

Homology and K -theory of dynamical systems II. Smale spaces with totally disconnected transversal

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Abstract. We apply our previous work on the relation between groupoid homology and K -theory to Smale spaces. More precisely, we consider the unstable equivalence relation of a Smale space with totally disconnected stable sets and prove that the associated spectral sequence shows Putnam’s stable homology groups on the second sheet. Moreover, this homology is in fact isomorphic to the groupoid homology of the unstable equivalence relation.

1. Introduction

Continuing our previous work on the study of groupoid homology and operator K -theory for ample groupoids, in this paper, we look at the groupoids arising from Smale spaces and Putnam’s homology [32].

The framework of Smale spaces was introduced by Ruelle [36], who designed them to model the basic sets of Axiom A diffeomorphisms [37]. This turned out to be a particularly nice class of hyperbolic topological dynamical systems, where Markov partitions provide a symbolic approximation of the dynamics.

Groupoids with a totally disconnected base (ample groupoids) arise from Smale spaces with totally disconnected stable sets. This is especially useful in the study of dynamical systems whose topological dimension is not zero, but whose dynamics is completely captured by a restriction to a totally disconnected transversal. Such spaces include generalized solenoids [40, 45], substitution tiling spaces [2, Theorem 3.3], and dynamical systems of self-similar group actions [25], and they can be characterized by certain inverse limits [43].

Beyond the theory of dynamical systems, these groupoids also play an important role in the theory of operator algebras, where they provide an invaluable source of examples of C^* -algebras. These are obtained by considering the (reduced) groupoid C^* -algebras $C_r^*(G)$ [35], generalizing the crossed product algebras for group actions on the Cantor set. The resulting C^* -algebras capture interesting aspects of the homoclinic and heteroclinic structure of expansive dynamics [17, 29, 39], extending the correspondence between shifts of finite type and the Cuntz–Krieger algebras.

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In [28], based on the Meyer–Nest theory of triangulated categorical structure on the equivariant KK-theory [20–22], we constructed a spectral sequence of the form

$$E_{pq}^2 = H_p(G, K_q(\mathbb{C})) \Rightarrow K_{p+q}(C_r^*(G)) \tag{1}$$

for étale groupoids G that have torsion-free stabilizers and satisfy a strong form of the Baum–Connes conjecture as in Tu’s work [41]. While this directly applies to the reduction of the unstable equivalence relation to the transversals of Smale spaces as above, there is another homology theory proposed by Putnam [32]. We show that one of the variants, H_\bullet^s , fits into this scheme for the groupoid $R^u(Y, \psi)$ of the unstable equivalence relation on the underlying space, as follows.

Theorem A (Theorem 4.9). *Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets and $R^u(Y, \psi)$ the groupoid of the unstable equivalence relation. Then, there is a convergent spectral sequence*

$$E_{pq}^2 = E_{pq}^3 = H_p^s(Y, \psi) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*(R^u(Y, \psi))).$$

This result gives a partial answer to a question raised by Putnam, who aimed to relate the K -theory of $C^*(R^u(Y, \psi))$ to his homology theory $H_\bullet^s(Y, \psi)$ [32, Question 8.4.1]. An immediate consequence is that the K -groups of $C^*(R^u(Y, \psi))$ are of a finite rank. Our proof of Theorem A above is based on a “relativized” analogue of the method we developed in [28].

Although we give an independent proof of Theorem A, it can also be obtained from the spectral sequence (1) and the result below.

Theorem B (Theorem 5.1). *Let G be an étale groupoid, and suppose it is Morita equivalent to $R^u(Y, \psi)$, we have an isomorphism $H_p^s(Y, \psi) \simeq H_p(G, \mathbb{Z})$.*

In order to prove the result above, we turn the definition of Putnam’s homology into a resolution of modules which computes groupoid homology. As a corollary, we obtain a Künneth formula for H_\bullet^s , generalizing a result in [8]. In the framework of substitution tiling spaces [2], this result, combined with those of [28], implies that $H_\bullet^s(\Omega, \omega)$ for the associated Smale space (Ω, ω) is isomorphic to the degree shift of the Čech cohomology of Ω (this partially solves [32, Question 8.3.2]).

This paper is organized as follows. In Section 2, we lay out the basic notation and definitions for all the background objects of the paper.

In Section 3, we discuss the multiple fibered product of groupoid homomorphisms, generalizing a construction described in [4]. This provides the spatial implementation of the groupoid bar complex in the case of the inclusion map $G^{(0)} \rightarrow G$ regarded as a groupoid homomorphism. Turning to Smale spaces, a key technical transversality result in Proposition 3.9 allows us to relate multiple fiber products of s -bijective maps $f: (\Sigma, \sigma) \rightarrow (Y, \psi)$ from shifts of finite type to the multiple groupoid fibered products. This is a crucial ingredient for Putnam’s homology.

In Section 4, we consider a simplicial object relating homology and K -groups of the groupoid following the scheme of [28]. We recall that the spectral sequence (1) appeared from the Moore complex of the simplicial object $(F(L^{n+1}A))_{n=0}^\infty$ with $L = \text{Ind}_X^G \text{Res}_X^G$, converging to $F(A)$, with the functor $F = K_\bullet(G \ltimes -)$. For the groupoid of the unstable equivalence relation on a Smale space (Y, ψ) with totally disconnected stable sets, we follow the same scheme but replace X by the subgroupoid coming from an s -bijective factor map from a shift of finite type. The resulting complex is isomorphic to the one defining Putnam’s homology $H_\bullet^s(Y, \psi)$.

In Section 5, we combine the previous sections and prove our main results.

Next, in Section 6, we explain how a similar method applies to the K -groups of Ruelle algebras. Here, we obtain a homology built out of Bowen–Franks groups closely following Putnam’s homology theory. In Section 7, we discuss some examples, including solenoids and self-similar group actions.

2. Preliminaries

In this section, we fix conventions and go over some preliminaries in order to clarify the basic notions that will be used in the rest of the paper. We generally follow our exposition in [28].

2.1. Locally compact groupoids

Throughout the paper, G denotes a topological groupoid with unit space $X = G^{(0)}$. We let $s, r: G \rightarrow X$ denote, respectively, the source and range maps. In addition, we let $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, and for a subset $A \subset X$, we write $G_A = \bigcup_{x \in A} G_x$, $G^A = \bigcup_{x \in A} G^x$, and $G|_A = G^A \cap G_A$.

Definition 2.1. A topological groupoid G is *étale* if s and r are local homeomorphisms and *ample* if it is étale and $G^{(0)}$ is totally disconnected.

If G is étale and $g \in G$, then by definition, for small enough neighborhoods U of $s(g)$, there is a neighborhood U' of g such that $s(U') = U$, and the restrictions of s and r to U' are homeomorphisms onto the images. When this is the case, we write $g(U) = r(U')$ and use g as a label for the map $U \rightarrow g(U)$ induced by the identification of $U \sim U' \sim g(U)$.

As our blanket assumption, we further assume that a topological groupoid is second countable, locally compact Hausdorff and admits a continuous Haar system $\lambda = (\lambda^x)_{x \in X}$ (i.e., an invariant continuous distribution of Radon measures on the spaces $(G^x)_{x \in X}$), so that its full and reduced groupoid C^* -algebras $C^*(G, \lambda)$, $C_r^*(G, \lambda)$ make sense. In general, these algebras may depend on λ , though different Haar systems lead to Morita equivalent C^* -algebras (see also [28, Remark 1.7]). In particular, G and X are σ -compact and paracompact. When dealing with étale groupoids, we take as usual the counting measure on G^x , and in this case, we suppress the notation λ and simply write $C_r^*(G)$ instead of

$C_r^*(G, \lambda)$. For a C^* -algebra A equipped with an action of G , the (reduced) crossed product of A by G in this paper will be equivalently denoted by $A \rtimes G$, $G \rtimes A$, or $C_r^*(G, A)$ depending on what is easier to read in context.

A locally compact groupoid G is *amenable* if there exists a net of probability measures on G^x for $x \in G^{(0)}$ which is approximately invariant; see [1]. In this case, we have $C^*(G, \lambda) = C_r^*(G, \lambda)$ for any Haar system λ . This covers all of our concrete examples.

The notion of Morita equivalence of groupoids in the sense of [23] plays an important role in this paper. We review it here for convenience. First, recall a topological groupoid G is *proper* if the map $(r \times s): G \rightarrow X \times X$ is proper. Furthermore, if Z is a locally compact, Hausdorff G -space, we say that G *acts properly* on Z if the transformation groupoid $G \times Z$ is proper. The map $Z \rightarrow G^{(0)}$ underlying the G -action is called the *anchor map*.

Definition 2.2. The groupoids G and H are *Morita equivalent* if there is a locally compact Hausdorff space Z such that

- Z is a free and proper left G -space with anchor map $\rho: Z \rightarrow G^{(0)}$;
- Z is a free and proper right H -space with anchor map $\sigma: Z \rightarrow H^{(0)}$;
- the actions of G and H on Z commute;
- $\rho: Z \rightarrow G^{(0)}$ induces a homeomorphism $Z/H \rightarrow G^{(0)}$;
- $\sigma: Z \rightarrow H^{(0)}$ induces a homeomorphism $G \backslash Z \rightarrow H^{(0)}$.

This can be conveniently packaged by a *bibundle* over G and H : that is, a topological space Z with G and H acting continuously from both sides with surjective and open anchor maps, such that the maps

$$\begin{aligned} G \times_{G^{(0)}} Z &\rightarrow Z \times_{H^{(0)}} Z, & (g, z) &\mapsto (gz, z), \\ Z \times_{H^{(0)}} H &\rightarrow Z \times_{G^{(0)}} Z, & (z, h) &\mapsto (z, zh) \end{aligned}$$

are homeomorphisms.

An important class of Morita equivalences comes from generalized transversals. For a topological space X and $x \in X$, let us denote the family of the open neighborhoods of x by $\mathcal{O}(x)$.

Definition 2.3 ([34]). Let G be a topological groupoid. A *generalized transversal* for G is given by a topological space T and an injective continuous map $f: T \rightarrow G^{(0)}$ such that

- $f(T)$ meets every orbit of G ;
- the *condition (Ar)* for the neighborhoods of x and $f^{-1}(rx)$ holds for all $x \in G$, i.e.,

$$\begin{aligned} \forall x \in G^{f(T)}, U_0 \in \mathcal{O}(x), V_0 \in \mathcal{O}(f^{-1}(rx)) \quad \exists U \in \mathcal{O}(x), V \in \mathcal{O}(f^{-1}(rx)): \\ U \subset U_0, V \subset V_0, \forall y \in U \quad \exists! z \in U, s(y) = s(z), r(z) \in f(V). \end{aligned}$$

Under the above setting, there is a finer topology on the subgroupoid $H = G|_{f(T)}$ such that H is étale and Morita equivalent to G [34, Theorem 3.6]. The equivalence is

implemented by the principal bibundle $G^{f(T)}$ with a natural finer topology from that of G and T .

2.2. Induction and restriction for groupoid KK-theory

Suppose G is an étale groupoid as in the previous subsection. We denote by KK^G the category of separable G - C^* -algebras with the equivariant KK-groups $\text{KK}^G(A, B)$ [15] as morphisms sets.

Let $H \subseteq G$ be an open subgroupoid with the same base space $X = G^{(0)} = H^{(0)}$. In particular, H is an étale groupoid over X , and the restriction of action gives a functor $\text{Res}_H^G: \text{KK}^G \rightarrow \text{KK}^H$. It admits a left adjoint, which is an analogue of induction, as follows. Full details will appear elsewhere in a joint work of the first named author with C. Bönicke.

Let B be an H - C^* -algebra, with structure map $\rho: C_0(X) \rightarrow Z(\mathcal{M}(B))$. As before, take the $C_0(G)$ -algebra

$$B' = C_0(G)^s \otimes_{C_0(X)} B,$$

where the superscript s indicates that we regard $C_0(G)$ as a $C_0(X)$ -algebra with respect to the source map. This has a right action of H , by the combination of the right translation on $C_0(G)$ and the action on B twisted by the inverse map of H . We then set

$$\text{Ind}_H^G(B) = B' \rtimes H = (C_0(G)^s \otimes_{C_0(X)} B) \rtimes_{\text{diag}} H.$$

This can be regarded as the crossed product of B' by the transformation groupoid $G \rtimes H$ for the right translation action of H on G . Moreover, notice that G also acts on B' by left translation on $C_0(G)$. This induces a continuous action of G on $\text{Ind}_H^G(B)$.

Let A be a G - C^* -algebra. Then, the Haar system on G induces an A -valued inner product on $C_c(G) \otimes_{C_0(X)} A$, and by completion, we obtain a right Hilbert A -module $E_A^G = L^2(G, A)$. We then have the following; see Appendix A for details.

Proposition 2.4. *Under the above setting, E_A^G implements an equivariant strong Morita equivalence between A and $\text{Ind}_G^G A$.*

Let κ denote the inclusion homomorphism

$$\text{Ind}_H^G \text{Res}_H^G(A) = (C_0(G)^s \otimes_{C_0(X)} A) \rtimes H \rightarrow (C_0(G)^s \otimes_{C_0(X)} A) \rtimes G = \text{Ind}_G^G A,$$

induced by $H \subseteq G$ because H is open, and let ι denote the map

$$\text{Ind}_H^H B = (C_0(H)^s \otimes_{C_0(X)} B) \rtimes H \rightarrow (C_0(G)^s \otimes_{C_0(X)} B) \rtimes H = \text{Res}_H^G \text{Ind}_H^G(B),$$

induced by the ideal inclusion $C_0(H) \subseteq C_0(G)$.

Theorem 2.5. *The functor Ind_H^G induces a functor $\text{KK}^H \rightarrow \text{KK}^G$, and there is a natural isomorphism*

$$\text{KK}^G(\text{Ind}_H^G B, A) \simeq \text{KK}^H(B, \text{Res}_H^G A)$$

defining an adjunction $(\varepsilon, \eta): \text{Ind}_H^G \dashv \text{Res}_H^G$ with counit and unit natural morphisms

$$\begin{aligned} \varepsilon_A &= [\kappa] \otimes_{\text{Ind}_G^G A} [E_A^G] \in \text{KK}^G(\text{Ind}_H^G \text{Res}_H^G A, A), \\ \eta_B &= [\tilde{E}_B^H] \otimes_{\text{Ind}_H^H B} [l] \in \text{KK}^H(B, \text{Res}_H^G \text{Ind}_H^G B). \end{aligned}$$

Example 2.6. If G is the transformation groupoid $\Gamma \times X$ and $H = X$, the previous theorem amounts to

$$\text{KK}^{\Gamma \times X}(C_0(\Gamma) \otimes B, A) \simeq \text{KK}^X(B, A)$$

for any $C_0(X)$ -algebra B and G -algebra A , where the Γ -action on $C_0(\Gamma) \otimes B$ is given by translation on the factor $C_0(\Gamma)$.

2.3. Triangulated categories and spectral sequences

Let us quickly recall the formalism of [28] behind the spectral sequence in (1). The main ingredients are triangulated categories \mathcal{S} and \mathcal{T} with countable direct sums, and exact functors $E: \mathcal{S} \rightarrow \mathcal{T}$ and $F: \mathcal{T} \rightarrow \mathcal{S}$ compatible with countable direct sums, with natural isomorphisms

$$\mathcal{S}(A, FB) \simeq \mathcal{T}(EA, B) \quad (A \in \mathcal{S}, B \in \mathcal{T}).$$

Let \mathcal{I} denote the kernel of F , that is, the collection of morphisms f in \mathcal{S} such that $Ff = 0$. An object $A \in \mathcal{T}$ is said to be \mathcal{I} -projective if any $f \in \mathcal{I}(A', A'')$ induces the zero map $\mathcal{T}(A, A') \rightarrow \mathcal{T}(A, A'')$. The category \mathcal{T} has two triangulated subcategories: one $\langle E\mathcal{S} \rangle$ generated by the image of E and another $\mathcal{N}_{\mathcal{I}}$ consisting of the objects N satisfying $FN = 0$.

Now, consider the endofunctor $L = EF$ on \mathcal{T} .

Proposition 2.7 ([28, Proposition 3.1]). *In the above setting, any object $A \in \mathcal{T}$ admits an \mathcal{I} -projective resolution P_\bullet consisting of $P_n = L^{n+1}A$. The pair of subcategories $(\langle E\mathcal{S} \rangle, \mathcal{N}_{\mathcal{I}})$ is complementary.*

In particular, for any $A \in \mathcal{T}$, there is an exact triangle

$$P \rightarrow A \rightarrow N \rightarrow \Sigma P$$

satisfying $P \in \langle E\mathcal{S} \rangle$ and $N \in \mathcal{N}_{\mathcal{I}}$. By [20, Theorems 4.3 and 5.1], we then get the following.

Theorem 2.8. *Let $K: \mathcal{T} \rightarrow \text{Ab}$ be a homological functor to the category of commutative groups, and write $K_n(A) = K(\Sigma^{-n}A)$. With notation as above, there is a convergent spectral sequence*

$$E_{pq}^r \Rightarrow K_{p+q}(P),$$

with the E^2 -sheet $E_{pq}^2 = H_p(K_q(P_\bullet))$.

The Baum–Connes conjecture for groupoids allows us to compare P and A . We are going to use the following fundamental result proved by J.-L. Tu.

Theorem 2.9 ([41]). *Suppose that G has the Haagerup property. Then, there exists a proper G -space Z with an open surjective structure morphism $Z \rightarrow X$ and a $G \ltimes Z$ - C^* -algebra P which is a continuous field of nuclear C^* -algebras over Z , such that $P \simeq C_0(X)$ in KK^G .*

For the case of $\mathcal{T} = \text{KK}^G$, $\mathcal{S} = \text{KK}^X$, $E = \text{Ind}_X^G$, and $F = \text{Res}_X^G$, we have the following.

Proposition 2.10 ([28]). *Let G be an étale groupoid with torsion-free stabilizers and satisfying the conclusions of Theorem 2.9. Any separable G - C^* -algebra A belongs to the localizing subcategory generated by the objects $\text{Ind}_X^G B$ for $C_0(X)$ -algebras B .*

2.4. Smale spaces

Next, let us recall basic definitions on Smale spaces, mostly following [32].

Definition 2.11. A *Smale space* (X, ϕ) is given by a compact metric space (X, d) and a homeomorphism $\phi: X \rightarrow X$ such that

- there exist constant $0 < \varepsilon_X$ and a continuous map

$$\{(x, y) \in X \times X \mid d(x, y) \leq \varepsilon_X\} \rightarrow X, \quad (x, y) \mapsto [x, y]$$

satisfying the *bracket axioms*

$$\begin{aligned} [x, x] &= x, & [x, [y, z]] &= [x, z], \\ [[x, y], z] &= [x, z], & \phi([x, y]) &= [\phi(x), \phi(y)], \end{aligned}$$

for any x, y, z in X when both sides are defined;

- there exists $0 < \lambda < 1$ satisfying the *contraction axioms*

$$\begin{aligned} [x, y] = y &\Rightarrow d(\phi(x), \phi(y)) \leq \lambda d(x, y), \\ [x, y] = x &\Rightarrow d(\phi^{-1}(x), \phi^{-1}(y)) \leq \lambda d(x, y), \end{aligned}$$

whenever the brackets are defined.

