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Noncommutative Hodge conjecture

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Abstract. The paper provides a version of the rational Hodge conjecture for dg categories. The noncommutative Hodge conjecture is equivalent to the version proposed by Perry (2022) for admissible subcategories. We obtain examples of evidence of the Hodge conjecture by techniques of noncommutative geometry. Finally, we show that the noncommutative Hodge conjecture for smooth proper connective dg algebras is true.

1. Introduction

Recently, G. Tabuada proposed a series of noncommutative counterparts of the celebrated conjectures, for example, Grothendieck standard conjecture of type C and type D, Voevodsky nilpotence conjecture, Tate conjecture, Weil conjecture, and so on. After proposing the noncommutative counterparts, he proved additivity with respect to the SODs (semi-orthogonal decomposition, see the notation in Section 1) for most of these conjectures. Then, he was able to give new evidence of the conjectures by a good knowledge of the semi-orthogonal decompositions of the derived category of varieties. For the details, the reader can refer to "Noncommutative counterparts of celebrated conjectures" [32].

In this paper, the author provides a version of the rational Hodge conjecture to the small dg categories. This new conjecture is equivalent to the classical Hodge conjecture when the dg category is $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$, where X is a projective smooth variety. It is equivalent to the version of the Hodge conjecture in [25] for the admissible subcategories of $\mathsf{D}^{\mathsf{b}}(\mathsf{X})$.

For $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$, $\mathsf{HH}_0(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})) \cong \bigoplus \mathsf{H}^{\mathsf{p},\mathsf{p}}(\mathsf{X}, \mathbb{C})$ by the HKR isomorphism. In order to generalize the Hodge conjecture, we need to find natural intrinsic rational Hodge classes in $\mathsf{HH}_0(\mathcal{A})$, and most importantly, it becomes the usual rational Hodge classes when $\mathcal{A} = \mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$. Classically, it is well known that the images of rational topological K-groups under topological Chern character recovers the rational Betti cohomology. The topological K-theory was generalized to the noncommutative spaces by A. Blanc [6], it turns out that the image of the rational topological K-group $\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}$ under the topological Chern character becomes the even rational Betti cohomology when $\mathcal{A} = \mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$.

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There is a functorial commutative diagram

$$\begin{tikzcd}
 & & \mathsf{HH}_0(\mathcal{A}) \\
\mathsf{K}_0(\mathcal{A}) \arrow[r, "\mathsf{Ch}"'] \arrow[urr, "\mathsf{Ch}"] \arrow[d] & \mathsf{HN}_0(\mathcal{A}) \arrow[ur, "\pi"'] \arrow[d, "\mathsf{j}"] & \\
\mathsf{K}_0^{\mathsf{top}}(\mathcal{A}) \arrow[r, "\mathsf{Ch}^{\mathsf{top}}"] & \mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}). &
\end{tikzcd}$$

Definition 1.1. Let $\mathcal{A}$ be a small dg category. The Hodge classes of $\mathcal{A}$ are defined as

$$\mathsf{Hodge}(\mathcal{A}) := \pi(\mathsf{j}^{-1}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}))) \subset \mathsf{HH}_0(\mathcal{A}).$$

Clearly, the Chern character $\mathsf{Ch}\colon \mathsf{K}_0(\mathcal{A}) \to \mathsf{HH}_0(\mathcal{A})$ maps $\mathsf{K}_0(\mathcal{A})$ to $\mathsf{Hodge}(\mathcal{A})$. We define the noncommutative Hodge conjecture for any dg category as follows.

Conjecture 1.2 (Noncommutative Hodge conjecture). *The Chern character* $\mathsf{Ch}\colon \mathsf{K}_0(\mathcal{A}) \mapsto \mathsf{HH}_0(\mathcal{A})$ *maps* $\mathsf{K}_0(\mathcal{A})_{\mathbb{Q}}$ *surjectively into the Hodge classes* $\mathsf{Hodge}(\mathcal{A})$.

For the smooth proper dg categories, we propose an equivalent version of the rational Hodge conjecture, for the reason that they are equivalent, see Remark 3.8. We write H as the isomorphism $\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}) \cong^{\mathsf{H}} \bigoplus \mathsf{HH}_{2n}(\mathcal{A})$ which is the Hodge decomposition by degeneration of the noncommutative Hodge-to-de Rham spectral sequence [10]. Note that we choose a splitting. Define the rational class in $\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A})$ as $\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A}))$. Then we define the Hodge classes in $\mathsf{HH}_0(\mathcal{A})$ as

$$\mathsf{Hodge}(\mathcal{A}) = \mathsf{Pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A}))).$$

Here the map Pr is the projection from $\bigoplus \mathsf{HH}_{2n}(\mathcal{A})$ to $\mathsf{HH}_0(\mathcal{A})$. Clearly the natural Chern character map $\mathsf{K}_0(\mathcal{A})_{\mathbb{Q}}$ to $\mathsf{Hodge}(\mathcal{A})$.

Definition 1.3 (= Definition 3.7). Hodge conjecture for smooth proper dg categories: the Chern character $\mathsf{Ch}\colon \mathsf{K}_0(\mathcal{A}) \to \mathsf{HH}_0(\mathcal{A})$ maps $\mathsf{K}_0(\mathcal{A})_{\mathbb{Q}}$ surjectively into the Hodge classes $\mathsf{Hodge}(\mathcal{A})$.

Remark 1.4. Since the noncommutative Hodge conjecture of smooth proper dg categories here is equivalent to Conjecture 1.2, see Remark 3.8, the formulation for smooth proper dg categories is independent of the choice of splitting.

The version of the Hodge conjecture is equivalent to the one in [25] for admissible subcategories of $\mathsf{D}^{\mathsf{b}}(\mathsf{X})$, see Theorem 3.5.

After establishing the language of the noncommutative Hodge conjecture, the author proves that the conjecture is additive for general SODs and the noncommutative motives.

Theorem 1.5 (= Theorem 3.20). *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be smooth and proper* dg *categories. Suppose there is a direct sum decomposition $\mathcal{U}(\mathcal{C})_{\mathbb{Q}} \cong \mathcal{U}(\mathcal{A})_{\mathbb{Q}} \oplus \mathcal{U}(\mathcal{B})_{\mathbb{Q}}$, see Section 3.2 for the definition of $\mathcal{U}(\bullet)$ and $\mathcal{U}(\bullet)_{\mathbb{Q}}$. We have the following:*

$$\begin{aligned} &\textit{noncommutative Hodge conjecture for } \mathcal{C} \\ &\quad \Leftrightarrow \textit{noncommutative Hodge conjecture for } \mathcal{A} \textit{ and } \mathcal{B}. \end{aligned}$$

Corollary 1.6 (= Theorem 3.13). *Suppose we have an* SOD, $\mathsf{D}^{\mathsf{b}}(\mathsf{X}) = \langle \mathcal{A}, \mathcal{B} \rangle$. *There are natural* dg *liftings $\mathcal{A}_{\mathsf{dg}}$, $\mathcal{B}_{\mathsf{dg}}$ of $\mathcal{A}$, $\mathcal{B}$ corresponding to the* dg *enhancement* $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$ *of* $\mathsf{D}^{\mathsf{b}}(\mathsf{X})$.

$$\textit{Hodge conjecture for } \mathsf{X} \;\Leftrightarrow\; \textit{noncommutative Hodge conjecture for } \mathcal{A}_{\mathsf{dg}} \textit{ and } \mathcal{B}_{\mathsf{dg}}.$$

Remark 1.7. This follows directly from Theorem 1.5. By Theorem 3.5, this version of the noncommutative Hodge conjecture for admissible categories is equivalent with the one proposed by Alex Perry, so the corollary was known by Alex Perry [25].

Let $\mathcal{A}$ be a sheaf of Azumaya algebras on X. Using the work of G. Tabuada and M. Van den Bergh on Azumaya algebras [33, Theorem 2.1], $\mathcal{U}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}, \mathcal{A}))_{\mathbb{Q}} \cong \mathcal{U}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))_{\mathbb{Q}}$. We have the following.

Theorem 1.8 (= Theorem 3.24). *Noncommutative Hodge conjecture for* $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}, \mathcal{A}) \Leftrightarrow$ *noncommutative Hodge conjecture for* $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$.

This formulation of the noncommutative Hodge conjecture is compatible with the semi-orthogonal decompositions. Therefore, good knowledge of semi-orthogonal decomposition of varieties can simplify the Hodge conjecture, and gives new evidence of the Hodge conjecture. The survey "Noncommutative counterparts of celebrated conjectures" [32, Section 2] provides many examples of the applications to the geometry for some conjectures via this approach. The examples also apply to the noncommutative Hodge conjecture, and we give some further examples which are combined in the theorem below.

