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Homomorphisms into simple $\mathcal{Z}$-stable C^*-algebras, II

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Abstract. Let A and B be unital finite separable simple amenable C^*-algebras which satisfy the UCT, and B is $\mathcal{Z}$-stable. Following Gong, Lin, and Niu (2020), we show that two unital homomorphisms from A to B are approximately unitarily equivalent if and only if they induce the same element in $KL(A, B)$, the same affine map on tracial states, and the same Hausdorffified algebraic K_1 group homomorphism. A complete description of the range of the invariant for unital homomorphisms is also given.

Dedicated to Professor Guoliang Yu on his sixtieth birthday.

1. Introduction

Let X and Y be two compact Hausdorff spaces, and denote by $C(X)$ (or $C(Y)$) the C^*-algebra of complex-valued continuous functions on X (or Y). Any continuous map $\lambda : Y \to X$ induces a homomorphism φ from the commutative C^*-algebra $C(X)$ into the commutative C^*-algebra $C(Y)$ by $\varphi(f) = f \circ \lambda$, and any homomorphism from $C(X)$ to $C(Y)$ arises this way (in this paper, by homomorphisms or isomorphisms between C^*-algebras, we mean $*$-homomorphisms or $*$-isomorphisms). It should be noted that, by the Gelfand transformation, every unital commutative C^*-algebra has the form $C(X)$ as above. Therefore, studying continuous maps from Y to X is equivalent to studying the homomorphisms from $C(X)$ to $C(Y)$.

We study the non-commutative version of this. In this paper, we consider only simple C^*-algebras. The paper is a continuation of [18]. The first part of the results can be stated as follows: let A and B be unital finite separable simple amenable C^*-algebras which satisfy the UCT such that B is $\mathcal{Z}$-stable. Let $\varphi, \psi : A \to B$ be two unital monomorphisms. Then, there exists a sequence of unitaries $\{u_n\} \subset B$ such that

$$\lim_{n\to\infty} u_n^* \psi(a) u_n = \varphi(a) \quad \text{for all } a \in A,$$

if and only if

$$[\varphi] = [\psi] \text{ in } KL(A, B), \quad \varphi_T = \psi_T \quad \text{and} \quad \varphi^\ddagger = \psi^\ddagger,$$

where $\varphi_T, \psi_T : T(B) \to T(A)$ are the continuous affine maps induced by φ and ψ, $T(A)$ and $T(B)$ are tracial state spaces of A and B, and $\varphi^\ddagger$ and $\psi^\ddagger$ are induced homomorphisms

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from $U(A)/CU(A)$ to $U(B)/CU(B)$, respectively, $U(A)$ and $U(B)$ are unitary groups of A and B, $CU(A)$ and $CU(B)$ are the closures of commutator subgroups of A and B, respectively (see Theorem 4.3 and [18] for the cases in which A may not be simple, and see earlier results in [16, 19]).

In the case that B is a unital purely infinite simple C^*-algebra, $T(B) = \emptyset$. Then, φ_T and ψ_T are both trivial maps. Also, by [8, Corollary 2.7], $U(B)/CU(B) = \{0\}$. One ignores the trivial maps $\varphi^\ddagger$ and $\psi^\ddagger$. Without assuming A is simple, the same result as in Theorem 4.3, in the case that B is purely infinite simple, is known as stated in [11, Theorem 6.7].

Theorem 4.3 is a generalization of [18, Theorem 5.8] at least in the case that A is simple. The proof also follows the same lines as described in [18, Remark 5.7] using the established results in [6, 7].

The second part of this research seeks the range of the invariant for the homomorphisms from A to B. Similar results were also obtained in [18]. Let $\kappa \in KL(A, B)$ be a strictly positive element (see Definition 2.6) with $\kappa([1_A]) = [1_B]$, $\kappa_T : T(B) \to T(A)$ a continuous affine map, and $\kappa_\gamma : U(A)/CU(A) \to U(B)/CU(B)$ a continuous homomorphism. As in [18], not all compatible triples $(\kappa, \kappa_T, \kappa_\gamma)$ are proved to be reached by unital homomorphisms. This is not just the limitation of the method. By the classification theorem in [15, 17], there is a unital separable simple $\mathcal{Z}$-stable C^*-algebra A with a unique tracial state which is locally approximated by sub-homogeneous C^*-algebras such that $(K_0(A), K_0(A)_+, [1_A]) = (\mathbb{Z}, \mathbb{Z}_+, 1)$ and $K_1(A) = \mathbb{Z}/p\mathbb{Z}$ for some prime number $p > 1$. By [18, Lemma 6.8], there is a unital homomorphism $\varphi : A \to \mathcal{Z}$ which induces identity on $K_0(A)$. We found that, up to approximately unitary equivalence, φ is the only homomorphism that induces $KL(\varphi)$. However, there are other continuous homomorphisms $\gamma : U(A)/CU(A) \to U(\mathcal{Z})/CU(\mathcal{Z})$ which are compatible to $KL(\varphi)$ and the identity map on the tracial state spaces (which has only one point for both C^*-algebras). In other words, there are compatible triples $(\kappa, \kappa_T, \kappa_\gamma)$ which cannot be reached by unital homomorphisms.

This is by no means an accident. Fix a compatible pair (κ, κ_T). Denote by

$$\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)$$

the subset of those homomorphisms in $\mathrm{Hom}(U(A)/CU(A), U(B)/CU(B))$ which are compatible to the pair (κ, κ_T). There is a bijection from

$$\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)$$

to the group $\mathrm{Hom}(K_1(A), \mathbb{T})$, where $\mathbb{T} = \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$ and $\rho_B : K_0(B) \to \mathrm{Aff}(T(B))$ (the space of all real continuous affine functions on the tracial state space $T(B)$ of B) is the usual pairing of $K_0(B)$ and $T(B)$.

Let $\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$ be the approximately unitary equivalence classes of unital homomorphisms φ such that φ induce the pair $(KL(\varphi), \varphi_T) = (\kappa, \kappa_T)$. We show that $\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$ is not empty. The uniqueness part of this paper gives an injective map from $\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$ to a subgroup of $\mathrm{Hom}(K_1(A), \mathbb{T})$. This subgroup is isomorphic to the group $\mathrm{Hom}(K_1(A)/\mathrm{Tor}(K_1(A)), \mathbb{T})$. Theorem 5.10 shows that there is a splitting short exact sequence which further describes this subgroup. It turns out that (see Proposition 5.15), whenever $\overline{\mathbb{R}\rho_B(K_0(B))} \neq \overline{\rho_B(K_0(B))}$ and $K_1(A)$ has a torsion, this subgroup is a proper subgroup of $\mathrm{Hom}(K_1(A), \mathbb{T})$. In those cases, there are compatible triples $(\kappa, \kappa_T, \kappa_\gamma)$ which cannot be reached by unital homomorphisms. We also show that there is another way to describe the range of the invariant of unital homomorphisms by considering a sequence of compatible triples which complements the description of the range of unital homomorphisms mentioned above (see Theorem 6.5 and Remark 6.6). The group $U(A)/CU(A)$ is also an essential part of the invariant set for the classification of non-simple C^*-algebras with ideal property (not just for homomorphisms); see [4, 5] (see also [21] for general information about C^*-algebras with ideal property).

2. Notations

Definition 2.1. Let A be a unital C^*-algebra. Denote by $A_{\mathrm{s.a.}}$ the self-adjoint part of A and A_+ the set of all positive elements of A. Denote by $U(A)$ the unitary group of A and $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries which are in the connected component of $U(A)$ containing 1_A. Denote by $DU(A)$ the commutator subgroup of $U_0(A)$ and $CU(A)$ the closure of $DU(A)$ in $U(A)$.

Definition 2.2. Let A be a unital C^*-algebra, and let $T(A)$ denote the simplex of tracial states of A, a compact subset of A^*, the dual of A, with the weak* topology. Denote by $\mathrm{Aff}(T(A))$ the space of real valued affine continuous functions on $T(A)$.

Let A be a unital stably finite C^*-algebra with $T(A) \neq \emptyset$. Let $\tau \in T(A)$. For each integer $n \geq 1$, we will continue to use τ for its extension $\tau \otimes \mathrm{Tr}$ on $M_n(A)$, where Tr is the standard trace on M_n.

Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the order-preserving homomorphism defined by $\rho_A([p])(\tau) = \tau(p)$ for any projection $p \in M_n(A), n = 1, 2, \ldots$ (see the convention above).

Suppose that B is another C^*-algebra with $T(B) \neq \emptyset$ and $\varphi : A \to B$ is a unital homomorphism. Then, φ induces a continuous affine map $\varphi_T : T(B) \to T(A)$ defined by $\varphi_T(\tau)(a) = \tau(\varphi(a))$ for all $a \in A$ and $\tau \in T(B)$. Denote by $\varphi_\sharp : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ the continuous map induced by φ_T.

Definition 2.3. Let A be a unital C^*-algebra, and let $u \in U_0(A)$. Let $u(t) \in C([0,1], A)$ be a piecewise smooth path of unitaries such that $u(0) = u$ and $u(1) = 1$. Then, the de La Harpe–Skandalis determinant of the path $\{u(t)\}_{0 \leq t \leq 1}$ is defined by

$$\mathrm{Det}\left(\{u(t)\}_{0\leq t\leq 1}\right)(\tau) = \frac{1}{2\pi i}\int_0^1 \tau\left(\frac{du(t)}{dt}u(t)^*\right)dt \quad \text{for all } \tau \in T(A),$$

which induces a homomorphism

$$\overline{\mathrm{Det}} : U_0(A) \to \mathrm{Aff}\left(T(A)\right)/\overline{\rho_A\left(K_0(A)\right)}.$$

The determinant $\overline{\mathrm{Det}}$ can be extended to a map from $U_0(M_\infty(A))$ into

$$\mathrm{Aff}\left(T(A)\right)/\overline{\rho_A\left(K_0(A)\right)}.$$

Definition 2.4. Suppose that A is a unital C^*-algebra with $T(A)\neq\emptyset$. Recall that $CU(A)$ is the closure of the commutator subgroup of $U_0(A)$. Let $u\in U(A)$. We shall use $\bar{u}$ to denote the image in $U(A)/CU(A)$. It was proved in [25] that there is a splitting short exact sequence

$$0\to \mathrm{Aff}\big(T(A)\big)/\overline{\rho_A\big(K_0(A)\big)}\to \bigcup_{n=1}^{\infty} U\big(M_n(A)\big)/\bigcup_{n=1}^{\infty} CU\big(M_n(A)\big)\xrightarrow{\Pi_A^{cu}} K_1(A)\to 0.$$

For each A, we will fix one splitting map

$$s_A: K_1(A)\to \bigcup_{n=1}^{\infty} U\big(M_n(A)\big)/\bigcup_{n=1}^{\infty} CU\big(M_n(A)\big)$$

such that $\Pi_A^{cu}\circ s_A=\mathrm{id}_{K_1(A)}$.

In the case that A has stable rank no more than k ($k\geq 1$), one may have

$$0\to \mathrm{Aff}\big(T(A)\big)/\overline{\rho_A\big(K_0(A)\big)}\to U\big(M_k(A)\big)/CU\big(M_k(A)\big)\underset{s_A}{\overset{\Pi_A^{cu}}{\rightleftarrows}} K_1(A)\to 0. \qquad (\mathrm{e}\,2.1)$$

Definition 2.5. Let $\Sigma_A:\mathrm{Aff}(T(A))\to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ be the quotient map.

Definition 2.6. Let A be a unital separable C^*-algebra and B a unital finite simple $\mathcal{Z}$-stable C^*-algebra. Denote by $KL_e(A,B)^{++}$ the subset of those elements κ in $KL(A,B)$ such that $\kappa(K_0(A)_+\setminus\{0\})\subset K_0(B)_+\setminus\{0\}$ and $\kappa([1_A])=[1_B]$.

Suppose, in addition, $T(A)\neq\emptyset$. Let $\kappa_T: T(B)\to T(A)$ be a continuous affine map. Then, κ_T induces an affine continuous map $\kappa_\sharp:\mathrm{Aff}(T(A))\to\mathrm{Aff}(T(B))$. The pair (κ,κ_T) is compatible if

$$\rho_B\big(\kappa(x)\big)(\tau)=\rho_A(x)\big(\kappa_T(\tau)\big)\quad \text{for all } x\in K_0(A),\ \tau\in T(B).$$

In particular, $\kappa_\sharp(\overline{\rho_A(K_0(B))})\subset\overline{\rho_B(K_0(B))}$. Thus, $\kappa_\sharp$ induces a homomorphism

$$\overline{\kappa_\sharp}:\mathrm{Aff}\big(T(A)\big)/\overline{\rho_A\big(K_0(A)\big)}\to \mathrm{Aff}\big(T(B)\big)/\overline{\rho_B\big(K_0(B)\big)}.$$

For convenience, let us also assume that A has stable rank at most n. Let

$$\kappa_\gamma: U\big(M_n(A)\big)/CU\big(M_n(A)\big)\to U\big(M_n(B)\big)/CU\big(M_n(B)\big)$$

be a continuous homomorphism. We say that $(\kappa,\kappa_T,\kappa_\gamma)$ is compatible if (κ,κ_T) is compatible, and the following diagram commutes

$$\begin{array}{ccccccccc}
0\longrightarrow & \mathrm{Aff}\big(T(A)\big)/\overline{\rho_A\big(K_0(A)\big)} & \longrightarrow & U\big(M_n(A)\big)/CU\big(M_n(A)\big) & \xrightarrow{\Pi_A^{cu}} & K_1(A) & \longrightarrow 0\\
 & \Big\downarrow{\scriptstyle \overline{\kappa_T}} & & \Big\downarrow{\scriptstyle \kappa_\gamma} & & \Big\downarrow{\scriptstyle \kappa|_{K_1(A)}} & \\
0\longrightarrow & \mathrm{Aff}\big(T(B)\big)/\overline{\rho_B\big(K_0(B)\big)} & \longrightarrow & U\big(M_n(B)\big)/CU\big(M_n(B)\big) & \xrightarrow{\Pi_B^{cu}} & K_1(B) & \longrightarrow 0,
\end{array} \qquad (\mathrm{e}\,2.2)$$

where $\overline{\kappa_T}$ is the homomorphism induced by κ_T.

Let $n > 1$, and let

$$j : U(A)/CU(A) \to U\big(M_n(A)\big)/CU\big(M_n(A)\big),$$
$$j_* : U(A)/U_0(A) \to U\big(M_n(A)\big)/U_0\big(M_n(A)\big),$$
$$j_\sharp : U_0(A)/CU(A) \to U_0\big(M_n(A)\big)/CU\big(M_n(A)\big)$$

be the homomorphisms induced by the map $u \mapsto \mathrm{diag}(u, 1_{n-1})$. Suppose that A has stable rank one. Then, by [22, Theorem 2.9] and [8, Corollary 3.11], the maps j_* and $j_\sharp$ are isomorphisms. Moreover, $K_1(A) = U(A)/U_0(A)$. Note that $\prod_A^{cu} \circ j = j_* \circ \prod_A^{cu}$. It follows that j is injective. Let $u \in U(M_n(A))$. There is $u_0 \in U(A)$ such that $u \cdot \mathrm{diag}(u_0^*, 1_{n-1}) \in U_0(M_n(A))$. It follows that

$$\overline{u \cdot \mathrm{diag}(u_0^*, 1_{n-1})} \in U_0\big(M_n(A)\big)/CU\big(M_n(A)\big).$$

By [8, Corollary 3.11], there is $v_0 \in U_0(A)$ such that $\overline{u \cdot \mathrm{diag}(u_0^*, 1_{n-1})} = \overline{\mathrm{diag}(v_0, 1_{n-1})}$. Thus, $\bar{u} = \overline{\mathrm{diag}(u_0 v_0, 1_{n-1})}$. In other words, the map $z \to \mathrm{diag}(z, 1_{n-1})$ from $U(A)/CU(A)$ to $U(M_n(A))/CU(M_n(A))$ is an isomorphism.

Definition 2.7. Let A and B be unital C^*-algebras with $T(A) \neq \emptyset$ and $T(B) \neq \emptyset$. Let $\varphi : A \to B$ be a unital homomorphism. Denote by $KK(\varphi)$ and $KL(\varphi)$ the elements in $KK(A, B)$ and $KL(A, B)$ induced by φ, respectively. We also use $[\varphi]$ for $KL(\varphi)$ whenever it is convenient.

Note that $\varphi_\sharp$ maps $\rho_A(K_0(A))$ to $\rho_B(K_0(B))$ and φ maps $CU(A)$ into $CU(B)$. Denote by $\varphi^\ddagger : U(A)/CU(A) \to U(B)/CU(B)$ the induced continuous homomorphism. Then, $(KL(\varphi), \varphi_T, \varphi^\ddagger)$ is compatible.

2.8. Let A and B be unital C^*-algebras such that $T(B) \neq \emptyset$. Let $\varphi, \psi : A \to B$ be two unital homomorphisms such that $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(B)$. Consider the mapping torus

$$M_{\varphi,\psi} = \big\{(b, a) \in C\big([0, 1], B\big) \oplus A : b(0) = \varphi(a),\ b(1) = \psi(a)\big\}.$$

Let $u = (u(t), a) \in U_0(M_n(M_{\varphi,\psi}))$ $(u(0) = \varphi(a), u(1) = \psi(a))$ such that $u(t)$ is piecewise smooth. Then, $u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_m)$, where $h_j \in M_n(M_{\varphi,\psi})_{\text{s.a.}}$. Moreover, one may choose $h_j(t)$ $(t \in [0, 1])$ so that $h_j(t)$ is piecewise smooth. One then computes that, for each $\tau \in T(B)$ (since $\tau \circ \varphi = \tau \circ \psi$),

$$\begin{aligned} R_{\varphi,\psi}\big(u(t)\big)(\tau) &= \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{du(t)}{dt}u^*(t)\Big)dt \\ &= \frac{1}{2\pi i}\int_0^1 \sum_{j=1}^m \tau\Big(\frac{dh_j(t)}{dt}\Big)dt \\ &= \frac{1}{2\pi i}\sum_{j=1}^m \big(\tau\big(h_j(0)\big) - \tau\big(h_j(1)\big)\big) = 0. \end{aligned}$$

As in [15, Definition 3.2 and Lemma 3.3], $R_{\varphi,\psi} : K_1(M_{\varphi,\psi}) \to \mathrm{Aff}(T(B))$ is a homomorphism. In fact, we have the following commutative diagram:

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\quad \iota \quad} & K_1(M_{\varphi,\psi}) \\ & {\scriptstyle \rho_B} \searrow \quad \swarrow {\scriptstyle R_{\varphi,\psi}} & \\ & \mathrm{Aff}\big(T(B)\big). & \end{array}$$

Definition 2.9. Let $\kappa \in KL_e(A, B)^{++}$ and $\kappa_T : T(B) \to T(A)$ be a continuous affine map such that (κ, κ_T) is a compatible pair. Let $\mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$ be the set of homomorphisms $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ such that $(\kappa, \kappa_T, \gamma)$ is compatible.

Fix $g \in \mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$. Then, for any

$$\beta \in \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big),$$

$g - \beta$ gives a homomorphism in

$$\mathrm{Hom}\big(U(A)/CU(A), U(B)/CU(B)\big)$$

which maps $U(A)/CU(A)$ to $\mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$ and vanishes on

$$\mathrm{Aff}\big(T(A)\big)/\overline{\rho_A\big(K_0(A)\big)}.$$

Thus,

$$\begin{aligned}&\big\{g - \beta : \beta \in \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)\big\}\\ &\quad = \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\big(T(A)\big)/\overline{\rho_A\big(K_0(A)\big)}\big).\end{aligned}$$

Let Γ^g be the bijection $\beta \mapsto g - \beta$ ($\beta \in \mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$) which gives a group structure on $\mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$. Note that the group is independent of the choice of g. In this way, we may view

$$\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)$$

as an abelian group. Denote by $\mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$ the subgroup of homomorphisms $\bar{h}$ in $\mathrm{Hom}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$ such that there is a sequence of homomorphisms $h_n \in \mathrm{Hom}(K_1(A), \mathrm{Aff}(T(B)))$ such that $\pi \circ h_n|_{G_n} = \bar{h}|_{G_n}$, where $G_n \subset G_{n+1} \subset K_1(A)$ is a finitely generated subgroup and $K_1(A) = \bigcup_{n=1}^{\infty} G_n$.

Let $\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$ be the set of approximately unitary equivalence classes of homomorphisms φ from A to B such that

$$\big(KL(\varphi), \varphi_T\big) = (\kappa, \kappa_T).$$

Let $\mathrm{Hom}_{\mathrm{alf}}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$ be the subgroup of homomorphisms $\bar{h}$ in $\mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$ such that

$$\bar{h}\big(K_1(A)\big) \subset \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}.$$

It is also a subgroup of those $\bar{h}$'s in $\mathrm{Hom}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$ such that there is a sequence of homomorphisms $h_n \in \mathrm{Hom}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))})$ such that $\pi \circ h_n|_{G_n} = \bar{h}|_{G_n}$, where $G_n \subset G_{n+1} \subset K_1(A)$ is a finitely generated subgroup and $K_1(A) = \bigcup_{n=1}^{\infty} G_n$.

Definition 2.10 ([6, Definition 9.2]). Let A be a unital simple C^*-algebra. We say A has generalized tracial rank at most one (gTR(A) ≤ 1) if the following property holds: let $\varepsilon > 0$, let $a \in A_+ \setminus \{0\}$, and let $\mathcal{F} \subseteq A$ be a finite set. There exists a non-zero projection $p \in A$ and a unital C^*-subalgebra C which is a subhomogeneous C^*-algebra whose spectrum has dimension at most one and with $1_C = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2) $\mathrm{dist}(pxp, C) < \varepsilon$ for all $x \in \mathcal{F}$,

(3) $1 - p \lesssim a$.

By [3, Theorem 4.10], every unital finite separable simple C^*-algebra with finite nuclear dimension which satisfies the UCT has the property that $\mathrm{gTR}(A \otimes U) \leq 1$ (see also [20, Theorem 3.4]) for every infinite-dimensional UHF-algebra U.

Now let A be a unital finite separable simple amenable C^*-algebra which satisfies the UCT and U a UHF-algebra of infinite type (so $U \cong U \otimes U$). By [1], $A \otimes U$ has finite nuclear dimension. From the previous paragraph,

$$gTR(A \otimes U) = gTR\big((A \otimes U) \otimes U\big) \leq 1.$$

This fact will be repeatedly used throughout the paper.

Definition 2.11. Throughout the paper, Q is the UHF-algebra such that

$$\big(K_0(Q), K_0(Q)_+, [1_Q]\big) = (\mathbb{Q}, \mathbb{Q}_+, 1).$$

Let $\mathfrak{r}$ be a supernatural number. Denote by $M_{\mathfrak{r}}$ the UHF-algebra associated with $\mathfrak{r}$.

Let $\mathfrak{p}$ and $\mathfrak{q}$ be a pair of relatively prime supernatural numbers of infinite type such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} = Q$. Let $j_{\mathfrak{p}} : M_{\mathfrak{p}} \to Q$ be defined by $j_{\mathfrak{p}}(a) = a \otimes 1_{M_{\mathfrak{q}}}$ and $j_{\mathfrak{q}} : M_{\mathfrak{q}} \to Q$ be defined by $j_{\mathfrak{q}}(b) = b \otimes 1_{M_{\mathfrak{q}}}$. Define

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \big\{(f, a, b) : (f, a, b) \in C\big([0,1], Q\big) \oplus (M_{\mathfrak{p}} \oplus M_{\mathfrak{q}}) : \\ f(0) = j_{\mathfrak{p}}(a), f(1) = j_{\mathfrak{q}}(b)\big\}.$$

Definition 2.12. Let A be a unital separable amenable C^*-algebra, and let $x \in A$. Suppose that $\|xx^* - 1\| < 1$ and $\|x^*x - 1\| < 1$. This "approximate unitary" is close to a unitary. In fact, $x|x|^{-1}$ is a unitary. Let us use $\langle x \rangle$ to denote $x|x|^{-1}$.

Let A and B be unital C^*-algebras, and let $\varphi : A \to B$ be a homomorphism and $v \in U(B)$. We refer to [6, Definition 2.14] for the definition of locally defined $\mathrm{bott}_0(\varphi, v)$, $\mathrm{bott}_1(\varphi, v)$, and $\mathrm{Bott}(\varphi, v)$ when φ and v almost commute. We also refer to [6, Definitions 2.12 and 2.14] for other related terminologies.

3. Homotopy lemmas, restated

Lemma 3.1 ([7, Lemma 25.4]). *Let $A = A_1 \otimes U_1$, where $gTR(A_1) \le 1$ and satisfies the UCT and U_1 is a UHF-algebra of infinite type. For any $1 > \varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\{p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_k\}$ of projections of A such that $Q := \{[p_1] - [q_1], [p_2] - [q_2], \dots, [p_k] - [q_k]\}$ generates a free abelian subgroup G_u of $K_0(A)$, and a finite subset $\mathcal{P} \subset \underline{K}(A)$, satisfying the following condition:*

Let $B = B_1 \otimes U_2$, where $gTR(B_1) \le 1$ and U_2 is a UHF-algebra of infinite type. Suppose that $\varphi : A \to B$ is a unital homomorphism.

If $u \in U(B)$ is a unitary such that

$$\big\|[\varphi(x), u]\big\| < \delta \quad \text{for all } x \in \mathcal{G}, \tag{e 3.1}$$

$$\mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0, \tag{e 3.2}$$

$$\mathrm{dist}\Big(\overline{\big\langle\big((1 - \varphi(p_i)) + \varphi(p_i)u\big)\big((1 - \varphi(q_i)) + \varphi(q_i)u^*\big)\big\rangle}, \bar{1}\Big) < \sigma, \tag{e 3.3}$$

$$\mathrm{dist}(\bar{u}, \bar{1}) < \sigma, \tag{e 3.4}$$

then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset U(B)$ such that

$$u(0) = u, \quad u(1) = 1_B,$$
$$\mathrm{dist}\big(u(t), CU(B)\big) < \varepsilon \quad \text{for all } t \in [0,1],$$
$$\big\|[\varphi(a), u(t)]\big\| < \varepsilon \quad \text{for all } a \in \mathcal{F},\ t \in [0,1],$$
$$\mathrm{length}\big(\{u(t)\}\big) \le 2\pi + \varepsilon.$$

Remark 3.2. The original statement of [7, Lemma 25.4] assumes $A_1 \in \mathcal{B}_0$. However, since $U_1 \otimes U_1 \cong U_1$, we may assume $A_1 = A_1 \otimes U_1$. If $gTR(A_1) \le 1$, by [6, Theorem 19.2], $A_1 \otimes Q \in \mathcal{B}_0$. Then, by [20, Theorem 3.4] (see also [6, Theorem 3.20]), $A_1 \otimes U_1 \in \mathcal{B}_0$. Thus, it suffices to assume that $gTR(A_1) \le 1$ as well as $gTR(B) \le 1$. Moreover, B does not need to be assumed to satisfy the UCT.

Let us also comment that the condition (e 3.4) may be dropped (if we choose sufficiently large set $\{p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_k\}$ and sufficiently small σ). To see this, one notes that one can always assume $[1_A] + x \in G_u$ for some $x \in K_0(A)$ with $mx = 0$ for some integer $m \ge 1$. Thus, $m[1_A] \in G_u$. Suppose that $m[1_A] = \sum_{j=1}^k m_j([p_j] - [q_j])$ for some integers m_j, $j = 1, 2, \dots, k$. Once this is done, let $K = (\sum_{j=1}^k |m_j|)$. For any $1 > \varepsilon > 0$, choose $\sigma = \varepsilon/K$. Then, one checks that the condition (e 3.3) implies that $\mathrm{dist}(\overline{u^m}, \bar{1}) < K\sigma$.

We claim that $\operatorname{dist}(\bar{u}, \bar{1}) < \varepsilon/m$. In fact, there exists a unitary $\omega_c \in CU(A)$ and $\omega \in U(A)$ such that $u\omega_c\omega = 1$ and $\|\omega - 1\| < K\sigma = \varepsilon < 1$. Therefore, $\omega = \exp(i\pi a)$ with $a \in A_{\text{s.a.}}$ and $\|a\| < \varepsilon/m$. Write $\omega_c = \prod_{k=1}^{k_2} \exp(ib_k)$ for some $b_k \in M_{\text{s.a.}}$, $k = 1, 2, \ldots, k_2$. Define $\omega_{c,0} = \prod_{k=1}^{k_2} \exp(ib_k/m)$ and $\omega_0 = \exp(ia/m)$. Then, $(\overline{u\omega_{c,0}\omega_0})^m = \bar{1} \in CU(A)$. By [6, Corollary 11.7], $U(A)/CU(A)$ is torsion-free. It follows that $u\omega_{c,0}\omega_0 \in CU(A)$ and $\omega_{c,0} \in CU(A)$. Hence,

$$\operatorname{dist}(\bar{u}, \bar{1}) \le \|\omega_0 - 1\| < \varepsilon/m.$$

Lemma 3.3 (cf. [7, Lemma 24.5]). *Let $A = A_1 \otimes U_1$, where A_1 is as in* [6, Theorem 14.10] *and $B = B_1 \otimes U_2$, where B_1 is a unital simple C^*-algebra with $gTR(B_1) \le 1$ and U_1, U_2 are two UHF-algebras of infinite type. Let $A = \lim_{n\to\infty}(C_n, \iota_n)$ be as described in* [6, Theorem 14.10]. *For any $\varepsilon > 0$, any $\sigma > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{P} \subset \underline{K}(A)$, and any projections $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k \in A$ such that $\{x_1, x_2, \ldots, x_k\}$ generates a free abelian subgroup G of $K_0(A)$, where $x_i = [p_i] - [q_i]$, $i = 1, 2, \ldots, k$, there exists an integer $n \ge 1$ such that $x_i \in \mathcal{P} \subset [\iota_{n,\infty}](\underline{K}(C_n))$ $(1 \le i \le k)$, there is a finite subset $\mathcal{Q} \subset K_1(C_n)$ which generates $K_1(C_n)$, and there exists $\delta > 0$ satisfying the following condition: let $\varphi : A \to B$ be a unital homomorphism, $\Gamma : G \to U(B)/CU(B)$ a homomorphism, and $\alpha \in KK(C_n \otimes C(\mathbb{T}), B)$ such that*

$$\begin{aligned} &\alpha\big(\boldsymbol{\beta}(g)\big) = \Pi_B^{cu}\big(\Gamma\big((\iota_{n,\infty})_{*0}(g)\big)\big) \quad \text{for all } g \in \iota_{n,\infty}^{-1}(G) \quad \big(\text{see (e 2.2)}\big), \\ &\big|\tau \circ \rho_B\big(\alpha\big(\boldsymbol{\beta}(x)\big)\big)\big| < \delta \quad \text{for all } x \in \mathcal{Q},\ \tau \in T(B). \end{aligned} \qquad \text{(e 3.5)}$$

Then, there exists a unitary $u \in B$ such that

$$\big\|\big[\varphi(x), u\big]\big\| < \varepsilon \text{ for all } x \in \mathcal{F}, \quad \operatorname{Bott}\big(\varphi \circ [\iota_{n,\infty}], u\big) = \alpha(\boldsymbol{\beta}),$$

and, for $i = 1, 2, \ldots, k$,

$$\operatorname{dist}\Big(\overline{\big\langle\big((1 - \varphi(p_i)) + \varphi(p_i)u\big)\big((1 - \varphi(q_i)) + \varphi(q_i)u^*\big)\big\rangle},\ \Gamma(x_i)\Big) < \sigma. \qquad \text{(e 3.6)}$$

Proof. As in Remark 3.2, we may assume that $B_1 \in \mathcal{B}_0$. The lemma follows from [7, Lemma 24.2 and Theorem 22.17]. In fact, for any $0 < \varepsilon_1 < \varepsilon/2$ and finite subset $\mathcal{F}_1 \supset \mathcal{F}$, by [7, Lemma 24.2], there exists an integer $n \ge 1$, a finite subset $\mathcal{Q} \subset K_1(C_n)$, and $\delta > 0$ as described above, and a unitary $u_1 \in U_0(B)$, such that

$$\begin{aligned} &\big\|\big[\varphi(x), u_1\big]\big\| < \varepsilon_1 \quad \text{for all } x \in \mathcal{F}_1, \\ &\operatorname{Bott}(\varphi \circ \iota_{n,\infty}, u_1) = \alpha(\boldsymbol{\beta})|_{\mathcal{P}}. \end{aligned}$$

Choosing a smaller ε_1 and a larger $\mathcal{F}_1$, if necessary, we may assume that the class

$$\overline{\big\langle\big((1 - \varphi(p_i)) + \varphi(p_i)u_1\big)\big((1 - \varphi(q_i)) + \varphi(q_i)u_1^*\big)\big\rangle} \in U(B)/CU(B)$$

is well defined for all $1 \le i \le k$. Define a map $\Gamma_1 : G \to U_0(B)/CU(B)$ by

$$\Gamma_1(x_i) = \overline{\big\langle\big((1 - \varphi(p_i)) + \varphi(p_i)u_1\big)\big((1 - \varphi(q_i)) + \varphi(q_i)u_1^*\big)\big\rangle}, \quad i = 1, 2, \ldots, k.$$

Choosing a large enough n, without loss of generality, we may assume that there are projections $p_1', p_2', \ldots, p_k', q_1', q_2,' \ldots, q_k' \in C_n$ such that $\imath_{n,\infty}(p_i') = p_i$ and $\imath_{n,\infty}(q_i') = q_i$, $i = 1, 2, \ldots, k$. Moreover, we may assume that $\mathcal{F}_1 \subset \imath_{n,\infty}(C_n)$. Let $\Gamma_2(x_i) = \Gamma_1(x_i)^*\Gamma(x_i)$, $i = 1, 2, \ldots, k$. By (e 3.5), $\Gamma_2(x_i) \in U_0(B)/CU(B)$. Hence, Γ_2 defines a map from G to $U_0(B)/CU(B)$. It follows by [7, Theorem 22.17] that a unitary $v \in U_0(B)$ such that

$$\|[\varphi(x), v]\| < \varepsilon/2 \quad \text{for all } x \in \mathcal{F}, \tag{e 3.7}$$

$$\mathrm{Bott}(\varphi \circ \imath_{n,\infty}, v) = 0, \tag{e 3.8}$$

$$\mathrm{dist}\left(\overline{\langle\big((1-\varphi(p_i)) + \varphi(p_i)v\big)\big((1-\varphi(q_i)) + \varphi(q_i)v^*\big)\rangle}, \Gamma_2(x_i)\right) < \sigma, \tag{e 3.9}$$

$i = 1, 2, \ldots, k$. Define $u = u_1 v$,

$$X_i = \overline{\langle\big((1-\varphi(p_i)) + \varphi(p_i)u_1\big)\big((1-\varphi(q_i)) + \varphi(q_i)u_1^*\big)\rangle},$$
$$Y_i = \overline{\langle\big((1-\varphi(p_i)) + \varphi(p_i)v\big)\big((1-\varphi(q_i)) + \varphi(q_i)v^*\big)\rangle},$$

$i = 1, 2, \ldots, k$. We then compute that

$$\begin{aligned}
&\|[\varphi(x), u]\| < \varepsilon_1 + \varepsilon/2 < \varepsilon \quad \text{for all } x \in \mathcal{F},\\
&\mathrm{Bott}(\varphi \circ \imath_{n,\infty}, u) = \mathrm{Bott}(\varphi \circ \imath_{n,\infty}, u_1) = \alpha(\boldsymbol{\beta}),\\
&\mathrm{dist}\left(\overline{\langle\big((1-\varphi(p_i)) + \varphi(p_i)u\big)\big((1-\varphi(q_i)) + \varphi(q_i)u^*\big)\rangle}, \Gamma(x_i)\right)\\
&\quad \le \mathrm{dist}\big(X_iY_i, \Gamma_1(x_i)Y_i\big) + \mathrm{dist}\big(\Gamma_1(x_i)Y_i, \Gamma(x_i)\big)\\
&\quad = \mathrm{dist}\big(X_i, \Gamma_1(x_i)\big) + \mathrm{dist}\big(Y_i, \Gamma_2(x_i)\big) < 0 + \sigma \quad \big(\text{by (e 3.9)}\big),
\end{aligned}$$

for $i = 1, 2, \ldots, k$. ∎

Remark 3.4. Lemma 3.3 also holds if $p_i, q_i \in M_N(A)$ for some given integer N. Then, (e 3.6) may be written as

$$\mathrm{dist}\left(\overline{\langle\big((1_N-\varphi(p_i)) + \varphi(p_i)(u\otimes 1_N)\big)\big((1-\varphi(q_i)) + \varphi(q_i)(u^*\otimes 1_N)\big)\rangle}, \Gamma(x_i)\right) < \sigma. \tag{e 3.10}$$

However, in (e 3.10), $\varphi(p_i) := (\varphi \otimes \mathrm{id}_{M_N})(p_i)$, $i = 1, 2, \ldots$. Note also that $\varphi(p_i)$ approximately commutes with $u \otimes 1_N$ within any previously prescribed small number, say η. By [8, Theorem 4.6] (see also [6, Lemma 11.9]), there is $z_i \in U(B)$ such that

$$\overline{\mathrm{diag}(z_i, 1_{N-1})} = \overline{\langle\big((1_N-\varphi(p_i)) + \varphi(p_i)(u\otimes 1_N)\big)\big((1-\varphi(q_i)) + \varphi(q_i)(u^*\otimes 1_N)\big)\rangle}.$$

In fact, by (e 3.6), we mean

$$\mathrm{dist}\big(\overline{z_i}, \Gamma(x_i)\big) < \sigma.$$

By [8, Theorem 4.6], $\overline{z_i}$ is unique.

