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Ring-theoretic blowing down II: Birational transformations

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Abstract. One of the major open problems in noncommutative algebraic geometry is the classification of noncommutative projective surfaces (or, slightly more generally, of noetherian connected graded domains of Gelfand–Kirillov dimension 3). Earlier work of the authors classified the connected graded noetherian subalgebras of Sklyanin algebras using a noncommutative analogue of blowing up. In a companion paper the authors also described a noncommutative version of blowing down and, for example, gave a noncommutative analogue of Castelnuovo's classic theorem that lines of self-intersection (-1) on a smooth surface can be contracted.

In this paper we will use these techniques to construct explicit birational transformations between various noncommutative surfaces containing an elliptic curve. Notably we show that Van den Bergh's quadrics can be obtained from the Sklyanin algebra by suitably blowing up and down, and we also provide a noncommutative analogue of the classical Cremona transform.

1. Introduction

Throughout the paper, $\mathbb{k}$ will denote an algebraically closed field, which in the introduction has characteristic zero, and all rings will be $\mathbb{k}$-algebras. A $\mathbb{k}$-algebra R is *connected graded* or *cg* if $R = \bigoplus_{n \geq 0} R_n$ is a finitely generated, $\mathbb{N}$-graded algebra with $R_0 = \mathbb{k}$. For such a ring R, the category of graded noetherian right R-modules will be denoted gr-R, with quotient category qgr-R obtained by quotienting out the Serre subcategory of finite-dimensional modules. We define qgr-R to be *smooth* (which is synonymous with nonsingular in our usage) if it has finite homological dimension. An effective intuition is to regard qgr-R as the category of coherent sheaves on the (nonexistent) space $\operatorname{Proj}(R)$. Thus, for example, we define qgr-R to be *a noncommutative surface* if R has Gelfand–Kirillov dimension GKdim $R = 3$.

One of the main open problems in noncommutative algebraic geometry is the classification of noncommutative projective surfaces (or, slightly more generally, of noetherian connected graded domains of Gelfand–Kirillov dimension 3). This has been solved in

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many particular cases and those solutions have led to some fundamental advances in the subject; see, for example, [2, 7, 13, 18, 21, 22] and the references therein. In [1], Artin conjectured that, birationally at least, there is a short list of such surfaces, with the generic case being a Sklyanin algebra. Here, the graded quotient ring $Q_{\mathrm{gr}}(R)$ of R is obtained by inverting the nonzero homogeneous elements and two such domains R, S are *birational* if $Q_{\mathrm{gr}}(R)_0 \cong Q_{\mathrm{gr}}(S)_0$; that is, if they have the same *function skewfield* (note that $Q_{\mathrm{gr}}(R)_0$ is a division ring). Although no solution of Artin's conjecture is in sight, it does lead naturally to the question of determining the surfaces within each birational class. We consider this problem in this paper.

In classical (commutative) geometry, birational projective surfaces can be obtained from each other by means of blowing up and down (monoidal transformations). The noncommutative notion of blowing up points on a noncommutative surface has been described in [21] with a rather more ring-theoretic approach given in [12] and [14]. Moreover, an analogue of blowing down (contracting) (-1)-curves is defined and explored in the companion paper to this one [15]. The latter paper proves, among other things, that there is a noncommutative analogue of Castelnuovo's classic theorem: curves of self-intersection (-1) can indeed be contracted in a smooth noncommutative surface.

The next step in understanding noncommutative surfaces is then to determine when two noncommutative surfaces are birational, and we take steps in this direction in the present paper. In particular, there are two known classes of generic minimal noncommutative surfaces: Sklyanin algebras themselves, which are thought of as coordinate rings of noncommutative projective planes, and the noncommutative quadrics defined by Van den Bergh [22]. (See [16] for discussion of the way in which these are noncommutative minimal models.) One of the main aims of this paper is to prove that, given that they have the appropriate invariants, these algebras can indeed be transformed into each other by means of blowing up and down; thereby giving a noncommutative version of the classical birational transform

$$\begin{array}{ccc} & \mathrm{Bl}_{p,p'}\,\mathbb{P}^2 \cong \mathrm{Bl}_r(\mathbb{P}^1\times\mathbb{P}^1) & \\ {\scriptstyle \pi_{p,p'}}\swarrow & & \searrow{\scriptstyle \pi_r} \\ \mathbb{P}^2 & \overset{\phi}{\dashleftarrow\dashleftarrow\dashleftarrow} & \mathbb{P}^1\times\mathbb{P}^1. \end{array} \tag{1.1}$$

In this paper, we apply the birational geometry of noncommutative projective surfaces as developed in [12, 15] to construct a noncommutative version of (1.1). Doing this involves developing substantial new methods in noncommutative geometry; in particular, establishing a noncommutative version of the isomorphism in the top line of (1.1) involves a recognition theorem for two-point blowups of a noncommutative $\mathbb{P}^2$ which is of independent interest.

In order to state these results formally we need some definitions. We first remark that as this paper is a continuation of [15] we will keep the same notation as in that paper, and will therefore refer the reader to that paper for more standard notation. In particular, given an automorphism τ of an elliptic curve E, we always assume that $|\tau| = \infty$ and, as in [15, Section 2], write $B = B(E, \mathcal{M}, \tau)$ for the *twisted homogeneous coordinate ring or TCR* of E relative to a line bundle $\mathcal{M}$. A domain R is called an *elliptic algebra* if it is a cg domain, with a central element $g \in R_1$ such that $R/gR \cong B(E, \mathcal{M}, \tau)$ for some choice of $\{E, \mathcal{M}, \tau\}$. We say that $E = E(R)$ and $\tau = \tau(R)$ are respectively the elliptic curve and the automorphism *associated to* R, and deg $\mathcal{M}$ is the *degree* of R. If we loosen our definition to allow τ to be the identity on E, then a commutative elliptic algebra is the same as the anticanonical ring of a (possibly singular) del Pezzo surface, and the degree of the algebra is the same as the commutative definition: the self-intersection of the canonical class.

Two examples of elliptic algebras are important to us. The first is the 3-Veronese ring $T = S^{(3)}$ of the Sklyanin algebra $S = S(E, \sigma)$, as defined in [15, (4.2)]. This is called a *Sklyanin elliptic algebra*, and the third Veronese is needed to ensure that $g \in T_1$. The second, called a *quadric elliptic algebra*, is the Veronese ring $Q^{(2)}$ of a *Van den Bergh quadric* $Q = Q_{\mathrm{VdB}} = Q_{\mathrm{VdB}}(E, \sigma, \Omega)$ for a second parameter $\Omega \in \mathbb{P}^1$. The latter is constructed in [22], and described in more detail in Example 7.1.

In this paper we only consider elliptic algebras R for which qgr-R *is smooth.* This is not a restriction for the Sklyanin elliptic algebra $S^{(3)}$, and also holds for any blowup or contraction that we construct. But, for each $\{E, \sigma\}$, it does exclude a discrete family of quadrics $Q_{\mathrm{VdB}}(E, \sigma, \Omega)$. However, these can easily be dealt with separately; see Remark 1.7 for the details.

The notions of blowing up an elliptic algebra R at a point $p \in E$ or blowing down a line L of self-intersection (-1) can be found in [15]. In brief, if R is an elliptic algebra of degree ≥ 2, one can blow up any point $p \in E$ to obtain a subring $\mathrm{Bl}_p(R)$. (For notational reasons, in the body of the paper we will usually write $R(p)$ in place of $\mathrm{Bl}_p(R)$.) In particular, $\mathrm{Bl}_p(R)/(g) = B(E, \mathcal{M}(-p), \tau)$, and so blowing up decreases the degree of an elliptic algebra by one. This is also what happens in the commutative setting: given a smooth surface X, the canonical divisor of $\mathrm{Bl}_p(X)$ is $\pi_p^* K_X + L_p$, where L_p is the exceptional divisor [6, Proposition V.3.3]. Thus the anticanonical ring of the blowup is a subring of the anticanonical ring of X. In the noncommutative setting, it follows from the construction that R and $\mathrm{Bl}_p(R)$ are also birational; indeed we even have $Q_{\mathrm{gr}}(R) = Q_{\mathrm{gr}}(\mathrm{Bl}_p(R))$.

There is a notion of the *exceptional line module* L_p of the noncommutative blowup $\mathrm{Bl}_p(R)$. Here, a *line module* is a cyclic R-module with the Hilbert series $(1-s)^{-2}$ of $\Bbbk[x, y]$. Moreover, as in the commutative setting, L_p has self-intersection (-1), where we are now using the noncommutative intersection notion of intersection theory due to Mori and Smith [9] and defined by

$$(L \bullet_{\mathrm{MS}} L') = \sum_{n \geq 0} (-1)^{n+1} \dim_{\Bbbk} \operatorname{Ext}^i_{\mathrm{qgr}\text{-}R}(L, L').$$

In [15, Theorem 1.4], we gave a noncommutative version of Castelnuovo's contraction criterion: Suppose that R is an elliptic algebra with qgr-R smooth, such that R has a line module L with $(L \bullet_{\mathrm{MS}} L) = -1$. Then we can contract L in the sense that there exists a second elliptic algebra R', called the *blowdown* or *contraction of R at L*, such that qgr-R'

is smooth, $R = \mathrm{Bl}_q(R')$ is the blowup of R' at some point $q \in E$, and L is the exceptional line module of this blowup.

The main result of this paper is the following noncommutative analogue of the classical diagram (1.1).

Theorem 1.2 (See Theorem 7.3). *Let $Q = Q_{\mathrm{VdB}}(E, \alpha, \Omega)$ be a Van den Bergh quadric such that* qgr-Q *is smooth and pick $r \in E$. Then there exist $\sigma \in \mathrm{Aut}(E)$ with $\sigma^3 = \alpha^2$ and $p, q \in E$ so that the blowups $\mathrm{Bl}_r(Q^{(2)})$ and $\mathrm{Bl}_{p,q}(S(E,\sigma)^{(3)})$ are isomorphic.*

Identify the rings $\mathrm{Bl}_r(Q^{(2)})$ and $\mathrm{Bl}_{p,q}(S(E,\sigma)^{(3)})$ in Theorem 1.2. As blowups are inclusions in our setting, the effect of that result is to show that there are inclusions

$$Q^{(2)} \supset \mathrm{Bl}_r(Q^{(2)}) = \mathrm{Bl}_{p,q}(S^{(3)}) \subset S^{(3)},$$

for $S = S(E,\sigma)$. Since we have seen that R and $\mathrm{Bl}_x(R)$ are always birational, we obtain the following corollary.

Corollary 1.3. *Any Van den Bergh quadric Q with* qgr-Q *smooth is birational to a Sklyanin algebra.* ■

We should emphasise that Theorem 1.2 is not the first proof of the birationality of $S^{(3)}$ and $Q^{(2)}$. Indeed, this was announced by Van den Bergh in [19, Theorem 13.4.1] with complete proofs given in [10]. The proof of Presotto and Van den Bergh is more geometric than the algebraic one given here. Our argument has the advantage of providing a more explicit identity, which should also be useful elsewhere.

In order to prove Theorem 1.2, we develop new techniques in noncommutative birational geometry which are of independent interest. Let Q, r be as in the statement of Theorem 1.2. The key problem in proving the theorem is to *recognise* $\mathrm{Bl}_r(Q^{(2)})$ as the two-point blowup of a Sklyanin algebra. We do this through noncommutative intersection theory.

In the commutative setting, the surface $\mathrm{Bl}_r(\mathbb{P}^1 \times \mathbb{P}^1)$ contains three lines of self-intersection (-1). These are the strict transforms L, L' of the "horizontal" and "vertical" lines through r on $\mathbb{P}^1 \times \mathbb{P}^1$, and the exceptional divisor L_r of the birational morphism π_r from (1.1). Their intersection theory is summarised in Figure 1: $L \cdot L' = 0$ while $L_r \cdot L = L_r \cdot L' = 1$. To realise $\mathrm{Bl}_r(\mathbb{P}^1 \times \mathbb{P}^1)$ as a two-point blowup of $\mathbb{P}^2$ as in (1.1), one observes that $\phi \circ \pi_r$ is defined everywhere on $\mathrm{Bl}_r(\mathbb{P}^1 \times \mathbb{P}^1)$ and contracts L and L' to distinct points $p, p' \in \mathbb{P}^2$. Further, $\phi \circ \pi_r = \pi_{p,p'}$ realises L_r as the strict transform of the line through p and p'.

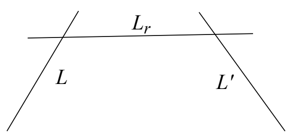


Figure 1. The intersection theory of $\mathrm{Bl}_r(\mathbb{P}^1 \times \mathbb{P}^1)$.

Proving Theorem 1.2 requires different techniques, since in the noncommutative context there is no good analogue of an "open set" on which there is a well-defined "morphism". One way to think about the problem is that we do not have an a priori inclusion of $S^{(3)}$ and $Q^{(2)}$ in the same graded quotient ring. Instead, we prove a recognition theorem for two-point blowups of Sklyanin elliptic algebras, which roughly says that the corresponding elliptic algebra is characterised by the intersection theory described by Figure 1. More specifically, we prove the following.

Theorem 1.4 (Theorem 6.1). *Let R be a degree 7 elliptic algebra such that* qgr-R *is smooth and set $E = E(R)$ and $\tau = \tau(R)$. Suppose that R has line modules with the following properties.*

(1) *There are right line modules L, L' and L_r satisfying the intersection theory,*

  (i) $(L \bullet_{\mathrm{MS}} L) = (L_r \bullet_{\mathrm{MS}} L_r) = (L' \bullet_{\mathrm{MS}} L') = -1$*;*

  (ii) $(L \bullet_{\mathrm{MS}} L') = 0$ *while* $(L_r \bullet_{\mathrm{MS}} L) = (L_r \bullet_{\mathrm{MS}} L') = 1$.

(2) *The points where E intersects L and L' are distinct.*

Then $R \cong \mathrm{Bl}_{p,q}(T)$, where $p \neq q \in E$, and $T = S^{(3)}$ for a Sklyanin algebra $S = S(E, \sigma)$, where $\sigma^3 = \tau$.

To prove Theorem 1.4, we first show that we can iteratively blow down the two lines L and L'. In order to show that this gives a noncommutative $\mathbb{P}^2$, we show how to construct noncommutative analogues of the twisting sheaves $\mathcal{O}(1)$ and $\mathcal{O}(-1)$. These determine a so-called *principal $\mathbb{Z}$-algebra*, from which we can recover the Sklyanin algebra $S(E, \sigma)$. Once we have proved Theorem 1.4, it is relatively straightforward to show that $\mathrm{Bl}_r(Q^{(2)})$ has the necessary intersection theory and hence to prove Theorem 1.2.

A converse to Theorem 1.4 is provided by the following proposition.

Proposition 1.5 (Proposition 8.4). *Let T be a Sklyanin elliptic algebra with associated elliptic curve E, and let $p \neq q \in E$. Then $R = \mathrm{Bl}_{p,q} T$ satisfies the hypotheses of Theorem 1.4.*

Theorem 1.4 is a useful new technique for understanding noncommutative surfaces. To demonstrate its utility, we use it to construct a noncommutative version of the Cremona transform

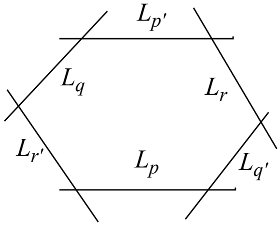
$$\mathrm{Cr} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x : y : z] \mapsto [yz : xz : xy],$$

which factorises as

$$\begin{array}{ccccc} & & \mathrm{Bl}_{p,q,r}\,\mathbb{P}^2 & & \\ & \swarrow & & \searrow & \\ \mathbb{P}^2 & & \overset{\mathrm{Cr}}{\dashrightarrow} & & \mathbb{P}^2 \end{array}$$

where $\{p, q, r\} = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$.

We prove our version by showing that the blowup of $T = S^{(3)}$ at three general points p, q, r has a hexagon



of (-1) lines, just as happens for the 3-point blowup of $\mathbb{P}^2$ in classical geometry. Using intersection theory and the noncommutative Castelnuovo criterion [15, Theorem 1.4] we show that each of the (-1) lines $L_{p'}$, $L_{q'}$ and $L_{r'}$ that are not exceptional for the original blowup can be contracted to give a ring to which Theorem 1.4 applies. This leads to the following theorem.

Theorem 1.6 (Theorem 9.1 and Remark 9.2). *Let $T = S^{(3)}$ be a Sklyanin elliptic algebra with associated curve E. Pick distinct points $p, q, r \in E$ satisfying minor conditions and let $R = \mathrm{Bl}_{p,q,r}\, T$ be the corresponding blowup. Then* qgr-R *is smooth, and there is a second elliptic algebra $T' \cong T$ obtained by blowing down the three lines that are not exceptional for the original blowup. Thus $R = \mathrm{Bl}_{p_1,q_1,r_1}\, T'$ for suitable points $p_1, q_1, r_1 \in E$.*

Once again, the first noncommutative analogue of the Cremona transform was announced in [19] and proved in detail in [11].

Remark 1.7. We end the introduction by commenting on the smoothness hypothesis. As noted in [18] there do exist quadrics $Q = Q_{\mathrm{VdB}}$ for which qgr-Q is not smooth. One can presumably extend the results of this paper to cover these examples, although this will require a more awkward notion of self-intersection, see [15, Section 6] and [15, Definition 6.11] in particular. However, for most questions, in particular for those concerned with birationality, this is unnecessary. Indeed if Q is a quadric for which qgr-Q is not smooth, then Q has a Morita context to a second quadric $\widetilde{Q}$ such that qgr-$\widetilde{Q}$ is smooth. More precisely, there exists a $(Q^{(2)}, \widetilde{Q}^{(2)})$-bimodule M, finitely generated and of Goldie rank one on each side, such that $Q^{(2)} = \mathrm{End}_{\widetilde{Q}^{(2)}}(M)$ and $\widetilde{Q}^{(2)} = \mathrm{End}_{Q^{(2)}}(M)$. This is proved by combining [16, Lemma 6.6 and Corollary 6.1] following results from [18, 20]. Clearly Q and $\widetilde{Q}$ are birational and so one can use $\widetilde{Q}$ to test birationality questions for Q. Thus, for example, Corollary 1.3 immediately extends to the non-smooth case.

Corollary 1.8. *Any Van den Bergh quadric Q is birational to a Sklyanin algebra.* ∎

2. The noncommutative geometry of elliptic algebras

In this section, we consider properties of the category qgr-R, where R is an elliptic algebra. We start however with some basic results and notation. As mentioned in the introduction, this paper is a continuation of [15], and so we refer the reader to that paper for basic notation. We fix an algebraically closed ground field $\Bbbk$. All algebras and schemes will be defined over $\Bbbk$, and all maps will be $\Bbbk$-linear unless otherwise specified. We begin with a few comments about the elliptic algebras defined in the introduction. We emphasise that *throughout the paper we only consider elliptic algebras R for which* qgr-R *is smooth.*

Notation 2.1. Let $R = \bigoplus_{n \geq 0} R_n$ be such an algebra, with central element $0 \neq g \in R_1$ such that $R/gR = B = B(E, \mathcal{M}, \tau)$ is a TCR over the elliptic curve E. Except when stated to the contrary, we always assume that $|\tau| = \infty$ and that $\deg R \geq 3$. If M is a graded R-module, the *g-torsion* submodule of M consists of elements annihilated by some power of g, and *g-torsionfree* has the obvious meaning. Finally, we note that, by [14, Proposition 2.4], an elliptic algebra R is always Auslander–Gorenstein and Cohen–Macaulay (CM) in the sense of [15, Definition 2.1].

Fix an elliptic algebra R as above. Since $\operatorname{GKdim}(R) = 3$, recall that a graded module M is *Cohen–Macaulay (CM)* if $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \neq 3 - \operatorname{GKdim}(M)$, while M is *maximal Cohen–Macaulay (MCM)* if in addition $\operatorname{GKdim}(M) = 3$.

For an elliptic algebra R, let Gr-R be the category of all graded right R-modules, with quotient category Qgr-R obtained by quotienting out direct limits of finite-dimensional modules. Let π : Gr-$R \to$ Qgr-R be the natural map, although we will typically abuse notation and simply denote πM by M for any $M \in$ Gr-R. By the adjoint functor theorem, π has a right adjoint ω : Qgr-$R \to$ Gr-R. A graded R-module M is called *saturated* if it is in the image of the section functor ω. By [4, (2.2.3)], this is equivalent to $\operatorname{Ext}^1_R(\Bbbk, M) = 0$. The subcategory of finitely generated (hence noetherian) modules in Gr-R is denoted gr-R; similarly, qgr-$R = \pi($gr-$R)$ is the subcategory of noetherian objects in Qgr-R. Below we will write

$$\mathcal{X} = \text{qgr-}R.$$

Note that qgr-$B \simeq \operatorname{coh} E$ by [3].

Certain standard localisations of R will be important. If $M \in$ gr-R, let

$$M^\circ = (M \otimes_R R[g^{-1}])_0.$$

By [15, Lemma 6.8], the hypothesis that qgr-R is smooth holds if and only if R° has finite global dimension. The image of an element or set under the canonical surjection $R \to B$ is written $x \mapsto \overline{x}$. Let $R_{(g)}$ be the Ore localisation $R\mathcal{C}^{-1}$, where $\mathcal{C}$ is the set of homogeneous elements of $R \smallsetminus gR$. Then $R_{(g)}/R_{(g)}g \cong Q_{\mathrm{gr}}(B)$. A right R-submodule M of $R_{(g)}$ is *g-divisible* if $M \cap gR_{(g)} = Mg$, and hence $\overline{M} \cong M/Mg$. This is equivalent to $R_{(g)}/M$ being g-torsionfree. If $M, M' \subseteq R_{(g)}$ are g-divisible, then so is $\operatorname{Hom}_R(M, M')$, by [13, Lemma 2.12].

Let $M = \bigoplus_n M_n \in$ Gr-R. We say M is *left (right) bounded* if $M_n = 0$ for $n \ll 0$ (respectively $n \gg 0$). If $M \in$ gr-R then M is left bounded. It is clear from calculating with a resolution by finite rank graded free modules that $\operatorname{Ext}^i_R(M, N)$ is left bounded for all i when $M, N \in$ gr-R. Given $k \in \mathbb{N}$, the shifted module $M[k]$ is defined by $M[k]_n = M_{n+k}$.

Set $\underline{\mathrm{Hom}}_{\mathcal{X}}(M, N) = \bigoplus_n \mathrm{Hom}_{\mathcal{X}}(M, N[n])$ and define $\underline{\mathrm{Ext}}^i_{\mathcal{X}}(M, N)$ similarly. We let $H^i(\mathcal{X}, M) = \mathrm{Ext}^i_{\mathcal{X}}(R, M)$ denote the i-th *(sheaf) cohomology* of M. By [14, Theorem 5.3], R satisfies the Artin–Zhang χ condition which, by [4, Corollary 4.6], implies that $\mathrm{Ext}^i_{\mathcal{X}}(M, N)$ is finite-dimensional for all i.

Suppose that $M \in \text{gr-}R$ is g-torsionfree and $N \in \text{gr-}B$. We often use the facts that

$$\mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_B(M/Mg, N) \quad \text{and} \quad \mathrm{Ext}^i_{\mathcal{X}}(M, N) \cong \mathrm{Ext}^i_{\text{qgr-}B}(M/Mg, N). \tag{2.2}$$

See [15, Lemma 4.7].

With this background in hand, we now prove some needed preliminary results. First, we show that maximal Cohen–Macaulay g-divisible submodules of $R_{(g)}$ have a nice characterization.

Proposition 2.3. *Let $M \subseteq R_{(g)}$ be a nonzero g-divisible finitely generated module over an elliptic algebra R. Then M is MCM if and only if $\overline{M}$ is a saturated B-module.*

Proof. We first claim that M is MCM over R if and only if $\overline{M}$ is MCM over B. Apply $\mathrm{Hom}_R(-, R)$ to the exact sequence $0 \to R \xrightarrow{\bullet g} R \to B \to 0$ and consider the corresponding long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(M, R)[-1] \xrightarrow{\bullet g} \mathrm{Ext}^i_R(M, R) \to \mathrm{Ext}^i_B(\overline{M}, B) \to \mathrm{Ext}^{i+1}_R(M, R)[-1] \to \dots \tag{2.4}$$

where we have used (2.2). Note that $\mathrm{GKdim}(M) = 3$ and $\mathrm{GKdim}(\overline{M}) = 2$. If M is MCM over R then it follows from (2.4) that $\overline{M}$ is MCM over B. Conversely, if $\overline{M}$ is MCM over B, then for all $i \geq 1$ multiplication by g gives a surjection of right R-modules $\mathrm{Ext}^i_R(M, R)[-1] \to \mathrm{Ext}^i_R(M, R)$. Since $\mathrm{Ext}^i_R(M, R)$ is left bounded, this forces $\mathrm{Ext}^i_R(M, R) = 0$. So M is MCM over R, and the claim is proved.

To finish the proof we show that for any finitely generated graded B-submodule $N \subseteq Q_{\mathrm{gr}}(B)$, N is MCM over B if and only if N is saturated. By multiplying by a homogeneous element of B to clear denominators we can assume that $N \subseteq B$. If N is saturated, then B/N is pure of GK dimension 1. The critical B-modules of dimension 1 are the shifted point modules [14, Lemma 2.8] and so B/N has a finite filtration with factors that are shifted point modules. Since a point module is CM by [15, Lemma 3.3], B/N is CM of GK dimension 1. Now, $\mathrm{Ext}^i_B(N, B) \cong \mathrm{Ext}^{i+1}_B(B/N, B) = 0$ for $i \geq 1$, so N is MCM of GK dimension 2. Conversely, if N is not saturated, then there is a graded module N' with $N \subsetneqq N' \subseteq B$, where N'/N is finite-dimensional and $N' = \omega\pi(N)$ is saturated. Since N' is MCM by the argument above, $\mathrm{Ext}^i_B(N', B) = 0$ for $i \geq 1$. Hence $\mathrm{Ext}^1_B(N, B) \cong \mathrm{Ext}^2_B(N'/N, B)$. But using that B is AS Gorenstein of dimension 2, we get $\mathrm{Ext}^2_B(N'/N, B) \neq 0$. Thus $\mathrm{Ext}^1_B(N, B) \neq 0$ and N is not CM. ∎

Note that a MCM R-module M is automatically reflexive: that is,

$$\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) = M.$$

This follows from the Gorenstein spectral sequence [8, Theorem 2.2].

Next, we show that Ext groups in the categories gr-R and $\mathcal{X}$ = qgr-R are equal in certain circumstances.

Proposition 2.5. *Let $M, N \in$ gr-R.*

(1) *If N is saturated and has zero socle, then* $\operatorname{Hom}_R(M, N) = \underline{\operatorname{Hom}}_{\mathcal{X}}(M, N)$.

(2) *If $N \subseteq R_{(g)}$ is g-divisible and MCM, then* $\operatorname{Ext}^1_R(M, N) = \underline{\operatorname{Ext}}^1_{\mathcal{X}}(M, N)$.