Theorem 1.9. *Combining Corollary 1.6, Theorem 1.5, and Theorem 1.8, we have:*

(1) Fractional Calabi–Yau categories.
Let X *be a hypersurface of degree $\leq n+1$ in $\mathbb{P}^n$. There is a semi-orthogonal decomposition*

$$\mathsf{Perf}(\mathsf{X}) = \langle \mathcal{T}(\mathsf{X}), \mathcal{O}_{\mathsf{X}}, \ldots, \mathcal{O}_{\mathsf{X}}(n - \mathsf{deg}(\mathsf{X})) \rangle.$$

$\mathcal{T}(\mathsf{X})$ *is a fractional Calabi–Yau of dimension* $\frac{(n+1)(\mathsf{deg}-2)}{\mathsf{deg}(\mathsf{X})}$ [18, Theorem 3.5]. *We write* $\mathcal{T}_{\mathsf{dg}}(\mathsf{X})$ *for the full* dg *subcategory of* $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$ *whose objects belong to* $\mathcal{T}(\mathsf{X})$. *Then*

$$\textit{Hodge conjecture of } \mathsf{X} \Leftrightarrow \textit{noncommutative Hodge conjecture of } \mathcal{T}_{\mathsf{dg}}(\mathsf{X}).$$

(2) Twisted scheme.

(A) *Let* X *be a cubic fourfold containing a plane. There is a semi-orthogonal decomposition*

$$\mathsf{Perf}(\mathsf{X}) = \langle \mathsf{Perf}(\mathsf{S}, \mathcal{A}), \mathcal{O}_{\mathsf{X}}, \mathcal{O}_{\mathsf{X}}(1), \mathcal{O}_{\mathsf{X}}(2)\rangle,$$

where S *is a* K_3 *surface, and* $\mathcal{A}$ *is a sheaf of the Azumaya algebra over* S [16, Theorem 4.3]. *Since the noncommutative Hodge conjecture is true for* $\mathsf{Per}_{\mathsf{dg}}(\mathsf{S}, \mathcal{A})$ *by Theorem* 1.8, *hence the Hodge conjecture is true for* X.

(B) *Let* $\mathsf{f}\colon \mathsf{X} \to \mathsf{S}$ *be a smooth quadratic fibration, for example, the smooth quadric in the relative projective space* $\mathbb{P}^{n+1}_{\mathsf{S}}$ [15]. *There is a semi-orthogonal decomposition*

$$\mathsf{Perf}(\mathsf{X}) = \langle \mathsf{Perf}(\mathsf{S}, \mathsf{Cl}_0), \mathsf{Perf}(\mathsf{S}), \dots, \mathsf{Perf}(\mathsf{S})\rangle,$$

where Cl_0 *is a sheaf of the Azumaya algebra over* S *if the dimension* n *of the fiber of* f *is odd.*

Thus, if n *is odd, the Hodge conjecture of* $\mathsf{X} \Leftrightarrow \mathsf{S}$. *Moreover, if* $\dim \mathsf{S} \leq 3$, *the Hodge conjecture for* X *is true.*

(3) HP duality.

We write $\mathsf{Hodge}(\bullet)$ *if the (noncommutative) Hodge conjecture is true for varieties (smooth and proper* dg *categories). Let* $\mathsf{Y} \to \mathbb{P}(\mathsf{V}^*)$ *be the* HP *dual of* $\mathsf{X} \to \mathbb{P}(\mathsf{V})$, *then* $\mathsf{Hodge}(\mathsf{X}) \Leftrightarrow \mathsf{Hodge}(\mathsf{Y})$. *Choose a linear subspace* $\mathsf{L} \subset \mathsf{V}^*$. *Let* $\mathsf{X}_{\mathsf{L}} = \mathsf{X} \times_{\mathbb{P}(\mathsf{V})} \mathbb{P}(\mathsf{L}^{\perp})$ *and* $\mathsf{Y}_{\mathsf{L}} = \mathsf{Y} \times_{\mathbb{P}(\mathsf{V}^*)} \mathbb{P}(\mathsf{L})$ *be the corresponding linear sections. Assume* X_{L} *and* Y_{L} *are of expected dimension and smooth. If we assume* $\mathsf{Hodge}(\mathsf{X})$, *then* $\mathsf{Hodge}(\mathsf{X}_{\mathsf{L}}) \Leftrightarrow \mathsf{Hodge}(\mathsf{Y}_{\mathsf{L}})$.

We can prove (3) directly from the description of HPD, see Theorem 3.27. For more examples constructed from HPD, see Example 3.29. Motivated from the noncommutative techniques, Theorem 1.9 (3), we expect that we can establish duality of the Hodge conjecture for certain linear sections of the projective dual varieties by classical methods of algebraic geometry.

Conjecture 1.10 (= Conjecture 3.31). *Let* $\mathsf{X} \subset \mathbb{P}(\mathsf{V})$ *be a projective smooth variety. Suppose the Hodge conjecture is true for* X. *Let* $\mathsf{Y} \subset \mathbb{P}(\mathsf{V}^*)$ *be the projective dual of* $\mathsf{X} \subset \mathbb{P}(\mathsf{V})$. *Choose a linear subspace* $\mathsf{L} \subset \mathsf{V}^*$. *Suppose the linear section* $\mathsf{X}_{\mathsf{L}} = \mathsf{X} \cap \mathbb{P}(\mathsf{L}^{\perp})$ *and* $\mathsf{Y}_{\mathsf{L}} = \mathsf{Y} \cap \mathbb{P}(\mathsf{L})$ *are both of expected dimension and smooth. Then, the Hodge conjecture of* X_{L} *is equivalent to the Hodge conjecture of* Y_{L}.

Finally, we obtain some results by the algebraic techniques. A dg algebra A is called connective if $\mathsf{H}^i(\mathsf{A}) = 0$ for $i > 0$. According to [26, Theorem 4.6], if A is a connective smooth proper dg algebra, then $\mathcal{U}(\mathsf{A})_{\mathbb{Q}} \cong \mathcal{U}(\mathsf{H}^0(\mathsf{A})/\mathsf{Jac}(\mathsf{H}^0(\mathsf{A})))_{\mathbb{Q}} \cong \bigoplus \mathcal{U}(\mathbb{C})_{\mathbb{Q}}$. Thus, we have the following.

Theorem 1.11. *The noncommutative Hodge conjecture is true for a smooth proper and connected* dg *algebra* A, *see Theorem 3.33. In particular, the noncommutative Hodge conjecture is true for smooth and proper algebras.*

We also provide another proof for the case of smooth and proper algebras, see Theorem 3.34. Theorem 1.11 implies that if a variety X admits a tilting bundle (or sheaf), then the Hodge conjecture is true for X, see Corollary 3.38 in the text.

Notation

We assume the varieties to be defined over $\mathbb{C}$. We write SOD for a semi-orthogonal decomposition of triangulated categories. We say a semi-orthogonal decomposition is geometric if its components are equivalent to some derived categories of projective smooth varieties. We always assume the dg categories to be small categories. We write k as the field $\mathbb{C}$ in some places without mentioning.

2. Preliminary

2.1. The classical Hodge conjecture

Given a projective smooth variety X, there is a famous Hodge decomposition

$$\mathsf{H}^{\mathsf{k}}(\mathsf{X}(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} \cong \bigoplus_{\mathsf{p}+\mathsf{q}=\mathsf{k}} \mathsf{H}^{\mathsf{p}}(\mathsf{X}, \Omega^{\mathsf{q}}_{\mathsf{X}})$$

where $\mathsf{H}^{\mathsf{p}}(\mathsf{X}, \Omega^{\mathsf{q}}_{\mathsf{X}})$ can be identified with the (p, q) classes in $\mathsf{H}^{\mathsf{p}+\mathsf{q}}(\mathsf{X}(\mathbb{C}), \mathbb{C})$. We define the rational (integral) Hodge classes as rational (integral) (p, p) classes. Namely, the rational Hodge classes are $\mathsf{H}^{\mathsf{even}}(\mathsf{X}(\mathbb{C}), \mathbb{Q}) \cap \bigoplus_{\mathsf{p}} \mathsf{H}^{\mathsf{p},\mathsf{p}}(\mathsf{X}, \mathbb{C})$, and the integral Hodge classes are $\mathsf{H}^{\mathsf{even}}(\mathsf{X}(\mathbb{C}), \mathbb{Z}) \cap \bigoplus_{\mathsf{p}} \mathsf{H}^{\mathsf{p},\mathsf{p}}(\mathsf{X}, \mathbb{C})$. By the Poincaré duality, there is a cycle map which relates the Chow group of X with its Betti cohomology

$$\mathsf{Cycle}\colon \mathsf{CH}^*(\mathsf{X}) \to \mathsf{H}^*(\mathsf{X}(\mathbb{C}), \mathbb{C}).$$

Clearly, the image lies in the $\mathsf{H}^{\mathsf{even}}(X(\mathbb{C}), \mathbb{Z})$. We obtain the rational cycle map when we tensor with $\mathbb{Q}$. The famous Hodge conjecture concerns whether the image of the (rational) cycle map is exactly the (rational) integral Hodge class. It is well known that the integral Hodge conjecture is not true in general [3], and the rational Hodge conjecture is still open. For more introductions to the classical Hodge conjecture, the reader can refer to the survey "Some aspects of the Hodge conjecture" [36].

Remark 2.1. The rational (and integral) Hodge conjecture is true for weight one by the Lefschetz one-one theorem. According to the Poincaré duality, the rational Hodge conjecture is true for weight $n-1$, n is the dimension of the variety. In particular, the rational Hodge conjecture is true for varieties of dimension less than or equal to 3.