To see that we can allow $p_i, q_i \in M_N(A)$, suppose that $U_1 = M_{\mathfrak{p}}$ and $U_2 = M_{\mathfrak{q}}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are supernatural numbers of infinite type. We identify $K_0(M_{\mathfrak{r}})$ with the dense subgroup $\mathbb{D}_{\mathfrak{r}}$ of $\mathbb{Q}$ given by the supernatural number $\mathfrak{r}$. Choose $N_1 \geq N$ such that $\frac{1}{N_1} \in \mathbb{D}_{\mathfrak{p}}$. There are mutually orthogonal and mutually unitarily equivalent projections $p_{i,1}, \ldots, p_{i,N_1} \in A \otimes U_1$ such that $p_i = \sum_{j=1}^{N_1} p_{i,j}$, and mutually orthogonal and mutually unitarily equivalent projections $q_{i,1}, \ldots, q_{i,N_1} \in A \otimes U_1$ such that

$$q_i = \sum_{j=1}^{k} q_{i,j}, \quad i = 1, 2, \ldots, N_1.$$

Put $x_i' := [p_{i,1}] - [q_{i,1}]$, $i = 1, 2, \ldots, k$. Let G_0 be generated by $\{x_1', x_2,' \ldots, x_k'\}$. Then, G_0 is also a free abelian group.

Let $\mathcal{P}_1 = \mathcal{P} \cup G_0$. Choose larger n so that $\mathcal{P}_1 \subset [\imath_{n,\infty}](\underline{K}(C_n))$. If Γ is given and fits α as (e 3.5), choose $y_1, y_2, \ldots, y_k \in K_0(C_n)$ such that $[j_{n,\infty}](y_i) = x_i'$, $i = 1, 2, \ldots, k$. Let $z_i = \alpha \circ \boldsymbol{\beta}(y_i)$. Then, $\prod_B^{cu}(\Gamma(x_i) s_B(z_i)^{-N_1}) = 0$ in $K_1(B)$ (see (e 2.1) for s_B). It follows that $f_i := \Gamma(x_i) s_B(z_i)^{-N_1} \in U_0(B)/CU(B)$. Recall that

$$U_0(B)/CU(B) = \operatorname{Aff}\big(T(B)\big)/\rho_B\big(K_0(B)\big)$$

is divisible. Define $\Gamma_0 : G_0 \to U(B)/CU(B)$ by

$$\Gamma_0(x_i') = \big((1/N_1) f_i\big) s_B(z_i), \quad i = 1, 2, \ldots, k.$$

Then, $\Gamma_0(x_i) = N_1 \Gamma_0(x_i') = \Gamma(x_i)$, $i = 1, 2, \ldots, k$. Then, we apply the current Lemma 3.3. We apply this for G_0 instead of G, Γ_0 instead of Γ, and $\sigma/2N_1$ instead of σ. We will have, among other things,

$$\operatorname{dist}\Big(\overline{\big\langle\big(\big(1 - \varphi(p_{i,1})\big) + \varphi(p_{i,1})u\big)\big(\big(1 - \varphi(q_{i,1})\big) + \varphi(q_{i,1})u^*\big)\big\rangle}, \Gamma_0(x_i')\Big) < \sigma/2N_1,$$

$1 \leq i \leq k$. We also assume that $gu \approx_{\sigma/(8N_1)^2} ug$ for all g in a finite subset $\mathcal{F}_1$ such that $\varphi(p_{i,j}), \varphi(q_{i,j}) \in \mathcal{F}_1$. One has (with sufficiently large $\mathcal{F}_1$)

$$\begin{aligned} &\Big(\overline{\big\langle\big(\big(1 - \varphi(p_{i,1})\big) + \varphi(p_{i,1})u\big)\big(\big(1 - \varphi(q_{i,1})\big) + \varphi(q_{i,1})u^*\big)\big\rangle}\Big)^{N_1} \\ &\quad = \overline{\big\langle\big(\big(1_N - \varphi(p_i)\big) + \varphi(p_i)(u \otimes 1_N)\big)\big(\big(1 - \varphi(q_i)\big) + \varphi(q_i)(u^* \otimes 1_N)\big)\big\rangle}. \end{aligned}$$

It follows that, for $i = 1, 2, \ldots, k$,

$$\operatorname{dist}\Big(\overline{\big\langle\big(\big(1_N - \varphi(p_i)\big) + \varphi(p_i)(u \otimes 1_N)\big)\big(\big(1 - \varphi(q_i)\big) + \varphi(q_i)(u^* \otimes 1_N)\big)\big\rangle}, \Gamma(x_i)\Big) < \sigma.$$

Remark 3.5. Let A_1 be a separable amenable simple C^*-algebra which satisfies the UCT. Then, $A_1 \otimes U_1$ is $\mathcal{Z}$-stable for any UHF-algebra U_1. If U_1 is a UHF-algebra of infinite type, then $A \cong (A_1 \otimes U) \otimes U_1$. By the classification theorem (see [7, Corollary 29.10 and also Remark 29.11]) and [3, Theorem 4.10], $A_1 \otimes U_1$ can be written as $A_1 \otimes U_1 = \lim_{n\to\infty}(C_n, \imath_n)$ as described by [6, Theorem 14.10]; namely, C_n is a direct sum of a homogeneous C^*-algebra in $\mathbf{H}$ and a unital 1-dimensional NCCW complex (see notation in [6, Section 14]). It follows that the assumption that A_1 is an inductive limit of the form in [6, Theorem 14.10] can be replaced by the assumption that A_1 is a separable finite amenable simple C^*-algebra satisfying the UCT.

The following is a slight improvement of [18, Lemma 6.6] for the current purposes.

Lemma 3.6 (cf. [18, Lemma 6.6]). *Let C and A be two unital separable stably finite C^*-algebras, and let $\mathfrak{p}, \mathfrak{q}$ be two relatively prime supernatural numbers of infinite type such that $Q = M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. Suppose that $\varphi_{\mathfrak{r}} : C \otimes M_{\mathfrak{r}} \to A \otimes M_{\mathfrak{r}}$ are unital homomorphisms such that*

$$[\varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}] = [\varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}}] \text{ in } KL(C \otimes Q, A \otimes Q) \text{ and } (\varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}})_{\sharp} = (\varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}})_{\sharp},$$

$\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$. Suppose that $\{U(t) : t \in [0, 1)\}$ is a continuous and piecewise smooth path of unitaries in $A \otimes Q$ such that $U(0) = 1$ and

$$\lim_{t\to 1} U^*(t)\big((\varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}})(u \otimes 1_Q)\big)U(t) = (\varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}})(u \otimes 1_Q)$$

for some $u \in U(C)$, and suppose that $\{Z(t, s)\}_s$ is a continuous and piecewise smooth (piecewise smooth with respect to s) path of unitaries in $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ (that is, for each fixed $s \in [0, 1]$, $Z(-, s) \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$) such that $Z(t, 1) = 1$ and

$$Z(t, 0) = U^*(t)\big((\varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}})(u \otimes 1_{M_{\mathfrak{p}}})\big)U(t)(w^* \otimes 1_Q) \quad \text{if } t \in [0, 1)$$

and $Z(1, 0) = (\varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}})(u)(w^* \otimes 1_Q)$ for some $w \in U(A)$. Suppose also that there exist $h \in \mathrm{Aff}(T(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))$, $f_0 \in \rho_{A\otimes M_{\mathfrak{p}}}(K_0(A \otimes M_{\mathfrak{p}}))$, and $f_1 \in \rho_{A\otimes M_{\mathfrak{q}}}(K_0(A \otimes M_{\mathfrak{q}}))$ such that

$$\mathrm{Det}(Z)(\tau \otimes \delta_j) = h(\tau) + f_j(\tau) \quad \text{for all } \tau \in T(A \otimes Q),\ j = 0, 1, \tag{e 3.11}$$

where δ_t is the extremal tracial state of $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ which factors through the point-evaluation at $t \in [0, 1]$.

Suppose also that there is a continuous and piecewise smooth path of unitaries $\{z(s) : s \in [0, 1]\}$ in $A \otimes M_{\mathfrak{p}} \otimes 1_{M_{\mathfrak{q}}}$ such that $z(0) = ((\varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}})(u \otimes 1_Q))(w^* \otimes 1_Q)$, $z(1) = 1$, and $f_{\mathfrak{p}} \in \rho_{A\otimes M_{\mathfrak{p}}}(K_0(A \otimes M_{\mathfrak{p}}))$, and

$$\frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{dz(s)}{ds} z(s)^*\Big) ds = h(\tau) + f_{\mathfrak{p}}(\tau) \quad \text{for all } \tau \in T(A \otimes Q). \tag{e 3.12}$$

Then, there is $f \in \rho_{A\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(K_0(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))$ such that

$$\Big(\frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{dZ(t,s)}{ds} Z(t,s)^*\Big) ds\Big)(\delta_t) = h(\tau) + f(\tau \otimes \delta_t)$$

for all $t \in [0, 1]$ and $\tau \in T(A)$.

Proof. Put $\varphi := \varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ and $\psi := \varphi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}}$. Define

$$Z_1(t, s) = \begin{cases} U^*(t - 2s)\varphi(u \otimes 1_Q)U(t - 2s)(w^* \otimes 1_Q) & \text{for } s \in [0, t/2) \\ \varphi(u \otimes 1_Q)(w^* \otimes 1_Q) & \text{for } s \in [t/2, 1/2) \\ z(2s - 1) & \text{for } s \in [1/2, 1] \end{cases}$$

for $t \in [0,1)$, and define

$$Z_1(1,s) = \begin{cases} \psi(u \otimes 1_Q)(w^* \otimes 1_Q) & \text{for } s = 0 \\ U^*(1-2s)\varphi(u \otimes 1_Q)U(1-2s)(w^* \otimes 1_Q) & \text{for } s \in (0,1/2) \\ z(2s-1) & \text{for } s \in [1/2,1]. \end{cases}$$

Thus, $\{Z_1(t,s) : s \in [0,1]\} \subset C([0,1], A \otimes Q)$ is a continuous path of unitaries such that $Z_1(t,0) = Z(t)$ and $Z_1(t,1) = 1$. This path may not be piecewise smooth (at $s = 0$). To compute $\mathrm{Det}(Z_1)$, we approximate it by a piecewise smooth path.

Let $1/2 > \varepsilon > 0$. Choose $\delta \in (0,1/8)$ such that

$$\|U^*(t)\varphi(u \otimes 1_Q)U(t)(w^* \otimes 1_Q) - \psi(u \otimes 1_Q)(w^* \otimes 1_Q)\| < \varepsilon/64 \text{ for all } t \in (1-\delta,1)$$

and

$$\|U^*(t)\varphi(u \otimes 1_Q)U(t)(w^* \otimes 1_Q) - U^*(t')\varphi(u \otimes 1_Q)U(t')(w^* \otimes 1_Q)\| < \varepsilon/64$$

whenever $|t - t'| < 2\delta$. There is $H \in (A \otimes Q)_{\text{s.a.}}$ such that

$$U^*(1-\delta)\varphi(u \otimes 1_Q)U(1-\delta)(w^* \otimes 1_Q) = \exp(iH)\psi(u \otimes 1_Q)(w^* \otimes 1_Q)$$

and $\|H\| < \varepsilon/16$. Define $W(t) = U^*(t)\varphi(u \otimes 1_Q)U(t)(w^* \otimes 1_Q)$ if $t \in [0,1-\delta)$ and $W(t) = \psi(u \otimes 1_Q)(w^* \otimes 1_Q)\exp(i(\frac{1-t}{\delta})H)$ if $t \in [1-\delta,1]$. Note that $W(t)$ is a continuous and piecewise smooth path of unitaries in $A \otimes Q$ and it is a unitary in $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. Moreover,

$$\sup\{\|W(t) - Z(t,0)\| : t \in [0,1]\} < \varepsilon/16.$$

There is $H_0 \in (A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})_{s,a}$ such that $Z(t,0) = W\exp(iH_0)$ with $\|H_0\| < \varepsilon/16$. In fact, $H_0(t) = 0$ if $t \in [0,1-\delta]$ and $H_0(t) = -\frac{1-t}{\delta}H$ if $t \in (1-\delta,1]$.

Define

$$Z_\varepsilon(t,s) = \begin{cases} W(t)\exp\big(i\big(\frac{\delta-s}{\delta}\big)H_0(t)\big) & \text{for } s \in [0,\delta), \\ W\big(t - \big(\frac{s-\delta}{1/2-\delta}\big)\big) & \text{for } s \in \big[\delta,(1/2-\delta)t+\delta\big) \\ \varphi(u \otimes 1_Q)(w^* \otimes 1_Q) & \text{for } s \in \big[(1/2-\delta)t+\delta,1/2\big) \\ z(2s-1) & \text{for } s \in [1/2,1] \end{cases}$$

for $t \in [0,1)$, and define

$$Z_\varepsilon(1,s) = \begin{cases} W(1)\exp\big(i\big(\frac{\delta-s}{\delta}\big)H_0(1)\big) & \text{for } s \in [0,\delta] \\ W\big(1 - \big(\frac{s-\delta}{1/2-\delta}\big)\big) & \text{for } s \in (\delta,1/2) \\ z(2s-1) & \text{for } s \in [1/2,1]. \end{cases}$$

Thus, $\{Z_\varepsilon(t,s) : s \in [0,1]\} \subset C([0,1], A \otimes Q)$ is a continuous and piecewise smooth path of unitaries such that $Z_\varepsilon(t,0) = Z(t)$ and $Z_\varepsilon(t,1) = 1$. Moreover,

$$\|Z_\varepsilon - Z_1\| < \varepsilon/8.$$

Thus, there is an element $g \in \rho_{A\otimes Q}(K_0(A\otimes Q)) \subset \mathrm{Aff}(T(A\otimes Q))$ such that

$$g(\tau\otimes\delta_t) = \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{dZ(t,s)}{ds}Z(t,s)^*\Big)ds - \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*\Big)ds$$

for all $\tau \in T(A)$ and $t \in [0,1]$, where δ_t is the extremal tracial state of $C([0,1], Q)$ factors through the point-evaluation at t.

On the other hand, for each $t \in [0,1)$, let $V(t) = U(t)^*\varphi(u\otimes 1_Q)U(t)$ and $V(1) = \psi(u\otimes 1_Q)$. For any $s \in [0,1)$, since $U(0) = 1$, $\{U(t)\}_{0\le t\le s} \in U_0(C([0,s], A\otimes Q))$. There are $a_1, a_2, \ldots, a_k \in U([0,s], A\otimes Q)_{\mathrm{s.a.}}$ such that

$$U(t) = \prod_{j=1}^{k} \exp\big(ia_j(t)\big) \quad \text{for all } t\in[0,s].$$

Then, a straightforward calculation (see [13, Lemma 4.2]) shows that, for each $t \in [0,1)$,

$$\tau\Big(\frac{dV(t)}{dt}V^*(t)\Big) = 0 \quad \text{for all } \tau \in T(A).$$

It follows that, for any $a \in [0,1)$,

$$\frac{1}{2\pi i}\int_0^a \tau\Big(\frac{dV(t)}{dt}V^*(t)\Big)dt = 0 \quad \text{for all } \tau\in T(A). \tag{e 3.13}$$

Hence, for $t \in [0, 1-\delta)$, $W(t) = V(t)(w^*\otimes 1_Q)$ and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\delta^{\delta+(1/2-\delta)t} \tau\Big(\frac{dZ_\varepsilon(t,s)}{ds}Z_\varepsilon(t,s)^*\Big)ds\\
&= \frac{1}{2\pi i}\int_\delta^{\delta+(1/2-\delta)t} \tau\Big(\frac{d}{ds}\Big(W\Big(t-\Big(\frac{s-\delta}{1/2-\delta}\Big)\Big)\Big)W^*\Big(t-\Big(\frac{s-\delta}{1/2-\delta}\Big)\Big)\Big)ds\\
&= \frac{1}{2\pi i}\int_\delta^{\delta+(1/2-\delta)t} \tau\Big(\frac{d}{ds}\Big(V\Big(t-\Big(\frac{s-\delta}{1/2-\delta}\Big)\Big)\Big)V^*\Big(t-\Big(\frac{s-\delta}{1/2-\delta}\Big)\Big)\Big)ds = 0.
\end{aligned}$$

If $t \in [1-\delta, 1]$, then, applying (e 3.13) again,

$$\begin{aligned}
&\Big|\int_\delta^{\delta+(1/2-\delta)t} \tau\Big(\frac{dZ_\varepsilon(t,s)}{ds}Z_\varepsilon(t,s)^*\Big)ds\Big|\\
&= \Big|\Big(\int_\delta^{\delta+(1/2-\delta)(t-1+\delta)} + \int_{\delta+(1/2-\delta)(t-1+\delta)}^{\delta+(1/2-\delta)t}\Big)\tau\Big(\frac{dZ_\varepsilon(t,s)}{ds}Z_\varepsilon(t,s)^*\Big)ds\Big|\\
&= \Big|\int_\delta^{\delta+(1/2-\delta)(t-1+\delta)} \tau\Big(\frac{i}{(1/2-\delta)\delta}H\Big)ds\Big| + 0\\
&= \Big|\frac{t-1+\delta}{\delta}\tau(H)\Big| < \varepsilon/16 \quad \text{for all } \tau \in T(A\otimes Q).
\end{aligned} \tag{e 3.14}$$

Moreover,

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_0^{1/2}\tau\left(\frac{dZ_\varepsilon(1,s)}{ds}Z_\varepsilon(1,s)^*\right)ds\right| \\
&\quad=\left|\frac{1}{2\pi i}\left(\int_0^\delta+\int_\delta^{\delta+\delta(1/2-\delta)}+\int_{\delta+\delta(1/2-\delta)}^{1/2}\right)\tau\left(\frac{dZ_\varepsilon(1,s)}{ds}Z_\varepsilon(1,s)^*\right)ds\right| \\
&\quad\le\frac{1}{2\pi}\left(\left|\tau\big(H_0(1)\big)\right|+\left|\tau(H)\right|+0\right)<\varepsilon/16\pi \quad \text{for all } \tau\in T(A).
\end{aligned} \tag{e 3.15}$$

One then computes that, for any $\tau\in T(A)$ and $t\in[0,1)$, by applying (e 3.14),

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^1\tau\left(\frac{dZ_\varepsilon(t,s)}{ds}Z_\varepsilon(t,s)^*\right)ds \\
&\quad=\frac{1}{2\pi i}\left(\int_0^\delta+\int_\delta^{\delta+(1/2-\delta)t}+\int_{\delta+(1/2-\delta)t}^{1/2}+\int_{1/2}^1\right)\tau\left(\frac{dZ_\varepsilon(t,s)}{ds}Z_\varepsilon(t,s)^*\right)ds \\
&\quad\approx_{\varepsilon/32\pi}\left(\frac{1}{2\pi i}\right)\left(\tau\big(iH_0(t)\big)+\int_{\delta+(1/2-\delta)t}^{1/2}\tau\left(\frac{dZ_\varepsilon(t,s)}{ds}Z_\varepsilon(t,s)^*\right)ds\right. \\
&\qquad\left.+\int_{1/2}^1\tau\left(\frac{dz(2s-1)}{ds}z(2s-1)^*\right)ds\right) \\
&\quad=\frac{1}{2\pi}\tau\big(H_0(t)\big)+0+\frac{1}{2\pi i}\int_{1/2}^1\tau\left(\frac{dz(2s-1)}{ds}z(2s-1)^*\right)ds \\
&\quad=\frac{1}{2\pi}\tau\big(H_0(t)\big)+\frac{1}{2\pi i}\int_0^1\tau\left(\frac{dz(s)}{ds}z(s)^*\right)ds\approx_{\varepsilon/32\pi}h(\tau)+f_{\mathfrak{p}}(\tau).
\end{aligned}$$

It then follows from (e 3.15) and (e 3.12) that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^1\tau\left(\frac{dZ_\varepsilon(1,s)}{ds}Z_\varepsilon(1,s)^*\right)ds \\
&\quad=\frac{1}{2\pi i}\left[\int_0^{1/2}\tau\left(\frac{dZ_\varepsilon(1,s)}{ds}Z_\varepsilon(1,s)^*ds\right)+\int_{1/2}^1\tau\left(\frac{dz(2s-1)}{ds}z(2s-1)^*\right)ds\right] \\
&\quad\approx_{\varepsilon/16\pi}\frac{1}{2\pi i}\int_0^1\tau\left(\frac{dz(s)}{ds}z(s)^*\right)ds=h(\tau)+f_{\mathfrak{p}}(\tau).
\end{aligned}$$

Note that if $Z_2(t,s)$ is any continuous and piecewise smooth path of unitaries in $C([0,1],A\otimes Q)$ with $Z_2(t,0)=Z_1(t,0)$ and $Z_2(t,1)=Z_1(t,1)=1$ as well as

$$\|Z_2-Z_1\|<\varepsilon,$$

then $Z_2Z_\varepsilon^*$ is a trivial loop and $\mathrm{Det}(Z_2)(\tau\otimes\delta_t)=\mathrm{Det}(Z_\varepsilon)(\tau\otimes\delta_t)$.

It follows that

$$\frac{1}{2\pi i}\int_0^t\tau\left(\frac{dZ_2(t,s)}{ds}Z_2(t,s)^*\right)ds=h(\tau)+f_{\mathfrak{p}}(\tau)\quad\text{for all } \tau\in T(A\otimes Q).$$

Thus, there is an element $g \in \rho_{A\otimes Q}(K_0(A\otimes Q)) \subset \mathrm{Aff}(T(A\otimes Q))$, such that

$$g(\tau\otimes\delta_t) = \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{dZ(t,s)}{ds}Z(t,s)^*\Big)ds - \frac{1}{2\pi i}\int_0^1 \tau\Big(\frac{dZ_2(t,s)}{ds}Z_2(t,s)^*\Big)ds$$

for all $\tau \in T(A)$ and $t \in [0,1]$. Thus, for any $t \in [0,1]$,

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\Big(\frac{dZ(t,s)}{ds}Z(t,s)^*\Big)ds = h(\tau) + f_{\mathfrak{p}}(\tau) + g(\tau\otimes\delta_t).$$

Put $f(\tau\otimes\delta_t) := f_{\mathfrak{p}}(\tau) + g(\tau\otimes\delta_t) \in \rho_{C([0,1],A\otimes Q)}(K_0(C([0,1],A\otimes Q)))$. Then, for fixed $\tau \in T(A\otimes Q)$, f is constant on $[0,1]$. By (e 3.11),

$$f(\tau\otimes\delta_i) = f_i(\tau) \quad \text{for all } \tau \in T(A\otimes Q),\ i = 0,1.$$

Recall $f_0 \in \rho_{A\otimes M_{\mathfrak{q}}}(K_0(A\otimes M_{\mathfrak{q}}))$ and $f_1 \in \rho_{A\otimes M_{\mathfrak{q}}}(K_0(A\otimes M_{\mathfrak{q}}))$. It follows that

$$f \in \rho_{A\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}\big(K_0(A\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})\big).$$

Then the lemma follows. ∎

4. Approximate unitary equivalence

Lemma 4.1 (cf. [18, Lemma 5.1]). *Let C_0 and A_1 be unital separable simple C^*-algebras, and let $C = C_0\otimes U_1$ and $A = A_1\otimes U_2$, where U_1 and U_2 are UHF-algebras of infinite type. Suppose that C satisfies the UCT and $gTR(A_1) \le 1$. Suppose further that $C = \lim_{n\to\infty}(C_n, \iota_n)$ as described in* [6, Theorem 14.10]. *If there are monomorphisms $\varphi, \psi : C \to A$ such that*

$$[\varphi] = [\psi] \text{ in } KL(C,A), \quad \varphi_\sharp = \psi_\sharp, \quad \varphi^\ddagger = \psi^\ddagger,$$

then, for any $2>\varepsilon>0$, any finite subset $\mathcal{F}\subseteq C$, and any finite subset of unitaries $\mathcal{P}\subset U(C)$, there exists a finite subset $\mathcal{G}\subset K_1(C)$ with $\overline{\mathcal{P}}\subseteq\mathcal{G}$ (where $\overline{\mathcal{P}}$ is the image of $\mathcal{P}$ in $K_1(C)$) and $\delta>0$ such that, for any map $\eta : G(\mathcal{G}) \to \mathrm{Aff}(\mathrm{T}(A))$ (where $G(\mathcal{G})$ is the subgroup generated by $\mathcal{G}$) with $|\eta(x)(\tau)| < \delta$ for all $\tau \in \mathrm{T}(A)$ and $\eta(x) - \overline{R}_{\varphi,\psi}(x) \in \rho_A(K_0(A))$ for all $x \in \mathcal{G}$, there is a unitary $u \in A$ such that

$$\big\|\varphi(f) - u^*\psi(f)u\big\| < \varepsilon \quad \text{for all } f \in \mathcal{F},$$

and

$$\tau\Big(\frac{1}{2\pi i}\log\big((\varphi\otimes \mathrm{id}_{M_n}(x^*))(u\otimes 1_{M_n})^*(\psi\otimes\mathrm{id}_{M_n}(x))(u\otimes 1_{M_n})\big)\Big) = \tau\big(\eta([x])\big)$$

for all $x \in \mathcal{P}$ and $\tau \in \mathrm{T}(A)$.

Proof. Without loss of generality, one may assume that any element in $\mathcal{F}$ has norm at most one. Let $\varepsilon > 0$. Choose θ with $\varepsilon > \theta > 0$ and a finite subset $\mathcal{F} \subset \mathcal{F}_0 \subset C$ satisfying the following: For all $x \in \mathcal{P}$, $\tau(\frac{1}{2\pi i}\log(\psi(x^*)w^*\psi(x)w))$ is well defined and

$$\tau\left(\frac{1}{2\pi i}\log\left(\psi(x^*)w^*\psi(x)w\right)\right) = \tau\big(\mathrm{bott}_1\left(w, \psi(x)\right)\big) \quad \text{for all } \tau \in T(B),$$

whenever $w \in U(B)$ and

$$\left\|w\psi(f) - \psi(f)w\right\| < \theta \quad \text{for all } f \in \mathcal{F}_0$$

(see the Exel formula in [9]), and for any unitaries z_1, z_2 in any unital C^*-algebra D, which satisfy

$$\|z_1 - 1\| < \theta, \quad \|z_2 - 1\| < \theta,$$

then

$$\tau\left(\frac{1}{2\pi i}\log(z_1 z_2)\right) = \tau\left(\frac{1}{2\pi i}\log(z_1)\right) + \tau\left(\frac{1}{2\pi i}\log(z_2)\right) \quad \text{for all } \tau \in T(D)$$

(see [12, Lemma 6.1]).

Let $n_0 \geq 1$ (in place of n), $\delta' > 0$ (in place of δ), and $\mathcal{G}' \subseteq K_1(C_{n_0})$ (in place of $\mathcal{Q}$) the constant and the finite subset with respect to C (in place of A), $\mathcal{F}_0$ (in place of $\mathcal{F}$), $\mathcal{P}$ (in place of $\mathcal{P}$), ψ (in place of φ), and $k = 1$, $p_1 = 1$, $q_1 = 0$, and $\sigma = 1$, required by Lemma 3.3. Put $\delta = \delta'/2$.

Fix a decomposition $(\iota_{n_0,\infty})_{*1}(K_1(C_{n_0})) = \mathbb{Z}^k \oplus \mathrm{Tor}((\iota_{n_0,\infty})_{*1}(C_{n_0}))$ (for some integer $k \geq 0$). Let $\mathcal{G}'' \subset U(C)$ (recall that, by [6, Theorem 9.7], C has stable rank one) be a finite subset containing a representative for each generator of $\mathbb{Z}^k$. Without loss of generality, one may assume that $\mathcal{P} \subseteq \mathcal{G}''$. By [6, Theorem 12.11 (a)], the maps φ and ψ are approximately unitary equivalent. Hence, for any finite subset $\mathcal{Q}$ and any δ_1, there is a unitary $v \in A$ such that

$$\left\|\varphi(f) - v^*\psi(f)v\right\| < \delta_1, \quad \forall f \in \mathcal{Q}.$$

By choosing $\mathcal{Q} \supseteq \mathcal{F}_0$ sufficiently large and $\delta_1 < \theta/2$ sufficiently small, the map

$$[x] \mapsto \tau\left(\frac{1}{2\pi i}\log\left(\varphi^*(x)v^*\psi(x)v\right)\right), \quad x \in \mathcal{G}'',$$

induces a homomorphism

$$\eta_1 : (\iota_{n_0,\infty})_{*1}\big(K_1(C_{n_0})\big) \to \mathrm{Aff}\left(\mathrm{T}(A)\right)$$

(note that $\eta_1(\mathrm{Tor}((\iota_{n_0,\infty})_{*1}(K_1(C_{n_0})))) = \{0\}$), and $\|\eta_1(x)\| < \delta$ for all $x \in \mathcal{G}$.

By [18, Lemma 3.8], the image of $\eta_1 - \overline{R}_{\varphi,\psi}$ is in $\rho(K_0(A))$. Since $\eta(x) - \overline{R}_{\varphi,\psi}(x) \in \rho_A(K_0(A))$ for all $x \in \mathcal{G}$, the image $(\eta - \eta_1)((\iota_{n_0,\infty})_{*1}(K_1(C')))$ is also in $\rho_A(K_0(A))$. Since $\eta - \eta_1$ factors through $\mathbb{Z}^k$, there is a homomorphism

$$h : (\iota_{n_0,\infty})_{*1}\big(K_1(C_{n_0})\big) \to K_0(A)$$

(which maps $\mathrm{Tor}((\iota_{n_0,\infty})_{*1}(K_1(C_{n_0})))$ to zero) such that $\eta - \eta_1 = \rho_A \circ h$. Note that $|\tau(h(x))| < 2\delta = \delta'$ for all $\tau \in \mathrm{T}(A)$ and $x \in \mathcal{G}$.

By the universal multi-coefficient theorem (see [2]), there is

$$\kappa \in \mathrm{Hom}_\Lambda\left(\underline{K}\big(C_{n_0} \otimes \mathrm{C}(\mathbb{T})\big), \underline{K}(A)\right) \quad \text{such that } \kappa \circ \boldsymbol{\beta}|_{K_1(C_{n_0})} = h \circ (\iota)_{*1}.$$

Applying Lemma 3.3, there is a unitary w such that

$$\big\|[w, \psi(f)]\big\| < \theta/2, \quad \forall f \in \mathcal{F}_0,$$

and $\mathrm{Bott}(w, \psi \circ \iota) = \kappa$. In particular, $\mathrm{bott}_1(w, \psi)(x) = h(x)$ for all $x \in \mathcal{P}$.

Set $u = wv$. One then has

$$\big\|\varphi(f) - u^*\psi(f)u\big\| < \theta, \quad \forall f \in \mathcal{F}_0,$$

and for any $x \in \mathcal{P}$ and any $\tau \in \mathrm{T}(A)$,

$$\begin{aligned}
&\tau\left(\frac{1}{2\pi i}\log\big(\varphi(x^*)u^*\psi(x)u\big)\right)\\
&\quad = \tau\left(\frac{1}{2\pi i}\log\big(\varphi(x)v^*w^*\psi(z)wv\big)\right)\\
&\quad = \tau\left(\frac{1}{2\pi i}\log\big(\varphi(x^*)v^*\psi(x)vv^*\psi(x^*)w^*\psi(x)wv\big)\right)\\
&\quad = \tau\left(\frac{1}{2\pi i}\log\big(\varphi(x^*)v^*\psi(x)v\big)\right) + \tau\left(\frac{1}{2\pi i}\log\big(\psi(x^*)w^*\psi(x)w\big)\right)\\
&\quad = \eta_1\big([x]\big)(\tau) + h\big([x]\big)(\tau) = \eta\big([x]\big)(\tau).
\end{aligned}$$

■

The proof of the following lemma is long and is taken from the proof of [18, Lemma 5.6]. The only modification has been outlined in [18, Remark 5.7]. Since the statements in Section 3 are slightly different from what were used in the proof of [18, Lemma 5.6], we provide a full proof for the convenience of the reader.