Suppose $x \in X$ and $0 < \varepsilon \leq \varepsilon_X$. We define the *local stable sets* and the *local unstable sets* around x as

$$\begin{aligned} X^s(x, \varepsilon) &= \{y \in X \mid d(x, y) < \varepsilon, [y, x] = x\}, \\ X^u(x, \varepsilon) &= \{y \in X \mid d(x, y) < \varepsilon, [x, y] = x\}. \end{aligned}$$

The bracket $[x, y]$ can be characterized as the unique element of $X^s(x, \varepsilon) \cap X^u(y, \varepsilon)$ when $2d(x, y) < \varepsilon < \varepsilon_X$. This means that, locally, we can choose coordinates so that (see [33, Theorem 4.1.4])

$$[-, -]: X^u(x, \varepsilon) \times X^s(x, \varepsilon) \rightarrow X$$

is a homeomorphism onto an open neighborhood of $x \in X$ for all ε smaller than some $\varepsilon'_X \leq \varepsilon_X/2$.

A point $x \in X$ is called *non-wandering* if for all open sets $U \subseteq X$ containing x there exists $N \in \mathbb{N}$ with $U \cap \phi^N(U) \neq \emptyset$. Periodic points are dense among the non-wandering points [33, Theorem 4.4.1]. We say that X is non-wandering if any point of X is non-wandering. We will set a blanket assumption that Smale spaces are non-wandering. This holds in virtually all interesting examples.

It can be shown that any non-wandering Smale space (X, ϕ) can be partitioned in a finite number of ϕ -invariant clopen sets X_1, \dots, X_n , in a unique way, such that $(X_k, \phi|_{X_k})$ is *irreducible* for $k = 1, \dots, n$ [30]. Irreducibility means that for every (ordered) pair U, V of nonempty open sets in X , there exists $N \in \mathbb{N}$ such that $U \cap \phi^N(V) \neq \emptyset$.

Example 2.12. A fundamental example of Smale space is given by *shift of finite type* (or *topological Markov shift*). This can be modeled by finite directed graphs, as follows. A *directed graph* $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, i, t)$ consists of finite sets \mathcal{G}^0 and \mathcal{G}^1 , called vertices and edges, and maps $i, t: \mathcal{G}^1 \rightarrow \mathcal{G}^0$. Thus, each edge $e \in \mathcal{G}^1$ represents a directed arrow from $i(e) \in \mathcal{G}^0$ to $t(e) \in \mathcal{G}^0$. Then, a shift of finite type $(\Sigma_{\mathcal{G}}, \sigma)$ is defined as the space of bi-infinite sequences of paths

$$\Sigma_{\mathcal{G}} = \{e = (e_k)_{k \in \mathbb{Z}} \in (\mathcal{G}^1)^{\mathbb{Z}} \mid t(e_k) = i(e_{k+1})\},$$

together with the left shift map $\sigma(e)_k = e_{k+1}$. The metric is defined by $d(e, f) = 2^{-n-1}$ for $e \neq f$, where n is the largest integer such that $e_k = f_k$ for $|k| \leq n$. In particular, $d(e, f) \leq 2^{-1}$ means that e, f share the central edge, i.e., $e_0 = f_0$. Then, we can define

$$[e, f] = (\dots, f_{-2}, f_{-1}, e_0, e_1, e_2, \dots).$$

The pair $(\Sigma_{\mathcal{G}}, \sigma)$ is a Smale space with constant $\varepsilon = 1/2$.

We are particularly interested in groupoids encoding the *unstable equivalence relation* of Smale spaces. Given $x, y \in X$, we say they are

- *stably equivalent*, denoted by $x \sim_s y$, if

$$\lim_{n \rightarrow \infty} d(\phi^n(x), \phi^n(y)) = 0;$$

- *unstably equivalent*, $x \sim_u y$, if

$$\lim_{n \rightarrow \infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0.$$

We denote the graphs of these relations as

$$\begin{aligned} R^s(X, \phi) &= \{(x, y) \in X \times X \mid x \sim_s y\}, \\ R^u(X, \phi) &= \{(x, y) \in X \times X \mid y \sim_u x\}, \end{aligned}$$

and treat them as groupoids, with source, range, and composition maps given by

$$s(x, y) = y, \quad r(x, y) = x, \quad (x, y) \circ (w, z) = (x, z) \quad \text{if } y = w.$$

The class of $x \in X$ under the stable (resp., unstable) equivalence relation is called the *global stable* (resp., *unstable*) *set* and is denoted by $X^s(x)$ (resp., $X^u(x)$). They satisfy the following identities:

$$X^s(x) = \bigcup_{n \geq 0} \phi^{-n}(X^s(\phi^n(x), \varepsilon)), \tag{2}$$

$$X^u(x) = \bigcup_{n \geq 0} \phi^n(X^s(\phi^{-n}(x), \varepsilon)), \tag{3}$$

for any fixed $\varepsilon < \varepsilon_X$.

This leads to locally compact Hausdorff topologies on the above groupoids [29]: consider the induced topology on

$$G_s^n = \{(x, y) \mid y \in \phi^{-n}(X^s(\phi^n(x), \varepsilon))\}, \quad G_u^n = \{(x, y) \mid y \in \phi^n(X^u(\phi^{-n}(x), \varepsilon))\}$$

as subsets of $X \times X$. Then, $R^u(X, \phi)$ is the union of the increasing sequence $(G_u^n)_n$, with *open* inclusion maps $G_u^n \rightarrow G_u^{n+1}$. This makes $G = R^u(X, \phi)$ a locally compact Hausdorff groupoid. Moreover, the *Bowen measure* defines a Haar system on G . Of course, analogous considerations make $R^s(X, \phi)$ a locally compact Hausdorff groupoid with a Haar system.

To get an étale groupoid, we can take a transversal $T \subset X$ and restrict the base space to T , putting $G|_T = G_T^T$. A convenient choice is to take $T = X^s(x)$ for some $x \in X$, with the inductive limit topology from (2), which is an example of generalized transversal. Slightly generalizing this, for a subset $P \subseteq X$, we write $X^s(P)$ meaning the union of all $X^s(x)$'s for $x \in P$, with the disjoint union topology. Analogously we define $X^u(P) = \bigcup_{x \in P} X^u(x)$. Let us put

$$R^s(X, P) = R^s(X, \phi)|_{X^u(P)}, \quad R^u(X, P) = R^u(X, \phi)|_{X^s(P)}. \tag{4}$$

Theorem 2.13 ([34, Theorem 1.1]). *The groupoids in (4) are amenable.*

2.5. Maps of Smale spaces

Given two Smale spaces (X, ϕ) and (Y, ψ) , a continuous and surjective map $f: X \rightarrow Y$ is called a *factor map* if it intertwines the respective self-maps, i.e.,

$$f \circ \phi = \psi \circ f. \tag{5}$$

Equation (5) is enough to guarantee that f preserves the local product structure. In particular, there is $\varepsilon_f > 0$ such that both $[x_1, x_2]$ and $[f(x_1), f(x_2)]$ are defined and $f([x_1, x_2]) = [f(x_1), f(x_2)]$ for all x_1, x_2 with $d(x_1, x_2) < \varepsilon_f$.

Given $\varepsilon > 0$ and $x \in X$, let us write $X(x, \varepsilon) = \{y \in X \mid d(y, x) < \varepsilon\}$.

Proposition 2.14 ([33, Lemma 5.1.11]). *Let $f: X \rightarrow Y$ be a factor map of Smale spaces and $y_0 \in Y$ a point whose preimage is finite. Then, given $\varepsilon > 0$, there exists $\delta = \delta(z, \varepsilon) > 0$*

such that

$$f^{-1}(Y(y_0, \delta)) \subseteq \bigcup_{i=1}^N X(x_i, \varepsilon)$$

where we write $f^{-1}(y_0) = \{x_1, \dots, x_N\}$.

The assumption is automatically satisfied when (Y, ψ) is non-wandering.

Definition 2.15. A factor map $f: (X, \phi) \rightarrow (Y, \psi)$ is called *s-resolving* if it induces an injective map from $X^s(x)$ to $Y^s(f(x))$ for each $x \in X$. It is called *s-bijective* if, moreover, these induced maps are bijective.

Theorem 2.16 ([31, Corollary 3]). *Let (X, ϕ) be an irreducible Smale space such that $X^s(x, \varepsilon)$ is totally disconnected for every $x \in X$ and $0 < \varepsilon < \varepsilon_X$. Then, there is an irreducible shift of finite type (Σ, σ) and an s-bijective factor map $f: (\Sigma, \sigma) \rightarrow (X, \phi)$.*

Theorem 2.17 ([33, Theorem 5.2.4]). *Let $f: X \rightarrow Y$ be an s-resolving map between Smale spaces. There is a constant $N \geq 1$ such that, for any $y \in Y$, there exist x_1, \dots, x_n in X , with $n \leq N$, satisfying*

$$f^{-1}(Y^u(y)) = \bigcup_{k=1}^n X^u(x_k).$$

For any $y \in Y$, the cardinality of the fiber $f^{-1}(y)$ is less than or equal to N .

Let us list several additional facts about s-resolving maps, which can be found in [33]. First, if each point in Y is non-wandering, then f is s-bijective. Second, the induced maps $X^s(x) \rightarrow Y^s(f(x))$ and $X^u(x) \rightarrow Y^u(f(x))$ are both continuous and proper in the inductive limit topology of the presentation in (2) and (3). If, moreover, f is s-bijective, the map $X^s(x) \rightarrow Y^s(f(x))$ is a homeomorphism. Assume that X and Y are irreducible, and P is an at most countable subset of X such that no two points of P are stably equivalent after applying f . Then,

$$f \times f: R^u(X, P) \rightarrow R^u(Y, f(P))$$

is a homeomorphism onto an open subgroupoid of $R^u(Y, f(P))$.

2.6. Putnam’s homology for Smale spaces

For any shift of finite type (Σ, σ) , the K_0 -group $K_0(C^*(R^u(\Sigma, \sigma)))$ can be described as Krieger’s *dimension group* $D^s(\Sigma, \sigma)$ [13]. This group is generated by the elements $[E]$ for compact open sets E in the stable orbits in Σ . We can restrict ourselves to a collection of stable orbits which form a generalized transversal and also assume that E is contained in a *local* stable orbit [27, Lemma 1.3].

Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets. Then, there is an irreducible shift of finite type (Σ, σ) and an s-bijective factor map $f: (\Sigma, \sigma) \rightarrow (Y, \psi)$. Then, we obtain a Smale space Σ_n by taking the fiber product of $(n + 1)$ -copies of Σ over Y and the diagonal action σ_n of σ on Σ_n , so that we have $\Sigma_0 = \Sigma$.

Then, we get a simplicial structure on the groups $(D^s(\Sigma_n, \sigma_n))_{n=0}^\infty$, whose face maps are induced by the maps $\delta_k^s: \Sigma_n \rightarrow \Sigma_{n-1}$ which delete the k -th entry of a point in Σ_n . This yields a well-defined map between the corresponding dimension groups, via the assignment $[E] \mapsto [\delta_k^s(E)]$. This way the groups $D^s(\Sigma_\bullet, \sigma_\bullet)$ form a simplicial object, and the associated homology $H_\bullet^s(Y, \psi)$, called *stable homology* of (Y, ψ) , does not depend on the choice of f [32, Section 5.5]; [27].

3. Fibered products of groupoids

We start by defining an appropriate notion of fibered product between groupoids which will be used in the following proofs.

Definition 3.1. Let $\alpha: H \rightarrow G$ be a homomorphism of groupoids, and $n \geq 2$. We define the n -th fibered product of H with respect to α as the groupoid $H^{\times_G n}$, as follows:

- the object space is the set

$$(H^{\times_G n})^{(0)} = \{(y_1, g_1, y_2, \dots, g_{n-1}, y_n) \mid y_k \in H^{(0)}, g_k \in G_{\alpha(y_{k+1})}^{\alpha(y_k)}\};$$

- the arrows from $(y_1, g_1, y_2, \dots, g_{n-1}, y_n)$ to $(y'_1, g'_1, y'_2, \dots, g'_{n-1}, y'_n)$ are given by the n -tuples $(h_1, \dots, h_n) \in H_{y_1}^{y'_1} \times \dots \times H_{y_n}^{y'_n}$ such that the squares in

$$\begin{array}{ccccccc} \alpha(y'_1) & \xleftarrow{g'_1} & \alpha(y'_2) & \xleftarrow{g'_2} & \dots & \xleftarrow{g'_{n-1}} & \alpha(y'_n) \\ \alpha(h_1) \uparrow & & \alpha(h_2) \uparrow & & & & \uparrow \alpha(h_n) \\ \alpha(y_1) & \xleftarrow{g_1} & \alpha(y_2) & \xleftarrow{g_2} & \dots & \xleftarrow{g_{n-1}} & \alpha(y_n) \end{array}$$

are all commutative.

Of course, we can put $H^{\times_G 1} = H$. We say that an arrow in $H^{\times_G n}$ is represented by the tuple $(h_1, g'_1, h_2, \dots, g'_{n-1}, h_n)$ in the above situation. This way we can think of $H^{\times_G n}$ as a subset of $H \times G \times \dots \times G \times H$, and in the setting of topological groupoids, this gives a compatible topology on $H^{\times_G n}$ (for example, local compactness passes to $H^{\times_G n}$).

Remark 3.2. The above definition makes sense for n -tuples of different homomorphisms $\alpha_k: H_k \rightarrow G$, so that we can define $H_1 \times_G \dots \times_G H_n$ as a groupoid. The case of $n = 2$ appears in [4].

We will need a slight generalization of Definition 3.1 falling under this more general setting, where $H_j = H$ except for one value $j = k$, with $H_k = G$.

Definition 3.3. In the setting of Definition 3.1, define a groupoid $G \times_G H^{\times_G n}$ as follows:

- the object space is the set

$$\begin{aligned} &(G \times_G H^{\times_G n})^{(0)} \\ &= \{(g_0, y_1, g_1, y_2, \dots, g_{n-1}, y_n) \mid y_k \in H^{(0)}, g_0 \in G_{\alpha(y_1)}, g_k \in G_{\alpha(y_{k+1})}^{\alpha(y_k)} (k \geq 1)\}; \end{aligned}$$

- a morphism from $(g_0, y_1, g_1, y_2, \dots, g_{n-1}, y_n)$ to $(g'_0, y'_1, g'_1, y'_2, \dots, g'_{n-1}, y'_n)$ is given by $k \in G_{rg'_0}^{rg'_0}$ and an n -tuple $(h_1, \dots, h_n) \in H_{y'_1}^{y'_1} \times \dots \times H_{y'_n}^{y'_n}$ such that the squares in

$$\begin{array}{ccccccc}
 rg'_0 & \xleftarrow{g'_0} & \alpha(y'_1) & \xleftarrow{g'_1} & \alpha(y'_2) & \xleftarrow{g'_2} & \dots & \xleftarrow{g'_{n-1}} & \alpha(y'_n) \\
 k \uparrow & & \alpha(h_1) \uparrow & & \alpha(h_2) \uparrow & & & & \uparrow \alpha(h_n) \\
 rg_0 & \xleftarrow{g_0} & \alpha(y_1) & \xleftarrow{g_1} & \alpha(y_2) & \xleftarrow{g_2} & \dots & \xleftarrow{g_{n-1}} & \alpha(y_n)
 \end{array}$$

are all commutative.

Again we say that an arrow of $G \times_G H^{\times G^n}$ is represented by $(k, g'_0, h_1, \dots, h_n)$ in the above situation. As in the case of $H^{\times G^n}$, this induces a compatible topology in the setting of topological groupoids.

Proposition 3.4. *Let $\alpha: H \rightarrow G$ be a homomorphism of topological groupoids. Then, $H^{\times G^n}$ and $G \times_G H^{\times G^n}$ are Morita equivalent as topological groupoids.*

Proof. Consider the space

$$Z = \{(g_0, h_1, g_1, h_2, \dots, g_{n-1}, h_n) \mid (g_0, \dots, g_{n-1}) \in G^{(n)}, \alpha(rh_k) = sg_{k-1}\}.$$

We define a left action of $G \times_G H^{\times G^n}$ as follows. The anchor map is

$$Z \rightarrow (G \times_G H^{\times G^n})^{(0)}, \quad (g_0, h_1, \dots, h_n) \mapsto (g_0, rh_1, g_1, \dots, rh_n),$$

and an arrow of $G \times_G H^{\times G^n}$ with source $(g_0, rh_1, g_1, \dots, rh_n)$ acts by

$$(k, g'_0, h'_1, \dots, h'_n) \cdot (g_0, h_1, \dots, h_n) = (g'_0, h'_1 h_1, g'_1, \dots, h'_n h_n).$$

On the other hand, there is a right action of $H^{\times G^n}$ defined as follows. The anchor map is

$$Z \rightarrow (H^{\times G^n})^{(0)}, \quad (g_0, h_1, \dots, h_n) \mapsto (sh_1, g'_1, \dots, sh_n), \quad (g'_k = \alpha(h_k)^{-1} g_k \alpha(h_{k+1})).$$

An arrow of $H^{\times G^n}$ with a range $(sh_1, g'_1, \dots, sh_n)$ acts by

$$(g_0, h_1, \dots, h_n) \cdot (h''_1, h''_2, \dots, h''_n) = (g_0, h_1 h''_1, g_1, \dots, h_n h''_n).$$

We claim that Z is a bibundle implementing the Morita equivalence (compatibility with topology will be obvious from the concrete “coordinate transform” formulas).

Comparing between $Z \times_{(G \times_G H^{\times G^n})^{(0)}} Z$ and $Z \times_{(H^{\times G^n})^{(0)}} H^{\times G^n}$ amounts to a comparison of pairs (h_k, h'_k) with $rh_k = rh'_k$ on the one hand and the composable pairs $(h_k, h''_k) \in H^{(2)}$ on the other. There is a bijective correspondence between the two sides, given by the coordinate transform $h'_k = h_k h''_k$. Comparing $Z \times_{(H^{\times G^n})^{(0)}} Z$ with $G \times_G H^{\times G^n} \times_{(G \times_G H^{\times G^n})^{(0)}} Z$ amounts to comparing the following:

- on the side of $Z \times_{(H \times_{G^n})^{(0)}} Z$: $((g_0, h_1), (g'_0, h'_1))$ with $(g_0, \alpha(h_1)), (g'_0, \alpha(h'_1)) \in G^{(2)}$ and $sh_1 = sh'_1$, and $(h_k, h'_k) \in H^2$ with $sh_k = sh'_k$ for $k \geq 2$;
- on the side of $G \times_G H^{\times_{G^n}} \times_{(G \times_G H^{\times_{G^n}})^{(0)}} Z$: $(k, g''_0) \in G^{(2)}$, $(h_1, h''_1) \in H^{(2)}$ with $sh_1 = sg''_0$, and $(h_k, h''_k) \in H^{(2)}$ for $k \geq 2$.