This paper focuses on the non-weighted rational Hodge conjecture. That is, we concern whether the rational cycle map maps $\mathsf{CH}^*(\mathsf{X})_{\mathbb{Q}}$ surjectively into the rational Hodge classes.

Theorem 2.2 (Part of Grothendieck–Riemann–Roch (SGA6, exp. XIV)[5]). *Let* X *be a smooth projective variety. There is a commutative diagram, where* $\mathsf{Ch}_{\mathbb{Q}}$ *are the certain Chern characters,* $\mathsf{K}_0(\mathsf{X})_{\mathbb{Q}}$ *is the rational* 0*-th algebraic* K*-group of the coherent sheaves,*

$$\begin{tikzcd} \mathsf{K}_0(\mathsf{X})_{\mathbb{Q}} \arrow[r, "\mathsf{Ch}_{\mathbb{Q}}"] \arrow[d, "\mathsf{Ch}_{\mathbb{Q}}"', "\cong"] & \mathsf{H}^*(\mathsf{X}, \mathbb{C}) \\ \mathsf{CH}^*(\mathsf{X})_{\mathbb{Q}}. \arrow[ur, "\mathsf{cycle}"'] & \end{tikzcd}$$

The image of the Chern character is in the rational Hodge classes, and the rational Hodge conjecture can be reformulated that $\mathsf{Ch}_{\mathbb{Q}}$ maps $\mathsf{K}_0(\mathsf{X})_{\mathbb{Q}}$ surjectively into the rational Hodge classes.

2.2. Noncommutative geometry

We briefly recall the theory of noncommutative spaces. We regard certain dg categories as noncommutative counterparts of varieties. We will recall the basic notions. For a survey of the dg categories, the reader can refer to the survey by B. Keller, "On differential graded categories" [12].

Definition 2.3. The $\mathbb{C}$-linear category $\mathcal{A}$ is called a dg category if $\mathsf{Mor}(\bullet, \bullet)$ are differential $\mathbb{Z}$-graded k-vector spaces. For every object $\mathsf{E}, \mathsf{F}, \mathsf{G} \in \mathcal{A}$, the compositions

$$\mathsf{Mor}(\mathsf{F}, \mathsf{E}) \otimes \mathsf{Mor}(\mathsf{G}, \mathsf{F}) \to \mathsf{Mor}(\mathsf{G}, \mathsf{E})$$

of complexes are associative. Furthermore, there is a unit $\mathsf{k} \to \mathsf{Mor}(\mathsf{E}, \mathsf{E})$. Note that the composition law implies that $\mathsf{Mor}(\mathsf{E}, \mathsf{E})$ is a differential graded algebra.

Example 2.4. A basic example of dg categories is $\mathsf{C}_{\mathsf{dg}}(\mathsf{k})$, whose objects are complexes of k-vector space. The morphism spaces are refined as follows:

Let $\mathsf{E}, \mathsf{F} \in \mathsf{C}_{\mathsf{dg}}(\mathsf{k})$ define the degree n piece of the morphism $\mathsf{Mor}(\mathsf{E}, \mathsf{F})$ to be

$$\mathsf{Mor}(\mathsf{E}, \mathsf{F})(n) := \Pi \mathsf{Hom}(\mathsf{E}_i, \mathsf{F}_{i+n}).$$

The n-th differential is given by $\mathsf{d}_n(\mathsf{f}) = \mathsf{d}_{\mathsf{E}} \circ \mathsf{f} - (-1)^n \mathsf{f} \circ \mathsf{d}_{\mathsf{F}}$, $\mathsf{f} \in \mathsf{Mor}(\mathsf{E}, \mathsf{F})(n)$.

Definition 2.5. We call $\mathsf{F}\colon \mathcal{C} \to \mathcal{D}$ a dg functor between dg categories if $\mathsf{F}\colon \mathsf{Hom}(\mathsf{E}, \mathsf{G}) \to \mathsf{Hom}(\mathsf{F}(\mathsf{E}), \mathsf{F}(\mathsf{G}))$ is in $\mathsf{C}(\mathsf{k})$ (morphisms are morphisms of chain complexes), $\mathsf{E}, \mathsf{G} \in \mathcal{C}$. We call F to be quasi-equivalent if F induces isomorphisms on homologies of morphisms and equivalences on their homotopic categories.

Definition 2.6. The dg functor $\mathsf{F}\colon \mathcal{A} \to \mathcal{B}$ is derived Morita equivalent if it induces an equivalence of derived categories by composition

$$\mathsf{F}^*\colon \mathsf{D}(\mathcal{B}) \cong \mathsf{D}(\mathcal{A}).$$

Note that if the dg functor $\mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then it is derived Morita equivalent, the reader can refer to "Categorical resolutions of irrational singularities" [19, Proposition 3.9] for an explicit proof.

We consider the category of small dg categories, whose morphisms are the dg functors. It is written as dg-cat. According to G. Tabuada [29], there is a model structure on dg-cat with derived Morita equivalent dg functors as weak equivalences. We write $\mathsf{Hmo}(\text{dg-cat})$ as the associated homotopy category for such model structure. Given two dg categories $\mathcal{A}$ and $\mathcal{B}$, we have a bijection $\mathsf{Hom}_{\mathsf{Hmo}}(\mathcal{A}, \mathcal{B}) \cong \mathsf{Iso}\,\mathsf{rep}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathsf{L}} \mathcal{B})$, where $\mathsf{rep}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathsf{L}} \mathcal{B})$ is the subcategory of $\mathsf{D}(\mathcal{A} \otimes^{\mathsf{L}} \mathcal{B})$ with bi-module X such that $\mathsf{X}(\mathcal{A}, \bullet)$ is a perfect $\mathcal{B}$ module. Linearizing the category, we obtain Hmo_0 whose morphism spaces become $\mathsf{K}_0(\mathsf{rep}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}))$. After $\mathbb{Q}$ linearization and idempotent completion, we get the category of the pre-noncommutative motive $\mathsf{PChow}_{\mathbb{Q}}$.

Definition 2.7. Any functor to an additive category $\mathcal{C}$, $\mathsf{F}\colon \text{dg-cat} \to \mathcal{C}$, is called an additive invariant in the sense of G. Tabuada [29] if

(1) it maps the Morita equivalences to isomorphisms;

(2) for pre-triangulated dg categories $\mathcal{A}$, $\mathcal{B}$ and X with natural morphism $\mathsf{i}\colon \mathcal{A} \to \mathsf{X}$ and $\mathsf{j}\colon \mathcal{B} \to \mathsf{X}$ which induce a semi-orthogonal decomposition of triangulated categories $\mathsf{Ho}(\mathsf{X}) = \langle \mathsf{Ho}(\mathcal{A}), \mathsf{Ho}(\mathcal{B})\rangle$, there is an isomorphism $\mathsf{F}(\mathsf{X}) \cong \mathsf{F}(\mathcal{A}) \oplus \mathsf{F}(\mathcal{B})$ which is induced by $\mathsf{F}(\mathsf{i}) + \mathsf{F}(\mathsf{j})$.

The following theorem is due to G. Tabuada.

Theorem 2.8 (G. Tabuada [29, Theorem 4.1]). *The functor* F *in Definition 2.7 that induces* $\mathsf{Hmo} \to \mathcal{A}$ *is an additive invariant if and only if it factors through* $\mathsf{Hmo} \to \mathsf{Hmo}_0 \to \mathcal{A}$. *That is,* Hmo_0 *plays a role as the usual motives and the additive invariants should be regarded as noncommutative Weil cohomology theories.*

Remark 2.9. Due to the work of many people, see the survey [31], the Hochschild homology, algebraic K-theory, (periodic) cyclic homology theory are all additive invariants. The Hochschild homology of proper smooth varieties is the noncommutative counterpart of the Hodge cohomology, and the periodic cyclic homology corresponds to the de Rham cohomology.

Given a proper smooth variety X, there is a natural dg enhancement $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$, which is a dg enhancement of $\mathsf{Perf}(\mathsf{X})$. In this sense, the dg categories can be regarded as noncommutative counterpart of varieties. In order to focus on the nice spaces, for example, the Chow motive concerns the proper smooth varieties, we restrict the dg-cat to the smooth proper dg categories.

Definition 2.10. A dg category $\mathcal{A}$ is called smooth if $\mathcal{A}$ is a perfect $\mathcal{A}-\mathcal{A}$ bi-module. It is called smooth and proper if $\mathcal{A}$ is derived Morita equivalent to a smooth dg algebra of finite type.

It is well known that the property of dg categories being smooth and proper is closed under derived Morita equivalence and tensor product [31, Chapter 1, Theorem 1.43]. People also define the properness as $\mathsf{Hom}_{\mathcal{A}}(\bullet,\bullet)$ being perfect k-mod. According to a book of G. Tabuada, "Noncommutative motive" [31, Proposition 1.45], such a definition of smooth and properness is equivalent to our definition.