Lemma 4.2 (cf. [18, Lemma 5.6]). *Let A be a unital finite separable simple amenable C^*-algebra which satisfies the UCT, and let B be a separable simple C^*-algebra. Suppose that $gTR(A \otimes Q) \le 1$ and $gTR(B \otimes Q) \le 1$.*

Suppose that there are two unital monomorphisms $\varphi, \psi : A \to B$ with

$$[\varphi] = [\psi] \text{ in } KL(A, B), \quad \varphi_\sharp = \psi_\sharp, \quad \varphi^\ddagger = \psi^\ddagger.$$

Let $\mathfrak{p}$ and $\mathfrak{q}$ be a pair of relatively prime supernatural numbers of infinite type with $M_\mathfrak{p} \otimes M_\mathfrak{q} = Q$. Then, for any finite subset $\mathcal{F} \subseteq A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, there exists a unitary $u \in B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that

$$\big\|(\varphi \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})(x) - u^*\big((\psi \otimes 1_{\mathcal{Z}_{p,q}})(x)\big)u\big\| < \varepsilon \quad \text{for all } x \in \mathcal{F}.$$

Proof. Let $\mathfrak{r}$ be a supernatural number. Denote by $\iota_{\mathfrak{r}} : A \to A \otimes M_{\mathfrak{r}}$ the embedding defined by $\iota_{\mathfrak{r}}(a) = a \otimes 1$ for all $a \in A$. Denote by $j_{\mathfrak{r}} : B \to B \otimes M_{\mathfrak{r}}$ the embedding defined by $j_{\mathfrak{r}}(b) = b \otimes 1$ for all $b \in B$. Without loss of generality, one may assume that

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 := \{x \otimes y : x \in \mathcal{F}_1,\ y \in \mathcal{F}_2\},$$

where $\mathcal{F}_1 \subseteq A$ and $\mathcal{F}_2 \subseteq \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ are finite subsets and $1_A \in \mathcal{F}_1$ and $1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}} \in \mathcal{F}_2$. Moreover, one may assume that any element in $\mathcal{F}_1$ or $\mathcal{F}_2$ has norm at most one.

We will also write $\mathbb{D}_{\mathfrak{r}} = K_0(M_{\mathfrak{r}})$ which is identified with a dense subgroup of $\mathbb{Q}$.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition of $[0, 1]$ such that

$$\left\|b(t) - b(t_i)\right\| < \varepsilon/4, \quad \forall b \in \mathcal{F}_2,\ \forall t \in [t_{i-1}, t_i],\ i = 1, \ldots, m. \tag{e 4.1}$$

Consider

$$\begin{aligned} \mathcal{E} &= \left\{a \otimes b(t_i);\ a \in \mathcal{F}_1,\ b \in \mathcal{F}_2,\ i = 0, \ldots, m\right\} \subseteq A \otimes Q, \\ \mathcal{E}_{\mathfrak{p}} &= \left\{a \otimes b(t_0);\ a \in \mathcal{F}_1,\ b \in \mathcal{F}_2\right\} \subseteq A \otimes M_{\mathfrak{p}} \subset A \otimes Q, \\ \mathcal{E}_{\mathfrak{q}} &= \left\{a \otimes b(t_m);\ a \in \mathcal{F}_1,\ b \in \mathcal{F}_2\right\} \subseteq A \otimes M_{\mathfrak{q}} \subset A \otimes Q. \end{aligned}$$

By [20], $gTR(A \otimes M_{\mathfrak{r}}) \le 1$ for any (infinite) supernatural number $\mathfrak{r}$. By [7, Theorem 21.9], we may write $A \otimes Q = \lim_{n\to\infty}(C_n, J_n)$ as described in [6, Theorem 14.10]. In particular, each C_n is isomorphic to a direct sum of a homogeneous C^*-algebra in **H** and an Elliott–Thomsen algebra with trivial K_1-group, and J_n is unital and injective.

Let $\mathcal{H} \subset A \otimes Q$ (in place of $\mathcal{G}$), $\mathcal{P} \subseteq \underline{K}(A \otimes Q)$, and $\mathcal{Q} = \{x_1, x_2, \ldots, x_m\} \subset K_0(A \otimes Q)$ which generates a free abelian subgroup of $K_0(A \otimes Q)$, where we may assume that $x_i = [p_i] - [q_i]$ and $p_i, q_i \in A \otimes Q$ are projections, and $\delta > 0$ and $\gamma > 0$ are the constants of Lemma 3.1 and Remark 3.2 (so condition (e 3.4) is not needed in Lemma 3.1) with respect to $\mathcal{E}$ (in place of $\mathcal{F}$) and $\varepsilon/8$ (in place of ε). We may assume $\mathcal{Q} \subset \mathcal{P}$ and $\delta < \varepsilon/4$.

Let $G^o_{u,\infty} \subseteq K_0(A \otimes Q)$ be the subgroup generated by $\mathcal{Q}$.

Note that we may assume that $\mathcal{P} \subset [J_{n_0,\infty}](\underline{K}(C_{n_0}))$ for some n_0 and

$$\mathcal{E}, \mathcal{E}_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{q}} \subseteq \mathcal{H}.$$

Denote by ∞ the supernatural number associated with $\mathbb{Q}$. Let $\mathcal{P}_i = \mathcal{P} \cap K_i(A \otimes Q)$, $i = 0, 1$. There is a finitely generated free subgroup $G(\mathcal{P})_{i,0} \subset K_i(A)$ such that if one sets

$$G(\mathcal{P})_{i,\infty,0} = G\big(\{gr : g \in (\iota_\infty)_{*i}\big(G(\mathcal{P})_{i,0}\big),\ r \in D_0\}\big),$$

where $D_0 \subset \mathbb{Q}$ ($1 \in D_0$) is a finite subset, then $G(\mathcal{P})_{i,\infty,0}$ contains the subgroup generated by $\mathcal{P}_i$, $i = 0, 1$. Moreover, we may assume that if $r = k/m$, where k and m are non-zero integers, and $r \in D_0$, then $1/m \in D_0$. Let $\mathcal{P}'_i \subset K_i(A)$ be a finite subset which generates $G(\mathcal{P})_{i,0}$, $i = 0, 1$. In addition, denote $\mathcal{P}' = \mathcal{P}'_0 \cup \mathcal{P}'_1$.

Write the subgroup generated by the image of $\mathcal{Q}$ in $K_0(A \otimes Q)$ as $\mathbb{Z}^k$ (for some integer $k \ge 1$). Choose $\{x'_1, \ldots, x'_k\} \subseteq K_0(A)$ and $\{r_{ij};\ 1 \le i \le m,\ 1 \le j \le k\} \subseteq \mathbb{Q}$

such that

$$x_i = \sum_{j=1}^{k} r_{ij}(j_\infty)_{*0}(x_j'), \quad 1 \le i \le m,\ 1 \le j \le k,$$

and $\{x_1', \dots, x_k'\}$ generates a free abelian subgroup G_u^o of $K_0(A)$ of rank k. Choose projections $p_j', q_j' \in M_N(A)$ (for some integer $N \ge 1$) such that $x_j' = [p_j'] - [q_j']$, $1 \le j \le k$. Choose an integer M such that Mr_{ij} are integers for $1 \le i \le m$ and $1 \le j \le k$. In particular, Mx_i is the linear combination of $(j_\infty)_{*0}(x_j')$ with integer coefficients.

Let $\bar{p}_i$ be an orthogonal direct sum of $Mr_{i,j}$ copies of $j_\infty(p_j')$ in $M_{N_1}(A \otimes Q)$ for some integer $N_1 \ge 1$. One can find M mutually orthogonal and mutually equivalent projections $e_{1,i}, \dots, e_{M,i}$ such that $\sum_{l=1}^{M} e_{1,i} = \bar{p}_i$. Since $p_i \in A \otimes Q$, by replacing p_j' by a unitarily equivalent projection, we may assume that $e_{1,i} = p_i$. In other words, we make the arrangement so that $\bar{p}_i$ is the direct sum of M copies of p_i.

Note also that the subgroup of $K_0(A \otimes Q)$ generated by

$$\left\{(\imath_\infty)_{*0}(x_1'), \dots, (\imath_\infty)_{*0}(x_k')\right\}$$

is isomorphic to $\mathbb{Z}^k$ and the subgroup of $K_0(A \otimes M_{\mathfrak{r}})$ generated by

$$\left\{(\imath_{\mathfrak{r}})_{*0}(x_1'), \dots, (\imath_{\mathfrak{r}})_{*0}(x_k')\right\}$$

has to be isomorphic to $\mathbb{Z}^k$, where $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$. We assume that $x_j' \in \mathcal{P}_0'$, $j = 1, 2, \dots, k$.

Since $gTR(A \otimes M_{\mathfrak{r}}) \le 1$, by [6, Theorem 21.9], one may write

$$A \otimes M_{\mathfrak{r}} = \lim_{n\to\infty} (C_n^{\mathfrak{r}}, J_n^{\mathfrak{r}})$$

as described in [6, Theorem 14.10]. In particular, each $J_n^{\mathfrak{r}} : C_n^{\mathfrak{r}} \to C_{n+1}^{\mathfrak{r}}$ is a unital embedding. We may assume that, for sufficiently large $n_{\mathfrak{r}}'$, $\mathcal{E}_{\mathfrak{r}} \subseteq J_{n_{\mathfrak{r}}',\infty}^{\mathfrak{r}}(C_{n_{\mathfrak{r}}'}^{\mathfrak{r}})$ and there are projections

$$\{p_{1,\mathfrak{r}}'', \dots, p_{k,\mathfrak{r}}'', q_{1,\mathfrak{r}}'', \dots, q_{k,\mathfrak{r}}''\} \subseteq M_N(C_{n_{\mathfrak{r}}'}^{\mathfrak{r}})$$

such that for any $1 \le j \le k$, with $p_{j,\mathfrak{r}}' = J_{n_{\mathfrak{r}}',\infty}^{\mathfrak{r}}(p_{j,\mathfrak{r}}'')$ and $q_{j,\mathfrak{r}}' = J_{n_{\mathfrak{r}}',\infty}^{\mathfrak{r}}(q_{j,\mathfrak{r}}'')$,

$$\|p_j' \otimes 1_{M_{\mathfrak{r}}} - p_{j,\mathfrak{r}}'\| < \gamma/2N\Big(64\Big(1 + \sum_{i,j'} |Mr_{ij'}|\Big)\Big) < 1, \tag{e 4.2}$$

$$\|q_j' \otimes 1_{M_{\mathfrak{r}}} - q_{j,\mathfrak{r}}'\| < \gamma/2N\Big(64\Big(1 + \sum_{i,j'} |Mr_{ij'}|\Big)\Big) < 1, \tag{e 4.3}$$

and $\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$.

Denote $x_{j,\mathfrak{r}}' = [p_{j,\mathfrak{r}}'] - [q_{j,\mathfrak{r}}']$ and $x_{j,\mathfrak{r}}'' = [p_{j,\mathfrak{r}}''] - [q_{j,\mathfrak{r}}'']$, $1 \le j \le k$, denote by $G_{\mathfrak{r}}$ the subgroup of $K_0(C_{n_{\mathfrak{r}}'}^{\mathfrak{r}})$ generated by $\{x_{1,\mathfrak{r}}'', \dots, x_{k,\mathfrak{r}}''\}$, and write $G_{\mathfrak{r}} = \mathbb{Z}^r \oplus \mathrm{Tor}(G_{\mathfrak{r}})$. Since $G_{\mathfrak{r}}$ is generated by k elements, one has that $r \le k$ and $r = k$ if and only if $G_{\mathfrak{r}}$ is torsion-free. Note that the image of $G_{\mathfrak{r}}$ in $K_0(A \otimes M_{\mathfrak{r}})$ is the group generated by $\{[p_1' \otimes 1_{M_{\mathfrak{r}}}] - [q_1' \otimes 1_{M_{\mathfrak{r}}}], \dots, [p_k' \otimes 1_{M_{\mathfrak{r}}}] - [q_k' \otimes 1_{M_{\mathfrak{r}}}]\}$, which is isomorphic to $\mathbb{Z}^k$ $\{[p_j' \otimes 1_{M_{\mathfrak{r}}}] - [q_j' \otimes 1_{M_{\mathfrak{r}}}];\ 1 \le j \le k\}$ as the standard generators. Hence $G_{\mathfrak{r}}$ is torsion-free and $r = k$.

Without loss of generality, one may assume that $\imath_{\mathfrak{r}}(\mathcal{P}') \subseteq [J^{\mathfrak{r}}_{n'_{\mathfrak{r}},\infty}](\underline{K}(C^{\mathfrak{r}}_{n'_{\mathfrak{r}}}))$.

Assume that $\mathcal{H}$ is sufficiently large and δ is sufficiently small such that for any homomorphism h from $A \otimes Q$ to $B \otimes Q$ and any unitary z_j $(j = 1, 2, 3, 4)$, if $\|[h(x), z_j]\| < \delta$ for any $x \in \mathcal{H}$, then the maps $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup generated by $\mathcal{P}$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup generated by $\mathcal{P}$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$.

By choosing larger $\mathcal{H}$ and smaller δ, one may also assume that

$$\big\|\big[h(p_i), z_j\big]\big\| < 1/16, \quad \big\|\big[h(q_i), z_j\big]\big\| < 1/16, \quad 1 \le i \le m,\ j = 1, 2, 3, 4, \qquad \text{(e 4.4)}$$

and, for any $1 \le i \le m$,

$$\mathrm{dist}\left(\widetilde{\zeta}^{M}_{i,z_1}, \prod_{j=1}^{k} (\zeta'_{j,z_1})^{Mr_{ij}}\right) < \gamma/64N, \qquad \text{(e 4.5)}$$

where (with $\mathbf{1} := 1_{A\otimes Q}$)

$$\zeta_{i,z_1} = \overline{\big\langle\big(\mathbf{1} - h(p_i) + h(p_i)z_1\big)\big(\mathbf{1} - h(q_i) + h(q_i)z_1^*\big)\big\rangle},$$
$$\widetilde{\zeta}_{i,z_1} = \overline{\big\langle\big(\mathbf{1} - h(p_i) + h(p_i)z_1\big)\big(\mathbf{1} - h(q_i) + h(q_i)z_1^*\big) \oplus \mathbf{1}_{N-1}\big\rangle},$$

and (with $\mathbf{1} := 1_{(A\otimes Q)}$)

$$\zeta'_{j,z_1} =$$
$$\overline{\big\langle\big(\mathbf{1}_N - h(p'_j \otimes 1_{A\otimes Q}) + h(p'_j \otimes 1_{A\otimes Q}) z_1^{(N)}\big)\big(\mathbf{1}_N - h(q'_j \otimes 1_{A\otimes Q}) + h(q'_j \otimes 1_{A\otimes Q}) z_1^{(N)^*}\big)\big\rangle},$$

where $z_1^{(N)} = z_1 \otimes 1_N$.

By choosing even smaller δ, without loss of generality, we may assume that

$$\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{H}^{\mathfrak{p}} \otimes \mathcal{H}^{\mathfrak{q}},$$

where $\mathcal{H}^0 \subset A$, $\mathcal{H}^{\mathfrak{p}} \subset M_{\mathfrak{p}}$, and $\mathcal{H}^{\mathfrak{q}} \subset M_{\mathfrak{q}}$ are finite subsets, and $1 \in \mathcal{H}^0$, $1 \in \mathcal{H}^{\mathfrak{p}}$, and $1 \in \mathcal{H}^{\mathfrak{q}}$.

Moreover, choose $\mathcal{H}^0$, $\mathcal{H}^{\mathfrak{p}}$, and $\mathcal{H}^{\mathfrak{q}}$ even larger and δ even smaller so that for any homomorphism $h_{\mathfrak{r}} : A \otimes M_{\mathfrak{r}} \to B \otimes M_{\mathfrak{r}}$ and unitaries $z_1, z_2 \in B \otimes M_{\mathfrak{r}}$ with $\|h_{\mathfrak{r}}(x), z_i\| < \delta$ for any $x \in \mathcal{H}_0 \otimes \mathcal{H}_{\mathfrak{r}}$, one has

$$\big\|\big[h_{\mathfrak{r}}(p'_{i,\mathfrak{r}}), z_j\big]\big\| < 1/16, \quad \big\|\big[h_{\mathfrak{r}}(q'_{i,\mathfrak{r}}), z_j\big]\big\| < 1/16, \quad 1 \le i \le k,\ j = 1, 2,$$

and

$$\mathrm{dist}\left(\zeta_{i,z_1z_2}, \overline{(1_{B\otimes M_{\mathfrak{r}}})_N}\right) < \mathrm{dist}(\zeta_{i,z_1^*}, \zeta_{i,z_2}) + \gamma/\Big(64N\Big(1 + \sum_{i',j} |Mr_{i'j}|\Big)\Big),$$

where

$$\zeta_{i,z'} = \overline{\langle (\mathbf{1}_N - h_{\mathfrak{r}}(p'_{i,\mathfrak{r}}) + h_{\mathfrak{r}}(p'_{i,\mathfrak{r}})z')(\mathbf{1}_N - h_{\mathfrak{r}}(q'_{i,\mathfrak{r}}) + h_{\mathfrak{r}}(q'_{i,\mathfrak{r}})(z')^*)\rangle},$$

and

$$z' = z_1 z_2 \otimes 1_N, z_1^* \otimes 1_N, z_2 \otimes 1_N.$$

Let us apply Lemma 3.3 to the sets $A \otimes Q$ and $B \otimes Q$ (playing the role of A and B, respectively), the homomorphism $\varphi \otimes \mathrm{id}_Q$ (playing the role of φ), the positive constant $\delta/4$ (playing the role of ε), the subsets $\mathcal{H} \subset A \otimes Q$ and $\mathcal{P} \subset \underline{K}(A \otimes Q)$ (playing the role of $\mathcal{F}$ and $\mathcal{P}$, respectively), and the free abelian subgroup $[\imath_\infty](G_u^o)$ of $K_0(A \otimes Q)$ (playing the role of G in Lemma 3.3). Then Lemma 3.3 tells us the existence of an integer n (assumed to satisfy $n \geq n_0$), a constant δ_2 (replacing δ in Lemma 3.3) and a finite subset $\mathcal{G} \subset K_1(C_n)$ (replacing $\mathcal{Q}$) with the stated properties. Without loss of generality, we may write that $n = n_0$.

Let $\mathcal{H}' \subseteq A \otimes Q$ be a finite subset, and assume that δ_2 is small enough such that for any homomorphism h from $A \otimes Q$ to $B \otimes Q$ and any unitary z_j $(j = 1, 2, 3, 4)$, the maps $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup $[J_{n_0,\infty}](\underline{K}(C_{n_0}))$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup $[J_{n_0,\infty}](\underline{K}(C_{n_0}))$ if $\|[h(x), z_j]\| < \delta_2$ for any $x \in \mathcal{H}'$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$. Furthermore, as above, one may assume, without loss of generality, that

$$\mathcal{H}' = \mathcal{H}^{0'} \otimes \mathcal{H}^{\mathfrak{p}'} \otimes \mathcal{H}^{\mathfrak{q}'},$$

where $\mathcal{H}^0 \subseteq \mathcal{H}^{0'} \subset A$, $\mathcal{H}^{\mathfrak{p}} \subseteq \mathcal{H}^{\mathfrak{p}'} \in M_{\mathfrak{q}}$ and $\mathcal{H}^{\mathfrak{q}} \subseteq \mathcal{H}^{\mathfrak{q}'} \subset M_{\mathfrak{q}}$ are finite subsets.

Let $\delta_2' > 0$ be a constant such that for any unitary with $\|u - 1\| < \delta_2'$, one has that $\|\log u\| < \delta_2/4$. Without loss of generality, one may assume that $\delta_2' < \delta_2/16 < \varepsilon/16$ and $\delta_2' < \delta$.

Let $n_{\mathfrak{r}} \in \mathbb{N}$ (in place of n) be the integer, $\mathcal{R}_{\mathfrak{r}} \subset K_1(C_{n_{\mathfrak{r}}}^{\mathfrak{r}})$ (in place of $\mathcal{Q}$) and $\delta_{\mathfrak{r}}$ (in place of δ) the finite subset and constant required by Lemma 3.3 with respect to $A \otimes M_{\mathfrak{r}}$ (in place of A), $B \otimes M_{\mathfrak{r}}$ (in place of B), $\varphi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$ (in place of h), $\mathcal{H}^{0'} \otimes \mathcal{H}^{\mathfrak{r}'}$ (in place of $\mathcal{F}$) and $(\imath_{\mathfrak{r}})_{*0}(\mathcal{P}_0') \cup (\imath_{\mathfrak{r}})_{*1}(\mathcal{P}_1')$ (in place of $\mathcal{P}$) and $\delta_2'/8$ (in place of ε), $[\imath_{\mathfrak{r}}](G_u^o)$ (in place of G), and $p'_{j,\mathfrak{r}}, q'_{j,\mathfrak{r}}$ (in place of p_j, q_j—see also Remark 3.4), $\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$. Let $\mathcal{R}_{\mathfrak{r}}^{(i)} = (\imath_{\mathfrak{r}})_{*i}(J_{n_{\mathfrak{r}},\infty}(K_i(C_{n_{\mathfrak{r}}}^{\mathfrak{r}})))$, $i = 0, 1$. There is a finitely generated subgroup $G_{i,0,\mathfrak{r}} \subset K_i(A)$ and a finitely generated subgroup $D_{0,\mathfrak{r}} \subseteq \mathbb{D}_{\mathfrak{r}}$ so that

$$G'_{i,0,\mathfrak{r}} := G(\{gr : g \in (\imath_{\mathfrak{r}})_{*i}(G_{i,0,\mathfrak{r}}),\ r \in D_{0,\mathfrak{r}}\})$$

contains the subgroup $\mathcal{R}_{\mathfrak{r}}^{(i)}$, $i = 0, 1$. Without loss of generality, one may assume that $D_{0,\mathfrak{p}} = \{\frac{k}{m_{\mathfrak{p}}};\ k \in \mathbb{Z}\}$ and $D_{0,\mathfrak{q}} = \{\frac{k}{m_{\mathfrak{q}}};\ k \in \mathbb{Z}\}$ for an integer $m_{\mathfrak{p}}$ divides $\mathfrak{p}$ and an integer $m_{\mathfrak{q}}$ divides $\mathfrak{q}$, and $n_{\mathfrak{r}}' = n_{\mathfrak{r}}$. It follows that

$$[\imath_{\mathfrak{r}}](x_j') \subset [\imath_{\mathfrak{r}}](\mathcal{P}_i) \subset \mathcal{R}_{\mathfrak{r}}^{(i)}, \quad j = 1, 2, \ldots, k. \tag{e 4.6}$$

In what follows, we also use φ_Q and ψ_Q for $\varphi \otimes \mathrm{id}_Q$ and $\psi \otimes \mathrm{id}_Q$, respectively. Moreover, if $\mathfrak{r}$ is a supernatural number, we also use $\varphi_{\mathfrak{r}}$ and $\psi_{\mathfrak{r}}$ for $\varphi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$ and $\psi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$, respectively. Let $\mathcal{R} \subset \underline{K}(A \otimes Q)$ be a finite subset which generates a subgroup containing

$$\frac{1}{m_{\mathfrak{p}} m_{\mathfrak{q}}}\big((\iota_{\mathfrak{p},\infty})_*(G'_{0,0,\mathfrak{p}} \cup G'_{1,0,\mathfrak{p}}) \cup (\iota_{\mathfrak{q},\infty})_*(G'_{0,0,\mathfrak{q}} \cup G'_{1,0,\mathfrak{q}})\big)$$

in $\underline{K}(A \otimes Q)$, where $\iota_{\mathfrak{r},\infty}$ is the canonical embedding $A \otimes M_{\mathfrak{r}} \to A \otimes Q$, $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$. Without loss of generality, we may also assume that $\mathcal{R} \supseteq (J_{n_0,\infty})_{*1}(\mathcal{G})$.

Let $\mathcal{H}_{\mathfrak{r}} \subset A \otimes M_{\mathfrak{r}}$ be a finite subset and $\delta_3 > 0$ such that for any homomorphism h from $A \otimes M_{\mathfrak{r}}$ to $B \otimes M_{\mathfrak{r}}$ ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$) and any unitary z_j $(j = 1, 2, 3, 4)$, the maps $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup $[J^{\mathfrak{r}}_{n_{\mathfrak{r}},\infty}](\underline{K}(C^{\mathfrak{r}}_{n_{\mathfrak{r}}}))$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup generated by $[J^{\mathfrak{r}}_{n_{\mathfrak{r}},\infty}](\underline{K}(C^{\mathfrak{r}}_{n_{\mathfrak{r}}}))$ if $\|[h(x), z_j]\| < \delta_3$ for any $x \in \mathcal{H}_{\mathfrak{r}}$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$. Without loss of generality, we assume that $\mathcal{H}^0 \otimes \mathcal{H}^{\mathfrak{p}} \subset \mathcal{H}_{\mathfrak{p}}$ and $\mathcal{H}^0 \otimes \mathcal{H}^{\mathfrak{q}} \subset \mathcal{H}_{\mathfrak{q}}$. Furthermore, we may also assume that

$$\mathcal{H}_{\mathfrak{r}} = \mathcal{H}_{0,0} \otimes \mathcal{H}_{0,\mathfrak{r}} \tag{e 4.7}$$

for some finite subsets $\mathcal{H}_{0,0}$ and $\mathcal{H}_{0,\mathfrak{r}}$ with $\mathcal{H}^{0'} \subset \mathcal{H}_{0,0} \subset A$, $\mathcal{H}^{\mathfrak{p}'} \subset \mathcal{H}_{0,\mathfrak{p}} \subset M_{\mathfrak{p}}$, and $\mathcal{H}^{\mathfrak{q}'} \subset \mathcal{H}_{0,\mathfrak{q}}$. In addition, we may also assume that $\delta_3 < \delta_2/2$.

Furthermore, one may assume that δ_3 is sufficiently small such that, for any unitaries z_1, z_2, z_3 in a C^*-algebra with tracial states, $\tau(\frac{1}{2\pi i}\log(z_i z_j^*))$ $(i, j = 1, 2, 3)$ is well defined and

$$\tau\bigg(\frac{1}{2\pi i}\log(z_1 z_2^*)\bigg) = \tau\bigg(\frac{1}{2\pi i}\log(z_1 z_3^*)\bigg) + \tau\bigg(\frac{1}{2\pi i}\log(z_3 z_2^*)\bigg)$$

for any tracial state τ, whenever $\|z_1 - z_3\| < \delta_3$ and $\|z_2 - z_3\| < \delta_3$.

To simplify notation, we also assume that, for any unitary z_j $(j = 1, 2, 3, 4)$, the maps $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup generated by $\mathcal{R}$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup generated by $\mathcal{R}$ if $\|[h(x), z_j]\| < \delta_3$ for any $x \in \mathcal{H}''$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, \ldots, 4$, and assume that

$$\mathcal{H}'' = \mathcal{H}_{0,0} \otimes \mathcal{H}_{0,\mathfrak{p}} \otimes \mathcal{H}_{0,\mathfrak{q}}.$$

Let $\mathcal{R}^i = \mathcal{R} \cap K_i(A \otimes Q)$. There is a finitely generated subgroup $G_{i,0}$ of $K_i(A)$, and there is a finite subset $D'_0 \subset \mathbb{Q}$ such that

$$G_{i,\infty} := G\big(\{gr : g \in (\iota_\infty)_{*i}(G_{i,0}) \text{ and } r \in D'_0\}\big)$$

contains the subgroup generated by $\mathcal{R}^i$, $i = 0, 1$. Without loss of generality, we may assume that $G_{i,\infty}$ is the subgroup generated by $\mathcal{R}^i$. Note that we may also assume that $G_{i,0} \supset G(\mathcal{P})_{i,0}$ and $1 \in D_0' \supset D_0$. Moreover, we may assume that if $r = k/m$, where m, k are relatively prime non-zero integers, and $r \in D_0'$, then $1/m \in D_0'$. We may also assume that $G_{i,0,\mathfrak{r}} \subseteq G_{i,0}$ for $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$ and $i = 0, 1$. Let $\mathcal{R}^{i'} \subset K_i(A)$ be a finite subset which generates $G_{i,0}$, $i = 0, 1$. Choose a finite subset $\mathcal{U} \subset U_{n_1}(A)$ for some n_1 such that for any element of $\mathcal{R}^{1'}$, there is a representative in $\mathcal{U}$. Let S be a finite subset of A such that if $(z_{i,j}) \in \mathcal{U}$, then $z_{i,j} \in S$.

Denote by δ_4 and $\mathcal{Q}_\mathfrak{r} \subset K_1(A \otimes M_\mathfrak{r}) \cong K_1(A) \otimes \mathbb{D}_\mathfrak{r}$ the constant and finite subset of Lemma 4.1 corresponding to $\mathcal{E}_\mathfrak{r} \cup \mathcal{H}_\mathfrak{r} \otimes 1 \cup \imath_\mathfrak{r}(S)$ (in place of $\mathcal{F}$), $\imath_\mathfrak{r}(\mathcal{U})$ (in place of $\mathcal{P}$), and $\frac{1}{n_1^2}\min\{\delta_2'/8, \delta_3/4\}$ (in place of ε) ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$). We may assume that $\mathcal{Q}_\mathfrak{r} = \{x \otimes r : x \in \mathcal{Q}' \text{ and } r \in D_\mathfrak{r}''\}$, where $\mathcal{Q}' \subset K_1(A)$ is a finite subset and $D_\mathfrak{r}'' \subset \mathbb{Q}_\mathfrak{r}$ is also a finite subset. Let $K = \max\{|r| : r \in D_\mathfrak{p}'' \cup D_\mathfrak{q}''\}$. Since $[\varphi] = [\psi]$ in $KL(A, B)$, $\varphi_\sharp = \psi_\sharp$, and $\varphi^\ddagger = \psi^\ddagger$, by [18, Lemma 3.5], $\overline{R}_{\varphi,\psi}(K_1(A)) \subseteq \overline{\rho_B(K_0(B))} \subset \mathrm{Aff}(\mathrm{T}(B))$. Therefore, there is a map $\eta : G(\mathcal{Q}') \to \overline{\rho_B(K_0(B))} \subset \mathrm{Aff}(\mathrm{T}(B))$ such that

$$(\eta - \overline{R}_{\varphi,\psi})([z]) \in \rho_B(K_0(B)), \quad \|\eta(z)\| < \frac{\delta_4}{1+K} \quad \text{for all } z \in \mathcal{Q}'. \tag{e 4.8}$$

Consider the map $\varphi_\mathfrak{r} = \varphi \otimes \mathrm{id}_{M_\mathfrak{r}}$ and $\psi_\mathfrak{r} = \psi \otimes \mathrm{id}_{M_\mathfrak{r}}$ ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$). Since η vanishes on the torsion part of $G(\mathcal{Q}')$, there is a homomorphism

$$\eta_\mathfrak{r} : G((\imath_\mathfrak{r})_{*1}(\mathcal{Q}')) \to \overline{\rho_{B\otimes M_\mathfrak{r}}(K_0(B \otimes M_\mathfrak{r}))} \subset \mathrm{Aff}\,(T(B \otimes M_\mathfrak{r}))$$

such that

$$\eta_\mathfrak{r} \circ (\imath_\mathfrak{r})_{*1} = \eta. \tag{e 4.9}$$

Since $\overline{\rho_{B\otimes M_\mathfrak{r}}(K_0(B\otimes M_\mathfrak{r}))} = \overline{\mathbb{R}\rho_B(K_0(B))}$ is divisible, one can extend $\eta_\mathfrak{r}$ so it is defined on $K_1(A) \otimes \mathbb{Q}_\mathfrak{r}$. We will continue to use $\eta_\mathfrak{r}$ for the extension. It follows from (e 4.8) that $\eta_\mathfrak{r}(z) - \overline{R}_{\varphi_\mathfrak{r},\psi_\mathfrak{r}}(z) \in \rho_{B\otimes M_\mathfrak{r}}(K_0(B\otimes M_\mathfrak{r}))$ and $\|\eta_\mathfrak{r}(z)\| < \delta_4$ for all $z \in \mathcal{Q}_\mathfrak{r}$. By Lemma 4.1, there exists a unitary $u_\mathfrak{p} \in B \otimes M_\mathfrak{p}$ such that

$$\|u_\mathfrak{p}^*(\varphi \otimes \mathrm{id}_{M_\mathfrak{p}})(c)u_\mathfrak{p} - (\psi \otimes \mathrm{id}_{M_\mathfrak{p}})(c)\| < \frac{1}{n_1^2}\min\{\delta_2'/8, \delta_3/4\} \tag{e 4.10}$$

for all $c \in \mathcal{E}_\mathfrak{p} \cup H_\mathfrak{p} \cup \imath_\mathfrak{p}(S)$, and

$$\tau\left(\frac{1}{2\pi i}\log\left(u_\mathfrak{p}^*(\varphi \otimes \mathrm{id}_\mathfrak{p})(z)u_\mathfrak{p}(\psi \otimes \mathrm{id}_\mathfrak{p})(z^*)\right)\right) = \eta_\mathfrak{p}([z])(\tau)$$

for all $z \in \imath_\mathfrak{p}(\mathcal{U})$, where we also use φ and ψ for $\varphi \otimes \mathrm{id}_{M_N}$ and $\psi \otimes \mathrm{id}_{M_N}$ and $u_\mathfrak{p}$ with $u_\mathfrak{p} \otimes 1_{M_N}$, respectively. Note that

$$\|u_\mathfrak{p}^*(\varphi \otimes \mathrm{id}_{M_\mathfrak{p}})(z)u_\mathfrak{p} - (\psi \otimes \mathrm{id}_{M_\mathfrak{p}})(z)\| < \delta_3 \quad \text{for any } z \in \mathcal{U}.$$

The same argument shows that there is a unitary $u_\mathfrak{q} \in B \otimes M_\mathfrak{q}$ such that

$$\|u_\mathfrak{q}^*(\varphi \otimes \mathrm{id}_{M_\mathfrak{q}})(c)u_\mathfrak{q} - (\psi \otimes \mathrm{id}_{M_\mathfrak{q}})(c)\| < \frac{1}{n_1^2}\min\{\delta_2'/8, \delta_3/4\} \tag{e 4.11}$$

for all $c \in \mathcal{E}_{\mathfrak{q}} \cup \mathcal{H}_{\mathfrak{q}} \cup \imath_{\mathfrak{p}}(S)$, and (recall $\varphi_{\mathfrak{r}} = \varphi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$ and $\psi_{\mathfrak{r}} = \psi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$)

$$\tau\left(\frac{1}{2\pi i}\log\left(u_{\mathfrak{q}}^*(\varphi_{\mathfrak{q}})(z)u_{\mathfrak{q}}(\psi_{\mathfrak{q}})(z^*)\right)\right) = \eta_{\mathfrak{q}}\big([z]\big)(\tau)$$

for all $z \in \imath_{\mathfrak{q}}(\mathcal{U})$, where we identify φ and ψ with $\varphi \otimes \mathrm{id}_{M_n}$ and $\psi \otimes \mathrm{id}_{M_n}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{q}} \otimes 1_{M_n}$, respectively. We will also identify $u_{\mathfrak{p}}$ with $u_{\mathfrak{p}} \otimes 1_{M_{\mathfrak{q}}}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{q}} \otimes 1_{M_{\mathfrak{p}}}$, respectively. Then, $u_{\mathfrak{p}}u_{\mathfrak{q}}^* \in A \otimes Q$, and one estimates that for any $c \in \mathcal{H}_{00} \otimes \mathcal{H}_{0,\mathfrak{p}} \otimes \mathcal{H}_{\mathfrak{q}}$,

$$\left\|u_{\mathfrak{q}}u_{\mathfrak{p}}^*(\varphi_Q(c))u_{\mathfrak{p}}u_{\mathfrak{q}}^* - (\varphi_Q)(c)\right\| < \delta_3,$$

and hence $\mathrm{Bott}(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(z)$ is well defined on the subgroup generated by $\mathcal{R}$. Moreover, for any $z \in \mathcal{U}$, by the Exel formula (see [9]) and applying (e 4.9),

$$\begin{aligned}
&\tau\big(\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\big((\imath_\infty)_{*1}([z])\big)\big)\\
&= \tau\big(\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\big(\imath_\infty(z)\big)\big)\\
&= \tau\left(\frac{1}{2\pi i}\log\left(u_{\mathfrak{p}}u_{\mathfrak{q}}^*(\varphi_Q)\big(\imath_\infty(z)\big)\right)u_{\mathfrak{q}}u_{\mathfrak{p}}^*(\varphi_Q)\big(\imath_\infty(z)\big)^*\right)\\
&= \tau\left(\frac{1}{2\pi i}\log\left(u_{\mathfrak{q}}^*(\varphi_Q)\big(\imath_\infty(z)\big)u_{\mathfrak{q}}(\psi_Q)\big(\imath_\infty(z^*)\big)\right)\right)\\
&\quad - \tau\left(\frac{1}{2\pi i}\log\left(u_{\mathfrak{p}}^*(\varphi_Q)\big(\imath_\infty(z)\big)u_{\mathfrak{p}}(\psi_Q)\big(\imath_\infty(z^*)\big)\right)\right)\\
&= \eta_{\mathfrak{q}}\big((\imath_{\mathfrak{q}})_{*1}\big([z]\big)\big)(\tau) - \eta_{\mathfrak{p}}\big((\imath_{\mathfrak{p}})_{*1}\big([z]\big)\big)(\tau)\\
&= \eta\big([z]\big)(\tau) - \eta\big([z]\big)(\tau) = 0 \quad \text{for all } \tau \in T(B),
\end{aligned}$$

where we also use φ_Q and ψ_Q for $\varphi_Q \otimes \mathrm{id}_{M_n}$ and $\psi_Q \otimes \mathrm{id}_{M_n}$ and $u_{\mathfrak{p}}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{p}} \otimes 1_{M_n}$ and $u_{\mathfrak{q}} \otimes 1_{M_n}$, respectively.

