c^\infty \sim d^\infty$, we have $d \in \Pi_c$, and so $c = d$, since c and $d \in C_s(R_n)$. We also note that $\varphi((c^*)^{k+1} * z) = \varphi(c^* * ((c^*)^k z)) = c^* * \varphi((c^*)^k z) = c^* * (az') = c^*(az') = ag_1(c)z'$. By induction, we may prove that

$$\varphi\big((c^*)^{k+i} * z\big) = \varphi\big((c^*)^{k+i} z\big) = ag_1(c)^i z'$$

for all $i \geq 0$, where $g_1(c)^0 := 1_{L_K(R_n)}$.

In $S_c^{f,p}$ we have $z = c^m f_1(c)^m z$ and $(c^*)^m = f_1(c)^m z$ for all $m \geq 1$, and $c * z = cz + e_{k_1} \cdots e_{k_{m-1}} e_1 pz = cz + p'z$, where $p' := e_{k_1} \cdots e_{k_{m-1}} e_1 p \in A_{R_n}(e_1, e_2)$, so

$$\begin{aligned} c * \big((c^*)^{k+1} z\big) &= c * \big(f_1(c)^{k+1} z\big) = c\big(f_1(c)^{k+1} z\big) + p'\big(f_1(c)^{k+1} z\big) \\ &= f_1(c)^k z + p'\big(f_1(c)^{k+1} z\big) = (c^*)^k z + p'\big((c^*)^{k+1} z\big) \\ &= (c^*)^k z + p' * \big((c^*)^{k+1} z\big), \end{aligned}$$

since $\sigma_p(p') = p'$ (by Corollary 2.8). This implies that

$$\begin{aligned} \varphi\big(c * \big((c^*)^{k+1} z\big)\big) &= \varphi\big((c^*)^k z\big) + \varphi\big(p' * \big((c^*)^{k+1} z\big)\big) = az' + p'ag_1(c)z' \\ &= az' + e_{k_1} \cdots e_{k_{m-1}} e_1 pag_1(c)z'. \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi\big(c * \big((c^*)^{k+1} z\big)\big) &= c * \varphi\big((c^*)^{k+1} z\big) = c * \big(ag_1(c)z'\big) \\ &= cag_1(c)z' + e_{k_1} \cdots e_{k_{m-1}} e_1 qag_1(c)z' \\ &= az' + e_{k_1} \cdots e_{k_{m-1}} e_1 qag_1(c)z', \end{aligned}$$

since $cag_1(c)z' = acg_1(c)z'$ and $z' = cg_1(c)z'$. From these observations, we have

$$e_{k_1} \cdots e_{k_{m-1}} e_1 pag_1(c)z' = e_{k_1} \cdots e_{k_{m-1}} e_1 qag_1(c)z'$$

in S_c^g, showing that $e_{k_1} \cdots e_{k_{m-1}} e_1(p-q)ag_1(c)z' = 0$ in S_c^g, and hence

$$(p-q)ag_1(c)z' = (e_{k_1} \cdots e_{k_{m-1}} e_1)^* e_{k_1} \cdots e_{k_{m-1}} e_1(p-q)ag_1(c)z' = 0$$

in S_c^g. By Theorem 3.2 (1), we have

$$S_c^g \cong L_K(R_n)/L_K(R_n)g(c),$$

as left $L_K(R_n)$-modules, via the map: $z \mapsto 1 + L_K(R_n)g(c)$. Therefore, $(p-q)ag_1(c) = bg(c)$ for some $b \in L_K(R_n)$. Since $ag_1(c)$ is a unit of $K[c]/K[c]g(c)$, there exist elements $\alpha, \beta \in K[c]$ such that $(ag_1(c))\alpha = 1 + \beta g(c)$, and so

$$bg(c)\alpha = (p-q)ag_1(c)\alpha = (p-q)\big(1 + \beta g(c)\big) = p - q + (p-q)\beta g(c).$$

This implies that

$$p - q = (b\alpha + q\beta - p\beta)g(c) = rg(c),$$

where $r := b\alpha + q\beta - p\beta \in L_K(R_n)$.