Definition 2.11 (Noncommutative Chow motive [31]). We write $\mathsf{Hmo}_0^{\mathsf{sp}}$ as a full subcategory of Hmo_0 whose objects are smooth proper dg categories. Then, $\mathbb{Q}$ linearizing the category $\mathsf{Hmo}_0^{\mathsf{sp}}$, that is, the morphisms become $\mathsf{K}_0(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})_{\mathbb{Q}}$ [31, Corollary 1.44], we obtain $\mathsf{Hmo}_{0,\mathbb{Q}}^{\mathsf{sp}}$. Then, we define $\mathsf{NChow}_{\mathbb{Q}}$ to be the idempotent completion of $\mathsf{Hmo}_{0,\mathbb{Q}}^{\mathsf{sp}}$.

There is a universal additive invariant

$$\mathcal{U}\colon \mathsf{dg\text{-}cat}^{\mathsf{sp}} \to \mathsf{NChow}\,.$$

Let $\underline{\mathbb{C}}$ be the category with one object whose morphism space is $\mathbb{C}$. Then for any $\mathcal{A}\in$ $\mathsf{dg\text{-}cat}$, $\mathsf{Hom}_{\mathsf{NChow}}(\mathcal{U}(\mathbb{C}),\mathcal{U}(\mathcal{A}))\cong \mathsf{K}_0(\mathsf{rep}(\mathcal{A}))\cong \mathsf{K}_0(\mathcal{A}):=\mathsf{K}_0(\mathsf{D}^c(\mathcal{A}))$. Since we have a functorial morphism $\mathsf{Hom}_{\mathsf{NChow}}(\mathcal{U}(\mathbb{C}),\mathcal{U}(\mathcal{A}))\to \mathsf{Hom}_{\mathbb{C}}(\mathsf{HH}_0(\mathbb{C}),\mathsf{HH}_0(\mathcal{A}))$, there is a Chern character map

$$\mathsf{Ch}\colon \mathsf{K}_0(\mathcal{A})\to \mathsf{HH}_0(\mathcal{A}).$$

Given any $\mathcal{A}$ module $X\in \mathsf{D}^c(\mathcal{A})$, it is defined via the following diagram of dg categories:

$$\begin{tikzcd} & \mathsf{Per}_{\mathsf{dg}}(\mathcal{A}) \\ \underline{\mathbb{C}} \arrow[ur, "X"] & \mathcal{A}. \arrow[u] \end{tikzcd}$$

It induces morphisms of Hochschild complexes naturally, and then an element in $\mathsf{HH}_0(\mathcal{A})$ via the isomorphism $\mathsf{HH}_0(\mathcal{A})\cong \mathsf{HH}_0(\mathsf{Per}_{\mathsf{dg}}(\mathcal{A}))$. The isomorphism is a derived Morita equivalence, because of the Yoneda embedding $\mathcal{A}\to\mathsf{Per}_{\mathsf{dg}}(\mathcal{A})$. Here, $\mathsf{Per}_{\mathsf{dg}}(\mathcal{A})$ is defined as a full subcategory of the dg$\mathcal{A}$ module whose objects are isomorphic to objects in $\mathsf{Perf}(\mathcal{A})$.

In general, given any additive invariant F with $\mathsf{F}(\mathsf{k})\cong\mathsf{k}$, we have a Chern character map $\mathsf{K}_0(\mathcal{A})\to\mathsf{F}(\mathcal{A})$, for example, the (periodic) cyclic homology, and the negative cyclic homology.

It is natural to ask what the relations between the Chow motive $\mathsf{Chow}_{\mathbb{Q}}$ and the noncommutative Chow motive $\mathsf{NChow}_{\mathbb{Q}}$ are. There is a nice answer due to remarkable works by M. Kontsevich and G. Tabuada.

Theorem 2.12 ([30, Theorem 1.1]). *There is a symmetric monoidal functor*

$$\phi\colon \mathsf{SmProjec}^{\mathrm{op}} \to \mathsf{dg\text{-}cat}^{\mathrm{op}}, \quad \mathsf{X} \mapsto \mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$$

such that the following natural diagram is commutative:

$$\begin{tikzcd}
\mathsf{SmProjec}^{\mathrm{op}} \arrow[rr, "\phi"] \arrow[d] & & \mathsf{dg\text{-}cat}^{\mathsf{sp}} \arrow[d] \\
\mathsf{Chow}_{\mathbb{Q}} \arrow[d] & & \mathsf{Hmo}_0^{\mathsf{sp}} \arrow[d] \\
\mathsf{Chow}_{\mathbb{Q}} / - \otimes \mathbb{Q}(1) \arrow[r, "\phi'"] & \mathsf{NChow}_{\mathbb{Q}} \subset & \mathsf{Hmo}_{0,\mathbb{Q}}^{*}.
\end{tikzcd}$$

With this commutative diagram, G. Tabuada was able to generalize some famous conjectures to the noncommutative spaces, see "Noncommutative counterparts of celebrated conjectures" [32].

3. Noncommutative Hodge conjecture

In this section, we propose the noncommutative Hodge conjecture, and prove that the noncommutative Hodge conjecture is additive for semi-orthogonal decomposition. We obtain more evidence of the Hodge conjecture via good knowledge of semi-orthogonal decompositions. Finally, we prove that the noncommutative Hodge conjecture is true for smooth proper connective dg algebras.

3.1. Formulation

Definition 3.1. Let $\mathscr{A}$ be a small dg category. The Hodge class of $\mathscr{A}$ is defined as

$$\mathsf{Hodge}(\mathscr{A}) := \pi(\mathsf{j}^{-1}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathscr{A})_{\mathbb{Q}}))) \subset \mathsf{HH}_0(\mathscr{A}),$$

$$\begin{tikzcd}
 & & \mathsf{HH}_0(\mathscr{A}) \\
\mathsf{K}_0(\mathscr{A}) \arrow[r, "\mathsf{Ch}"'] \arrow[rru, "\mathsf{Ch}"] \arrow[d] & \mathsf{HN}_0(\mathscr{A}) \arrow[ru, "\pi"'] \arrow[d, "\mathsf{j}"] & \\
\mathsf{K}_0^{\mathsf{top}}(\mathscr{A}) \arrow[r, "\mathsf{Ch}^{\mathsf{top}}"] & \mathsf{HC}_0^{\mathsf{per}}(\mathscr{A}). &
\end{tikzcd}$$

Conjecture 3.2 (Noncommutative Hodge conjecture). *The Chern character* $\mathsf{Ch}\colon \mathsf{K}_0(\mathscr{A}) \mapsto \mathsf{HH}_0(\mathscr{A})$ *maps* $K_0(\mathscr{A})_{\mathbb{Q}}$ *surjectively into the Hodge class* $\mathsf{Hodge}(\mathscr{A})$.

Remark 3.3. Note that we obtain the abstract rational Hodge class in $\mathsf{HH}_0(\mathcal{A})$. Classically, the Hodge conjecture concerns the weight. However, to the author's knowledge, we do not know how to obtain the weight of the abstract Hodge class. In this paper, we always assume the conjecture as a non-weighted Hodge conjecture.

Example 3.4. Let $\mathsf{X} = \mathsf{Spec}\,\mathsf{A}$ be a smooth affine connected variety over $\mathbb{C}$, and $\mathcal{A} = \mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$. Then we have

$$\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}) = \mathsf{HN}_0(\mathcal{A}) \cong \mathbb{C} \oplus \bigoplus_{i \geq 1} \mathsf{H}_{\mathsf{dR}}^{2i}(X, \mathbb{C}).$$

The projection $\pi \colon \mathsf{HN}_0(\mathcal{A}) \to \mathsf{HH}_0(\mathcal{A})$ maps $\bigoplus_{i \geq 1} \mathsf{H}_{\mathsf{dR}}^{2i}(X, \mathbb{C})$ to 0, and $\mathbb{C}$ to $\mathsf{HH}_0(\mathcal{A}) = \mathsf{A}$ as inclusion of functions. The Hodge classes are $\pi(\mathbb{Q}) = \mathbb{Q} \subset \mathsf{A}$. Clearly, $\mathsf{Ch}(\mathsf{K}_0(\mathcal{A})_{\mathbb{Q}}) = \mathbb{Q}$ which is exactly the Hodge class. Thus, the noncommutative Hodge conjecture is true for $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$.

Theorem 3.5. *Conjecture 3.2 is equivalent to the one in Alex Perry's paper* [25, Conjecture 5.11] *for admissible subcategories of* $\mathsf{D}^{\mathsf{b}}(\mathsf{X})$.