Now suppose that $g \in G_{1,\infty}$. Then, $g = (k/m)(\imath_\infty)_{*1}([z])$ for some $z \in \mathcal{U}$, where k, m are non-zero integers. It follows that

$$\tau\big(\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(mg)\big) = k\tau\big(\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\big([z]\big)\big) = 0$$

for all $\tau \in T(B)$. Since $\mathrm{Aff}(T(B))$ is torsion-free, it follows that

$$\tau\big(\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(g)\big) = 0$$

for all $g \in G_{1,\infty}$ and $\tau \in T(B)$. Therefore, the image of $\mathcal{R}^1$ under $\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)$ is in $\ker \rho_{B\otimes Q}$. One may write

$$G_{1,0} = \mathbb{Z}^r \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s\mathbb{Z},$$

where r is a non-negative integer and $p_1, \ldots, p_s$ are powers of prime numbers. Since $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime, one then has the decomposition

$$G_{1,0} = \mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}(G_{1,0}) \oplus \mathrm{Tor}_{\mathfrak{q}}(G_{1,0}) \subseteq K_1(A),$$

where $\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ consists of the torsion-elements whose order divides $\mathfrak{p}$, and analogously for $\mathfrak{q}$ instead of $\mathfrak{p}$. Fix this decomposition.

Note that the restriction of $(\iota_{\mathfrak{p}})_{*1}$ to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{q}}(G_{1,0})$ is injective and the restriction to $\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ is zero, and the restriction of $(\iota_{\mathfrak{q}})_{*1}$ to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ is injective and the restriction to $\mathrm{Tor}_{\mathfrak{q}}(G_{1,0})$ is zero.

Moreover, using the assumption that $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime again, for any element $k \in (\iota_{\mathfrak{q}})_{*1}(\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}(G_{1,0}))$ and any non-zero integer q which divides $\mathfrak{q}$, the element k/q is well defined in $K_1(A \otimes M_{\mathfrak{q}})$; that is, there is a unique element $s \in K_1(A \otimes M_{\mathfrak{q}})$ such that $qs = k$.

Denote by $e_1, \dots, e_r$ the standard generators of $\mathbb{Z}^r$. It is also clear that

$$(\iota_\infty)_{*1}\big(\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})\big) = (\iota_\infty)_{*1}\big(\mathrm{Tor}_{\mathfrak{q}}(G_{1,0})\big) = 0.$$

Recall that $D_{0,\mathfrak{p}} = \{k/m_{\mathfrak{p}};\ k \in \mathbb{Z}\} \subset \mathbb{D}_{\mathfrak{p}}$ and $D_{0,\mathfrak{q}} = \{k/m_{\mathfrak{q}};\ k \in \mathbb{Z}\} \subset \mathbb{D}_{\mathfrak{q}}$ for an integer $m_{\mathfrak{p}}$ dividing $\mathfrak{p}$ and an integer $m_{\mathfrak{q}}$ dividing $\mathfrak{q}$. Put $m_\infty = m_{\mathfrak{p}} m_{\mathfrak{q}}$.

Consider $\frac{1}{m_\infty}\mathbb{Z}^r \subseteq K_1(A \otimes Q)$, and for each e_i, $1 \le i \le r$, consider

$$\frac{1}{m_\infty}\,\mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*)\big((\iota_\infty)_{*1}(e_i)\big) \in \ker \rho_{B\otimes Q}.$$

Note

$$\ker \rho_{B\otimes Q} \cong (\ker \rho_B) \otimes \mathbb{Q},\ \ker \rho_{B\otimes M_{\mathfrak{p}}} \cong (\ker \rho_B) \otimes \mathbb{D}_{\mathfrak{p}},\ \text{and}\ \ker \rho_{B\otimes M_{\mathfrak{q}}} \cong (\ker \rho_B) \otimes \mathbb{D}_{\mathfrak{q}}.$$

Since $\ker \rho_{A\otimes Q}$ is torsion-free, $\mathrm{bott}_1(\psi \otimes \mathrm{id}_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*)$ maps $\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ to zero. Suppose that $(\frac{1}{m_\infty})\,\mathrm{bott}_1(\psi \otimes \mathrm{id}_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*)$ maps $\iota_\infty(e_i)$ to $\sum_{j=1}^{m_i} x_{i,j} \otimes r_{i,j}$, where $x_{i,j} \in \ker \rho_B$ and $r_{i,j} \in \mathbb{Q}$, $j = 1, 2, \dots, m_i$ and $i = 1, 2, \dots, r$. Since $\mathfrak{p}$ and $\mathfrak{q}$ are relative prime, any rational number r can be written as $r = r_{\mathfrak{p}} - r_{\mathfrak{q}}$ with $r_{\mathfrak{p}} \in \mathbb{Q}_{\mathfrak{p}}$ and $r_{\mathfrak{q}} \in \mathbb{Q}_{\mathfrak{q}}$ (see, for example, [18, Section 2.6]). Hence, there are $r_{i,j,\mathfrak{p}} \in \mathbb{Q}_{\mathfrak{p}}$ and $r_{i,j,\mathfrak{q}} \in \mathbb{Q}_{\mathfrak{q}}$ such that $r_{i,j} = r_{i,j,\mathfrak{p}} - r_{i,j,\mathfrak{q}}$, $j = 1, 2, \dots, m_i$ and $i = 1, 2, \dots, r$. Choose $g_{i,\mathfrak{p}} = \sum_{j=1}^{m_i} x_{i,j} \otimes r_{i,j,\mathfrak{p}}$ and $g_{i,\mathfrak{q}} = \sum_{j=1}^{m_i} x_{i,j} \otimes r_{i,j,\mathfrak{q}}$. Then, $g_{i,\mathfrak{p}} \in \ker \rho_{B\otimes M_{\mathfrak{p}}}$ and $g_{i,\mathfrak{q}} \in \ker \rho_{B\otimes M_{\mathfrak{q}}}$. Moreover,

$$\mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*)\left(\frac{1}{m_\infty}\big((\iota_\infty)_{*1}(e_i)\big)\right) = (j_{\mathfrak{p}})_{*0}(g_{i,\mathfrak{p}}) - (j_{\mathfrak{q}})_{*0}(g_{i,\mathfrak{q}}), \tag{e 4.12}$$

where $g_{i,\mathfrak{p}}$ and $g_{i,\mathfrak{q}}$ are identified as their images in $K_0(A \otimes Q)$.

Note that the subgroup $(\iota_{\mathfrak{p}})_{*1}(G_{1,0})$ in $K_1(A \otimes M_{\mathfrak{p}})$ is isomorphic to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{q}}$ and $\frac{1}{m_{\mathfrak{p}}}(\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{q}})$ is well defined in $K_1(A \otimes M_{\mathfrak{p}})$, and the subgroup $(\iota_{\mathfrak{q}})_{*1}(G_{1,0})$ in $K_1(B \otimes M_{\mathfrak{q}})$ is isomorphic to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}$ and $\frac{1}{m_{\mathfrak{q}}}(\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}})$ is well defined in $K_1(A \otimes M_{\mathfrak{q}})$. One then defines the maps

$$\theta_{\mathfrak{p}} : \frac{1}{m_{\mathfrak{p}}}(\iota_{\mathfrak{p}})_{*1}(G_{1,0}) \to \ker \rho_{B\otimes M_{\mathfrak{p}}}, \quad \theta_{\mathfrak{q}} : \frac{1}{m_{\mathfrak{q}}}(\iota_{\mathfrak{q}})_{*1}(G_{1,0}) \to \ker \rho_{B\otimes M_{\mathfrak{q}}}$$

by

$$\theta_{\mathfrak{p}}\left(\frac{1}{m_{\mathfrak{p}}}(\iota_{\mathfrak{p}})_{*1}(e_i)\right) = m_{\mathfrak{q}} g_{i,\mathfrak{p}}, \quad \theta_{\mathfrak{q}}\left(\frac{1}{m_{\mathfrak{q}}}(\iota_{\mathfrak{q}})_{*1}(e_i)\right) = m_{\mathfrak{p}} g_{i,\mathfrak{q}}$$

for $1 \le i \le r$ and

$$\theta_{\mathfrak{p}}|_{\mathrm{Tor}((\iota_{\mathfrak{p}})_{*1}(G_{1,0}))} = 0, \quad \theta_{\mathfrak{q}}|_{\mathrm{Tor}((\iota_{\mathfrak{q}})_{*1}(G_{1,0}))} = 0.$$

Then, for each e_i, by (e 4.12), one has

$$\begin{aligned}
&(j_{\mathfrak{p}})_{*0} \circ \theta_{\mathfrak{p}} \circ (\iota_{\mathfrak{p}})_{*1}(e_i) - (j_{\mathfrak{q}})_{*0} \circ \theta_{\mathfrak{q}} \circ (\iota_q)_{*1}(e_i) \\
&\quad = m_{\mathfrak{p}}\left(\frac{1}{m_{\mathfrak{p}}}(j_{\mathfrak{p}})_{*0} \circ \theta_{\mathfrak{p}} \circ (\iota_{\mathfrak{p}})_{*1}(e_i)\right) - m_{\mathfrak{q}}\left(\frac{1}{m_{\mathfrak{q}}}(j_{\mathfrak{q}})\right)_{*0} \circ \theta_{\mathfrak{q}} \circ (\iota_q)_{*1}(e_i) \\
&\quad = m_{\mathfrak{p}} m_{\mathfrak{q}}\big((j_{\mathfrak{p}})_{*0}(g_{i,p}) - (j_{\mathfrak{q}})_{*0}(g_{i,q})\big) \\
&\quad = m_\infty \,\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) \circ (\iota_\infty)_{*1}(e_i/m_\infty) \\
&\quad = \mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) \circ (\iota_\infty)_{*1}(e_i),
\end{aligned}$$

where $\varphi_Q = \varphi \otimes \mathrm{id}_Q$. Since the restrictions of $\theta_{\mathfrak{p}} \circ (\iota_{\mathfrak{p}})_{*1}$, $\theta_{\mathfrak{q}} \circ (\iota_{\mathfrak{q}})_{*1}$ and $\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) \circ (\iota_\infty)_{*1}$ to the torsion part of $G_{1,0}$ are zero, one has

$$\begin{aligned}
&\mathrm{bott}_1(\varphi_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) \circ (\iota_\infty)_{*1} \\
&\quad = (j_{\mathfrak{p}})_{*0} \circ \theta_{\mathfrak{p}} \circ (\iota_{\mathfrak{p}})_{*1} - (j_{\mathfrak{q}})_{*0} \circ \theta_{\mathfrak{q}} \circ (\iota_q)_{*1} \quad \text{on } G_{1,0}. \qquad \text{(e 4.13)}
\end{aligned}$$

The same argument shows that there also exist maps

$$\alpha_{\mathfrak{p}} : \frac{1}{m_{\mathfrak{p}}}\big((\iota_{\mathfrak{p}})_{*0}(G_{0,0})\big) \to K_1(B \otimes M_{\mathfrak{p}}),$$

$$\alpha_{\mathfrak{q}} : \frac{1}{m_{\mathfrak{q}}}\big((\iota_{\mathfrak{q}})_{*0}(G_{0,0})\big) \to K_1(B \otimes M_{\mathfrak{q}})$$

such that

$$\begin{aligned}
&\mathrm{bott}_0(\varphi_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) \circ (\iota_\infty)_{*0} \\
&\quad = (j_{\mathfrak{p}})_{*1} \circ \alpha_{\mathfrak{p}} \circ (\iota_{\mathfrak{p}})_{*0} - (j_{\mathfrak{q}})_{*1} \circ \alpha_{\mathfrak{q}} \circ (\iota_{\mathfrak{q}})_{*0} \quad \text{on } G_{0,0}. \qquad \text{(e 4.14)}
\end{aligned}$$

Note that $G_{i,0,\mathfrak{r}} \subseteq G_{i,0}$, $i = 0, 1$, $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$. In particular, one has that

$$(\iota_{\mathfrak{r}})_{*i}(G_{i,0,\mathfrak{r}}) \subseteq (\iota_{\mathfrak{r}})_{*i}(G_{i,0}),$$

and therefore

$$G'_{1,0,\mathfrak{p}} \subseteq \frac{1}{m_{\mathfrak{p}}}(\iota_{\mathfrak{p}})_{*0}(G_{1,0}), \quad G'_{1,0,\mathfrak{q}} \subseteq \frac{1}{m_{\mathfrak{q}}}(\iota_{\mathfrak{q}})_{*0}(G_{1,0}).$$

Then, the maps $\theta_{\mathfrak{p}}$ and $\theta_{\mathfrak{q}}$ can be restricted to $G'_{1,0,\mathfrak{p}}$ and $G'_{1,0,\mathfrak{q}}$, respectively. Since the group $G'_{i,0,\mathfrak{r}}$ contains $(J^{\mathfrak{r}}_{n_{\mathfrak{r}},\infty})_{*i}(K_i(C^{\mathfrak{r}}_{n_{\mathfrak{r}}}))$, the maps $\theta_{\mathfrak{p}}$ and $\theta_{\mathfrak{q}}$ can be restricted further to $(J^{\mathfrak{p}}_{n_{\mathfrak{p}},\infty})_{*1}(K_1(C^{\mathfrak{p}}_{n_{\mathfrak{p}}}))$ and $(J^{\mathfrak{q}}_{n_{\mathfrak{q}},\infty})_{*1}(K_1(C^{\mathfrak{q}}_{n_{\mathfrak{q}}}))$, respectively.

For the same reason, the maps $\alpha_{\mathfrak{p}}$ and $\alpha_{\mathfrak{q}}$ can be restricted to $(J^{\mathfrak{p}}_{n_{\mathfrak{p}},\infty})_{*0}(K_0(C^{\mathfrak{p}}_{n_{\mathfrak{p}},\infty}))$ and $(J^{\mathfrak{q}}_{n_{\mathfrak{q}},\infty})_{*0}(K_0(C^{\mathfrak{q}}_{n_{\mathfrak{q}},\infty}))$, respectively. We keep the same notation for the restrictions of these maps $\alpha_{\mathfrak{p}}$, $\alpha_{\mathfrak{q}}$, $\theta_{\mathfrak{p}}$, and $\theta_{\mathfrak{q}}$.

By the universal multi-coefficient theorem (see [2]), there is

$$\kappa_{\mathfrak{p}} \in \mathrm{Hom}_{\Lambda}\left(\underline{K}\big(C_{n_{\mathfrak{p}}}^{\mathfrak{p}} \otimes \mathrm{C}(\mathbb{T})\big), \underline{K}(B \otimes M_{\mathfrak{p}})\right)$$

such that

$$\begin{aligned} \kappa_{\mathfrak{p}}|_{\boldsymbol{\beta}(K_1(C'_{\mathfrak{p}}))} &= -\theta_{\mathfrak{p}} \circ (J^{\mathfrak{p}}_{n_{\mathfrak{p}},\infty})_{*1} \circ \boldsymbol{\beta}^{-1}, \\ \kappa_{\mathfrak{p}}|_{\boldsymbol{\beta}(K_0(C^{\mathfrak{p}}_{n_{\mathfrak{p}}}))} &= -\alpha_{\mathfrak{p}} \circ (J^{\mathfrak{p}}_{n_{\mathfrak{p}},\infty})_{*0} \circ \boldsymbol{\beta}^{-1}, \end{aligned} \tag{e 4.15}$$

where $\boldsymbol{\beta} : \underline{K}(\cdot) \to \underline{K}(\cdot \otimes \mathrm{C}(\mathbb{T}))$ is defined by the identification

$$\underline{K}\big(\cdot \otimes \mathrm{C}(\mathbb{T})\big) = \underline{K}(\cdot) \oplus \boldsymbol{\beta}\big(\underline{K}(\cdot)\big).$$

Similarly, there exists $\kappa_{\mathfrak{q}} \in \mathrm{Hom}_{\Lambda}(\underline{K}(C^{\mathfrak{q}}_{n_{\mathfrak{q}}} \otimes \mathrm{C}(\mathbb{T})), \underline{K}(B \otimes M_{\mathfrak{q}}))$ such that

$$\begin{aligned} \kappa_{\mathfrak{q}}|_{\boldsymbol{\beta}(K_1(C'_{\mathfrak{q}}))} &= -\theta_{\mathfrak{q}} \circ (J^{\mathfrak{q}}_{n_{\mathfrak{q}},\infty})_{*1} \circ \boldsymbol{\beta}^{-1}, \\ \kappa_{\mathfrak{q}}|_{\boldsymbol{\beta}(K_0(C^{\mathfrak{q}}_{n_{\mathfrak{q}}}))} &= -\alpha_{\mathfrak{q}} \circ (J^{\mathfrak{q}}_{n_{\mathfrak{q}},\infty})_{*0} \circ \boldsymbol{\beta}^{-1}. \end{aligned} \tag{e 4.16}$$

Define

$$\begin{aligned} &\zeta_{x'_j,u,b} \\ &= \overline{\big\langle\big(\mathbf{1}_N - \varphi_Q(p'_j \otimes 1_Q) + \varphi_Q(p'_j)u_{\mathfrak{p}}u^*_{\mathfrak{q}}\big)\big(\mathbf{1}_N - \varphi_Q(q'_j \otimes 1_Q) + \varphi_Q(q'_j \otimes 1_Q)u_{\mathfrak{q}}u^*_{\mathfrak{p}}\big)\big\rangle}. \end{aligned}$$

Recall that we use $u_{\mathfrak{p}} := (u_{\mathfrak{p}} \otimes 1_{M_{\mathfrak{q}}}) \otimes 1_N$ and $u_{\mathfrak{q}} := (u_{\mathfrak{q}} \otimes 1_{M_{\mathfrak{p}}}) \otimes 1_N$ above. Choose the unique $\zeta_{x'_j,u,b} \in U(B)/CU(B)$ which is represented by a unitary $z_{x'_j,u} \in U(B)$ such that $\overline{\mathrm{diag}(z_{x'_j,u,b}, 1_{N-1})} = \zeta_{x'_j,u,b}$ (see [6, Theorem 11.10]). Choose $z_{x'_j,\mathfrak{r}} \in U(B \otimes M_{\mathfrak{r}})$ such that

$$[z_{x'_{j,\mathfrak{p}}}] = \alpha_{\mathfrak{p}}(x'_{j,\mathfrak{p}}), \quad [z_{x'_j,\mathfrak{q}}] = -\alpha_{\mathfrak{q}}(x'_{j,\mathfrak{p}}). \tag{e 4.17}$$

Then, by (e 4.14),

$$f_j := \zeta_{x'_j,u,b}\overline{(z_{x'_{j,\mathfrak{p}}} \otimes 1_{M_{\mathfrak{q}}})^*(z_{x'_{j,\mathfrak{q}}} \otimes 1_{M_{\mathfrak{p}}})^*} \in U_0(B \otimes Q)/CU(B \otimes Q).$$

Identify $U_0(B)/CU(B)$ with

$$\mathrm{Aff}\big(T(B \otimes Q)\big)/\overline{\rho_{B\otimes Q}\big(K_0(B \otimes Q)\big)} = \mathrm{Aff}\big(T(B \otimes M_{\mathfrak{p}})\big)/\overline{\rho_{B\otimes M_{\mathfrak{p}}}\big(K_0(B \otimes M_{\mathfrak{p}})\big)}.$$

So we may also view $f_j \in U_0(B \otimes M_{\mathfrak{p}})/CU(B \otimes M_{\mathfrak{p}})$. Define

$$\zeta_{x'_{j,\mathfrak{p}},u_{\mathfrak{p}}} = (f_j\overline{z_{x'_j,\mathfrak{p}}}), \quad \zeta_{x'_{j,\mathfrak{q}},u_{\mathfrak{q}}} = \overline{z_{x'_{j,\mathfrak{q}}}}.$$

Note that

$$\zeta_{x'_j,u,b} = \big(j_{\mathfrak{p}}{}^{\ddagger}(\zeta_{x'_j\mathfrak{p},u_{\mathfrak{p}}})\big)\big(j_{\mathfrak{q}}{}^{\ddagger}(\zeta_{x'_{j,\mathfrak{q}},u_{\mathfrak{q}}})\big). \tag{e 4.18}$$

Define the map $\Gamma_{\mathfrak{r}} : \mathbb{Z}^k \to U(B \otimes M_{\mathfrak{r}})/CU(B \otimes M_{\mathfrak{r}})$ by

$$\Gamma_{\mathfrak{r}}(x'_{j,\mathfrak{r}}) = \zeta_{x'_{j,\mathfrak{r}},u_{\mathfrak{r}}}, \quad 1 \le j \le k.$$

Note that, by (e 4.6), (e 4.15), (e 4.16), and (e 4.17),

$$\Pi^{cu}_{B\otimes M_{\mathfrak{p}}}\big(\Gamma_{\mathfrak{p}}(x'_{j,\mathfrak{p}})\big) = -\kappa_{\mathfrak{p}}\circ\boldsymbol{\beta}(x''_{j,\mathfrak{p}}), \quad \Pi^{cu}_{B\otimes M_{\mathfrak{q}}}\big(\Gamma_{\mathfrak{q}}(x'_{j,\mathfrak{q}})\big) = \kappa_{\mathfrak{q}}\circ\boldsymbol{\beta}(x''_{j,\mathfrak{q}}),$$

where the map $\Pi^{cu}_{B\otimes M_{\mathfrak{r}}}$ is defined in Definition 2.4. Since $g_{i,\mathfrak{r}}\in\ker\rho_{A\otimes M_{\mathfrak{r}}}$, we have $\kappa_{\mathfrak{r}}(\boldsymbol{\beta}(K_1(C^{\mathfrak{r}}_{n_{\mathfrak{r}}})))\subseteq\ker\rho_{B\otimes M_{\mathfrak{r}}}$, $\mathfrak{r}=\mathfrak{p}$ or $\mathfrak{r}=\mathfrak{q}$. By Lemma 3.3, there exist unitaries $w_{\mathfrak{p}}\in B\otimes M_{\mathfrak{p}}$ and $w_{\mathfrak{q}}\in B\otimes M_{\mathfrak{q}}$ such that

$$\big\|[w_{\mathfrak{p}},\varphi_{\mathfrak{p}}(x)]\big\| < \delta_2'/8, \quad \big\|[w_{\mathfrak{q}},\varphi_{\mathfrak{q}}(y)]\big\| < \delta_2'/8, \tag{e 4.19}$$

for any $x\in\mathcal{H}^{\mathfrak{0}'}\otimes\mathcal{H}^{\mathfrak{p}'}$ and $y\in\mathcal{H}^{\mathfrak{0}'}\otimes\mathcal{H}^{\mathfrak{q}'}$,

$$\mathrm{Bott}(\varphi_{\mathfrak{p}},w_{\mathfrak{p}})\circ[J^{\mathfrak{p}}_{n_{\mathfrak{p}},\infty}] = \kappa_{\mathfrak{p}}\circ\boldsymbol{\beta}, \quad \mathrm{Bott}(\varphi_{\mathfrak{q}},w_{\mathfrak{q}})\circ[J^{\mathfrak{q}}_{n_{\mathfrak{q}},\infty}] = \kappa_{\mathfrak{q}}\circ\boldsymbol{\beta}, \tag{e 4.20}$$

and

$$\mathrm{dist}\,\big(\zeta_{x'_{j,\mathfrak{p}},w^*_{\mathfrak{p}}},\Gamma_{\mathfrak{r}}(x'_{j,\mathfrak{p}})\big) \le \gamma\Big/\Big(64N\Big(1+\sum_{i,j}|Mr_{ij}|\Big)\Big), \tag{e 4.21}$$

$$\mathrm{dist}\,\big(\zeta_{x'_{j,\mathfrak{q}},w_{\mathfrak{q}}},\Gamma_{\mathfrak{r}}(x'_{j,\mathfrak{q}})\big) \le \gamma\Big/\Big(64N\Big(1+\sum_{i,j}|Mr_{ij}|\Big)\Big) \quad 1\le j\le k, \tag{e 4.22}$$

where

$$\zeta_{x'_{j,\mathfrak{r}},w^*_{\mathfrak{r}}} = \overline{\langle(\mathbf{1}_N-(\varphi_{\mathfrak{r}})(p'_{j,\mathfrak{r}})+((\varphi_{\mathfrak{r}})(p'_{j,\mathfrak{r}}))w^{(N)}_{\mathfrak{r}})^*(\mathbf{1}_N-(\varphi_{\mathfrak{r}})(q'_{j,\mathfrak{r}})+((\varphi_{\mathfrak{r}})(q'_{j,\mathfrak{r}}))w^{(N)}_{\mathfrak{r}})\rangle},$$

where $w^{(N)}_{\mathfrak{r}} = w_{\mathfrak{r}}\otimes 1_N$ and $\mathfrak{r}=\mathfrak{p},\mathfrak{q}$. Define

$$\xi_{x'_{j,\mathfrak{r}},w^*_{\mathfrak{r}}} = \overline{\langle(\mathbf{1}_N-(\varphi_{\mathfrak{r}})(p'_j)+(\varphi_{\mathfrak{r}}(p'_j))w^{(N)}_{\mathfrak{r}})^*(\mathbf{1}_N-(\varphi_{\mathfrak{r}})(q'_j)+((\varphi_{\mathfrak{r}})(q'_j))w^{(N)}_{\mathfrak{r}})\rangle},$$

where $w^{(N)}_{\mathfrak{r}} = w_{\mathfrak{r}}\otimes 1_N$, and define (with $w_{\mathfrak{p}} := w_{\mathfrak{p}}\otimes 1_{M_{\mathfrak{q}}}$ and $w_{\mathfrak{q}} := w_{\mathfrak{q}}\otimes 1_{M_{\mathfrak{p}}}$)

$$\zeta_{x_i,w^*_{\mathfrak{r}}} = \overline{\langle(\mathbf{1}-\varphi_Q(p_i)+\varphi_Q(p_i)w^*_{\mathfrak{r}})(\mathbf{1}-\varphi_Q(q_i)+\varphi_Q(q_i)w_{\mathfrak{r}})\rangle},$$

$$\tilde{\zeta}_{x_i,w^*_{\mathfrak{r}}} = \overline{\langle(\mathbf{1}-\varphi_Q(p_i)+\varphi_Q(p_i)w^*_{\mathfrak{r}})(\mathbf{1}-\varphi_Q(q_i)+\varphi_Q(q_i)w_{\mathfrak{r}})\oplus\mathbf{1}_{N-1}\rangle},$$

$\mathfrak{r}=\mathfrak{p},\mathfrak{q}$. Also, define

$$\zeta_{x_i,u} = \overline{\langle(\mathbf{1}-\varphi_Q(p_i)+\varphi_Q(p_i)u_{\mathfrak{p}}u^*_{\mathfrak{q}})((\mathbf{1}-\varphi_Q(q_i)+\varphi_Q(q_i)u_{\mathfrak{q}}u^*_{\mathfrak{p}}))\rangle},$$

$$\tilde{\zeta}_{x_i,u} = \overline{\langle(\mathbf{1}-\varphi_Q(p_i)+\varphi_Q(p_i)u_{\mathfrak{p}}u^*_{\mathfrak{q}})((\mathbf{1}-\varphi_Q(q_i)+\varphi_Q(q_i)u_{\mathfrak{q}}u^*_{\mathfrak{p}})\oplus\mathbf{1}_{N-1})\rangle}.$$

By the choice of $\mathcal{H}$ and δ, and by (e 4.5) (see also [6, Lemma 11.9]),

$$\mathrm{dist}\left(\zeta^M_{i,u},\prod_{j=1}^k\zeta^{Mr_{i,j}}_{x'_j,u,b}\right) < \gamma/32, \tag{e 4.23}$$

and, together with (e 4.21), (e 4.22), (e 4.2), and (e 4.3),

$$\operatorname{dist}\left(\zeta_{i,w_{\mathfrak{p}}^*}^{M}, \prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{p}},u_{\mathfrak{p}}}^{Mr_{i,j}}\right) < \gamma/8, \quad \operatorname{dist}\left(\zeta_{i,w_{\mathfrak{q}}}^{M}, \prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{q}},u_{\mathfrak{q}}}^{Mr_{i,j}}\right) < \gamma/8. \tag{e 4.24}$$

Put $v_{\mathfrak{r}} = w_{\mathfrak{r}} u_{\mathfrak{r}}$. In what follows, we will also write $v_{\mathfrak{p}}$ for $v_{\mathfrak{p}} \otimes 1_{M_{\mathfrak{q}}} \in A \otimes Q$ and $v_{\mathfrak{q}}$ for $v_{\mathfrak{q}} \otimes 1_{M_{\mathfrak{p}}} \in A \otimes Q$, whenever it is convenient.

We have, by (e 4.24), (e 4.23), and (e 4.18),

$$\begin{aligned}
&\operatorname{dist}\big(\overline{\langle(1-(\varphi\otimes \mathrm{id}_Q)(p_i)+(\varphi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{p}}v_{\mathfrak{q}}^*)(1-(\varphi\otimes \mathrm{id}_Q)(q_i)+(\varphi\otimes \mathrm{id}_Q)(q_i)v_{\mathfrak{q}}v_{\mathfrak{p}}^*)\rangle}^{M}, \overline{(1_{B\otimes Q})}\big)\\
&\quad= \operatorname{dist}\big(\zeta_{x_i,w_{\mathfrak{p}}}^M \zeta_{x_i,u}^M \zeta_{x_i,w_{\mathfrak{q}}^*}^M, \overline{1_{B\otimes Q}}\big)\\
&\quad= \operatorname{dist}\left(\left(\zeta_{x_j,w_{\mathfrak{p}}}^M \prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{p}},u_{\mathfrak{p}}}^{Mr_{i,j}}\right)\left(\prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{p}},u_{\mathfrak{p}}}^{-Mr_{i,j}} \zeta_{x_j,u}^M \prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{q}},u_{\mathfrak{q}}}^{-Mr_{i,j}}\right)\left(\prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{q}},u_{\mathfrak{q}}}^{Mr_{i,j}} \zeta_{x_j,w_{\mathfrak{q}}^*}^M\right), \overline{1_{B\otimes Q}}\right)\\
&\quad\le \operatorname{dist}\left(\zeta_{x_j,w_{\mathfrak{p}}}^M, \prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{p}},u_{\mathfrak{p}}}^{-Mr_{i,j}}\right) + \operatorname{dist}\left(\left(\prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{p}},u_{\mathfrak{p}}}^{-Mr_{i,j}} \zeta_{x_j,u}^M \prod_{j=1}^{k} \zeta_{x'_{j,\mathfrak{q}},u_{\mathfrak{q}}}^{-Mr_{i,j}}\right), \overline{1_{B\otimes Q}}\right)\\
&\qquad+ \operatorname{dist}\left(\zeta_{x'_{j,\mathfrak{q}},w_{\mathfrak{q}}^*}^M, \prod_{j=1}^{k} \zeta_{x_{j',\mathfrak{q}},u_{\mathfrak{q}}}^{-Mr_{i,j}}\right)\\
&\quad< \gamma/8 + \gamma/32 + \gamma/8 < \gamma/3.
\end{aligned}$$

That is,

$$\operatorname{dist}(\zeta_{x_i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*}^{-M}, \overline{1_{B\otimes Q}}) < \gamma/3,$$

where

$$\zeta_{x_i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*} = \overline{\langle(1-\varphi_Q(p_i)+\varphi_Q(p_i)v_{\mathfrak{q}}v_{\mathfrak{p}}^*)(1-\varphi_Q(q_i)+\varphi_Q(q_i)v_{\mathfrak{p}}v_{\mathfrak{q}}^*)\rangle}.$$

By the second part of Remark 3.2,

$$\operatorname{dist}(\zeta_{x_i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*}, \overline{1_{B\otimes Q}}) = \operatorname{dist}(\zeta_{x_i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*}^{-1}, \overline{1_{B\otimes Q}}) < \gamma/3. \tag{e 4.25}$$

Then, by (e 4.7) and the line below it and by (e 4.10), (e 4.11), and (e 4.19), one also has

$$\left\|\psi\otimes \mathrm{id}_Q(x) - v_{\mathfrak{p}}^*(\varphi\otimes \mathrm{id}_Q)(x)v_{\mathfrak{p}}\right\| < \delta_2'/4, \quad \forall x \in \mathcal{H}^{0'}\otimes\mathcal{H}^{p'}\otimes\mathcal{H}^{\mathfrak{q}'}, \tag{e 4.26}$$

$$\left\|\psi\otimes \mathrm{id}_Q(x) - v_{\mathfrak{q}}^*(\varphi\otimes \mathrm{id}_Q)(x)v_{\mathfrak{q}}\right\| < \delta_2'/4, \quad \forall x \in \mathcal{H}^{0'}\otimes\mathcal{H}^{p'}\otimes\mathcal{H}^{q'}. \tag{e 4.27}$$

Hence,

$$\left\|[v_{\mathfrak{p}}v_{\mathfrak{q}}^*, \varphi\otimes \mathrm{id}_Q]\right\| < \delta_2'/2 < \delta_2, \quad \forall x \in \mathcal{H}'.$$

Thus, $\mathrm{Bott}(\varphi_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)$ is well defined on the subgroup generated by $\mathcal{P}$. Moreover, a direct calculation shows that

$$\begin{aligned}
&\mathrm{bott}_1(\varphi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z)\\
&\quad= \mathrm{bott}_1(\varphi\otimes \mathrm{id}_Q, w_{\mathfrak{p}})\circ(\imath_\infty)_{*1}(z) + \mathrm{bott}_1(\varphi\otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z)\\
&\qquad+ \mathrm{bott}_1(\varphi\otimes \mathrm{id}_Q, w_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z)
\end{aligned}$$

$$
\begin{aligned}
&= (j_{\mathfrak{p}})_{*0} \circ \mathrm{bott}_1(\varphi \otimes \mathrm{id}_{M_{\mathfrak{p}}}, w_{\mathfrak{p}}) \circ (\imath_{\mathfrak{p}})_{*1}(z) + \mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) \circ (\imath_\infty)_{*1}(z) \\
&\quad + (j_{\mathfrak{q}})_{*0} \circ \mathrm{bott}_1(\varphi \otimes \mathrm{id}_{M_{\mathfrak{q}}}, w_{\mathfrak{q}}^*) \circ (\imath_{\mathfrak{q}})_{*1}(z) \\
&= -(j_{\mathfrak{p}})_{*0} \circ \theta_{\mathfrak{p}} \circ (\imath_{\mathfrak{p}})_{*1}(z) + \big((j_{\mathfrak{p}})_{*0} \circ \theta_{\mathfrak{p}} \circ (\imath_p)_{*1} - (j_{\mathfrak{q}})_{*0} \circ \theta_{\mathfrak{q}} \circ (\imath_q)_{*1}\big) \\
&\quad - \big(- (j_{\mathfrak{q}})_{*0} \circ \theta_{\mathfrak{q}} \circ (\imath_q)_{*1}(z)\big) \quad \big(\text{see (e 4.20), (e 4.13), (e 4.16)}\big) \\
&= 0 \quad \text{for all } z \in G(\mathcal{P})_{1,0}.
\end{aligned}
$$

The same argument shows that $\mathrm{bott}_0(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} v_{\mathfrak{q}}^*) = 0$ on $G(\mathcal{P})_{0,0}$. Now, for any $g \in G(\mathcal{P})_{1,\infty,0}$, there is $z \in G(\mathcal{P})_{1,0}$ and integers k, m such that $(k/m)z = g$. From the above,

$$\mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} v_{\mathfrak{q}}^*)(mg) = k\, \mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} v_{\mathfrak{q}}^*)(z) = 0.$$

Since $K_0(B \otimes Q)$ is torsion-free, it follows that

$$\mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} v_{\mathfrak{q}}^*)(g) = 0$$

for all $g \in G(\mathcal{P})_{1,\infty,0}$. Therefore, it vanishes on $\mathcal{P} \cap K_1(A \otimes Q)$. Similarly,

$$\mathrm{bott}_0(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} v_{\mathfrak{q}}^*)|_{\mathcal{P} \cap K_0(A \otimes Q)} = 0$$

on $\mathcal{P} \cap K_0(A \otimes Q)$.