Write $f(x) = 1 - a_1 x - \cdots - a_s x^s$. We then have $(1 - a_1 c - \cdots - a_s c^s)z = 0$ and $((c^*)^{k+s} - a_1(c^*)^{k+s-1} - \cdots - a_s(c^*)^k)z = (c^*)^{k+s}(1 - r_1 c - \cdots - a_s c^s)z = 0$ in $S_c^{f,p}$, and so

$$\begin{aligned} ag_1(c)^s f(c)z' &= ag_1(c)^s(1 - a_1 c - \cdots - a_s c^s)z' \\ &= ag_1(c)^s z' - a_1 ag_1(c)^{s-1} z' - \cdots - a_s z' \\ &= \varphi\big(((c^*)^{k+s} - a_1(c^*)^{k+s-1} - \cdots - a_s(c^*)^k)z\big) = \varphi(0) = 0 \end{aligned}$$

in S_c^g. By repeating the same argument described above, we obtain that $f(c) = rg(c)$ for some $\gamma \in L_K(R_n)$. Write $\gamma = \sum_{i=1}^d k_i \alpha_i \beta_i^*$, where $k_i \in K \setminus \{0\}$ and α_i, β_i are paths in R_n. Let $m = \max\{|\alpha_i|,\ |\beta_i| \mid 1 \le i \le d\}$. We then have

$$(c^*)^m \gamma c^m = \sum_{i=1}^d k_i (c^*)^m \alpha_i \beta_i^* c^m \in K[c]$$

and

$$\begin{aligned} f(c) &= (c^*)^m c^m f(c) = (c^*)^m f(c) c^m \\ &= (c^*)^m \Big(\sum_{i=1}^d k_i \alpha_i \beta_i^*\Big) g(c) c^m = \Big(\sum_{i=1}^d k_i (c^*)^m \alpha_i \beta_i^* c^m\Big) g(c) \end{aligned}$$

in $K[c]$, and so $f = g$, since $f, g \in \mathrm{Irr}(K[x])$.

Conversely, assume that $f = g$, $c = d$, and $p - q = rf(c)$ for some $r \in L_K(R_n)$. We use induction to claim that $\sigma_q(\sigma_p^{-1}(c^m))z = c^m z$ for all $m \geq 1$. For $m = 1$, by Corollary 2.8, $\sigma_p^{-1}(c) = e_{k_1} \cdots e_{k_{m-1}} e_2 - e_{k_1} \cdots e_{k_{m-1}} e_1 p$, and so

$$\begin{aligned}\sigma_q\big(\sigma_p^{-1}(c)\big) &= \sigma_q(e_{k_1} \cdots e_{k_{m-1}} e_2) - \sigma_q(e_{k_1} \cdots e_{k_{m-1}} e_1 p)\\ &= c + e_{k_1} \cdots e_{k_{m-1}} e_1 q - e_{k_1} \cdots e_{k_{m-1}} e_1 p\\ &= c + e_{k_1} \cdots e_{k_{m-1}} e_1 (q - p)\\ &= c - e_{k_1} \cdots e_{k_{m-1}} e_1 r f(c).\end{aligned}$$

Then, since $f(c)z = 0$, we have $\sigma_q(\sigma_p^{-1}(c))z = cz$. For $m > 1$, we have

$$\begin{aligned}\sigma_q\big(\sigma_p^{-1}(c^{m+1})\big)z &= \sigma_q\big(\sigma_p^{-1}(c)\sigma_p^{-1}(c^m)\big)z = \sigma_q\big(\sigma_p^{-1}(c)\big)\sigma_q\big(\sigma_p^{-1}(c^m)\big)z\\ &= \sigma_q\big(\sigma_p^{-1}(c)\big)c^m z = \big(c - e_{k_1} \cdots e_{k_{m-1}} e_1 r f(c)\big)(c^m z)\\ &= c^{m+1} z - e_{k_1} \cdots e_{k_{m-1}} e_1 r f(c) c^m z\\ &= c^{m+1} z - e_{k_1} \cdots e_{k_{m-1}} e_1 r c^m f(c) z = c^{m+1} z,\end{aligned}$$

as desired. This shows that $\sigma_q(\sigma_p^{-1}(f(c)))z = f(c)z$.