Proof. For the admissible subcategory $\mathcal{A} \subset \mathsf{D}^{\mathsf{b}}(\mathsf{X})$, Alex Perry defines the Hodge classes of $\mathcal{A}$ as the classes of $\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}} \subset \mathsf{K}_0^{\mathsf{top}}(\mathcal{A}) \otimes \mathbb{C}$ that lie in $\mathsf{HH}_0(\mathcal{A})$ under the isomorphism

$$\mathsf{K}_0^{\mathsf{top}}(\mathcal{A}) \otimes \mathbb{C} \xrightarrow{\mathsf{Ch}^{\mathsf{top}}} \mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}) \xrightarrow{\cong} \bigoplus_i \mathsf{HH}_{2i}(\mathcal{A}) .$$

Note that there is a natural choice of splitting of the Hodge filtration of $\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A})$ induced from that of X, and then we have a natural decomposition $\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}) \cong \bigoplus_i \mathsf{HH}_{2i}(\mathcal{A})$. The map $\mathsf{j} \colon \mathsf{HN}_0(\mathcal{A}) \to \mathsf{HC}_0^{\mathsf{per}}(\mathcal{A})$ is injective by degeneration of the noncommutative Hodge-to-de Rham spectral sequence. Choose the natural splitting of the Hodge decomposition of $\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A})$ (the one in [25]), and induce a splitting for $\mathsf{HN}_0(\mathcal{A})$, we get a commutative diagram,

$$\begin{tikzcd}
 & & & \mathsf{HH}_0(\mathcal{A}) \arrow[ddd, "="] \\
\mathsf{K}_0(\mathcal{A}) \arrow[r] \arrow[urrr, "\mathsf{Ch}"] \arrow[d] & \mathsf{HN}_0(\mathcal{A}) \arrow[r, "\cong", "\mathsf{H}"'] \arrow[urr, "\pi"] \arrow[d, "\mathsf{j}"] & \bigoplus_{i \leq 0} \mathsf{HH}_{2i}(\mathcal{A}) \arrow[ur, "\mathsf{Pr}"] \arrow[d] & \\
\mathsf{K}_0^{\mathsf{top}}(\mathcal{A}) \arrow[r, "\mathsf{Ch}^{\mathsf{top}}"] & \mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}) \arrow[r, "\cong", "\mathsf{H}"'] & \bigoplus_i \mathsf{HH}_{2i}(\mathcal{A}) \arrow[dr, "\mathsf{Pr}"] & \\
 & & & \mathsf{HH}_0(\mathcal{A}).
\end{tikzcd}$$

Note that the projection $\mathsf{Pr} \circ \mathsf{H} \colon \mathsf{HN}_0(\mathcal{A}) \to \mathsf{HH}_0(\mathcal{A})$ is naturally the morphism π. We show that the Hodge class defined in [25] is isomorphic to the image $\mathrm{pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A})))$ in $\mathsf{HH}_0(\mathcal{A})$. It is true if replacing $\mathcal{A}$ with $\mathsf{D}^{\mathsf{b}}(\mathsf{X})$: identify $\mathsf{HC}_0^{\mathsf{per}}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))$ with

$\mathsf{H}^{\mathsf{even}}(X, \mathbb{C})$, the Hodge class defined in [25] and the space $\mathrm{pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))))$ are both rational Hodge classes $\mathsf{H}^{\mathsf{even}}(\mathsf{X}, \mathbb{Q}) \cap \bigoplus_p \mathsf{H}^{\mathsf{p,p}}(\mathsf{X}, \mathbb{C})$. Therefore, the Hodge class defined in [25] for the admissible subcategory $\mathcal{A} \subset \mathsf{D}^{\mathsf{b}}(\mathsf{X})$ is isomorphic to $\mathrm{pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A})))$ by additivity. According to the commutative diagram above, $\mathrm{pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A})))$ is exactly the class

$$\pi(\mathsf{j}^{-1}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}))) \subset \mathsf{HH}_0(\mathcal{A}). \qquad \blacksquare$$

Lemma 3.6. *Let $\mathcal{A}$ be a smooth proper* dg *category, the noncommutative Hodge-to-de Rham spectral sequence degenerates* [10].

Let $\mathcal{A}$ be a smooth proper dg category. By Lemma 3.6, we can choose a splitting of the Hodge filtration of $\mathsf{HC}_0^{\mathsf{per}}(\mathcal{A})$, and then we have an isomorphism $\mathsf{H}\colon \mathsf{HC}_0^{\mathsf{per}}(\mathcal{A}) \cong \bigoplus_{2i} \mathsf{HH}_{2i}(\mathcal{A})$.

Definition 3.7 (Hodge conjecture for smooth proper dg categories). Define the Hodge classes in $\mathsf{HH}_0(\mathcal{A})$ as $\mathrm{pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A})))$. The Hodge conjecture of $\mathcal{A}$ is that the Chern character $\mathsf{Ch}\colon \mathsf{K}_0(\mathcal{A}) \to \mathsf{HH}_0(\mathcal{A})$ maps $\mathsf{K}_0(\mathcal{A})_{\mathbb{Q}}$ surjectively into the Hodge classes.

Remark 3.8. This is equivalent to Conjecture 3.2 by identifying $\mathrm{pr} \circ \mathsf{H}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}) \cap \mathsf{j}(\mathsf{HN}_0(\mathcal{A})))$ with $\pi(\mathsf{j}^{-1}(\mathsf{Ch}^{\mathsf{top}}(\mathsf{K}_0^{\mathsf{top}}(\mathcal{A})_{\mathbb{Q}}))) \subset \mathsf{HH}_0(\mathcal{A})$. Thus, the conjecture is independent of the choice of splitting H.

Remark 3.9. Let X be a smooth projective variety. Combining Theorem 3.5 and Perry's work [25], the Hodge conjecture for $\mathsf{X} \Leftrightarrow$ the noncommutative Hodge conjecture for $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$. For completeness, we provide the proof here. The rational Hodge conjecture claims that the Chern character $\mathsf{Ch}\colon \mathsf{K}_0(\mathsf{X})_{\mathbb{Q}} \to \bigoplus_{\mathsf{p}} \mathsf{H}^{\mathsf{p,p}}(\mathsf{X}, \mathbb{C})$ maps $\mathsf{K}_0(\mathsf{X})_{\mathbb{Q}}$ surjectively to the rational Hodge classes. The noncommutative Hodge conjecture claims that the map $\mathsf{Ch}_{\mathbb{Q}}\colon \mathsf{K}_0(\mathsf{X})_{\mathbb{Q}} = \mathsf{K}_0(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))_{\mathbb{Q}} \to \mathsf{Hodge}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))$ is surjective. There is a commutative diagram

$$\begin{tikzcd}
\mathsf{HH}_0(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})) \arrow[ddd, "\cong"] & & \mathsf{K}_0^{\mathsf{top}}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})) \arrow[dr] \arrow[ddd] & \\
& \mathsf{K}_0(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})) \arrow[ul, "\mathsf{Ch}"'] \arrow[ur] \arrow[r, "\mathsf{Ch}"] \arrow[d, "\cong"] & \mathsf{HN}_0(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})) \arrow[r, hook] \arrow[ull, "\pi"'] \arrow[d, "\cong"] & \mathsf{HC}_0^{\mathsf{per}}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})) \arrow[d, "\cong"] \\
& \mathsf{K}_0(\mathsf{X}) \arrow[r, "\mathsf{Ch}"] \arrow[dl, "\mathsf{Ch}"'] \arrow[dr] & \bigoplus_{i \leq 0} \mathsf{H}^{\mathsf{p,p}-2i}(\mathsf{X}, \mathbb{C}) \arrow[r, hook] \arrow[dll, "\pi"] & \mathsf{H}_{\mathsf{dR}}^{\mathsf{even}}(\mathsf{X}, \mathbb{C}) \\
\bigoplus_{\mathsf{p}} \mathsf{H}^{\mathsf{p,p}}(\mathsf{X}, \mathbb{C}) & & \mathsf{K}_0^{\mathsf{top}}(\mathsf{X}). \arrow[ur] &
\end{tikzcd}$$

We explain the commutative diagram. There is a natural quasi-isomorphism of double complexes of the periodic cyclic homology $\mathsf{Tot}^{\bullet,\bullet}(\mathsf{Per}_{dg}(X)) \to \mathsf{Tot}^{\bullet,\bullet}(\mathsf{R\Gamma}(\bigoplus \Omega_X^i[i]))$ which is described by B. Keller in [11].

After identifying $\mathsf{HC}_0^{per}(\mathsf{Per}_{dg}(X))$ with $\mathsf{H}_{dR}^{even}(X, \mathbb{C})$, the noncommutative Chern character becomes the usual Chern character. The reader can refer to C. Weibel [37, Proposition 3.8.1] or [6, Proposition 4.32]. Hence, the noncommutative Chern character maps $\mathsf{K}_0(X)_{\mathbb{Q}}$ surjectively to the noncommutative rational Hodge classes if and only if the Chern character maps $\mathsf{K}_0(X)_{\mathbb{Q}}$ surjectively to the rational Hodge classes.

Theorem 3.10. *Suppose* $\mathsf{F}\colon \mathscr{A} \to \mathscr{B}$ *is a derived Morita equivalence, then the Hodge conjecture is true for* $\mathscr{A}$ *if and only if it is true for* $\mathscr{B}$.