Proof. (2) We claim that $\operatorname{Ext}^2_R(\Bbbk, N) = 0$. Consider the exact sequence $0 \to N[-1] \xrightarrow{\bullet g} N \to \overline{N} \to 0$ and the corresponding exact sequence

$$\operatorname{Ext}^1_R(\Bbbk, \overline{N}) \to \operatorname{Ext}^2_R(\Bbbk, N)[-1] \to \operatorname{Ext}^2_R(\Bbbk, N). \tag{2.6}$$

Suppose that $\operatorname{Ext}^1_R(\Bbbk, \overline{N}) \neq 0$. Let $\overline{N} \subseteq P$ represent a nonzero class α in this extension group, where $P/\overline{N} \cong \Bbbk$. Necessarily $\overline{N}$ is essential in P. If P is not a B-module, then $Pg \neq 0$. Since $\overline{N}g = 0$, in this case $Pg \cong \Bbbk$. By essentiality, $Pg \cap \overline{N} \neq 0$. This contradicts the fact that $\overline{N} \subseteq Q_{\mathrm{gr}}(B)$, so that $\overline{N}$ has no socle. Hence $P \in$ gr-B. This shows that $\alpha \in \operatorname{Ext}^1_B(\Bbbk, \overline{N}) = 0$, because $\overline{N}$ is saturated by Proposition 2.3. This contradiction proves that $\operatorname{Ext}^1_R(\Bbbk, \overline{N}) = 0$.

Now (2.6) implies that there is a graded injective vector space map $\operatorname{Ext}^2_R(\Bbbk, N)[-1] \hookrightarrow \operatorname{Ext}^2_R(\Bbbk, N)$. However, $\operatorname{Ext}^2_R(\Bbbk, N)$ is finite-dimensional by the χ condition. This forces $\operatorname{Ext}^2_R(\Bbbk, N) = 0$ as claimed.

Next note that $\operatorname{Ext}^1_R(\Bbbk, N) = 0$, since g-divisibility implies that N is a saturated R-module. Thus $\operatorname{Ext}^i_R(M/M_{\geq n}, N) = 0$ for $i = 1, 2$ and for all $n \in \mathbb{Z}$. Given that $\underline{\operatorname{Ext}}^1_{\mathcal{X}}(M, N) = \lim_{n\to\infty} \operatorname{Ext}^1_R(M_{\geq n}, N)$, the result follows from the standard long exact sequence.

(1) Again $\operatorname{Ext}^1_R(\Bbbk, N) = 0$ since N is saturated, and $\operatorname{Hom}_R(\Bbbk, N) = 0$ since N has no socle. Now an argument analogous to the proof of part (2) gives the result. ■

Notation 2.7. Since line modules play a vital role in noncommutative monoidal transformations, we recall some of their properties from [15, Section 5]. If L is a right (left) line module, then $L^\vee = \operatorname{Ext}^1_R(L, R)[1]$ is a left (right) line module, referred to as the *dual line module*. Moreover, $L \cong L^{\vee\vee}$. There is a unique right (left) ideal J of R such that $L \cong R/J$, and, since $\operatorname{Ext}^1_R(L, R)_1 = \Bbbk$, a unique module $M \subseteq Q_{\mathrm{gr}}(R)$ such that $R \subseteq M$ with $M/R \cong L[-1]$. We refer to J as the *line ideal* of L and M as the *line extension* of L. Note that line ideals and line extensions are MCM, and thus reflexive by [15, Lemma 5.6 (2)]. Further, the line ideal of $L^\vee$ is the reflexive dual of the line extension of L [15, Lemma 5.6 (3)].

For $M, N \in \mathcal{X}$, following Mori and Smith, one defines the *intersection number*

$$(M \bullet_{\mathrm{MS}} N) = \sum_{n\geq 0} (-1)^{n+1} \dim_{\Bbbk} \operatorname{Ext}^i_{\text{qgr-}R}(M, N)$$

(see [9, Definition 8.4] or [15, Definition 6.1]). The following result follows easily from this definition.

Lemma 2.8. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence in* $\mathcal{X}$ *and let* $N \in \mathcal{X}$. *Then*

$$(M \bullet_{\mathrm{MS}} N) = (M' \bullet_{\mathrm{MS}} N) + (M'' \bullet_{\mathrm{MS}} N)$$

and

$$(N \bullet_{\mathrm{MS}} M) = (N \bullet_{\mathrm{MS}} M') + (N \bullet_{\mathrm{MS}} M''). \qquad \blacksquare$$

The intersections between lines are especially important. By [15, Lemma 6.8 and Corollary 6.6] and the fact that $\mathcal{X} = \text{qgr-}R$ is smooth, we get the following identities. If L, L' are R-line modules, then

$$\begin{aligned}(L \bullet_{\mathrm{MS}} L') &= \operatorname{grk} \operatorname{Ext}^1_R(L, L') - \operatorname{grk} \operatorname{Hom}_R(L, L') \\ &= \operatorname{grk} \underline{\operatorname{Ext}}^1_{\mathcal{X}}(L, L') - \operatorname{grk} \underline{\operatorname{Hom}}_{\mathcal{X}}(L, L'),\end{aligned} \tag{2.9}$$

where $\operatorname{grk} V$ denotes the torsionfree rank of V as a $\Bbbk[g]$-module.

Next, we give several versions of Serre duality for $\mathcal{X}$. In the next two results, for a vector space V, the notation V^* means the usual vector space dual $\operatorname{Hom}_{\Bbbk}(V, \Bbbk)$.

Lemma 2.10. *The space* $\mathcal{X}$ *has cohomological dimension* $\operatorname{cd} \mathcal{X} = 2$ *in the sense that* $\operatorname{Ext}^i_{\mathcal{X}}(R, \_\_) = 0$ *for* $i \geq 3$. *Further,* $R[-1]$ *is the dualizing sheaf for* $\mathcal{X}$ *in the sense of* [23, (4-4)]. *Finally,*

$$\operatorname{Ext}^i_{\mathcal{X}}(\_\_, R[-1]) \cong \operatorname{Ext}^{2-i}_{\mathcal{X}}(R, \_\_)^*, \qquad \textit{for all } i. \tag{2.11}$$

Proof. By [15, Proposition 4.3], R is AS Gorenstein, in the sense of [15, Definition 2.1]. Thus there exists $e \in \mathbb{Z}$ so that $\operatorname{Ext}^i_R(\Bbbk, R) = \delta_{i,3}\Bbbk[e]$ in gr-R. Then by [23, Corollary 4.3], $R[-e]$ is the dualizing sheaf for $\mathcal{X}$, and $\operatorname{Ext}^i_{\mathcal{X}}(\_\_, R[-e]) \cong \operatorname{Ext}^{2-i}_{\mathcal{X}}(R, \_\_)^*$. Using (2.2) we have a long exact sequence

$$\cdots \to \operatorname{Ext}^i_{\mathcal{X}}(R, R[-1]) \to \operatorname{Ext}^i_{\mathcal{X}}(R, R) \to \operatorname{Ext}^i_{\text{qgr-}B}(B, B) \to \cdots .$$

Together with the fact that B is the dualizing sheaf for qgr-B, (equivalently, $\mathcal{O}_E$ is the dualizing sheaf for coh E) this implies that $e = 1$. $\blacksquare$

For g-torsionfree modules we have the following stronger version of this result.

Proposition 2.12. *Assume that* $N \in \operatorname{gr} R$ *is* g*-torsionfree. Then* $\operatorname{Ext}^i_{\mathcal{X}}(N, \_\_) = 0$ *for* $i \geq 3$. *For* $0 \leq i \leq 2$, $\operatorname{Ext}^i_{\mathcal{X}}(N, \_\_)^*$ *and* $\operatorname{Ext}^{2-i}_{\mathcal{X}}(\_\_, N[-1])$ *are naturally isomorphic functors from* $\mathcal{X} \to \text{Mod-}\Bbbk$.

Proof. We first show that if $G \in \mathcal{X}$ is g-torsion, then $\operatorname{Ext}^{\geq 2}_{\mathcal{X}}(N, G) = 0$. Since G is noetherian there is some n such that $Gg^n = 0$ and, by induction, it suffices to prove that $\operatorname{Ext}^{\geq 2}_{\mathcal{X}}(N, G) = 0$ holds when $Gg = 0$. Let $i \geq 2$ and let $B = R/Rg$. By (2.2) we have $\operatorname{Ext}^i_{\mathcal{X}}(N, G) \cong \operatorname{Ext}^i_{\text{qgr-}B}(N/Ng, G)$, which is zero since qgr-$B \simeq \operatorname{coh}(E)$ for the elliptic curve E.

Now let $F \in \mathcal{X}$ be noetherian and g-torsionfree. Let $i \geq 3$. As in the proof of [15, Lemma 6.8], $\operatorname{Ext}^i_{\mathcal{X}}(N, F)$ is g-torsionfree and hence by [15, Lemma 6.2] the natural map

$$\operatorname{Ext}^i_{\mathcal{X}}(N, F) \to \operatorname{Ext}^i_{\mathcal{X}}(N, F) \otimes_{\Bbbk[g]} \Bbbk[g, g^{-1}] \cong \operatorname{Ext}^i_{R^\circ}(N^\circ, F^\circ) \otimes_{\Bbbk} \Bbbk[g, g^{-1}]$$

is injective. As R° is CM, the proof of [15, Lemma 6.8] implies that R° has gldim $R^\circ \leq 2$. Thus $\operatorname{Ext}^i_{\mathcal{X}}(N, F) = 0$.

Let $M \in \mathcal{X}$ be arbitrary; then there is an exact sequence $0 \to G \to M \to F \to 0$ in $\mathcal{X}$, where G is g-torsion and F is g-torsionfree. From the previous two paragraphs, we see that $\operatorname{Ext}^{\geq 3}_{\mathcal{X}}(N, M) = 0$.

By (2.11), we have $\operatorname{Ext}^2_{\mathcal{X}}(N, R[-n-1])^* \cong \operatorname{Hom}_{\mathcal{X}}(R, N[n])$. Thus, if $N \neq 0$ then $\operatorname{Ext}^2_{\mathcal{X}}(N, \_\_) \neq 0$. Let $C = \bigoplus_{n \geq 0} \operatorname{Ext}^2_{\mathcal{X}}(N, R[-n])^*$, which is a right R-module. Using (2.11), we compute that

$$C \cong \bigoplus_{n \geq 0} \operatorname{Hom}_{\mathcal{X}}(R, N[n-1]) = \omega\pi N[-1]_{\geq 0}.$$

By Proposition 2.5, $\underline{\operatorname{Hom}}_{\mathcal{X}}(N, R)$ and $\underline{\operatorname{Ext}}^1_{\mathcal{X}}(N, R)$ are left bounded. Therefore, by [23, Theorem 2.2],

$$\operatorname{Ext}^i_{\mathcal{X}}(\_\_, C) \cong \operatorname{Ext}^i_{\mathcal{X}}(\_\_, N[-1])$$

is naturally isomorphic to $\operatorname{Ext}^{2-i}_{\mathcal{X}}(N, \_\_)^*$. ■

We remark that, as $\mathcal{X}$ is smooth, it is surely the case that hd $\mathcal{X} = 2$ and so Proposition 2.12 will hold for all modules in $\mathcal{X}$; however, we have not been able to prove this.

Given a $\mathbb{Z}$-graded vector space $M = \bigoplus_n M_n$, write M^* for the *graded dual* $M^* = \bigoplus_n \operatorname{Hom}_{\Bbbk}(M_{-n}, \Bbbk)$. The previous result has the following immediate extension to graded Ext groups.

Corollary 2.13. *Let $N \in \text{gr-}R$ be g-torsionfree. For all $M \in \text{gr-}R$ and $0 \leq i \leq 2$,*

$$\underline{\operatorname{Ext}}^i_{\mathcal{X}}(N, M)^* \cong \underline{\operatorname{Ext}}^{2-i}_{\mathcal{X}}(M, N)[-1]$$

as graded vector spaces. ■

We also have the following corollary for intersection numbers of lines.

Corollary 2.14. *If L, L' are R-line modules, then*

(1) $(L \bullet_{\mathrm{MS}} L'[k]) = (L \bullet_{\mathrm{MS}} L')$ *for all $k \in \mathbb{Z}$.*

(2) *Moreover,* $(L \bullet_{\mathrm{MS}} L') = (L' \bullet_{\mathrm{MS}} L) = (L^\vee \bullet_{\mathrm{MS}} (L')^\vee) = ((L')^\vee \bullet_{\mathrm{MS}} L^\vee)$.

Proof. (1) This is [15, Proposition 6.4 (1)].

(2) The first and third equalities follow from Proposition 2.12, together with part (1). The second equality follows from [15, Lemma 5.6 (4)] and (2.9). ■

Finally, we compute the cohomology of a line module.

Lemma 2.15. *Let L_R be a line module. For all $n \in \mathbb{Z}$, we have*

$$\dim H^0(\mathcal{X}, L[n]) = \max(n+1, 0), \qquad \dim H^1(\mathcal{X}, L[n]) = \max(-n-1, 0),$$

and

$$H^2(\mathcal{X}, L[n]) = 0.$$

Thus $(R \bullet_{\mathrm{MS}} L[n]) = -n-1$ *and* $(L[n] \bullet_{\mathrm{MS}} R) = -n$.

Proof. First, $\underline{\mathrm{Hom}}_{\mathcal{X}}(L, R) = \mathrm{Hom}_R(L, R) = 0$ by Proposition 2.5. As L is Goldie torsion, $H^2(\mathcal{X}, L[n]) = 0$ follows from (2.11). Thus, by again using Proposition 2.5 and [15, Lemma 5.6(1)],

$$\dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{X}}(L[n], R) = \dim_{\Bbbk} \mathrm{Ext}^1_{\mathrm{gr}\text{-}R}(L[n], R) = \dim_{\Bbbk} \mathrm{Ext}^1_R(L, R)_{-n}$$
$$= \begin{cases} -n & n \le 0, \\ 0 & \text{else.} \end{cases}$$

By (2.11), we have $\dim \mathrm{Ext}^2_{\mathcal{X}}(L[n], R) = \dim \mathrm{Hom}_{\mathcal{X}}(R, L[n-1])$. Note that a line module L is saturated and g-torsionfree [15, Lemmata 5.2 and 5.6 (5)]. In particular, L has zero socle. By Proposition 2.5,

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{X}}(R, L[n-1]) = \dim_{\Bbbk} \mathrm{Hom}_{\mathrm{gr}\text{-}R}(R, L[n-1]) = \begin{cases} n & n \ge 0 \\ 0 & \text{else.} \end{cases}$$

Now using (2.11) and the definition of the intersection product, all of the claims follow. ■

3. Twisting sheaves for elliptic algebras of degree nine

In this section and the next, we study elliptic algebras R of degree 9 and develop a theory which will allow us to recognise when such an algebra is a Sklyanin elliptic algebra $T = S^{(3)}$ for $S = S(E, \sigma)$.

If T is such a Sklyanin elliptic algebra, then qgr-$T \simeq$ qgr-S, and as qgr-S is a noncommutative projective plane, it has objects $\{\mathcal{O}(k) \mid k \in \mathbb{Z}\}$ which play the role of the Serre twisting sheaves on $\mathbb{P}^2$. The objects $\mathcal{O}(k)$ correspond to the right T-modules $\bigoplus_{n \in \mathbb{Z}} S_{3n+k}$; in particular, we have the T-modules $H = \bigoplus_{n \ge 1} S_{3n-1}$ and $M = \bigoplus_{n \ge 0} S_{3n+1}$. Ignoring degree shifts, it is easy to check that $\mathrm{End}_T(H) = \mathrm{End}_T(M) = T$, while $H^* = TS_1$, and $M^* = TS_2$. Here, we have returned to the standard notation $H^* = \mathrm{Hom}_T(TS_1, T)$, etc. Similarly, $\mathrm{Hom}_T(H, M) = MH^* = S_1 T S_1 = H$ and $\mathrm{Hom}_T(M, H) = M$. Consider $H \oplus T \oplus M$ as a column vector and $\mathrm{End}_T(H \oplus T \oplus M)$ as a subalgebra of $M_{3\times 3}(S)$. Taking Hilbert series of the above, and with the correct degree shifts, it follows routinely that

$$\mathsf{hilb}\,\mathrm{End}_T(H \oplus T \oplus M) = \begin{pmatrix} \frac{1+7s+s^2}{(1-s)^3} & \frac{6s+3s^2}{(1-s)^3} & \frac{3s+6s^2}{(1-s)^3} \\ \frac{3+6s}{(1-s)^3} & \frac{1+7s+s^2}{(1-s)^3} & \frac{6s+3s^2}{(1-s)^3} \\ \frac{6+3s}{(1-s)^3} & \frac{3+6s}{(1-s)^3} & \frac{1+7s+s^2}{(1-s)^3} \end{pmatrix}. \tag{3.1}$$

The main result of this section, Proposition 3.3, gives sufficient conditions for a degree 9 elliptic algebra T to have right modules H, M for which (3.1) holds, and hence such that their images in qgr-T play the role of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$.

Before stating that result, we need a few observations from [3]. Given an elliptic algebra T, set $B = T/gT = B(E, \mathcal{N}, \tau)$; thus $B = \bigoplus B_n$, where $B_n = H^0(E, \mathcal{N}_n)$ and $\mathcal{N}_n = \mathcal{N} \otimes \cdots \otimes \mathcal{N}^{\tau^{n-1}}$, under the notation $\mathcal{F}^\tau = \tau^*\mathcal{F}$ for a sheaf $\mathcal{F} \in \mathrm{coh}(E)$. The isomorphism classes of B-point modules are in one-to-one correspondence with the closed points of E; explicitly, if $p \in E$ has skyscraper sheaf $\mathcal{O}_p \cong \mathcal{O}_E/\mathcal{I}_p$, then $p \in E$ corresponds to the point module $M_p = \bigoplus_{n \geq 0} H^0(E, \mathcal{O}_p \otimes \mathcal{N}_n) \cong B/I_p$ for the *point ideal* $I_p = \bigoplus_{n\geq 0} H^0(E, \mathcal{I}_p \otimes \mathcal{N}_n) \subseteq B$. The images $\pi(M_p)$ of the point modules M_p are the simple objects in Qgr-B. We will also frequently use the fact that $M_p[n]_{\geq n} \cong M_{\tau^n p}$ for any n (see, for example, [15, (3.1)]). Of course simple objects in B-Qgr are also parameterised by closed points of E, and we denote these left point modules by M_p^ℓ, where $M_p^\ell := \bigoplus_{n\geq 0} H^0((\mathcal{O}_p)^{\tau^{n-1}} \otimes \mathcal{N}_n)$.

If M is a g-torsionfree right T-module such that $\pi(M/Mg)$ has finite length with composition factors $\pi(M_{p_1}), \dots, \pi(M_{p_n})$, we say that the *divisor of* M is $\mathrm{Div}\, M = p_1 + \cdots + p_n$. In particular, if L is a line module then $\mathrm{Div}\, L$ is a single point. Finally, graded vector spaces V, V' are called *numerically equivalent*, written $V \equiv V'$, if $\mathsf{hilb}\, V = \mathsf{hilb}\, V'$. We often write $V \equiv \mathsf{hilb}\, V$.

Notation 3.2. Let T be an elliptic algebra of degree 9, with $B = T/gT = B(E, \mathcal{N}, \tau)$. We will be interested in the following three conditions on T.

(i) T has a g-divisible right ideal H with

$$\overline{H} = \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n(-a-b-c))$$

for some points $a, b, c \in E$, such that $\mathsf{hilb}\, \mathrm{End}_T(H) = \mathsf{hilb}\, T$.

(ii) T has a g-divisible right module M with $T \subseteq M \subseteq T_{(g)}$ such that

$$\overline{M} = \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n(d+e+f))$$

where $d, e, f \in E$, such that $\mathsf{hilb}\, \mathrm{End}_T(M) = \mathsf{hilb}\, T$.

(ii)′ T has a g-divisible left ideal $H^\vee$ with

$$\overline{H^\vee} = \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n(-\tau^{-n}d - \tau^{-n}e - \tau^{-n}f))$$

where $d, e, f \in E$, and such that $\mathsf{hilb}\, \mathrm{End}_T(H^\vee) = \mathsf{hilb}\, T$.

Proposition 3.3. *Let T be an elliptic algebra of degree 9, set $B = T/gT = B(E, \mathcal{N}, \tau)$ and consider the conditions (i), (ii) and (ii)′ of Notation 3.2.*

(1) *Assume that (i), (ii) hold and set $N^{(1)} = H$, $N^{(2)} = T$, $N^{(3)} = M$. Then*

$$\overline{\mathrm{Hom}_T(N^{(i)}, N^{(j)})} = \mathrm{Hom}_B(\overline{N^{(i)}}, \overline{N^{(j)}}) \qquad \text{for all } i, j \in \{1,2,3\}. \tag{3.4}$$

Moreover, (3.1) holds; that is,

$$\mathsf{hilb}\left(\mathrm{Hom}_T(N^{(j)}, N^{(i)})\right)_{ij} = \begin{pmatrix} \frac{1+7s+s^2}{(1-s)^3} & \frac{6s+3s^2}{(1-s)^3} & \frac{3s+6s^2}{(1-s)^3} \\ \frac{3+6s}{(1-s)^3} & \frac{1+7s+s^2}{(1-s)^3} & \frac{6s+3s^2}{(1-s)^3} \\ \frac{6+3s}{(1-s)^3} & \frac{3+6s}{(1-s)^3} & \frac{1+7s+s^2}{(1-s)^3} \end{pmatrix}. \tag{3.5}$$

(2) *Conditions (ii) and (ii)′ are equivalent. Indeed, M satisfies (ii) if and only if $H^\vee = \mathrm{Hom}_T(M, T)$ satisfies (ii)′. Similarly, $H^\vee$ satisfies (ii)′ if and only if $M = \mathrm{Hom}_T(H^\vee, T)$ satisfies (ii).*

Proof. (1) The $N^{(i)}$ are g-divisible by hypothesis, and hence, by [13, Lemma 2.12], so is each $\mathrm{Hom}_T(N^{(i)}, N^{(j)})$. By hypothesis, we may write $\overline{N^{(i)}} = \bigoplus_{n\geq 0} H^0(E, \mathcal{G}_i \otimes \mathcal{N}_n)t^n$ where $\mathcal{G}_1 = \mathcal{O}_E(-a-b-c)$, $\mathcal{G}_2 = \mathcal{O}_E$, and $\mathcal{G}_3 = \mathcal{O}_E(d+e+f)$. Then [15, Lemma 2.3] implies that

$$\mathrm{Hom}_B(\overline{N^{(j)}}, \overline{N^{(i)}}) = \bigoplus_{n\geq 0} H^0(E, (\mathcal{G}_j^{\tau^n})^{-1}\mathcal{G}_i \otimes \mathcal{N}_n)t^n \subseteq k(E)[t, t^{-1}; \tau] \tag{3.6}$$

for all i, j. Further, a straightforward calculation, using [15, Lemma 2.2], shows that $\mathsf{hilb}(\mathrm{Hom}_B(\overline{N^{(j)}}, \overline{N^{(i)}}))_{ij}$ is equal to $(1-s)$ times the matrix on the right hand side of (3.5). Since $\mathrm{Hom}_T(N^{(j)}, N^{(i)})$ is g-divisible, this shows that equality in the (i, j)-entry of (3.5) is equivalent to having $\overline{\mathrm{Hom}_T(N^{(j)}, N^{(i)})} = \mathrm{Hom}_B(\overline{N^{(j)}}, \overline{N^{(i)}})$. Thus we just need to prove (3.4).

Using (2.2), there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_T(N^{(j)}, N^{(i)})[-1] &\xrightarrow{\bullet g} \mathrm{Hom}_T(N^{(j)}, N^{(i)}) \\ &\to \mathrm{Hom}_B(\overline{N^{(j)}}, \overline{N^{(i)}}) \xrightarrow{\alpha} \mathrm{Ext}^1_T(N^{(j)}, N^{(i)})[-1] \end{aligned} \tag{3.7}$$

and so for each i, j it is sufficient to prove that $\mathrm{Ext}^1_T(N^{(j)}, N^{(i)}) = 0$. Now each $N^{(i)}$ is g-divisible, and it is easy to see that $\overline{N^{(i)}}$ is saturated by hypotheses (i), (ii). So each $N^{(i)}$ is MCM by Proposition 2.3. Therefore, by Proposition 2.5 and Corollary 2.13, we have

$$\mathrm{Ext}^1_T(N^{(j)}, N^{(i)}) = \underline{\mathrm{Ext}}^1_{\mathcal{X}}(N^{(j)}, N^{(i)}) = \underline{\mathrm{Ext}}^1_{\mathcal{X}}(N^{(i)}, N^{(j)})^*[1] = \mathrm{Ext}^1_T(N^{(i)}, N^{(j)})^*[1].$$

Thus for each (i, j) we have $\mathrm{Ext}^1_T(N^{(j)}, N^{(i)}) = 0$ if and only if $\mathrm{Ext}^1_T(N^{(i)}, N^{(j)}) = 0$.

By assumption, $T \equiv \mathrm{End}_T(M) \equiv \mathrm{End}_T(H)$ and so (3.4) holds when $i = j$. Trivially, $\mathrm{Ext}^1_T(T, N^{(i)}) = 0$, and so $\mathrm{Ext}^1_T(N^{(i)}, T) = 0$ also holds for all i. (This also follows from the CM property of $N^{(i)}$.) As

$$\mathrm{Ext}^1_T(M, H) = 0 \iff \mathrm{Ext}^1_T(H, M) = 0,$$

in order to prove part (1), it just remains to prove that $\mathrm{Ext}^1_T(M, H) = 0$.

Since $\operatorname{Ext}^1_T(M,T)=0$, we have the exact sequence

$$0\to \operatorname{Hom}_T(M,H)\to \operatorname{Hom}_T(M,T)\xrightarrow{\beta}\operatorname{Hom}_T(M,T/H)\to \operatorname{Ext}^1_T(M,H)\to 0. \quad (3.8)$$

We would like to understand the Hilbert series of $\operatorname{Hom}_T(M,T/H)$. For this, we consider also the following exact sequence, where we are applying (2.2),

$$0\to \operatorname{Hom}_T(M,T/H)[-1]\xrightarrow{\bullet g}\operatorname{Hom}_T(M,T/H)\to \operatorname{Hom}_B(\overline{M},\overline{T}/\overline{H})\to\cdots \quad (3.9)$$

First we prove that $\operatorname{Hom}_B(\overline{M},\overline{T}/\overline{H})_0=0$. By (i), there is a surjection $\beta:\overline{T}/\overline{H}\twoheadrightarrow M_a$. But any nonzero, degree 0 map $\theta:\overline{M}\to\overline{T}/\overline{H}$ is surjective, since $\overline{T}/\overline{H}$ is cyclic. Thus $\rho=\beta\circ\theta$ gives a surjection $\rho:\overline{M}\twoheadrightarrow M_a$. Since $\overline{M}$ is saturated, it clearly follows from the equivalence qgr-$B\simeq\operatorname{coh}E$ that $\ker\rho=\bigoplus_{n\geq 0}H^0(E,\mathcal{N}_n(d+e+f-a))$. Therefore, by [15, Lemma 2.2], $\ker\rho$ is generated in degree 0. On the other hand, by construction, $(\ker\rho)_0=(\ker\theta)_0$ and so $\ker\rho=\ker\theta$. This is impossible, so no such θ can exist. Thus $\operatorname{Hom}_B(\overline{M},\overline{T}/\overline{H})_0=0$, as desired.