Again we have a bijective correspondence by $h'_k = h''_k^{-1}$, $g_0 = g''_0 \alpha(h_1)^{-1}$, and $g'_0 = kg''_0 \alpha(h''_1)$. ■

A slight generalization is obtained by considering the groupoid $H^{\times_{G^a}} \times_G G \times_G H^{\times_{G^b}}$ for $a, b \geq 0$. This is defined as $H^{\times_G(a+b+1)}$ in Definition 3.1, with the difference that h_{a+1} is not in $H^{y_{a+1}}$ and instead in $G^{\alpha(y'_{a+1})}$.

Proposition 3.5. *The groupoid $H^{\times_{G^a}} \times_G G \times_G H^{\times_{G^b}}$ is Morita equivalent to $H^{\times_G(a+b)}$.*

Proof. Recall the construction in the proof of Proposition 3.4 for the Morita equivalence between $G \times_G H^{\times_{G^b}}$ and $H^{\times_{G^b}}$: we have the space

$$Z = \{(g_0, h_1, g_1, \dots, h_b) \mid (g_0, \dots, g_{b-1}) \in G^{(b)}, \alpha(rh_k) = rg_k\},$$

which is a bimodule between these groupoids. Based on this, put

$$\tilde{Z} = \{(h_1, g_1, h_2, \dots, g_a, g_{a+1}, h_{a+1}, g_{a+2}, \dots, h_{a+b}) \mid (g_1, \dots, g_{a+b}) \in G^{(a+b)}, \alpha(rh_k) = rg_k (k \leq a), \alpha(rh_k) = sg_k (k > a)\}.$$

This has obvious “composition” actions of $H^{\times_{G^a}} \times_G G \times_G H^{\times_{G^b}}$ from the left and $H^{\times_G(a+b)}$ from the right. By a similar argument as before, we can see that \tilde{Z} implements a Morita equivalence. ■

Next, let us show the compatibility of fiber products and generalized transversals.

Proposition 3.6. *Let $\alpha: H \rightarrow G$ be a homomorphism of topological groupoids and $f: T \rightarrow H^{(0)}$ a generalized transversal. Consider the space*

$$\tilde{T} = \{(t_1, g_1, t_2, \dots, t_n) \mid t_k \in T, g_k \in G_{f(t_{k+1})}^{f(t_k)}\}$$

with the induced topology from the natural embedding into $T^n \times G^{n-1}$. The map

$$\tilde{f}: \tilde{T} \rightarrow (H^{\times_{G^n}})^{(0)}, \quad (t_1, g_1, t_2, \dots, t_n) \mapsto (f(t_1), g_1, f(t_2), \dots, f(t_n))$$

is a generalized transversal for $H^{\times_{G^n}}$.

Proof. Let us check the conditions in Definition 2.3. First, \tilde{T} meets all orbits of $H^{\times_{G^n}}$. Indeed, if we take a point $(y_1, g_1, y_2, \dots, g_{n-1}, y_n) \in (H^{\times_{G^n}})^{(0)}$, we can find $t_k \in T$ and $h_k \in H_{y_k}^{f(t_k)}$ for $k = 1, \dots, n$. Then, there are unique g'_k such that (h_1, \dots, h_n) represents an arrow from $(y_1, g_1, y_2, \dots, g_{n-1}, y_n)$ to $(f(t_1), g'_1, \dots, f(t_n))$.

Next, we turn to checking condition (Ar). Thus, we consider an arrow x represented by $(h_1, g_1, h_2, \dots, g_{n-1}, h_n)$ with the range $rx = (f(t_1), g_1, f(t_2), \dots, g_{n-1}, f(t_n))$, open

neighborhood U_0 of x , and another V_0 of rx . We may assume that these neighborhoods are of the form

$$U_0 = (U'_1 \times U''_1 \times U'_2 \times \dots \times U'_n) \cap H^{\times G^n}, \quad (U'_k \in \mathcal{O}(h_k), U''_k \in \mathcal{O}(g_k))$$

$$V_0 = (V'_1 \times V''_1 \times V'_2 \times \dots \times V'_n) \cap \tilde{T}, \quad (V'_k \in \mathcal{O}(t_k), V''_k \in \mathcal{O}(g_k)).$$

Then, for each k , we can find $\tilde{U}_k \in \mathcal{O}(h_k)$ with $\tilde{U}_k \subset U'_k$, $\tilde{V}_k \in \mathcal{O}(t_k)$ with $\tilde{V}_k \subset V'_k$, realizing the condition (Ar). We claim that

$$U = (\tilde{U}_1 \times U''_1 \times \dots \times \tilde{U}'_n) \cap H^{\times G^n}, \quad V = (\tilde{V}_1 \times V''_1 \times \dots \times \tilde{V}_n) \cap \tilde{T}$$

do the job. Indeed, if $y = (\tilde{h}_1, \tilde{g}_1, \dots, \tilde{h}_n) \in U$, another element $z = (\tilde{h}'_1, \tilde{g}'_1, \dots, \tilde{h}'_n)$ as the same source as y if and only if $s\tilde{h}_k = s\tilde{h}'_k$ and $f(\tilde{h}_k)^{-1}\tilde{g}_k f(\tilde{h}_{k+1}) = f(\tilde{h}'_k)^{-1}\tilde{g}'_k f(\tilde{h}'_{k+1})$ hold for all k . Moreover, $rz \in \tilde{T}$ if and only if $rh'_k \in f(T)$ for all k . The elements \tilde{g}'_k are determined by the \tilde{h}'_k , and we can find such \tilde{h}'_k uniquely by condition (Ar) for U'_k and V'_k . ■

Suppose $f: T \rightarrow H^{(0)}$ is a generalized transversal for H such that $\alpha f: T \rightarrow G^{(0)}$ is also a transversal for G . Then, α induces a homomorphism of étale groupoids from $H' = H|_{f(T)}$ to $G' = G|_{\alpha f(T)}$.

Corollary 3.7. *In the setting above, $H^{\times G^n}$ is Morita equivalent to $H'^{\times G'^n}$.*

Proof. The construction of Proposition 3.6 gives a generalized transversal for $\tilde{f}: \tilde{T} \rightarrow (H^{\times G^n})^{(0)}$. The étale groupoid obtained by this is isomorphic to $H'^{\times G'^n}$. ■

Now, let (Y, ψ) be a non-wandering Smale space with totally disconnected unstable sets and $f: (\Sigma, \sigma) \rightarrow (Y, \psi)$ an s -resolving (hence, s -bijective) factor map from a shift of finite type.

Let Σ_n denote the fibered product of $n + 1$ copies of Σ with respect to f . Then, $\sigma_n = \sigma \times \dots \times \sigma|_{\Sigma_n}$ defines a Smale space, which is again a shift of finite type. If $a = (a^0, \dots, a^n)$ and $b = (b^0, \dots, b^n)$ are points of Σ_n , they are unstably (resp., stably) equivalent if and only if a^k is unstably (resp., stably) equivalent to b^k for all k .

Theorem 3.8. *In the setting above, set $G = R^u(Y, \psi)$, $H = R^u(\Sigma, \sigma)$, and $\alpha = f \times f: H \rightarrow G$ as the induced groupoid homomorphism. Then, $H^{\times G^{n+1}}$ is Morita equivalent to $R^u(\Sigma_n, \sigma_n)$ as a locally compact groupoid.*

We will apply this to the s -bijective maps from Theorem 2.16. A key step is the following proposition, which is our first technical result.

Proposition 3.9. *Let $f: (X, \phi) \rightarrow (Y, \psi)$ be an s -resolving map of Smale spaces. Suppose a_0, \dots, a_n in X are points such that $f(a_0) \sim_u f(a_k)$ for all k . Then, there are points b_0, \dots, b_n in X satisfying*

$$a_k \sim_u b_k, \quad f(b_0) = f(b_k)$$

for $k = 0, \dots, n$.

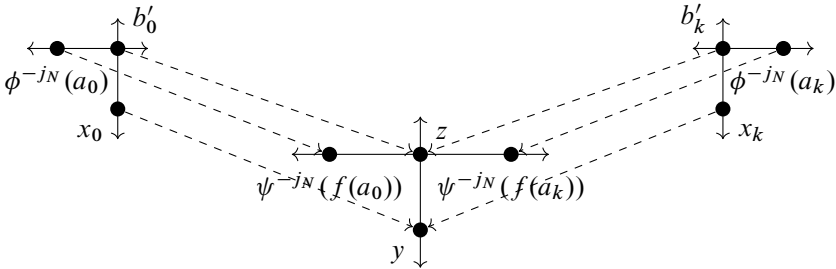


Figure 1. The configuration of points in the proof of Proposition 3.9. The vertical direction represents the stable direction, while the horizontal direction represents the unstable direction.

Proof. The configuration of points we consider in this proof is displayed in Figure 1. By compactness, there is a sequence of nonnegative integers $(j_n)_n$, $j_n \rightarrow \infty$, such that $\psi^{-j_n}(f(a_0))$ converges to a point $y \in Y$. Since $f(a_k) \sim_u f(a_0)$, we must have the convergence $\psi^{-j_n}(f(a_k)) \rightarrow y$ for $k = 1, \dots, n$ as well.

By Theorem 2.17, y has a finite preimage in X . Let $\varepsilon = 1/2 \cdot \max\{\varepsilon_X, \varepsilon_f\}$, and choose $\delta = \delta(y, \varepsilon)$ given by Proposition 2.14, so that we have

$$f^{-1}(Y(y, \delta)) \subseteq \bigcup_{x \in f^{-1}(y)} X(x, \varepsilon).$$

Choose a big enough N such that

$$\psi^{-j_N}(f(a_k)) \in Y^u(\psi^{-j_N}(f(a_0)), \varepsilon'), \quad d(\psi^{-j_N}(f(a_k)), y) < \delta$$

hold for $k = 0, \dots, n$ and some $\varepsilon' < \varepsilon_Y$. Since $\psi^{-j_N}(f(a_k)) = f(\phi^{-j_N}(a_k))$, we have $\phi^{-j_N}(a_k) \in X(x_k, \varepsilon)$ where $x_k \in X$ is a point satisfying $f(x_k) = y$.

Because of the choice of ε , we can construct $b'_k = [x_k, \phi^{-j_N}(a_k)]$, and f commutes with the bracket of these points. Then, $z = f(b'_k) = [y, \psi^{-j_N}(f(a_k))]$ is independent of k by our choice of j_N . We also have $b'_k \sim_u \phi^{-j_N}(a_k)$ by construction. Then, $b_k = \phi^{j_N}(b'_k)$ is unstably equivalent to a_k , and $f(b_k) = \psi^{j_N}(z)$ does not depend on k . ■

Let us note in passing that instead of the s -resolving condition one could assume something else that still implies that f is finite-to-one for the above proof to work.

Proof of Theorem 3.8. We have an embedding of the groupoid $R^u(\Sigma_n, \sigma_n)$ into $H^{\times G^{n+1}}$ by the correspondence

$$(a^0, \dots, a^n) \mapsto (a^0, \text{id}_y, a^1, \dots, \text{id}_y, a^n) \quad (y = f(a^0) = \dots = f(a^n))$$

at the level of objects and by

$$((a^0, \dots, a^n), (b^0, \dots, b^n)) \mapsto ((a^0, b^0), \dots, (a^n, b^n))$$

at the level of arrows. Proposition 3.9 implies that $\Sigma_n \subset (H^{\times G^{n+1}})^{(0)}$ meets all orbits of $H^{\times G^{n+1}}$. Moreover, $a, b \in \Sigma_n$ are connected by an arrow in $(H^{\times G^{n+1}})^{(0)}$ if and only if

they are connected in $R^u(\Sigma_n, \sigma_n)$. Thus, $R^u(\Sigma_n, \sigma_n) \curvearrowright (H^{\times_{G^{n+1}}})^{\Sigma_n} \curvearrowleft H^{\times_{G^{n+1}}}$ gives a Morita equivalence between the two groupoids. It is a routine task to see that this is compatible with the topology on the two groupoids. ■

Combining Proposition 3.4, Corollary 3.7, and Theorem 3.8, we obtain the following.

Theorem 3.10. *In addition to $f: (\Sigma, \sigma) \rightarrow (Y, \psi)$ as above, let $f': T \rightarrow \Sigma$ be a generalized transversal for the locally compact groupoid $R^u(\Sigma, \sigma)$ such that $f \circ f': T \rightarrow Y$ defines a generalized transversal for $R^u(Y, \psi)$. Denote the corresponding étale groupoids by*

$$H = R^u(\Sigma, \sigma)|_{f'(T)}, \quad G = R^u(Y, \psi)|_{f \circ f'(T)}.$$

The groupoid $G \times_G H^{\times_{G^{n+1}}}$ with respect to the natural inclusion $H \rightarrow G$ is Morita equivalent to $R^u(\Sigma_n, \sigma_n)$ as a topological groupoid.

3.1. Semidirect products and transversality

Let Γ be a discrete group and G a groupoid admitting a left action of Γ by groupoid automorphisms $(\phi_\gamma)_{\gamma \in \Gamma}$. We denote the corresponding semidirect product by $\Gamma \ltimes G$. Thus, an arrow in $\Gamma \ltimes G$ is represented by pairs (γ, g) for $\gamma \in \Gamma$ and $g \in G$, with range $\phi_\gamma(r(g))$ and source $s(g)$, and composition is given by

$$(\gamma, g)(\gamma', g') = (\gamma\gamma', \phi_{\gamma'^{-1}}(g)g') \quad (s(g) = \phi_{\gamma'}(r(g'))).$$

Let H be another groupoid with an action of Γ by automorphisms $(\psi_\gamma)_{\gamma \in \Gamma}$, and let $\alpha: H \rightarrow G$ be a homomorphism commuting with ϕ and ψ . Then, we get a groupoid homomorphism $\Gamma \ltimes H \rightarrow \Gamma \ltimes G$.

Proposition 3.11. *Suppose that the fiber product of base $H^{(0) \times_G (0)^n}$ meets all orbits in the groupoid pullback $H^{\times_{G^n}}$. Then, $H^{(0) \times_G (0)^n}$ meets all orbits in $(\Gamma \ltimes H)^{\times_{\Gamma \ltimes G^n}}$.*

Proof. Let $z = (y_1, (\gamma_1, g_1), y_2, \dots, (\gamma_{n-1}, g_{n-1}), y_n)$ be a point of $(\Gamma \ltimes H)^{\times_{\Gamma \ltimes G^n}}^{(0)}$. Note that we have

$$\alpha(y_1) = \phi_{\gamma_1}(r(g_1)), \quad \alpha(y_2) = s(g_1) = \phi_{\gamma_2}(r(g_2)), \quad \dots, \quad \alpha(y_n) = s(g_{n-1});$$

thus, the tuple

$$z' = (\psi_{(\gamma_1 \cdots \gamma_{n-1})^{-1}}(y_1), \phi_{(\gamma_2 \cdots \gamma_{n-1})^{-1}}(g_1), \psi_{(\gamma_2 \cdots \gamma_{n-1})^{-1}}(y_2), \dots, g_{n-1}, y_n)$$

is a point in $(H^{\times_{G^n}})^{(0)}$. Moreover, the arrows

$$(\gamma_1 \cdots \gamma_{n-1}, \text{id}_{\psi_{(\gamma_1 \cdots \gamma_{n-1})^{-1}}(y_1)}), \quad \dots, \quad (\gamma_{n-1}, \text{id}_{\psi_{\gamma_{n-1}^{-1}}(y_{n-1})}), \quad (e, \text{id}_{y_n})$$

in $\Gamma \ltimes H$ give an arrow from z' to z in $(\Gamma \ltimes H)^{\times_{\Gamma \ltimes G^n}}$ up to the embedding $G \rightarrow \Gamma \ltimes G$, $g \mapsto (e, g)$, etc. By assumption, there is $z'' \in H^{(0) \times_G (0)^n}$ and an arrow from z'' to z' ; hence, z is also in the orbit of z'' . ■

4. Spectral sequence for Smale spaces

Let us start with the following generalization of Proposition 2.10.

Theorem 4.1. *Suppose that G is an étale groupoid with torsion-free stabilizers satisfying the conclusion of Theorem 2.9 and that $H \subseteq G$ is an étale subgroupoid with the same base space X . Any G - C^* -algebra in the category KK^G belongs to the localizing subcategory generated by the image of $\text{Ind}_H^G: \text{KK}^H \rightarrow \text{KK}^G$.*

In Theorem 4.1, H is an open subgroupoid of G because $H^{(0)} = X$ and H is étale.

Proof of Theorem 4.1. Consider the functors

$$\text{Res}_H^G: \text{KK}^G \rightarrow \text{KK}^H, \quad \text{Ind}_H^G: \text{KK}^H \rightarrow \text{KK}^G$$

as in Section 2.2. By Proposition 2.7, we have a complementary pair $(\langle \mathcal{P}_I \rangle, \mathcal{N}_I)$ for $I = \ker \text{Res}_H^G$, with $\langle \mathcal{P}_I \rangle$ being generated by the image of Ind_H^G as a localizing subcategory.

Moreover, we have a natural isomorphism of functors $\text{Ind}_X^G \simeq \text{Ind}_H^G \text{Ind}_X^H$. Concretely, if A is a $C_0(X)$ -algebra,

$$\text{Ind}_X^G A = C_0(G) \otimes_{C_0(X)} A, \quad \text{Ind}_H^G \text{Ind}_X^H A = (C_0(G) \otimes_{C_0(X)}^r C_0(H) \otimes_{C_0(X)} A) \rtimes H$$

are G -equivariantly strongly Morita equivalent via a Hilbert C^* -bimodule completion of $C_c(G) \otimes_{C_0(X)}^r C_c(H) \otimes_{C_0(X)} A$. Combined with Proposition 2.10, we obtain that A belongs to $\langle \mathcal{P}_I \rangle$. ■

Corollary 4.2. *Let G, H , and A be as in Theorem 4.1. Let $P_H(A) \in \langle \text{Ind}_H^G \text{KK}^H \rangle$ be the algebra appearing in the exact triangle*

$$P_H(A) \rightarrow A \rightarrow N \rightarrow \Sigma P_H(A)$$

that we get by applying Proposition 2.7. Then, we have $P_H(A) \simeq A$ in KK^G .