We note that since $S_c^{f,p}$ is a simple left $L_K(R_n)$-module, every element of $S_c^{f,p}$ may be written in the form $\sigma_p(s)z$, where $s \in L_K(R_n)$. Define $\varphi : S_c^{f,p} \to S_c^{f,q}$ as follows: $\sigma_p(s)z \mapsto \sigma_q(s)z$. We claim that φ is well defined. Indeed, let s and t be two elements in $L_K(R_n)$ such that $\sigma_p(s)z = \sigma_p(t)z$ in S_c^f. By Theorem 3.2 (1), $\sigma_p(s - t) = \sigma_p(s) - \sigma_p(t) = bf(c)$ for some $b \in L_K(R_n)$, so $s - t = \sigma_p^{-1}(bf(c)) = \sigma_p^{-1}(b)\sigma_p^{-1}(f(c))$. This implies that

$$\begin{aligned}\big(\sigma_q(s) - \sigma_q(t)\big)z &= \sigma_q(s - t)z = \sigma_q\big(\sigma_p^{-1}(b)\big)\sigma_q\big(\sigma_p^{-1}\big(f(c)\big)\big)z\\ &= \sigma_q\big(\sigma_p^{-1}(b)\big)\big(f(c)z\big) = 0\end{aligned}$$

in S_c^f, thus proving the claim.

It is obvious that φ is a nonzero $L_K(R_n)$-homomorphism (since $\varphi(z) = z$), so φ is an isomorphism.

(3) (⇒) Assume that $S_c^{f,p} \cong S_{\Pi_\alpha}^f$. We then have $S_c^{f,p} \cong S_\alpha^f$. By repeating the same method described in the proof of the direction (⇒) of Theorem 3.2 (2), we obtain that $\alpha \in \Pi_c$. This implies that $S_c^{f,p} \cong S_c^f = S_c^{f,0}$, and so $p = rf(c)$ for some $r \in L_K(R_n)$, by item (2).

(⇐) It immediately follows from item (2).

(4) Let $\varphi : S_c^{f,p} \to S_c^{f,p}$ be a nonzero $L_K(R_n)$-homomorphism. Since $S_c^{f,p}$ is a simple left $L_K(R_n)$-module, φ is an isomorphism. Similar to item (2), we have $\varphi((c^*)^k z) = az$ for some nonzero element $a \in K[c]/K[c]f(c)$ and some positive integer k. Therefore, $\varphi(r * ((c^*)^k z)) = r * (az)$ for all $r \in L_K(R_n)$. Conversely, let a be a nonzero element of $K[c]/K[c]f(c)$. Since $S_c^{f,p}$ is a simple left $L_K(R_n)$-module, $S_c^{f,p} = L_K(R_n) * ((c^*)^k z)$. We claim that the map $\mu : S_c^{f,p} \to S_c^{f,p}$, defined by $\mu(r * ((c^*)^k z)) = r * (az)$, is a