Now, $(\overline{T}/\overline{H})_{\geq 1}$ is filtered by the shifted point modules $(M_{\tau(a)})[-1]$, $(M_{\tau(b)})[-1]$, and $(M_{\tau(c)})[-1]$. Since $\operatorname{hilb}\operatorname{Hom}_B(\overline{M},M_p)=1/(1-s)$ for any point p such that $\operatorname{Hom}_B(\overline{M},M_p)\neq 0$, it follows that $\operatorname{hilb}\operatorname{Hom}_B(\overline{M},\overline{T}/\overline{H})\leq(3s)/(1-s)$. Using (3.9), we then get $\operatorname{hilb}\operatorname{Hom}_T(M,T/H)\leq(3s)/(1-s)^2$.

Now, $\operatorname{hilb}\operatorname{Hom}_B(\overline{M},\overline{H})=(3s+6s^2)/(1-s)^2$, by (3.6). So, considering (3.7) for $(j,i)=(3,1)$, gives the upper bound $\operatorname{hilb}\operatorname{Hom}_T(M,H)\leq(3s+6s^2)/(1-s)^3$. On the other hand,

$$(3s)/(1-s)^2+(3s+6s^2)/(1-s)^3=(6s+3s^2)/(1-s)^3=\operatorname{hilb}\operatorname{Hom}_T(M,T),$$

where the last equality follows since we have already proven that the $(3,2)$ entry of (3.5) is correct. By considering the first three terms of (3.8) and the upper bounds for $\operatorname{hilb}\operatorname{Hom}_T(M,T/H)$ and $\operatorname{hilb}\operatorname{Hom}_T(M,H)$, this forces both bounds to be equalities. In particular,

$$\operatorname{hilb}\operatorname{Hom}_T(M,H)=(3s+6s^2)/(1-s)^3=\operatorname{hilb}\operatorname{Hom}(T)-\operatorname{hilb}(M,T/H)$$

and so the first three terms of (3.8) form a short exact sequence. Thus $\operatorname{Ext}^1_T(M,H)=0$, and the proof of part (1) is complete.

(2) Suppose that hypothesis (ii) holds and define $H^\vee=M^*=\operatorname{Hom}_T(M,T)$. Since M is MCM, it is reflexive, so $M=\operatorname{Hom}_T(H^\vee,T)$. It follows that

$$\operatorname{End}_T(M)=\{x\in Q_{\mathrm{gr}}(T)\mid xM\subseteq M\}=\{x\mid H^\vee xM\subseteq T\}$$

and, similarly, $\operatorname{End}_T(H^\vee)=\{y\in Q_{\mathrm{gr}}(T)\mid H^\vee y\subseteq H^\vee\}=\{y\mid H^\vee yM\subseteq T\}$. Thus $\operatorname{End}_T(M)=\operatorname{End}_T(H^\vee)$. Since $\operatorname{End}_T(M)\equiv T$ by assumption, we also get $\operatorname{End}_T(H^\vee)\equiv T$.

Since T is g-divisible, the module $P=M/T$ is g-torsionfree. By hypothesis (ii),

$$P/Pg=\overline{M}/B\equiv(2+s)/(1-s)$$

is filtered by the shifted right point modules $M_{\tau(d)}[-1] \cong (M_d)_{\geq 1}$, M_e, and M_f. Therefore, [15, Lemma 5.4] applies to P. By that lemma, if $N = \mathrm{Ext}^1_T(P,T)$, then N/Ng is filtered by the shifted left point modules $M^\ell_{\tau^{-1}d}$, $M^\ell_{\tau^{-2}e}[-1]$, and $M^\ell_{\tau^{-2}f}[-1]$. Now consider the exact sequence

$$0 \to H^\vee \to T \xrightarrow{\beta} N \to \mathrm{Ext}^1_T(M,T),$$

and note that $\mathrm{Ext}^1_T(M,T) = 0$ was proved within the proof of part (1). Thus $N \cong T/H^\vee$, and so $N/Ng \cong B/\overline{H^\vee}$, as $H^\vee$ is g-divisible. Since we know the point modules in a filtration of N/Ng, this forces $\overline{H^\vee} = \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n(-\tau^{-n}d - \tau^{-n}e - \tau^{-n}f))$. Thus (ii) implies (ii)$'$.

The converse (ii)$'$ $\Rightarrow$ (ii) is proved similarly, using the left-sided versions of [15, Lemmata 4.5 and 5.4]. The relevant exact sequence is

$$0 \to T \to \mathrm{Hom}_T(H^\vee, T) \to \mathrm{Ext}^1_T(T/H^\vee, T) \to 0,$$

where there is no extra Ext^1 term to worry about. We leave the details to the reader. ∎

4. Recognition I: Recognising Sklyanin elliptic algebras

We continue to consider when an elliptic algebra T of degree 9 can be shown to be the 3-Veronese of a Sklyanin algebra. Let T be such an elliptic algebra with $T/gT \cong B(E,\mathcal{N},\tau)$, and recall that we are always assuming that qgr-T is smooth. Suppose, further, that T has right modules H and M satisfying the conditions from Notation 3.2. The main result of this section is that, under an additional condition on $\mathrm{Div}\, T/H$ and $\mathrm{Div}\, M/T$, $T = S^{(3)}$ is the 3-Veronese of a Sklyanin algebra $S = S(E,\sigma)$. Note that, as discussed at the beginning of Section 3, this also implies that $T \cong \mathrm{End}_T(H) \cong \mathrm{End}_T(M)$. Neither of these conclusions is obvious a priori.

To explain the condition on the divisors of T/H and M/T, suppose for the moment that T is a Sklyanin elliptic algebra; that is, $T = S^{(3)}$ where S is the Sklyanin algebra with $S/gS = B(E,\mathcal{L},\sigma)$ and $\mathcal{L}_3 = \mathcal{L}\otimes\mathcal{L}^\sigma\otimes\mathcal{L}^{\sigma^2} \cong \mathcal{N}$. If $0 \neq x \in S_1 = H^0(E,\mathcal{L})$ vanishes at $a,b,c \in E$, then $H := xS_2T$ satisfies $\overline{H} = \bigoplus_{n\geq 1} H^0(E,\mathcal{N}_n(-a-b-c))$. Recall that the isomorphism class of an invertible sheaf on E is determined by its degree and its image under the natural map from $\mathrm{Pic}\, E \to E$ [6, Example IV.1.3.7]. Thus $\mathcal{L} \cong \mathcal{O}_E(a+b+c)$ and so

$$\mathcal{N} \cong \mathcal{O}_E(\tau^{-1}a+b+c)^{\otimes 3}. \tag{4.1}$$

Further, if we set $H^\vee = TS_2x$ then $\overline{H^\vee} = \bigoplus_{n\geq 1} H^0(E, \mathcal{N}(-\tau^{-n}d - \tau^{-n}e - \tau^{-n}f))$, for $d = \sigma a$, $e = \sigma b$, and $f = \sigma c$. Thus

$$\mathcal{O}_E(\tau a + b + c) \cong \mathcal{O}_E(d+e+f). \tag{4.2}$$

The following theorem, which is the main result of this section, shows that the necessary conditions (4.1), (4.2) for T to be a Sklyanin elliptic algebra are also sufficient.

Theorem 4.3. *Let T be an elliptic algebra of degree 9, such that* qgr-T *is smooth, and with $T/Tg \cong B(E, \mathcal{N}, \tau)$. Suppose that T satisfies the conditions from Notation 3.2, where the points a, b, c, d, e, f satisfy (4.1) and (4.2). Then $T \cong S^{(3)}$, where $S \cong S(E, \sigma)$ is a Sklyanin algebra with $\sigma^3 = \tau$.*

We prove this theorem by means of $\mathbb{Z}$-algebras. Recall that a $\mathbb{Z}$*-algebra* is a $\Bbbk$-algebra A of the form $\bigoplus_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} A_{i,j}$ such that $A_{i,j}A_{k,\ell} = 0$ if $j \neq k$, and $A_{i,j}A_{j,\ell} \subseteq A_{i,\ell}$. The $\mathbb{Z}$-algebra A is *lower triangular* if $A_{ij} = 0$ for all $i < j$, and *connected* if $A_{n,n} = \Bbbk$ for all n.

We say that a homomorphism $\phi : A \to A$ of a $\mathbb{Z}$-algebra A has *degree d* if $\phi(A_{i,j}) \subseteq A_{i+d,j+d}$ for all i, j. The $\mathbb{Z}$-algebra A is called *d-periodic* if it has an automorphism of degree d. A 1-periodic $\mathbb{Z}$-algebra A is also called *principal*. Given any $\mathbb{Z}$-graded algebra $B = \bigoplus_{n\in\mathbb{Z}} B_n$, we can form a $\mathbb{Z}$-algebra $A = \widehat{B}$ by putting $A_{ij} = B_{j-i}$ for all i, j, with the natural multiplication induced by the multiplication of B. Clearly $\widehat{B}$ is principal. Conversely, if a $\mathbb{Z}$-algebra A is principal then there exists a $\mathbb{Z}$-graded algebra B such that $A \cong \widehat{B}$, by [17, Proposition 3.1]; further, the multiplication on B is determined by the given degree 1 automorphism of A. Given a $\mathbb{Z}$-algebra A, $m \in \mathbb{Z}$ and $d \geq 2$, the corresponding *d-th-Veronese* of A is the $\mathbb{Z}$-algebra $A' = A^{(d)}$ where $A'_{i,j} = A_{m+di,m+dj}$, with multiplication induced by the multiplication in A. Clearly A has d distinct d-th-Veronese algebras, depending on the choice of m. We call the $\mathbb{Z}$-algebra A a *quasi-domain* if for all $0 \neq x \in A_{i,j}, 0 \neq y \in A_{j,k}$, we have $0 \neq xy \in A_{i,k}$. If B is a $\mathbb{Z}$-graded domain, the corresponding principal $\mathbb{Z}$-algebra $\widehat{B}$ is a quasi-domain.

Consider the Sklyanin algebra $S = S(E, \sigma)$ with its factor TCR $B = B(E, \mathcal{L}, \sigma) = S/gS$ for an invertible sheaf $\mathcal{L}$ of degree 3 and the central element $g \in S_3$, as described in [15, (4.2)]. Then S has a graded presentation $S \cong \Bbbk\{x_1, x_2, x_3\}/(r_1, r_2, r_3)$, where $\deg x_i = 1$ and $\deg r_j = 2$ for each i, j. So B has a graded presentation $B \cong \Bbbk\{x_1, x_2, x_3\}/(r_1, r_2, r_3, r_4)$, where r_4 corresponds to the central element g and hence $\deg r_4 = 3$. We now give a $\mathbb{Z}$-algebra version of this observation; more specifically, we study $\mathbb{Z}$-algebras satisfying the following properties.

Assumption 4.4. *Let A be a connected lower triangular $\mathbb{Z}$-algebra which is a quasi-domain. Assume there is an ideal of I which is generated by elements $\{0 \neq g_n \in A_{n+3,n} \mid n \in \mathbb{Z}\}$, such that there is a degree 0 isomorphism $\rho : A/I \cong \widehat{B}$, where $B = B(E, \mathcal{L}, \sigma)$ for some elliptic curve E, invertible sheaf $\mathcal{L}$ of degree 3, and translation automorphism $\sigma \in \mathrm{Aut}(E)$. Let ϕ be the degree 1 automorphism of A/I corresponding under ρ to the canonical degree 1 automorphism of $\widehat{B}$. Finally, we can identify $(A/I)_{n,n-1} = A_{n,n-1}$ for all n and we assume that $g_n x = \phi^3(x) g_{n-1}$ for all $x \in A_{n,n-1}$ under this identification.*

Proposition 4.5. *Let A be a $\mathbb{Z}$-algebra satisfying Assumption 4.4. Then there is a graded ring S and a degree 0 isomorphism $\gamma : \widehat{S} \to A$, such that either*

(i) *$S = S(E, \sigma)$ is a Sklyanin algebra, or else*

(ii) *$S = B[z]$, where z is a degree 3 indeterminate.*

Moreover, the principal automorphism ϕ of A/I lifts to a principal automorphism $\widetilde{\phi}$ of A, which corresponds under γ to the canonical principal automorphism of $\widehat{S}$.

Remark 4.6. Although we have a blanket assumption that qgr-R is smooth for an elliptic algebra R, we note for use elsewhere that Proposition 4.5 and its proof hold without any smoothness assumptions. It is also worth noting that qgr-$B[z]$ is not smooth (see the final step in the proof of Theorem 4.3).

Proof. First, since hilb $B = (1+s+s^2)(1-s)^{-2}$, we have $\dim_{\Bbbk}(A/I)_{n+m,n} = 3m$ for $m \geq 1$, while $\dim_{\Bbbk}(A/I)_{n,n} = 1$. Because of the relations $g_n x = \phi^3(x) g_{n-1}$, the ideal I is generated by the $\{g_n\}$ as a right ideal of A. Then, since A is a quasi-domain, we easily get $\dim_{\Bbbk} A_{n+m,n} = \binom{m+2}{2}$ for all $m \geq 0$.

Let $F = \Bbbk\langle V\rangle$ be the free algebra on a 3-dimensional $\Bbbk$-space V, and fix surjections

$$F \to S(E,\sigma) = F/(r_1, r_2, r_3) \to B = B(E, \mathcal{L}, \sigma) = F/(r_1, r_2, r_3, r_4)$$

as above, where r_1, r_2, r_3 are quadratic relations and r_4 is cubic. Then we get induced surjective maps of $\mathbb{Z}$-algebras $\psi : \widehat{F} \to S(\widehat{E}, \sigma)$ and $\theta : S(\widehat{E}, \sigma) \to \widehat{B}$. Let A' be the sub-$\mathbb{Z}$-algebra of A generated by $\{A_{n+1,n} \mid n \in \mathbb{Z}\}$, and let $\mu : A' \to A/I$ be given by the natural inclusion $A' \to A$ followed by the natural surjection $A \to A/I$. The isomorphism $\rho : A/I \to \widehat{B}$ gives an isomorphism

$$A'_{n+1,n} = A_{n+1,n} = (A/I)_{n+1,n} \xrightarrow{\rho} \widehat{B}_{n+1,n} = \widehat{F}_{n+1,n}$$

for each n. Then there is a clearly unique surjection of $\mathbb{Z}$-algebras $\nu : \widehat{F} \to A'$ with $\rho\mu\nu = \theta\psi$.

We claim that ν factors through $S(\widehat{E}, \sigma)$. Since I is generated in degrees $(n+3, n)$, we have isomorphisms

$$A'_{n+2,n} = A_{n+2,n} = (A/I)_{n+2,n} \xrightarrow{\rho} \widehat{B}_{n+2,n}$$

for each n. (The first equality follows since B is generated in degree 1, which forces $A_{n+2,n+1}A_{n+1,n} = A_{n+2,n}$ since A agrees with $\widehat{B}$ in low degree.) Thus $\rho\mu$ is an isomorphism in degree $(n+2, n)$. It follows that ν must kill the same degree $(n+2, n)$ relations that $\theta\psi$ does. This shows that there is a map $\chi : S(\widehat{E}, \sigma) \to A'$ such that $\nu = \chi\psi$; in other words, ν factors as claimed. Now $\theta\psi = \rho\mu\nu = \rho\mu\chi\psi$ and since ψ is surjective, $\theta = \rho\mu\chi$.

To distinguish it from the elements $g_n \in A$, let $h \in S(E, \sigma)_3$ be the image of the degree 3 relation $r_4 \in F$ and let $h_n = h \in S(\widehat{E}, \sigma)_{n+3,n}$. Since h is central in $S(E, \sigma)$, we have

$$S(\widehat{E}, \sigma)_{n+4,n+3} h_n = h_{n+1} S(\widehat{E}, \sigma)_{n+1,n}, \qquad \text{for all } n \in \mathbb{Z}. \tag{4.7}$$

Since the image of $\chi(h_n)$ in A/I is 0, we have $\chi(h_n) \in I$. As I is generated by the elements $\{g_n\}$, this forces $\chi(h_n) = \lambda_n g_n$ for some scalars $\lambda_n \in \Bbbk$. Using that A is a quasi-domain, applying χ to (4.7) shows that $\lambda_n = 0 \iff \lambda_{n+1} = 0$. By induction, if some $\lambda_i = 0$ then $\lambda_n = 0$ for all n.

Consider the case that $\lambda_n \neq 0$ for all $n \in \mathbb{Z}$. In this case, we have $g_n \in A'$ for all n. Since A/I is generated by $\{(A/I)_{n+1,n} \mid n \in \mathbb{Z}\}$, and I is generated by the g_n, clearly A is generated as an algebra by $\{A_{n+1,n} \mid n \in \mathbb{Z}\}$ and $\{g_n \mid n \in \mathbb{Z}\}$. Thus $A = A'$ in this case and χ is a surjection $S(\widehat{E}, \sigma) \to A$. As

$$\dim_{\Bbbk} A_{n+m,n} = \binom{m+2}{2} = \dim_{\Bbbk} S(E, \sigma)_m = \dim_{\Bbbk} S(\widehat{E}, \sigma)_{n+m,n}$$

for all $n \geq \mathbb{Z}$ and $m \geq 0$, χ is a degree 0 isomorphism from $S(\widehat{E}, \sigma)$ to A. So (i) holds with $\gamma = \chi$. There is a unique principal automorphism $\widetilde{\phi}$ of A corresponding under γ to the canonical degree 1 automorphism of $S(\widehat{E}, \sigma)$. The identity $\theta = \rho\mu\chi$ implies that $\widetilde{\phi}$ lifts the given automorphism ϕ of A/I.

Otherwise, $\lambda_n = 0$ for all $n \in \mathbb{Z}$. In this case, $\chi(h_n) = 0$ for all n, and so χ factors through $\widehat{B}$ to give a map $\overline{\chi} : \widehat{B} \to A'$ such that $\overline{\chi}\theta = \chi$. Then $\theta = \rho\mu\chi = \rho\mu\overline{\chi}\theta$ and since θ is surjective, $\rho\mu\overline{\chi} = 1_{\widehat{B}}$. In particular, the surjection $\overline{\chi} : \widehat{B} \to A'$ must be an isomorphism, and then so is $\mu : A' \to A/I$.

Let z be a central indeterminate of degree 3 and let $z_n = z \in \widehat{B[z]}_{n+3,n}$. We claim that we can extend the isomorphism $(\rho\mu)^{-1} : \widehat{B} \to A'$ to a homomorphism $\gamma : \widehat{B[z]} \to A$ by defining $\gamma(z_n) = g_n$. For each $n \in \mathbb{Z}$, define a map $V \to \widehat{B}_{n+1,n}$ by the identification $V = \widehat{F}_{n+1,n} \xrightarrow{\theta\psi} \widehat{B}_{n+1,n}$. Write the image of $v \in V$ under this map as v_n. To show that γ is well defined, we need to check that $\gamma(z_{n+1}v_n) = g_{n+1}(\rho\mu)^{-1}(v_n)$ is equal to $\gamma(v_{n+3}z_n) = (\rho\mu)^{-1}(v_{n+3})g_n$ for all $n \in \mathbb{Z}$ and $v \in V$. But this follows from the equation $g_{n+1}x = \phi^3(x)g_n$, since ρ intertwines ϕ with the canonical principal automorphism of $\widehat{B}$.

Similarly to the first case, A is generated by A' and the g_n, so that γ is surjective, and comparing Hilbert series, we see that γ is an isomorphism. Thus case (ii) holds. Again, there is a unique principal automorphism $\widetilde{\phi}$ of A corresponding under γ to the canonical degree 1 automorphism of $\widehat{B[z]}$. If $\pi : \widehat{B[z]} \to \widehat{B}$ is the canonical surjection, by the definition of γ we have $\pi = \rho\mu\gamma$. From this equation it follows similarly to the first case that $\widetilde{\phi}$ lifts ϕ. ■

In order to apply the proposition above, we need to be able to recognise when a $\mathbb{Z}$-algebra is 1-periodic and isomorphic to $\widehat{B}$ for a TCR B. The next lemma will help us to do this. It is useful to consider $\mathbb{Z}$-algebras of the following form.

Definition 4.8. Let $K = \Bbbk(E)$ for an elliptic curve, let $\tau \in \mathrm{Aut}_{\Bbbk}(K)$, and consider the skew-Laurent ring $Q = K[t, t^{-1}; \tau]$. We call a $\mathbb{Z}$-algebra A *standard* if

(i) it is connected, lower triangular, and generated by $\{A_{n+1,n} \mid n \in \mathbb{Z}\}$;

(ii) each piece $A_{i,j}$ has the form $V_{i,j}t^{d_{i,j}} \subseteq Q$ for some finite-dimensional $\Bbbk$-subspace $V_{i,j} \subseteq K$ and $d_{i,j} \in \mathbb{Z}$; and

(iii) for each $i \geq j \geq k$ the multiplication map $A_{i,j} \otimes A_{j,k} \to A_{i,k}$ is given by the multiplication in Q.

It is easy to see that once the ring $Q = K[t, t^{-1}; \tau]$ is fixed, given arbitrary choices of $V_n t^{d_n}$ with $V_n \subseteq K$ and $d_n \in \mathbb{Z}$ for each $n \in \mathbb{Z}$, there is a unique standard $\mathbb{Z}$-algebra A with $A_{n+1,n} = V_n t^{d_n}$.

We show now that every standard $\mathbb{Z}$-algebra is isomorphic to one where each graded piece is contained in K and no automorphism is involved in the multiplication.

Lemma 4.9. *Fix K and τ as in Definition 4.8. Let A be a standard $\mathbb{Z}$-algebra associated to this data. Then there is a standard algebra $\widetilde{A}$ with $\widetilde{A}_{n+1,n} \subseteq K$ for all n and a degree 0 isomorphism $A \cong \widetilde{A}$.*

Proof. Write $A_{n+1,n} = V_n t^{d_n}$ where $V_n \subseteq K$. Let $e_1 = 0$, for each $n \geq 2$ let $e_n = -\sum_{i=1}^{n-1} d_i$, and for $n \leq 0$ let $e_n = \sum_{i=n}^{0} d_i$. Let $\widetilde{A}$ be the unique standard $\mathbb{Z}$-algebra with $\widetilde{A}_{n+1,n} = \tau^{e_{n+1}}(V_n) \subseteq K$ for each n. Write $A_{i,j} = V_{i,j} t^{d_{i,j}}$ and $\widetilde{A}_{i,j} = W_{i,j}$ for each i, j with $i \geq j$, for some $V_{i,j}, W_{i,j} \subseteq K$. We have $d_{i,i} = 0$ because A is connected.

We claim that the map $\phi : A \to \widetilde{A}$, given on the graded piece $A_{i,j} = V_{i,j} t^{d_{i,j}} \to \widetilde{A}_{i,j} = W_{i,j}$ by the formula $vt^{d_{i,j}} \mapsto \tau^{e_i}(v)$, is an isomorphism of $\mathbb{Z}$-algebras. It is not obvious that ϕ actually has image in $\widetilde{A}$, but it is at least a function from A to the $\mathbb{Z}$-algebra which has every piece equal to K. Thus we first check ϕ is a homomorphism and then it will be clear that it lands in $\widetilde{A}$.

Given $u \in V_{i,j}$ and $v \in V_{j,k}$, we have $x = ut^{d_{i,j}} \in A_{i,j}$ and $y = vt^{d_{j,k}} \in A_{j,k}$. Then $\phi(x)\phi(y) = \tau^{e_i}(u)\tau^{e_j}(v)$ while

$$\phi(xy) = \phi(u\tau^{d_{i,j}}(v)t^{d_{i,j}+d_{j,k}}) = \tau^{e_i}(u\tau^{d_{i,j}}(v)) = \tau^{e_i}(u)\tau^{e_i+d_{i,j}}(v).$$

Thus to see that ϕ is a homomorphism, we need $e_i + d_{i,j} = e_j$ for all $i \geq j$. Note that $d_{n+1,n} = d_n$ by definition and that $d_{i,j} = d_{i-1} + \cdots + d_j$ for all $i > j$ since $A_{i,j} = A_{i,i-1} \cdots A_{j+1,j}$. By the definition of the e_n we also have $d_{i-1} + \cdots + d_j = -e_i + e_j$. Thus ϕ is a homomorphism, so $\phi(A_{n+1,n}) = \widetilde{A}_{n+1,n}$ by definition. Since A is generated as an algebra by the $\{A_{n+1,n}\}$ and $\widetilde{A}$ is generated by the $\widetilde{A}_{n+1,n}$, the homomorphism ϕ does have image in $\widetilde{A}$ and is even surjective. Then, ϕ is an isomorphism since it is injective on each graded piece by definition. ■

We are now ready for the proof of Theorem 4.3, for which we need the following notation.

Notation 4.10. Let T be an elliptic algebra of degree 9 with $T/Tg \cong B(E, \mathcal{N}, \tau)$ that satisfies conditions (i) and (ii) of Notation 3.2. Thus, we have right T-modules $N^{(1)} = H \subseteq N^{(2)} = T \subseteq N^{(3)} = M$. Set

$$\mathbb{E} = \operatorname{End}_T(N^{(1)} \oplus N^{(2)} \oplus N^{(3)}) = \left(\operatorname{Hom}_T(N^{(j)}, N^{(i)})\right)_{ij}.$$

Define a $\mathbb{Z}$-algebra $\mathbb{S}$ by $\mathbb{S}_{3m+i,3n+j} = (\mathbb{E}_{i,j})_{m-n}$, for all $m, n \in \mathbb{Z}$ and $i, j \in \{1, 2, 3\}$, where the multiplication is induced from that in $\mathbb{E}$. Let $\varphi : \mathbb{S} \to \mathbb{S}$ be the degree 3 automorphism given by the identifications $\mathbb{S}_{3m+i,3n+j} = (\mathbb{E}_{i,j})_{m-n} = \mathbb{S}_{3m+3+i,3n+3+j}$.

Proof of Theorem 4.3. We assume Notation 4.10. Assume in addition that the points a, b, c, d, e, f in Notation 3.2 (i), (ii) satisfy (4.1) and (4.2).