Corollary 4.3. *Let G, H , and A be as in Theorem 4.1. Then, we have a convergent spectral sequence*

$$E_{pq}^2 = H_p(K_q(G \rtimes L^{\bullet+1} A)) \Rightarrow K_{p+q}(G \rtimes A), \tag{6}$$

where $L^n A = (\text{Ind}_H^G \text{Res}_H^G)^n(A)$.

Proof. The reduced crossed product functor

$$\text{KK}^G \rightarrow \text{KK}, \quad A \mapsto G \rtimes A$$

is exact and compatible with direct sums, while

$$\text{KK} \rightarrow \text{Ab}, \quad B \mapsto K_0(B)$$

is a homological functor. Thus, their composition

$$K_0(G \rtimes -): \text{KK}^G \rightarrow \text{Ab}$$

is a homological functor, cf. [22, Examples 13 and 15]. Now we can apply Theorem 2.8 to get a spectral sequence

$$H_p(K_q(G \ltimes P_\bullet)) \Rightarrow K_{p+q}(G \ltimes P_H(A)),$$

where P_\bullet is a $(\ker \text{Res}_H^G)$ -projective resolution of A . The $(\ker \text{Res}_H^G)$ -projective resolution from Proposition 2.7 gives the left-hand side of (6). Now the claim follows from Corollary 4.2. ■

Remark 4.4. It would be an interesting question to cast the above constructions to the groupoid equivariant E -theory [26] since we mostly use the formal properties of KK^G . However, since some parts of our constructions involve reduced crossed products, there are some details to be checked. (Note that H need not be a proper subgroupoid.)

4.1. Projective resolution from subgroupoid induction

Put $P_n = L^{n+1}A$ for the functor $L = \text{Ind}_H^G \text{Res}_H^G: \text{KK}^G \rightarrow \text{KK}^G$. This is a simplicial object in KK^G , and the associated complex structure on P_\bullet is given by the differential

$$\delta_n = \sum_{i=0}^n (-1)^i d_i^n: P_n \rightarrow P_{n-1}, \tag{7}$$

together with the augmentation morphism $\delta_0 = \varepsilon: P_0 = LA \rightarrow A$. This makes P_\bullet a $(\ker \text{Res}_H^G)$ -projective resolution of A [28, Proposition 2.1].

Suppose G is a second countable locally compact Hausdorff étale groupoid, and H is an open subgroupoid with the same base space. Let us analyze the chain complex in (7) more concretely. Let $s_n: G^{(n)} \rightarrow X$ be the map $(g_1, \dots, g_n) \mapsto sg_n$.

Lemma 4.5. *Let A be an H - C^* -algebra. The $C_0(G^{(n)})$ -algebra s_n^*A is endowed with a continuous action of the groupoid $G \times_G H^{\times G^n}$.*

Proof. We use $(C_0(G) \otimes_{\min} A)_{\Delta(X)}$ as a model of $C_0(G) \otimes_{C_0(X)} A$ and analogous models for other relative C^* -algebra tensor products as well. Recall that the arrow set of $G \times_G H^{\times G^n}$ can be identified with the set of tuples $(g, g_1, h_1, \dots, g_n, h_n)$, where $(g, g_1, \dots, g_n) \in G^{(2)}$, $h_i \in H$, and $s(g_i) = s(h_i)$. Then,

$$\begin{aligned} & C_0(G \times_G H^{\times G^n}) \text{}^s \otimes_{C_0(G^{(n)})} (C_0(G^{(n)}) \text{}^s \otimes_{C_0(X)} A) \\ & \simeq (C_0(G \times_G H^{\times G^n} \times G^{(n)}) \otimes A)_Y, \end{aligned}$$

where Y is the space of tuples $(g, g_1, h_1, \dots, g_n, h_n, g_1, \dots, g_n, x)$ with $(g, g_1, h_1, \dots, g_n, h_n)$ as above and $x = s(g_n)$. Let us set $K = G^{(n-1)}$ and think of $G^{(n)}$ as $K \times_r G$. We have

$$\begin{aligned} & C_0(G \times_G H^{\times G^{(n-1)}} \times_K G^{(n)}) \text{}^s \otimes_{C_0(X)} C_0(H) \text{}^s \otimes_{C_0(G)} A \\ & \simeq (C_0(G \times_G H^{\times G^{(n-1)}} \times_K G^{(n)} \times H) \otimes A)_Z, \end{aligned}$$

where Z is the space of tuples $(g, g_1, h_1, \dots, g_{n-1}, h_{n-1}, g_n, h_n, x)$ where the components are related as above. Via the obvious homeomorphism between Y and Z , we have the identification of these algebras. The structure map $\alpha: C_0(H)^s \otimes_{C_0(G)} A \rightarrow C_0(H)^r \otimes_{C_0(G)} A$ of the H - C^* -algebra induces an isomorphism onto

$$\begin{aligned} &C_0(G \times_G H^{\times_G(n-1)} \times_K G^{(n)} \otimes_{C_0(X)}^s C_0(H)^r \otimes_{C_0(G)} A) \\ &\simeq (C_0(G \times_G H^{\times_G(n-1)} \times_K G^{(n)} \times H) \otimes A)_{Z'}, \end{aligned}$$

where Z' is the space of tuples $(g, g_1, h_1, \dots, g_{n-1}, h_{n-1}, g_n, h_n, y)$ as above and $y = r(h_n)$. Finally, we have

$$\begin{aligned} &C_0(G \times_G H^{\times_G n})^r \otimes_{C_0(G^{(n)})} (C_0(G^{(n)}) \otimes_{C_0(X)} A) \\ &= C_0(G \times_G H^{\times_G n} \times G^{(n)} \otimes A)_{Y'}, \end{aligned}$$

where Y' is the space of tuples $(g, g_1, h_1, \dots, g_n, h_n, g'_1, \dots, g'_n, y)$ as above, where $g'_1 = gg_1h_1^{-1}$, $g'_i = h_{i-1}g_ih_i^{-1}$ for $i > 1$, and $y = s(g'_n) = r(h_n)$. Again the obvious bijection between Y' and Z' induces an isomorphism of the last two algebras, and combining everything, we obtain an isomorphism

$$C_0(G \times_G H^{\times_G n})^s \otimes_{C_0(G^{(n)})} s_n^* A \rightarrow C_0(G \times_G H^{\times_G n})^r \otimes_{C_0(G^{(n)})} s_n^* A$$

which is the desired structure morphism of the $G \times_G H$ - C^* -algebra. ■

Proposition 4.6. *In the setting above, we have*

$$G \times L^n A \simeq (G \times_G H^{\times_G n}) \times s_n^* A.$$

Proof. We have $L^n A = H^{\times_G n} \times s_n^* A$ by expanding the definitions. ■

Using the Morita equivalence between $G \times_G H^{\times_G n}$ and $H^{\times_G n}$, we can replace the formula above with $H^{\times_G n} \times s_{n-1}^* A$. This enables us to transport the simplicial structure on $(G \times L^{n+1} A)_{n=0}^\infty$ to $(H^{\times_G(n+1)} \times s_n^* A)_{n=0}^\infty$. The proof is again straightforward from definitions.

Proposition 4.7. *The induced simplicial structure on $(H^{\times_G(n+1)} \times s_n^* A)_{n=0}^\infty$ has face maps d_i^n represented by the composition of KK-morphisms*

$$C_r^*(H^{\times_G(n+1)}, s_n^* A) \rightarrow C_r^*(H^{\times_G i} \times_G G \times_G H^{\times_G(n-i)}, s_n^* A) \rightarrow C_r^*(H^{\times_G n}, s_{n-1}^* A),$$

where the first morphism is induced by the inclusion

$$H^{\times_G(n+1)} \rightarrow H^{\times_G i} \times_G G \times_G H^{\times_G(n-i)}$$

as an open subgroupoid, and the second morphism is given by the Morita equivalence of Proposition 3.5.

For a suitable choice of $P \subseteq \Sigma$ as in Section 2.5, we have an open inclusion of étale groupoid

$$f \times f: R^u(\Sigma, P) \rightarrow R^u(Y, f(P)).$$

We set $G = R^u(Y, f(P))$ and $H = (f \times f)(R^u(\Sigma, P))$. Notice that G is an ample groupoid and H is approximately finite-dimensional (AF) [33, 39].

Proposition 4.8. *There is an isomorphism of chain complexes*

$$\begin{aligned} (K_0(G \rtimes L^{\bullet+1}C_0(X)), \delta_\bullet) &\simeq (D^s(\Sigma_\bullet, \sigma_\bullet), d^s(f)_\bullet), \\ (K_1(G \rtimes L^{\bullet+1}C_0(X)), \delta_\bullet) &\simeq 0. \end{aligned}$$

Before going into the proof, let us recall the concept of *correspondences* between groupoids. In general, if G and H are topological groupoids, a correspondence from G to H is a topological space Z together with commuting proper actions $G \curvearrowright Z \curvearrowleft H$, such that the anchor map $Z \rightarrow H^{(0)}$ is open (surjective) and induces a homeomorphism $G \setminus Z \simeq H^{(0)}$. Of course, one source of such correspondence is Morita equivalence. Another example is provided by continuous homomorphisms $f: G \rightarrow H$, where one puts $Z = \{[g, h] \mid f(sg) = rh\}$ with the relation $[g_1g_2, h] = [g_1, f(g_2)h]$.

If G and H are (second countable) locally compact Hausdorff groupoids with Haar systems, a correspondence Z induces a right Hilbert $C_r^*(H)$ -module $C_r^*(Z)_{C_r^*(H)}$ with a left action of $C_r^*(G)$ [16]. If the action of $C_r^*(G)$ is in $\mathcal{K}(C_r^*(Z)_{C_r^*(H)})$, we obtain a map $K_\bullet(C_r^*(G)) \rightarrow K_\bullet(C_r^*(H))$. While finding a good characterization of this condition in terms of Z seems to be somewhat tricky, in concrete examples as below, it is not too difficult. On the other hand, the composition of such Hilbert modules is easy to describe. If H' is another topological groupoid with the Haar system, and Z' is a correspondence from H to H' , we have the identification

$$C_r^*(Z)_{C_r^*(H)} \otimes_{C_r^*(H)} C_r^*(Z')_{C_r^*(H')} \simeq C_r^*(Z \times_H Z')_{C_r^*(H')}.$$

Proof of Proposition 4.8. As before, consider the functor $L = \text{Ind}_H^G \text{Res}_H^G: \text{KK}^G \rightarrow \text{KK}^G$. Then, we have

$$G \rtimes L^n A \simeq (G \times_G H^{\times n}) \rtimes_{s_n^*} A;$$

see [28]. By Theorem 3.10, the C^* -algebra $G \rtimes L^{n+1}C_0(X)$ is strongly Morita equivalent to $C^*(R^u(\Sigma_n, \sigma_n))$. From this, we have the identification of the underlying modules, and it remains to compare the corresponding simplicial structures. Let us give a concrete comparison of the maps $K_0(C_r^*(H^{\times n+1})) \rightarrow D^s(\Sigma_{n-1}, \sigma_{n-1})$ corresponding to the 0-th face maps, as the general case is completely parallel.

Let us put $\tilde{G} = R^u(Y, \psi)$, $\tilde{H} = R^u(\Sigma, \sigma)$, take a (generalized) transversal T' for $R^u(\Sigma_n, \sigma_n)$, and put $K = R^u(\Sigma_n, \sigma_n)|_{T'}$, $K' = R^u(\Sigma_{n-1}, \sigma_{n-1})|_{\delta_0(T')}$ so that we have

$$D^s(\Sigma_n, \sigma_n) \simeq K_0(C_r^*(K)), \quad D^s(\Sigma_{n-1}, \sigma_{n-1}) \simeq K_0(C_r^*(K')).$$

We denote the generalized transversal of $\tilde{H}^{\times n}$ induced by P , as in Proposition 3.6, by \tilde{T}_n .

The map δ_0 induces a groupoid homomorphism $K \rightarrow K'$ and hence a correspondence Z_{δ_0} from K to K' . Composing this with the Morita equivalence bibundle $\tilde{\tau}_{n+1}(\tilde{H}^{\times \tilde{G}^{n+1}})_{T'}$, we obtain a correspondence

$$\tilde{\tau}_{n+1}(\tilde{H}^{\times \tilde{G}^{n+1}})_{T'} \times_K Z_{\delta_0} \tag{8}$$

from $H^{\times G^{n+1}}$ to K' representing the effect of δ_0 on the K -groups.

As for the 0-th face map d_0 of $H^{\times G^{\bullet+1}}$, let Z be the Morita equivalence bibundle between $G \times_G H^{\times G^n}$ and $H^{\times G^n}$ from Proposition 3.4. Since $H^{\times G^{n+1}}$ is an open subgroupoid of $G \times_G H^{\times G^n}$, Z becomes a correspondence from $H^{\times G^{n+1}}$ to $H^{\times G^n}$. Composing this with the Morita equivalence $\tilde{\tau}_n(\tilde{H}^{\times \tilde{G}^n})_{\delta_0(T')}$ between $H^{\times G^n}$ and K' , we obtain the correspondence

$$Z \times_{H^{\times G^n}} \tilde{\tau}_n(\tilde{H}^{\times \tilde{G}^n})_{\delta_0(T')} \tag{9}$$

from $H^{\times G^{n+1}}$ to K' representing the effect of d_0 .

It remains to check that the above correspondences are isomorphic, hence giving isomorphic Hilbert modules. Expanding the ingredients of (9), we obtain the space

$$W = \{(g_0, h_1, g_1, h_2, \dots, g_{n-1}, h_n) \mid (g_0, \dots, g_{n-1}) \in G^{(n)}, \\ h_k \in \tilde{H}^{sg_{k-1}}, (sh_1, \dots, sh_n) \in \delta_0(T')\}.$$

On the other hand, (8) gives $W \times_K K'$ with

$$W' = \{(h_1, g_1, h_2, \dots, g_n, h_{n+1}) \mid (g_1, \dots, g_n) \in G^{(n)}, \\ h_k \in \tilde{H}^{rg_k}, h_{n+1} \in \tilde{H}^{sg_n}, (sh_1, \dots, sh_{n+1}) \in T'\}.$$

The operation $- \times_K K'$ “kills” the component h_1 , and we obtain the identification with W . ■

Thus, we obtain the isomorphisms of homology groups

$$H_p(K_q(G \times L^{\bullet+1} C_0(X)), \delta_\bullet) \simeq H_p^s(Y, \psi) \otimes K_q(\mathbb{C}).$$

Combining this with Corollary 4.3 and Proposition 4.8, we obtain the following main result of this paper.

Theorem 4.9. *Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets. Then, there is a convergent spectral sequence*

$$E_{pq}^r \Rightarrow K_{p+q}(C^*(R^u(Y, \psi))),$$

with $E_{pq}^2 = E_{pq}^3 = H_p^s(Y, \psi) \otimes K_q(\mathbb{C})$.

Corollary 4.10. *The K -groups $K_i(C^*(R^u(Y, \psi)))$ have a finite rank.*

Proof. By the above theorem, for $i = 0, 1$, the rank of $K_i(C^*(R^u(Y, \psi)))$ is bounded by that of $\bigoplus_k H_{i+2k}^s(Y, \psi)$. The latter is of finite rank by [32, Theorem 5.1.12]. ■

Remark 4.11. By the Pimsner–Voiculescu exact sequence, the same can be said for the unstable Ruelle algebra $C^*(R^u(Y, \psi)) \rtimes_{\psi} \mathbb{Z}$. If the stable relation $R^s(Y, \psi)$ also has finite-rank K -groups, the Ruelle algebras will have finitely generated K -groups by [12].

5. Comparison of homologies

In fact, Putnam’s homology is isomorphic to groupoid homology, as follows.

Theorem 5.1. *We have $H_{\bullet}^s(Y, \psi) \simeq H_{\bullet}(G, \mathbb{Z})$.*

The homology groups on the right-hand side of Theorem 5.1 are a special case of groupoid homology with coefficients in equivariant sheaves [4].

When G is ample (as is the case in the theorem), such G -sheaves are represented by unitary $C_c(G, \mathbb{Z})$ -modules [38]. Here, we consider the convolution product on $C_c(G, \mathbb{Z})$, and a module M over $C_c(G, \mathbb{Z})$ is said to be unitary if it has the factorization property $C_c(G, \mathbb{Z})M = M$. The correspondence is given by $\Gamma_c(U, F) = C_c(U, \mathbb{Z})M$ for compact open sets $U \subset X$ if F is the sheaf corresponding to such a module M .

The homology of G with coefficient in F , denoted by $H_{\bullet}(G, F)$, is the homology of the chain complex $(C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X, \mathbb{Z})} M)_{n=0}^{\infty}$ with differentials coming from the simplicial structure as above. Concretely, the differential is given by

$$\begin{aligned} \partial_n: C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X, \mathbb{Z})} M &\rightarrow C_c(G^{(n-1)}, \mathbb{Z}) \otimes_{C_c(X, \mathbb{Z})} M, \\ \partial_n(f \otimes m) &= \sum_{i=0}^{n-1} (-1)^i (d_i^n)_* f \otimes m + (-1)^n \alpha_n(f \otimes m), \end{aligned}$$

where α_n is the concatenation of the last leg of $C_c(G^{(n)}, \mathbb{Z})$ with M induced by the module structure map $C_c(G, \mathbb{Z}) \otimes M \rightarrow M$. This definition agrees with the one given in [4] as there is no need to take c -soft resolutions of equivariant sheaves (see [28, Proposition 1.8]).

More generally, if F_{\bullet} is a chain complex of G -sheaves modeled by a chain complex of unitary $C_c(G, \mathbb{Z})$ -modules M_{\bullet} , we define $\mathbb{H}_{\bullet}(G, F_{\bullet})$, the hyperhomology with coefficient F_{\bullet} , as the homology of the double complex $(C_c(G^{(p)}, \mathbb{Z}) \otimes_{C_c(X, \mathbb{Z})} M_q)_{p,q}$. As usual, a chain map of complexes of G -sheaves $f: F_{\bullet} \rightarrow F'_{\bullet}$ is a quasi-isomorphism if it induces quasi-isomorphisms on the stalks. When F_{\bullet} and F'_{\bullet} are bounded below, such maps induce an isomorphism of the hyperhomology [4, Lemma 3.2].