nonzero $L_K(R_n)$-homomorphism. Indeed, assume that $r * (c^*)^k z = s * (c^*)^k z$, where $r, s \in L_K(R_n)$. We then have $\sigma_p(r) f_1(c)^k z = \sigma_p(s) f_1(c)^k z$ in S_c^f. By Theorem 3.2 (1), we obtain that $(\sigma_p(r) - \sigma_p(s)) f_1(c)^k = bf(c)$ for some $b \in L_K(R_n)$, and so $r * (az) - s * (az) = \sigma_p(r)az - \sigma_p(s)az = (\sigma_p(r) - \sigma_p(s))az = (\sigma_p(r) - \sigma_p(s))ac^k f_1(c)^k z = (\sigma_p(r) - \sigma_p(s)) f_1(c)^k ac^k z = bf(c)ac^k z = bac^k f(c)z = 0$ (since $f(c)z = 0$), that means, $\mu(r * ((c^*)^k z)) = \mu(s * ((c^*)^k z))$. This implies that μ is well defined. It is not hard to check that μ is an $L_K(R_n)$-homomorphism. From these observations, we have $\mathrm{End}_{L_K(R_n)}(S_c^{f,p}) \cong K[c]/K[c]f(c) \cong K[x]/K[x]f(x)$.

(5) We first note that $S_c^{f,p} = L_K(R_n) * z$; i.e., every element of $S_c^{f,p}$ is of the form $r * z = \sigma_p(r)z$, where $r \in L_K(R_n)$. We next compute $\mathrm{ann}_{L_K(R_n)}(z)$. Indeed, let $r \in \mathrm{ann}_{L_K(R_n)}(z)$. We then have $\sigma_p(r)z = r * z = 0$ in S_c^f. By Theorem 3.2 (1), $\sigma_p(r) = bf(c)$ for some $b \in L_K(R_n)$, and so $r = \sigma_p^{-1}(b)\sigma_p^{-1}(f(c)) = \sigma_p^{-1}(b) f(\sigma_p^{-1}(c))$, since σ_p^{-1} is an endomorphism of the K-algebra $L_K(R_n)$. This implies that

$$\mathrm{ann}_{L_K(R_n)}(z) \subseteq L_K(R_n) f\left(\sigma_p^{-1}(c)\right).$$

Conversely, assume that $r \in L_K(R_n) f(\sigma_p^{-1}(c))$; i.e., $r = \beta f(\sigma_p^{-1}(c))$, where $\beta \in L_K(R_n)$. We then have $r * z = \sigma_p(r)z = \sigma_p(\beta) f(c)z = 0$ (since $f(c)z = 0$), and so $r \in \mathrm{ann}_{L_K(R_n)}(z)$, showing that $L_K(R_n) f(\sigma_p^{-1}(c)) \subseteq \mathrm{ann}_{L_K(R_n)}(z)$. Hence $\mathrm{ann}_{L_K(R_n)}(z) = L_K(R_n) f(\sigma_p^{-1}(c))$. This implies that

$$S_c^{f,p} \cong L_K(R_n)/L_K(R_n) f\left(\sigma_p^{-1}(c)\right),$$

thus finishing the proof. ■

For clarification, we illustrate Theorem 3.6 by presenting the following example.

Examples 3.7. Let K be a field and R_2 the rose with 2 petals. We then have $C_s(R_2) = \{e_2,\ e_1^m e_2 \mid m \in \mathbb{Z},\ m \geq 1\}$, and $A_{R_2}(e_1, e_2)$ is the K-subalgebra of $L_K(R_2)$ generated by v, e_1, e_2^*, which means that

$$A_{R_2}(e_1, e_2) = \left\{ \sum_{i=1}^{n} k_i e_1^{m_i} (e_2^*)^{l_i} \mid n \geq 1,\ k_i \in K,\ m_i, l_i \geq 0 \right\},$$

where $e_1^0 = v = (e_2^*)^0$ and $K[e_1] \subset A_{R_2}(e_1, e_2)$. Let $f(x) = 1 - x \in \mathrm{Irr}(K[x])$. By the grading on $L_K(R_2)$, we always have $p \neq r(1 - c)$ for all $c \in C_s(R_2)$, $p \in K[e_1] \setminus \{0\}$, and $r \in L_K(R_2)$, and so the set

$$\left\{S_c^{f,p} \mid c \in C_s(R_2),\ p \in K[e_1]\right\}$$

consists of pairwise non-isomorphic simple left $L_K(R_2)$-modules, by Theorem 3.6.