The main step is to show that $\mathbb{S}$ satisfies the hypotheses of Proposition 4.5. For any $n \in \mathbb{Z}$ we have $\mathbb{S}_{n+3,n} = \operatorname{End}_T(N^{(i)}, N^{(i)})_1$ (for some i), which contains the element g. Thus we set $g_n = g \in \mathbb{S}_{n+3,n}$ for each n. Note that the multiplication in $\mathbb{S}$ is induced by the multiplication in $T_{(g)}$; in particular, $\mathbb{S}$ is a quasi-domain. Let I be the ideal of $\mathbb{S}$ generated by $\{g_n \mid n \in \mathbb{Z}\}$. Since all of the $N^{(i)}$ are g-divisible, so is each $\operatorname{Hom}_T(N^{(j)}, N^{(i)})$ by [15, Lemma 4.4]. Thus given $i, j \in \{1, 2, 3\}$ and $m, n \in \mathbb{Z}$ with $m - n \geq 3$ we have

$$\begin{aligned} I_{3m+i,3n+j} \supseteq \mathbb{S}_{3m+i,3(n+1)+j}\, g_{3n+j} &= \operatorname{Hom}_T(N^{(j)}, N^{(i)})_{m-n-1}\, g \\ &= \operatorname{Hom}_T(N^{(j)}, N^{(i)})_{m-n} \cap gT_{(g)}. \end{aligned}$$

Conversely, since each g_n is a copy of g and the multiplication of $\mathbb{S}$ is induced by multiplication in $T_{(g)}$, clearly $I_{3m+i,3n+j} \subseteq gT_{(g)}$. Thus

$$I_{3m+i,3n+j} = \operatorname{Hom}_T(N^{(j)}, N^{(i)})_{m-n} \cap gT_{(g)}.$$

We see that $\overline{\mathbb{S}} := \mathbb{S}/I$ is the $\mathbb{Z}$-algebra with $\overline{\mathbb{S}}_{3m+i,3n+j} = (\overline{\mathbb{E}_{i,j}})_{m-n}$ for $i, j \in \{1, 2, 3\}$ where, by Proposition 3.3,

$$\overline{\mathbb{E}_{i,j}} = \overline{\operatorname{Hom}_T(N^{(j)}, N^{(i)})} = \operatorname{Hom}_B(\overline{N^{(j)}}, \overline{N^{(i)}}).$$

Clearly the multiplication in $\overline{\mathbb{S}}$ is induced by the multiplication in $\Bbbk(E)[t, t^{-1}; \tau] = T_{(g)}/gT_{(g)}$.

As in the proof of Proposition 3.3, for each i we can write

$$\overline{N^{(i)}} = \bigoplus_{n \geq 0} H^0(E, \mathcal{G}_i \otimes \mathcal{N}_n)t^n,$$

where $\mathcal{G}_1 = \mathcal{O}_E(-a - b - c)$, $\mathcal{G}_2 = \mathcal{O}_E$, and $\mathcal{G}_3 = \mathcal{O}_E(d + e + f)$, and then

$$\operatorname{Hom}_B(\overline{N^{(j)}}, \overline{N^{(i)}}) = \bigoplus_{n \geq 0} H^0(E, (\mathcal{G}_j^{\tau^n})^{-1}\mathcal{G}_i \otimes \mathcal{N}_n)t^n \subseteq \Bbbk(E)[t, t^{-1}; \tau]$$

for all $i, j \in \{1, 2, 3\}$. In particular,

$$\overline{\mathbb{S}}_{3n+i,3n+i-1} = \begin{cases} H^0(E, \mathcal{N}(-\tau^{-1}(d) - \tau^{-1}(e) - \tau^{-1}(f) - a - b - c))t, & i = 1, \\ H^0(E, \mathcal{O}_E(a + b + c)), & i = 2, \\ H^0(E, \mathcal{O}_E(d + e + f)), & i = 3. \end{cases}$$

It is now easy to see that $\overline{\mathbb{S}}$ is generated as an algebra by $\{\overline{\mathbb{S}}_{n+1,n} \mid n \in \mathbb{Z}\}$, using [15, Lemma 2.2]. We have now checked that $\overline{\mathbb{S}}$ is a standard algebra in the sense of Definition 4.8.

We would like to show now that $\overline{\mathbb{S}}$ is isomorphic to $\widehat{C}$ where $C = B(E, \mathcal{L}, \sigma)$, with $\mathcal{L}$ some sheaf of degree 3 and $\sigma \in \operatorname{Aut}(E)$ such that $\sigma^3 = \tau$. Let $\mathcal{L}$ be any sheaf of degree 3 and σ any automorphism for the moment, and write $C = B(E, \mathcal{L}, \sigma) = \bigoplus_{n\geq 0} H^0(E, \mathcal{L}_n)u^n \subseteq \Bbbk(E)[u, u^{-1}; \sigma]$. We use Lemma 4.9 to change $\overline{\mathbb{S}}$ and $\widehat{C}$ to isomorphic standard algebras involving only multiplication in $\Bbbk(E)$. Writing $\overline{\mathbb{S}}_{n+1,n} = V_n t^{d_n}$ with $V_n \subseteq \Bbbk(E)$, we have $d_n = 1$ when n is a multiple of 3, $d_n = 0$ otherwise. Thus by the proof of Lemma 4.9 there is a degree 0 isomorphism $\overline{\mathbb{S}} \cong D$, where D is the standard algebra with $D_{n+1,n} = \tau^{-\lfloor n/3\rfloor}(V_n) = H^0(E, \mathcal{D}_n)$ where the sheaf $\mathcal{D}_n$ is given by

$$\begin{aligned}
\mathcal{D}_n &= [\mathcal{N}(-\tau^{-1}(d) - \tau^{-1}(e) - \tau^{-1}(f) - a - b - c)]^{\tau^{-(n/3)}} && n \equiv 0 \mod 3,\\
\mathcal{D}_n &= [\mathcal{O}_E(a + b + c)]^{\tau^{-(n-1)/3}} && n \equiv 1 \mod 3,\\
\mathcal{D}_n &= [\mathcal{O}_E(d + e + f)]^{\tau^{-(n-2)/3}} && n \equiv 2 \mod 3.
\end{aligned}$$

Similarly, writing $\widehat{C}_{n+1,n} = W_n u$, where $W_n = H^0(E, \mathcal{L})$ for all n, we get a degree 0 isomorphism $\widehat{C} \to F$ for the standard algebra F with $F_{n+1,n} = \sigma^{-n}(W_n) = H^0(E, \mathcal{F}_n)$, where $\mathcal{F}_n = \mathcal{L}^{\sigma^{-n}}$.

Now it is easy to see that if for all $n \in \mathbb{Z}$ there is an isomorphism of sheaves $\mathcal{F}_n \cong \mathcal{D}_n$, then there will be a degree 0 isomorphism $F \cong D$. Let

$$\mathcal{L} = \mathcal{D}_0 = \mathcal{N}(-\tau^{-1}(d) - \tau^{-1}(e) - \tau^{-1}(f) - a - b - c).$$

We now show that we can choose σ so that $\mathcal{F}_n = \mathcal{L}^{\sigma^{-n}} \cong \mathcal{D}_n$ for all $n \in \mathbb{Z}$.

To have $\mathcal{F}_n \cong \mathcal{D}_n$ for $n = 1, 2$, we need that $\mathcal{L}^{\sigma^{-1}} \cong \mathcal{D}_1 = \mathcal{O}_E(a + b + c)$, and $\mathcal{L}^{\sigma^{-2}} \cong \mathcal{D}_2 = \mathcal{O}_E(d + e + f)$. Fix a group operation $\oplus$ on E and let $t \in E$ be such that $\tau : E \to E$ is given by $x \mapsto x \oplus t$. (Such t exists because τ is infinite order.) Let $\sum :$ Pic $E \to E$ be the natural map, so $\sum \mathcal{O}_E(a_1 + \cdots + a_n) = a_1 \oplus \cdots \oplus a_n$. Two invertible sheaves $\mathcal{M}, \mathcal{M}'$ on E are isomorphic if and only if $\deg \mathcal{M} = \deg \mathcal{M}'$ and $\sum \mathcal{M} = \sum \mathcal{M}'$.

Equations (4.1) and (4.2) are equivalent to

$$\sum \mathcal{N} \oplus 3t = 3a \oplus 3b \oplus 3c \tag{4.11}$$

and

$$a \oplus b \oplus c \oplus t = d \oplus e \oplus f. \tag{4.12}$$

Since the group structure on an elliptic curve is divisible, we can choose $s \in E$ such that $3s = t$; then the translation automorphism $\sigma(x) = x \oplus s$ satisfies $\sigma^3 = \tau$. From (4.12) and (4.11) we obtain

$$a \oplus b \oplus c \ominus t = \sum \mathcal{N} \ominus d \ominus e \ominus f \ominus a \ominus b \ominus c \oplus 3t = \sum \mathcal{L},$$

and thus $\mathcal{L}^{\sigma^{-1}} \cong \mathcal{O}(a + b + c) = \mathcal{D}_1$. Then by (4.12),

$$\mathcal{L}^{\sigma^{-2}} \cong \mathcal{D}_1^{\sigma^{-1}} \cong \mathcal{O}_E(\sigma a + \sigma b + \sigma c) \cong \mathcal{O}_E(d + e + f) \cong \mathcal{D}_2,$$

as needed.

Fix these choices of σ and $\mathcal{L}$ and the isomorphisms $\mathcal{D}_r \cong \mathcal{F}_r$ for $r = 0, 1, 2$ given above. Then for each $n \in \mathbb{Z}$, writing $n = 3q + r$ with $r \in \{0, 1, 2\}$ we get an induced isomorphism $\mathcal{D}_n = \mathcal{D}_r^{\tau^{-q}} \cong \mathcal{F}_r^{\sigma^{-3q}} \cong \mathcal{F}_n$. We conclude that for the choice of σ and $\mathcal{L}$

as above, there are degree 0 isomorphisms of $\mathbb{Z}$-algebras $\mathbb{S}/I = \overline{\mathbb{S}} \cong D \cong F \cong \widehat{C}$. Let ϕ be the degree 1 automorphism of $\overline{\mathbb{S}}$ that corresponds under this chain of isomorphisms to the canonical principal automorphism of $\widehat{C}$.

In order to apply Proposition 4.5, the last hypothesis that remains to be checked is that identifying $\overline{\mathbb{S}}_{n+1,n} = \mathbb{S}_{n+1,n}$, we have $g_{n+1}x = \phi^3(x)g_n$ for all $x \in \mathbb{S}_{n+1,n}$. Since by construction $\mathbb{S}_{n+1,n}$ and $\mathbb{S}_{n+4,n+3}$ are naturally identified, and g_n is equal to the copy of the central element g in $S_{n+3,n}$ for each n, the needed equation amounts to showing that $\phi^3 : \mathbb{S}_{n+1,n} \to \mathbb{S}_{n+4,n+3}$ is simply the natural identification for each n; in other words, showing that $\phi^3 = \varphi$ on degree $(n+1,n)$ elements. In other words, we must show for all n that the diagram

$$\begin{array}{ccccc} \mathbb{S}_{n+1,n} & \longrightarrow & \overline{\mathbb{S}}_{n+1,n} & \longrightarrow & \widehat{C}_{n+1,n} \\ \| & & \| & & \| \\ \mathbb{S}_{n+4,n+1} & \longrightarrow & \overline{\mathbb{S}}_{n+4,n+1} & \longrightarrow & \widehat{C}_{n+4,n+1} \end{array}$$

commutes, where the horizontal arrows are the maps constructed above.

Let $n = 3q + r$, where $0 \le r \le 2$. Let $\alpha_n : D_{n+1,n} \to F_{n+1,n}$ be such that the map $\overline{\mathbb{S}}_{n+1,n} \to \widehat{C}_{n+1,n}$ is given by the composition

$$\overline{\mathbb{S}}_{n+1,n} = V_n t^{d_n} \xrightarrow{\tau^{-q}} D_{n+1,n} \xrightarrow{\alpha_n} F_{n+1,n} \xrightarrow{\sigma^n} \widehat{C}_{n+1,n}.$$

Now, by construction, α_n is induced by pullback from our choice of α_r: in other words, $\alpha_n = \tau^{-q}\alpha_r\tau^q$. Thus the composition $\overline{\mathbb{S}}_{n+1,n} \to \widehat{C}_{n+1,n}$ is $\sigma^r\alpha_r$ and depends only on the residue of n mod 3, as we need.

We may now apply Proposition 4.5 to $\mathbb{S}$. We get immediately from that proposition that there is a degree 0 isomorphism $\gamma : \widehat{S} \to \mathbb{S}$, where S is either the Sklyanin algebra $S(E,\sigma)$ or else $B[z]$. In addition, the canonical principal automorphism of $\widehat{S}$ corresponds under this isomorphism to a degree 1 automorphism $\widetilde{\phi}$ of $\mathbb{S}$, which lifts the automorphism ϕ of $\mathbb{S}/I$. Since $\phi^3 = \varphi$ on elements in $\mathbb{S}_{n+1,n}$, we see that $(\widetilde{\phi})^3 = \varphi$ as automorphisms of $\mathbb{S}$.

By construction there is a 3-Veronese $\mathbb{S}^{(3)}$, where

$$\mathbb{S}^{(3)}_{m,n} = \mathbb{S}_{3m+2,3n+2} = \operatorname{Hom}_T(T,T)_{m-n} = T_{m-n}.$$

By the definition of the multiplication in $\mathbb{S}$ we have an identification of $\mathbb{Z}$-algebras $\mathbb{S}^{(3)} = \widehat{T}$. Moreover, the degree 3 automorphism φ of $\mathbb{S}$ induces a principal automorphism φ of $\mathbb{S}^{(3)}$ which is just the canonical principal automorphism of $\widehat{T}$ under this identification. It follows from [17, Proposition 3.1] that there is an isomorphism of $\mathbb{Z}$-graded algebras $S^{(3)} \cong T$.

To conclude the proof we just need to rule out the case $S = B[z]$. In this case $T = B^{(3)}[z]$, where z now has degree 1. By the smoothness assumption, T° has finite global dimension. On the other hand, $T^\circ = T[z^{-1}]_0 \cong B(E,\mathcal{N},\tau)$, which has infinite global dimension (use the proof of [12, Theorem 5.4]). Thus this case does not occur. ■

Remark 4.13. We remark that we do not know of a degree 9 elliptic algebra T (with qgr-T smooth) which is not isomorphic to an algebra S as in Theorem 4.3. However, we do not see how to prove that (4.1) and (4.2) always hold.

5. Iterating blowing down

Our ultimate goal is to give a recognition theorem for 2-point blowups of a noncommutative $\mathbb{P}^2$. In order to do this, we need to study the process of iterating blowing down, and we do that in this section. Throughout the section, we continue to have a standing assumption that qgr-R is smooth for any elliptic algebra R.

Our first result is more general: we study how to "blow down" a reflexive right ideal of an elliptic algebra. For motivation, suppose that $T = S^{(3)}$ is the 3-Veronese of a Sklyanin algebra $S = S(E, \sigma)$. Let $p \neq q \in E$, and consider how we would construct the right T-module H in Proposition 3.3 from the blowup $R = T(p + q)$. Besides the line modules that come from blowing up p and q, there is a third R-line module constructed as follows. Let $x \in S_1$ be the line that vanishes at p, q. Then $xS_2 \subseteq R_1$ and one may compute that R/xS_2R is a line module. If we let $J = xS_2R$, then $J_1T = H$. The next lemma generalises this process.

Lemma 5.1. *Let R be an elliptic algebra with $R/gR = B = B(E, \mathcal{M}, \tau)$, and fix a line module $L = R/J$ satisfying $(L \bullet_{\mathrm{MS}} L) = -1$. Let $K \subseteq R$ be a g-divisible MCM right ideal, generated in degree* 1 *as an R-module, with* $\dim_{\Bbbk} K_1 \geq 2$.

Let $\widetilde{R}$ be the blowdown of R at L, as constructed by [15, Theorem 1.4]. *As in* [15, (8.1)], *let*

$$\widetilde{K} = \sum_{\alpha} \{N_\alpha \mid K \subseteq N_\alpha \subset Q_{\mathrm{gr}}(R) \text{ with } N_\alpha/K \cong L[-i_\alpha] \text{ for some } i_\alpha \in \mathbb{Z}\}.$$

Then:

1. *We have* $\mathsf{hilb}\,\mathrm{Ext}^1_R(L, K) = s^i/(1-s)^2$, *for $i \in \{0, 1, 2\}$; thus* $\mathsf{hilb}\,\widetilde{K} = \mathsf{hilb}\,K + s^i/(1-s)^3$. *Also, $i = 0$ if and only if $K = J$. In any case,* $\widetilde{K} = \mathrm{Hom}_R(J, K)R$.
2. *$\widetilde{K}$ is a g-divisible MCM right $\widetilde{R}$-module.*
3. *If $i = 1$ or $i = 2$ then $\widetilde{K}$ is also generated in degree* 1 *as a right $\widetilde{R}$-module. In particular, if $i = 2$ then $\widetilde{K} = K_1\widetilde{R}$. If $i = 0$ then $\widetilde{K} = \widetilde{R}$.*

Proof. We first note that since K is MCM it is reflexive, and so $\widetilde{K} = \mathrm{Hom}_R(J, K)R$ holds by [15, Lemma 8.2]. Also, by Proposition 2.3 $\overline{K}$ is a saturated B-module.

(1) As vector spaces, $\mathrm{Ext}^1_R(L, K) \cong \mathrm{Hom}_R(J, K)/K$. Note that

$$\overline{J} = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(-p)),$$

for $p = \mathrm{Div}\, L$. Similarly, since $\overline{K}$ is saturated, $\overline{K} = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(-D))$ for some effective divisor D. If $D = 0$, then $\overline{K}_0 \neq 0$, a contradiction. The g-divisible proper right ideal K cannot contain g, and so the hypothesis $\dim_{\Bbbk} K_1 \geq 2$ implies that $\dim_{\Bbbk} \overline{K}_1 \geq 2$ as well. In conclusion, we have $0 < \deg D \leq \deg \mathcal{M} - 2$.

Now [15, Lemma 2.3] gives

$$\operatorname{Hom}_B(\overline{J}, \overline{K}) = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(-D + \tau^{-n}(p))).$$

Thus $\mathsf{hilb}\operatorname{Hom}_B(\overline{J}, \overline{K})/\overline{K} = s^j/(1-s)$, where either $j = 0$ if $D = p$ or $j = 1$ otherwise. By [15, Lemma 4.4 (1)], $\operatorname{Hom}_R(J, K)$ is g-divisible since J and K are, and so

$$\mathsf{hilb}[\overline{\operatorname{Hom}_R(J, K)}/\overline{K}]/(1-s) = \mathsf{hilb}\operatorname{Hom}_R(J, K)/K = \mathsf{hilb}\operatorname{Ext}^1_R(L, K).$$

We have $\overline{K} \subseteq \overline{\operatorname{Hom}_R(J, K)} \subseteq \operatorname{Hom}_B(\overline{J}, \overline{K})$. Since J is a line ideal, it is g-divisible and reflexive by [15, Lemma 5.6 (2)]. Thus [15, Lemma 4.8] applies and shows that $\overline{\operatorname{Hom}_R(J, K)}$ is equal in large degree to $\operatorname{Hom}_B(\overline{J}, \overline{K})$. Since $\mathsf{hilb}\operatorname{End}_R(J) = \mathsf{hilb}\, R$ by [15, Theorem 7.1],

$$\overline{\operatorname{End}_R(J)} = \operatorname{End}_B(\overline{J}) = B(E, \mathcal{M}(-p + \tau^{-1}(p)), \tau)$$

is a full TCR, and $\overline{\operatorname{Hom}_R(J, K)}$ is a right module over this ring. Using $\deg D \leq \deg \mathcal{M} - 2$ as mentioned above, $\mathcal{M}(-D + \tau^{-1}(p))$ has degree at least 3. By [15, Lemma 2.3], for any $i \geq 1$ we have $\operatorname{Hom}_B(\overline{J}, \overline{K})_i \operatorname{End}_B(\overline{J}) = \operatorname{Hom}_B(\overline{J}, \overline{K})_{\geq i}$. This equation also clearly holds for $i = 0$ in case $D = p$ and $\overline{J} = \overline{K}$. Now if for some $i \geq j$ we have $\overline{\operatorname{Hom}_R(J, K)}_i = \operatorname{Hom}_B(\overline{J}, \overline{K})_i$, then we will have $\overline{\operatorname{Hom}_R(J, K)}_{\geq i} = \operatorname{Hom}_B(\overline{J}, \overline{K})_{\geq i}$ by multiplying on the right by $\overline{\operatorname{End}_R(J)}$. We conclude that the actual value of $\mathsf{hilb}\,\overline{\operatorname{Hom}_R(J, K)}/\overline{K}$ must be $s^i/(1-s)$ for some $i \geq j$. This implies that $\overline{\operatorname{Hom}_R(J, K)} = \overline{K} + \operatorname{Hom}_B(\overline{J}, \overline{K})_{\geq i}$. By g-divisibility we have $\mathsf{hilb}\operatorname{Hom}_R(J, K)/K = \mathsf{hilb}\operatorname{Ext}^1_R(L, K) = s^i/(1-s)^2$.

Now the case $i = 0$ can occur only if $j = 0$, in which case $D = p$ and $\overline{J} = \overline{K}$, and also $\operatorname{Hom}_R(J, K)_0 \neq 0$. But $\operatorname{Hom}_R(J, K) \subseteq \operatorname{Hom}_R(J, R)$, and we know by [15, Lemma 5.6 (3)] that $\operatorname{Hom}_R(J, R)_0 = \Bbbk$. If $\Bbbk J \subseteq K$ then $J \subseteq K$. Since J and K are g-divisible and $\overline{J} = \overline{K}$ this forces $J = K$.

Assume now that $J \neq K$ and so $i \geq 1$. Now $\widetilde{K}$ is a right $\widetilde{R}$-module by [15, Corollary 8.5], so $\overline{\widetilde{K}}$ must be a right module over $\overline{\widetilde{R}} = B(E, \mathcal{M}(\tau^{-1}(p)), \tau)$. If $i > 2$, then

$$\begin{aligned} \overline{\widetilde{K}}_1 \overline{\widetilde{R}}_1 &= H^0(E, \mathcal{M}_1(-D)) H^0(E, \mathcal{M}_1^{\tau}(\tau^{-2}(p))) \\ &= H^0(E, \mathcal{M}_2(-D + \tau^{-2}(p))) \not\subseteq H^0(E, \mathcal{M}_2(-D)) = \overline{K}_2 = \overline{\widetilde{K}}_2, \end{aligned}$$

a contradiction. Thus $i = 1$ or $i = 2$. Clearly the case $i = 1$ occurs if $\operatorname{Hom}_R(J, K)_1 \supsetneqq K_1$, while $i = 2$ occurs if $\operatorname{Hom}_R(J, K)_1 = K_1$.

The Hilbert series of $\widetilde{K}$ follows directly from $\mathsf{hilb}\operatorname{Ext}^1_R(L, K) = s^i/(1-s)^2$ and [15, Lemma 8.2].

(2) As noted above, $\widetilde{K}$ is a right $\widetilde{R}$-module. Write $\overline{\widetilde{R}} = B' = B(E, \mathcal{N}, \tau)$, where $\mathcal{N} = \mathcal{M}(\tau^{-1}(p))$.

If $i = 0$, then $K = J$ and so $\widetilde{K} = \operatorname{Hom}(J, J)R = \widetilde{R}$, which is certainly g-divisible and MCM. If $i \geq 1$, then

$$\overline{\widetilde{K}} = \overline{K} + \operatorname{Hom}_B(\overline{J}, \overline{K})_{\geq i}\overline{R} = \overline{K} + \operatorname{Hom}_B(\overline{J}, \overline{K})_i \operatorname{End}_B(\overline{J})\overline{R}$$
$$= \overline{K} + \operatorname{Hom}_B(\overline{J}, \overline{K})_i \overline{\widetilde{R}}.$$

If $i = 1$ then $\overline{\widetilde{K}}_i = H^0(E, \mathcal{M}(-D + \tau^{-1}(p))) = H^0(E, \mathcal{N}(-D))$, while if $i = 2$, then we have $\overline{\widetilde{K}}_i = H^0(E, \mathcal{M}_2(-D + \tau^{-2}(p))) = H^0(E, \mathcal{N}_2(-D - \tau^{-1}(p)))$. Using [15, Lemma 2.3], in either case we get

$$\overline{\widetilde{K}} = \overline{K} + \bigoplus_{n \geq i} H^0(E, \mathcal{M}_n(-D + \tau^{-i}(p) + \cdots + \tau^{-n}(p))). \tag{5.2}$$

It follows from (5.2) that $\operatorname{hilb} \overline{\widetilde{K}} = \operatorname{hilb} \overline{K} + s^i/(1-s)^2$. On the other hand, we showed $\operatorname{hilb} \widetilde{K} = \operatorname{hilb} K + s^i/(1-s)^3$ in part (1). This forces $\operatorname{hilb} \widetilde{K} = \operatorname{hilb}(\overline{\widetilde{K}})/(1-s)$ and thus $\widetilde{K}$ is g-divisible.

Further, considering the formula for $\overline{\widetilde{K}}_i$ from above (5.2), either $i = 1$ and $\overline{\widetilde{K}} = \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n(-D))$, or else $i = 2$ and $\overline{\widetilde{K}} = \bigoplus_{n \geq 0} H^0(E, \mathcal{N}_n(-D - \tau^{-1}(p)))$. In either case, $\overline{\widetilde{K}}$ is saturated as a B'-module. Thus $\widetilde{K}$ is MCM by Proposition 2.3.

(3) When $i = 1$ or 2, then $\overline{\widetilde{K}}$ is generated in degree 1 as a right $\overline{\widetilde{R}}$-module, by the calculations in part (2) and [15, Lemma 2.3]. Since $\widetilde{K}$ is g-divisible, $\widetilde{K}$ is therefore generated in degree 1 as a $\widetilde{R}$-module by the graded Nakayama lemma. When $i = 2$, $\widetilde{K}_1 = K_1$ by construction and so $\widetilde{K} = K_1\widetilde{R}$. ∎

By combining Lemma 5.1 with results from [15], we get the following useful characterizations of the intersection numbers of two distinct lines.

Lemma 5.3. *Let R be an elliptic algebra with* $\deg R \geq 3$. *Let $L \not\cong L'$ be line modules, with line ideals J and J', respectively. Assume that* $(L \bullet_{\mathrm{MS}} L) = -1$. *Then*

$$(L \bullet_{\mathrm{MS}} L') = 1 \iff \dim \operatorname{Hom}_R(J, J')_1 = \dim R_1 - 2 \iff \operatorname{Hom}_R(J, J')_1 = J'_1,$$

while

$$(L \bullet_{\mathrm{MS}} L') = 0 \iff \dim \operatorname{Hom}_R(J, J')_1 = \dim R_1 - 1 \iff \operatorname{Hom}_R(J, J')_1 \neq J'_1.$$

Proof. By [15, Lemma 5.6], line ideals are g-divisible, MCM, and generated in degree 1, and so we can apply Lemma 5.1 with $K = J'$. Since $L \not\cong L'$, we have $J' \neq J$ and so, in the notation of Lemma 5.1, $i = 1$ or $i = 2$. Examining the proof of that lemma, $\operatorname{hilb} \operatorname{Hom}_R(J, J')/J' = s^i/(1-s)^2$. Thus if $i = 1$ we have

$$\operatorname{hilb} \operatorname{Hom}_R(J, J') = \operatorname{hilb} R - 1/(1-s),$$

and so $\dim \operatorname{Hom}_R(J, J')_1 = \dim R_1 - 1 \neq \dim J'_1$. If $i = 2$ then

$$\operatorname{hilb} \operatorname{Hom}_R(J, J') = \operatorname{hilb} R - (1+s)/(1-s).$$

Thus $\dim \operatorname{Hom}_R(J, J')_1 = \dim R_1 - 2 = \dim(J')_1$ and so $\operatorname{Hom}_R(J, J')_1 = (J')_1$. This gives the second equivalence on each line.