Proof. Let us consider $G^{(n+1)}$ as an $H^{\times_G(n+1)}$ -space by the anchor map

$$(g_0, \dots, g_n) \mapsto (g_1, \dots, g_n) \in G^{(n)} = (H^{\times_G(n+1)})^{(0)}$$

and the action map

$$(h_1, g_1, h_2, \dots, h_n)(g_0, \dots, g_n) = (g_0 h_1^{-1}, g'_1, \dots, g'_n)$$

in the same notation used in Definition 3.1. Then, $H_0(H^{\times_G(n+1)}, C_c(G^{(n+1)}, \mathbb{Z}))$ is a unitary $C_c(X, \mathbb{Z})$ -module by the action from the left, and the associated sheaf F_n on X is a G -sheaf by the left translation action of G . At $x \in X$, the stalk can be presented as

$$(F_n)_x = H_0(H^{\times_G(n+1)}, C_c((G^{(n+1)})^x, \mathbb{Z})) = C_c((G^{(n+1)})^x, \mathbb{Z})_{H^{\times_G(n+1)}}. \tag{10}$$

Indeed, the sheaf corresponding to the $C_c(X, \mathbb{Z})$ -module $C_c(G^{(n+1)}, \mathbb{Z})$ has the stalk $C_c((G^{(n+1)})^x, \mathbb{Z})$ at x , and taking coinvariants by $H^{\times_G(n+1)}$ commutes with taking stalks.

We then have

$$H_0(G, F_n) \simeq H_0(G \times_G H^{\times_G(n+1)}, \mathbb{Z}) \simeq H_0(H^{\times_G(n+1)}, \mathbb{Z}).$$

The simplicial structure on $(G \times_G H^{\times_G(n+1)})_n$ leads to the complex of G -sheaves

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0, \tag{11}$$

and $H_*^s(Y, \psi)$ is the homology of the complex obtained by applying the functor $H_0(G, -)$ to (11).

We first claim that the augmented complex

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \underline{\mathbb{Z}} \rightarrow 0 \tag{12}$$

is exact. It is enough to check the exactness at the level of stalks. In terms of the presentation (10), we have the chain complex

$$\dots \rightarrow C_c((G^{(2)})^x, \mathbb{Z})_{H^{\times_{G^2}}} \rightarrow C_c(G^x, \mathbb{Z})_H \rightarrow \mathbb{Z}$$

with differential

$$\begin{aligned} d((g_1, \dots, g_{n+1})) &= (g_1g_2, g_3, \dots, g_{n+1}) - (g_1, g_2g_3, \dots, g_{n+1}) + \dots \\ &\quad + (-1)^{n-1}(g_1, \dots, g_n g_{n+1}) + (g_1, \dots, g_n), \end{aligned}$$

where $(g_1, \dots, g_n) \in (G^{(n)})^x$ represents the image of $\delta_{(g_1, \dots, g_n)} \in C_c((G^{(n)})^x, \mathbb{Z})$ in the coinvariant space, and the augmentation is given by $d(g) = 1$ at $n = 0$. This has a contracting homotopy given by $\mathbb{Z} \rightarrow C_c(G^x, \mathbb{Z})_H, a \rightarrow a(\text{id}_x)$ and

$$C_c((G^{(n)})^x, \mathbb{Z})_{H^{\times_{G^n}}} \rightarrow C_c((G^{(n+1)})^x, \mathbb{Z})_{H^{\times_{G^{n+1}}}}, \quad (g_1, \dots, g_n) \rightarrow (\text{id}_x, g_1, \dots, g_n);$$

hence, (12) is indeed exact.

We next claim that $H_k(G, F_n) = 0$ for $k > 0$. Let H_{n+1} be a subgroupoid of G which is Morita equivalent to $H^{\times_G(n+1)}$ (this exists by choosing a transversal for (Σ_n, σ_n) that maps bijectively to a transversal of Σ). Then, the module $H_0(H^{\times_G(n+1)}, C_c(G^{(n+1)}, \mathbb{Z}))$ representing F_n is isomorphic to $H_0(H_{n+1}, C_c(G, \mathbb{Z}))$. Thus, it is enough to check the claim when $n = 0$.

Let us write $M = H_0(H, C_c(G, \mathbb{Z}))$, and consider the double complex of modules $C_c(G^{(p+1)} \times_X H^{(q)}, \mathbb{Z})$ for $p, q \geq 0$, with differentials coming from the simplicial structures on $(G^{(p)})_{p=0}^\infty$ and $(H^{(q)})_{q=0}^\infty$, cf. [4, Theorem 4.4]. For fixed p , this is a resolution of $C_c(G^{(p)}, \mathbb{Z}) \otimes_{C_c(X, \mathbb{Z})} M$; hence, the double complex computes $H_\bullet(G, F)$. For fixed q , this is a resolution of $H_0(G, C_c(G \times_X H^{(q)}, \mathbb{Z})) \simeq C_c(H^{(q)}, \mathbb{Z})$, and this double complex also computes $H_\bullet(H, \mathbb{Z})$. Since H is Morita equivalent to an AF groupoid, $H_k(H, \mathbb{Z}) = 0$ by [18, Theorem 4.11]. We thus obtain $H_k(G, F_n) = 0$.

Finally, consider the hyperhomology $\mathbb{H}_\bullet(G, F_\bullet)$. On the one hand, by the above vanishing of $H_k(G, F_n)$, this is isomorphic to the homology of the complex $(H_0(G, F_n))_n$, i.e., $H_\bullet^s(Y, \psi)$. On the other hand, since F_\bullet is quasi-isomorphic to $\underline{\mathbb{Z}}$ concentrated in degree 0, we also have $\mathbb{H}_\bullet(G, F_\bullet) \simeq H_\bullet(G, \mathbb{Z})$. ■

As a consequence, we then have the following Künneth formula from the corresponding result for groupoid homology [19, Theorem 2.4].

Corollary 5.2. *Let (Y_1, ψ_1) and (Y_2, ψ_2) be Smale spaces with totally disconnected stable sets. Then, we have a split exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{a+b=k} H_a^s(Y_1, \psi_1) \otimes H_b^s(Y_2, \psi_2) &\rightarrow H_k^s(Y_1 \times Y_2, \psi_1 \times \psi_2) \\ &\rightarrow \bigoplus_{a+b=k-1} \text{Tor}(H_a^s(Y_1, \psi_1), H_b^s(Y_2, \psi_2)) \rightarrow 0. \end{aligned}$$

Remark 5.3. As usual, the splitting is not canonical. This generalizes [8, Theorem 6.5], in which one of the factors is assumed to be a shift of finite type. Indeed, if (Y_1, ψ_1) is a shift of finite type, the first direct sum reduces to $D^s(Y_1, \psi_1) \otimes H_k^s(Y_2, \psi_2)$, while the second direct sum of torsion groups vanishes as the dimension group $D^s(Y_1, \psi_1)$, being torsion-free, is flat.

Remark 5.4. Theorem 5.1 holds in general without the assumption of total disconnectedness on stable sets. We plan to expand on this direction in a forthcoming work.

Remark 5.5. Consider the Smale space (Ω, ω) of a substitution tiling system. As we observed in [28, Section 4], if G is an étale groupoid Morita equivalent to $R^u(\Omega, \omega)$, its groupoid homology $H_k(G, \mathbb{Z})$ is isomorphic to the Čech cohomology $\check{H}^{d-k}(\Omega)$ for the constant sheaf $\underline{\mathbb{Z}}$ on Ω . Combined with Theorem 5.1, we obtain

$$H_k^s(\Omega, \omega) \simeq \check{H}^{d-k}(\Omega),$$

giving a positive answer to [32, Question 8.3.2] in the case of tiling spaces.

6. K-theory of Ruelle algebra

6.1. Semidirect product and homology

Recall the following result for semidirect products.

Proposition 6.1 ([28, Proposition 3.8]). *Suppose that Γ is torsion-free and satisfies the strong Baum–Connes conjecture and that G is an ample groupoid with torsion-free stabilizers satisfying the strong Baum–Connes conjecture. Then, any separable $(\Gamma \times G)$ - C^* -algebra A belongs to the localizing subcategory generated by the image of*

$$\text{Ind}_X^{\Gamma \times G} : \text{KK}^X \rightarrow \text{KK}^{\Gamma \times G} .$$

Let G and Γ be as above, and let A be a separable $(\Gamma \times G)$ - C^* -algebra. We can choose a $(\ker \text{Res}_X^G)$ -projective resolution P_\bullet of A such that each P_n is a $(\Gamma \times G)$ - C^* -algebra and the structure morphisms of $P_\bullet \rightarrow A$ are a restriction of morphisms in $\text{KK}^{\Gamma \times G}$. For example, the standard choice

$$P_n = (\text{Ind}_X^G \text{Res}_X^G)^{n+1} A = C_0(G^{(n+1)}) \otimes_{C_0(X)} A$$

satisfies this assumption.

Then, we have a complex of Γ -modules $K_q(G \times P_\bullet)$. We claim that the hyperhomology of Γ with this coefficient is the E^2 -sheet of our spectral sequence.

Proposition 6.2. *In the above setting, we have a spectral sequence*

$$E_{pq}^r \Rightarrow K_{p+q}(\Gamma \times G \times A)$$

with $E_{pq}^2 = \mathbb{H}_p(\Gamma, K_q(G \times P_\bullet))$.

Proof. Put $L_G = \text{Ind}_X^G \text{Res}_X^G$ and $L_\Gamma B = C_0(\Gamma) \otimes B$. Then, L_Γ is a model of the endofunctor $\text{Ind}_G^{\Gamma \times G} \text{Res}_G^{\Gamma \times G}$ on $\text{KK}^{\Gamma \times G}$. We then put $Q_{a,b} = L_\Gamma^{a+1} P_b$, and consider the natural bicomplex structure on $(Q_{a,b})_{a,b=0}^\infty$.

Let us show that the total complex $\text{Tot } Q_{\bullet,\bullet}$ is a $(\ker \text{Res}_X^{\Gamma \times G})$ -resolution of A . We have

$$Q_{a,b} \simeq \text{Ind}_X^{\Gamma \times G} \text{Res}_X^{\Gamma \times G} Q_{a-1,b-1};$$

hence, $Q_{a,b}$ is $(\ker \text{Res}_X^{\Gamma \times G})$ -projective. Moreover, we claim that the augmented complex $\text{Tot } Q_{\bullet,\bullet} \rightarrow A$ is $(\ker \text{Res}_X^{\Gamma \times G})$ -exact. Indeed, after applying $\text{Res}_G^{\Gamma \times G}$ to the bicomplex $Q_{\bullet,\bullet}$, each row gives the complex $Q_{\bullet,b}$ consisting of the terms $C_0(\Gamma^{a+1}) \otimes P_b$ for $a \geq 0$. Then, the standard argument shows that $Q_{\bullet,b}$ is chain homotopic to P_b concentrated at degree 0; hence, $\text{Res}_G^{\Gamma \times G} \text{Tot } Q_{\bullet,\bullet}$ is chain-homotopic to P_\bullet .

We now know that there is a spectral sequence converging to $K_{p+q}(\Gamma \times G \times A)$ with $E_{pq}^2 = H_p(K_q(\Gamma \times G \times \text{Tot}(Q_{\bullet,\bullet})))$. Observe that

$$G \times Q_{a,b} \simeq C_0(\Gamma^{a+1}) \otimes G \times P_b,$$

where Γ is acting by translation on the first coordinate of Γ^{a+1} . Then, we have

$$K_q(\Gamma \times G \times Q_{a,b}) = \mathbb{Z}[\Gamma^a] \otimes K_q(G \times P_b).$$

Unpacking the differential, we see that $K_q(\Gamma \times G \times \text{Tot}(Q_{\bullet,\bullet}))$ is the complex computing the hyperhomology as in the claim. ■

6.2. Application to Ruelle algebra

Let (Y, ψ) be a Smale space with totally disconnected stable sets. Then, the groupoid $\mathbb{Z} \ltimes_{\psi} R^u(Y, \psi)$ behind the *unstable Ruelle algebra*

$$\mathcal{R}_u(Y, \psi) = C^*(R^u(Y, \psi)) \rtimes_{\psi} \mathbb{Z}$$

fits into the setting we considered for the semidirect products of étale groupoids (note the notational difference: \mathcal{R}_u indicates the Ruelle algebra, while R^u indicates the unstable equivalence relation). Indeed, as a generalized transversal of $R^u(Y, \psi)$, take $T = Y^s(P)$ for some set P of periodic points of ψ , so ψ induces an automorphism of the étale groupoid $G = R^u(Y, \psi)|_T$ as in (4). Then, $\mathbb{Z} \ltimes_{\psi} G$ is Morita equivalent to $\mathbb{Z} \ltimes_{\psi} R^u(Y, \psi)$.

Let $\pi: (\Sigma_{\mathcal{G}}, \sigma) \rightarrow (Y, \psi)$ be a (regular) s -bijective map of Smale spaces. Then, we have the graph $\mathcal{G}_N(\pi)$ modeling the $(N + 1)$ -fold fiber product of $\Sigma_{\mathcal{G}}$ over X , so $\Sigma_{\mathcal{G}_N(\pi)} = (\Sigma_{\mathcal{G}})_N$. Combined with functoriality, we get simplicial groups

$$(\text{BF}(\gamma_{\mathcal{G}_N(\pi)}^s))_{N=0}^{\infty}, \quad (\ker(1 - \gamma_{\mathcal{G}_N(\pi)}^s))_{N=0}^{\infty}.$$

Proposition 6.3. *There is a spectral sequence $E_{pq}^r \Rightarrow K_{p+q}(\mathcal{R}_u(Y, \psi))$ with*

$$E_{N,2k}^1 = \text{BF}(\gamma_{\mathcal{G}_N(\pi)}^s), \quad E_{N,2k+1}^1 = \ker(1 - \gamma_{\mathcal{G}_N(\pi)}^s),$$

with the E^1 -differential $E_{p,q}^1 \rightarrow E_{p-1,q}^1$ given by the simplicial structure.

Proof. When $f: (\Sigma, \sigma) \rightarrow (Y, \psi)$ is an s -bijective map and $X' \subset \Sigma$ is a σ -invariant transversal such that $f(X') = X$, we get a subgroupoid $\mathbb{Z} \ltimes H \subseteq \mathbb{Z} \ltimes G$ from the reduction of $\mathbb{Z} \ltimes R^u(\Sigma, \sigma)$ to X' . By Proposition 6.1, $C_0(X)$ is in the triangulated subcategory generated by the image of $\text{Ind}_{\mathbb{Z} \ltimes H}^{\mathbb{Z} \ltimes G}$. We thus have a convergent spectral sequence

$$E_{pq}^1 = K_q((\mathbb{Z} \ltimes G) \rtimes (\text{Ind}_{\mathbb{Z} \ltimes H}^{\mathbb{Z} \ltimes G} \text{Res}_{\mathbb{Z} \ltimes H}^{\mathbb{Z} \ltimes G})^{p+1} C_0(X)) \Rightarrow K_{p+q}(C^*(\mathbb{Z} \ltimes G)).$$

Moreover, by Proposition 3.11, we have

$$E_{pq}^1 \simeq K_q(C^*(R^u(\Sigma_p, \sigma_p)) \rtimes \mathbb{Z}) \simeq K_q(\mathcal{R}_u(\Sigma_p, \sigma_p)).$$

When (Σ, σ) is presented by a graph \mathcal{G} , these can be interpreted as $K_q(\mathcal{R}_u(\Sigma_{\mathcal{G}_p(\pi)}, \sigma))$, and the associated complex is the E^1 -sheet of the above spectral sequence.

Finally, for any graph \mathcal{G} , we have

$$K_0(\mathcal{R}_u(\Sigma, \sigma)) \simeq \text{BF}(\gamma_{\mathcal{G}}^s), \quad K_1(\mathcal{R}_u(\Sigma, \sigma)) \simeq \ker(1 - \gamma_{\mathcal{G}}^s)$$

for the connecting map $\gamma_{\mathcal{G}}^s: \mathbb{Z}\mathcal{G}^0 \rightarrow \mathbb{Z}\mathcal{G}^0$; see Appendix B. ■

There is another spectral sequence which plays nicer for low-dimensional examples. Consider the subsets $B_n \subset \mathcal{G}_n(\pi)$ for normalization, which is a complete system of representatives of the free S_{n+1} -orbits. We have an induced map γ_{B_n} on $\mathbb{Z}B_n$, and the

inductive limit groups $D_n = \varinjlim_{\gamma_{B_n}} \mathbb{Z} B_n$ form a chain complex that is quasi-isomorphic to $D^s(\mathcal{G}_\bullet(\pi))$ [32, Section 4.2]. Then, we have complexes

$$C_\bullet = (\text{BF}(\gamma_{B_n}))_{k=0}^\infty, \quad C'_\bullet = (\ker(1 - \gamma_{B_n}))_{k=0}^\infty,$$

endowed with induced differentials.

Proposition 6.4. *There is a spectral sequence $E_{pq}^r \Rightarrow K_{p+q}(\mathcal{R}_u(Y, \psi))$ such that $E_{pq}^2 = 0$ for odd q and $E_{pq}^2 = E_{p0}^2$ for even q , sitting in a long exact sequence*

$$\cdots \rightarrow H_{p-1}(C'_\bullet) \rightarrow E_{p0}^2 \rightarrow H_p(C_\bullet) \rightarrow H_{p-2}(C'_\bullet) \rightarrow E_{p-1,0}^2 \rightarrow \cdots \quad (13)$$

Proof. We keep the notations G and X from a previous proposition, but this time we use the spectral sequence given by Proposition 6.2 for the resolution $P_n = C_0(G^{(n+1)})$ of $C_0(X)$.