Since $K_i(B \otimes Q, \mathbb{Z}/m\mathbb{Z}) = \{0\}$ for all $m \geq 2$, we conclude that

$$\mathrm{Bott}(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} v_{\mathfrak{q}}^*)|_{\mathcal{P}} = 0$$

on the subgroup generated by $\mathcal{P}$.

Since $[\varphi] = [\psi]$ in $KL(A, B)$, $\varphi_\sharp = \psi_\sharp$, and $\varphi^\ddagger = \psi^\ddagger$, one has that

$$
\begin{gathered}
[\varphi \otimes \mathrm{id}_Q] = [\psi \otimes \mathrm{id}_Q] \quad \text{in } KL(A \otimes Q, B \otimes Q), \\
(\varphi \otimes \mathrm{id}_Q)_\sharp = (\psi \otimes \mathrm{id}_Q)_\sharp, \quad (\varphi \otimes \mathrm{id}_Q)^\ddagger = (\psi \otimes \mathrm{id}_Q)^\ddagger.
\end{gathered}
$$

Therefore, by [6, Theorem 12.11 (a)], $\varphi \otimes \mathrm{id}_Q$ and $\psi \otimes \mathrm{id}_Q$ are approximately unitarily equivalent. Thus, there exists a unitary $u \in B \otimes Q$ such that

$$\big\| u^*(\varphi \otimes \mathrm{id}_Q)(c)u - (\psi \otimes \mathrm{id}_Q)(c) \big\| < \delta_2'/8 \quad \text{for all } c \in \mathcal{H}'. \tag{e 4.28}$$

It follows from (e 4.26) that

$$\big\| u v_{\mathfrak{p}}^* (\varphi \otimes \mathrm{id}_Q)(c) v_{\mathfrak{p}} u^* - (\varphi \otimes \mathrm{id}_Q)(c) \big\| < \delta_2'/2 + \delta_2'/8 \quad \forall c \in \mathcal{H}'.$$

By the choice of δ_2' and $\mathcal{H}'$, $\mathrm{Bott}(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} u^*)$ is well defined on $[J_{n_0,\infty}](\underline{K}(C_{n_0}))$, and

$$\big| \tau\big(\mathrm{bott}_1(\varphi \otimes \mathrm{id}_Q, v_{\mathfrak{p}} u^*)(z)\big) \big| < \delta_2/2, \quad \forall \tau \in \mathrm{T}(B), \forall z \in \mathcal{G}.$$

For each $1 \leq i \leq m$, define (see (e 4.4))

$$\zeta_{x_i, u v_{\mathfrak{p}}^*} = \overline{\big\langle \big(\mathbf{1} - (\varphi_Q)(p_i) + \big((\varphi_Q)(p_i)\big) u v_{\mathfrak{p}}^*\big)\big(\mathbf{1} - (\varphi_Q)(q_i) + \big((\varphi_Q)(q_i)\big) v_{\mathfrak{p}} u^*\big) \big\rangle},$$

and define the map $\Gamma : \mathbb{Z}^m = G^o_{u,\infty} \to U(B \otimes Q)/CU(B \otimes Q)$ by $\Gamma(x_i) = \zeta_{x_i, uv_{\mathfrak{q}}^*}$. Note that $\prod_{B\otimes Q}^{cu} \circ \Gamma(x_i) = \mathrm{Bott}(\varphi_Q, v_{\mathfrak{p}} u^*)(x_i)$. By Lemma 3.3, there exists a unitary $y_{\mathfrak{p}} \in B \otimes Q$ such that

$$\|[y_{\mathfrak{p}}, \varphi_Q(h)]\| < \delta/2, \quad \forall h \in \mathcal{H}, \qquad (\mathrm{e}\,4.29)$$
$$\mathrm{Bott}(\varphi_Q, y_{\mathfrak{p}}) = \mathrm{Bott}(\varphi_Q, v_{\mathfrak{p}} u^*)$$

on the subgroup generated by $\mathcal{P}$, and

$$\mathrm{dist}\left(\zeta_{x_i, y_{\mathfrak{p}}^*}, \Gamma(x_i)\right) \le \gamma/2,$$

where

$$\zeta_{x_i, y_{\mathfrak{p}}^*} = \overline{\langle (\mathbf{1} - (\varphi_Q)(p_i) + (\varphi_Q)(p_i) y_p^*)(\mathbf{1} - (\varphi_Q)(q_i) + (\varphi_Q)(q_i) y_{\mathfrak{p}}) \rangle}.$$

Considering the unitary $v = y_{\mathfrak{p}} u$, one has that

$$\|[vv_{\mathfrak{p}}^*, (\varphi \otimes \mathrm{id}_Q)(h)]\| < \delta, \quad \text{for all } h \in \mathcal{H}, \ \mathrm{Bott}(\varphi \otimes \mathrm{id}_Q, vv_{\mathfrak{p}}^*) = 0$$

on the subgroup generated by $\mathcal{P}$, and for any $1 \le i \le m$,

$$\mathrm{dist}(\zeta_{x_i, vv_{\mathfrak{p}}^*}, \bar{1}) < \gamma/2, \qquad (\mathrm{e}\,4.30)$$

where

$$\zeta_{x_i, vv_{\mathfrak{p}}^*} = \overline{\langle (\mathbf{1} - (\varphi_Q)(p_i) + ((\varphi_Q)(p_i)) vv_{\mathfrak{p}}^*)(\mathbf{1} - (\varphi_Q)(q_i) + ((\varphi_Q)(q_i)) v_{\mathfrak{p}} v^*) \rangle}.$$

Applying Lemma 3.1 to $A \otimes Q$ and $\varphi_Q = \varphi \otimes \mathrm{id}_Q$, one obtains a continuous path of unitaries $z_{\mathfrak{p}}(t)$ in $B \otimes Q$ such that $z_{\mathfrak{p}}(0) = 1$ and $z_{\mathfrak{p}}(t_1) = vv_{\mathfrak{p}}^*$, and

$$\|[z_{\mathfrak{p}}(t), (\varphi \otimes \mathrm{id}_Q)(c)]\| < \varepsilon/8 \quad \forall c \in \mathcal{E}, \ \forall t \in [0, t_1]. \qquad (\mathrm{e}\,4.31)$$

Note that

$$\begin{aligned} \mathrm{Bott}(\varphi_Q, v_{\mathfrak{q}} v^*) &= \mathrm{Bott}(\varphi_Q, v_{\mathfrak{q}} v_{\mathfrak{p}}^* v_{\mathfrak{p}} v^*) \\ &= \mathrm{Bott}(\varphi_Q, v_{\mathfrak{q}} v_{\mathfrak{p}}^*) + \mathrm{Bott}(\varphi_Q, v_{\mathfrak{p}} v^*) \\ &= 0 + 0 = 0 \end{aligned}$$

on the subgroup generated by $\mathcal{P}$, and for any $1 \le i \le m$,

$$\mathrm{dist}(\zeta_{x_i, v_{\mathfrak{q}} v^*}, \bar{1}) \le \mathrm{dist}(\zeta_{x_i, v_{\mathfrak{q}} v_{\mathfrak{p}}^*}, \bar{1}) + \mathrm{dist}(\zeta_{x_i, v_{\mathfrak{p}} v^*}, \bar{1}) = \gamma \quad (\text{by } (\mathrm{e}\,4.25), (\mathrm{e}\,4.30)),$$

where

$$\zeta_{x_i, v_q v^*} = \overline{\langle (1 - \varphi_Q(p_i) + (\varphi_Q)(p_i) v_{\mathfrak{q}} v^*)(1 - (\varphi_Q)(q_i) + (\varphi_Q)(q_i) vv_{\mathfrak{q}}^*) \rangle}.$$

Since

$$\|[vv_{\mathfrak{q}}^*, (\varphi \otimes \mathrm{id}_Q)(c)]\| < \delta, \quad \forall c \in \mathcal{H},$$

Lemma 3.1 implies that there is a continuous path of unitaries $z_{\mathfrak{q}}(t) : [t_{m-1}, 1] \to \mathrm{U}(B \otimes Q)$ such that $z_{\mathfrak{q}}(t_{m-1}) = vv_{\mathfrak{q}}^*$, $z_{\mathfrak{q}}(1) = 1$, and

$$\|[z_{\mathfrak{q}}(t), (\varphi \otimes \mathrm{id}_Q)(c)]\| < \varepsilon/8, \quad \forall t \in [t_{m-1}, 1], \ \forall c \in \mathcal{E}.$$

Consider the unitary

$$v(t) = \begin{cases} z_{\mathfrak{p}}(t)v_{\mathfrak{p}}, & \text{if } 0 \le t \le t_1, \\ v, & \text{if } t_1 \le t \le t_{m-1}, \\ z_{\mathfrak{q}}(t)v_{\mathfrak{q}}, & \text{if } t_{m-1} \le t \le t_m. \end{cases}$$

Then, for any t_i, $0 \le i \le m-1$ by (e 4.29) and (e 4.28) (recall $\mathcal{E} \subset \mathcal{H} \subset \mathcal{H}'$), one has that, for any $c \in \mathcal{E}$,

$$\begin{aligned} \|v^*(t_i)(\varphi_Q)(c)v(t_i) - (\psi_Q)(c)\| &= \|u^* y_{\mathfrak{p}}^*(\varphi_Q)(c)y_{\mathfrak{p}}u - (\psi_Q)(c)\| \\ &\le \|u^*(\varphi_Q)(c)u - (\psi_Q)(c)\| + \delta/2 \\ &\le \delta_2'/8 + \delta/2 < 3\varepsilon/4. \end{aligned}$$

Thus, for any $t \in [t_j, t_{j+1}]$ with $1 \le j \le m-2$, one has, by (e 4.1), for any $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$,

$$\begin{aligned} &\|v^*(t)(\varphi \otimes \mathrm{id}\,(a \otimes b(t)))v(t) - \psi \otimes \mathrm{id}\,(a \otimes b(t))\| \\ &\quad = \|v(t_j)^*(\varphi(a) \otimes b(t))v(t_j) - \psi(a) \otimes b(t)\| \\ &\quad < \|v(t_j)^*(\varphi(a) \otimes b(t_j))v(t_j) - \psi(a) \otimes b(t_j)\| + \varepsilon/4 \\ &\quad < 3\varepsilon/4 + \varepsilon/4 < \varepsilon. \end{aligned}$$

For any $t \in [0, t_1]$, by (e 4.1), (e 4.31), and (e 4.26) (note that $\delta_2' < \varepsilon/16$ and $\mathcal{E}_{\mathfrak{p}} \subseteq \mathcal{H}$), one has that, for any $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$,

$$\begin{aligned} &\|v^*(t)(\varphi \otimes \mathrm{id}\,(a \otimes b(t)))v(t) - \psi \otimes \mathrm{id}\,(a \otimes b(t))\| \\ &\quad = \|v_{\mathfrak{p}}^* z_{\mathfrak{p}}^*(t)(\varphi(a) \otimes b(t))z_{\mathfrak{p}}(t)v_{\mathfrak{p}} - \psi(a) \otimes b(t)\| \\ &\quad < \|v_{\mathfrak{p}}^* z_{\mathfrak{p}}^*(t)(\varphi(a) \otimes b(t_0))z_{\mathfrak{p}}(t)v_{\mathfrak{p}} - \psi(a) \otimes b(t_0)\| + \varepsilon/2 \\ &\quad < \|v_{\mathfrak{p}}^*(\varphi(a) \otimes b(t_0))v_{\mathfrak{p}} - \psi(a) \otimes b(t_0)\| + \varepsilon/8 + \varepsilon/2 \\ &\quad < \varepsilon/16 + 5\varepsilon/8 < \varepsilon. \end{aligned}$$

The same argument shows that, for any $t \in [t_{m-1}, 1]$, one has that, for any $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$,

$$\|v^*(t)((\varphi \otimes \mathrm{id}_Q)(a \otimes b(t)))v(t) - (\psi \otimes \mathrm{id}_Q)(a \otimes b(t))\| < \varepsilon.$$

Therefore, one has

$$\|v(\varphi \otimes \mathrm{id}(f))v - \psi \otimes \mathrm{id}(f)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}. \qquad \blacksquare$$

Theorem 4.3. *Let A and B be unital separable simple C^*-algebras. Suppose that A is finite and amenable and satisfies the UCT, and suppose that B is $\mathcal{Z}$-stable and $gTR(B \otimes M_{\mathfrak{r}}) \leq 1$ for all supernatural number $\mathfrak{r}$ of infinite type. Let $\varphi, \psi : A \to B$ be two unital monomorphisms. Then, there exists a sequence of unitaries $\{u_n\} \subset B$ such that*

$$\lim_{n\to\infty} u_n^* \psi(c) u_n = \varphi(c) \quad \text{for all } c \in A$$

if and only if

$$[\varphi] = [\psi] \text{ in } KL(A, B), \quad \varphi_\sharp = \psi_\sharp, \quad \varphi^\ddagger = \psi^\ddagger.$$

Proof. Note that, by the remark at the end of Definition 2.10, $gTR(A \otimes M_{\mathfrak{r}}) \leq 1$ for any supernatural number $\mathfrak{r}$ of infinite type. In what follows, we let $B = B \otimes \mathcal{Z}$. Choose a pair of relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ of infinite type. Let $\mu : Z_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}$ and $\lambda : \mathcal{Z} \to \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be unital embeddings given by [24, Proposition 3.5]. Then, $\mu \circ \lambda : \mathcal{Z} \to \mathcal{Z}$ is a unital embedding. Therefore, $\mu \circ \lambda$ and $\mathrm{id}_{\mathcal{Z}}$ are approximately unitarily equivalent (see [10, Theorem 7.6]). Let $j_D : D \to D \otimes \mathcal{Z}$ be the unital embedding $d \mapsto d \otimes 1_{\mathcal{Z}}$, and let $E_D : D \to D \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be the unital embedding $d \mapsto d \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}$ for any unital C^*-algebra D.

Then, $j_B \circ \varphi = (\varphi \otimes \mathrm{id}_{\mathcal{Z}}) \circ j_A$ and $(\mathrm{id}_B \otimes \lambda) \circ j_B \circ \varphi = (\varphi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}) \circ E_A$. In addition, $(\mathrm{id}_B \otimes \lambda) \circ j_B \circ \psi = (\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}) \circ E_A$.

By Lemma 4.2 (together with the remark at the end of Definition 2.10), $(\mathrm{id}_B \otimes \lambda) \circ j_B \circ \varphi$ and $(\mathrm{id}_B \otimes \lambda) \circ j_B \circ \psi$ are approximately unitarily equivalent. It follows that $(\mathrm{id}_B \otimes \mu) \circ (\mathrm{id}_B \otimes \lambda) \circ j_B \circ \varphi$ and $(\mathrm{id}_B \otimes \mu) \circ (\mathrm{id}_B \otimes \lambda) \circ j_B \circ \psi$ are approximately unitarily equivalent. As $\mu \circ \lambda$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{Z}}$, $j_B \circ \varphi$ and $j_B \otimes \psi$ are approximately unitarily equivalent.

Recall $B = B \otimes \mathcal{Z}$ and the unital embedding $j_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{Z}}$. We conclude that φ and ψ are approximately unitarily equivalent. ■

Remark 4.4. The condition that $gTR(B \otimes M_{\mathfrak{r}}) \leq 1$ in Theorem 4.3 may be replaced by the condition that B is amenable and satisfies the UCT (see [3]).

5. The range

Theorem 5.1. *Let A be a separable amenable C^*-algebra which satisfies the UCT with a fixed splitting map s_A as in Definition 2.4, and let B be a unital C^*-algebra such that $T(B) \neq \emptyset$. Suppose that there are two unital homomorphisms $\varphi, \psi : A \to B$ such that $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(B)$.*

(1) *Suppose that $KK(\varphi) = KK(\psi)$. Then, there is a homomorphism $\delta : K_1(A) \to$ $\mathrm{Aff}(T(B))$ such that*

$$(\varphi^\ddagger - \psi^\ddagger) \circ s_A = \Sigma_B \circ \delta,$$

where $\Sigma_B : \mathrm{Aff}(T(B)) \to \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$ is the quotient map.

(2) *Suppose that $KL(\varphi) = KL(\psi)$. Let $K_1(A) = \bigcup_{n=1}^{\infty} G_n$, where $G_n \subset G_{n+1} \subset K_1(A)$ is a finitely generated subgroup. Then, for each n, there is a homomorphism $\delta_n : K_1(A) \to \mathrm{Aff}(T(B))$ such that*

$$(\varphi^{\ddagger} - \psi^{\ddagger}) \circ s_A|_{G_n} = \Sigma_B \circ \delta_n|_{G_n}.$$

Proof. Let $z \in K_1(A)$ be represented by a unitary $u \in M_n(A)$ for some integer $n \geq 1$. As before, we will continue to use φ and ψ for the extensions $\varphi \otimes \mathrm{id}_{M_n}$ and $\psi \otimes \mathrm{id}_{M_n}$, respectively. Then, $[\varphi(u)\psi(u)^*] = 0$ in $K_1(B)$. By replacing u by $u \oplus 1_k$ in M_{n+k} for some integer $k \in \mathbb{N}$ and n by $n + k$, without loss of generality, we may assume that $\varphi(u)\psi(u)^* \in U_0(M_n(B))$. It follows that there is a continuous and piecewise smooth path $\{v(t) : t \in [0, 1]\} \subset M_n(B)$ such that $v(0) = \varphi(u)\psi(u)^*$ and $v(1) = 1_{M_n(B)}$. Put $w(t) = v(t)\psi(v)$. Then, $w(0) = \varphi(u)$ and $w(1) = \psi(u)$.

Then, in $\mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))}$,

$$\begin{aligned}(\varphi^{\ddagger} - \psi^{\ddagger}) \circ s_A\big([z]\big) &= \frac{1}{2\pi i}\int_0^1 \tau\bigg(\frac{dv(t)}{dt}v^*(t)\bigg)dt + \overline{\rho_B\big(K_0(B)\big)} \\ &= \frac{1}{2\pi i}\int_0^1 \tau\bigg(\frac{dw(t)}{dt}w^*(t)\bigg)dt + \overline{\rho_B\big(K_0(B)\big)} \qquad \big(\tau \in T(B)\big).\end{aligned}$$

Let

$$M_{\varphi,\psi} = \big\{(b, a) \in C\big([0, 1], B\big) \oplus A : b(0) = \varphi(a),\ b(1) = \psi(a)\big\}$$

be the mapping torus. Since $\tau \circ \varphi = \tau \circ \psi$, as in 2.8,

$$R_{\varphi,\psi}\big(\big[w(t)\big]\big) = \frac{1}{2\pi i}\int_0^1 \tau\bigg(\frac{dw(t)}{dt}w^*(t)\bigg)dt$$

gives a homomorphism $R_{\varphi,\psi} : K_1(M_{\varphi,\psi}) \to \mathrm{Aff}(T(B))$.

If $KK(\varphi) = KK(\psi)$, as in [15, Definition 3.4], there is a splitting map $\theta : K_1(A) \to K_1(M_{\varphi,\psi})$ such that $\theta(z) - [w(t)] \in \iota_{*1}(K_0(B))$, where $\iota : B \to M_{\varphi,\psi}$ is the embedding (see also [15, Lemma 3.3]). Then,

$$R_{\varphi,\psi}\big(\theta(z) - \big[w(t)\big]\big) \in \rho_B\big(K_0(B)\big).$$

Define

$$\delta := R_{\varphi,\psi} \circ \theta : K_1(A) \to \mathrm{Aff}\big(T(B)\big).$$

One then has

$$(\varphi^{\ddagger} - \psi^{\ddagger}) \circ s_A(z) = \Sigma_B \circ \delta(z).$$

This proves case (i).

For case (ii), let $KL(\varphi) = KL(\psi)$. Then, for each n, there is a homomorphism $\theta_n : G_n \to K_1(M_{\varphi,\psi})$ such that $(\pi_e)_{*0} \circ \theta_n = \mathrm{id}_{G_n}$, where $\pi_e : M_{\varphi,\psi} \to A$ is the quotient map, $n = 1, 2, \ldots$. Since $\mathrm{Aff}(T(B))$ is divisible, there is $\delta_n : K_1(A) \to \mathrm{Aff}(T(B))$ such that $\delta_n|_{G_n} = R_{\varphi,\psi} \circ \theta_n$, $n = 1, 2, \ldots$. Note that if $z \in G_n$, then $\theta_n(z) - [w(t)] \in \iota_{*1}(K_0(B))$.

The computation above shows that

$$R_{\varphi,\psi}\big(\theta_n(z) - [w(t)]\big) \in \overline{\rho_B\big(K_0(B)\big)}.$$

It follows that

$$(\varphi^{\ddagger} - \psi^{\ddagger}) \circ s_A(z) = \Sigma_B \circ \delta_n(z).$$

This proves case (ii). ∎

Lemma 5.2 (cf. [18, Lemma 6.8]). *Let A and B be unital separable simple C^*-algebras such that A is finite and amenable and satisfies the UCT, and $gTR(B \otimes M_{\mathfrak{r}}) \le 1$ for any supernatural number $\mathfrak{r}$ of infinite type. Suppose also that B is $\mathcal{Z}$-stable. Let $\kappa \in KL_e(A, B)^{++}$ and $\lambda : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ an affine homomorphism, Then, there exists a unital homomorphism $\Psi : A \to B$ such that*

$$[\Psi] = \kappa, \quad (\Psi)_\sharp = \lambda.$$

Moreover, if $\gamma \in \bigcup_{n=1}^{\infty} U(M_n(A))/CU(M_n(A)) \to U(B)/CU(B)$ is a continuous homomorphism which is compatible with κ and λ, then one may also require that

$$\Psi^{\ddagger}|_{U(A)_0/CU(A)} = \gamma|_{U(A)_0/CU(A)}, \quad (\Psi)^{\ddagger} \circ s_A = \gamma \circ s_A - \bar{h},$$

where $s_A : K_1(A) \to U(A)/CU(A)$ is a splitting map (see Definition 2.4), and

$$\bar{h} : K_1(A) \to \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}$$

is a homomorphism.

Recall that B has stable rank one (see [23, Theorem 6.7]). By the last part of Definition 2.6, the map $\bar{u} \to \mathrm{diag}(\bar{u}, \bar{1}_m) : U(B)/CU(B) \to U(M_m(B))/CU(M_m(B))$ is an isomorphism.

In the following proof and rest of the paper, we will use E_D to denote the homomorphism $E_D : D \to D \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ defined by $d \mapsto d \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}$ for all $d \in D$ and for any C^*-algebra D.

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two relative prime supernatural numbers of infinite type such that $Q = M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. Let $A_{\mathfrak{p}} = A \otimes M_{\mathfrak{p}}$, $A_{\mathfrak{q}} = A \otimes M_{\mathfrak{q}}$, $B_{\mathfrak{p}} = B \otimes M_{\mathfrak{p}}$, and $B_{\mathfrak{q}} = B \otimes M_{\mathfrak{q}}$. Note, by the second part of Definition 2.10, $gTR(A_{\mathfrak{r}}) \le 1$ for any supernatural number $\mathfrak{r}$, and by the assumption, $gTR(B_{\mathfrak{r}}) \le 1$. Let $\kappa_{\mathfrak{r}} \in KL(A_{\mathfrak{r}}, B_{\mathfrak{r}})$, $\lambda_{\mathfrak{r}} : \mathrm{Aff}(T(A_{\mathfrak{r}})) \to \mathrm{Aff}(T(B_{\mathfrak{r}}))$, and $\gamma_{\mathfrak{r}} : U(A_{\mathfrak{r}})/CU(A_{\mathfrak{r}}) \to U(B_{\mathfrak{r}})/CU(B_{\mathfrak{r}})$ be induced by κ, λ, and γ, respectively (see [18, Lemma 6.1] for $\gamma_{\mathfrak{r}}$) for infinite supernatural number $\mathfrak{r}$, including the supernatural number ∞ (recall $M_\infty = Q$). Moreover, $M_{\mathfrak{r}} \cong M_{\mathfrak{r}} \otimes M_{\mathfrak{r}}$ for any supernatural number $\mathfrak{r}$ of infinite type. It follows from [7, Corollary 24.4] that there is a unital homomorphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ such that

$$[\varphi_{\mathfrak{p}}] = \kappa_{\mathfrak{p}} \text{ in } KL(A_{\mathfrak{p}}, B_{\mathfrak{p}}), \quad (\varphi_{\mathfrak{p}})^{\ddagger} = \gamma_{\mathfrak{p}}, \quad (\varphi_{\mathfrak{p}})_\sharp = \lambda_{\mathfrak{p}}. \tag{e 5.1}$$

For the same reason, there is also a unital homomorphism $\psi_{\mathfrak{q}} : A_{\mathfrak{q}} \to B_{\mathfrak{q}}$ such that

$$[\psi_{\mathfrak{q}}] = \kappa_{\mathfrak{q}} \text{ in } KL(A_{\mathfrak{q}}, B_{\mathfrak{q}}), \quad (\psi_{\mathfrak{q}})^{\ddagger} = \gamma_{\mathfrak{q}}, \quad (\psi_{\mathfrak{q}})_{\sharp} = \lambda_{\mathfrak{q}}. \tag{e 5.2}$$

Define $\varphi = \varphi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ and $\psi = \psi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}} : A \otimes Q \to B \otimes Q$. From above, one has that

$$[\varphi] = [\psi] \text{ in } KL(A \otimes Q, B \otimes Q), \quad \varphi_{\sharp} = \psi_{\sharp}, \quad \varphi^{\ddagger} = \psi^{\ddagger} = \gamma_{\infty}. \tag{e 5.3}$$

Since both $K_i(B \otimes Q)$ are divisible ($i = 0, 1$), one actually has

$$[\varphi] = [\psi] \quad \text{in } KK(A \otimes Q, B \otimes Q).$$

As in the proof of [7, Theorem 28.7] (see also [7, the proof of Theorem 28.3 and (e.28.6)]), there is $\beta \in \overline{\mathrm{Inn}}(\psi(A \otimes Q), B \otimes Q)$ with $KK(\beta) = KK(\imath_{\psi(A\otimes Q)})$ (where $\imath_{\psi(A\otimes Q)}$ is the embedding of $\psi(A \otimes Q)$ into $B \otimes Q$), $(\beta \circ \psi)_T = \psi_T$, $(\beta \circ \psi)^{\dagger} = \psi^{\dagger}$, and $\overline{R}_{\psi,\beta\circ\psi} = -\overline{R}_{\varphi,\psi}$. It follows that $\overline{R}_{\varphi,\beta\circ\psi} = 0$ (see also [7, the proof of Theorem 28.7]). Then, by [7, Theorem 27.5], φ and $\beta \circ \psi$ are asymptotically unitarily equivalent. Since $K_1(B \otimes Q)$ is divisible and $K_0(A \otimes Q)$ is torsion-free, $H_1(K_0(A \otimes Q), K_1(B \otimes Q)) = K_1(B \otimes Q)$ (see [7, Definition 28.10] for a notation). It follows that φ and $\beta \circ \psi$ are strongly asymptotically unitarily equivalent.

Note that one may identify $T(B_{\mathfrak{q}})$, $T(B_{\mathfrak{p}})$, and $T(B \otimes Q)$. Moreover,

$$\overline{\rho_{B\otimes Q}\big(K_0(B \otimes Q)\big)} = \overline{\mathbb{R}\rho_B\big(K_0(B)\big)} = \overline{\rho_{B_{\mathfrak{q}}}\big(K_0(B_{\mathfrak{q}})\big)}.$$

Denote by $\imath_{\mathfrak{p}} : B_{\mathfrak{q}} \to B \otimes Q$ the embedding $a \mapsto a \otimes 1_{\mathfrak{p}}$ (where $1_{\mathfrak{r}} := 1_{M_{\mathfrak{r}}}$), and note that the image of $\imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}$ is in the image of ψ. Thus, by [18, Lemma 3.5], $R_{\beta\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}},\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}}$ is

$$\mathrm{Hom}\,\big(K_1(M_{\beta\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}},\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}}), \overline{\rho_{B_{\mathfrak{q}}}\big(K_0(B_{\mathfrak{q}})\big)}\big).$$

Note that

$$[\beta \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}] = [\imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}] \quad \text{in } KK(A_{\mathfrak{q}}, B_{\mathfrak{q}}).$$

By [7, Theorem 28.3], there exists $\alpha \in \overline{\mathrm{Inn}}(\psi_{\mathfrak{q}}(A_{\mathfrak{q}}), B_{\mathfrak{q}})$ such that

$$[\alpha] = [\imath_{\psi_{\mathfrak{q}}(A_{\mathfrak{q}})}] \quad \text{in } KK\big(\psi_{\mathfrak{q}}(A_{\mathfrak{q}}), B_{\mathfrak{q}}\big), \tag{e 5.4}$$

where $\imath_{\psi_{\mathfrak{q}}(A_{\mathfrak{q}})}$ is the embedding of $\psi_{\mathfrak{q}}(A_{\mathfrak{q}})$ into $B_{\mathfrak{q}}$, and

$$\overline{R}_{\alpha,\imath_{\psi_{\mathfrak{q}}(A_{\mathfrak{q}})}} = -\overline{R}_{\beta\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}},\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}}.$$

One computes (just as [18, Lemma 6.5]) that

$$[\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q] = [\beta \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}] \quad \text{in } KK(A_q, B \otimes Q),$$
$$(\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q)_{\sharp} = (\beta \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}})_{\sharp}, \quad (\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q)^{\ddagger} = (\beta \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}})^{\ddagger},$$

and

$$\overline{R}_{\imath_{\mathfrak{p}}\circ\alpha\circ\psi_q,\beta\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}} = 0.$$

It follows from [7, Theorems 27.5 and 28.13] that $\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q$ and $\beta \circ \imath_{\mathfrak{p}} \circ \psi_q$ are strongly asymptotically unitarily equivalent.

We will show that $\beta \circ \psi$ and $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. Define $\beta_1 = (\beta \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}} : A \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{p}} \to B \otimes Q \otimes M_{\mathfrak{p}}$. Let $j : Q \to Q \otimes M_{\mathfrak{p}}$ be defined by $j(b) = b \otimes 1_{\mathfrak{p}}$. Consider the C^*-subalgebra

$$C = \beta \circ \psi(1_{A \otimes M_{\mathfrak{q}}} \otimes M_{\mathfrak{p}}) \otimes M_{\mathfrak{p}} = \beta(1_{B_{\mathfrak{q}}} \otimes M_{\mathfrak{p}}) \otimes M_{\mathfrak{p}} \subset B \otimes Q \otimes M_{\mathfrak{p}}. \qquad \text{(e 5.5)}$$

Note $\psi(1_{A\otimes M_{\mathfrak{q}}}) = 1_{B_{\mathfrak{q}}}$ and $\psi(1_{A\otimes M_{\mathfrak{q}}} \otimes M_{\mathfrak{p}}) = 1_{B_{\mathfrak{q}}} \otimes M_{\mathfrak{p}}$. Since $K_1(C) = \{0\}$, in C, $(\mathrm{id}_B \otimes j) \circ (\beta|_{1_{B_{\mathfrak{q}}} \otimes M_{\mathfrak{p}}})$ and j_0 are strongly asymptotically unitarily equivalent, where $j_0 : M_{\mathfrak{p}} \to C$ is defined by $j_0(a) = 1_{B\otimes Q} \otimes a$ for all $a \in M_{\mathfrak{p}}$. In particular, there exists a continuous path of unitaries $\{v(t) : t \in [0, 1)\} \subset C$ such that

$$\lim_{t\to 1} \mathrm{Ad}\, v(t) \circ (\mathrm{id}_B \otimes j) \circ (\beta \circ \psi)(1_{A_{\mathfrak{q}}} \otimes a) = 1_{B\otimes Q} \otimes a \quad \text{for all } a \in M_{\mathfrak{p}}.$$

Note that, for $a_q \in A_{\mathfrak{q}}$, $(\mathrm{id}_B \otimes j)(\beta \circ \psi(a_q \otimes 1_{\mathfrak{p}})) = \beta(\psi_q(a_q) \otimes 1_{\mathfrak{p}}) \otimes 1_{\mathfrak{p}}$. Then, by (e 5.5), $v(t)$ commutes with $(\mathrm{id}_B \otimes j)(\beta \circ \psi(a_q \otimes 1_{\mathfrak{p}}))$. It follows that $(\mathrm{id}_B \otimes j)\circ\beta \circ \psi$ and β_1 are strongly asymptotically unitarily equivalent. Since $\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q$ and $\beta \circ \imath_{\mathfrak{p}} \circ \psi_q$ are strongly asymptotically unitarily equivalent, one concludes that $(\mathrm{id}_B \otimes j) \circ \beta \circ \psi$ and $(\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. There is a homomorphism $\theta : Q \otimes M_{\mathfrak{p}} \to Q$ such that $\theta \circ j : Q \to Q$ is strongly asymptotically unitarily equivalent to id_Q. Consequently, $(\mathrm{id}_B \otimes\theta) \circ (\mathrm{id}_B \otimes j) \circ \beta \circ \psi$ is strongly asymptotically unitarily equivalent $\beta \circ \psi$, and $(\mathrm{id}_B \otimes\theta) \circ ((\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q) \otimes \mathrm{id}_{M_{\mathfrak{p}}})$ is strongly asymptotically unitarily equivalent $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$. Therefore, $\beta \circ \psi$ and $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent.