Since qgr-R is smooth, J° and $(J')^\circ$ are projective by [15, Remark 7.2]. Thus by [15, Theorem 7.6], we have $(L \bullet_{\mathrm{MS}} L') = 1$ if and only if $\mathsf{hilb}\, R - \mathsf{hilb}\, \operatorname{Hom}_R(J, J') = (1+s)/(1-s)$. Since $(L \bullet_{\mathrm{MS}} L') \in \{0, 1\}$ in any case by [15, Lemma 7.4], this gives the first equivalence on each line. ∎

The following technical lemma will also help us in our analysis of intersection numbers of lines.

Lemma 5.4. *Let R be an elliptic algebra with line modules $L = R/J$, $L' = R/J'$, with $L \not\cong L'$. Let K be a reflexive graded right R-submodule of $Q_{\mathrm{gr}}(R)$. Suppose that* $\operatorname{Hom}_R(J, K)_1 \subseteq \operatorname{Hom}_R(J', K)_1$. *Then* $\operatorname{Hom}_R(J, K)_1 = K_1$.

Proof. Since $L \not\cong L'$, we have $J \neq J'$. Given $x \in \operatorname{Hom}_R(J, K)_1$, then $x(J + J') \subseteq K$. Let $I := J + J'$. Since R/J is a line module, it is 2-critical and so $J \neq J'$ forces GKdim $R/I \leq 1$. Then $xI \subseteq K$ implies that $\operatorname{GKdim}(xR + K)/K \leq 1$, but since K is reflexive, this forces $x \in K$ by [15, Lemma 4.5]. Consequently, we conclude that $\operatorname{Hom}_R(J, K)_1 \subseteq K_1$. The reverse inclusion is immediate. ∎

We now study the process of blowing down two lines in succession. Recall that $U \equiv V$ if U, V are graded vector spaces with $\mathsf{hilb}\, U = \mathsf{hilb}\, V$. Similarly, $U \equiv \mathsf{hilb}\, U$.

Lemma 5.5. *Fix an elliptic algebra R with $R/gR = B(E, \mathcal{M}, \tau)$. Let $L_p = R/J_p$ and $L_q = R/J_q$ be line modules for R with $(L_p \bullet_{\mathrm{MS}} L_p) = (L_q \bullet_{\mathrm{MS}} L_q) = -1$, where* $\operatorname{Div} L_p = p$ *and* $\operatorname{Div} L_q = q$. *Suppose further that $(L_p \bullet_{\mathrm{MS}} L_q) = 0$.*

(1) *Let $\widetilde{R}$ be the blowdown of R along the line L_p. Let $\widetilde{J}_q$ be the blowdown to $\widetilde{R}$ of the right ideal J_q. Then $\widetilde{L}_q = \widetilde{R}/\widetilde{J}_q$ is a line module over $\widetilde{R}$ with* $\operatorname{Div} \widetilde{L}_q = q$. *Moreover, $(\widetilde{L}_q \bullet_{\mathrm{MS}} \widetilde{L}_q) = -1$; in particular, we can blow down $\widetilde{L}_q$ starting from $\widetilde{R}$ to obtain a ring T.*

(2) *We have $(L_q \bullet_{\mathrm{MS}} L_p) = 0$ and so part* (1) *also applies with the roles of p and q reversed, leading to a ring T'. Then $T' = T$; thus the order in which one blows down the two lines is irrelevant.*

Proof. (1) Note that the conditions on the intersection numbers force $L_p \not\cong L_q$; thus $J_p \neq J_q$. Applying Lemma 5.1 with $J = J_p$ and $K = J_q$, as in the proof of Lemma 5.3, the condition $(L_p \bullet_{\mathrm{MS}} L_q) = 0$ means we are in the case $i = 1$. Then Lemma 5.1 shows that

$$\begin{aligned} \mathsf{hilb}\, \widetilde{J}_q &= \mathsf{hilb}\, J_q + s/(1-s)^3 = \mathsf{hilb}\, \widetilde{R} - 1/(1-s)^3 + s/(1-s)^3 \\ &= \mathsf{hilb}\, \widetilde{R} - 1/(1-s)^2. \end{aligned}$$

Thus $\widetilde{L}_q = \widetilde{R}/\widetilde{J}_q$ is a line module for $\widetilde{R}$ as claimed.

Note that $\overline{J_q} = \bigoplus_n H^0(E, \mathcal{M}_n(-q))$. Thus $\overline{\widetilde{J_q}} = \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n(-q))$, where $\overline{\widetilde{R}} = B' = B(E, \mathcal{N}, \tau)$ with $\mathcal{N} = \mathcal{M}(\tau^{-1}(p))$, as it was calculated in the proof of Lemma 5.1 (2). This shows that Div $\widetilde{L}_q = q$.

By [15, Corollary 9.2], $\widetilde{R}$ is again an elliptic algebra for which qgr-$\widetilde{R}$ is smooth. Next we want to show that $(\widetilde{L}_q \bullet_{\mathrm{MS}} \widetilde{L}_q) = -1$ which, by [15, Remark 7.2 and Theorem 7.1], is equivalent to $\mathrm{End}_{\widetilde{R}}(\widetilde{J}_q) \equiv \widetilde{R}$.

By [15, Lemma 2.3], we have

$$\overline{\mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)} \subseteq \mathrm{End}_{B'}(\overline{\widetilde{J}_q}, \overline{\widetilde{J}_q}) = B(E, \mathcal{N}(-q + \tau^{-1}(q)), \tau).$$

Since $\widetilde{J}_q$ is g-divisible by Lemma 5.1, $\mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)$ is also g-divisible, and it suffices to prove that the inclusion above is an equality. Since the TCR $B(E, \mathcal{N}(-q + \tau^{-1}(q)), \tau)$ is generated in degree 1, it is enough to show that

$$\dim_{\Bbbk} \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1 = \dim_{\Bbbk} \widetilde{R}_1 = \dim_{\Bbbk} R_1 + 1 = \dim_{\Bbbk} \mathrm{End}_R(J_q)_1 + 1.$$

Now we claim that $\mathrm{End}_R(J_q) \subseteq \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)$. Recalling that $\widetilde{J}_q = \mathrm{Hom}_R(J_p, J_q)R$, if $xJ_q \subseteq J_q$, then $x\,\mathrm{Hom}_R(J_p, J_q) \subseteq \mathrm{Hom}_R(J_p, J_q)$ since $x\,\mathrm{Hom}_R(J_p, J_q)J_p \subseteq xJ_q \subseteq J_q$. Therefore, $x\,\mathrm{Hom}_R(J_p, J_q)R \subseteq \mathrm{Hom}_R(J_p, J_q)R$ as well, proving the claim. Thus it suffices to show that $\mathrm{End}_R(J_q)_1 \neq \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1$.

Suppose, instead, that $\mathrm{End}_R(J_q)_1 = \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1$. Since $\widetilde{J}_q \subseteq \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)$, certainly

$$(\widetilde{J}_q)_1 = \mathrm{Hom}_R(J_p, J_q)_1 \subseteq \mathrm{End}_R(J_q)_1 = \mathrm{Hom}_R(J_q, J_q)_1.$$

Since we know that $J_p \neq J_q$, Lemma 5.4 gives $\mathrm{Hom}_R(J_p, J_q)_1 = (J_q)_1$. But then the hypothesis $(L_p \bullet_{\mathrm{MS}} L_q) = 0$ contradicts Lemma 5.3.

(2) We have $(L_p \bullet_{\mathrm{MS}} L_q) = (L_q \bullet_{\mathrm{MS}} L_p)$ by Corollary 2.14, so we can indeed apply part (1) with the roles of p and q reversed to produce a ring T'. We know that T and T' are elliptic of degree at least 3 and so generated as algebras in degree 1, so it suffices to prove that $T_1 = T'_1$. The argument of part (1) showed that $\mathrm{Hom}_R(J_p, J_q)_1 \not\subseteq \mathrm{End}_R(J_q)_1$. Since we saw that $\dim \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1 = \dim \mathrm{End}_R(J_q)_1 + 1$, this implies that

$$\mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1 = \mathrm{End}_R(J_q)_1 + \mathrm{Hom}_R(J_p, J_q)_1 = \mathrm{End}_R(J_q)_1 + (\widetilde{J}_q)_1.$$

Now, by [15, Theorem 8.3], since T is the blowdown of $\widetilde{R}$ along the line $\widetilde{L}_q = \widetilde{R}/\widetilde{J}_q$, we have $T = \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q, \widetilde{J}_q)\widetilde{R}$, so in particular $T_1 = \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1 + \widetilde{R}_1$. Similarly, $\widetilde{R}_1 = \mathrm{End}_R(J_p)_1 + R_1$. Thus

$$\begin{aligned} T_1 &= \widetilde{R}_1 + \mathrm{End}_{\widetilde{R}}(\widetilde{J}_q)_1 = \widetilde{R}_1 + (\widetilde{J}_q)_1 + \mathrm{End}_R(J_q)_1 = \widetilde{R}_1 + \mathrm{End}_R(J_q)_1 \\ &= R_1 + \mathrm{End}_R(J_p)_1 + \mathrm{End}_R(J_q)_1. \end{aligned}$$

A symmetric argument shows that T'_1 is equal to the same vector space. ■

We next ask when we can begin with a line ideal of an elliptic algebra R and blow down twice to obtain a right ideal with the properties of the right ideal $H = xS_2T$ in a Sklyanin elliptic algebra. Note that we do not assume that $\deg R = 7$.

Lemma 5.6. *Let R be an elliptic algebra with three line modules $L_p = R/J_p$, $L_q = R/J_q$, $L_r = R/J_r$, where* $\operatorname{Div} L_p = p$, $\operatorname{Div} L_q = q$, $\operatorname{Div} L_r = r$ *with $p \neq q$. Assume that*

(a) $(L_z \bullet_{\mathrm{MS}} L_z) = -1$ *for all $z \in \{p, q, r\}$;*

(b) $(L_p \bullet_{\mathrm{MS}} L_q) = 0$*; and*

(c) $(L_p \bullet_{\mathrm{MS}} L_r) = (L_q \bullet_{\mathrm{MS}} L_r) = 1$.

As in Lemma 5.5, blow down R along L_p to obtain a ring $\widetilde{R}$, and then blow down $\widetilde{R}$ along $\widetilde{L}_q$ to obtain a ring T. Let $K = (J_r)_1 T$. Then K is a g-divisible MCM right ideal of T with $\mathsf{hilb}\, K = \mathsf{hilb}\, T - (1+2s)/(1-s)^2$, $\operatorname{Div} T/K = r + \tau^{-1}(p) + \tau^{-1}(q)$*, and* $\operatorname{End}_T(K) \equiv T$.

Proof. We first show that $K = (J_r)_1 T$ is precisely the right ideal obtained by blowing down the right ideal J_r to the ring $\widetilde{R}$ and then blowing down the resulting right ideal to T. This will produce the desired Hilbert series for K automatically, and the value of $\operatorname{Div}(T/K)$ will also follow immediately.

By [15, Remark 7.2 and Theorem 7.6], hypothesis (c) is equivalent to

$$\mathsf{hilb}\, \operatorname{Hom}_R(J_p, J_r) = \mathsf{hilb}\, \operatorname{Hom}_R(J_q, J_r) = \mathsf{hilb}\, R - (1+s)/(1-s).$$

We have $\operatorname{Ext}^1_R(L_p, J_r) = \operatorname{Hom}_R(J_p, J_r)/J_r$ so $\mathsf{hilb}\, \operatorname{Ext}^1_R(L_p, J_r) = s^2/(1-s)^2$. In this case, J_r satisfies the hypotheses of Lemma 5.1 with $i = 2$. Thus if $\widetilde{R}$ is the blowdown of R along L_p, then we may apply Lemma 5.1 to obtain a right ideal $\widetilde{J}_r$ of $\widetilde{R}$, which satisfies $\mathsf{hilb}\, \widetilde{J}_r = \mathsf{hilb}\, \widetilde{R} - (1+s)/(1-s)^2$ and $\widetilde{J}_r = (J_r)_1 \widetilde{R}$. Moreover, Lemma 5.1 shows that $\widetilde{J}_r$ is again g-divisible, MCM, and generated in degree 1 as an $\widetilde{R}$-module. Write $\widetilde{L}_q = \widetilde{R}/\widetilde{J}_q$. If we now blow down $\widetilde{R}$ along $\widetilde{L}_q$ using Lemma 5.5, obtaining the ring T, then Lemma 5.1 again applies to blow down the right ideal $\widetilde{J}_r$ to a right ideal K of T. We have $\mathsf{hilb}\, \operatorname{Ext}^1_{\widetilde{R}}(\widetilde{L}_q, \widetilde{J}_r) = s^i/(1-s)^2$, for $i \in \{0, 1, 2\}$, or equivalently $\mathsf{hilb}\, \operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_q, \widetilde{J}_r) = \mathsf{hilb}\, \widetilde{R} + (s^i - s - 1)/(1-s)^2$. Now if $i = 0$ then $\widetilde{J}_q = \widetilde{J}_r$, which is not true since $\widetilde{J}_r$ is not a line ideal; so $i \in \{1, 2\}$.

We claim that $i = 2$. Suppose that $x \in \operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_q, \widetilde{J}_r)_1$. Since $(\widetilde{J}_q)_1 = \operatorname{Hom}_R(J_p, J_q)_1$ by Lemma 5.1 (i), $x \operatorname{Hom}_R(J_p, J_q)_1 \subseteq (\widetilde{J}_r)_2 = [\operatorname{Hom}_R(J_p, J_r)R]_2$. The assumption on the Hilbert series of $\operatorname{Hom}_R(J_p, J_r)$ implies that $\operatorname{Hom}_R(J_p, J_r)_1 = (J_r)_1$. Thus

$$[\operatorname{Hom}_R(J_p, J_r)R]_2 = \operatorname{Hom}_R(J_p, J_r)_2 + (J_r)_1 R_1 = \operatorname{Hom}_R(J_p, J_r)_2.$$

Hence $x \operatorname{Hom}_R(J_p, J_q)_1 J_p \subseteq J_r$. Let $I = \operatorname{Hom}_R(J_p, J_q)_1 J_p$, a right R-module contained in J_q. Looking at the images in $\overline{R}$, since $\overline{\operatorname{Hom}_R(J_p, J_q)_1} = H^0(E, \mathcal{M}(-q + \tau^{-1}(p)))$ and $\overline{J_p} = \bigoplus_n H^0(E, \mathcal{M}_n(-p))$, one sees that $\overline{I}_{\geq 2} = (\overline{J_q})_{\geq 2}$. Since J_q is g-divisible and $\overline{J_q}$ and $\overline{I}$ agree in large degree, this implies that $\operatorname{GKdim} J_q/I \leq 1$. But now since J_r is reflexive, by [15, Lemma 4.5] it follows that $xI \subseteq J_r$ implies that $xJ_q \subseteq J_r$. Thus $x \in \operatorname{Hom}_R(J_q, J_r)_1$. This proves that $\operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_q, \widetilde{J}_r)_1 = \operatorname{Hom}_R(J_q, J_r)_1$ and so the proof of Lemma 5.1 implies that $i = 2$ as claimed.

Thus by Lemma 5.1, again, $K = (\widetilde{J_r})_1 T = (J_r)_1 \widetilde{R} T = (J_r)_1 T$, and $\mathsf{hilb}\, K = \mathsf{hilb}\, T - (1+2s)/(1-s)^2$. Now $T/gT = B(E, \mathcal{N}, \tau)$ for $\mathcal{N} = \mathcal{M}(\tau^{-1}(p) + \tau^{-1}(q))$, and

$$\overline{K} = \overline{(J_r)_1 T} = H^0\big(E, \mathcal{N}(-r - \tau^{-1}(p) - \tau^{-1}(q))\big) \bigoplus_{n\geq 0} H^0(E, \mathcal{N}_n^{\tau})$$
$$= \bigoplus_{n\geq 1} H^0\big(E, \mathcal{N}_n(-r - \tau^{-1}(p) - \tau^{-1}(q))\big).$$

Thus $\operatorname{Div} T/K = r + \tau^{-1}(p) + \tau^{-1}(q)$. The right ideal K is also g-divisible and MCM by Lemma 5.1.

It remains to prove that $\operatorname{End}_T(K) \equiv T$. Since $K = (J_r)_1 T$, clearly $\operatorname{Hom}_T(K, K) = \operatorname{Hom}_R(J_r, K)$. Consider the exact sequence

$$0 \to \operatorname{End}_R(J_r) \to \operatorname{Hom}_R(J_r, K) \to \operatorname{Hom}_R(J_r, K/J_r) \to \operatorname{Ext}^1_R(J_r, J_r) \to \cdots.$$

Since $(L_r \bullet_{\mathrm{MS}} L_r) = -1$, it follows that $\operatorname{Ext}^1_R(J_r, J_r) = 0$ by [15, Theorem 7.1]. We have calculated $\overline{K}$ above, and it easily follows from [15, Lemma 2.3] that $\mathsf{hilb}\, \operatorname{End}_{\overline{T}}(\overline{K}) = \mathsf{hilb}\, \overline{T}$. Since K is g-divisible, $\operatorname{End}_T(K)$ is g-divisible; thus it suffices to prove that $\overline{\operatorname{End}_T(K)} = \operatorname{End}_{\overline{T}}(\overline{K})$. Since $\operatorname{End}_{\overline{T}}(\overline{K})$ is a TCR generated in degree 1, it will suffice to show that $\dim_{\Bbbk} \operatorname{End}_T(K)_1 \geq \dim_{\Bbbk} \operatorname{End}_R(J_r)_1 + 2$. Thus from the exact sequence above, we need to prove that $\dim_{\Bbbk} \operatorname{Hom}_R(J_r, K/J_r)_1 \geq 2$.

Now K/J_r contains the R-submodule $\widetilde{J_r}/J_r$, where $\mathsf{hilb}\, \widetilde{J_r}/J_r = s^2/(1-s)^3$ by Lemma 5.1. Then by [15, Lemma 8.2], $\widetilde{J_r}/J_r \cong \bigoplus_{i\geq 2} L_p[-i]$ as right R-modules. In particular, K/J_r contains a submodule isomorphic to $L_p[-2]$. Now we could have done all of part (1) by blowing down in the other order, as we saw in Lemma 5.5. In particular, if R' is the blowdown of R along L_q, then $(J_r)_1 R'$ is the blowdown of the right ideal J_r to the ring R'. Since T contains R' we see that K/J_r also contains $[(J_r)_1 R']/J_r$ which is isomorphic to $\bigoplus_{i\geq 2} L_q[-i]$. Thus K/J_r also contains a submodule isomorphic to $L_q[-2]$.

By hypothesis (c) and [15, Theorem 7.6],

$$\operatorname{Hom}_R(J_r, L_p[-2])_1 \neq 0 \neq \operatorname{Hom}_R(J_r, L_q[-2])_1.$$

Note that $0 \neq \theta \in \operatorname{Hom}_R(J_r, L_p[-2])_1$ provides a surjection $J_r[-1] \to L_p[-2]$, since both modules are generated in degree 2; following θ by the embedding of $L_p[-2]$ in K/J_r gives a nonzero element $\theta_p \in \operatorname{Hom}_R(J_r, K/J_r)_1$. Similarly, we get $0 \neq \theta_q \in \operatorname{Hom}_R(J_r, K/J_r)_1$ from $L_q[-2] \hookrightarrow K/J_r$. Now the image of θ_p is isomorphic to $L_p[-2]$ and the image of θ_q is isomorphic to $L_q[-2]$, while $L_p \not\cong L_q$ by hypothesis (b). Thus θ_p and θ_q are linearly independent. Thus $\dim_{\Bbbk} \operatorname{Hom}_R(J_r, K/J_r)_1 \geq 2$ as required. ∎

6. Recognition II: Two-point blowups of Sklyanin elliptic algebras

The goal of this section is to prove Theorem 1.4 from the introduction, which we state here in full detail.

Theorem 6.1. *Let R be a degree 7 elliptic algebra with $R/gR \cong B(E, \mathcal{M}, \tau)$ and such that* qgr-R *is smooth. Suppose that R satisfies the following:*

(1) *there are right line modules $L_p = R/J_p$, $L_q = R/J_q$ and $L_r = R/J_r$ satisfying:*
  (a) $(L_x \bullet_{\mathrm{MS}} L_x) = -1$ *for* $x \in \{p, q, r\}$*;*
  (b) $(L_p \bullet_{\mathrm{MS}} L_q) = 0$*;*
  (c) $(L_r \bullet_{\mathrm{MS}} L_y) = 1$ *for* $y \in \{p, q\}$*;*

(2) $p = \operatorname{Div} L_p \neq \operatorname{Div} q = L_q$.

Then $R \cong T(\tau^{-1}p + \tau^{-1}q)$, where T is the 3-Veronese of the Sklyanin algebra $S(E, \sigma)$ for some $\sigma \in \operatorname{Aut} E$ with $\sigma^3 = \tau$.

We begin, however, with a weaker recognition theorem for 2-point blowups which follows quickly from the results of the previous section and from Theorem 4.3.

Proposition 6.2. *Let R be an elliptic algebra that satisfies the hypotheses and notation of Theorem 6.1, with* $\operatorname{Div} L_r = r$*, and in addition assume that*

$$\mathcal{M}(\tau^{-1}p + \tau^{-1}q) \cong \mathcal{O}_E(\tau^{-1}p + \tau^{-1}q + \tau^{-1}r)^{\otimes 3}. \tag{6.3}$$

Then $R \cong T(\tau^{-1}p + \tau^{-1}q)$, where T is the 3*-Veronese of the Sklyanin algebra $S(E, \sigma)$ for some $\sigma \in \operatorname{Aut} E$ with $\sigma^3 = \tau$.*

Proof. All the hypotheses of Lemma 5.5 hold, and we can apply [15, Theorem 1.4] to successively blow down L_p and L_q to obtain an elliptic algebra T. By [15, Theorems 7.1 and 8.3], $R \cong T(\tau^{-1}p + \tau^{-1}q)$ is the iterated blowup of T at $\tau^{-1}p$ and $\tau^{-1}q$; thus $T/gT \cong B(E, \mathcal{N}, \tau)$ for $\mathcal{N} = \mathcal{M}(\tau^{-1}p + \tau^{-1}q)$. We will show that T is a Sklyanin elliptic algebra.

Lemma 5.6 shows that if $H = (J_r)_1 T$, then H is a g-divisible MCM right ideal of T with $\overline{H} = \bigoplus_{n \geq 0} H^0(E, \mathcal{N}_n(-a - b - c))$, where $a = r$, $b = \tau^{-1}p$ and $c = \tau^{-1}q$. Moreover, $\operatorname{End}_T(H) \equiv T$. Thus condition (i) of Notation 3.2 is satisfied.

By Corollary 2.14, the left line modules $L_p^\vee, L_q^\vee, L_r^\vee$ satisfy the same intersection theory as L_p, L_q, L_r; that is, they satisfy the hypotheses from Theorem 6.1 (i) (a)–(c). We can therefore use the left hand analogue of the first paragraph of this proof to blow down the left line modules $L_p^\vee$ and $L_q^\vee$. By [15, Theorem 8.3], this gives the same ring T. Further, $H^\vee = TJ_r^\vee$ is a left ideal of T with $\overline{H^\vee} = \bigoplus_{n \geq 0} H^0(E, \mathcal{N}_n(-\tau^{-n}d - \tau^{-n}e - \tau^{-n}f))$ for some d, e, f, and $\operatorname{End}_T(H^\vee) \equiv T$. Thus we also have condition (ii)′ of Notation 3.2. By Proposition 3.3 (2), $M = \operatorname{Hom}_T(H^\vee, T)$ satisfies condition (ii).

To compute d, e, f, note that by [15, Lemma 5.4],

$$\overline{(J_r^\vee)_1} = H^0(E, \mathcal{M}(-\tau^{-2}r)) = H^0(E, \mathcal{N}(-\tau^{-2}r - \tau^{-1}p - \tau^{-1}q)).$$

So we can take $d = \tau^{-1}r$, $e = p$, and $f = q$. Thus

$$\mathcal{O}_E(\tau a + b + c) = \mathcal{O}_E(\tau r + \tau^{-1}p + \tau^{-1}q) \cong \mathcal{O}_E(\tau^{-1}r + p + q) = \mathcal{O}_E(d + e + f)$$

and (4.2) is satisfied. Note that (4.1) follows directly from (6.3). Finally, by [15, Theorem 9.1], T is an elliptic algebra with qgr-T smooth. Thus all hypotheses of Theorem 4.3 are satisfied. By that theorem we have $T \cong S^{(3)}$, where $S = S(E, \sigma)$ is a Sklyanin algebra for some σ with $\sigma^3 = \tau$. ∎

We will now use the intersection theory developed in Section 2 to prove Theorem 6.1; in other words, we will show that the final condition (6.3) of Proposition 6.2 is superfluous.

For the rest of the section assume that R is a degree 7 elliptic algebra satisfying the hypotheses of Theorem 6.1. In addition, set $\mathcal{X} = \text{qgr-}R$, let $B = R/Rg \cong B(E, \mathcal{M}, \tau)$ and let $L = L_r$ and $J = J_r$. Let $r = \operatorname{Div} L$. Fix a group law $\oplus$ on E and let τ be given by translating by $t \in E$. We will show under these hypotheses that (6.3) is automatic, or, equivalently, that

$$\sum \mathcal{M} = 2p \oplus 2q \oplus 3r \ominus 7t. \tag{6.4}$$

We will prove this by constructing two different R-submodules of $Q_{\text{gr}}(R)$, which we will show are isomorphic. Factoring out g will give us (6.4).

To understand our strategy, consider for a moment the projective surface X which is the blowup of $\mathbb{P}^2$ at the points $p \neq q$. We use $\sim$ to denote linear equivalence of divisors on X. The exceptional lines L_p and L_q, together with the pullback H of a line in $\mathbb{P}^2$, form a basis for the divisor group $\operatorname{Div}(X)$. The third (-1) line on X is the strict transform L of the line through p and q, and we have $L \sim H - L_p - L_q$. Further, the canonical class $K = K_X$ satisfies $K \sim -3H + L_p + L_q \sim -3L - 2L_p - 2L_q$. Thus

$$\mathcal{O}_X(2L + L_p + L_q) \cong \mathcal{O}_X(-L - L_p - L_q - K). \tag{6.5}$$

We want to show that $R \cong T(\tau^{-1}p + \tau^{-1}q)$, so R deforms the anticanonical coordinate ring of X. The sheaves in (6.5) should therefore deform to give graded R-modules. We will construct in Lemma 6.7 an R-module Z that corresponds to $\mathcal{O}_X(2L + L_p + L_q)$, and in Lemma 6.9 a module Y corresponding to $\mathcal{O}_X(-L - L_p - L_q)$, and then show that $Z \cong Y[1]$.

We note the following easy result.