As before, by transversality, we have

$$K_0(G \times P_n) \simeq D^s(\mathcal{G}_n(\pi)), \quad K_1(G \times P_n) \simeq 0.$$

As above, take the complex D_\bullet with $D_n = \varinjlim_{\gamma_{B_n}} \mathbb{Z} B_n$. The standard quasi-isomorphism from $D^s(\mathcal{G}_\bullet(\pi))$ to D_\bullet as given in [32, Section 4.2] intertwines the actions of $\gamma_{\mathcal{G}_n}$ and γ_{B_n} by construction. Generally, when $D'_\bullet \rightarrow D_\bullet$ is a quasi-isomorphism of complexes of Γ -modules, it induces an isomorphism of hyperhomology $\mathbb{H}_p(\Gamma, D_\bullet) \simeq \mathbb{H}_p(\Gamma, D'_\bullet)$. We thus have

$$E_{pq}^2 = \mathbb{H}_p(\mathbb{Z}, D^s(\mathcal{G}_\bullet(\pi))) \simeq \mathbb{H}_p(\mathbb{Z}, D_\bullet).$$

It remains to put $\mathbb{H}_p(\mathbb{Z}, D_\bullet)$ in a long exact sequence of the form (13). Recall that the trivial $\mathbb{Z}[\mathbb{Z}]$ -module \mathbb{Z} has a free resolution of length 1, given by

$$0 \rightarrow \mathbb{Z}[\mathbb{Z}] \xrightarrow{1-\tau_1} \mathbb{Z}[\mathbb{Z}] \rightarrow \mathbb{Z},$$

where τ_1 is the translation by 1. Thus, the hyperhomology $\mathbb{H}_\bullet(\mathbb{Z}, D_\bullet)$ can be computed from the double complex with two rows of the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \longrightarrow D_0 \\ & & \downarrow 1-\gamma_{B_n} & & \downarrow 1-\gamma_{B_{n-1}} & & \downarrow 1-\gamma_{B_0} \\ \cdots & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \longrightarrow D_0. \end{array}$$

It contains a subcomplex given by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \ker(1 - \gamma_{B_n}) & \longrightarrow & \ker(1 - \gamma_{B_{n-1}}) & \longrightarrow & \cdots \longrightarrow \ker(1 - \gamma_{B_0}) \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \longrightarrow 0, \end{array}$$

and the quotient is quasi-isomorphic to the complex of $(\text{BF}(\gamma_{B_n}))_n$. From this, we indeed obtain a long exact sequence of the form (13). ■

Remark 6.5. Recall that Y is of the form $\varprojlim Y_0$ for a projective system of some compact metric space Y_0 and a suitable self-map $g: Y_0 \rightarrow Y_0$ as the connecting map at each step [43, Theorem B], analogous to the standard presentation of the m^∞ -solenoid in the next section. Suppose further that g is open and the groupoid C^* -algebra of the stable relation $R^s(Y, \psi)$ has finite-rank K -groups. Then, combining [12, Theorem 1.1] and [7, Corollary 5.5], we see that $K_\bullet(\mathcal{R}_u(Y, \psi))$ fits in an exact sequence

$$\begin{CD}
 K^{*+1}(C(Y_0)) @>{1-[E_g]}>> K^{*+1}(C(Y_0)) @>>> K_\bullet(\mathcal{R}_u(Y, \psi)) @>>> K^\bullet(C(Y_0)) \\
 @. @. @. @VVV \\
 @. @. @. K^\bullet(C(Y_0)),
 \end{CD}$$

where E_g is the $C(Y_0)$ -bimodule associated with g . It would be an interesting problem to compare the two ways to compute $K_\bullet(\mathcal{R}_u(Y, \psi))$.

7. Examples

7.1. Solenoid

One class of examples is that of *one-dimensional solenoids* [42, 44]. Let us first explain the easiest example, the m^∞ -solenoid. Consider the space

$$Y = \{(z_0, z_1, \dots) \mid z_k \in S^1, z_k = z_{k+1}^m\},$$

which is the projective limit of

$$S^1 \xleftarrow{z^m \mapsto z} S^1 \xleftarrow{z^m \mapsto z} S^1 \xleftarrow{z^m \mapsto z} \dots \tag{14}$$

A compatible metric is given by

$$d((z_k)_k, (z'_k)_k) = \sum_k m^{-k} d_0(z_k, z'_k),$$

where d_0 is any metric on S^1 compatible with its topology; for example, one may take the arc-length metric $d_0(e^{is}, e^{it}) = |s - t|$ when $|s - t| \leq \pi$.

There is a natural “shift” self-homeomorphism

$$\phi: Y \rightarrow Y, \quad (z_0, z_1, \dots) \mapsto (z_0^m, z_1^m = z_0, z_2^m = z_1, \dots),$$

with inverse given by $\phi^{-1}((z_0, z_1, \dots)) = (z_1, z_2, \dots)$. Then, (Y, ϕ) is a Smale space [32, 44].

Denote by π the canonical projection $Y \rightarrow S^1$ on the first factor. As each step of (14) is an m -to-1 map, $\pi^{-1}(z_0)$ can be identified with the Cantor set $\Sigma = \prod_{n=1}^\infty \{0, 1, \dots, m - 1\}$ for any $z_0 \in S^1$. This allows us to write local stable and unstable sets around $z = (z_k)_k$, as

$$Y^s(z, \varepsilon) = \pi^{-1}(z_0) \cong \Sigma, \quad Y^u(z, \varepsilon) = \{(e^{itm^{-k}} z_k)_{k=0}^\infty \mid |t| < \delta_\varepsilon\}$$

for small enough $\varepsilon > 0$, with $\delta_\varepsilon > 0$ depending on ε . Note that π defines a fiber bundle with fiber Σ , and $Y^u(z, \varepsilon) \times \Sigma \rightarrow Y$ corresponding to the bracket map gives local trivializations.

Now, the groupoid $R^u(Y, \phi)$ is the transformation groupoid $\mathbb{R} \ltimes_\alpha Y$ for the flow

$$\alpha_t(z_0, z_1, \dots) = (e^{it} z_0, e^{itm^{-1}} z_1, \dots, e^{itm^{-k}} z_k, \dots) \quad (t \in \mathbb{R}).$$

Restricted to the transversal $\pi^{-1}(1)$, we obtain the ‘‘odometer’’ transformation groupoid $\mathbb{Z} \ltimes_\beta \Sigma$, where Σ is identified with $\varprojlim_k \mathbb{Z}_{m^k}$, and the generator $1 \in \mathbb{Z}$ acts by the $+1$ map on \mathbb{Z}_{m^k} .

There is a well-known factor map from the two-sided full shift on m letters onto (Y, ϕ) . Namely, writing

$$\Sigma' = \{0, 1, \dots, m - 1\}^{\mathbb{Z}} = \{(a_n)_{n=-\infty}^\infty \mid 0 \leq a_n < m\},$$

we have a continuous map $f: \Sigma' \rightarrow Y$ by

$$f((a_n)_n) = (z_k)_{k=0}^\infty, \quad z_k = \exp\left(2\pi i \sum_{j=0}^\infty m^{-j-1} a_{j-k}\right).$$

Then, we have $f\sigma = \phi f$ for $\sigma: \Sigma' \rightarrow \Sigma'$ defined by $\sigma((a_n)_n) = (a_{n+1})_n$.

This allows us to compute all relevant invariants separately. As for the K -groups, [46] gives

$$K_0(C^*(R^u(Y, \phi))) \simeq \mathbb{Z}\left[\frac{1}{m}\right], \quad K_1(C^*(R^u(Y, \phi))) \simeq \mathbb{Z}.$$

(These can be also obtained through the Connes–Thom isomorphism $K_\bullet(C^*(R^u(Y, \phi))) \simeq K^{\bullet+1}(Y)$.) As for groupoid homology, we have

$$H_\bullet(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) \simeq H_\bullet(\mathbb{Z}, C(\Sigma, \mathbb{Z})),$$

where the right-hand side is the groupoid homology of \mathbb{Z} with coefficient $C(\Sigma, \mathbb{Z})$ endowed with the \mathbb{Z} -module structure induced by β . This leads to

$$\begin{aligned} H_0(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) &\simeq C(\Sigma, \mathbb{Z})_\beta \simeq \mathbb{Z}\left[\frac{1}{m}\right], \\ H_1(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) &\simeq C(\Sigma, \mathbb{Z})^\beta \simeq \mathbb{Z}, \end{aligned}$$

with coinvariants and invariants of β , while $H_n(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) = 0$ for $n > 1$. The computation for $H_n^s(Y, \phi)$ will be more involved, but one finds [32, Section 7.3] that

$$H_0^s(Y, \phi) \simeq D^s(\Sigma', \sigma) \simeq \mathbb{Z}\left[\frac{1}{m}\right], \quad H_1^s(Y, \phi) \simeq \mathbb{Z},$$

and $H_n^s(Y, \phi) = 0$ for $n > 1$. Thus, the spectral sequence of Theorem 4.9 collapses at the E^2 -sheet, and there is no extension problem.

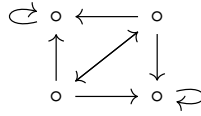


Figure 2. $\mathcal{G}_1^2(\pi)$ for $m = 2$.

Now let us look at the unstable Ruelle algebra. From the Pimsner–Voiculescu exact sequence, we get

$$K_0(C^*(R^u(Y, \phi)) \rtimes \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}/(1 - m)\mathbb{Z}, \quad K_1(C^*(R^u(Y, \phi)) \rtimes \mathbb{Z}) \simeq \mathbb{Z}.$$

Let us relate our spectral sequence to these groups.

As in [32, Section 7.3], there is an s -bijective map $\pi: \Sigma_{\mathcal{G}} \rightarrow Y$ for the graph \mathcal{G} with one vertex and m loops around it. Since $\Sigma_{\mathcal{G}^2} \rightarrow Y$ is regular, we use this to compute fiber products. Now \mathcal{G}^2 is the complete graph with m vertices. We thus have

$$\text{BF}(\gamma_{B_0^2}) \simeq \mathbb{Z}/(m - 1)\mathbb{Z}, \quad \ker(1 - \gamma_{B_0^2}) \simeq 0.$$

At the next step, $\mathcal{G}_1^2(\pi)$ is a graph with $m + 2$ vertices.

There is a single free S_2 -orbit, as the nontrivial element $g \in S_2$ acts by flipping across the 45° in Figure 2. Thus, the subset B_1^2 is a singleton (given by a choice of vertex with a loop), and the induced map $\gamma_{B_1^2}$ on $\mathbb{Z}B_1^2$ is the identity map. We thus have

$$\text{BF}(\gamma_{B_1^2}) \simeq \mathbb{Z}, \quad \ker(1 - \gamma_{B_1^2}) \simeq \mathbb{Z}.$$

For $n > 1$, we have $B_n^2 = \emptyset$; thus, $\text{BF}(\gamma_{B_n^2}) = 0 = \ker(1 - \gamma_{B_n^2})$ for trivial reasons, which are the torsion-free parts of $K_0(\mathbb{Z} \rtimes C^*(R^u(Y, \phi)))$.

Again from the computation [32, Section 7.3], the boundary map $\text{BF}(\gamma_{B_1^2}) \rightarrow \text{BF}(\gamma_{B_0^2})$ is zero. Then, Proposition 6.2 gives a spectral sequence with

$$E_{0q}^2 \simeq \mathbb{Z}/(m - 1)\mathbb{Z}, \quad E_{1q}^2 \simeq \mathbb{Z}, \quad E_{2q}^2 \simeq \mathbb{Z}, \quad E_{pq}^2 \simeq 0 \quad (p > 2)$$

for even q , and $E_{pq}^2 = 0$ for odd q . From this, we get an exact sequence

$$0 \rightarrow \mathbb{Z}/(m - 1)\mathbb{Z} \rightarrow K_0(C^*(R^u(Y, \phi))) \rightarrow \mathbb{Z} \rightarrow 0$$

and isomorphism $K_1(C^*(R^u(Y, \phi))) \simeq \mathbb{Z}$, as expected.

7.2. Self-similar group action

Let us explain an application to the Smale spaces of self-similar group actions. First, let us recall the setting. We refer to [24, 25] for unexplained terminology. Let X be a finite set of symbols, with $d = |X|$, and let X^* denote the set of words with letters in X with the structure of a rooted tree. A faithful action of a group Γ on X^* (by tree automorphisms) is *self-similar* if it satisfies the condition

$$\forall x \in X, \gamma \in \Gamma \quad \exists \gamma' \in \Gamma \quad \forall w \in X^*: \gamma(xw) = \gamma(x)\gamma'(w).$$

Writing $\gamma|_x = \gamma'$ in the above setting, we get a cocycle $(\gamma, x) \mapsto \gamma|_x$ for the action of Γ on X . This extends to a cocycle $\gamma|_u$ for $u \in X^*$.

Consider the full shift space $(\Sigma, \hat{\sigma})$ where $\Sigma = X^{\mathbb{Z}}$ and $\hat{\sigma}(x)_k = x_{k-1}$ (note the opposite convention). Thus, $x, y \in \Sigma$ are stably equivalent if and only if there is some N such that $x_k = y_k$ holds for $k < N$, while they are unstably equivalent if and only if this holds for $k > N$.

Moreover, $x, y \in \Sigma$ are said to be *asymptotically equivalent* if there is a finite set $F \subset \Gamma$ and a sequence $(\gamma_n)_{n \geq 0}$ in F , such that

$$\gamma_n(x_{-n}x_{-n+1} \cdots) = y_{-n}y_{-n+1} \cdots \quad \text{for } n \geq 0.$$

The quotient space $S_{\Gamma, X}$ by this relation is called the *limit solenoid*, and it is equipped with the homeomorphism induced by the shift map, denoted by $\hat{\sigma}$. When (Γ, X) is *contracting*, *recurrent* (or *self-replicating*), and *regular*, then $(S_{\Gamma, X}, \hat{\sigma})$ is a Smale space with totally disconnected stable sets [25, Proposition 6.10] (alternatively, one can use [25, Proposition 2.6] in conjunction with the main result of [43]).

In the rest of the section, we assume that (Γ, X) satisfies the above assumptions. Let us give a more detailed description of the ingredients of the spectral sequence. Let us fix $x \in \Sigma$, and let $\Sigma^s(x)$ denote its stable equivalence class. We denote by T_0 the set of points $y \in \Sigma^s(x)$ such that $y_n = x_n$ for $n < 0$.

Proposition 7.1. *There is no point $y \neq x$ in $\Sigma^s(x)$ which is asymptotically equivalent to x .*

Proof. Suppose that $y \in \Sigma^s(x)$ is asymptotically equivalent to y , and take a finite subset $F \subset \Gamma$ satisfying the defining condition of asymptotic equivalence between x and y above.

By contractibility assumption, there is a finite subset $\mathcal{N} \subset \Gamma$ containing e , and $N_0 \in \mathbb{N}$ such that $|u| \geq N_0$ implies $\gamma|_u \in \mathcal{N}$ for $\gamma \in F$. Moreover, by regularity and contractibility, there is $N_1 \in \mathbb{N}$ such that, for any $\gamma, \gamma' \in \mathcal{N}$ and $u \in X^{N_1}$, one has either $\gamma(u) \neq \gamma'(u)$ or $\gamma|_u = \gamma'|_u$ [25, Proposition 6.2]. By assumption $x \sim_s y$, there is $N \in \mathbb{N}$ such that $x_k = y_k$ for $k \leq -N$. Set $M = N + N_0 + N_1$, and $u = x_{-M}x_{-M+1} \cdots x_{-(N+N_1)} \in X^{N_0}$. Then, on the one hand, $\gamma' = \gamma_M|_u$ is in \mathcal{N} , and on the other hand, γ' should fix $v = x_{-(N+N_1)+1} \cdots x_{-N} \in X^{N_1}$. Thus, we get $\gamma'|_v = e|_v$; hence, $\gamma' = e$. This implies that $x_k = y_k$ for $k > -N$, and consequently, $x = y$. ■

Corollary 7.2. *The projection map $(\Sigma, \hat{\sigma}) \rightarrow (S_{\Gamma, X}, \hat{\sigma})$ is s -bijective.*

We can thus compute the homology groups $H_*^s(S_{\Gamma, X}, \hat{\sigma})$ using the dimension groups of iterated fiber products of Σ over $S_{\Gamma, X}$.

Proposition 7.3. *We have $H_0^s(S_{\Gamma, X}, \hat{\sigma}) \simeq \mathbb{Z}[1/d]$.*

Proof. Since $D^s(\Sigma, \hat{\sigma}) \simeq \mathbb{Z}[1/d]$, it is enough to show that the boundary map

$$D^s(\Sigma_1, \hat{\sigma}_1) \rightarrow D^s(\Sigma, \hat{\sigma})$$

is trivial. Note that Σ_1 is the graph of the asymptotic equivalence relation.

Recall that $\Sigma^s(x)$ admits a unique regular Borel measure μ which is invariant under the unstable equivalence relation and normalized as $\mu(T_0) = 1$. Namely, with the identification $T_0 \simeq X^{\mathbb{N}}$, μ is the infinite product of uniform measure on X . Then, the isomorphism $D^s(\Sigma, \hat{\sigma}) \rightarrow \mathbb{Z}[1/d]$ is given by $[E] \mapsto \mu(E)$ for compact open sets $E \subset \Sigma^s(x)$.

Then, it is enough to show that the asymptotic equivalence relation on $\Sigma^s(x)$ preserves μ . Let $y, z \in \Sigma^s(x)$ be two asymptotically equivalent points. Thus, there is a sequence $(\gamma_n)_{n=0}^\infty$ in Γ such that $\gamma_n(y_{-n}y_{-n+1}\cdots) = z_{-n}z_{-n+1}\cdots$ holds for all n . For some fixed N , consider the compact open neighborhood $E = \{y' \mid y'_k = y_k \ (k < N)\}$ of y in $\Sigma^s(x)$, which has measure $\mu(E) = d^{-N}$. Then,

$$\tilde{E} = \{(y', z') \mid y' \in E, \forall n : \gamma_n(y'_{-n}y'_{-n+1}\cdots) = z'_{-n}z'_{-n+1}\cdots\}$$

is a compact open neighborhood of (y, z) in $\Sigma^s_1((y, z))$, and the projection to the first factor is a bijection $\tilde{E} \rightarrow E$. Moreover, its projection to the second factor, $E' = \{z' \mid \exists y' : (y', z') \in \tilde{E}\}$, can be written as $E' = \{z' \mid z'_k = z_k \ (k < N)\}$. This has the same measure as E ; hence, the asymptotic equivalence relation indeed preserves μ . ■

Let us next look at an étale groupoid model. Let T be the image of T_0 in $S_{\Gamma, X}$. We denote the étale groupoid $R^u(S_{\Gamma, X}, \hat{\sigma})|_T$ by $M_{\Gamma, X}$, so that $C^*(R^u(S_{\Gamma, X}, \hat{\sigma}))$ is strongly Morita equivalent to $C^*(M_{\Gamma, X})$. As for the Ruelle algebra, the groupoid $D_{\Gamma, X} = \mathbb{Z} \times M_{\Gamma, X}$ is the groupoid of germs of partially defined homeomorphisms on $X^{\mathbb{N}}$, of the form $x_1 \cdots x_m z \mapsto x'_1 \cdots x'_n \gamma z$ for $z \in X^{\mathbb{N}}$, with $x_i, x'_i \in X$ and $\gamma \in \Gamma$ (notice we range from 1 to m in the domain and from 1 to n in the codomain). The associated C^* -algebra $\mathcal{O}_{\Gamma, X} = C^*(D_{\Gamma, X}) = \mathcal{R}_u(S_{\Gamma, X}, \hat{\sigma})$ has a presentation similar to Cuntz–Pimsner algebras. The structure of these algebras, including their K -groups and equilibrium states, have been previously studied in [9, 14].

Proposition 7.4 ([25, Proposition 6.8]). *The image of $y \in T_0$ in $S_{\Gamma, X}$ is unstably equivalent to the image of x if and only if $\gamma(x_k x_{k+1} \cdots) = y_k y_{k+1} \cdots$ for some $\gamma \in \Gamma$ and $k \geq 0$.*

This allows further recursive description of the groupoid homology, as follows. Let $M_{\Gamma, X}^n$ be the groupoid of germs of partially defined transformations on $X^{\mathbb{N}}$ of the form $x_1 \cdots x_n z \mapsto x'_1 \cdots x'_n \gamma z$ for $z \in X^{\mathbb{N}}$, with $x_i, x'_i \in X$ and $\gamma \in \Gamma$. These form an increasing sequence of subgroupoids of $M_{\Gamma, X}$ such that $\bigcup_n M_{\Gamma, X}^n = M_{\Gamma, X}$. By the self-similarity assumption, $M_{\Gamma, X}^n$ is isomorphic to the product groupoid $R_n \times M_{\Gamma, X}^0$, where R_n is the complete equivalence relation on X^n .