Finally, we conclude that $(\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}$ and φ are strongly asymptotically unitarily equivalent. Note that, by (e 5.4), $\alpha \circ \psi_{\mathfrak{q}}$ is an isomorphism which induces $\Gamma_{\mathfrak{q}}$.

Thus, there is a continuous path of unitaries $\{u(t) : t \in [0, 1)\}$ in $B \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ (it can be made piecewise smooth—see [15, Lemma 4.1]) such that $u(0) = 1$ and

$$\lim_{t\to 1} \mathrm{ad}\, u(t) \circ \varphi(a) = \big((\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}\big)(a) \quad \text{for all } a \in A \otimes Q.$$

Note that if $a \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, then $a(0) \in A \otimes M_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$, $\varphi(a(0)) \in B_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$, $a(1) \in A \otimes M_{\mathfrak{q}} \otimes 1_{\mathfrak{p}}$, and

$$\big((\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}\big)\big(a(1)\big) \in B_{\mathfrak{q}} \otimes 1_{\mathfrak{p}}.$$

This provides a unital homomorphism $\Phi : A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that, for each $t \in (0, 1)$,

$$\pi_t \circ \Phi(a) = \mathrm{ad}\, u(t) \circ \varphi\big(a(t)\big) \quad \text{for all } a \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}.$$

Denote by C_k a commutative C^*-algebra with $K_0(C_k) = \mathbb{Z}/k\mathbb{Z}$ and $K_1(C_k) = \{0\}$, $2, 3, \ldots,$ and $C_0 = \mathbb{C}$. Therefore, one identifies $K_i(A \otimes C_k)$ with $K_i(A, \mathbb{Z}/k\mathbb{Z})$ $(i = 0, 1)$.

Note that $[E_A] : \underline{K}(A) \to \underline{K}(A \otimes Z_{\mathfrak{p},\mathfrak{q}})$ is an isomorphism in

$$\mathrm{Hom}_\Lambda\left(\underline{K}(A), \underline{K}(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})\right)$$

(recall $K_0(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}) = \mathbb{Z}$ and $K_1(Z_{\mathfrak{p},\mathfrak{q}}) = \{0\}$). Denote by $[E_A]^{-1}$ the inverse of $[E_A]$ and $\kappa^Z \in KL(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}, B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$ the composition $[E_B] \circ \kappa \circ [E_A]^{-1}$. One computes, applying the Künneth formula, that

$$\kappa^Z\big(g \otimes [1_{Z_{\mathfrak{p},\mathfrak{q}}}]_0\big) = \kappa(g) \otimes [1_{Z_{\mathfrak{p},\mathfrak{q}}}]_0$$

for all $g \in K_i(A \otimes C_k)$, $k = 0, 2, \ldots$ and $i = 0, 1$.

Thus, we have the commutative diagrams

$$\begin{array}{ccccc} K_i(A \otimes C_k) & \xrightarrow{[E_{A\otimes C_k}]} & K_i(A \otimes C_k \otimes Z_{\mathfrak{p},\mathfrak{q}}) & \xrightarrow{[\pi_e]} & K_i(A \otimes C_k \otimes M_{\mathfrak{p}}) \oplus K_i(A \otimes C_k \otimes M_{\mathfrak{q}}) \\ \downarrow{\scriptstyle \kappa|_{K_0(A,\mathbb{Z}/k\mathbb{Z})}} & & \downarrow{\scriptstyle \kappa^Z|_{K_i(A\otimes C_k\otimes Z_{\mathfrak{p},\mathfrak{q}})}} & & \downarrow{\scriptstyle \kappa_{\mathfrak{p}}\oplus\kappa_{\mathfrak{q}}} \\ K_i(B \otimes C_k) & \xrightarrow{[E_{B\otimes C_k}]} & K_i(B \otimes C_k \otimes Z_{\mathfrak{p},\mathfrak{q}}) & \xrightarrow{[\pi_e]} & K_i(B \otimes C_k \otimes M_{\mathfrak{p}}) \oplus K_i(B \otimes C_k \otimes M_{\mathfrak{q}}), \end{array} \tag{e 5.6}$$

$$\begin{array}{ccc} K_i(A \otimes C_k \otimes Z_{\mathfrak{p},\mathfrak{q}}) & \xrightarrow{[\pi_e]} & K_i(A \otimes C_k \otimes M_{\mathfrak{p}}) \oplus K_i(A \otimes C_k \otimes M_{\mathfrak{q}}) \\ \downarrow{\scriptstyle [\Phi]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})}} & & \downarrow{\scriptstyle [\varphi_{\mathfrak{p}}]\oplus[\psi_{\mathfrak{q}}]} \\ K_i(B \otimes C_k \otimes Z_{\mathfrak{p},\mathfrak{q}}) & \xrightarrow{[\pi_e]} & K_i(B \otimes C_k \otimes M_{\mathfrak{p}}) \oplus K_i(B \otimes C_k \otimes M_{\mathfrak{q}}) \end{array} \tag{e 5.7}$$

(recall that $E_D : D \to D \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is defined by $E_D(d) = d \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}$), where

$$\pi_e : D \otimes Z_{\mathfrak{p},\mathfrak{q}} \to (D \otimes M_{\mathfrak{p}}) \oplus D \otimes M_{\mathfrak{q}}$$

denotes the quotient map (for $D = A \otimes C_k$ and $D = B \otimes C_k$). Recall, by (e 5.1) and (e 5.2), $[\varphi_{\mathfrak{p}}] = \kappa_{\mathfrak{p}}$ and $[\psi_{\mathfrak{q}}] = \kappa_{\mathfrak{q}}$. Note (since $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime) that $[\pi_e]$ is injective (see [26, Proposition 5.2]) and $[j_D]$ is an isomorphism. Therefore, from the commutative diagrams (e 5.6) and (e 5.7), one concludes that $KL(\Phi) = \kappa^Z$.

Let $\eta : Z_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}$ be the unital embedding given by [24, Proposition 3.3]. Define $\Psi : A \to B \otimes \mathcal{Z}$ by $(\mathrm{id}_B \otimes \eta) \circ \Phi \circ E_A$. Note $[\mathrm{id}_B \otimes \eta] = [E_B]^{-1}$. Then, Ψ is a unital homomorphism such that $KL(\Psi) = [E_B]^{-1} \circ \kappa^Z \circ E_A = \kappa$. For each t, and $\tau \in T(B)$, $\tau(\Phi_t(a)) = \lambda(\hat{a})(\tau)$ for all $a \in A$. One then checks that $\tau(\Psi(a)) = \lambda(\hat{a})(\tau)$ for all $a \in A_{\mathrm{s.a.}}$ and all $\tau \in T(B)$. In fact, one has that

$$\Phi_\sharp(a \otimes b)(\tau \otimes \mu) = \lambda\big(a(\tau)\big)\mu(b) \quad \text{for all } a \in A_{\mathrm{s.a.}},\ b \in (\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})_{\mathrm{s.a.}} \tag{e 5.8}$$

for any $\tau \in T(B)$ and $\mu \in T(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$.

Note that it follows from (e 5.8) that

$$(\Phi \circ E_A)^\ddagger|_{U_0(A)/CU(A)} = E_B^\ddagger \circ \gamma|_{U_0(A)/CU(A)}. \tag{e 5.9}$$

Then, one has, for $t \in (0, 1)$,

$$(\pi_t \circ \Phi \circ E_A)^\ddagger = \gamma_Q = \imath_Q^\ddagger \circ \gamma,$$

where $\imath_Q : B \to B \otimes Q$ is defined by

$$\imath_Q(a) = a \otimes 1_Q, \quad \gamma_Q : U(A \otimes Q)/CU(A \otimes Q) \to U(B \otimes Q)/CU(B \otimes Q)$$

(see [18, Lemma 6.1]). On the other hand, for each $z \in U(M_k(A))/CU(M_k(A))$ for some integer $k \geq 1$, let $w_0 \in U(B)$ be such that its image $\bar{w}_0 = \gamma(z)$. Put $w_1 = w_0 \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}} \in B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ and $w = \mathrm{diag}(w_1, 1_{k-1})$.

In what follows, we will use H for $H \otimes \mathrm{id}_{M_k}$ (for a map H) and $U(t)$ for $U(t) \otimes 1_{M_k}$; in particular, this includes the case $H = \varphi$.

Then,

$$\pi_t(w) = \pi_{t'}(w) \text{ for all } t, t' \in [0,1], \quad E_B^{\ddagger} \circ \gamma(z) = \bar{w}.$$

Since $\pi_t(w) \in B$ is constant, one may use w for its evaluation at t. Let $v_0 \in U(M_k(A))$ be such that $\overline{v_0} = z$.

Let $Z = \Phi(E_A(v_0))w^*$. Then, for any $t \in (0,1)$,

$$Z(t) = \pi_t \circ \Phi\big(E_A(v_0)\big)w^* = u(t)^*\varphi(v_0)u(t)w^*.$$

Since $(\kappa, \lambda, \gamma)$ is compatible, in $K_1(B \otimes Z_{\mathfrak{p},\mathfrak{q}})$,

$$\begin{aligned}
[Z] &= \big[\Phi\big(E_A(v_0)\big)w^*\big] \\
&= \big[\kappa^Z\big(E_A(v_0)\big)\big][w_0^* \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}] \\
&= \big[\kappa\big([v_0]\big) \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}\big][w_0^* \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}] = 0.
\end{aligned}$$

It follows that $\mathrm{diag}(Z, 1_m) \in U_0(M_{m+1}(M_k(B) \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))$. Let $Z_1(t,s)$ be a piecewise smooth continuous path of unitaries in $U_0(M_{m+1}(M_k(B) \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))$ such that $Z_1(t,0) = Z_1(t)$ and $Z_1(t,1) = 1$. Denote by τ_0 the unique tracial state in $\mathrm{T}(Q)$, where $\mathfrak{r}$ is a supernatural number. For each $s_\mu \in T(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$, one may write

$$s_\mu(a) = \int_0^1 \tau_0\big(a(t)\big)d\mu(t),$$

where μ is a probability Borel measure on $[0,1]$.

To apply [18, Lemma 6.6], put

$$\begin{aligned}
V(t) &= \mathrm{diag}\,\big(u(t), 1_m\big), \\
\varphi^{(m+1)}(a) &= \mathrm{diag}\,\big(\varphi(a), \varphi(a), \ldots, \varphi(a)\big), \\
w_1 &= \mathrm{diag}\,\big(w, \varphi(v_0), \ldots, \varphi(v_0)\big)
\end{aligned}$$

as well as (for $a \in A$)

$$\psi^{(m+1)}(a) = \mathrm{diag}\,\big(\big((\alpha \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}}\big)(a), \varphi(a), \ldots, \varphi(a)\big).$$

Then, $Z_1(t) = V(t)^*\varphi^{(m+1)}(v_0)V(t)w_1^*$ for all $t \in [0,1)$, and

$$\lim_{t \to 1} V(t)^*\varphi^{(m+1)}(v_0)V(t)w_1^* = \mathrm{diag}\,\big(u(t)^*\varphi(v_0)u(t)w^*, 1_m\big).$$

Then, for $\tau \in T(B)$ and $s_\mu \in T(Z_{\mathfrak{p},\mathfrak{q}})$, by applying [18, Lemma 6.6],

$$\begin{aligned}
&\mathrm{Det}(Z_1)(\tau \otimes s_\mu) && \text{(e 5.10)}\\
&= \frac{1}{2\pi\sqrt{-1}}\int_0^1 (\tau\otimes s_\mu)\Big(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*\Big)ds\\
&= \frac{1}{2\pi\sqrt{-1}}\int_0^1\int_0^1 (\tau\otimes \tau_0)\Big(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*\Big)d\mu(t)ds\\
&= \int_0^1\Big(\frac{1}{2\pi\sqrt{-1}}\int_0^1 (\tau\otimes \tau_0)\Big(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*\Big)ds\Big)d\mu(t)\\
&= \int_0^1 \mathrm{Det}\,\big(\varphi(v_0)w_0^*\big)(\tau)d\mu(t) + f(\tau) \quad \text{for some } f\in\rho_B\big(K_0(B\otimes Q)\big)\\
&= \mathrm{Det}\,\big(\varphi(v_0)w_0^*\big)(\tau) + f(\tau), && \text{(e 5.11)}
\end{aligned}$$

where μ is a Borel probability measure on [0, 1] associated with s_μ. Note that if $p, q \in M_n(B\otimes Q)$ are two projections, then there are projections $p_0, q_0 \in M_m(B)$ (for some integer m) and $r \in Q$ such that $[p]-[q] = r([p_0]-[q_0])$. By (e 5.3), for any $\varepsilon > 0$, there are projections $p_0, q_0 \in M_m(B)$ and $r \in \mathbb{Q}$ such that

$$\sup\big\{\big|g(\tau) - r\big(\tau(p)-\tau(q)\big)\big| : \tau\in T(B)\big\} < \varepsilon,$$

where $g(\tau) = \mathrm{Det}(\varphi(v_0)w_0^*)(\tau)$ for all $\tau \in T(B)$. Put $p_1 = p_0 \otimes 1_{Z_{\mathfrak{p},\mathfrak{q}}}$ and $q_1 = q_0\otimes 1_{Z_{\mathfrak{p},\mathfrak{q}}}$. By (e 5.11), for $g_1(\tau\otimes s_\mu) = \mathrm{Det}(Z_1)(\tau\otimes s_\mu)$,

$$\big|g_1(\tau\otimes s_\mu) - r\big((\tau\otimes s_\mu)(p_1) + (\tau\otimes s_\mu)(q_1)\big)\big| < \varepsilon$$

for all $\tau\in T(B)$ and $s_\mu \in T(Z_{\mathfrak{p},\mathfrak{q}})$.

Therefore, the map

$$T(B\otimes Z_{\mathfrak{p},\mathfrak{q}}) \ni \tau\otimes s_\mu \mapsto \mathrm{Det}(Z_1)(\tau\otimes s_\mu) = \mathrm{Det}\,\big(\varphi(v_0)w_0^*\big)(\tau) + f(\tau)$$

defines an element in $\overline{\mathbb{R}\rho_B(K_0(B))} \subseteq \mathrm{Aff}(T(B\otimes Z_{\mathfrak{p},\mathfrak{q}}))$.

Thus,

$$(\Phi\circ E_A)^{\ddagger}(z)\big(E_B\circ\gamma(z)^*\big)$$

defines a homomorphism from the group $U(A)/CU(A)$ into $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ which will be denoted by h_0. By (e 5.9),

$$h_0|_{U_0(A)/CU(A)} = 0.$$

Thus, $-h_0$ induces a homomorphism $\bar{h} : K_1(A) \to \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$. Since all unital endomorphisms on $\mathcal{Z}$ are approximately inner (see [10, Theorem 7.6]),

$$\begin{aligned}
\Psi^{\ddagger}\big(s_A(x)\big)\gamma\big(s_A(x)\big)^* &= \big(\big((\iota\otimes\eta)\circ\Phi\circ E_A\big)^{\ddagger}\big(s_A(x)\big)\big)\gamma\big(s_A(x)^{-1}\big)\\
&= -\bar{h}(x) \quad \text{for all } x\in K_1(A).
\end{aligned}$$

In other words,

$$\Psi^{\ddagger}\circ s_A = \gamma\circ s_A - \bar{h}. \qquad \blacksquare$$

Lemma 5.3. *Let A and B be two unital separable simple C^*-algebras such that A is finite and amenable and satisfies the UCT, and $gTR(B \otimes M_{\mathfrak{r}}) \le 1$ for any supernatural number $\mathfrak{r}$ of infinite type. Suppose that B is $\mathcal{Z}$-stable. Let $\psi : A \to B$ be a unital homomorphism. Suppose that*

$$\bar{h} \in \mathrm{Hom}\left(K_1(A), \overline{\mathbb{R}\rho_B\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}\right)$$

such that there exists $h \in \mathrm{Hom}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))})$ with $\bar{h} = \Sigma_B \circ h$. Then, there exists a homomorphism $\varphi : A \to B$ such that

$$KL(\psi) = KL(\varphi), \quad \psi_T = \varphi_T, \quad (\psi^\ddagger - \varphi^\ddagger) \circ s_A = \bar{h}.$$

Proof. First, recall, by the second part of Definition 2.10, that $gTR(A \otimes M_{\mathfrak{r}}) \le 1$ for any supernatural number $\mathfrak{r}$ of infinite type. Fix a splitting map

$$s_A : K_1(A) \to U\left(M_\infty(A)\right)/CU\left(M_\infty(A)\right)$$

as defined in Definition 2.4. Let $\gamma : U(M_\infty(A))/CU(M_\infty(A)) \to U(B)/CU(B)$ be homomorphism such that

$$\gamma|_{\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}} = \psi^\ddagger|_{\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}}, \quad \gamma \circ s_A = \psi^\ddagger \circ s_A + \bar{h} \circ s_A.$$

Therefore,

$$(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})^\ddagger \circ E_A^\ddagger \circ s_A = E_B^\ddagger \circ \gamma \circ s_A - \bar{h}, \quad \Pi_B^{cu} \circ \gamma \circ s_A = \psi_{*1}. \tag{e 5.12}$$

In what follows, we will identify $T(B)$ with $T(B \otimes M_{\mathfrak{r}})$ whenever it is necessary. There is a homomorphism $h_{\mathfrak{r}} : K_1(A \otimes M_{\mathfrak{r}}) \to \overline{\rho_B(K_0(B \otimes M_{\mathfrak{r}}))} = \overline{\mathbb{R}\rho_B(K_0(B))}$ such that

$$h = h_{\mathfrak{r}} \circ (\iota_{A,\mathfrak{r}})_{*1}, \tag{e 5.13}$$

where $\iota_{A,\mathfrak{r}} : A \to A \otimes M_{\mathfrak{r}}$ is the embedding so that $\iota_{A,\mathfrak{r}}(a) = a \otimes 1_{\mathfrak{r}}$ for all $a \in A$ ($\mathfrak{r}$ is a supernatural number, including ∞ which corresponds to $\mathbb{Q}$).

Choose a pair of relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ of infinite type. We also require that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} = Q$. Put $A'_{\mathfrak{r}} = (\psi \otimes \mathrm{id}_{M_{\mathfrak{r}}})(A \otimes M_{\mathfrak{r}})$, where $\mathfrak{r}$ is a supernatural number.

It follows from [7, Theorem 28.3] that there is a monomorphism $\beta_0 \in \overline{\mathrm{Inn}}(A'_{\mathfrak{p}}, B_{\mathfrak{p}})$ such that

$$[\beta_0] = [\iota_{A'_{\mathfrak{p}}}] \text{ in } KK(A'_{\mathfrak{p}}, B_{\mathfrak{p}}), \quad (\beta_0)_\sharp = \iota_{A'_{\mathfrak{p}}\sharp}, \quad \beta_0^\ddagger = \iota_{A'_{\mathfrak{p}}}{}^\ddagger, \tag{e 5.14}$$

$$\overline{R}_{\mathrm{id}_{A'_{\mathfrak{p}}},\beta_0} = h_{\mathfrak{p}} + (\rho_{B_{\mathfrak{p}}} \circ f_{\mathfrak{p}}), \tag{e 5.15}$$

where $\iota_{A'_{\mathfrak{p}}}$ is the embedding of $A'_{\mathfrak{p}}$ and $f_{\mathfrak{p}} \in \mathrm{Hom}(K_1(A_{\mathfrak{p}}), K_0(B \otimes M_{\mathfrak{p}}))$. Recall, here,

$$\overline{R}_{\mathrm{id}_{A'_{\mathfrak{p}}},\beta_0} \in \mathrm{Hom}\left(K_1(A_{\mathfrak{p}}), \mathrm{Aff}\left(T(B_{\mathfrak{p}})\right)\right)/\mathcal{R}_0,$$

where $\mathcal{R}_0$ is the subgroup of those $\lambda \in \mathrm{Hom}(K_1(A_{\mathfrak{p}}), \mathrm{Aff}(T(B_{\mathfrak{p}})))$ such that

$$\lambda_0 \in \mathrm{Hom}\left(K_1(A_{\mathfrak{p}}), K_0(B_{\mathfrak{p}})\right)$$

and $\lambda = \rho_{B_\mathfrak{p}} \circ \lambda_0$ (see [15, Definition 3.4]). Put

$$f'_\mathfrak{p} := \rho_{B_\mathfrak{p}} \circ f_\mathfrak{p} \in \mathrm{Hom}\big(K_1(A_\mathfrak{p}), \rho_{B_\mathfrak{p}}\big(K_0(B_\mathfrak{p})\big)\big).$$

Denote by $\widetilde{\psi}_\mathfrak{r} : A \to B_\mathfrak{r}$ the map defined by

$$\widetilde{\psi}_\mathfrak{r}(a) = (\psi \otimes \mathrm{id}_{M_\mathfrak{r}})(a \otimes 1_{M_\mathfrak{r}}) \quad \text{for all } a \in A,\ \mathfrak{r} = \mathfrak{p}, \mathfrak{q}.$$

Thus,

$$\overline{R}_{\iota_\mathfrak{p}\circ\widetilde{\psi}_\mathfrak{p}, \iota_\mathfrak{p}\circ\beta_0\circ\widetilde{\psi}_\mathfrak{p}} = h + \big(f'_\mathfrak{p} \circ (\iota_{A,\mathfrak{p}})_{*1}\big), \tag{e 5.16}$$

where $\iota_\mathfrak{p} : B_\mathfrak{p} \to B \otimes Q$ is the embedding defined by $\iota_\mathfrak{p}(b) = b \otimes 1_{M_\mathfrak{q}}$. Note that

$$\iota_\mathfrak{p} \circ (\psi \otimes \mathrm{id}_{M_\mathfrak{p}}) = \psi \otimes \mathrm{id}_Q\,.$$

Similarly, there is a monomorphism $\beta_1 \in \overline{\mathrm{Inn}}(A'_\mathfrak{q}, B_\mathfrak{q})$ such that

$$[\beta_1] = [\iota_{A'_\mathfrak{q}}] \text{ in } KK(A'_\mathfrak{q}, B_\mathfrak{q}), \quad (\beta_1)_\sharp = {\iota_{A'_\mathfrak{q}}}_\sharp, \quad \beta_1^\ddagger = {\iota_{A'_\mathfrak{q}}}^\ddagger,$$
$$\overline{R}_{\iota_\mathfrak{q}\circ\widetilde{\psi}_\mathfrak{q}, \iota_\mathfrak{q}\circ\beta_1\circ\widetilde{\psi}_\mathfrak{q}} = h + f'_\mathfrak{q} \circ (\iota_{A,\mathfrak{q}})_{*1},$$

where $\iota_\mathfrak{q} : B_\mathfrak{q} \to B \otimes Q$ is the embedding defined by $\iota_\mathfrak{q}(b) = b \otimes 1_{M_\mathfrak{p}}$, and $f'_\mathfrak{q} := \rho_{B_\mathfrak{q}} \circ f_\mathfrak{q}$ for some $f_\mathfrak{q} \in \mathrm{Hom}(K_1(A_\mathfrak{q}), K_0(B_\mathfrak{q}))$.

Denote $\psi_0 = \iota_\mathfrak{p} \circ \beta_0 \circ (\psi \otimes \mathrm{id}_{M_\mathfrak{p}})$ and $\psi_1 = \iota_\mathfrak{q} \circ \beta_1 \circ (\psi \otimes \mathrm{id}_{M_\mathfrak{q}})$. Consider

$$\psi_0 \otimes \mathrm{id}_{M_\mathfrak{q}} : A_\mathfrak{p} \otimes M_\mathfrak{q} (= A \otimes Q) \to B \otimes Q \otimes M_\mathfrak{q} (= B \otimes Q),$$
$$\psi_1 \otimes \mathrm{id}_{M_\mathfrak{p}} : A_\mathfrak{q} \otimes M_\mathfrak{p} (= A \otimes Q) \to B \otimes Q \otimes M_\mathfrak{p} (= B \otimes Q).$$

We have

$$KK(\psi_0 \otimes \mathrm{id}_{M_\mathfrak{q}}) = KK(\psi_1 \otimes \mathrm{id}_{M_\mathfrak{p}}), \quad (\psi_0 \otimes \mathrm{id}_{M_\mathfrak{q}})_\sharp = (\psi_1 \otimes \mathrm{id}_{M_\mathfrak{q}})_\sharp$$

and

$$(\psi_0 \otimes \mathrm{id}_{M_\mathfrak{q}})^\ddagger = (\psi_1 \otimes \mathrm{id}_{M_\mathfrak{q}})^\ddagger.$$

We also compute that

$$\overline{R}_{\psi_0\otimes\mathrm{id}_{M_\mathfrak{q}}, \psi_1\otimes\mathrm{id}_{M_\mathfrak{p}}} = \overline{R_{\psi_0\otimes\mathrm{id}_{M_\mathfrak{q}}, \psi\otimes\mathrm{id}_Q}} + \overline{R_{\psi\otimes\mathrm{id}_Q, \psi_1\otimes\mathrm{id}_{M_\mathfrak{q}}}} = -h_\infty + h_\infty = \bar{0}.$$

It follows from [7, Theorems 27.5 and 28.13] that there is a continuous path of unitaries $\{U(t) : t \in [0, 1)\} \subset U(B \otimes Q)$ with $U(0) = 0$ such that

$$\lim_{t\to 1} U(t)^*(\psi_0 \otimes \mathrm{id}_{M_\mathfrak{q}})(a)U(t) = (\psi_1 \otimes \mathrm{id}_{M_\mathfrak{p}})(a).$$

By [15, Lemma 4.1], we may also assume that $\{U(t) : t \in [0, 1)\}$ is piecewise smooth.

Let $\Phi : A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be defined by

$$\Phi(a \otimes b)(t) = U^*(t)\big((\psi_0 \otimes \mathrm{id}_{M_\mathfrak{q}})\big(a \otimes b(t)\big)\big)U(t) \quad \text{for all } t \in [0, 1),$$
$$\Phi(a \otimes b)(1) = \psi_1 \otimes \mathrm{id}_{M_\mathfrak{p}}\,\big(a \otimes b(1)\big),$$

for all $a \otimes b \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. Exactly the same argument in the proof of Lemma 5.2 around (e 5.6) and (e 5.7) shows that

$$KL(\Phi) = [E_B] \circ KL(\psi) \circ [E_A]^{-1}. \tag{e 5.17}$$

We claim that

$$\Phi^\ddagger \circ E_A^\ddagger \circ s_A = (E_B)^\ddagger \circ \gamma \circ s_A. \tag{e 5.18}$$

To compute $\Phi^\ddagger$, let $x \in s_A(K_1(A))$ and $v_0 \in U(M_k(A))$ (for some integer $k \geq 1$) such that $\overline{v_0} = x$. Let $w_0 \in U(B)$ such that its image $\bar{w}_0 = \gamma(x)$. Put $w_1 = w_0 \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}} \in B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. Then, $w(t) = w(t')$ for all $t, t' \in [0,1]$ and

$$E_B^\ddagger \circ \gamma \circ s_A(x) = \bar{w}.$$

Put $w := \mathrm{diag}(w, 1_{k-1})$. In what follows, we will use H for $H \otimes \mathrm{id}_{M_k}$ (for a map H) and $U(t)$ for $U(t) \otimes 1_{M_k}$. Let $Z = (\Phi \circ E_A(v_0))w^* \in M_k(B) \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. By the second part of (e 5.12) and by (e 5.17), $[Z] = 0$. Suppose that there is a piecewise smooth continuous path $\{Z_1(t,s) : s \in [0,1]\} \subset M_{m+1}(M_k(B) \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$ such that $Z_1(t,0) = \mathrm{diag}(Z(t), 1_m)$ and $Z_1(t,1) = 1_{m+1}$. Then, in $\mathrm{Aff}(T(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))/\rho_{B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(K_0(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))$, by (e 5.12) and $\bar{w}_0 = \gamma(x)$,

$$\begin{aligned} &\mathrm{Det}\big(Z_1(t,s)\big) \\ &= \mathrm{Det}\big(Z_1(t,s)\big(w\big(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}\big(E_A(v_0)^*\big)\big)\big)\big) + \mathrm{Det}\big((\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})\big(E_A(v_0)\big)\big(E_B(w_0^*)\big)\big) \\ &= \mathrm{Det}\big(Z_1(t,s)\big(w\big(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}\big(E_A(v_0)^*\big)\big)\big)\big) + h \circ \big([v_0]\big), \end{aligned} \tag{e 5.19}$$

where we identify $T(B)$ with $T(B \otimes Q)$, and $(h \circ s_A(x))(\tau \otimes \delta_t) = h([v_0])(\tau)$ for all $\tau \in T(B \otimes Q)$ and $t \in [0,1]$. By (e 5.15) (see also (e 5.16)), there is a continuous and piecewise smooth path $\{z(t) : t \in [0,1]\}$ in $U_0(M_k(B_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) \otimes 1_{M_{\mathfrak{q}}})$ such that

$$z(0) = \big(\beta_0\big(\psi(v_0) \otimes 1_{M_{\mathfrak{p}}}\big) \otimes 1_{M_{\mathfrak{q}}}\big)\big(\big(\psi(v_0) \otimes 1_{M_{\mathfrak{p}}}\big) \otimes 1_{M_{\mathfrak{q}}}\big), \quad z(1) = 1,$$

and

$$\frac{1}{2\pi i}\int_0^1 \tau\left(\frac{dz(s)}{ds}z(s)^*\right)ds = -h\big([v_0]\big)(\tau) + \big(f_{\mathfrak{p}}'(x)\big)(\tau) \quad \text{for all } \tau \in T(A \otimes M_{\mathfrak{p}}).$$

Define $Z_2(t,s) = Z_1(t,s)(w(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(v_0)^*)(E_A(v_0)^*))$. We also have

$$\begin{aligned} Z_2(t,0) &= \mathrm{diag}\,\big(U^*(t)\big(\big(\beta_0\big(\psi(v_0) \otimes 1_{M_{\mathfrak{p}}}\big)\big) \otimes 1_{M_{\mathfrak{q}}}\big)U(t)\big(\psi(v_0) \otimes 1_Q\big)^*, 1_m\big), \\ Z_2(0,0) &= \mathrm{diag}\,\big(\big(\big(\beta_0\big(\psi(v_0) \otimes 1_{M_{\mathfrak{p}}}\big)\big) \otimes 1_{M_{\mathfrak{q}}}\big)\big(\psi(v_0) \otimes 1_Q\big)^*, 1_m\big), \\ Z_2(1,0) &= \mathrm{diag}\,\big(\big(\big(\beta_1\big(\psi(v_0) \otimes 1_{M_{\mathfrak{q}}}\big)\big) \otimes 1_{M_{\mathfrak{p}}}\big)\big(\psi(v_0) \otimes 1_Q\big)^*, 1_m\big). \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{Det}(Z_2)(\tau \otimes \delta_0) &= -h\big([v_0]\big)(\tau) + h_{00}(\tau), \\ \mathrm{Det}(Z_2)(\tau \otimes \delta_1) &= -h\big([v_0]\big)(\tau) + h_{1,0}(\tau) \end{aligned}$$

for some $h_{00} \in \rho_{B_{\mathfrak{p}}}(K_0(B_{\mathfrak{p}}))$ and $h_{1,0} \in \rho_{B_{\mathfrak{q}}}(K_0(B_{\mathfrak{q}}))$. Recall that we have identified $T(B)$ with $T(B \otimes M_{\mathfrak{r}})$ as well as $T(B \otimes Q)$. It follows from Lemma 3.6 that (see also the lines above (e 5.10)) there is $f \in \rho_{B\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(K_0(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
&\mathrm{Det}\,\big(Z_1(t,s)w\big((\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})\big(E_A(v_0)^*\big)\big)\big)(\tau \otimes \delta_t)\\
&\quad = \mathrm{Det}(Z_2)(\tau \otimes \delta_t) = \left(\frac{1}{2\pi i}\int_0^1 \tau\left(\frac{dZ_2(t,s)}{ds}Z_2(t,s)^*\right)ds\right)(\delta_t)\\
&\quad = -h\big(s_A(x)\big)(\tau) + f(\tau \otimes \delta_t).
\end{aligned}$$

Therefore, by (e 5.13) and (e 5.19), the map

$$T(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}) \ni \tau \otimes s_\mu \ \big(\text{where } \tau \in T(B),\ s_\mu \in T(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})\big) \mapsto \mathrm{Det}\,\big(Z_1(t,s)\big)(\tau \otimes s_\mu),$$

where $\tau \in T(B)$, $s_\mu \in T(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$, defines an element in $\overline{\rho_{B\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(K_0(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}))}$. Hence, $\Phi^{\ddagger} \circ E_A^{\ddagger}(x) = \bar{w} = (E_B)^{\ddagger} \circ \gamma(x)$. This proves the claim.

Denote by $\eta : \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}$ the unital embedding given by [24, Proposition 3.3]. Consider

$$\varphi = (\mathrm{id}_B \otimes \eta) \circ \Phi \circ E_A.$$

One then checks that

$$[\varphi] = [\psi] \quad \text{in } KL(A,B), \quad \varphi_\sharp = \psi_\sharp.$$

Since B is $\mathcal{Z}$-stable, $\mathcal{Z}$ itself is strongly absorbing and every unital endomorphism of $\mathcal{Z}$ is approximately inner [10, Theorems 7.6 and 8.7], $(\mathrm{id}_B \otimes \eta)^{\ddagger} \circ E_B^{\ddagger} = \mathrm{id}$. By (e 5.18),

$$\varphi^{\ddagger} \circ s_A = (\mathrm{id}_B \otimes \eta)^{\ddagger} \circ \Phi^{\ddagger} \circ E_A^{\ddagger} \circ s_A = (\mathrm{id}_B \otimes \eta)^{\ddagger} \circ (E_B)^{\ddagger} \circ \gamma \circ s_A = \gamma \circ s_A,$$

which implies $\varphi^{\ddagger} = \gamma$. ■

Theorem 5.4. *Let A be a unital finite separable amenable simple $\mathcal{Z}$-stable C^*-algebra which satisfies the UCT. Then, there exists a sequence of unital separable amenable simple $\mathcal{Z}$-stable C^*-algebras A_n such that $K_i(A)$ are finitely generated $(i = 0, 1)$ and a sequence of homomorphisms $\varphi_n : A_n \to A_{n+1}$ such that*

$$A = \lim_{n\to\infty}(A_n, \varphi_n), \quad \varphi_{n*i} : K_i(A_n) \to K_i(A_{n+1})$$

is injective.