Proof. The topological and algebraic K-theory, Hochschild homology, periodic (negative) cyclic homology are all additive invariants. We have a commutative diagram,

$$\begin{tikzcd}
\mathsf{K}_0^{top}(\mathscr{A}) \arrow[rrr, "\cong"] \arrow[dddr] & & & \mathsf{K}_0^{top}(\mathscr{B}) \arrow[dddl] \\
& \mathsf{K}_0(\mathscr{A}) \arrow[ul] \arrow[r, "\cong"] \arrow[d] \arrow[dddl, "\mathsf{Ch}"'] & \mathsf{K}_0(\mathscr{B}) \arrow[ur] \arrow[d] \arrow[dddr, "\mathsf{Ch}"] & \\
& \mathsf{HN}_0(\mathscr{A}) \arrow[r, "\cong"] \arrow[d] \arrow[ddl, "\pi"] & \mathsf{HN}_0(\mathscr{B}) \arrow[d] \arrow[ddr, "\pi"'] & \\
& \mathsf{HC}_0^{per}(\mathscr{A}) \arrow[r, "\cong"] & \mathsf{HC}_0^{per}(\mathscr{B}) & \\
\mathsf{HH}_0(\mathscr{A}) \arrow[rrr, "\cong"] & & & \mathsf{HH}_0(\mathscr{B})
\end{tikzcd}$$

whose rows are isomorphisms. It is clear that any morphism of dg categories induces a morphism of Hodge classes: write ϕ as the corresponding morphism from additive invariants of $\mathscr{A}$ to $\mathscr{B}$. Let $x \in \mathsf{Hodge}(\mathscr{A})$, this implies that there is $x' \in \mathsf{HN}_0(\mathscr{A})$ such that $\pi(x') = x$, and $y \in \mathsf{K}_0^{top}(\mathscr{A})_{\mathbb{Q}}$ such that $j(x') = \mathsf{Ch}_{\mathbb{Q}}^{top}(y)$. Applying ϕ, we get $\phi(x) = \pi(\phi(x'))$, and $j(\phi(x')) = \mathsf{Ch}_{\mathbb{Q}}^{top}(\phi(y))$, that is, $\phi(x) \in \mathsf{Hodge}(\mathscr{B})$. There is a commutative diagram,

$$\begin{tikzcd}
\mathsf{K}_0(\mathscr{A}) \arrow[r, "\cong"] \arrow[d, "\mathsf{Ch}"] & \mathsf{K}_0(\mathscr{B}) \arrow[d, "\mathsf{Ch}"] \\
\mathsf{Hodge}(\mathscr{A}) \arrow[r, "\cong"] & \mathsf{Hodge}(\mathscr{B}).
\end{tikzcd}$$

The isomorphism of Hodge classes is as follows: take $z \in \mathsf{Hodge}(\mathscr{B})$, since ϕ induces an isomorphism $\mathsf{HH}_0(\mathscr{A}) \cong \mathsf{HH}_0(\mathscr{B})$, there exists a unique $x \in \mathsf{HH}_0(\mathscr{A})$ such that $\phi(x) = z$. It can be shown that $x \in \mathsf{Hodge}(\mathscr{A})$ by diagram chasing. ∎

Corollary 3.11. *For the unique enhanced triangulated categories, we can define its Hodge conjecture via* dg *enhancement. The Hodge conjecture does not depend on the* dg *enhancement.*

Proof. This is because two dg enhancements of the unique enhanced triangulated categories are connected by a chain of quasi-equivalences, and the corollary follows from Theorem 3.10. ∎

Remark 3.12. For a projective smooth variety X, $\mathsf{D^b(X)} \cong \mathsf{Perf(X)}$ is a unique enhanced triangulated category. Thus, it suffices to check whether the conjecture is true for any pre-triangulated dg enhancement of $\mathsf{D^b(X)}$.

Combining Theorem 3.5 and Perry's work [25], we get the following theorem. Here we provide an independent proof.

Theorem 3.13. *Suppose we have an* SOD, $\mathsf{D^b(X)} = \langle \mathcal{A}, \mathcal{B} \rangle$. *There are natural* dg *enhancements* $\mathcal{A}_{\mathsf{dg}}$, $\mathcal{B}_{\mathsf{dg}}$ *of* $\mathcal{A}$, $\mathcal{B}$ *corresponding to the* dg *enhancement* $\mathsf{Per_{dg}(X)}$ *of* $\mathsf{D^b(X)}$.

$$\text{Hodge conjecture for } \mathsf{X} \Leftrightarrow \text{noncommutative Hodge conjecture for } \mathcal{A}_{\mathsf{dg}} \text{ and } \mathcal{B}_{\mathsf{dg}}.$$

Proof. We still write $\mathcal{A}$ and $\mathcal{B}$ as dg categories corresponding to the natural dg enhancement again. We can lift the semi-orthogonal decomposition to the dg world by [19, Proposition 4.10]. That is, there is a diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, shift right, "\mathsf{i}"'] & \mathsf{D} \arrow[l, shift right, "\mathsf{R}"'] \arrow[r, shift right, "\mathsf{L}"'] & \mathcal{A} \arrow[l, shift right, "\mathsf{j}"'] \end{tikzcd}$$

where D is a certain gluing of $\mathcal{A}$ and $\mathcal{B}$ and it is quasi-equivalent to $\mathsf{Per_{dg}(X)}$. Therefore, we still have a diagram such that $\mathsf{i} + \mathsf{j}$ induces an isomorphism of K-groups, and $\mathsf{i_H} + \mathsf{j_H}$ induces

$$\begin{tikzcd} \mathsf{K_0}(\mathcal{B}) \arrow[r, shift right, "\mathsf{i}"'] \arrow[d, "\mathsf{Ch}"] & \mathsf{K_0(D)} \arrow[l, shift right, "\mathsf{R}"'] \arrow[r, shift right, "\mathsf{L}"'] \arrow[d, "\mathsf{Ch}"] & \mathsf{K_0}(\mathcal{A}) \arrow[l, shift right, "\mathsf{j}"'] \arrow[d, "\mathsf{Ch}"] \\ \mathsf{Hodge}(\mathcal{B}) \arrow[r, shift right, "\mathsf{i_H}"'] & \mathsf{Hodge(D)} \arrow[l, shift right, "\mathsf{R_H}"'] \arrow[r, shift right, "\mathsf{L_H}"'] & \mathsf{Hodge}(\mathcal{A}). \arrow[l, shift right, "\mathsf{j_H}"'] \end{tikzcd}$$

Hence $\mathsf{Ch_{D,\mathbb{Q}}}$ maps $\mathsf{K_0}(\mathcal{D})_{\mathbb{Q}}$ surjectively to $\mathsf{Hodge(D)}$ if and only if $\mathsf{Ch}_{\mathcal{B},\mathbb{Q}}$ and $\mathsf{Ch}_{\mathcal{A},\mathbb{Q}}$ map $\mathsf{K_0}(\mathcal{B})_{\mathbb{Q}}$ and $\mathsf{K_0}(\mathcal{A})_{\mathbb{Q}}$ surjectively to $\mathsf{Hodge}(\mathcal{B})$ and $\mathsf{Hodge}(\mathcal{A})$, respectively. But the noncommutative Hodge conjecture is true for D if and only if it is true for the Hodge conjecture of X by Remark 3.9 and Theorem 3.10. Thus, the statement follows. ∎

Remark 3.14. The statement is still true if there are more than two components for SODs.

Corollary 3.15. *Let* X *be a projective smooth variety, suppose there is an* SOD, $\mathsf{D^b(X)} = \langle \mathsf{D^b(Z)}, \mathsf{D^b(Y)} \rangle$. *Then the Hodge conjecture is true for* X *if and only if for* Z *and* Y. *In particular, the Hodge conjecture is a derived invariant.*

Proof. The Hodge conjecture is true for X if and only if it is true for the corresponding dg enhancement of $\mathsf{D}^{\mathsf{b}}(\mathsf{Z})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{Y})$. Since $\mathsf{D}^{\mathsf{b}}(\mathsf{Z})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{Y})$ are unique enhanced triangulated categories [20], the Hodge conjecture is true for X if and only if for Z and Y. ∎

Corollary 3.16. *Consider the blow-up X of Y with smooth center Z, according to Orlov's blow-up formula* [7, Theorem 4.2], *we have an* SOD, $\mathsf{D}^{\mathsf{b}}(\mathsf{X}) = \langle \mathsf{D}^{\mathsf{b}}(\mathsf{Z}), \ldots, \mathsf{D}^{\mathsf{b}}(\mathsf{Z}), \mathsf{D}^{\mathsf{b}}(\mathsf{Y})\rangle$. *Hence the Hodge conjecture is true for X if and only if for Z and Y.*

Remark 3.17. It was known by a classical method. We can even write down the Chow groups with respect to the blow-up, for explicit details, the reader can refer to the book of C. Voisin, "Hodge theory and complex algebraic geometry II" [35, Theorem 9.27]

For low dimensional varieties, Hodge conjecture is a birational invariant. We use the following lemma.

Lemma 3.18 ([1, Theorem 0.1.1]). *Let X and Y be proper smooth varieties. If X is birational to Y, then there is a chain of blow-ups and blow-downs of smooth centers connecting X and Y,*

$$\begin{tikzcd}[column sep=small]
 & \mathsf{X}_1 \arrow[dl, dashed] \arrow[dr, dashed] & & \cdots & & \mathsf{X}_3 \arrow[dl, dashed] \arrow[dr, dashed] & \\
\mathsf{X} & & \mathsf{X}_2 & & & & \mathsf{Y}.
\end{tikzcd}$$

The following may be well known for the experts, see also [22]. Here, we use the noncommutative techniques to prove the results.

Theorem 3.19. *Since the Hodge conjecture is true for* 0-, 1-, 2- *and* 3-*dimensional varieties, the Hodge conjecture is a birational invariant for* 4- *and* 5-*dimensional varieties.*

Proof. Combine Corollary 3.16 and Lemma 3.18, and observe that X and Y are connected by a chain of blow-ups of smooth center whose dimension is less or equal to 3. ∎

3.2. Application to geometry and examples

The survey "Noncommutative counterparts of celebrated conjecture" [32, Section 2] provides many examples of the applications to the geometry for some celebrated conjectures. The examples also apply to the noncommutative Hodge conjecture. In this subsection, we still show some interesting examples.