Proposition 7.5. *The natural homomorphism from the group $H_0(M_{\Gamma, X}, \mathbb{Z}) \simeq \mathbb{Z}[1/d]$ to $K_0(C^*(M_{\Gamma, X}))$ is injective.*

Proof. First, we have $H_0(M_{\Gamma, X}, \mathbb{Z}) \simeq \mathbb{Z}[1/d]$ by Proposition 7.3 and Theorem 5.1. By the above discussion, the image of this group in $K_0(C^*(M_{\Gamma, X}))$ is equal to the image of $K_0(M_{d^\infty}(\mathbb{C}))$, where $M_{d^\infty}(\mathbb{C})$ is the UHF algebra that appears as the gauge invariant part of $\mathcal{O}_d \subset \mathcal{O}_{\Gamma, X}$. The measure μ in the proof of Proposition 7.3 defines a tracial state τ on $C^*(M_{\Gamma, X})$. Thus, on the image of $H_0(M_{\Gamma, X}, \mathbb{Z})$, τ induces isomorphism to $\mathbb{Z}[1/d]$. ■

Turning to higher groupoid homology, we have

$$H_k(M_{\Gamma, X}, \mathbb{Z}) \simeq \varinjlim_n H_k(M_{\Gamma, X}^n, \mathbb{Z})$$

from the compatibility of homology and increasing sequence of complexes. Moreover, by the invariance of groupoid homology under Morita equivalence, we have

$$H_k(M_{\Gamma, X}^n, \mathbb{Z}) \simeq H_k(M_{\Gamma, X}^0, \mathbb{Z}).$$

Up to this isomorphism, the connecting maps $H_k(M_{\Gamma, X}^n, \mathbb{Z}) \rightarrow H_k(M_{\Gamma, X}^{n+1}, \mathbb{Z})$ are all the same.

If, moreover, the action of Γ on $X^{\mathbb{N}}$ is topologically free¹, then the groupoid $M_{\Gamma, X}^0$ (which is defined by the germs of the G -action on $X^{\mathbb{N}}$) can be identified with the transformation groupoid $\Gamma \ltimes X^{\mathbb{N}}$. Note that $\Gamma \ltimes X^{\mathbb{N}}$ can be regarded as the projective limit of the groupoids $\Gamma \ltimes X^n$ for increasing n . This again gives a presentation

$$H_k(\Gamma \ltimes X^{\mathbb{N}}, \mathbb{Z}) \simeq \varinjlim_n H_k(\Gamma \ltimes X^n, \mathbb{Z}).$$

By the recurrence assumption, the groupoid $\Gamma \ltimes X^n$ is isomorphic to $R_n \times \Gamma$, and we have

$$M_{\Gamma, X}^0 = M_{\Gamma, X}^k = M_{\Gamma, X}$$

by [25, Proposition 5.2]. In particular, $H_k(M_{\Gamma, X}, \mathbb{Z})$ becomes the inductive limit of the ordinary group homology $H_k(\Gamma, \mathbb{Z})$, with respect to the map induced by the *matrix recursion*

$$\phi: \mathbb{Z}\Gamma \rightarrow \text{End}_{\mathbb{Z}\Gamma}(\mathbb{Z}(X \cdot \Gamma)) \simeq \mathbb{Z}(R_1 \times \Gamma)$$

for the *permutation bimodule* $X \cdot \Gamma$, and the identification $H_k(\Gamma, \mathbb{Z}) \simeq H_k(R_n \times \Gamma, \mathbb{Z})$. Thus, under the above additional assumptions, our spectral sequence gives a computation of the K -groups for the C^* -algebra $C^*(R^u(S_{\Gamma, X}, \hat{\sigma}))$ in terms of the direct limits of sequences with the terms $H_k(\Gamma, \mathbb{Z})$.

Analogously, for the K -theory, we have

$$K_i(C^*(M_{\Gamma, X})) \simeq \varinjlim_n K_i(C^*(\Gamma \ltimes X^n)),$$

cf. [25, Proposition 3.8].

Example 7.6. As a concrete example, let us look at the binary adding machine from [24, Section 1.7.1]. This one is given by $\Gamma = \mathbb{Z}$, $X = \{0, 1\}$, and the action of $1 \in \mathbb{Z}$ on X^* is presented as

$$a(0w) = 1w, \quad a(1w) = 0a(w)$$

¹This assumption is equivalent to the injectivity of the associated virtual endomorphisms of Γ and forces the group Γ to be of polynomial growth [24, Section 6.1].

for $w \in X^*$. To define the transversal, we choose $x = (x_k)_k \in X^{\mathbb{Z}}$ with $x_k = 0$ for all k . Then, the induced equivalence relation on T_0 by Proposition 7.4 agrees with the orbit equivalence relation of $\mathbb{Z} \ltimes \varprojlim \mathbb{Z}_{\gamma^k}$ from the previous section (up to a change of convention of the shift map). Turning to homology, on $H_0(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z} \simeq H_1(\mathbb{Z}, \mathbb{Z})$, the matrix recursion induces multiplication by 2, while it acts as the identity on $H_1(\mathbb{Z}, \mathbb{Z})$. This gives a direct computation of $H_0(M_{\mathbb{Z}, X}, \mathbb{Z}) \simeq \mathbb{Z}[1/2]$ and $H_1(M_{\mathbb{Z}, X}, \mathbb{Z}) \simeq \mathbb{Z}$, while $H_k(M_{\mathbb{Z}, X}, \mathbb{Z}) = 0$ for $k > 1$.

Next, let us look at higher dimensional odometer actions in more detail. Let us take $B \in M_N(\mathbb{Z}) \cap GL_N(\mathbb{Q})$ with its eigenvalues all bigger than 1. Put $\Gamma = \mathbb{Z}^N$, and take a set X of representatives of the $B\Gamma$ -cosets in Γ . Then, we have a contracting, regular, and recurrent action of Γ on X^* such that the natural action of Γ on $\varprojlim_n \Gamma/B^n\Gamma$ can be interpreted as the induced action on $X^{\mathbb{N}}$; see [24] for details. Thus, the groupoid homology $H_k(M_{\Gamma, X}, \mathbb{Z})$ can be identified with

$$H_k(\Gamma, C(\varprojlim_n \Gamma/B^n\Gamma, \mathbb{Z})) \simeq \varinjlim_n H_k(\Gamma, C(\Gamma/B^n\Gamma, \mathbb{Z})).$$

The first step connecting map is $H_k(\Gamma, \mathbb{Z}) \rightarrow H_k(\Gamma, C(\Gamma/B\Gamma, \mathbb{Z}))$, induced by the Γ -equivariant embedding of \mathbb{Z} into $C(\Gamma/B\Gamma, \mathbb{Z})$ as constant functions.

Now, recall that $H_{N-k}(\Gamma, \mathbb{Z})$ is isomorphic to $\bigwedge^k \Gamma'$, where $\Gamma' = \text{Hom}(\Gamma, \mathbb{Z})$ is the group of homomorphisms from Γ to \mathbb{Z} . Let us describe an invariant formula for this.

Suppose that $(v_i)_{i=1}^N$ is a basis of Γ , and let $(v^i)_i$ be its dual basis in Γ' . Then, we get a graded commutative $\mathbb{Z}[\Gamma]$ -algebra

$$\Omega_{v_*}^\bullet = \mathbb{Z}[\Gamma] \otimes \bigwedge^\bullet \{a(v^1), \dots, a(v^N)\},$$

where $a(v^i)$ are formal symbols subject to the rule $a(v^i) \wedge a(v^j) = -a(v^j) \wedge a(v^i)$. In particular, the degree k part $\bigwedge^k \{a(v^1), \dots, a(v^N)\}$ is a free commutative group with basis $a(v^{i_1}) \wedge \dots \wedge a(v^{i_k})$ for $1 \leq i_1 < i_2 < \dots < i_k \leq N$. Then, multiplication by the homogeneous element of degree 1,

$$D_{v_*} = \sum_{i=1}^N (1 - \lambda_{v_i}) \otimes a(v^i),$$

defines a cochain complex structure on $\Omega_{v_*}^\bullet$.

The degree shifted chain complex $P_{\bullet}^{v_*}$ with $P_k^{v_*} = \Omega_{v_*}^{N-k}$ is a free resolution of \mathbb{Z} as a $\mathbb{Z}[\Gamma]$ -module. Then, $H_\bullet(\Gamma, \mathbb{Z})$ can be computed as the homology of $\mathbb{Z} \otimes_{\Gamma} P_{\bullet}^{v_*}$, which is just the degree shift of $\bigwedge^\bullet \{a(v^1), \dots, a(v^N)\}$ with trivial differential, hence it is equal to the group $\bigwedge^\bullet \{a(v^1), \dots, a(v^N)\}$ itself.

Let $f_{v_*}: \bigwedge^\bullet \{a(v^1), \dots, a(v^N)\} \rightarrow \bigwedge^\bullet \Gamma'$ be the isomorphism of graded commutative rings characterized by $a(v^i) \mapsto v^i$. Then, if $(w_i)_i$ is another choice of basis in Γ , the map of resolutions $P_{\bullet}^{v_*} \rightarrow P_{\bullet}^{w_*}$ is compatible with f_{v_*} and f_{w_*} (up to chain-homotopy, this map is unique).

Now we are ready to identify the connecting map

$$\phi_k: H_k(\Gamma, \mathbb{Z}) \rightarrow H_k(\Gamma, C(\Gamma/B\Gamma, \mathbb{Z})) \simeq H_k(\Gamma, \mathbb{Z}).$$

Proposition 7.7. *With respect to the above identification $H_{N-k}(\Gamma, \mathbb{Z}) \simeq \bigwedge^k \Gamma'$, the map ϕ_{N-k} is equal to $\bigwedge^k B^t$, where $B^t: \Gamma' \rightarrow \Gamma'$ is the adjoint of B .*

Proof. Note that $C(\Gamma/B\Gamma, \mathbb{Z})$ can be identified with $\text{Ind}_{B\Gamma}^\Gamma \mathbb{Z}$. Let us use the latter presentation to compute homology. By the Smith normal form, there is a basis $(v_i)_i$ of Γ and positive integers $(m_i)_i$ such that $(m_i v_i)_i$ is a basis of $B\Gamma$. Let us put $w_i = B^{-1} m_i v_i$, so that $(w_i)_i$ is another basis of Γ . Then, the cochain complex $\text{Ind}_{B\Gamma}^\Gamma \Omega_{w_*}^\bullet$ is given by the same $\mathbb{Z}[\Gamma]$ -module $\Omega_{w_*}^\bullet$ but the differential is the multiplication by

$$D' = \sum_i (1 - \lambda_{m_i v_i}) \otimes a(w^i).$$

Its degree shift $\text{Ind}_{B\Gamma}^\Gamma P_{\bullet}^{w_*}$ gives a free resolution of $\text{Ind}_{B\Gamma}^\Gamma \mathbb{Z}$.

Thus, $H_k(\Gamma, \text{Ind}_{B\Gamma}^\Gamma \mathbb{Z})$ is computed from the complex $\mathbb{Z} \otimes \text{Ind}_{B\Gamma}^\Gamma P_{\bullet}^{w_*}$, which is again the degree shift of $\bigwedge^\bullet \{a(v^1), \dots, a(v^N)\}$ with trivial differential. The identification

$$H_k(\Gamma, \text{Ind}_{B\Gamma}^\Gamma \mathbb{Z}) \simeq \bigwedge^{N-k} \{a(v^1), \dots, a(v^N)\} \simeq H_k(\Gamma, \mathbb{Z})$$

is the canonical isomorphism $H_k(\Gamma, \text{Ind}_{B\Gamma}^\Gamma \mathbb{Z}) \simeq H_k(\Gamma, \mathbb{Z})$. (One can see this by interpreting the k -th homology group $H_k(\Gamma, \text{Ind}_{B\Gamma}^\Gamma \mathbb{Z})$ as the k -th derived functor of $M \mapsto \mathbb{Z} \otimes_{\Gamma} \text{Ind}_{B\Gamma}^\Gamma M$ and noting that $\text{Ind}_{B\Gamma}^\Gamma$ is an exact functor which preserves projectivity, i.e., it send projective modules to projective modules.)

We can concretely lift the embedding $\mathbb{Z} \rightarrow \text{Ind}_{B\Gamma}^\Gamma \mathbb{Z}$ as a map of chain complexes $P_{\bullet}^{v_*} \rightarrow \text{Ind}_{B\Gamma}^\Gamma P_{\bullet}^{w_*}$ as the $\mathbb{Z}[\Gamma]$ -linear extension of

$$a(v^{i_1}) \wedge \dots \wedge a(v^{i_k}) \mapsto \sum_{0 \leq c_{i_j} < m_{i_j}} \lambda_{c_{i_1} v_{i_1} + \dots + c_{i_k} v_{i_k}} \otimes a(w^{i_1}) \wedge \dots \wedge a(w^{i_k}).$$

Then, the induced map $H_{N-k}(\Gamma, \mathbb{Z}) \rightarrow H_{N-k}(\Gamma, \text{Ind}_{B\Gamma}^\Gamma \mathbb{Z})$ becomes a transformation on $\bigwedge^k \Gamma'$ characterized by

$$v^{i_1} \wedge \dots \wedge v^{i_k} \mapsto m_{i_1} \dots m_{i_k} w^{i_1} \wedge \dots \wedge w^{i_k},$$

which is exactly $\bigwedge^k B^t$. ■

Corollary 7.8. *In the above setting, we have*

$$H_{N-k}(M_{\Gamma, X}) \simeq \varinjlim_n \bigwedge^k \Gamma',$$

where the connecting map $\bigwedge^k \Gamma' \rightarrow \bigwedge^k \Gamma'$ at each step is given by $\bigwedge^k B^t$.

Remark 7.9. More generally, when B_1, B_2, \dots are the endomorphisms of Γ of rank N , we get

$$H_{N-k}(\Gamma \times \varprojlim_n \Gamma / B_1 \cdots B_n \Gamma, \mathbb{Z}) \simeq \varinjlim_n \bigwedge^k \Gamma',$$

where the right hand is with respect to the inductive system with connecting maps $\bigwedge^k (B_1^t), \bigwedge^k (B_2^t), \dots$ on $\bigwedge^k \Gamma'$.

Back to the higher dimensional odometer action of $\Gamma = \mathbb{Z}^N$, we have

$$K_i(C^*(\Gamma \times X^n)) \simeq \mathbb{Z}^{2^{N-1}} \simeq \bigoplus_k H_{i+2k}(\Gamma, C(\Gamma/B^n \Gamma, \mathbb{Z})),$$

which implies that the spectral sequence for the Γ -action on the coefficient algebra $C(X^n)$ collapses at the E^2 -sheet. From Proposition 7.7, combined with the naturality of these spectral sequences, we obtain that the maps

$$K_i(C^*(\Gamma \times X^n)) \rightarrow K_i(C^*(\Gamma \times X^{n+1}))$$

are injective. As a consequence, the group $K_i(C^*(M_{\Gamma, X}))$ has rank (equal to) 2^{N-1} for $i = 0, 1$. Since $\bigoplus_k H_{i+2k}(M_{\Gamma, X}, \mathbb{Z})$ has the same rank and is torsion-free, we conclude that the spectral sequence for $M_{\Gamma, X}$ also collapses at the E^2 -sheet.

A. Induction functor for subgroupoids

Suppose that a groupoid G acts freely and properly from the right on a second countable, locally compact, Hausdorff space Y . Then, the transformation groupoid $Y \rtimes G$ is Morita equivalent to the quotient space Y/G as a groupoid, with Y being the bibundle implementing the equivalence. This induces the strong Morita equivalence between $G \rtimes C_0(Y) \simeq C^*(Y \rtimes G)$ and $C_0(Y/G)$. In particular, for the case $Y = G$ and the action given by right translation, we get an isomorphism between $G \rtimes C_0(G)$ and $\mathcal{K}(L_r^2(G))$, where $L_r^2(G)$ is the right Hilbert $C_0(X)$ -module completion of $C_c(G)$ with $C_0(X)$ -module structure from $r^*: C_0(X) \rightarrow C_b(G)$ and inner product from the Haar system.

Proof of Proposition 2.4. As in the assertion, let A be a G - C^* -algebra. We have two actions of G : on the one hand, it acts on s^*A by the combination of right translation on G and the original action on A , while on the other hand, it acts on r^*A by the right translation on G and trivially on A . Then, the structure morphism $\alpha: s^*A \rightarrow r^*A$ of the action intertwines these two actions. Morally, s^*A can be thought of as a space of sections $f(g) \in A_{sg}$ for $g \in G$, with the action of G given by $f^{g'}(g) = g'^{-1}f(gg')$ for $(g, g') \in G^{(2)}$, while r^*A can be thought as a space of sections $f(g) \in A_{rg}$, with G acting by $f^{g'}(g) = f(gg')$ for $(g, g') \in G^{(2)}$. We have $(\alpha f)(g) = gf(g)$ for the sections of the first kind, and these formulas give $(\alpha f^{g'})(g) = gf(gg') = (\alpha f)^{g'}(g)$.

Now, $\text{Ind}_G^G \text{Res}_G^G(A)$ is the crossed product of s^*A by G , while $\mathcal{K}(L_r^2(G)) \otimes_{C_0(X)} A$ is the crossed product of r^*A by G . Consequently, we get an isomorphism between these algebras. The extra action of G on $\text{Ind}_G^G \text{Res}_G^G(A)$ comes from the action of G on s^*A given by the combination of the left translation on G and the trivial action on A . Under the above isomorphism, this corresponds to the action on r^*A given by the combination of left translation on G and the original action on A . Thus, it corresponds to the diagonal action of G on $\mathcal{K}(L_r^2(G)) \otimes_{C_0(X)} A$. ■

More generally, the same argument gives an isomorphism

$$\phi: \text{Ind}_H^G \text{Res}_H^G A \simeq (C_0(G) \rtimes H)^r \otimes_{C_0(X)} A,$$

where G acts diagonally on the algebra on the right.

The functor $\text{Ind}_H^G: C_H^* \rightarrow C_G^*$ preserves split extensions, respects equivariant Morita equivalence, and is compatible with homotopy. Then, it extends to a functor

$$\text{Ind}_H^G: \text{KK}^H \rightarrow \text{KK}^G$$

by the universal property of KK^H [28, Section A.3], analogous to the case of KK [10], see also [3]. Let us give a more concrete description at the level of Kasparov cycles.