Proof. Let $G_n^0 \subset K_0(A)$ be a sequence of finitely generated subgroups satisfying

$$[1_A] \in G_1^0 \subset G_2^0 \subset \cdots \subset G_n^0 \subset \cdots, \quad K_0(A) = \cup G_n^0,$$

and let $G_n^1 \subset K_1(A)$ be a sequence of finitely generated subgroups satisfying

$$G_1^1 \subset G_2^1 \subset \cdots \subset G_n^1 \subset \cdots, \quad K_1(A) = \cup G_n^1.$$

Recall that the Elliott invariant of A is described as

$$\big((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), r_A\big),$$

where $r_A : T(A) \to S(K_0(A))$ is the canonical map. Let $\Delta = T(A)$ and $r = r_A$. Define $r_n : \Delta \to S(G_n^0)$ by $r_n(\tau) = r(\tau)|_{G_n^0}$.

By [6, Corollary 13.51], there is a separable simple amenable unital $\mathcal{Z}$-stable C^*-algebra A_n such that

$$\begin{aligned}&\big((K_0(A_n), K_0(A_n)_+, [1_{A_n}]), K_1(A_n), T(A_n), r_{A_n}\big)\\ &\quad = \big((G_n^0, G_n^0 \cap K_0(A)_+, [1_A]), G_n^1, \Delta, r_n\big).\end{aligned}$$

By Lemma 5.2 above, there is a homomorphism $\varphi_n : A_n \to A_{n+1}$ such that

$$\begin{aligned}(\varphi_n)_{*,0} &: K_0(A_n) = G_n^0 \to K_0(A_{n+1}) = G_{n+1}^0,\\ (\varphi_n)_{*,1} &: K_1(A_n) = G_n^1 \to K_1(A_{n+1}) = G_{n+1}^1\end{aligned}$$

are the inclusion maps, and $(\varphi_n)_T : T(A_{n+1}) = \Delta \to T(A_n) = \Delta$ is the identity map. Let $B := \lim_{n\to\infty}(A_n, \varphi_n)$. Then, from the construction, A and B have the same Elliott invariant and therefore are isomorphic to each other by [6, Corollary 29.9] and [3, Theorem 4.10]. ∎

Theorem 5.5. *Let A and B be unital finite separable simple $\mathcal{Z}$-stable C^*-algebras. Suppose that A is amenable and satisfies the UCT and $gTR(B \otimes Q) \le 1$. Fix a splitting map $s_A : K_1(A) \to U(A)/CU(A)$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a finite subset $\mathcal{P} \subset \underline{K}(A)$ and a finite subset $\mathcal{U} \subset U(A)$ such that, for any two unital homomorphisms $\varphi, \psi : A \to B$, if*

$$KL(\varphi)|_{\mathcal{P}} = KL(\psi)|_{\mathcal{P}}, \quad \varphi_T = \psi_T, \tag{e 5.20}$$

$$(\varphi^{\ddagger} \circ s_A)|_{\mathcal{P}\cap K_1(A)} = (\psi^{\ddagger} \circ s_A)|_{\mathcal{P}\cap K_1(A)}, \tag{e 5.21}$$

then there exists a unitary $u \in B$ such that

$$\big\|u^*\varphi(a)u - \psi(a)\big\| < \varepsilon \quad \textit{for all } a \in \mathcal{F}.$$

Proof. It follows from Theorem 5.4 that we may write $A = \lim_{n\to\infty}(A_n, \iota_n)$, where each A_n is a unital separable amenable simple $\mathcal{Z}$-stable C^*-algebra with finitely generated $K_i(A_n)$ $(i = 0, 1)$, and $\iota_n : A_n \to A_{n+1}$ is a unit monomorphism. Therefore, there exists an increasing sequence $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots$ of finite subsets of A such that there are finite subsets $\mathcal{G}_n \subset A_n$ with the property $\iota_{n,\infty}(\mathcal{G}_n) = \mathcal{F}_n$ $(n = 1, 2, \ldots)$ and $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in A.

Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$ be given. Without loss of generality, we may assume that $\mathcal{F} \subset \mathcal{F}_n$ for some integer $n \ge 1$. Since $K_i(A_n)$ is finitely generated $(i = 0, 1)$, by [2, Corollary 2.11], there is a finitely generated subgroup $F \subset \underline{K}(A)$ such that if $\kappa_1, \kappa_2 \in KL(A_n, B)$ and $\kappa_1|_F = \kappa_2|_F$, then $\kappa_1 = \kappa_2$. Let $\mathcal{Q} \subset F$ be a finite generating set. Define $\mathcal{P} = [\iota_{n,\infty}](\mathcal{Q})$.

Now suppose that $\varphi, \psi : A \to B$ are two unital homomorphisms which satisfy (e 5.20) and (e 5.21). Then,

$$KL(\varphi \circ \iota_{n,\infty}) = KL(\psi \circ \iota_{n,\infty}), \quad (\varphi \circ \iota_{n,\infty})_\sharp = (\psi \circ \iota_{n,\infty})_\sharp, \quad \varphi \circ \iota^\ddagger_{n,\infty} = \psi \circ \iota^\ddagger_{n,\infty}.$$

It follows from Theorem 4.3 that there exists a unitary $u \in B$ such that

$$\left\| u^* \varphi \circ \iota_{n,\infty}(g) u - \psi \circ \iota_{n,\infty}(g) \right\| < \varepsilon \quad \text{for all } a \in \mathcal{G}_n.$$

It follows that

$$\left\| u^* \varphi(a) u - \psi(a) \right\| < \varepsilon \quad \text{for all } a \in \mathcal{F}_n. \qquad \blacksquare$$

Theorem 5.6. *Let A and B be two unital finite separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Let $\varphi : A \to B$ be a unital homomorphism. Suppose that*

$$\bar{h} \in \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right)$$

(see Definition 2.9 for the notation). Then, there exists a homomorphism $\psi : A \to B$ such that

$$KL(\psi) = KL(\varphi), \quad \psi_T = \varphi_T, \quad (\varphi^\ddagger - \psi^\ddagger) \circ s_A = \bar{h}.$$

Proof. Let $\gamma := \varphi^\ddagger - \bar{h} \circ \Pi^{su}_A$. Then, $([\varphi], \varphi_T, \gamma)$ is compatible. By Lemma 5.2 (see also the second part of Definition 2.10), there is a unital homomorphism $\psi' : A \to B$ such that $KL(\psi') = KL(\varphi)$, $(\psi')_T = \varphi_T$, and

$$\bar{h}_0 := \left((\psi')^\ddagger - \gamma\right) \circ s_A \in \mathrm{Hom}\left(K_1(A), \overline{\mathbb{R}\rho_B\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}\right).$$

Then,

$$\left((\psi')^\ddagger - \varphi^\ddagger\right) \circ s_A = \left((\psi')^\ddagger - (\gamma + \bar{h} \circ \Pi^{su}_A)\right) \circ s_A = \bar{h}_0 - \bar{h}.$$

It follows from Theorem 5.1 that $\bar{h}_0 - \bar{h} \in \mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$. Since $\bar{h}$ is in $\mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$, so is $\bar{h}_0$.

Let $K_1(A) = \bigcup_{n=1}^\infty G_n$, where $G_n \subset G_{n+1}$ is an increasing sequence of finitely generated subgroups. Since $\bar{h}_0 \in \mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$, there are homomorphisms $h_n : K_1(A) \to \mathrm{Aff}(T(B))$ such that $\Sigma_B \circ h_n|_{G_n} = -\bar{h}_0|_{G_n}$, $n = 1, 2, \dots$ (see Definition 2.5 for Σ_B). Since $\bar{h}_0 \in \mathrm{Hom}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$, $h_n|_{G_n} \in \mathrm{Hom}(G_n, \overline{\mathbb{R}\rho_B(K_0(B))})$. Since $\overline{\mathbb{R}\rho_B(K_0(B))}$ is divisible, there exists homomorphism $h_{0,n} : K_1(A) \to \overline{\mathbb{R}\rho_B(K_0(B))}$ such that $h_{0,n}|_{G_n} = h_n|_{G_n}$.

By the second part of Definition 2.10, $gTR(A \otimes M_\mathfrak{r}) \le 1$ and $gTR(B \otimes M_\mathfrak{r}) \le 1$ for any supernatural number $\mathfrak{r}$ of infinite type. By Lemma 5.2, there is a homomorphism $\varphi_n : A \to B$ such that

$$KL(\varphi_n) = KL(\psi') = KL(\varphi), \quad \varphi_{nT} = \varphi_T, \quad (\psi'^\ddagger - \varphi_n^\ddagger) \circ s_A = \bar{h}_n. \qquad \text{(e 5.22)}$$

Let $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ be a sequence of finite subsets of A such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A. By applying Theorem 5.5, we obtain a subsequence $\{\varphi_{n_k}\}$ and a sequence of unitaries

$\{u_k\}$ of B such that

$$\|u_{k+1}^*\varphi_{n_{k+1}}(a)u_{k+1} - \psi_k(a)\| < 1/2^{k+1} \quad \text{for all } a \in \mathcal{F}_k,$$

where $\psi_1 = \varphi_1$, and $\psi_{j+1} = \operatorname{Ad} u_{j+1} \circ \varphi_{n_{j+1}}$, $j = 1, 2, \ldots$.

Then, $\{\psi_k(a)\}$ is a Cauchy sequence for each $a \in A$. Let $\psi(a) = \lim_{k\to\infty} \psi_k(a)$ for $a \in A$. Then, ψ defines a unital homomorphism from A to B. Since $KL(\varphi_n) = KL(\varphi)$ and $\varphi_{n\,T} = \varphi_T$ for all $n \in \mathbb{N}$, one concludes that

$$KL(\psi) = KL(\varphi), \quad \psi_T = \varphi_T.$$

Note that $\bar{h}_n|_{G_n} = \bar{h}_0|_{G_n}$. By (e 5.22),

$$(\psi'^{\ddagger} - \psi^{\ddagger}) \circ s_A|_{G_n} = -\bar{h}_0|_{G_n}, \quad n = 1, 2, \ldots.$$

It follows that

$$(\psi'^{\ddagger} - \psi^{\ddagger}) \circ s_A = -\bar{h}_0.$$

Finally,

$$(\varphi^{\ddagger} - \psi^{\ddagger}) \circ s_A = (\varphi^{\ddagger} - \psi'^{\ddagger}) \circ s_A + (\psi'^{\ddagger} - \psi^{\ddagger}) \circ s_A = -(\bar{h}_0 - \bar{h}) - \bar{h}_0 = \bar{h}. \quad \blacksquare$$

Definition 5.7. Let A be a unital separable C^*-algebra with stable rank at most n such that $T(A) \neq \emptyset$. Let

$$j : \operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to U_0(M_n(A))/CU(M_n(A)) \subset U(M_n(A))/CU(M_n(A))$$

be the embedding.

Define $\mathbb{R}^0 := j(\overline{\mathbb{R}\rho_A(K_0(A))})$. Denote by $U(A)/CU(A)^{\mathbb{R}}$ the quotient group

$$(U(M_n(A))/CU(M_n(A)))/\mathbb{R}^0$$

and

$$\Pi_A^{\mathbb{R},cu} : U(M_n(A))/CU(M_n(A)) \to U(M_n(A))/\mathbb{R}^0 = U(M_n(A))/CU(M_n(A))^{\mathbb{R}}$$

is the quotient map. Denote by $\pi_A^{\mathbb{R}cu} : U(M_n(A))/CU(M_n(A))^{\mathbb{R}} \to K_1(A)$ and $\lambda_A^{\mathbb{R}} : \operatorname{Aff}(T(A)) \to \operatorname{Aff}(T(A))/\overline{\mathbb{R}\rho_A(K_0(A))}$ the quotient map. Since $\overline{\mathbb{R}\rho_A(K_0(A))}$ is a divisible subgroup (a real subspace of $\operatorname{Aff}(T(B))$, in fact), there is a splitting map

$$s_A^{\gamma} : U(M_n(A))/CU(M_n(A))^{\mathbb{R}} \to U(M_n(A))/CU(M_n(A)) \tag{e 5.23}$$

such that $\Pi_A^{\mathbb{R},cu} \circ s_A^{\gamma} = \operatorname{id}_{U(M_n(A))/CU(M_n(A))^{\mathbb{R}}}$.

If B is another separable C^*-algebra with stable rank at most n such that $T(A) \neq \emptyset$ and $\varphi : A \to B$ is a unital homomorphism, then φ induces a continuous homomorphism $\varphi^{\mathbb{R}\ddagger} : U(M_n(A))/CU(M_n(A))^{\mathbb{R}} \to U(M_n(B))/CU(M_n(B))^{\mathbb{R}}$.

Let $\kappa \in KL_e(A, B)^{++}$ and $\kappa_T : T(B) \to T(A)$ be a continuous affine map.

Let $\gamma^{\mathbb{R}} : U(M_n(A))/CU(M_n(A))^{\mathbb{R}} \to U(M_n(B))/CU(M_n(B))^{\mathbb{R}}$ be a homomorphism. We say $(\kappa, \kappa_T, \gamma^{\mathbb{R}})$ is compatible if (κ, κ_T) is compatible, $\gamma^{\mathbb{R}}|_{\mathrm{Aff}(T(A))/\overline{\mathbb{R}\rho_A(K_0(A))}}$ is induced by κ_T, and $\pi_B^{\mathbb{R}cu} \circ \gamma^{\mathbb{R}} = \kappa|_{K_1(A)} \circ \pi_A^{\mathbb{R}cu}$.

Denote by $\mathrm{Hom}_{\kappa,\kappa_T}(U(M_n(A))/CU(M_n(A))^{\mathbb{R}}, U(M_n(B))/CU(M_n(B))^{\mathbb{R}})$ the set of all homomorphisms

$$\gamma^{\mathbb{R}} : U\big(M_n(A)\big)/CU\big(M_n(A)\big)^{\mathbb{R}}, U\big(M_n(B)\big)/CU(M_n(B)\big)^{\mathbb{R}}$$

which are compatible with (κ, κ_T). Fix

$$\bar{g} \in \mathrm{Hom}_{\kappa,\kappa_T}\left(U\big(M_n(A)\big)/CU\big(M_n(A)\big)^{\mathbb{R}}, U\big(M_n(B)\big)/CU\big(M_n(B)\big)^{\mathbb{R}}\right).$$

Then,

$$\begin{aligned}&\left\{\bar{g} - \beta : \beta \in \mathrm{Hom}_{\kappa,\kappa_T}\left(U\big(M_n(A)\big)/CU\big(M_n(A)\big)^{\mathbb{R}},\ U\big(M_n(B)\big)/CU\big(M_n(B)\big)^{\mathbb{R}}\right)\right\}\\ &\quad = \mathrm{Hom}\left(K_1(A), \mathrm{Aff}\big(T(B)\big)/\overline{\mathbb{R}\rho_B\big(K_0(B)\big)}\right).\end{aligned}$$

We use the notation $\Gamma^{\bar{g}}$ for the bijection $\beta \mapsto \bar{g} - \beta$. Thus, we will view it as an abelian group.

For the simplicity of notation, we will use $U(A)$, $CU(A)$, and $CU(A)^{\mathbb{R}}$ for $U(M_n(A))$, $CU(M_n(A))$, and $CU(M_n(A))^{\mathbb{R}}$, or simply assume that the algebras A and B have stable rank 1, and therefore $n = 1$.

Proposition 5.8. *Let (κ, κ_T) be a compatible pair. Then, there is a splitting short exact sequence:*

$$\begin{aligned}0 &\to \mathrm{Hom}\left(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\right)\\ &\to \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)\\ &\to \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}}\big) \to 0.\end{aligned}$$

Proof. For each $\zeta \in \mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$, consider $\Pi_B^{cu} \circ \zeta(x)$ for all $x \in U(A)/CU(A)$. Since

$$\zeta\left(\overline{\mathbb{R}\rho_A\big(K_0(A)\big)}/\overline{\rho_A\big(K_0(A)\big)}\right) \subset \overline{\mathbb{R}\rho_A\big(K_0(B)\big)}/\overline{\rho_A\big(K_0(B)\big)}$$

as ζ is compatible with (κ, κ_T), $\Pi_B^{cu} \circ \zeta$ vanishes on $\overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}$ which uniquely defines a homomorphism

$$\Pi^{H,\mathbb{R}}(\zeta) \in \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}}\big).$$

Fix

$$g \in \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)$$

and let $\bar{g} := \Pi^{H,\mathbb{R}}(g) \in \mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}})$. Using Γ^g and $\Gamma^{\bar{g}}$ and viewing

$$\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big), \quad \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}}\big)$$

as abelian groups as described in Definitions 2.9 and 5.7, then

$$\Pi^{H,\mathbb{R}} : \mathrm{Hom}_{\kappa,\kappa_T}\left(U(A)/CU(A), U(B)/CU(B)\right) \\ \to \mathrm{Hom}_{\kappa,\kappa_T}\left(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}}\right)$$

defines a homomorphism. If $\Pi_B^{\mathbb{R},cu} \circ (g-\zeta) = 0$, then

$$g(x) - \zeta(x) \in \overline{\mathbb{R}\rho_A(K_0(B))}/\overline{\rho_A(K_0(B))} \quad \text{for all } x \in U(A)/CU(A).$$

Since g and ζ are both compatible with (κ, κ_T), $g-\zeta$ defines a homomorphism from $K_1(A)$ to $\overline{\mathbb{R}\rho_A(K_0(B))}/\overline{\rho_A(K_0(B))}$. Conversely, if $g-\zeta$ defines a homomorphism from $K_1(A)$ into $\overline{\mathbb{R}\rho_A(K_0(B))}/\overline{\rho_A(K_0(B))}$ (not just into $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(B))}$), then

$$\Pi^{H,\mathbb{R}}(g-\zeta) = 0.$$

It follows that

$$\ker \Pi^{H,\mathbb{R}} = \mathrm{Hom}\left(K_1(A), \overline{\mathbb{R}\rho_B\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}\right).$$

For each $\xi \in \mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}})$, define a homomorphism $\zeta : U(A)/CU(A) \to U(B)/CU(B)$ by $\zeta = s_B^{\gamma} \circ \xi \circ \Pi_A^{\mathbb{R},cu}$. Since (see the line below (e 5.23))

$$\Pi_B^{\mathbb{R},cu}(\zeta) = (\Pi_B^{\mathbb{R},cu} \circ s_B^{\gamma})(\xi \circ \Pi_A^{\mathbb{R},cu}) = \xi \circ \Pi_A^{\mathbb{R},cu},$$

we have

$$\Pi^{H,\mathbb{R}}(s_B^{\gamma} \circ \xi \circ \Pi_A^{\mathbb{R},cu}) = \xi. \tag{e 5.24}$$

This implies that $\Pi^{H,\mathbb{R}}$ is surjective. Define

$$S^{H,\mathbb{R}} : \mathrm{Hom}_{\kappa,\kappa_T}\left(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}}\right) \\ \to \mathrm{Hom}_{\kappa,\kappa_T}\left(U(A)/CU(A), U(B)/CU(B)\right)$$

by $S^{H,\mathbb{R}}(\xi) = s_B^{\gamma} \circ \Pi_A^{\mathbb{R},cu}$. Then, by (e 5.24), $S^{H,\mathbb{R}}$ is the splitting map. ∎

Proposition 5.9. *Consider (see Definition 2.9 for the notations)*

$$\mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right) \\ = \mathrm{Hom}\left(K_1(A)/\,\mathrm{Tor}\left(K_1(A)\right), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right),$$

$$\mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \overline{\mathbb{R}\rho_B\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}\right) \\ = \mathrm{Hom}\left(K_1(A)/\,\mathrm{Tor}\left(K_1(A)\right), \overline{\mathbb{R}\rho_B\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}\right).$$

Proof. Suppose that $\xi \in \mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$. Write

$$K_1(A) = \bigcup_{n=1}^{\infty} G_n,$$

where $G_n \subset G_{n+1}$ and each G_n is finitely generated. For any $x \in \mathrm{Tor}(K_1(A))$, there is an integer $n \geq 1$ such that $x \in G_n$. Choose $h_n : K_1(A) \to \mathrm{Aff}(T(B))$ such that $\Sigma_B \circ h_n|_{G_n} = \xi|_{G_n}$. Since $\mathrm{Aff}(T(B))$ is torsion-free, $h_n(x) = 0$. It follows that $\xi(x) = 0$. In other words, $\xi|_{\mathrm{Tor}(K_1(A))} = 0$. Therefore, ξ gives a unique homomorphism $\overline{\xi}$ in

$$\mathrm{Hom}\left(K_1(A)/\,\mathrm{Tor}\left(K_1(A)\right), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right).$$

The map

$$\begin{aligned} G : \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right) \\ \to \mathrm{Hom}\left(K_1(A)/\,\mathrm{Tor}\left(K_1(A)\right), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right) \end{aligned}$$

given by $\xi \mapsto \overline{\xi}$ is an injective group homomorphism.

To see the surjectivity, let $\overline{\zeta} \in \mathrm{Hom}(K_1(A)/\,\mathrm{Tor}(K_1(A)), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$. Define

$$\zeta : K_1(A) \to \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}$$

by $\zeta := \overline{\zeta} \circ q$, where $q : K_1(A) \to K_1(A)/\,\mathrm{Tor}(K_1(A))$ is the quotient map.

For each $n \in \mathbb{N}$, let $\overline{G}_n$ be the image of G_n in $K_1(A)/\,\mathrm{Tor}(K_1(A))$. Then, $\overline{G}_n$ is a free abelian group. Therefore, there exists a homomorphism $\lambda_n : \overline{G}_n \to \mathrm{Aff}(T(B))$ such that $\Sigma_B \circ \lambda_n = \overline{\zeta}|_{\bar{G}_n}$. Since $\mathrm{Aff}(T(B))$ is divisible, there is an extension

$$\widetilde{\lambda}_n : K_1(A)/\,\mathrm{Tor}\left(K_1(A)\right) \to \mathrm{Aff}\left(T(B)\right)$$

such that $\widetilde{\lambda}_n|_{\bar{G}_n} = \lambda_n$.

Define $\gamma_n : K_1(A) \to \mathrm{Aff}(T(B))$ by $\gamma_n' := \widetilde{\lambda}_n \circ q$. Then,

$$\zeta|_{G_n} = \Sigma_B \circ \gamma_n|_{G_n}.$$

This implies that

$$\zeta \in \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right).$$

However, $G(\zeta) = \overline{\zeta}$. Therefore, the map is surjective.

The second identity follows from the first one. ■

Theorem 5.10. *Let A and B be unital finite separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Then, for every compatible pair (κ, κ_T), where $\kappa \in KL_e(A, B)^{++}$ and $\kappa_T : T(B) \to T(A)$ is an affine continuous map, there exists a splitting short exact sequence*

$$\begin{aligned} 0 &\to \mathrm{Hom}\left(K_1(A)/\,\mathrm{Tor}\left(K_1(A)\right), \overline{\mathbb{R}\rho_B\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}\right) \\ &\to \mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B) \\ &\to \mathrm{Hom}_{\kappa,\kappa_T}\left(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(A)^{\mathbb{R}}\right) \to 0. \end{aligned}$$

Proof. By Theorem 4.3 and Lemma 5.2, for each compatible pair (κ, κ_T), there is a one-to-one map

$$\Gamma : \mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B) \to \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big) \tag{e 5.25}$$

which is not void. Hence, $\Gamma(\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B))$ is a subset of

$$\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big).$$

Choosing a splitting map $s_A^\gamma : U(A)/CU(A)^{\mathbb{R}} \to U(A)/CU(A)$, by Lemma 5.2 and Proposition 5.8, the quotient map

$$\begin{aligned}\Pi^{H,\mathbb{R}} : &\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)\\ &\to \mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A)^{\mathbb{R}}, U(B)/CU(B)^{\mathbb{R}}\big)\end{aligned}$$

restricting on $\Gamma(\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B))$ is surjective. Fix $[\varphi] \in \mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$. If $[\psi] \in \mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$ and $\Pi^{H,\mathbb{R}}(\Gamma([\varphi])) - \Pi^{H,\mathbb{R}}(\Gamma(([\psi]))) = 0$, then, by Theorem 5.1,

$$\bar{h} := \Gamma\big([\varphi]\big) - \Gamma\big([\psi]\big) \in \mathrm{Hom}_{\mathrm{alf}}\big(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\big).$$

By Theorem 5.6, there is $\psi_1 : A \to B$ such that

$$KL(\psi_1) = KL(\varphi), \quad \psi_{1T} = \kappa_T, \quad (\varphi^\ddagger - \psi^\ddagger) \circ s_A = \bar{h}.$$

This implies, applying Theorem 4.3, that $\mathrm{Hom}_{\mathrm{alf}}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$ is a subgroup in the subset $\Gamma(\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B))$ of an abelian group. Since

$$\Pi^{H,\mathbb{R}} \circ \Gamma\big(\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)\big)$$

is a group, we conclude that $\Gamma(\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B))$ is a subgroup of

$$\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big).$$

Thus, we obtain the short exact sequence, applying also Proposition 5.9.

To show that the short exact sequence splits, it suffices to show that

$$\mathrm{Hom}\big(K_1(A)/\operatorname{Tor}\big(K_1(A)\big), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\big)$$

is divisible. However, this is immediate since $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ is divisible. ∎

Corollary 5.11. *Let A and B be two finite separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Then,*

$$\begin{aligned}&\mathrm{Hom}_{\kappa,\kappa_T}\big(U(A)/CU(A), U(B)/CU(B)\big)/\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)\\ &\cong \mathrm{Hom}\big(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\big)/\mathrm{Hom}_{\mathrm{alf}}\big(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\big).\end{aligned}$$

Theorem 5.12. *Let A and B be finite unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Suppose (κ, κ_T) is a compatible pair, where $\kappa \in KL_e(A,B)^{++}$ and $\kappa_T : T(B) \to T(A)$ is an affine continuous map. Then, there exists a unital homomorphism $\varphi : A \to B$ such that $(KL(\varphi), \varphi_T) = (\kappa, \kappa_T)$. Moreover,*

$$\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A,B) \cong \mathrm{Hom}\left(K_1(A)/\operatorname{Tor}\left(K_1(A)\right), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right).$$

Proof. By Lemma 5.2, there exists a unital homomorphism $\varphi : A \to B$ such that

$$\left(KL(\varphi), \varphi_T\right) = (\kappa, \kappa_T).$$

Let $\Gamma : \mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A,B) \to \mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$ be the one-to-one map introduced in (e 5.25). Put $g := \Gamma(\varphi)$. If $\psi : A \to B$ is another unital homomorphism with $(KL(\psi), \psi_T) = (\kappa, \kappa_T)$, Then, by Theorem 5.1,

$$g - \psi^{\ddagger} \in \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right).$$

In other words,

$$\Gamma^g \circ \Gamma\left(\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A,B)\right) \subset \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right).$$

Note that Γ^g is also one-to-one. It follows from Theorem 5.6 that $\Gamma^g \circ \Gamma$ is surjective. Hence,

$$\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A,B) \cong \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \mathrm{Aff}(T(B))/\overline{\rho_B\left(K_0(B)\right)}\right).$$

Applying Proposition 5.9, one obtains

$$\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A,B) \cong \mathrm{Hom}\left(K_1(A)/\operatorname{Tor}\left(K_1(A)\right), \mathrm{Aff}\left(T(B)\right)/\overline{\rho_B\left(K_0(B)\right)}\right). \qquad \blacksquare$$

Corollary 5.13. *Let A and B be unital finite separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Then, for any compatible triple $(\kappa, \kappa_T, \kappa_\gamma)$, where $\kappa \in KL_e(A,B)^{++}$, $\kappa_T : T(B) \to T(A)$ is an affine continuous map, and*

$$\kappa_\gamma : U(A)/CU(A) \to U(B)/CU(B)$$

is a continuous homomorphism, there is a unital homomorphism $\varphi : A \to B$ such that

$$KL(\varphi) = \kappa, \quad \varphi_T = \kappa_T, \quad \varphi^{\ddagger} = \kappa_\gamma$$

if one of the following holds:

(1) $TR(B) \le 1$,

(2) $\overline{\rho_B(K_0(B))} = \overline{\mathbb{R}\rho_B(K_0(B))}$,

(3) $\mathrm{Hom}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$
$= \mathrm{Hom}_{\mathrm{alf}}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$,

(4) *$K_1(A)$ is torsion-free.*

Proof. Note that (2) follows from Theorem 5.10 immediately. Therefore, by [14, Proposition 3.6], (1) follows.

Moreover, (3) follows from Corollary 5.11 and (4) follows from Theorem 5.12. ∎

Remark 5.14. There are plenty of examples of

$$\mathrm{Hom}\big(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\big)/\,\mathrm{Hom}_{\mathrm{alf}}\big(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\big) \neq \{0\}.$$

In those cases, there are compatible triples $(\kappa, \kappa_T, \kappa_\gamma)$ which cannot be represented by homomorphisms from A to B.

To illustrate this, let us consider a simple example. By [6, Theorem 13.50], there is a unital separable simple amenable $\mathcal{Z}$-stable C^*-algebra A with a unique tracial state τ_A satisfying the UCT such that $(K_0(A), K_1(A)_+, [1]) = (\mathbb{Z}, \mathbb{Z}_+, 1)$ and $K_1(A) = \mathbb{Z}/m\mathbb{Z}$ for some prime number $m \geq 2$. Note that one has the following splitting short exact sequence:

$$0 \to \mathbb{R}/\mathbb{Z} \to U(A)/CU(A) \to \mathbb{Z}/m\mathbb{Z} \to 0.$$

Let $B = \mathcal{Z}$ be the Jiang–Su algebra. Note that $KL(A, B) = KK(A, B) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z})$. Let $\kappa \in KL_e(A, \mathcal{Z})^{++}$ with $\kappa([1_A]) = [1_{\mathcal{Z}}]$ (there are m such elements, and we will fix one). By Lemma 5.2, there is a unital homomorphism $\varphi : A \to \mathcal{Z}$. Then, $\varphi^\ddagger : U(A)/CU(A) \to \mathbb{R}/\mathbb{Z}$ is a homomorphism which is compatible with (κ, ι), where ι induces the identity map on $\mathbb{R}$. It follows that $\ker \varphi^\ddagger \cong \mathbb{Z}/m\mathbb{Z}$. One may also write

$$U(A)/CU(A) = \mathbb{R}/\mathbb{Z} \oplus \ker \varphi^\ddagger.$$

Note that

$$\mathrm{Hom}_{\kappa,\iota}\big(U(A)/CU(A), U(\mathcal{Z})/CU(\mathcal{Z})\big) = \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}.$$

Since $K_1(A) = \mathbb{Z}/m\mathbb{Z}$ is a torsion group,

$$\mathrm{Hom}_{\mathrm{alf}}\big(K_1(A), \mathbb{R}/\mathbb{Z}\big) = \mathrm{Hom}\big(K_1(A)/\,\mathrm{Tor}\big(K_1(A)\big), \mathbb{R}/\mathbb{Z}\big) = \{0\}.$$

Therefore, by Theorem 5.10, $\mathrm{Hom}_{\kappa,\iota}(A, B)$ has only a single point. Thus,

$$\mathrm{Hom}_{\kappa,\iota,\mathrm{app}}(A, B) \neq \mathrm{Hom}_{\kappa,\iota}\big(U(A)/CU(A), U(\mathcal{Z})/CU(\mathcal{Z})\big).$$

Proposition 5.15. *Let B be a unital finite separable simple amenable $\mathcal{Z}$-stable C^*-algebra which satisfies the UCT such that $\overline{\mathbb{R}\rho_B(K_0(B))} \neq \overline{\rho_B(K_0(B))}$. Then, for any unital separable simple amenable $\mathcal{Z}$-stable C^*-algebra which satisfies the UCT with* $\mathrm{Tor}(K_1(A)) \neq \{0\}$, *and for any compatible pair (κ, κ_T), where $\kappa \in KL_e(A, B)^{++}$ and $\kappa_T : T(B) \to T(A)$ is a continuous affine homeomorphism, there is a compatible triple $(\kappa, \kappa_T, \kappa_\gamma)$, such that no unital homomorphism $\varphi : A \to B$ has the property that*

$$\big(KL(\varphi), \varphi_T, \varphi^\ddagger\big) = (\kappa, \kappa_T, \kappa_\gamma).$$

Proof. Fix a compatible triple (κ, κ_T), where $\kappa \in KL_e(A, B)^{++}$ and $\kappa_T : T(B) \to T(A)$ is a continuous affine homeomorphism. It follows from Lemma 5.2 that there is a unital homomorphism $\psi : A \to B$ such that $KL(\psi) = \kappa$ and $\psi_T = \kappa_T$.

Let $x \in K_1(A) \setminus \{0\}$ such that $px = 0$ for some prime number $p > 1$.

Since $\overline{\mathbb{R}\rho_B(K_0(B))} \neq \overline{\rho_B(K_0(B))}$, there is $y \neq 0$ in $\overline{\rho_B(K_0(B))}$ such that

$$\left\{r \in \mathbb{Q} : ry \in \overline{\rho_B\big(K_0(B)\big)}\right\}$$

is not dense in $\mathbb{R}$. Note $\mathbb{D}_{\mathfrak{p}} = \{\frac{m}{p^n} : n \in \mathbb{N} \cup \{0\},\ m \in \mathbb{Z}\}$ is dense in $\mathbb{Q}$. Therefore, there must be an integer $n \in \mathbb{N}$ such that

$$(1/p^{n+1})y \notin \overline{\rho_B\big(K_0(B)\big)}, \quad (1/p^n)y \in \overline{\rho_B\big(K_0(B)\big)}.$$

Put $z_0 = (1/p^{n+1})y$. Then,

$$pz_0 \in \overline{\rho_B\big(K_0(B)\big)}.$$

Let z be the image of z_0 in $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$. Then, $z \neq 0$ and $pz = 0$. Let G_x be the subgroup of $K_1(A)$ generated by x. Then, $G_x \cong \mathbb{Z}/p\mathbb{Z}$. Define a homomorphism $h_x : G_x \to \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ by $h_x(x) = z$. Since $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ is divisible, there is a homomorphism $\bar{h} : K_1(A) \to \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ such that $\bar{h}|_{G_x} = h_x$.

Now define

$$\kappa_\gamma : U(A)/CU(A) \to U(B)/CU(B)$$

by $\kappa_\gamma := \psi^\ddagger + \bar{h} \circ \Pi_A^{cu}$. Then, $(\kappa, \kappa_T, \kappa_\gamma)$ is compatible. If there was a unital homomorphism $\varphi : A \to B$ such that $\varphi^\ddagger = \kappa_\gamma$, then, for a fixed splitting map s_A,

$$(\varphi^\ddagger - \psi^\ddagger) \circ s_A \in \mathrm{Hom}_{\mathrm{alf}}\left(K_1(A), \overline{\mathbb{R}\rho_B\big(K_0(B)\big)}/\overline{\rho_B\big(K_0(B)\big)}\right).$$

However,

$$(\varphi^\ddagger - \psi^\ddagger) \circ s_A = \bar{h} \circ \Pi_A^{cu} \circ s_A = \bar{h}$$

which is not in $\mathrm{Hom}_{\mathrm{alf}}(K_1(A), \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))})$. A contradiction. ∎

Remark 5.16. Note that if A is a unital separable simple C^*-algebra such that

$$gTR(A \otimes M_{\mathfrak{p}}) \leq 1$$

for some supernatural number $\mathfrak{p}$ of infinite type, then A as a C^*-subalgebra of $A \otimes M_{\mathfrak{p}}$ must have a finite faithful trace. In particular, A is stably finite.