There is a universal functor

$$\mathcal{U}\colon \mathsf{dg\text{-}cat} \to \mathsf{NChow}.$$

We call $\mathcal{U}(\mathcal{A})$ the noncommutative Chow motive that corresponds to $\mathcal{A}$. We write the image of $\mathcal{U}(\mathcal{A})$ in $\mathsf{NChow}_{\mathbb{Q}}$ as $\mathcal{U}(\mathcal{A})_{\mathbb{Q}}$. Similar to works by G. Tabuada, the noncommutative Hodge conjecture is compatible with the direct sum decomposition of the noncommutative Chow motives.

Theorem 3.20. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be smooth and proper* dg *categories. Suppose there is a direct sum decomposition $\mathcal{U}(\mathcal{C})_{\mathbb{Q}} \cong \mathcal{U}(\mathcal{A})_{\mathbb{Q}} \oplus \mathcal{U}(\mathcal{B})_{\mathbb{Q}}$, then the noncommutative Hodge conjecture holds for $\mathcal{C}$ if and only if it holds for $\mathcal{A}$ and $\mathcal{B}$.*

Proof. This follows from the fact that the periodic (negative) cyclic homology and rational (topological or algebraic) K-theory are all additive invariants, and the corresponding target categories are idempotent complete. The proof is similar to Theorem 3.13. ∎

Example 3.21. Suppose we have a semi-orthogonal decomposition

$$\mathsf{H}^0(\mathcal{C}) = \langle \mathsf{H}^0(\mathcal{A}), \mathsf{H}^0(\mathcal{B}) \rangle,$$

then $\mathcal{U}(\mathcal{C}) \cong \mathcal{U}(\mathcal{A}) \oplus \mathcal{U}(\mathcal{B})$.

3.2.1. Fractional Calabi–Yau categories.

Theorem 3.22 ([18, Theorem 3.5]). *Let X be a hypersurface of degree $\leq n+1$ in $\mathbb{P}^n$. There is a semi-orthogonal decomposition*

$$\mathsf{Perf}(\mathsf{X}) = \langle \mathcal{T}(\mathsf{X}), \mathcal{O}_{\mathsf{X}}, \ldots, \mathcal{O}_{\mathsf{X}}(n - \mathsf{deg}(\mathsf{X})) \rangle,$$

where $\mathcal{T}(\mathsf{X})$ is a fractional Calabi–Yau of dimension $\frac{(n+1)(\mathsf{deg}(\mathsf{X})-2)}{\mathsf{deg}(\mathsf{X})}$. Then

$$\mathcal{U}(\mathsf{X}) \cong \mathcal{U}(\mathcal{T}_{\mathsf{dg}}(\mathsf{X})) \oplus \mathcal{U}(\mathsf{k}) \oplus \cdots \oplus \mathcal{U}(\mathsf{k}).$$

Therefore, the Hodge conjecture of X $\Leftrightarrow$ the noncommutative Hodge conjecture of $\mathcal{T}_{\mathsf{dg}}(\mathsf{X})$.

3.2.2. Twisted scheme.

Definition 3.23. Let X be a scheme with structure sheaf $\mathcal{O}_{\mathsf{X}}$. $\mathcal{A}$ is a sheaf of the Azumaya algebra over X. We call the derived category of a perfect $\mathcal{A}$ module $\mathsf{Perf}(\mathsf{X}, \mathcal{A})$ the twisted scheme.

Theorem 3.24. *Noncommutative Hodge conjecture for $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}, \mathcal{A})$ $\Leftrightarrow$ noncommutative Hodge conjecture for $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$.*

Proof. According to [33, Theorem 2.1], $\mathcal{U}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}, \mathcal{A}))_{\mathbb{Q}} \cong \mathcal{U}(\mathsf{Per}_{\mathsf{dg}}(\mathsf{X}))_{\mathbb{Q}}$. Thus, by Theorem 3.20, the statement follows. ∎

3.2.3. Cubic fourfold containing a plane.

Example 3.25. Let X be a cubic fourfold containing a plane. There is a semi-orthogonal decomposition [16, Theorem 4.3]

$$\mathsf{Perf}(\mathsf{X}) = \langle \mathsf{Perf}(\mathsf{S}, \mathcal{A}), \mathcal{O}_{\mathsf{X}}, \mathcal{O}_{\mathsf{X}}(1), \mathcal{O}_{\mathsf{X}}(2) \rangle,$$

where S is a K_3 surface, and $\mathcal{A}$ is a sheaf of the Azumaya algebra over S. Since the noncommutative Hodge conjecture is true for $\mathsf{Per}_{\mathsf{dg}}(\mathsf{S}, \mathcal{A})$ which is unique enhanced, the Hodge conjecture is true for X.

3.2.4. Quadratic fibration.

Example 3.26. Let $f\colon X \to S$ be a smooth quadratic fibration, for example, the smooth quadric in the relative projective space $\mathbb{P}^n_S$. There is a semi-orthogonal decomposition

$$\mathrm{Perf}(X) = \langle \mathrm{Perf}(S, \mathrm{Cl}_0), \mathrm{Perf}(S), \dots, \mathrm{Perf}(S)\rangle,$$

where Cl_0 is a sheaf of the Azumaya algebra over S if the dimension n of the fiber of f is odd [15]. Thus, the Hodge conjecture of $X \Leftrightarrow S$. Moreover, if $\dim S \leq 3$, the Hodge conjecture for X is true.

3.2.5. HP duality. Let X be a projective smooth variety with morphism $f\colon X \to \mathbb{P}(V)$. Set $\mathcal{O}_X(1) = f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Assume there is an SOD

$$D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{m-1}(m-1)\rangle$$

where $\mathcal{A}_{m-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0$. Define $H := X \times_{\mathbb{P}(V)} Q$, where Q is the incidence quadric in $\mathbb{P}(V) \times \mathbb{P}(V^*)$. Then, there is an SOD

$$D^b(H) = \langle \mathcal{L}, \mathcal{A}_{1,\mathbb{P}(V^*)}(1), \dots, \mathcal{A}_{m-1,\mathbb{P}(V^*)}(m-1)\rangle.$$

The projective smooth variety Y with morphism $g\colon Y \to \mathbb{P}(V^*)$ is called homological projective dual of X if there is an object $\mathcal{E} \in D^b(H \times_{\mathbb{P}(V^*)} Y)$ which induces an equivalence from $D^b(Y)$ into $\mathcal{L}$.

We refer to [17, Section 2.3] or Kuznetsov's original paper [14]. Let (Y, g) be an HP dual of (X, f), then

(1) There is an SOD

$$D^b(Y) = \langle \mathcal{B}_{n-1}(1-n), \dots, \mathcal{B}_1(-1), \mathcal{B}_0\rangle$$

where $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$. Moreover, $\mathcal{A}_0 \cong \mathcal{B}_0$ via a Fourier–Mukai functor.

(2) (Symmetry) (X, f) is an HP dual of (Y, g).

(3) For any subspace $L \subset V^*$, define $X_L = X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L = Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$. If we assume that they have the expected dimension, $\dim X_L = \dim X - \dim L$, $\dim Y_L = \dim Y - (\dim V - \dim L)$, and write $\dim L = r$, $\dim V = N$, then there are SOD such that $\mathcal{L}_{X,L} \cong \mathcal{L}_{Y,L}$,

$$D^b(X_L) = \langle \mathcal{L}_{X,L}, \mathcal{A}_r(r), \dots, \mathcal{A}_{m-1}(m-1)\rangle,$$
$$D^b(Y_L) = \langle \mathcal{B}_{n-1}(1-n), \dots, \mathcal{B}_{N-r}(r-N), \mathcal{L}_{Y,L}\rangle.$$

Theorem 3.27. *We write* $\mathrm{Hodge}(\bullet)$ *if the (noncommutative) Hodge conjecture is true for varieties (smooth and proper* dg *categories). Then,*

$$\mathrm{Hodge}(X) \Leftrightarrow \mathrm{Hodge}(\mathcal{A}_0) \Leftrightarrow \mathrm{Hodge}(\mathcal{B}_0) \Leftrightarrow \mathrm{Hodge}(Y).$$

If we assume $\mathrm{Hodge}(X)$*, then* $\mathrm{Hodge}(X_L) \Leftrightarrow \mathrm{Hodge}(Y_L)$.

Proof. The midterm equivalence $\mathsf{Hodge}(\mathcal{A}_0) \Leftrightarrow \mathsf{Hodge}(\mathcal{B}_0)$ is because $\mathcal{A}_0 \cong \mathcal{B}_0$ via a Fourier–Mukai functor, and then there is an isomorphism of natural dg enhancements $\mathcal{A}_{\mathsf{dg},0} \cong \mathcal{B}_{\mathsf{dg},0}$ in Hmo, see a proof in [4, Section 9]. Since $\mathcal{L}_{\mathsf{X,L}} \cong \mathcal{L}_{\mathsf{Y,L}}$ via a Fourier–Mukai functor, the statement $\mathsf{Hodge}(\mathsf{X_L}) \Leftrightarrow \mathsf{Hodge}(\mathsf{Y_L})$ follows from the same argument. ∎

Remark 3.28. The HPD can be generalized to the noncommutative version, see the discussion in [17, Section 3.4] or the paper by Alexander Perry, “Noncommutative homological projective duality” [24].