Lemma 6.6. *Let $M \subseteq R_{(g)}$ be reflexive, and let L be a shifted line module. If $0 \to M \to N \xrightarrow{\alpha} L \to 0$ is a nonsplit extension in* Gr-R*, then N is Goldie torsionfree and may be regarded as a submodule of $R_{(g)}$.*

Proof. This is similar to the proof of [15, Lemma 8.2 (1)]. Let Z be the Goldie torsion submodule of N. Since M is Goldie torsionfree, $Z \cap M = 0$ and so α gives an injection $Z \hookrightarrow L$. By definition, N/Z is Goldie torsionfree and there is an exact sequence $0 \to M \to N/Z \to L/\alpha(Z) \to 0$, which is necessarily nonsplit. If $Z \cong \alpha(Z)$ is nonzero, then as L is 2-critical, $\operatorname{GKdim} L/\alpha(Z) \leq 1$. But by [15, Lemma 4.5], $Q_{\text{gr}}(R)/M$ is 2-pure, a contradiction. Thus $Z = 0$.

Now we obtain that $M \to N$ is an essential extension in Gr-R; else there is a nonzero submodule P of N such that $P \cap M = 0$, so that P is isomorphic to a submodule of

$N/M \cong L$ and so is Goldie torsion, a contradiction. Thus we may regard N as a submodule of the graded quotient ring $Q_{\mathrm{gr}}(R)$ of R. Since $Q_{\mathrm{gr}}(R)/R_{(g)}$ is g-torsion, we get $N \subseteq R_{(g)}$. ∎

Lemma 6.7. *Let M be the line extension of L. There are right R-modules $M \subset N \subset Z \subset R_{(g)}$ so that N is MCM, $N/M \cong L_p \oplus L_q$, and $Z/N \cong L$.*

Proof. We first construct N. By definition of the line extension, we have an exact sequence of graded modules $0 \to R \to M \to L[-1] \to 0$. Let $y \in \{p, q\}$. By Lemma 2.15, $(L_y \bullet_{\mathrm{MS}} R) = 0$. Thus by assumption, Lemma 2.8, and Corollary 2.14, $(L_y \bullet_{\mathrm{MS}} M) = (L_y \bullet_{\mathrm{MS}} R) + (L_y \bullet_{\mathrm{MS}} L) = 1$. By definition,

$$(L_y \bullet_{\mathrm{MS}} M) = -\dim \mathrm{Hom}_{\mathcal{X}}(L_y, M) + \dim \mathrm{Ext}^1_{\mathcal{X}}(L_y, M) - \dim \mathrm{Ext}^2_{\mathcal{X}}(L_y, M)$$

and so $\mathrm{Ext}^1_{\mathcal{X}}(L_y, M) \neq 0$. By [15, Lemma 5.6 (3)], M is g-divisible and MCM. By Proposition 2.5, $\mathrm{Ext}^1_{\mathrm{gr}\text{-}R}(L_y, M) \neq 0$. Thus there is a nonsplit extension

$$0 \to M \to N_y \to L_y \to 0$$

of graded R-modules, and by Lemma 6.6 we have $N_y \subset R_{(g)}$.

By construction, $\overline{N_y} = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(\tau^{-1} r + y))$. Thus

$$\begin{aligned} \mathsf{hilb}\, \overline{N_y}/(1-s) &= (2+5s)/(1-s)^3 = (1+5s+s^2)/(1-s)^3 + (1+s)/(1-s)^2 \\ &= \mathsf{hilb}\, N_y, \end{aligned}$$

and so N_y is g-divisible.

Certainly, $M \subseteq N_p \cap N_q$; we claim that $N_p \cap N_q = M$. We prove this by induction on degree: to start we have $(N_p \cap N_q)_{-1} = M_{-1} = 0$. So suppose that $M_k = (N_p \cap N_q)_k$. As N_p and N_q are g-divisible, so is $N_p \cap N_q$: thus $(N_p \cap N_q)_{k+1} \cap gR_{(g)} = g(N_p \cap N_q)_k = gM_k$. Now,

$$\begin{aligned} \overline{(N_p \cap N_q)}_{k+1} &\subseteq (\overline{N_p} \cap \overline{N_q})_{k+1} \\ &= H^0(E, \mathcal{M}_{k+1}(\tau^{-1} r + p)) \cap H^0(E, \mathcal{M}_{k+1}(\tau^{-1} r + q)) \\ &= H^0(E, \mathcal{M}_{k+1}(\tau^{-1} r)), \end{aligned}$$

where the last equality is because $p \neq q$. Thus $\overline{(N_p \cap N_q)}_{k+1} \subseteq \overline{M}_{k+1}$ and as

$$(N_p \cap N_q)_{k+1} \cap gR_{(g)} = g(N_p \cap N_q)_k = gM_k = M_{k+1} \cap gR_{(g)}$$

we have $(N_p \cap N_q)_{k+1} \subseteq M_{k+1}$. This proves that $N_p \cap N_q = M$.

Now let $N = N_p + N_q$. We have

$$\mathsf{hilb}\, N/M = \mathsf{hilb}\, N_p + \mathsf{hilb}\, N_q - 2\, \mathsf{hilb}\, M = \mathsf{hilb}\, L_p \oplus L_q.$$

There is a surjection $N_p/M \oplus N_q/M \cong L_p \oplus L_q \twoheadrightarrow N/M$. Comparing Hilbert series, we get $L_p \oplus L_q \cong N/M$. Note that

$$\overline{N} = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(\tau^{-1}r + p + q)). \tag{6.8}$$

From (6.8) it is easy to see that $\operatorname{hilb} N = \operatorname{hilb} \overline{N}/(1-t)$, so that N is g-divisible. It is also clear from (6.8) that $\overline{N}$ is saturated, so N is MCM by Proposition 2.3.

As before, we have

$$(L \bullet_{\text{MS}} N) = (L \bullet_{\text{MS}} R) + (L \bullet_{\text{MS}} L) + (L \bullet_{\text{MS}} L_p) + (L \bullet_{\text{MS}} L_q) = 1$$

and so $\operatorname{Ext}^1_{\mathcal{X}}(L, N) = \operatorname{Ext}^1_{\text{gr-}R}(L, N) \neq 0$, using Proposition 2.5. Thus, as before, there is a nonsplit extension $0 \to N \to Z \to L \to 0$, and by Lemma 6.6, $Z \subset R_{(g)}$. ■

Lemma 6.9. *Let J be the line ideal of L. There is a MCM graded right ideal Y of R such that $Y \subseteq J$ with $J/Y \cong L_p[-1] \oplus L_q[-1]$.*

Proof. As we noticed in the proof of Proposition 6.2, the left line modules $L_p^\vee, L_q^\vee, L^\vee$ have the same intersection theory as the line modules L_p, L_q, L_r. We may therefore prove a left handed version of Lemma 6.7, to obtain MCM left modules $M^\vee$ and $N^\vee$ with $M^\vee/R \cong L^\vee[-1]$ and $N^\vee/M^\vee \cong L_p^\vee \oplus L_q^\vee$; in fact, $M^\vee = J^* = \operatorname{Hom}_R(J, R)$. Applying $\operatorname{Hom}_R(-, R)$ to the exact sequence $0 \to M^\vee \to N^\vee \to L_p^\vee \oplus L_q^\vee \to 0$ gives an exact sequence

$$0 \to (N^\vee)^* \to (M^\vee)^* \to (L_p \oplus L_q)[-1] \to 0,$$

where we have used that $\operatorname{Ext}^1_R(N^\vee, R) = 0$ as $N^\vee$ is MCM, and that $(L_x^\vee)^\vee \cong L_x$ by [15, Lemma 5.6].

Finally, noting that $(M^\vee)^* = J^{**} = J$, the lemma holds for $Y = (N^\vee)^*$. ■

Proposition 6.10. *Let Z, Y be as constructed in Propositions 6.7 and 6.9. Then $Z \cong Y[1]$ as right R-modules.*

Proof. Since $\operatorname{hilb} R = (1 + 5s + s^2)/(1-s)^3$, we first note that

$$\operatorname{hilb} Z = \operatorname{hilb} R + (3+s)/(1-s)^2 = (4+3s)/(1-s)^3,$$

and

$$\operatorname{hilb} Y = \operatorname{hilb} R - (1+2s)/(1-s)^2 = (4s+3s^2)/(1-s)^3.$$

Thus $\operatorname{hilb} Z = \operatorname{hilb} Y[1]$.

Now using the construction of Z and Y and Lemma 2.8, we compute:

$$\begin{aligned}(Z \bullet_{\text{MS}} Y[1]) = (R \bullet_{\text{MS}} R[1]) &+ ((L[-1] \oplus L \oplus L_p \oplus L_q) \bullet_{\text{MS}} R[1]) \\ &- (R \bullet_{\text{MS}} (L[1] \oplus L_p \oplus L_q)) \\ &- ((L \oplus L \oplus L_p \oplus L_q) \bullet_{\text{MS}} (L \oplus L_p \oplus L_q)).\end{aligned}$$

Note that we have used Corollary 2.14 (1) to remove the shifts in the final term.

We next want to compute $(R \bullet_{\mathrm{MS}} R[1])$. By Proposition 2.5, $\underline{\mathrm{Hom}}_{\mathcal{X}}(R, R) = R$ for $\mathcal{X} = \text{qgr-}R$ and $\underline{\mathrm{Ext}}^1_{\mathcal{X}}(R, R) = 0$. Thus, (2.11) gives

$$\dim \mathrm{Ext}^2_{\mathcal{X}}(R, R[1]) = \dim \mathrm{Hom}_{\mathcal{X}}(R[1], R[-1]) = 0$$

and hence

$$(R \bullet_{\mathrm{MS}} R[1]) = -\dim \mathrm{Hom}_{\mathcal{X}}(R, R[1]) - \dim \mathrm{Ext}^2_{\mathcal{X}}(R, R[1]) = -\dim R_1 = -8,$$

as desired. By Lemma 2.15,

$$((L[-1] \oplus L \oplus L_p \oplus L_q) \bullet_{\mathrm{MS}} R[1]) = 5 \qquad \text{and} \qquad (R \bullet_{\mathrm{MS}} (L[1] \oplus L_p \oplus L_q)) = -4.$$

Finally, our intersection theory assumptions give

$$((L \oplus L \oplus L_p \oplus L_q) \bullet_{\mathrm{MS}} (L \oplus L_p \oplus L_q)) = 2,$$

so $(Z \bullet_{\mathrm{MS}} Y[1]) = -8 + 5 + 4 - 2 = -1$. Thus $\mathrm{Hom}_{\mathcal{X}}(Z, Y[1]) \oplus \mathrm{Ext}^2_{\mathcal{X}}(Z, Y[1]) \neq 0$.

By Proposition 2.12 we have $\dim \mathrm{Ext}^2_{\mathcal{X}}(Z, Y[1]) = \dim \mathrm{Hom}_{\mathcal{X}}(Y[1], Z[-1])$. The module Z is certainly saturated (for example, since N and $Z/N \cong L$ are) and so $\mathrm{Hom}_{\mathcal{X}}(Y[1], Z[-1]) = \dim \mathrm{Hom}_R(Y[1], Z[-1])$ by Proposition 2.5 (1). Note that for all $n \geq 0$ we have $\dim Z_{n-1} < \dim Z_n$; this can be seen either by directly computing the formula for $\dim Z_n$ or by noting that the analogous property holds both for $\mathsf{hilb}\, R$ and $\mathsf{hilb}\, Z/R$. Thus $\dim Y[1]_n = \dim Z_n > \dim Z_{n-1} = \dim Z[-1]_n$ for all $n \geq 0$ and there are no degree 0 injective maps from $Y[1] \to Z[-1]$. Since Y and Z are both Goldie rank 1 and torsionfree, all nonzero maps must be injective, so $\mathrm{Ext}^2_{\mathcal{X}}(Z, Y[1]) = 0$. Thus $\mathrm{Hom}_{\mathcal{X}}(Z, Y[1]) = \mathrm{Hom}_R(Z, Y[1]) \neq 0$, where we have used Proposition 2.5 and that Y is saturated again.

As above, all nonzero elements of $\mathrm{Hom}_R(Z, Y[1])$ are injective, but as $\mathsf{hilb}\, Z = \mathsf{hilb}\, Y[1]$, any injective map must be an isomorphism. Thus $Z \cong Y[1]$ in gr-R. ∎

Proof of Theorem 6.1. We must show that given the hypotheses of Theorem 6.1, condition (6.4) follows. Let Z, Y be the modules constructed above. By construction,

$$Z/Zg = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(\tau^{-1} r + r + p + q))$$

and

$$Y[1]/Y[1]g = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_{n+1}(-r - \tau^{-1} p - \tau^{-1} q)^{\tau^{-1}}).$$

By Proposition 6.10, $\mathcal{M}(-r - \tau^{-1} p - \tau^{-1} q)^{\tau^{-1}} \cong \mathcal{O}(\tau^{-1} r + r + p + q)$; equivalently,

$$\sum \mathcal{M} \ominus r \ominus p \ominus q \oplus 6t = 2r \oplus p \oplus q \ominus t,$$

establishing (6.4). Thus by Proposition 6.2, we obtain the result. ∎

7. Transforming a noncommutative quadric surface to a noncommutative $\mathbb{P}^2$

Here we will prove Theorem 1.2 from the introduction: if one blows up a point on a smooth noncommutative quadric and then blows down two lines of self-intersection (-1), one obtains a noncommutative $\mathbb{P}^2$. As was discussed in the introduction, this is a noncommutative version of the standard commutative result (1.1). In this section, we will also assume that $\operatorname{char} \Bbbk = 0$. This is simply because [20] and [18, Section 10] make that assumption; we conjecture that our results hold in arbitrary characteristic.

For the reader's convenience we recall the definition of the Van den Bergh quadrics.

Example 7.1. Let $\mathbb{A}$ denote a 4-dimensional Sklyanin algebra; thus $\mathbb{A}$ is the $\Bbbk$-algebra with 4 generators $x_0, \dots, x_3$ and 6 relations

$$x_0x_i - x_ix_0 = \alpha_i(x_{i+1}x_{i+2} + x_{i+2}x_{i+1}), \quad x_0x_i + x_ix_0 = x_{i+1}x_{i+2} - x_{i+2}x_{i+1},$$

where $i \in \{1, 2, 3\}$ mod 3 and the α_i satisfy $\alpha_1\alpha_2\alpha_3 + \alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\{\alpha_i\} \cap \{0, \pm 1\} = \emptyset$. The ring $\mathbb{A}$ has a two-dimensional space of central homogeneous elements $V \subset \mathbb{A}_2$. The factor $\mathbb{A} / \mathbb{A}V \cong B(E, \mathcal{A}, \alpha)$, where E is an elliptic curve, with a line bundle $\mathcal{A}$ of degree 4, and α is an automorphism. We always assume that $|\alpha| = \infty$. Fix an arbitrary group law $\oplus$ on E. The automorphism α is then translation by a point in E, which will be denoted by a.

Given $0 \neq \Omega \in V$, *the Van den Bergh quadric* is $Q_{\mathrm{VdB}} = Q_{\mathrm{VdB}}(\Omega) = \mathbb{A} / \mathbb{A}\Omega$. Then qgr-$Q_{\mathrm{VdB}}(\Omega)$ is smooth for generic Ω, with a precise description of the smooth cases given by [18, Theorem 10.2]. We always assume below that Ω is chosen so that qgr-$Q_{\mathrm{VdB}}(\Omega)$ is smooth. As was discussed in Remark 1.7 and is stated explicitly in Corollary 7.4 below, this implies the birationality result for arbitrary quadrics.

Let $A = \mathbb{A} / \mathbb{A}\Omega$ as above. Fix a basis element $g \in A_2$ for the image of $Z(\mathbb{A})_2$ in A_2. As usual, we write $\overline{x}$ for the image of $x \in A$ under the map $A \mapsto A/gA = B(E, \mathcal{A}, \alpha)$. Note that $T' := A^{(2)}$ is an elliptic algebra, called *a quadric elliptic surface*. Moreover, qgr-$T' \simeq$ qgr-A is again smooth and $T'/gT' \cong B(E, \mathcal{A}, \alpha)^{(2)} \cong B(E, \mathcal{P}, \tau)$, where $\mathcal{P} = \mathcal{A} \otimes \mathcal{A}^{\alpha}$ and $\tau = \alpha^2$.

Basic Facts 7.2. We recall some facts about $A = Q_{\mathrm{VdB}}$, mostly drawn from [18]. Identify $A_1 = \mathbb{A}_1 = H^0(E, \mathcal{A})$. Given an effective divisor D on E, set $V(D) = H^0(E, \mathcal{A}(-D)) \subseteq A_1 = \mathbb{A}_1$. For any point $p \in E$, $A/V(p)A$ is a point module for A. These are usually the only point modules for A, although for a discrete family of Ω there will exist one extra point module. These extra modules are described, for example, in [16, Lemma 6.6] and will play no further rôle in the present discussion. An analogous result holds on the left, and $V(p)A_1 = \{x \in A_2 \mid \overline{x} \in H^0(E, \mathcal{A}_2(-p))\} = A_1V(\alpha p)$ (use [15, Remark 3.2]).

Similarly, let L be a line module over $\mathbb{A}$. Then there are points p, t on E so that $L = \mathbb{A} / V(p+t)\mathbb{A}$; and any two points $p, t \in E$ give a line module (see [18, (10.3)]). There are two points $z, z' \in E$ so that $p \oplus t \in \{z, z'\} \Leftrightarrow \Omega \in V(p+t)\mathbb{A}$ (this is proved for left line modules in [18, (10.3)], but by [15, Remark 3.2], again, the same result holds for right line modules). Thus for fixed Ω, t, there are two points $p, q \in E$ so that $y \in \{p, q\} \iff \Omega \in V(y+t)\,\mathbb{A}$. By [18, Theorem 10.2], $p \neq q \iff z \neq z' \iff$ qgr-A has finite homological dimension; thus this will always be the case under our assumption on Ω. If $\Omega \in V(y+t)\,\mathbb{A}$ then $A/V(y+t)A \cong \mathbb{A}\,/\,V(y+t)\,\mathbb{A}$ is also a line module over A. In particular,

if $y \in \{p, q\}$, then $A/V(y+t)A$ is a line module,

and we write $I_y = V(y+t)A$ for the corresponding line ideal of A. Using [18, (10.3)] and [15, Remark 3.2], one can even describe q in terms of p, which we leave to the interested reader.

The rest of this section is devoted to the proof of the following theorem.

Theorem 7.3. *Assume that* char $\Bbbk = 0$ *and let $T' = Q^{(2)}$ for a Van den Bergh quadric Q such that* qgr-Q *is smooth. Keep the notation, and in particular the point $t \in E$ (which determines $\{p, q\} \subset E$) from Basic Facts 7.2.*

Then there exists $\sigma \in \mathrm{Aut}_{\Bbbk}(E)$ with $\sigma^3 = \tau$ so that $T'(t) \cong T(\tau^{-1}p + \tau^{-1}q)$, where $T \cong S^{(3)}$ is the 3*-Veronese of the Sklyanin algebra $S = S(E, \sigma)$.*

As was noted in Corollary 1.8, this theorem can also be applied to quadrics Q for which qgr-Q is not smooth. More precisely, combining the theorem with the discussion from Remark 1.7 gives the following corollary.

Corollary 7.4. *Any Van den Bergh quadric Q defined over a field of characteristic zero is birational to a Sklyanin algebra.* ■

We will prove Theorem 7.3 by appealing to Theorem 6.1, so the proof will be through a series of lemmata to show that $R = T'(t)$ satisfies the hypotheses of that result. We note that $R_1 = V(t)A_1$, since point ideals of $\mathbb{A}$ are generated in degree 1. To match the notation of Theorem 6.1, let $\mathcal{M} = \mathcal{P}(-t)$, so $\overline{R} = B(E, \mathcal{M}, \tau)$.

We first construct the three lines on R. Two of the lines are induced from the two rulings on A. For $y \in \{p, q\}$, let $K_y = (I_y)^{(2)} = V(y+t)A_1T'$.

Lemma 7.5. *Let $y \in \{p, q\}$ and let $J_y := (K_y)_1R$, for $R = T'(t)$. Then J_y is a line ideal of R.*

Proof. The proof is similar to the proof of [12, Theorem 5.2]. We consider $J_y = V(y+t)A_1T'(t)$ as a subspace of $A^{(2)}$ which depends on t, and still write $R = T'(t)$. (Note that since we always have $y \oplus t \in \{z, z'\}$, therefore y also varies with t.) By [12, Lemma 3.1],

$$\overline{J_y} = \overline{V(y+t)A_1}\,\overline{R} = \bigoplus_{n \geq 0} H^0(E, \mathcal{M}_n(-y)) \tag{7.6}$$

is a point ideal of $\overline{R}$, and so

$$\mathsf{hilb}\, J_y \geq \mathsf{hilb}\, R - 1/(1-s)^2 = \mathsf{hilb}\, T' - s/(1-s)^3 - 1/(1-s)^2,$$

with equality if and only if J_y is g-divisible.

Suppose now that $y \neq \tau^{-j}t$ for any $j \geq 0$, that is, $2t \not\in \{\tau^j z, \tau^j z' \mid j \in \mathbb{Z}\}$. Let $K = K_y$ and let $J = J_y = K_1 R$. We claim for all $n \geq 0$ that, first, $K_n \cap R_n = J_n$, and, second, this has codimension $n+1$ in R_n. Both are trivial for $n = 0, 1$. So assume it is true for n. Note that K is g-divisible (since T'/K is g-torsionfree), as is R. By induction, then, we have $K \cap R_{n+1} \cap gT' = gK \cap gR_n = gJ_n$. By induction, this is codimension $n+1$ in gR_n.

Working modulo g, we have

$$H^0(E, \mathcal{M}_{n+1}(-y)) = \overline{K_{n+1}} \cap \overline{R_{n+1}} \supseteq \overline{K_{n+1} \cap R_{n+1}} \supseteq \overline{J_{n+1}} = H^0(E, \mathcal{M}_{n+1}(-y)).$$

Thus all are equal, and this vector space clearly has codimension 1 in $\overline{R_{n+1}}$. Since $J \subseteq K \cap R$, the claim is proved.

Thus for fixed $n \geq 1$, we have $\dim(J_y)_n = \dim T'_n - \binom{n+1}{2} - (n+1)$ for a dense set of $t \in E$. By lower semi-continuity, $\mathsf{hilb}\, J_y \leq \mathsf{hilb}\, T' - s/(1-s)^3 - 1/(1-s)^2$ for all $t \in E$. Combined with the first paragraph, this proves the lemma. ■

We thus obtain two line modules $L_y := R/J_y$ for $y \in \{p, q\}$ for R coming from the two rulings on A. Since (7.6), respectively Basic Facts 7.2, implies that $\operatorname{Div} L_y = y$, respectively $z \neq z'$, *hypothesis (2) of Theorem 6.1 holds.*

Let $r = \tau(t)$. The third line module comes from the blowup $R = T'(t) \subseteq T'$. By construction this produces an exceptional line module $L = L_r$ such that $(R + T'_1 R)/R \cong L[-1]$ and $(R + RT'_1)/R \cong L^\vee[-1]$, where the divisor of L_r is r, as the notation suggests. See [15, Theorems 8.3 and 8.6] for the details. Write $L_r = R/J_r$. Then $J_r = (R + RT'_1)^* := \operatorname{Hom}_R(R + RT'_1, R)$ by [15, Lemma 5.6]. For future reference we note that [15, Lemma 5.6] also implies that $J^* = R + RT'_1$. The line ideal J_r also has the following explicit description.

Lemma 7.7. *The equation $J_r = V(r)V(\alpha^{-1}r)R$ holds.*

Proof. Certainly,

$$\begin{aligned} T'_1 V(r)V(\alpha^{-1}r)R &= A_1 A_1 V(r)V(\alpha^{-1}r) = V(\alpha^{-2}r)A_1 V(\alpha^{-2}r)A_1 R \\ &= V(t)A_1 V(t)A_1 R \subseteq R. \end{aligned}$$

Thus $V(r)V(\alpha^{-1}r)R \subseteq J_r$. On the other hand,

$$\overline{V(r)V(\alpha^{-1}r)R} = \bigoplus_{n \geq 1} H^0(E, \mathcal{M}_n(-\alpha^{-2}(r))),$$

which has Hilbert series $\mathsf{hilb}\, R - 1/(1-t)$. Thus $\mathsf{hilb}\, V(r)V(\alpha^{-1}r)R \geq \mathsf{hilb}\, {-1/(1-t)^2}$. Since J_r is a line ideal, we conclude that $J_r = V(r)V(\alpha^{-1}r)R$ as claimed. ■

Lemma 7.8. *The category* qgr-R *is smooth and* $(L_r \bullet_{\mathrm{MS}} L_r) = -1$.

Proof. Set $J = J_r$; thus $J^* = R + RT'_1$ from the discussion above. Hence

$$JJ^* \supseteq J_1 T'_1 = V(r)A_1 V(r)A_1 = T'(r)_2.$$

Recall that $N^\circ = N[g^{-1}]_0$ for an R-module N. Thus $J^\circ(J^*)^\circ = T'(r)^\circ$. Thus J° is projective by the Dual Basis Lemma. By [15, Theorem 8.6], $\mathrm{End}_R(J) = T'(r)$ and so $(L_r \bullet_{\mathrm{MS}} L_r) = -1$ follows from [15, Theorem 7.1]. By our choice of Ω, qgr-$A \simeq$ qgr-T' has finite homological dimension. As $\mathrm{pdim}\, L_r^\circ = 1$, it follows that qgr-$R$ is smooth by [15, Theorem 9.1]. ∎

The next result verifies hypothesis (1) (c) of Theorem 6.1.

Lemma 7.9. *Let $y \in \{p, q\}$. Then $(L_r \bullet_{\mathrm{MS}} L_y) = 1$ and $\mathrm{Hom}_R(J_r, J_y) \equiv \mathsf{hilb}\, R - \frac{1+s}{1-s}$.*

Proof. First, note that $(J_y)_1 T' = K_y$ is g-divisible, while $(J_r)_1 T' \supseteq (J_r)_1 T'_1 T' = T'(r)_2 T' \supseteq g^2 T'$ (as calculated in Lemma 7.8) and so $(J_r)_1 T'$ is not. Thus $J_r \neq J_y$ and so $L_r \not\cong L_y$. By Lemma 5.1, $\mathrm{Ext}^1_R(L_r, J_y) \equiv s^i/(1-s)^2$ for $i \in \{0, 1, 2\}$, where $i \neq 0$ since $J_r \neq J_y$.

Let $\widetilde{J}_y$ be the right ideal of T' given by applying Lemma 5.1 to blow down J_y at L_r. By that lemma,

$$\widetilde{J}_y = \mathrm{Hom}_R(J_r, J_y)R \subseteq \mathrm{Hom}_R(J_r, R)R = T'.$$

(For the final equality, see [15, Lemma 8.2].) If $i = 1$, then from Lemma 5.1, $T'/\widetilde{J}_y \equiv 1/(1-s)^2 \equiv R/J_y$. As T' has no left line modules by [16, Lemma 6.14], this is impossible. So $i = 2$.