Theorems 5.6, 5.10, and 5.12, Corollaries 5.11 and 5.13, and Proposition 5.15 all hold if we replace the condition that B is amenable and satisfies the UCT by $gTR(B) \leq 1$, since we only use that before Theorem 5.6. If we further assume that $K_i(A)$ is finitely generated $(i = 0, 1)$, then the condition that A is $\mathcal{Z}$-stable can be replaced by $gTR(A \otimes Q) \leq 1$ as we do not need Theorems 5.4 and 5.5.

6. The sequence of maps, another description

Definition 6.1. Set

$$\mathbb{I}_k := \{f \in C_0((0,1], M_k) : f(1) \in \mathbb{C}1_k\}.$$

Put $A^{(k)} = A \otimes \mathbb{I}_k$ and $\widetilde{A^{(k)}}$, the unitization of $A^{(k)}$. We may identify

$$\widetilde{A^{(k)}} = \{f \in C([0,1], A \otimes M_k) : f(0) \in \mathbb{C}1_k,\ f(1) \in A \otimes 1_k\}.$$

Note that $K_i(A \otimes \mathbb{I}_k)$ is identified with $K_{i+1}(A, \mathbb{Z}/k\mathbb{Z})$ (see [2, 1.2]). Let $\eta_k : A^{(k)} \to A$ be the homomorphism defined by $\eta_k(f) = f(1)$ for all $f \in A^{(k)}$. Consider the short exact sequence

$$0 \to S(M_k(A)) \to A \otimes \mathbb{I}_k \xrightarrow{\eta_k} A \to 0,$$

where $S(M_k(A)) = \{f \in C([0,1], M_k) : f(0) = f(1) = 0\}$. It gives the following six-term exact sequence (see [2, equation (1.6)]):

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A,\mathbb{Z}/k\mathbb{Z}) & \xrightarrow{\eta_{k*1}} & K_1(A) \\ \uparrow{\scriptstyle \times k} & & & & \downarrow{\scriptstyle \times k} \\ K_0(A) & \xleftarrow{\eta_{k*0}} & K_1(A,\mathbb{Z}/k\mathbb{Z}) & \longleftarrow & K_1(A). \end{array} \tag{e 6.1}$$

One also has a (unnaturally) splitting short exact sequence

$$0 \to K_0(A)/kK_0(A) \to K_0(A, \mathbb{Z}/k\mathbb{Z}) \to \mathrm{Tor}(K_1(A), \mathbb{Z}/k\mathbb{Z}) \to 0,$$

where $\mathrm{Tor}(K_1(A), \mathbb{Z}/k\mathbb{Z}) = \{y \in K_1(A) : ky = 0\}$. Fix a splitting map

$$j_k : \mathrm{Tor}(K_1(A), \mathbb{Z}/k\mathbb{Z}) \to K_0(A, \mathbb{Z}/k\mathbb{Z}).$$

Then, combining (e 6.1),

$$\eta_{k*1} \circ j_k = \mathrm{id}_{\mathrm{Tor}(K_1(A),\mathbb{Z}/k\mathbb{Z})}. \tag{e 6.2}$$

Let $\pi^{\mathbb{C}}_{A^{(k)}} : \widetilde{A^{(k)}} \to \mathbb{C}$ be the quotient map and

$$(\pi^{\mathbb{C}}_{M_n(A^{(k)})})^u : U(M_n(\widetilde{A^{(k)}})) \to U(M_n(\mathbb{C}))$$

the induced group homomorphism. Define, for any $n \geq 1$,

$$U(M_n(A \otimes \mathbb{I}_k))^\iota = \ker(\pi^{\mathbb{C}}_{M_n(A^{(k)})})^u = \{u \in M_n(\widetilde{A^{(k)}}) : \pi^{\mathbb{C}}_{A^{(k)}} \otimes \mathrm{id}_n(u) = 1_n\}.$$

Denote $CU(M_n(A \otimes \mathbb{I}_k))^\iota := CU(M_n(\widetilde{A \otimes \mathbb{I}_k})) \cap U(M_n(A \otimes \mathbb{I}_k))^\iota$.

Definition 6.2. Let D be a non-unital C^*-algebra with $T(D) \neq \emptyset$. For each $f \in \mathrm{Aff}(T(D))$, define $\iota^\sharp_D : \mathrm{Aff}(T(D)) \to \mathrm{Aff}(T(\widetilde{D}))$ by $\iota^\sharp_D(f)(\alpha t_{\mathbb{C}} + (1-\alpha)\tau) = (1-\alpha)f(\tau)$ for all $0 \leq \alpha \leq 1$, $\tau \in T(D)$, and $t_{\mathbb{C}} \in T(\widetilde{D})$ such that $t_{\mathbb{C}}|_D = 0$, where $t_{\mathbb{C}} := t \circ \pi^{\mathbb{C}}_D$ such that $\pi^{\mathbb{C}}_D : \widetilde{D} \to \mathbb{C}$ is the quotient map and t is the unique tracial state on $\mathbb{C}$. Put

$$\mathrm{Aff}(T(D))^\iota = \{\iota^\sharp_D(f) : f \in \mathrm{Aff}(T(D))\} \subset \mathrm{Aff}(T(\widetilde{D})).$$

Definition 6.3. Let $k = 0, 2, \ldots$. Suppose that A is a unital C^*-algebra and has stable rank at most $n-1$. Note that since $K_0((A \otimes \mathbb{I}_k))$ is a torsion group, $\rho_{\widetilde{A^{(k)}}}(K_0(A \otimes \mathbb{I}_k)^{\sim}) = \mathbb{Z} \cdot 1_T$, where 1_T represents the constant affine function on $T(\widetilde{A^{(k)}})$ with value 1. Then,

$$\overline{\rho_{\widetilde{A^{(k)}}}\big(K_0(A \otimes \mathbb{I}_k)^{\sim}\big)} = \mathbb{Z} \cdot 1_T.$$

One then checks that $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^{\iota} \cap \overline{\rho_{\widetilde{A^{(k)}}}(K_0((A \otimes \mathbb{I}_k)^{\sim}))} = \{0\}$.

On the other hand, let $x \in K_1(A \otimes \mathbb{I}_k)$ and $u \in U(M_n(A \otimes \mathbb{I}_k))$ a unitary which represents x. Suppose that

$$\pi^{\mathbb{C}}_{M_n(\widetilde{A^{(k)}})}(u) = z$$

which is a scalar unitary. Let $Z \in M_n(\mathbb{C} \cdot 1_{\widetilde{A^{(k)}}})$ be the same scalar matrix in $M_n(\widetilde{A^{(k)}})$. Put $v = uZ^*$. Then, $[v] = [u]$ in $K_1(A \otimes \mathbb{I}_k)$ and $v \in U(A \otimes \mathbb{I}_k)^{\iota}$. This implies that $\Pi^{cu}_{A^{(k)}}$, the restriction of $\Pi^{cu}_{\widetilde{A^{(k)}}}$ on $U(A \otimes \mathbb{I}_k)^{\iota}/CU(A \otimes \mathbb{I}_k)^{\iota}$, is surjective. Thus, from the short exact sequence

$$0 \to \mathrm{Aff}\big(T(\widetilde{A^{(k)}})\big)/\mathbb{Z} \to U\big(M_n(\widetilde{A^{(k)}})\big)/CU\big(M_n(\widetilde{A^{(k)}})\big) \xrightarrow{\Pi^{cu}_{\widetilde{A^{(k)}}}} K_1(A^{(k)}) \to 0,$$

one obtains the following short exact sequence:

$$\begin{aligned} 0 \to \mathrm{Aff}\big(T(A \otimes \mathbb{I}_k)\big)^{\iota} &\to U\big(M_n(A \otimes \mathbb{I}_k)\big)^{\iota}/CU\big(M_n(A \otimes \mathbb{I}_k)\big)^{\iota} \\ &\xrightarrow{\Pi^{cu}_{A^{(k)}}} K_1(A \otimes \mathbb{I}_k) \to 0. \end{aligned} \tag{e 6.3}$$

Let A and B be unital C^*-algebras of stable rank no more than $n-1$. Fix a compatible pair (κ, κ_T). Then, κ_T induces an affine continuous map $\tau \otimes s \mapsto \kappa_T(\tau) \otimes s$ from $T(B \otimes \mathbb{I}_k)$ to $T(A \otimes \mathbb{I}_k)$ which in turn induces an affine continuous map

$$\overline{\kappa}^{(k)}_{\sharp} : \mathrm{Aff}(T(A \otimes \mathbb{I}_k))^{\iota} \to \mathrm{Aff}\,(T(B \otimes \mathbb{I}_k))^{\iota}.$$

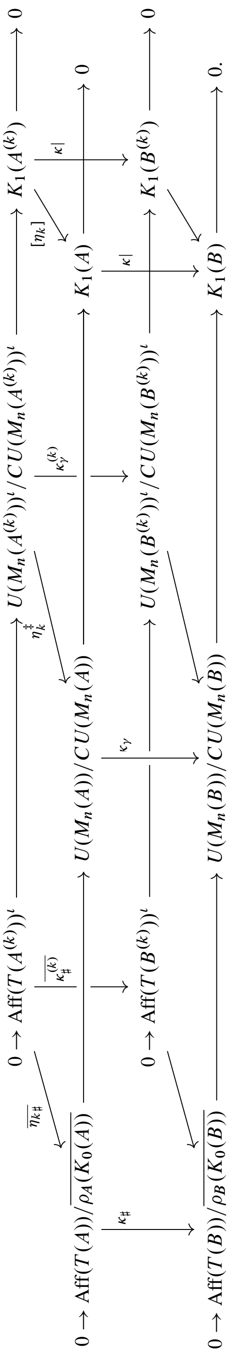
Let $\kappa^{(k)}_{\gamma} : U(M_n(A \otimes \mathbb{I}_k))^{\iota}/CU(M_n(A \otimes \mathbb{I}_k))^{\iota} \to U(M_n(B \otimes \mathbb{I}_k))^{\iota}/CU(M_n(B \otimes \mathbb{I}_k))^{\iota}$ be a continuous homomorphism. We say that $(\kappa, \kappa_T, \kappa^{(k)}_{\gamma})$ is compatible if the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Aff}\big(T(A \otimes \mathbb{I}_k)\big)^{\iota} & \longrightarrow & U\big(M_n(A \otimes \mathbb{I}_k)\big)^{\iota}/CU\big(M_n(A \otimes \mathbb{I}_k)\big)^{\iota} & \xrightarrow{\Pi^{cu}_{A^{(k)}}} & K_1(A \otimes \mathbb{I}_k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\kappa}^{(k)}_{\sharp}} & & \downarrow{\scriptstyle \kappa^{(k)}_{\gamma}} & & \downarrow{\scriptstyle \kappa|_{K_0(A,\mathbb{Z}/k\mathbb{Z})}} & & \\ 0 & \longrightarrow & \mathrm{Aff}\big(T(B \otimes \mathbb{I}_k)\big)^{\iota} & \longrightarrow & U\big(M_n(B \otimes \mathbb{I}_k)\big)^{\iota}/CU\big(M_n(B \otimes \mathbb{I}_k)\big)^{\iota} & \xrightarrow{\Pi^{cu}_{B^{(k)}}} & K_1(B \otimes \mathbb{I}_k) & \longrightarrow & 0. \end{array} \tag{e 6.4}$$

Recall that $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^{\iota}$ is identified with $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^{\iota}/\overline{\rho_{\widetilde{A^{(k)}}}(K_0(A \otimes \mathbb{I}_k{}^{\sim}))}$. Note that the homomorphism η_k also gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Aff}(T(A \otimes \mathbb{I}_k))^{\iota} & \longrightarrow & U(M_n(A \otimes \mathbb{I}_k))^{\iota}/CU(M_n(A \otimes \mathbb{I}_k))^{\iota} & \xrightarrow{\Pi^{cu}_{A^{(k)}}} & K_1(A \otimes \mathbb{I}_k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\eta_{k\sharp}}} & & \downarrow{\scriptstyle \eta^{\ddagger}_k} & & \downarrow{\scriptstyle [\eta_k]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})}} & & \\ 0 & \longrightarrow & \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} & \longrightarrow & U(M_n(A))/CU(M_n(A)) & \xrightarrow{\Pi^{cu}_{A^{(k)}}} & K_1(A) & \longrightarrow & 0. \end{array}$$

Denote by $\kappa_\Gamma := \{\kappa_\gamma^{(k)}\}_{k=0,2,...}$ a sequence of homomorphisms. We say $(\kappa, \kappa_T, \kappa_\Gamma)$ is *totally* compatible if, for each $k \geq 2$, the following diagram commutes:



Note that the assumption that C^*-algebras have finite stable rank is not necessary. We put it here for convenience so we do not need to draw infinite matrices.

Proposition 6.4. (1) *The short exact sequence of* (e 6.3) *splits uniquely for* $k \geq 2$.

(2) *Suppose* $(\kappa, \kappa_T, \kappa'_\Gamma)$ *and* $(\kappa, \kappa_T, \kappa''_\Gamma)$ *are totally compatible, where* $\kappa'_\Gamma = \{\kappa'_\gamma, \kappa_\gamma^{(k)'} : k \geq 2\}$ *and* $\kappa''_\Gamma = \{\kappa''_\gamma, \kappa_\gamma^{(k)''} : k \geq 2\}$, *and*

$$\kappa'_\gamma, \kappa''_\gamma : U\big(M_n(A)\big)/CU\big(M_n(A)\big) \to U\big(M_n(B)\big)/CU\big(M_n(B)\big)$$

and

$$\kappa_\gamma^{(k)'}, \kappa_\gamma^{(k)''} : U\big(M_n(A^{(k)})\big)^\iota/CU\big(M_n(A^{(k)})\big)^\iota \to U\big(M_n(B^{(k)})\big)^\iota/CU\big(M_n(B^{(k)})\big)^\iota$$

are continuous homomorphisms. Then, $(\kappa'_\gamma - \kappa''_\gamma)$ *induces a homomorphism in*

$$\mathrm{Hom}\,\big(K_1(A)/\,\mathrm{Tor}\,\big(K_1(A)\big), \mathrm{Aff}\,\big(T(B)\big)/\overline{\rho_B\big(K_0(B)\big)}\big).$$

(3) *In* (2), *if* $\kappa'_\gamma = \kappa''_\gamma$, *then* $\kappa'_\Gamma = \kappa''_\Gamma$.

Proof. (1) The fact that the short exact sequence splits follows from the fact that $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^\iota$ is divisible. Let $s_{A^{(k)}}$ be a splitting map. Then, $\Pi^{cu}_{A^{(k)}} \circ s_{A^{(k)}} = \mathrm{id}_{K_1(A^{(k)})}$. Suppose that $s : K_1(A \otimes \mathbb{I}_k) \to U(M_n(A \otimes \mathbb{I}_k))^\iota/CU(M_n(A \otimes \mathbb{I}_k))^\iota$ is another splitting map. Then, $s_{A^{(k)}} - s$ maps $K_1(A^{(k)})$ to $\mathrm{Aff}(A \otimes \mathbb{I}_k)^\iota$. However, $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^\iota$ is torsion-free and $K_1(A \otimes \mathbb{I}_k) \cong K_0(A, \mathbb{Z}/k\mathbb{Z})$ is a torsion group. It follows that $s_{A^{(k)}} - s = 0$.

(2) Let $x \in \mathrm{Tor}(K_1(A))$. Choose an integer $k \geq 2$ such that $kx = 0$. Let

$$\mathrm{Tor}\,\big(K_1(A), \mathbb{Z}/k\mathbb{Z}\big) = \big\{y \in K_1(A) : ky = 0\big\}.$$

Recall that $K_1(A \otimes \mathbb{I}_k) = K_0(A, \mathbb{Z}/k\mathbb{Z})$. Let

$$s_{A^{(k)}} : K_1(A \otimes \mathbb{I}_k) \to U\big(M_n(A \otimes \mathbb{I}_k)\big)^\iota/CU\big(M_n(A \otimes \mathbb{I}_k)\big)^\iota$$

be the unique splitting map.

Suppose that $(\kappa, \kappa_T, \kappa'_\Gamma)$ and $(\kappa, \kappa_T, \kappa''_\Gamma)$ are totally compatible. Then,

$$(\kappa_\gamma^{(k)'} - \kappa_\gamma^{(k)''})|_{\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}} = 0. \tag{e 6.5}$$

Moreover, the map

$$(\kappa_\gamma^{(k)'} - \kappa_\gamma^{(k)''}) \circ s_{A^{(k)}} : K_1(A \otimes \mathbb{I}_k) \to \mathrm{Aff}\,\big(T(A \otimes \mathbb{I}_k)\big)^\iota$$

has to be zero, as $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^\iota$ is torsion-free. In particular, for any element z with finite order,

$$(\kappa_\gamma^{(k)'} - \kappa_\gamma^{(k)''}) \circ s_{A^{(k)}}(z) = 0.$$

By the 12-term commutative diagram above at the end of Definition 6.3, for any $y \in U(M_n(A \otimes \mathbb{I}_k))^\iota/CU(M_n(A \otimes \mathbb{I}_k))^\iota$, one has

$$(\kappa'_\gamma - \kappa''_\gamma) \circ {\eta_k}^\ddagger(y) = {\eta_k}^B_\sharp\big(\kappa_\gamma^{(k)'} - \kappa_\gamma^{(k)''}\big)(y),$$

where $\eta_k^B : B \otimes \mathbb{I}_k \to B$ is defined by $\eta_k^B(f) = f(1)$ for all $f \in B \otimes \mathbb{I}_k$ as η_k defined. As mentioned above, since $\mathrm{Aff}\,(T(B \otimes \mathbb{I}_k))^\iota$ is torsion-free, if y has finite order,

$$(\kappa'_\gamma - \kappa''_\gamma) \circ {\eta_k}^{\ddagger}(y) = {\eta_k}^B_\sharp \big(\kappa_\gamma^{(k)\prime} - \kappa_\gamma^{(k)\prime\prime}\big)(y) = 0.$$

By the 12-term diagram above, again, since $s_{A^{(k)}}$ is unique,

$$\eta_{k*1} = \Pi_A^{cu} \circ \eta_k^\ddagger \circ s_{A^{(k)}}.$$

Therefore (recall (e 6.2)),

$$\Pi_A^{cu}\big(\ker(\kappa'_\gamma - \kappa''_\gamma)\big) \supset \mathrm{Tor}\,\big(K_1(A), \mathbb{Z}/k\mathbb{Z}\big).$$

Consequently, for any splitting map s_A (see also (e 6.5)),

$$J\big(\mathrm{Aff}\,(T(A))/\overline{\rho_A(K_0(A))}\big) + s_A\big(\mathrm{Tor}\,\big(K_1(A), \mathbb{Z}/k\mathbb{Z}\big)\big) \subset \ker(\kappa'_\gamma - \kappa''_\gamma),$$

where $J : \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to U(A)/CU(A)$ is given by the (inverse of) determinant map. This implies that $(\kappa'_\gamma - \kappa''_\gamma)(x) = 0$. Hence,

$$(\kappa'_\gamma - \kappa''_\gamma)|_{\mathrm{Tor}(K_1(A))} = 0.$$

This proves (2).

(3) For any $k \geq 2$, by (1), since $s_{A^{(k)}}$ is unique, the diagram (e 6.4) becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Aff}\,(T(A \otimes \mathbb{I}_k))^\iota & \longrightarrow & \mathrm{Aff}(T(A \otimes \mathbb{I}_k))^\iota \oplus s_{A^{(k)}}(K_1(A \otimes \mathbb{I}_k)) & \xrightarrow{\Pi^{cu}_{A^{(k)}}} & K_1(A \otimes \mathbb{I}_k) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \overline{\kappa}_\sharp^{(k)}} & & \Big\downarrow{\scriptstyle \kappa_\gamma^{(k)}} & & \Big\downarrow{\scriptstyle \kappa|_{K_0(A, \mathbb{Z}/k\mathbb{Z})}} & & \\
0 & \longrightarrow & \mathrm{Aff}(T(B \otimes \mathbb{I}_k))^\iota & \longrightarrow & \mathrm{Aff}(T(B \otimes \mathbb{I}_k))^\iota \oplus s_{B^{(k)}}(K_1(B \otimes \mathbb{I}_k)) & \xrightarrow{\Pi^{cu}_{B^{(k)}}} & K_1(B \otimes \mathbb{I}_k) & \longrightarrow & 0.
\end{array}$$

Then, as $\mathrm{Aff}(T(A \otimes \mathbb{I}_k))^\iota$ and $\mathrm{Aff}(T(B \otimes \mathbb{I}_k))^\iota$ are torsion-free, and $K_1(A \otimes \mathbb{I}_k)$ and $K_1(B \otimes \mathbb{I}_{(k)})$ are torsion, we may write the decomposition $\kappa_\gamma^{(k)} = \overline{\kappa_\sharp}^{(k)} \oplus \kappa|_{K_1(A^{(k)})}$ which is uniquely determined by κ and κ_T. Thus, (3) follows. ∎

Theorem 6.5. *Let A and B be unital finite separable simple amenable $\mathcal{Z}$-stable C^*-algebras such that A and B satisfy the UCT. Then, for any totally compatible triple $(\kappa, \kappa_T, \kappa_\Gamma)$, where $\kappa \in KL_e(A, B)^{++}$, $\kappa_T : T(B) \to T(A)$ is a continuous affine homomorphism, and $\kappa_\Gamma = \{\kappa_\gamma^{(k)}\}_{k=0,2,\ldots}$ is as defined in Definition 6.3, there is a unital homomorphism $\varphi : A \to B$ such that $(KL(\varphi), \varphi_T, \varphi_\Gamma^\ddagger) = (\kappa, \kappa_T, \kappa_\Gamma)$.*

Proof. Fix a compatible pair (κ, κ_T). By Lemma 5.2, there is a unital homomorphism $\psi : A \to B$ such that $KL(\psi) = \kappa$, and $\psi_T = \kappa_T$. Clearly, $(KL(\psi), \psi_T, \psi_\Gamma^\ddagger)$ is totally compatible.

Suppose that $(\kappa, \kappa_T, \kappa_\Gamma)$ is totally compatible. Then, by Proposition 6.4, $\psi^\ddagger - \kappa_\gamma$ induces a homomorphism $\bar{h} \in \mathrm{Hom}(K_1(A)/\mathrm{Tor}(K_1(A)), \mathrm{Aff}(T(B))/\overline{\rho_B(K_0(B))})$. It follows from Theorem 5.12 that there is a unique (up to approximately unitarily equivalent) unital homomorphism $\varphi : A \to B$ such that $KL(\varphi) = \kappa$, $\varphi_T = \kappa_T$, and $\psi^\ddagger - \varphi^\ddagger = \bar{h}$. It follows that $\varphi^\ddagger = \kappa_\gamma$. Since $(\kappa, \kappa_T, \varphi_\Gamma^\ddagger)$ is totally compatible, by Proposition 6.4, $\varphi_\Gamma^\ddagger = \kappa_\Gamma$. ∎

Remark 6.6. Theorem 6.5 provides a complement to Theorem 5.12 and is a consequence of Theorem 5.12 as the proof presented. It also provides a seemingly more functorial description. However, $\varphi_\Gamma^\ddagger$ is really a sequence of maps, and, by (3) of Proposition 6.4, most of the data are redundant. It does not appear to fit Theorem 4.3, the uniqueness theorem, well enough as Theorem 4.3 only requires one map $\varphi^\ddagger$ from the list of $\varphi_\Gamma^\ddagger$.

By (1) of Proposition 6.4 (and its proof), for $k \geq 2$, the splitting map

$$s_{A^{(k)}} : K_0(A, \mathbb{Z}/k\mathbb{Z}) = K_1(A \otimes \mathbb{I}_k) \to U\big(M_n(A \otimes \mathbb{I}_k)\big)^\iota / CU\big(M_n(A \otimes \mathbb{I}_k)\big)^\iota$$

is a natural map. It follows that the composition

$$\zeta_k^A := \eta_k^\ddagger \circ s_{A^{(k)}} : K_0(A, \mathbb{Z}/k\mathbb{Z}) \to U\big(M_n(A)\big)/CU\big(M_n(A)\big)$$

is also natural. By the last large diagram in Definition 6.3, in which $(\kappa, \kappa_T, \kappa_\Gamma)$ is totally compatible, it is equivalent to say that $(\kappa, \kappa_T, \kappa_\gamma)$ is compatible together with

$$\zeta_k^B \circ \kappa|_{K_0(A,\mathbb{Z}/k\mathbb{Z})} = \kappa_\gamma \circ \zeta_k^A \quad \text{for each } k \geq 2.$$

In Section 5, we state that, for a fixed compatible pair (κ, κ_T), $\mathrm{Hom}_{\kappa,\kappa_T,\mathrm{app}}(A, B)$ is a subset of $\mathrm{Hom}_{\kappa,\kappa_T}(U(A)/CU(A), U(B)/CU(B))$. Theorems 5.10 and 5.12 provide a complete description of this subset. One advantage of Theorems 5.10 and 5.12 is that they provide Corollary 5.13 which could not be seen from Theorem 6.5 as easily. More importantly, they reveal that it is the subgroup $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ that prevents some of the compatible triples $(\kappa, \kappa_T, \kappa_\gamma)$ from being realized by homomorphisms (see also Proposition 5.15).

Theorem 6.7 (cf. [7, Theorem 29.5]). *Let A and B be unital finite separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Suppose that there is an isomorphism $\gamma_i : K_i(A) \to K_i(B)$ $(i = 0, 1)$ and an affine homeomorphism $\kappa_T : T(B) \to T(A)$ such that $\gamma_0([1_A]) = [1_B]$ and $(\rho_B \circ \gamma_0(x))(\tau) = \rho_A(x)(\kappa_T)$ for all $x \in K_0(A)$ and $\tau \in T(B)$. Then, there exists an isomorphism $\Phi : A \to B$ such that Φ induces γ_i $(i = 0, 1)$ and κ_T. Moreover, if there is a totally compatible triple $(\kappa, \kappa_T, \kappa_\Gamma)$, where $\kappa \in KL(A, B)$ such that $\kappa([1_A]) = [1_B]$, κ induces isomorphisms from $K_i(A)$ onto $K_i(B)$ $(i = 0, 1)$, κ_T is an affine homeomorphism, and $\kappa_\Gamma = \{\kappa_\gamma^{(k)}\}_{k=0,2,\ldots}$ is as defined in Definition 6.3, then there is an isomorphism $\psi : A \to B$ such that $(KL(\psi), \psi_T, \psi_\Gamma^\ddagger) = (\kappa, \kappa_T, \kappa_\Gamma)$.*

Proof. The first part of the statement follows from [7, Theorem 29.5] (and the last two sentences of the proof). However, the first part also follows from the second part which can be proved using the results of this paper. In fact, there is $\kappa \in KL(A, B)$ which induces γ_i $(i = 0, 1)$. By Lemma 5.2, there is a unital homomorphism $\varphi : A \to B$ such that $(KL(\varphi), \varphi_T, \varphi_\Gamma^\ddagger) = (\kappa, \kappa_T, \varphi_\Gamma^\ddagger)$, which is totally compatible. Hence, the first part follows from the second part.

For the second part, by the UCT, there is a $\kappa^{-1} \in KL(B, A)$ so that $\kappa \times \kappa^{-1} = KL(\mathrm{id}_A)$ and $(\kappa^{-1}, \kappa_T^{-1})$ is compatible. It follows from Lemma 5.2 again that there is a unital homomorphism $\Psi : B \to A$ such that $KL(\Psi) = \kappa^{-1}$ and $\Psi_T = \kappa_T^{-1}$. Consider the endomorphism $\Psi \circ \varphi : A \to A$. Then, $KL(\Psi \circ \varphi) = KL(\mathrm{id}_A)$ and $(\Psi \circ \varphi)_T = \mathrm{id}_{T(A)}$. By Theorem 5.1, there is $h \in \mathrm{Hom}_{\mathrm{alf}}(K_1(A), \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))})$ such that

$$\left(\mathrm{id}_A^{\ddagger} - (\Psi \circ \varphi)^{\ddagger}\right) \circ s_A = h.$$

It follows from Theorem 5.6 that there is a unital homomorphism $H' : A \to A$ such that $KL(H') = KL(\mathrm{id}_A)$, $H'_T = \mathrm{id}_{T(A)}$, and $((H')^{\ddagger} - \mathrm{id}_A^{\ddagger}) \circ s_A = h$. Then, $KL(H \circ \Psi \circ \varphi) = KL(\mathrm{id}_A)$, $(H' \circ \Psi \circ \varphi)_T = \mathrm{id}_{T(A)}$, and

$$(H' \circ \Psi \circ \varphi)^{\ddagger} = \mathrm{id}_{U(A)/CU(A)}.$$

Put $H := H' \circ \Psi : B \to A$. Then, $KL(H) = \kappa^{-1}$, $H_T = \kappa_T^{-1}$, and $H^{\ddagger} = \kappa_\gamma^{-1}$. It follows from Theorem 4.3 that $H \circ \varphi$ is approximately unitarily equivalent to id_A and $\varphi \circ H$ is approximately unitarily equivalent to id_B. By the standard Elliott approximately intertwining argument, one obtains an isomorphism $\psi : A \to B$ such that $KL(\psi) = KL(\varphi)$, $\varphi_T = \kappa_T$, and $\varphi^{\ddagger} = \kappa_\gamma$. It follows from (3) of Proposition 6.4 that $\varphi_\Gamma^{\ddagger} = \kappa_\Gamma$. ∎

Remark 6.8. Our method heavily depends on [6, 7] and is in the same lines of those of [18]. Two ingredients of the proof are Winter's deformation method [26] and the asymptotic unitary equivalence of homomorphisms (see [15]). As this note was being drafted, we were aware that a general result of this type has been announced by J. Carrion, J. Gabe, C. Schafhauser, A. Tikuisis, and S. White which we understand do not use Winter's deformation method [26] and the asymptotic unitary equivalence of homomorphisms.

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References

[1] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, Nuclear dimension of simple C*-algebras. *Invent. Math.* **224** (2021), no. 1, 245–290 Zbl 1467.46055 MR 4228503

[2] M. Dadarlat and T. A. Loring, A universal multicoefficient theorem for the Kasparov groups. *Duke Math. J.* **84** (1996), no. 2, 355–377 Zbl 0881.46048 MR 1404333

[3] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, On the classification of simple amenable C^*-algebras with finite decomposition rank, II. 2015, arXiv:1507.03437v1

[4] G. Gong, C. Jiang, and L. Li, Hausdorffified algebraic K_1-groups and invariants for C^*-algebras with the ideal property. *Ann. K-Theory* **5** (2020), no. 1, 43–78 Zbl 1441.19010 MR 4078225

[5] G. Gong, C. Jiang, and L. Li, A classification of inductive limit C^*-algebras with ideal property. *Trans. London Math. Soc.* **9** (2022), no. 1, 158–236 MR 4535661

[6] G. Gong, H. Lin, and Z. Niu, A classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, I: C^*-algebras with generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Can.* **42** (2020), no. 3, 63–450 MR 4215379

[7] G. Gong, H. Lin, and Z. Niu, A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, II: C*-algebras with rational generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Can.* **42** (2020), no. 4, 451–539 MR 4215380

[8] G. Gong, H. Lin, and Y. Xue, Determinant rank of C^*-algebras. *Pacific J. Math.* **274** (2015), no. 2, 405–436 Zbl 1335.46049 MR 3332910

[9] J. Hua and H. Lin, Rotation algebras and the Exel trace formula. *Canad. J. Math.* **67** (2015), no. 2, 404–423 Zbl 1328.46042 MR 3314840

[10] X. Jiang and H. Su, On a simple unital projectionless C^*-algebra. *Amer. J. Math.* **121** (1999), no. 2, 359–413 Zbl 0923.46069 MR 1680321

[11] H. Lin, A separable Brown–Douglas–Fillmore theorem and weak stability. *Trans. Amer. Math. Soc.* **356** (2004), no. 7, 2889–2925 Zbl 1053.46032 MR 2052601

[12] H. Lin, AF-embedding of crossed products of AH-algebras by $\mathbb{Z}$ and asymptotic AF-embedding. *Indiana Univ. Math. J.* **57** (2008), no. 2, 891–944 Zbl 1155.46022 MR 2414337

[13] H. Lin, Asymptotically unitary equivalence and asymptotically inner automorphisms. *Amer. J. Math.* **131** (2009), no. 6, 1589–1677 Zbl 1209.46029 MR 2567503

[14] H. Lin, Unitaries in a simple C^*-algebra of tracial rank one. *Internat. J. Math.* **21** (2010), no. 10, 1267–1281 Zbl 1217.46036 MR 2748190

[15] H. Lin, Asymptotic unitary equivalence and classification of simple amenable C^*-algebras. *Invent. Math.* **183** (2011), no. 2, 385–450 Zbl 1255.46031 MR 2772085

[16] H. Lin, Homomorphisms from AH-algebras. *J. Topol. Anal.* **9** (2017), no. 1, 67–125 Zbl 1370.46035 MR 3594607

[17] H. Lin and Z. Niu, The range of a class of classifiable separable simple amenable C^*-algebras. *J. Funct. Anal.* **260** (2011), no. 1, 1–29 Zbl 1213.46057 MR 2733569

[18] H. Lin and Z. Niu, Homomorphisms into simple $\mathcal{Z}$-stable C^*-algebras. *J. Operator Theory* **71** (2014), no. 2, 517–569 Zbl 1389.46075 MR 3214650

[19] H. Lin and Z. Niu, Asymptotic unitary equivalence in C^*-algebras. *Russ. J. Math. Phys.* **22** (2015), no. 3, 336–360 Zbl 1344.46039 MR 3392522

[20] H. Lin and W. Sun, Tensor products of classifiable C^*-algebras. *J. Topol. Anal.* **9** (2017), no. 3, 485–504 Zbl 1380.46047 MR 3661652

[21] C. Pasnicu, Shape equivalence, nonstable K-theory and AH algebras. *Pacific J. Math.* **192** (2000), no. 1, 159–182 Zbl 1092.46513 MR 1741023

[22] M. A. Rieffel, The homotopy groups of the unitary groups of noncommutative tori. *J. Operator Theory* **17** (1987), no. 2, 237–254 Zbl 0656.46056 MR 887221

[23] M. Rørdam, The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras. *Internat. J. Math.* **15** (2004), no. 10, 1065–1084 Zbl 1077.46054 MR 2106263

[24] M. Rørdam and W. Winter, The Jiang–Su algebra revisited. *J. Reine Angew. Math.* **642** (2010), 129–155 Zbl 1209.46031 MR 2658184

[25] K. Thomsen, Traces, unitary characters and crossed products by **Z**. *Publ. Res. Inst. Math. Sci.* **31** (1995), no. 6, 1011–1029 Zbl 0853.46073 MR 1382564

[26] W. Winter, Localizing the Elliott conjecture at strongly self-absorbing C^*-algebras. *J. Reine Angew. Math.* **692** (2014), 193–231 Zbl 1327.46058 MR 3274552

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