Consider an H -equivariant right Hilbert module \mathcal{E} over B . By using an approximate unit in B , we can equip \mathcal{E} with a compatible $C_0(X)$ -action. We can form the Hilbert module $C_0(G) \otimes \mathcal{E}$ over $G_0(G) \otimes B$ and have a restriction on the diagonal to get $s^*\mathcal{E} = (C_0(G) \otimes \mathcal{E})_{\Delta(X)}$ over $s^*B \simeq (C_0(G) \otimes B)_{\Delta(X)}$. This still has an action of H , analogous to the right action of H on s^*B .

Assume, moreover, (π, \mathcal{E}, T) is an equivariant Kasparov module between H - C^* -algebras. Therefore, \mathcal{E} is a graded right Hilbert module over B , T is an odd adjointable (or self-adjoint) endomorphism, and $\pi: C \rightarrow \mathcal{L}(\mathcal{E})$ is a $*$ -representation, with commutation relations as in [15]. Then, $s^*\mathcal{E}$ is a right Hilbert module over s^*B , with a left module structure over s^*C . Moreover, we can extend T to s^*T on $s^*\mathcal{E}$ as the restriction of $1_{C_0(G)} \otimes T$, with the right commutation properties (they hold before restriction to $\Delta(X)$). Finally, we take the crossed product by the right action of H ,

$$\text{Ind}_H^G(\pi, \mathcal{E}, T) = j_H(s^*\pi, s^*\mathcal{E}, s^*T).$$

This way, we obtain a map $\text{Ind}_H^G: \text{KK}^H(C, B) \rightarrow \text{KK}^G(\text{Ind}_H^G C, \text{Ind}_H^G B)$, realizing the extension of Ind_H^G to KK^H .

B. Bowen–Franks groups

Given $A \in \text{End}(\mathbb{Z}^n)$, we call

$$\text{BF}(A) = \text{cok}(1 - A) = \mathbb{Z}^n / (1 - A)\mathbb{Z}^n$$

the *Bowen–Franks group* of A . For a square matrix $A \in M_n(\mathbb{Z})$, we define $\text{BF}(A)$ by interpreting A as an element of $\text{End}(\mathbb{Z}^n)$ by representing $v \in \mathbb{Z}^n$ by column vectors and computing Av in the usual way.

Given $B \in \text{End}(\mathbb{Z}^m)$ and $f: \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ satisfying $Bf = fA$, we have the induced maps $\text{BF}(A) \rightarrow \text{BF}(B)$, $\ker(1 - A) \rightarrow \ker(1 - B)$. This correspondence is covariant. Taking the transpose of f , we also have contravariant maps $\text{BF}(B^t) \rightarrow \text{BF}(A^t)$, etc.

Given a square matrix $A \in M_n(\{0, 1\})$, the associated shift of finite type (Σ_A, σ) is given by

$$\Sigma_A = \{(x_k)_k \in \{1, \dots, n\}^{\mathbb{Z}} \mid A(x_k, x_{k+1}) = 1\}, \quad \sigma(x)_k = x_{k+1} \quad (x = (x_k)_k \in \Sigma_A),$$

where we write $A_{i,j} = A(i, j)$. Then, we have

$$\mathbb{Z} \rtimes C^*(R^s(\Sigma_A, \sigma)) \simeq \mathcal{O}_A \otimes \mathcal{K}, \quad \mathbb{Z} \rtimes C^*(R^u(\Sigma_A, \sigma)) \simeq \mathcal{O}_{A^t} \otimes \mathcal{K}$$

for the Cuntz–Krieger algebra \mathcal{O}_A , and we also have

$$K_0(\mathcal{O}_{A^t}) \simeq \text{BF}(A) \simeq K^1(\mathcal{O}_A), \quad K_1(\mathcal{O}_{A^t}) \simeq \ker(I_n - A) \simeq K^0(\mathcal{O}_A);$$

see [5, 6, 11].

Given $A \in \text{End}(\mathbb{Z}^n)$, let us look at the inductive system

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \dots$$

and its inductive limit $\varinjlim_A \mathbb{Z}^n$. Following [32], we represent the elements of $\varinjlim_A \mathbb{Z}^n$ by $[v, i]$ for $v \in \mathbb{Z}^n$ and $i \in \mathbb{N}$ (“ v at the i -th copy of \mathbb{Z}^n ”), subject to the rules $[v, i] = [Av, i + 1]$ and $[v, i] + [w, i] = [v + w, i]$. We have $[v, i] = 0$ if and only if $A^j v = 0$ holds for some j .

We then have the automorphism α on $\varinjlim_A \mathbb{Z}^n$ defined by

$$\alpha([v, i]) = [v, i + 1].$$

Lemma B.1. *Under the above setting, we have*

$$\text{cok}(1 - \alpha) \simeq \text{BF}(A), \quad \ker(1 - \alpha) \simeq \ker(1 - A).$$

Proof. The inductive limit functor is exact; hence, we have

$$\ker(1 - \alpha) \simeq \varinjlim_A \ker(1 - A), \quad \text{cok}(1 - \alpha) \simeq \varinjlim_A \text{BF}(A).$$

Thus, it is enough to check that A induces automorphisms on $\ker(1 - A)$ and $\text{BF}(A)$ (we will get the identity maps).

For $\ker(1 - A)$, we have $v \in \ker(1 - A)$ if and only if $v = Av$; hence, $A: \mathbb{Z}^n \mapsto \mathbb{Z}^n$ is restricted to a bijective map (identity map) on $\ker(1 - A)$. For $\text{BF}(A)$, again $[Av] = [v]$ holds in $\text{BF}(A)$; hence, A induces the trivial isomorphism. ■

Combined with the Pimsner–Voiculescu sequence, we get a direct computation of

$$K_0(\mathbb{Z} \rtimes C^*(R^s(\Sigma_A, \sigma))) \simeq \text{BF}(A), \quad K_1(\mathbb{Z} \rtimes C^*(R^s(\Sigma_A, \sigma))) \simeq \ker(1 - A).$$

Let G be an oriented graph, again following the convention of [32]. Thus, the associated shift of finite type is given by

$$\Sigma_G = \{e = (e^k)_{k \in \mathbb{Z}} \mid e^k \in G^1, t(e^k) = i(e^{k+1})\}, \quad \sigma(e)^k = e^{k+1}.$$

If we take the graph G^2 , we have $\Sigma_{G^2} = \Sigma_A$ for the matrix $A \in M_{G^1}(\{0, 1\})$ defined by

$$A_{e',e} = \begin{cases} 1 & (t(e') = i(e) \Leftrightarrow (e', e) \in G^2) \\ 0 & \text{otherwise.} \end{cases}$$

Denoting the corresponding endomorphism on $\mathbb{Z}G^1$ by $\gamma_{G^2}^s$, we have

$$D^s(G^2) = \varinjlim_{\gamma_{G^2}^s} \mathbb{Z}G^1.$$

We have $D^s(G) \simeq D^s(G^2) \simeq D^s(G^3) \simeq \dots$ for the higher block presentations.

Lemma B.2. *Let $\alpha_{G^k}^s$ be the map α as above, induced by $\gamma_{G^k}^s$ on the inductive limit*

$$D^s(G^k) = \varinjlim_{\gamma_{G^k}^s} \mathbb{Z}G^{k-1}.$$

Then, the isomorphism $D^s(G) \simeq D^s(G^k)$ intertwines α_G^s to $\alpha_{G^k}^s$.

Proof. This follows from the concrete form of isomorphism

$$D^s(G) \simeq D^s(G^k), \quad [v, i] \mapsto [(t^k)^*(v), i];$$

see [32, Theorem 3.2.3]. ■

Combining this with Lemma B.1, we obtain

$$\text{BF}(\gamma_G^s) \simeq \text{BF}(\gamma_{G^k}^s) \quad \text{and} \quad \ker(1 - \gamma_G^s) \simeq \ker(1 - \gamma_{G^k}^s).$$

Finally, let $\pi: H \rightarrow G$ be a graph homomorphism that induces an s -bijective map $\Sigma_H \rightarrow \Sigma_G$. Then, there is an integer K such that, for any k , the linear extension of

$$\pi^{s,K}: H^{k-1} \rightarrow \mathbb{Z}G^{k+K-1}, \quad q \mapsto \sum_{q' \in (t^K)^{-1}(q)} \pi(q')$$

implements the induced map $D^s(H) \rightarrow D^s(G)$ up to the identification $D^s(H) \simeq D^s(H^k)$ and $D^s(G) \simeq D^s(G^{k+K})$; see [32, Section 3.4]. Moreover, $\pi^{s,K}$ commutes with $\gamma_{H^k}^s$ and $\gamma_{G^{k+K}}^s$. Combined with Lemmas B.1 and B.2, we see that $\pi^{s,K}$ also induces maps

$$\text{BF}(\gamma_H^s) \rightarrow \text{BF}(\gamma_G^s), \quad \ker(1 - \gamma_H^s) \rightarrow \ker(1 - \gamma_G^s).$$

These maps are again functorial for graph homomorphisms that induce s -bijective maps.

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References

- [1] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids. With a foreword by Georges Skandalis and Appendix B by E. Germain*. Monogr. Enseign. Math. 36, L'Enseignement Mathématique, Geneva, 2000 Zbl [0960.43003](#) MR [1799683](#)
- [2] J. E. Anderson and I. F. Putnam, [Topological invariants for substitution tilings and their associated \$C^*\$ -algebras](#). *Ergodic Theory Dynam. Systems* **18** (1998), no. 3, 509–537 Zbl [1053.46520](#) MR [1631708](#)
- [3] C. Bönicke and V. Proietti, [Categorical approach to the Baum–Connes conjecture for étale groupoids](#). 2022, arXiv:2202.08067v2
- [4] M. Crainic and I. Moerdijk, [A homology theory for étale groupoids](#). *J. Reine Angew. Math.* **521** (2000), 25–46 Zbl [0954.22002](#) MR [1752294](#)
- [5] J. Cuntz, [A class of \$C^*\$ -algebras and topological Markov chains. II. Reducible chains and the Ext-functor for \$C^*\$ -algebras](#). *Invent. Math.* **63** (1981), no. 1, 25–40 Zbl [0461.46047](#) MR [608527](#)
- [6] J. Cuntz and W. Krieger, [A class of \$C^*\$ -algebras and topological Markov chains](#). *Invent. Math.* **56** (1980), no. 3, 251–268 Zbl [0434.46045](#) MR [561974](#)
- [7] R. J. Deeley, M. Goffeng, B. Mesland, and M. F. Whittaker, [Wieler solenoids, Cuntz–Pimsner algebras and \$K\$ -theory](#). *Ergodic Theory Dynam. Systems* **38** (2018), no. 8, 2942–2988 Zbl [1401.37015](#) MR [3868019](#)
- [8] R. J. Deeley, D. B. Killough, and M. F. Whittaker, [Dynamical correspondences for Smale spaces](#). *New York J. Math.* **22** (2016), 943–988 Zbl [1381.37038](#) MR [3576278](#)
- [9] R. Exel, A. an Huef, and I. Raeburn, [Purely infinite simple \$C^*\$ -algebras associated to integer dilation matrices](#). *Indiana Univ. Math. J.* **60** (2011), no. 3, 1033–1058 Zbl [1254.46067](#) MR [2985865](#)
- [10] N. Higson, [A characterization of \$KK\$ -theory](#). *Pacific J. Math.* **126** (1987), no. 2, 253–276 Zbl [0645.46054](#) MR [869779](#)
- [11] J. Kaminker and I. Putnam, [\$K\$ -theoretic duality of shifts of finite type](#). *Comm. Math. Phys.* **187** (1997), no. 3, 509–522 Zbl [0887.19002](#) MR [1468312](#)

- [12] J. Kaminker, I. F. Putnam, and M. F. Whittaker, K -theoretic duality for hyperbolic dynamical systems. *J. Reine Angew. Math.* **730** (2017), 263–299 Zbl 1379.37016 MR 3692021
- [13] W. Krieger, On dimension functions and topological Markov chains. *Invent. Math.* **56** (1980), no. 3, 239–250 Zbl 0431.54024 MR 561973
- [14] M. Laca, I. Raeburn, J. Ramage, and M. F. Whittaker, Equilibrium states on the Cuntz–Pimsner algebras of self-similar actions. *J. Funct. Anal.* **266** (2014), no. 11, 6619–6661 Zbl 1305.46059 MR 3192463
- [15] P.-Y. Le Gall, Théorie de Kasparov équivariante et groupoïdes. I. *K-Theory* **16** (1999), no. 4, 361–390 Zbl 0932.19004 MR 1686846
- [16] M. Macho Stadler and M. O’uchi, Correspondence of groupoid C^* -algebras. *J. Operator Theory* **42** (1999), no. 1, 103–119 Zbl 0999.46035 MR 1694789
- [17] K. Matsumoto, Topological conjugacy of topological Markov shifts and Ruelle algebras. *J. Operator Theory* **82** (2019), no. 2, 253–284 Zbl 1438.46077 MR 4015953
- [18] H. Matui, Homology and topological full groups of étale groupoids on totally disconnected spaces. *Proc. Lond. Math. Soc. (3)* **104** (2012), no. 1, 27–56 Zbl 1325.19001 MR 2876963
- [19] H. Matui, Étale groupoids arising from products of shifts of finite type. *Adv. Math.* **303** (2016), 502–548 Zbl 1352.19003 MR 3552533
- [20] R. Meyer, Homological algebra in bivariant K -theory and other triangulated categories. II. *Tbil. Math. J.* **1** (2008), 165–210 Zbl 1161.18301 MR 2563811
- [21] R. Meyer and R. Nest, The Baum–Connes conjecture via localisation of categories. *Topology* **45** (2006), no. 2, 209–259 Zbl 1092.19004 MR 2193334
- [22] R. Meyer and R. Nest, Homological algebra in bivariant K -theory and other triangulated categories. I. In *Triangulated categories*, pp. 236–289, London Math. Soc. Lecture Note Ser. 375, Cambridge University Press, Cambridge, 2010 Zbl 1234.18008 MR 2681710
- [23] P. S. Muhly, J. N. Renault, and D. P. Williams, Equivalence and isomorphism for groupoid C^* -algebras. *J. Operator Theory* **17** (1987), no. 1, 3–22 Zbl 0645.46040 MR 873460
- [24] V. Nekrashevych, *Self-similar groups*. Math. Surveys Monogr. 117, American Mathematical Society, Providence, RI, 2005 Zbl 1166.37005 MR 2162164
- [25] V. Nekrashevych, C^* -algebras and self-similar groups. *J. Reine Angew. Math.* **630** (2009), 59–123 Zbl 1175.46048 MR 2526786
- [26] R. Popescu, Equivariant E -theory for groupoids acting on C^* -algebras. *J. Funct. Anal.* **209** (2004), no. 2, 247–292 Zbl 1059.46054 MR 2044224
- [27] V. Proietti, A note on homology for Smale spaces. *Groups Geom. Dyn.* **14** (2020), no. 3, 813–836 Zbl 1485.37027 MR 4167022
- [28] V. Proietti and M. Yamashita, Homology and K -theory of dynamical systems I. Torsion-free ample groupoids. *Ergodic Theory Dynam. Systems* **42** (2022), no. 8, 2630–2660 Zbl 1492.19005 MR 4448401
- [29] I. F. Putnam, C^* -algebras from Smale spaces. *Canad. J. Math.* **48** (1996), no. 1, 175–195 Zbl 0855.46034 MR 1382481
- [30] I. F. Putnam, Functoriality of the C^* -algebras associated with hyperbolic dynamical systems. *J. London Math. Soc. (2)* **62** (2000), no. 3, 873–884 Zbl 1021.46042 MR 1794291
- [31] I. F. Putnam, Lifting factor maps to resolving maps. *Israel J. Math.* **146** (2005), 253–280 Zbl 1089.37011 MR 2151603
- [32] I. F. Putnam, A homology theory for Smale spaces. *Mem. Amer. Math. Soc.* **232** (2014), no. 1094, viii+122 Zbl 1350.37035 MR 3243636
- [33] I. F. Putnam, Lecture notes on smale spaces. 2015, http://www.math.uvic.ca/faculty/putnam/ln/Smale_spaces.pdf

- [34] I. F. Putnam and J. Spielberg, *The structure of C^* -algebras associated with hyperbolic dynamical systems*. *J. Funct. Anal.* **163** (1999), no. 2, 279–299 Zbl [0932.46059](#) MR [1680475](#)
- [35] J. Renault, *A groupoid approach to C^* -algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl [0433.46049](#) MR [584266](#)
- [36] D. Ruelle, *Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics*. 2nd edn., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004 Zbl [1062.82001](#) MR [2129258](#)
- [37] S. Smale, *Differentiable dynamical systems*. *Bull. Amer. Math. Soc.* **73** (1967), 747–817 Zbl [0202.55202](#) MR [228014](#)
- [38] B. Steinberg, *Modules over étale groupoid algebras as sheaves*. *J. Aust. Math. Soc.* **97** (2014), no. 3, 418–429 Zbl [1333.22001](#) MR [3270778](#)
- [39] K. Thomsen, *C^* -algebras of homoclinic and heteroclinic structure in expansive dynamics*. *Mem. Amer. Math. Soc.* **206** (2010), no. 970, x+122 Zbl [1206.46001](#) MR [2667385](#)
- [40] K. Thomsen, *The homoclinic and heteroclinic C^* -algebras of a generalized one-dimensional solenoid*. *Ergodic Theory Dynam. Systems* **30** (2010), no. 1, 263–308 Zbl [1194.37003](#) MR [2586354](#)
- [41] J.-L. Tu, *La conjecture de Baum–Connes pour les feuilletages moyennables*. *K-Theory* **17** (1999), no. 3, 215–264 Zbl [0939.19001](#) MR [1703305](#)
- [42] D. van Dantzig, *Über topologisch homogene kontinua*. *Fundamenta Mathematicae* **15** (1930), no. 1, 102–125 Zbl [56.1130.01](#)
- [43] S. Wieler, *Smale spaces via inverse limits*. *Ergodic Theory Dynam. Systems* **34** (2014), no. 6, 2066–2092 Zbl [1351.37054](#) MR [3272784](#)
- [44] R. F. Williams, *One-dimensional non-wandering sets*. *Topology* **6** (1967), 473–487 Zbl [0159.53702](#) MR [217808](#)
- [45] R. F. Williams, *Expanding attractors*. *Inst. Hautes Études Sci. Publ. Math.* (1974), no. 43, 169–203 Zbl [0279.58013](#) MR [348794](#)
- [46] I. Yi, *K -theory of C^* -algebras from one-dimensional generalized solenoids*. *J. Operator Theory* **50** (2003), no. 2, 283–295 Zbl [1061.46049](#) MR [2050130](#)

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