Now as we saw in the proof of Lemma 5.1, $i = 2$ implies that

$$\begin{aligned} s^2/(1-s)^2 &\equiv \mathrm{Ext}^1_R(L_r, J_y) \equiv \mathsf{hilb}\,\mathrm{Hom}(J_r, J_y)/J_y \\ &= \mathsf{hilb}\,\mathrm{Hom}(J_r, J_y) - (\mathsf{hilb}\, R - 1/(1-s)^2) \end{aligned}$$

since J_y is a line ideal in R. Thus $\mathsf{hilb}\,\mathrm{Hom}(J_r, J_y) = \mathsf{hilb}\, R - (1+s)/(1-s)$. Since qgr-$R$ is smooth and $(L_r \bullet_{\mathrm{MS}} L_r) = -1$ by Lemma 7.8, it therefore follows from [15, Remark 7.2 and Theorem 7.6] that $(L_r \bullet_{\mathrm{MS}} L_y) = 1$. ∎

Corollary 7.10. *Let $y \in \{p, q\}$. Then $\mathrm{Hom}_R(J_y, L_r) \equiv s^{-1}/(1-s)^2$.*

Proof. By Corollary 2.14 and Lemma 7.9, $(L_y \bullet_{\mathrm{MS}} L_r) = 1$. Now use Lemma 7.8 combined with [15, Remark 7.2 and Theorem 7.6]. ∎

Together with Lemma 7.8, the next result verifies hypothesis (1) (a) of Theorem 6.1.

Lemma 7.11. *Let $y \in \{p, q\}$. Then $(L_y \bullet_{\mathrm{MS}} L_y) = -1$.*

Proof. By [15, Theorem 7.1] and Lemma 7.8, it suffices to show that $\mathsf{hilb}\,\mathrm{End}_R(J_y) \equiv \mathsf{hilb}\,R$. We saw in the proof of Lemma 7.9 that $\mathrm{Ext}^1_R(L_r, J_y) \equiv s^2/(1-s)^2$. Recall that $K_y = (I_y)^{(2)}$, so that K_y is generated in degree 1 as a right ideal of T'. By Lemma 7.5, $(J_y)_1 = (K_y)_1$ and $J_y = (J_y)_1 R$. Then applying Lemma 5.1 to blow down J_y along L_r, we obtain $\widetilde{J_y} = (J_y)_1 T' = K_y$, since $i = 2$. By [15, Lemma 8.2 (2) (4)], $K_y/J_y \cong \bigoplus_{i\geq 2} L_r[-i]$ and $\mathrm{Ext}^1_R(L_r, K_y) = 0$.

As K_y/J_y is Goldie torsion we have the exact sequence

$$0 \to \mathrm{End}_R(K_y) \to \mathrm{Hom}_R(J_y, K_y) \to \bigoplus_{i\geq 2} \mathrm{Ext}^1_R(L_r[-i], K_y), \tag{7.12}$$

and the final term in (7.12) is zero by the above paragraph. Thus $\mathrm{Hom}_R(J_y, K_y) = \mathrm{End}_R(K_y) = \mathrm{End}_{T'}(K_y)$. This is the 2-Veronese of $\overline{\mathrm{End}_A(I_y)}$, and it follows from [20] (see [16, Lemma 5.7] for the explicit statement) that $\overline{\mathrm{End}_{T'}(K_y)}$ is the TCR of a degree 8 line bundle on E and that $\mathrm{End}_{T'}(K_y) \equiv T'$. In particular, $\dim_{\Bbbk} \mathrm{End}_{T'}(K_y)_1 = 9$.

We also have the exact sequence

$$0 \to \mathrm{End}_R(J_y) \to \mathrm{Hom}_R(J_y, K_y) \to \bigoplus_{i\geq 2} \mathrm{Hom}_R(J_y, L_r[-i]). \tag{7.13}$$

By Corollary 7.10, $\mathsf{hilb} \bigoplus_{i\geq 2} \mathrm{Hom}_R(J_y, L[-i]) = s/(1-s)^3$, and so (7.13) induces an exact sequence

$$0 \to \mathrm{End}_R(J_y)_1 \to \mathrm{End}_R(K_y)_1 \to \Bbbk.$$

Consequently, $\dim_{\Bbbk} \mathrm{End}_R(J_y)_1$ is 8 or 9. On the other hand, by (7.6), $\mathrm{End}_B(\overline{J_y}) = B(E, \mathcal{M}(-y + \tau^{-1}(y)), \tau)$, and so

$$\dim_{\Bbbk} \mathrm{End}_R(J_y)_1 \leq 1 + \dim_{\Bbbk} \mathrm{End}_B(\overline{J_y})_1 = 8.$$

Thus $\dim_{\Bbbk} \mathrm{End}_R(J_y)_1 = 8$ and so $(\overline{\mathrm{End}_R(J_y)})_1 = \mathrm{End}_B(\overline{J_y})_1$. Since $\mathrm{End}_B(\overline{J_y})$ is a TCR, it follows that $\overline{\mathrm{End}_R(J_y)} = \mathrm{End}_B(\overline{J_y})$. Finally, as $\mathrm{End}_R(J_y)$ is g-divisible and $\mathsf{hilb}\,\overline{\mathrm{End}_R(J_y)} = \mathsf{hilb}\,\overline{R}$, it follows that $\mathsf{hilb}\,\mathrm{End}_R(J_y) \equiv \mathsf{hilb}\,R$, as required. ∎

The next result proves hypothesis (1) (b) of Theorem 6.1.

Lemma 7.14. *Let $y \neq x \in \{p, q\}$. Then $(L_y \bullet_{\mathrm{MS}} L_x) = 0$.*

Proof. By [15, Lemma 7.4], $(L_y \bullet_{\mathrm{MS}} L_x) \in \{0, 1\}$. By [15, Theorem 7.7], $(L_y \bullet_{\mathrm{MS}} L_x) = 0 \iff \mathsf{hilb}\,\mathrm{Hom}_R(J_y, J_x) = h_R - \frac{1}{1-s}$. Similarly, as $L_x \not\cong L_y$, [15, Theorem 7.6] implies that $(L_y \bullet_{\mathrm{MS}} L_x) = 1 \iff \mathsf{hilb}\,\mathrm{Hom}_R(J_y, J_x) = h_R - \frac{1+s}{1-s}$. Thus it suffices to prove that $\dim_{\Bbbk} \mathrm{Hom}_R(J_y, J_x)_1 \geq \dim R_1 - 1 = 7$. But

$$V(x+s)V(\alpha^{-1}s)(K_y^*)_0(J_y)_1 \subseteq V(x+s)V(\alpha^{-1}s)T_1' = V(x+s)A_1V(s)A_1 = (J_x)_2,$$

and so $\mathrm{Hom}_R(J_y, J_x)_1 \supseteq V(x+s)V(\alpha^{-1}s)(K_y^*)_0$. By the discussion after [16, (6.13)], $\dim(K_y^*)_0 = 2$. Since $\overline{V(x+s)V(\alpha^{-1}s)(K_y^*)_0}$ is a product of spaces of global sections, it is therefore easy to compute that its dimension is $2 + 3 + 2 = 7$. The result follows. ∎

Proof of Theorem 7.3. We have proved that R satisfies all of the hypotheses of Theorem 6.1, from which Theorem 7.3 follows. ∎

8. A converse result

In this short section, we prove Proposition 1.5 from the introduction, thereby giving the converse to Theorem 1.4.

We will need some preliminary results. The following result gives a method for proving that a right ideal is a line ideal, without explicitly calculating its Hilbert series.

Lemma 8.1. *Let R be an elliptic algebra. Suppose that $I \subseteq R$ is a right ideal and that $\overline{I}$ is a point ideal in $B = R/gR$. If $I^* \neq R$, then I is a line ideal. In particular, I is g-divisible.*

Proof. We first show that I is g-divisible. So, let $J = \tilde{I} = \{x \in R \mid xg^n \in I \text{ for some } n \geq 1\}$ be the g-divisible hull of I and suppose that $I \subsetneq J$. We first claim that $\overline{J} \supsetneq \overline{I}$. Indeed, pick a homogeneous element $a \in J \smallsetminus I$ of minimal degree. If $\overline{J} = \overline{I}$, then $a = b + gc$, where $b \in I$ with $\deg b \leq \deg a$. Since $gc \in J$ and J is g-divisible, $c \in J$. The minimality of $\deg a$ implies that $c \in I$ and hence $a \in I$, giving the required contradiction. Hence $\overline{J} \supsetneq \overline{I}$. As $\overline{I}$ is a point ideal, $\overline{R}/\overline{I}$ is 1-critical and so $\overline{R}/\overline{J}$ is finite-dimensional. As $gR \cap J = gJ$, it follows that GKdim $R/J \leq 1$. Since R is CM (see Notation 2.1), this implies that $J^* = R$.

As $I^* \neq R$, we may pick a homogeneous element $x \in Q_{\mathrm{gr}}(R) \smallsetminus R$ such that $xI \subseteq R$. As J is finitely generated, $g^m J \subseteq I$ and hence $xg^m J \subseteq R$ for some $m \geq 1$. Since $J^* = R$, we conclude that $xg^m \in R$. Thus $x = yg^{-r}$ for some $y \in R$ and $r \geq 1$, where we may also assume that $y \notin gR$. However, this implies that $yI = g^r xI \subseteq g^r R \subseteq gR$. Since $y \notin gR$ and gR is a completely prime ideal, it follows that $I \subseteq gR$, contradicting the fact that $\overline{I} \neq 0$. This proves that I is indeed g-divisible.

Finally, as $gI = gR \cap I$ and $\mathsf{hilb}\, \overline{R}/\overline{I} = (1-s)^{-1}$ it follows that $\mathsf{hilb}\, R/I = (1-s)^{-2}$, as required. ∎

Let $S = S(E, \sigma)$ be a Sklyanin algebra, where $|\sigma| = \infty$, and write

$$S/(g) = B(E, \mathcal{L}, \sigma) =: B$$

for some degree 3 invertible sheaf $\mathcal{L}$ on E.

Definition 8.2. Let $X \subseteq S_m$ be a subspace. We say that X is *defined by vanishing conditions on E* if $\overline{X} = H^0(E, \mathcal{L}_m(-p_1 - \cdots - p_n))$ for some $p_1, \ldots, p_n \in E$, and $X \supseteq gS_{m-3}$. Note that if X is defined by vanishing conditions on E, where $n < \deg \mathcal{L}_m$, then $\dim_{\Bbbk} X$ is immediately determined to be $\dim_{\Bbbk} S_m - n = \binom{m+2}{2} - n$.

For $a, b_1, \ldots, b_n \in E$ write

$$W(a) = H^0(E, \mathcal{L}(-a)) \subset S_1$$

and

$$V(b_1 + \cdots + b_n) = H^0(E, \mathcal{L}_2(-b_1 - \cdots - b_n)) \subseteq S_2.$$

Recall from [12, Lemma 4.1] that $S_1 W(a) = W(\sigma^{-1}a)S_1$, a fact that will be used without further comment. Similarly, we will use [12, Lemma 3.1] to compute products in B without comment.

We next investigate products of the spaces above and when they are defined by vanishing conditions. We say $a, b, c \in E$ are *collinear* if there is an $x \in S_1$ that vanishes at a, b, c; equivalently, if $\mathcal{L} \cong \mathcal{O}_E(a + b + c)$.

Lemma 8.3. *Let $a, b, c \in E$.*

(1) *If $c \neq \sigma^{-2}b$, then $W(b)W(c)$ and $W(b)W(c)S_1$ are defined by vanishing conditions on E; in particular, $W(b)W(c) = V(b + \sigma^{-1}c)$,* $\dim W(b)W(c) = 4$, *and* $\dim W(b)W(c)S_1 = 8$. *On the other hand,* $\dim W(b)W(\sigma^{-2}b) = 3$ *and* $\dim W(b)W(\sigma^{-2}b)S_1 = 7$.

(2) *$V(b + c)S_1 = S_1 V(\sigma b + \sigma c)$, and this space is defined by vanishing conditions on E. Moreover, $V(a + b + c)S_1$ is defined by vanishing conditions on E if and only if a, b, c are not collinear, while $S_1 V(a + b + c)$ is defined by vanishing conditions on E if and only if σa, σb, σc are not collinear.*

(3) *$W(a)V(b + c)$ is defined by vanishing conditions on E if and only if $\sigma^{-2}a \notin \{b, c\}$; otherwise* $\dim W(a)V(b + c) = 6$. *Similarly $V(a + b)W(c)$ is defined by vanishing conditions on E if and only if $\sigma c \notin \{a, b\}$, else* $\dim V(a + b)W(c) = 6$.

Proof. (1) The dimensions of these spaces are given in [12, Lemmata 4.1 and 4.6] while, from the proof of [12, Lemma 4.1], $W(b)W(c)$ is defined by vanishing conditions. The other claims follow easily.

(2) The first sentence follows from (1) once one notes that at least one of $V(b + c) = W(b)W(\sigma c)$ or $V(b + c) = W(c)W(\sigma b)$ must hold.

For the second sentence, we prove the first claim, as the other follows symmetrically. If a, b, c are collinear then $V(a + b + c) = xS_1$ for some $x \in S_1$. Thus $V(a + b + c)S_1 = xS_2$ and $\dim V(a + b + c)S_1 = 6$; in particular, this space is not equal to $\{x \in S_3 \mid \overline{x} \in H^0(E, \mathcal{L}_3(-a - b - c))\}$ which has dimension 7.

It therefore suffices to prove that $\dim V(a + b + c)S_1 = 7$ if a, b, c are not collinear. Indeed, since $\dim \overline{V(a + b + c)S_1} = 6$, it is enough to show that $g \in V(a + b + c)S_1$.

Let $V = V(a + b + c)$ and let $d \in E \smallsetminus \{\sigma^{-1}a, \sigma^{-1}b, \sigma^{-1}c\}$. We claim that $g \in VW(d)$. To see this, write $W(d) = \Bbbk x + \Bbbk y$ where $wy + xz = 0$ and $\{w, x\}$ is a basis of $W(\sigma^2 d)$. Then $Vy \cap Vz = Ywy$ where

$$Y = \{r \in S_1 \mid rW(\sigma^2 d) \subseteq V\}.$$

Since $\sigma d \notin \{a, b, c\}$, clearly $Y = W(a) \cap W(b) \cap W(c) = 0$, since a, b, c are not collinear. Thus $\dim VW(d) = 6$. As $\overline{VW(d)} = H^0(E, \mathcal{L}_3(-a - b - c - \sigma^{-2}d))$, which has dimension 5, we have $g \in VW(d) \subset VS_1$, as required.

(3) By symmetry it suffices to determine $\dim W(a)V(b + c)$, for which we follow the proof of [12, Lemma 4.6]. Let $\{w, x\}$ be a basis of $W(a)$ and let $\{y, z\}$ be a basis of

$W(\sigma^{-2}a)$ so that $wy + xz = 0$. Then $wS \cap xS = wyS = xzS$ and so

$$wV(b+c) \cap xV(b+c) = wyY \qquad \text{for } Y = \{r \in S_1 \mid W(\sigma^{-2}a)r \subseteq V(b+c)\}.$$

If $\sigma^{-2}a \in \{b, c\}$ then without loss of generality $b = \sigma^{-2}a$ and $Y = W(\sigma c)$. In this case

$$\dim W(a)V(b+c) = 2\dim V(b+c) - \dim Y = 6.$$

Otherwise, $\dim Y = 1$ and $\dim W(a)V(b+c) = 7$. ■

We can now prove the converse to Theorem 1.4.

Proposition 8.4. *Let T be a Sklyanin elliptic algebra with associated elliptic curve E, and let $p \neq q \in E$. Then $R := T(p+q)$ satisfies the hypotheses of Theorem 1.4.*

Proof. Certainly, R is a degree 7 elliptic algebra. We now change our earlier notation and write L_p for the exceptional line module obtained from writing R as the blowup at p of $T(q)$, with line ideal J_p. This differs from the notation in the earlier sections since now, by [12, Lemma 9.1], $\operatorname{Div} L_p = \tau(p)$.

Define $L_q = R/J_q$ analogously. Write $T = S^{(3)}$ where S is a Sklyanin algebra with $S/(g) = B(E, \mathcal{L}, \sigma)$. Let $\sigma^3 = \tau$ and write $T/(g) = B(E, \mathcal{M}, \tau)$. By our standing conventions on elliptic algebras, τ and hence also σ have infinite order.

Let $x \in S_1$ generate $H^0(E, \mathcal{L}(-p-q))$ and let $I_x = xS_2R \subseteq R$. It is easy to calculate that $\overline{I_x}$ is a point ideal in B, associated to the third point of E where x vanishes. We claim that I_x is a line ideal in R. To see this, define $U = V(p+q)V(\sigma^{-1}p + \sigma^{-1}q) \subseteq S_4$ and note that

$$US_2 = V(p+q)V(\sigma^{-1}p + \sigma^{-1}q)S_2 = V(p+q)S_1V(p+q)S_1 = R_2,$$

by Lemma 8.3 (2). Then $(Ux^{-1})(I_x)_1 = Ux^{-1}xS_2 = US_2 \subseteq R$. Moreover, if $Ux^{-1} \subseteq R_1$ were to hold, then $U \subseteq R_1x$, which is not true by looking at the images in $S/(g)$. Thus we can choose $y \in Ux^{-1} \smallsetminus R$ such that $yI_x \subseteq R$. By Lemma 8.1, I_x is a line ideal, as claimed. Write $I_x = J_r$, with $L_r = R/J_r$ for the corresponding line module L_r, where $r = \operatorname{Div}(L_r) \in E$.

By [5, Proposition 4.5.3], gldim $R^\circ < \infty$ and so by [15, Lemma 6.8], qgr-R is smooth. We have $(J_p)_1 = \{z \in R_1 \mid T(q)_1 z \subseteq R\}$ since $T(q)_1R/R \cong L_p[-1]$ by [15, Theorem 8.3]. Since

$$W(q)S_2W(\tau p)V(\sigma p + \sigma q) = W(q)W(\sigma p)S_1V(p+q)S_1 \subseteq R_2,$$

we have $W(\tau p)V(\sigma p + \sigma q) \subseteq (J_p)_1$. However, from Lemma 8.3 (3) we see

$$\dim W(\tau p)V(\sigma p + \sigma q) = \dim(J_p)_1,$$

so

$$J_p = W(\tau p)V(\sigma p + \sigma q)R. \tag{8.5}$$

From this and Lemma 8.3 (2) we get

$$(J_p)_1 T_1 = W(\tau p)V(\sigma p + \sigma q)T_1 = W(\tau p)S_2 V(\tau p + \tau q)S_1 \ni g^2.$$

It follows that $\mathrm{GKdim}_T\, T/J_pT = 1$, and so the T-module double dual $(J_pT)^{**} = T$, by the CM condition on T (see Notation 2.1). On the other hand, $(J_rT)^{**} = (xS_2T)^{**} = xS_2T$. Therefore

$$\begin{aligned}\mathrm{Hom}_R(J_p, J_r) &\subseteq \mathrm{Hom}_T(J_pT, J_rT)\\ &\subseteq \mathrm{Hom}_T((J_pT)^{**}, (J_rT)^{**}) = \mathrm{Hom}_T(T, xS_2T) = xS_2T.\end{aligned}$$

Consequently, $\mathrm{Hom}_R(J_p, J_r)_1 = xS_2 = (J_r)_1$ and by Lemma 5.3 and Corollary 2.14, $(L_r \bullet_{\mathrm{MS}} L_p) = 1$. Likewise, $(L_r \bullet_{\mathrm{MS}} L_q) = 1$.

Let $X = W(\tau q)W(\sigma q)S_1$. Then

$$W(p)S_2X = W(p)W(\sigma q)S_1W(q)S_2 \subseteq R_1W(q)S_2.$$

From the discussion before Lemma 7.7, $J_p^* = RT(q)_1 + R$ and $J_q^* = RT(p)_1 + R$. Therefore, $X \subseteq \mathrm{Hom}_R(J_q^*, J_p^*)_1 = \mathrm{Hom}_R(J_p, J_q)_1$. As $\dim X = 7$ by Lemma 8.3 (1), it follows from Lemma 5.3 that $(L_p \bullet_{\mathrm{MS}} L_q) = 0$.

Since qgr-R is smooth, [15, Proposition 1.3 (2)] implies that

$$(L_p \bullet_{\mathrm{MS}} L_p) = (L_q \bullet_{\mathrm{MS}} L_q) = -1.$$

It is easy to calculate that $\mathrm{End}_R(J_r) = xT(\sigma^{-2}p + \sigma^{-2}q)x^{-1}$, which has the same Hilbert series as R, and so $(L_r \bullet_{\mathrm{MS}} L_r) = -1$, by [15, Theorem 7.1]. ∎

9. The noncommutative Cremona transform

In this section, we use Theorem 1.4 to give a noncommutative version of the classical Cremona transform. Recall that if X is the blowup of $\mathbb{P}^2$ at three non-collinear points a, b, c, then X contains a hexagon of (-1) lines, given by the three exceptional lines and the strict transforms of the lines through two of a, b, c; further, blowing down the strict transforms of the lines gives a birational map from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The next theorem is our version of this construction.

Theorem 9.1. *Let $T = S^{(3)}$ be a Sklyanin elliptic algebra with associated elliptic curve $E = E(T)$. Let $p, q, r \in E$ be distinct points such that σp, σq, σr are not collinear in $\mathbb{P}(S_1^*)$. Set $R = T(p + q + r)$. Then* qgr-*R is smooth, and there is a subring $T' \cong T$ of $T_{(g)}$ such that $R = T'(p_1, q_1, r_1)$ for points $p_1, q_1, r_1 \in E$.*

Assume that σx, σy, σz are not collinear in $\mathbb{P}(S_1^*)$, for any $x, y, z \in \{p, q, r\}$, including possible repetitions. Then the 6 points $\{p, q, r, p_1, q_1, r_1\}$ are pairwise distinct.

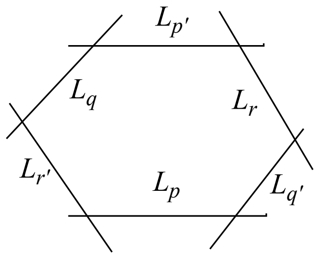


Figure 2. The lines on qgr-R.

Remark 9.2. As will be apparent from the proof, R contains a hexagon of lines of self-intersection (-1), as described in Figure 2. Then T' is obtained by blowing down the three lines in R that are not exceptional for the first blowup.

The proof of this theorem will be through a series of subsidiary results that will take the whole section. The strategy is to first show that R has six line modules of self-intersection (-1), each of which can be contracted. After constructing these line modules over R and computing their intersection theory (see Figure 2), we then contract one of these lines to give an overring $\widehat{R}$ of R. We then compute the intersection theory of $\widehat{R}$ (see Proposition 9.10) and show that $\widehat{R}$ satisfies the hypotheses of Theorem 1.4. Thus, by that result, we can then contract two further line modules to give a ring isomorphic to T.

As usual, write $S/(g) = B(E, \mathcal{L}, \sigma)$, and $T/(g) = B(E, \mathcal{M}, \tau)$ where $\mathcal{M} = \mathcal{L}_3 := \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \sigma^{2*}\mathcal{L}$ and $\tau = \sigma^3$. Fix a group law $\oplus$ on E so that three collinear points sum to zero and define $p' := \ominus q \ominus r$, $q' := \ominus p \ominus r$, and $r' := \ominus p \ominus q$. We continue to use the notations $W(a)$ and $V(a+b)$ from Definition 8.2.

We first construct the six lines on R. Let L_p be the exceptional line module obtained by writing R as the blowup of $T(q+r)$ at p, with line ideal J_p. Likewise, construct $L_q = R/J_q$ and $L_r = R/J_r$.

Lemma 9.3. *Let $x \in S_1$ define the line through q, r and define $J_{p'} = xW(\sigma p)S_1R$. Then $L_{p'} := R/J_{p'}$ is a line module with divisor $p' = \operatorname{Div} L_{p'}$.*

Proof. It is easy to check by using [12, Lemma 3.1] that $\overline{J_{p'}}$ is a point ideal in $\overline{R}$. Let

$$U = V(p+q+r)V(\sigma^{-1}q + \sigma^{-1}r) \subseteq S_4.$$

Then using Lemma 8.3 (2) we have

$$\begin{aligned} Ux^{-1}(J_{p'})_1 &= V(p+q+r)V(\sigma^{-1}q+\sigma^{-1}r)W(\sigma p)S_1 \\ &= V(p+q+r)S_1V(q+r)W(\sigma^2 p) \subseteq R_2. \end{aligned}$$

If $Ux^{-1} \subseteq R$, then $U \subseteq Rx$, which is not true by considering the images in $\overline{S}$. Thus $J_{p'}^* \supsetneqq R$ and so $J_{p'}$ is a line ideal by Lemma 8.1. Since $p' \oplus q \oplus r = 0$, clearly x vanishes at these three points and so $p' = \operatorname{Div} L_{p'}$ as $\overline{J_{p'}}$ is the point ideal of p'. ∎

Likewise, we construct $L_{q'}$, with divisor q', and $L_{r'}$, with divisor r'. We caution the reader that, by [12, Lemma 9.1], Div $L_p = \tau(p)$ and similarly for L_q and L_r. On the other hand, Div $L_{p'} = p'$ and similarly for $L_{q'}$ and $L_{r'}$.

Lemma 9.4. *The six lines L_p, L_q, L_r, $L_{p'}$, $L_{q'}$, $L_{r'}$ all have self-intersection* (-1).

Proof. For L_p, L_q, and L_r this follows from [15, Theorem 1.4]. We give the proof for $L_{p'}$. By [15, Theorem 7.1], it is enough to show that $\operatorname{hilb}\operatorname{End}_R(J_{p'}) = \operatorname{hilb} R$, and for this it suffices to prove that $\dim \operatorname{End}_R(J_{p'})_1 \geq \dim R_1$.

Using Lemma 8.3 and the fact that $\dim T(\sigma p + \sigma^{-2} q + \sigma^{-2} r)_1 = 7$, we calculate

$$\begin{aligned}
&(xT(\sigma p + \sigma^{-2} q + \sigma^{-2} r)_1 x^{-1}) \cdot (xW(\sigma p)S_1) \\
&\quad = xT(\sigma p + \sigma^{-2} q + \sigma^{-2} r)_1 W(\sigma p) S_1 \\
&\quad = xW(\sigma p)V(\sigma^{-1} q + \sigma^{-1} r)W(\sigma p)S_1 \subseteq xW(\sigma p)T(\sigma^{-1} p + \sigma^{-1} q + \sigma^{-1} r)_1 S_1 \\
&\quad = xW(\sigma p)S_1 T(p+q+r)_1 = (xW(\sigma p)S_1)R_1.
\end{aligned}$$

Thus $xT(\sigma p + \sigma^{-2} q + \sigma^{-2} r)_1 x^{-1} \subseteq \operatorname{End}_R(J_{p'})_1$ and $\dim \operatorname{End}_R(J_{p'})_1 \geq \dim T(\sigma p + \sigma^{-2} q + \sigma^{-2} r)_1 = \dim R_1$, as required. ∎

Since by hypothesis $\sigma p, \sigma q, \sigma r$ are not collinear, it follows from Lemma 8.3 (2) that $V(\sigma p + \sigma q + \sigma r)S_1$ is defined by vanishing conditions on E, in the sense of Definition 8.2. We now give explicit generators of J_p, J_q, and J_r. It suffices to do this for J_p.