Let $p = e_1$ and $q = e_1 e_2^* \in A_{R_2}(e_1, e_2)$. We then have

$$q - p = e_1 e_2^* - e_1 = e_1 e_2^*(1 - e_2),$$

so $S_{e_2}^{f,p} \cong S_{e_2}^{f,q}$ as left $L_K(R_2)$-modules, by Theorem 3.6.

Using Theorems 3.1, 3.2, and 3.6, we obtain a list of pairwise non-isomorphic simple modules for the Leavitt path algebra $L_K(R_n)$. Before doing so, we need some useful notions. For each pair $(f, c) \in \operatorname{Irr}(K[x]) \times C_s(R_n)$, we define a relation $\equiv_{f,c}$ on $A_{R_n}(e_1, e_n)$ as follows. For all $p, q \in A_{R_n}(e_1, e_n)$, $p \equiv_{f,c} q$ if and only if $p - q = rf(c)$ for some $r \in L_K(R_n)$. It is obvious that $\equiv_{f,c}$ is an equivalence on $A_{R_n}(e_1, e_n)$. We denote by $[p]$ the $\equiv_{f,c}$ equivalent class of p.

Theorem 3.8. *Let K be a field, $n \geq 2$ a positive integer, and R_n the rose with n petals. Then, the set*

$$\{V_{[\alpha]} \mid \alpha \in (R_n)^\infty_{\mathrm{irr}}\} \sqcup \{S^f_{\Pi_c} \mid c \in \mathrm{SCP}(R_n),\ f \in \operatorname{Irr}\big(K[x]\big)\}$$
$$\sqcup \{S^{f,p}_d \mid d \in C_s(R_n),\ f \in \operatorname{Irr}\big(K[x]\big),\ [0] \neq [p] \in A_{R_n}(e_1, e_2)/\equiv_{f,d}\}$$

consists of pairwise non-isomorphic simple left $L_K(R_n)$-modules.

Proof. All $V_{[\alpha]}$ $(\alpha \in (R_n)^\infty_{\mathrm{irr}})$ are pairwise non-isomorphic by Theorem 3.1. All $S^f_{\Pi_c}$ $(c \in \mathrm{SCP}(R_n),\ f \in \operatorname{Irr}(K[x]))$ are pairwise non-isomorphic by Theorem 3.2. All $S^{f,p}_d$ $(d \in C_s(R_n),\ f \in \operatorname{Irr}(K[x]),\ [0] \neq [p] \in A_{R_n}(e_1, e_2)/\equiv_{f,d})$ are pairwise non-isomorphic by Theorem 3.6.

By Theorem 3.1 and 3.6, respectively, $S^f_{\Pi_c}$ $(c \in \mathrm{SCP}(R_n),\ f \in \operatorname{Irr}(K[x]))$ and $S^{f,p}_d$ $(d \in C_s(R_n),\ f \in \operatorname{Irr}(K[x]),\ [0] \neq p \in A_{R_n}(e_1, e_2))$ are finitely presented. While by [7, Corollary 3.5], $V_{[\alpha]}$ is not finitely presented for all $\alpha \in (R_n)^\infty_{\mathrm{irr}}$. Therefore, each $V_{[\alpha]}$ is neither isomorphic to any $S^f_{\Pi_c}$ nor any $S^{f,p}_d$.

By Theorem 3.6 (3), each $S^{f,p}_d$ $(d \in C_s(R_n),\ f \in \operatorname{Irr}(K[x]),\ [0] \neq [p] \in A_{R_n}(e_1, e_2)/\equiv_{f,d})$ is not isomorphic to any $S^f_{\Pi_c}$ $(c \in \mathrm{SCP}(R_n),\ f \in \operatorname{Irr}(K[x]))$, thus finishing the proof. ∎

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