Lemma 9.5. *We have the identities*

$$(J_p)_1 = W(\tau p)V(\sigma p + \sigma q + \sigma r) \qquad \textit{and} \qquad J_p = W(\tau p)V(\sigma p + \sigma q + \sigma r)R.$$

Proof. Let $C = W(\tau p)V(\sigma p + \sigma q + \sigma r) \subseteq R_1$. By the discussion before Lemma 7.7, $\operatorname{Hom}_R(J_p, R) =: J_p^* = RV(q+r)S_1 + R$. Note that $R_1 \subseteq V(q+r)S_1$ and so $(J_p^*)_1 = V(q+r)S_1$. Thus

$$(J_p^*)_1 C = V(q+r)W(\sigma^2 p)S_1 V(\sigma p + \sigma q + \sigma r).$$

As R_1 is defined by vanishing conditions on E, both $V(q+r)W(\sigma^2 p)$ and $S_1 V(\sigma p + \sigma q + \sigma r)$ are contained in R_1, and so $(J_p^*)_1 C \subseteq R_2$ and $C \subseteq (J_p)_1$. Since $\dim C \geq \dim \overline{C} = 5 = \dim (J_p)_1$, we see that $C = (J_p)_1$. The fact that $J_p = (J_p)_1 R$ follows from [15, Lemma 5.6 (2)]. ∎

Lemma 9.6. *The category* qgr-R *is smooth.*

Proof. Using the fact that $\sigma p, \sigma q, \sigma r$ are not collinear, we compute

$$\begin{aligned}
(J_p J_p^*)_2 &\supseteq W(\tau p)V(\sigma p + \sigma q + \sigma r)V(q+r)S_1 && \text{(by Lemma 9.5)} \\
&= W(\tau p)V(\sigma p + \sigma q + \sigma r)S_1 V(\sigma q + \sigma r) && \text{(by Lemma 8.3 (2))} \\
&\supseteq W(\tau p) g V(\sigma q + \sigma r) && \text{(by Lemma 8.3 (2))} \\
&\ni g^2 && \text{(by Lemma 8.3 (3),} \\
& && \text{using that } p \notin \{q, r\}).
\end{aligned}$$

Therefore, by the Dual Basis Lemma, J_p° is projective. Since $q \neq r$, qgr-$T(q+r)$ is smooth by [5, Proposition 4.5.3] and so by [15, Theorem 9.1] qgr-R is smooth as well. ■

Lemma 9.7. *We have the identities*

(1) $(L_p \bullet_{\mathrm{MS}} L_q) = (L_p \bullet_{\mathrm{MS}} L_r) = (L_q \bullet_{\mathrm{MS}} L_r) = 0.$

(2) $\mathrm{Hom}_R(J_p, J_q)_1 = W(\tau q)V(\sigma q + \sigma r).$

Proof. (1) We compute $(L_p \bullet_{\mathrm{MS}} L_q)$. By Lemma 8.3 (3), $\dim W(\tau q)V(\sigma q + \sigma r) = 6 = \dim R_1 - 1$, therefore, by Lemma 5.3, it suffices to prove that $W(\tau q)V(\sigma q + \sigma r) \subseteq \mathrm{Hom}_R(J_p, J_q)_1$. We compute

$$\begin{aligned} &W(\tau q)V(\sigma q + \sigma r)(J_p)_1 \\ &\qquad = W(\tau q)V(\sigma q + \sigma r)W(\tau p)V(\sigma p + \sigma q + \sigma r) && \text{(by Lemma 9.5)} \\ &\qquad \subseteq W(\tau q)V(\sigma p + \sigma q + \sigma r)S_1 V(\sigma p + \sigma q + \sigma r) \\ &\qquad \subseteq (J_q)_1 R_1 = (J_q)_2 && \text{(by Lemma 9.5).} \end{aligned}$$

(2) This follows from the proof of part (1), combined with Lemma 5.3. ■

Lemma 9.8. *We have*

(1) $(L_p \bullet_{\mathrm{MS}} L_{p'}) = (L_q \bullet_{\mathrm{MS}} L_{q'}) = (L_r \bullet_{\mathrm{MS}} L_{r'}) = 0.$

(2) *Moreover,* $\mathrm{Hom}_R(J_p, J_{p'})_1 = xS_2$.

(3) *Similarly,* $(L_{p'} \bullet_{\mathrm{MS}} L_{q'}) = (L_{p'} \bullet_{\mathrm{MS}} L_{r'}) = (L_{q'} \bullet_{\mathrm{MS}} L_{r'}) = 0.$

Proof. (1), (2) We compute $(L_p \bullet_{\mathrm{MS}} L_{p'})$. Recall that $x \in S_1$ defines the line through q, r, p' and that $J_{p'} = xW(\sigma p)S_1 R$. Since $(J_p^*)_1 \subseteq T_1$, the calculation in the proof of Lemma 9.6 shows that $g^2 \in J_p T_1$. Thus GKdim $T/J_pT \le 1$ and so, by [15, Lemma 4.5 (1)], $(J_pT)^* := \mathrm{Hom}_T(J_pT, T) = T$. Hence

$$\begin{aligned} \mathrm{Hom}_R(J_p, J_{p'}) \subseteq \mathrm{Hom}_T(J_pT, J_{p'}T) &\subseteq \mathrm{Hom}_T((J_pT)^{**}, (J_{p'}T)^{**}) \\ &\subseteq \mathrm{Hom}_T(T, xS_2T) = xS_2T. \end{aligned}$$

Since R_1 is defined by vanishing conditions on E, Lemmata 9.3 and 8.3 (2) imply that

$$\begin{aligned} xS_2(J_p)_1 = xS_2W(\tau p)V(\sigma p + \sigma q + \sigma r) &= (xW(\sigma p)S_1)(S_1V(\sigma p + \sigma q + \sigma r)) \\ &\subseteq (J_{p'})_1 R_1. \end{aligned}$$

Thus $xS_2 \subseteq \mathrm{Hom}_R(J_p, J_{p'})_1$ and hence $\mathrm{Hom}_R(J_p, J_{p'})_1 = xS_2$. By Lemma 5.3, it follows that $(L_p \bullet_{\mathrm{MS}} L_{p'}) = 0$.

(3) We show that $(L_{p'} \bullet_{\mathrm{MS}} L_{q'}) = 0$. As in part (1), $J_{p'} = xW(\sigma p)S_1 R$ while $J_{q'} = yW(\sigma q)S_1 R$, where y defines the line through p, q', r. By Lemmata 5.3 and 8.3 (3) it is enough to show that

$$yW(\sigma q)V(\sigma^{-1}q + \sigma^{-1}r)x^{-1} \subseteq \mathrm{Hom}_R(J_{p'}, J_{q'})_1.$$

This follows from a familiar computation,

$$\begin{aligned}&\big(yW(\sigma q)V(\sigma^{-1}q+\sigma^{-1}r)x^{-1}\big)(xW(\sigma p)S_1)\\&\quad= yW(\sigma q)S_1(V(q+r)W(\sigma^2 p))\subseteq yW(\sigma q)S_1R_1=(J_{q'})_2,\end{aligned}$$

as required. ■

The final piece of intersection theory needed is to determine the lines that intersect with multiplicity 1.

Lemma 9.9. *If* $a\neq b\in\{p,q,r\}$, *then* $(L_{a'}\bullet_{\mathrm{MS}}L_b)=1$.

Proof. Without loss of generality, we compute $(L_{p'}\bullet_{\mathrm{MS}}L_q)$. Write $\widetilde{R}=T(q+r)$, which is the blowdown of R at L_p. By Lemmata 9.7 and 9.8 (1), $(L_p\bullet_{\mathrm{MS}}L_q)=0=(L_p\bullet_{\mathrm{MS}}L_{p'})$. Thus by Lemma 5.5 (1), we may blow down the line ideals J_q and $J_{p'}$ at L_p, to obtain line ideals $\widetilde{J}_q$ and $\widetilde{J}_{p'}$ in $\widetilde{R}$ such that the line modules $\widetilde{L}_q=\widetilde{R}/\widetilde{J}_q$ and $\widetilde{L}_{p'}=\widetilde{R}/\widetilde{J}_{p'}$ again have self-intersection (-1).

By Lemmata 5.1 (3) and 9.7 (2), $\widetilde{J}_q=\mathrm{Hom}_R(J_p,J_q)_1\widetilde{R}=W(\tau q)V(\sigma q+\sigma r)\widetilde{R}$. Note that by (8.5), $\widetilde{L}_q$ is the exceptional line module that comes from writing $\widetilde{R}$ as the blowup of $T(r)$ at q. Likewise, by Lemmata 5.1 (2) and 9.8 (2), $\widetilde{J}_{p'}=\mathrm{Hom}_R(J_p,J_{p'})_1\widetilde{R}=xS_2\widetilde{R}$. Thus $\widetilde{L}_{p'}$ is the line module denoted by L_r in the proof of Proposition 8.4.

Finally, Proposition 8.4 shows that $(\widetilde{L}_{p'}\bullet_{\mathrm{MS}}\widetilde{L}_q)=1$ and so $(L_{p'}\bullet_{\mathrm{MS}}L_q)=1$ by Proposition 9.10 below. ■

We can now almost complete the proof of Theorem 9.1, modulo proving one final result.

Proof of Theorem 9.1. Lemmata 9.7, 9.8, 9.4, and 9.9 together establish that R has a hexagon of (-1) lines with the intersection theory indicated in Figure 2. We will use these computations without further comment.

Since $(L_{p'}\bullet_{\mathrm{MS}}L_{p'})=-1$ we may, by [15, Theorem 8.3], blow down $L_{p'}$ to obtain an overring $\widehat{R}$ of R so that $R=\widehat{R}(\tau^{-1}(p'))$. By that result, $\widehat{R}$ is a degree 7 elliptic algebra while, by [15, Theorem 9.1], qgr-$\widehat{R}$ is smooth.

Now $(L_{p'}\bullet_{\mathrm{MS}}L_{q'})=(L_{p'}\bullet_{\mathrm{MS}}L_p)=(L_{p'}\bullet_{\mathrm{MS}}L_{r'})=0$ while $L_{q'}$, L_p, $L_{r'}$ have self-intersection (-1). So, by Lemma 5.5 there are induced $\widehat{R}$-line modules $\widehat{L}_{q'}$, $\widehat{L}_p$, and $\widehat{L}_{r'}$, each of which has self-intersection (-1). Moreover, $\mathrm{Div}\,\widehat{L}_{q'}=q'$ and $\mathrm{Div}\,\widehat{L}_{r'}=r'$ while, by construction, $q'=\ominus p\ominus r\neq r'=\ominus p\ominus q$. Thus, by Proposition 9.10 below, $(\widehat{L_{q'}}\bullet_{\mathrm{MS}}\widehat{L}_{r'})=(L_{q'}\bullet_{\mathrm{MS}}L_{r'})=0$ and $(\widehat{L}_p\bullet_{\mathrm{MS}}\widehat{L}_{q'})=(\widehat{L}_p\bullet_{\mathrm{MS}}\widehat{L}_{r'})=1$.

Therefore, after appropriately renaming the line modules, $\widehat{R}$ satisfies the hypotheses of Theorem 1.4 and thus there is an overring T' of $\widehat{R}$ such that $T'\cong T$ and $\widehat{R}=T'(\tau^{-1}q'+\tau^{-1}r')$. In other words, $R=T(p_1,q_1,r_1)$, for $p_1=\tau^{-1}(p')$, $q_1=\tau^{-1}(q')$ and $r_1=\tau^{-1}(r')$.

It remains to check that the 6 points $\{p,q,r,p_1,q_1,r_1\}$ are distinct, under the additional assumption that $\sigma x,\sigma y,\sigma z$ are not collinear for any $x,y,z\in\{p,q,r\}$ chosen with possible repetition. This is a routine computation, combining the definition of p', q', r' with the hypotheses of the theorem and the fact that points $x, y, z \in E$ are collinear if and only if $x \oplus y \oplus z = 0$. We leave details to the reader. ∎

In order to complete the proof of Theorem 9.1, it remains to prove the following result, which generalises Lemma 5.5 (1). The point of the result is that contracting a line L does not affect the interaction of other lines which are disjoint from L, as one would hope.

Proposition 9.10. *Let R be an elliptic algebra such that* qgr-R *is smooth and let L, L_p, L_q be line modules with line ideals J, J_p, J_q respectively, with* Div $L_p = p$ *and* Div $L_q = q$. *(We allow $L_p \cong L_q$ here.) Assume that*

(1) $(L \bullet_{\text{MS}} L) = -1$;

(2) *for* $x \in \{p, q\}$, $(L_x \bullet_{\text{MS}} L_x) = -1$;

(3) *for* $x \in \{p, q\}$, $(L \bullet_{\text{MS}} L_x) = 0$.

Let $\widetilde{R}$ be the blowdown of R along the line L. As in Lemma 5.5 (1), *for $x \in \{p, q\}$ let $\widetilde{J}_x$ be the blowdown to $\widetilde{R}$ of J_x, and let $\widetilde{L}_x = \widetilde{R}/\widetilde{J}_x$, which by Lemma* 5.5 *is a line module over $\widetilde{R}$. Then*

$$(\widetilde{L}_p \bullet_{\text{MS}} \widetilde{L}_q) = (L_p \bullet_{\text{MS}} L_q),$$

where the intersection product on the left hand side is in qgr-$\widetilde{R}$, *and on the right hand side is in* qgr-R.

Remark 9.11. We note that the proposition still holds without hypothesis (2), although the proof is more complicated and is omitted.

Proof of Proposition 9.10. Write $r = \operatorname{Div} L$, and let $R/gR = B(E, \mathcal{M}, \tau)$. Throughout the proof a statement involving x is being asserted to hold for both $x = p$ and $x = q$.

First, if $L_p \cong L_q$, then obviously $\widetilde{L}_p \cong \widetilde{L}_q$. In this case, since $(L_p \bullet_{\text{MS}} L_p) = -1$, we have $(\widetilde{L}_p \bullet_{\text{MS}} \widetilde{L}_q) = (\widetilde{L}_p \bullet_{\text{MS}} \widetilde{L}_p) = -1$ by Lemma 5.5, as required. So from now on we can and will assume that $L_p \not\cong L_q$ and hence $J_p \neq J_q$. Note that $L \not\cong L_x$ and hence $J \neq J_x$ by comparing hypotheses (1) and (3).

Case I. Assume that $r \neq \tau^j(q)$ for $j \geq 0$.

The point of this assumption is that it allows us to prove the following.

Sublemma 9.12. *Keep the hypotheses of the proposition and assume that $r \neq \tau^j(q)$ for $j > 0$. Then $\widetilde{J}_q \cap R = J_q$.*

Proof. Since $J \neq J_q$, we have $i \geq 1$ in Lemma 5.1. Hence $\widetilde{J}_q \neq \widetilde{R}$ by part (3) of that lemma. Write $X := \widetilde{J}_q \cap R \subseteq N \subseteq \widetilde{J}_q$, where N/X is some finitely generated graded R-submodule of $\widetilde{J}_q/X$ with $\operatorname{GKdim}(N/X) \leq 1$. Then $\operatorname{GKdim}(N + R)/R = 1$ and so, as R is reflexive, [15, Lemma 4.5] implies that $N \subseteq R$ and hence $N = X$. Therefore, $\widetilde{J}_q/X$ is 2-pure as an R-module.

By [15, Lemma 8.2], $Z := \widetilde{J}_q/J_q = \bigoplus_{i\in\mathbb{I}} L[-a_i]$, where $a_i \geq 0$ for all i. Let $Y := X/J_q$, which embeds in Z. Now apply [15, Proposition 8.1]: since $Z/Y \cong \widetilde{J}_q/X$ is 2-pure, there is an internal direct sum $Z = Y \oplus (\bigoplus_{i\in\mathbb{J}} L[-a_i])$ for some subset $\mathbb{J}$ of $\mathbb{I}$, and thus $Y \cong \bigoplus_{i\in\mathbb{I}\smallsetminus\mathbb{J}} L[-a_i]$. Now $Y = X/J_q \subseteq R/J_q \cong L_q$ is a submodule of a line module and is also isomorphic to a direct sum of shifts of line modules. Comparing Hilbert series, there must be only one line module in the sum, so $Y \cong L[-c]$ for some $c \geq 0$.

Finally, $L[-c] \cong Y$ embeds in the line module $R/J_q \cong L_q$, and by [15, Lemma 5.5] this forces $r = \tau^j(q)$ for some $j \geq 0$. Necessarily, $j > 0$ since otherwise $L \cong L_q$, which we have excluded. So $r = \tau^j(q)$ for some $j > 0$, contradicting the hypothesis of the sublemma. ■

We now return to the proof of the proposition. Suppose first that $\widetilde{L}_p \cong \widetilde{L}_q$. Then $\widetilde{J}_p = \widetilde{J}_q$ and hence, by the sublemma, $J_p \subseteq \widetilde{J}_p \cap R = \widetilde{J}_q \cap R = J_q$. Since $\operatorname{hilb} R/J_p = 1/(1-s)^2 = \operatorname{hilb} R/J_q$ this forces $J_p = J_q$; a contradiction. We conclude that $\widetilde{L}_p \ncong \widetilde{L}_q$. Since $(L_p \bullet_{\mathrm{MS}} L_p) = -1$, we also have $(\widetilde{L}_p \bullet_{\mathrm{MS}} \widetilde{L}_p) = -1$ by Lemma 5.5. In particular, Lemma 5.3 implies that $(L_p \bullet_{\mathrm{MS}} L_q) \in \{0, 1\}$ and $(\widetilde{L}_p \bullet_{\mathrm{MS}} \widetilde{L}_q) \in \{0, 1\}$.

By Lemma 5.1 (1),

$$\operatorname{Hom}_R(J_p, J_q)(\widetilde{J}_p) = \operatorname{Hom}_R(J_p, J_q)(\operatorname{Hom}_R(J, J_p)R) \subseteq \operatorname{Hom}_R(J, J_q)R = \widetilde{J}_q.$$

Thus $\operatorname{Hom}_R(J_p, J_q) \subseteq \operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q)$.

Suppose next that $(L_p \bullet_{\mathrm{MS}} L_q) = 0$ but $(\widetilde{L}_p \bullet_{\mathrm{MS}} \widetilde{L}_q) = 1$. First, recall that $\dim \widetilde{R}_1 = \dim R_1 + 1$ by [15, Theorem 8.3]. By Lemma 5.3, twice, $(J_q)_1 \subsetneq \operatorname{Hom}_R(J_p, J_q)_1$ with $\dim \operatorname{Hom}_R(J_p, J_q)_1 = \dim R_1 - 1$ while $\operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q)_1 = (\widetilde{J}_q)_1$ with

$$\dim \operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q)_1 = \dim \widetilde{R}_1 - 2 = \dim R_1 - 1.$$

It therefore follows from the last paragraph that

$$\operatorname{Hom}_R(J_p, J_q)_1 = \operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q)_1 = (\widetilde{J}_q)_1 = \operatorname{Hom}_R(J, J_q)_1.$$

Now since $J \neq J_p$, Lemma 5.4 implies that $\operatorname{Hom}_R(J_p, J_q)_1 = (J_q)_1$. By Lemma 5.3 this contradicts $(L_p \bullet_{\mathrm{MS}} L_q) = 0$.

Finally, assume that $(L_p \bullet_{\mathrm{MS}} L_q) = 1$ but $(\widetilde{L}_p \bullet_{\mathrm{MS}} \widetilde{L}_q) = 0$. By Sublemma 9.12, $\widetilde{J}_q \cap R = J_q$ and so

$$\begin{aligned}\operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q) \cap R &\subseteq \operatorname{Hom}_R(\widetilde{J}_p \cap R, \widetilde{J}_q \cap R)\\ &\subseteq \operatorname{Hom}_R(J_p, \widetilde{J}_q \cap R) = \operatorname{Hom}_R(J_p, J_q).\end{aligned}$$

In particular, using Lemma 5.3,

$$(\operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q) \cap R)_1 \subseteq \operatorname{Hom}_R(J_p, J_q)_1 = (J_q)_1.$$

Further, the same lemma shows that $\dim(\operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q))_1 = \dim \widetilde{R}_1 - 1$. Since $\dim R_1 = \dim \widetilde{R}_1 - 1$, we get $\dim(\operatorname{Hom}_{\widetilde{R}}(\widetilde{J}_p, \widetilde{J}_q) \cap R)_1 \geq \dim \widetilde{R}_1 - 2$, while $\dim(J_q)_1 = \dim R_1 - 2 = \dim \widetilde{R}_1 - 3$. This is a contradiction.

Therefore, the only possibility is that $(L_p \bullet_{\mathrm{MS}} L_q) = (\widetilde{L}_p \bullet_{\mathrm{MS}} \widetilde{L}_q)$ and Case I is complete.

Case II. Assume that $r = \tau^j(q)$ for some $j > 0$.

This part of the proof will actually work whenever $r \neq \tau^j(q)$ for $j < 0$. This case will parallel that of Case I, except that we will pass from right to left modules.

Write $L_r = L$ and let $y \in \{r, p, q\}$ and $x \in \{p, q\}$. By [15, Lemma 5.6], $L_y^\vee = \mathrm{Ext}^1_R(L_y, R)[1]$ is a left line module, and we write $N_y = L_y^\vee \cong R/K_y$ for the left line ideal K_y. The relationship between the left line ideal K_y and right line ideal J_y is most easily expressed in terms of the R-linear duals.

Sublemma 9.13. *Let J be a right line ideal and K a left line ideal in R. Then $R/K \cong (R/J)^\vee$ if and only if* $\mathrm{Hom}_R(J, R)_1 = \mathrm{Hom}_R(K, R)_1$.

Proof. By [15, Lemma 5.6 (3)], $R \subseteq M = \mathrm{Hom}_R(J, R)$ is the unique extension of R up to isomorphism such that $M/R \cong (R/J)^\vee[1]$. In particular, if $(R/J)^\vee \cong R/K$ then $M_1 K \subseteq R$ and so $\mathrm{Hom}_R(J, R)_1 \subseteq \mathrm{Hom}_R(K, R)_1$; the other inclusion follows analogously. The converse is similar. ■

By Corollary 2.14, the $L_y^\vee$ satisfy the same intersection theory as the L_y. Similarly, since $L_y^{\vee\vee} \cong L_y$ by [15, Lemma 5.4], we also have $L_x^\vee \not\cong L_r^\vee$. The left hand analogue of [15, Lemma 8.2] defines left modules $\widetilde{K}_x$ by attaching all possible copies of $L_r^\vee$ on top of K_x. Crucially, by [15, Theorem 8.3], blowing down R on the left along $L_r^\vee$ leads to the same overring $\widetilde{R}$ as blowing down R along L. In particular, $\widetilde{K}_x$ is a left line ideal of $\widetilde{R}$ and, by the left-right analogue of Lemma 5.5, $\widetilde{N}_x = \widetilde{R}/\widetilde{K}_x$ is a line module for $\widetilde{R}$.

The analogue of Sublemma 9.12 is the following.

Sublemma 9.14. *Assume that $r \neq \tau^j(q)$ for $j < 0$. Then $\widetilde{K}_q \cap R = K_q$.*

Proof. By [15, Lemma 5.4], $\mathrm{Div}\, L_r^\vee = \tau^{-2}(r)$ and $\mathrm{Div}\, L_x^\vee = \tau^{-2}(x)$. Using [15, Remark 3.2], the left hand analogue of [15, Lemma 5.5] asserts that, if $L_r^\vee[-a]$ embeds into $L_q^\vee$ for some $a \geq 1$, then $\tau^{-2}(r) = \mathrm{Div}\, L^\vee = \tau^{-j}(\mathrm{Div}\, L_q^\vee) = \tau^{-j-2}(q)$ for some $j \geq 0$. Therefore, the proof of Sublemma 9.12 will also work here, provided that this observation is used in place of the final paragraph of that proof. ■

We claim that $\widetilde{N}_x := \widetilde{R}/\widetilde{K}_x \cong (\widetilde{L}_x)^\vee$, where the dual is taken with respect to the ring $\widetilde{R}$. By Sublemma 9.13, since $R/K_x \cong (R/J_x)^\vee$, we have $X := \mathrm{Hom}_R(J_x, R)_1 = \mathrm{Hom}_R(K_x, R)_1$, where $\dim X = \dim R_1 + 1$. If $X \subseteq \widetilde{R}$, then $\mathrm{Hom}_R(J_x, R)_1 \subseteq \widetilde{R}_1 = \mathrm{Hom}_R(J, R)_1$. Since $J \neq J_x$, Lemma 5.4 gives $X = \mathrm{Hom}_R(J_x, R)_1 = R_1$, which is a contradiction. Now choose $z \in X$ such that $X = \Bbbk z + R_1$. Then $\widetilde{X} := \Bbbk z + \widetilde{R}_1$ has $\dim \widetilde{X} = \dim \widetilde{R} + 1$, because $z \notin \widetilde{R}$. By a similar argument to the one used earlier in Case I, we have $\mathrm{Hom}_R(J_x, R) \subseteq \mathrm{Hom}_{\widetilde{R}}(\widetilde{J}_x, \widetilde{R})$ and $\mathrm{Hom}_R(K_x, R) \subseteq \mathrm{Hom}_{\widetilde{R}}(\widetilde{K}_x, \widetilde{R})$. Thus $\widetilde{X} \subseteq \mathrm{Hom}_{\widetilde{R}}(\widetilde{J}_x, \widetilde{R})$ and since $\widetilde{J}_x$ is a right line ideal in $\widetilde{R}$, $\dim \mathrm{Hom}_{\widetilde{R}}(\widetilde{J}_x, \widetilde{R})_1 = \dim \widetilde{R}_1 + 1$, so $\widetilde{X} = \mathrm{Hom}_{\widetilde{R}}(\widetilde{J}_x, \widetilde{R})_1$. An analogous argument on the left gives $\widetilde{X} = \mathrm{Hom}_{\widetilde{R}}(\widetilde{K}_x, \widetilde{R})_1$. By Sublemma 9.13, $\widetilde{R}/\widetilde{K}_x \cong (\widetilde{R}/\widetilde{J}_x)^\vee = (\widetilde{L}_x)^\vee$, as claimed.

Now follow the proof of Case I on the left to prove that $(N_p \bullet_{\mathrm{MS}} N_q) = (\widetilde{N}_p \bullet_{\mathrm{MS}} \widetilde{N}_q)$. Since $\widetilde{N}_x \cong (\widetilde{L}_x)^\vee$, it therefore follows from Corollary 2.14, twice, that

$$\begin{aligned}(\widetilde{L}_p \bullet_{\mathrm{MS}} \widetilde{L}_q) &= \big((\widetilde{L}_p)^\vee \bullet_{\mathrm{MS}} (\widetilde{L}_q)^\vee\big) = (\widetilde{N}_p \bullet_{\mathrm{MS}} \widetilde{N}_q) = (N_p \bullet_{\mathrm{MS}} N_q)\\ &= (L_p^\vee \bullet_{\mathrm{MS}} L_q^\vee) = (L_p \bullet_{\mathrm{MS}} L_q),\end{aligned}$$

and the proof is complete. ∎

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