Example 3.29. One of the nontrivial examples of the homological projective duality comes from the Grassmannian–Pfaffian duality. Let W be a dimension n vector space, $\mathsf{X} = \mathsf{Gr}(2, \mathsf{W})$ the Grassmannian of 2-dimensional sub-vector spaces of W. Consider the projective space $\mathbb{P}(\wedge^2 \mathsf{W}^*)$, there is a natural filtration called the Pfaffian filtration: $\mathsf{Pf}(2, \mathsf{W}^*) \subset \mathsf{Pf}(4, \mathsf{W}^*) \cdots \subset \mathbb{P}(\wedge^2 \mathsf{W}^*)$,

$$\mathsf{Pf}(2\mathsf{k}, \mathsf{W}^*) = \{\omega \in \mathbb{P}(\wedge^2 \mathsf{W}^*) \mid \mathsf{rank}(\omega) \leq 2\mathsf{k}\}.$$

The intermediate Pfaffians are no longer smooth but with singularities. The singularity of $\mathsf{Pf}(2\mathsf{k}, \mathsf{W}^*)$ is $\mathsf{Pf}(2\mathsf{k}-2, \mathsf{W}^*)$. Classically, it was known that $\mathsf{Y} = \mathsf{Pf}(2\lfloor \frac{n}{2} \rfloor - 2, \mathsf{W}^*)$ is the classical projective dual of $\mathsf{X} = \mathsf{Gr}(2, \mathsf{W})$ via the Plücker embedding. For $n \leq 7$, the noncommutative categorical resolution of $\mathsf{Pf}(2\lfloor \frac{n}{2} \rfloor - 2, \mathsf{W}^*)$ is the homological projective dual of $\mathsf{Gr}(2, \mathsf{W})$. However, it was not known for the cases $n \geq 8$. The interested reader can refer to a survey [17, Section 4.4, Conjecture 4.4] or Kuznetsov’s original paper [13].

The known nontrivial Grassmannian–Pfaffian duality are the cases $n = 6, 7$. In these cases, the Hodge conjecture is true for X since it has a full exceptional collection, then the noncommutative Hodge conjecture is true for the noncommutative categorical resolution of the Pfaffians. However, the Hodge conjecture is trivial for the noncommutative category since it automatically has full exceptional collections, or the geometric resolution of the Pfaffians are of the form $\mathbb{P}_{\mathsf{Gr}(2,\mathsf{W})}(\mathsf{E})$ [13, Section 4] for some vector bundle E. It has a full exceptional collection too.

We expect to obtain duality of the Hodge conjecture for $\mathsf{X_L}$ and $\mathsf{Y_L}$ when they are smooth, and have the expected dimension. According to the Lefschetz hyperplane theorem, there is a commutative diagram for $i \leq \dim \mathsf{X_L} - 1$,

$$\begin{tikzcd}
\mathsf{CH}^i(\mathsf{X_L})_{\mathbb{Q}} \arrow[r] & \mathsf{H}^i(\mathsf{X_L}, \mathbb{Q}) \\
\mathsf{CH}^i(\mathsf{Gr}(2, \mathsf{W}))_{\mathbb{Q}} \arrow[u] \arrow[r, "\cong"] & \mathsf{H}^i(\mathsf{Gr}(2, \mathsf{W}), \mathbb{Q}). \arrow[u, "\cong"']
\end{tikzcd}$$

The Hodge conjecture is true for a weight less than $\dim \mathsf{X_L}$. By the hard Lefschetz isomorphism, it is still true for weights greater than $\dim \mathsf{X_L}$. Thus, if $\dim \mathsf{X_L}$ is odd, the Hodge conjecture for $\mathsf{X_L}$ is true.

The following examples for $n = 6, 7$ are from paper [13, Section 10].

(I). We have $n = 6$, $\dim \mathsf{X_L} = 8 - \dim \mathsf{L}$, $\dim \mathsf{Y_L} = \dim \mathsf{L} - 2$. When $\dim \mathsf{L} = 6$, the expected dimension of $\mathsf{X_L}$ is 2 while the expected dimension of $\mathsf{Y_L}$ is 4. This is the duality between the Pfaffian cubic fourfold and the K_3 surface [13]. When $\dim \mathsf{L} = 5$, $\dim \mathsf{X_L} = \dim \mathsf{Y_L} = 3$, the Hodge conjecture is true by a dimension reason. When $\dim \mathsf{L} = 4$, $\mathsf{Y_L} = \mathsf{Pf}(4,6) \cap \mathbb{P}^3$ is a cubic surface. Then $\mathsf{X_L} = \mathsf{Gr}(2,6) \cap \mathbb{P}^{10}$ has a full exceptional collection. Further, $\mathsf{X_L}$ is a rational Fano 4-fold [38, Section 2.2, Theorem 2.2.1]. Hence, the Hodge conjecture is true for $\mathsf{X_L}$ by the weak factorization theorem [1, Theorem 0.1.1]. When $\dim \mathsf{L} = 3$, $\dim \mathsf{X_L} = 5$, the Hodge conjecture is true for $\mathsf{X_L}$. When $\dim \mathsf{L} = 2$, $\mathsf{X_L}$ admits a full exceptional collection. We obtain Table 1.

$\dim \mathsf{L}$	$\dim \mathsf{X_L}$	$\dim \mathsf{Y_L}$	classically
2	6	0	
3	5	1	known
4	4	2	known, $\mathsf{X_L}$ is a rational Fano 4-fold
5	3	3	known, they are 3-fold
6	2	4	known, $\mathsf{Y_L}$ is a cubic 4-fold

Table 1. The case $n = 6$.

(II). We have $n = 7$, $\dim \mathsf{X_L} = 10 - \dim \mathsf{L}$, $\dim \mathsf{Y_L} = \dim \mathsf{L} - 4$. For example, take $\dim \mathsf{L} = 7$. The expected dimensions of $\mathsf{X_L}$ and $\mathsf{Y_L}$ are both 3. The Hodge conjecture is true for them by a dimension reason. When $\dim \mathsf{L} = 5$, $\dim \mathsf{X_L} = 5$, the Hodge conjecture is true for $\mathsf{X_L}$. When $\dim \mathsf{L} = 6$, $\dim \mathsf{X_L} = 4$, it is a Fano 4-fold. When $\dim \mathsf{L} = 8$, $\dim \mathsf{Y_L} = 4$, it is a Fano 4-fold. Since Fano varieties are uniruled, the Hodge conjecture is true for Fano 4-folds [8]. When $\dim \mathsf{Y_L} = 9$, $\mathsf{Y_L}$ is a Fano 5-fold, the Hodge conjecture is true for Fano 5-folds by [2]. When $\dim \mathsf{L} = 10$, $\mathsf{Y_L}$ admits a full exceptional collection. We obtain Table 2.

$\dim \mathsf{L}$	$\dim \mathsf{X_L}$	$\dim \mathsf{Y_L}$	classically
5	5	1	known, since the dimension of $\mathsf{X_L}$ is odd
6	4	2	known, $\mathsf{X_L}$ is a Fano 4-fold
7	3	3	known by a dimension reason
8	2	4	known, $\mathsf{Y_L}$ is a Fano 4-fold
9	1	5	known, $\mathsf{Y_L}$ is a Fano 5-fold
10	0	6	

Table 2. The case $n = 7$.

Remark 3.30. We thank Claire Voisin pointing out to the author a classical result that the Hodge conjecture is true for uniruled 4-folds [8]. Even though most examples here can be proved by classical methods, we hope that we can use geometry of dual varieties to prove the Hodge conjecture of these examples, see also Conjecture 3.31 below. We leave the blanks in the tables since it is not known for the author whether the Hodge conjecture is proved for these cases previously.

(III). For $n \geq 8$, the HPD is not constructed. However, when $n = 10$, there is an interesting picture inspired by the Mirror Symmetry which was constructed by E. Segal and R. P. Thomas [28, Theorem A].

Let L be a 5-dimensional subspace of $\wedge^2 \mathsf{W}^*$, $\mathsf{L}^\perp \subset \wedge^2 \mathsf{W}$. Write $\mathsf{X} = \mathsf{Gr}(2, 10) \subset \mathbb{P}^{44}$ and $\mathsf{Y} = \mathsf{Pf}(8, 10) \subset \mathbb{P}^{44}$; $\mathsf{X}_\mathsf{L} = \mathbb{P}(\mathsf{L}^\perp) \cap \mathsf{X}$, $\mathsf{Y}_\mathsf{L} = \mathbb{P}(\mathsf{L}) \cap \mathsf{Y}$. We choose a general linear subspace L such that both X_L and Y_L are smooth. In particular, Y_L is a quintic 3-fold and X_L is a Fano 11-fold. According to E. Segal and R. P. Thomas [28, Theorem A], there is a fully faithful embedding

$$\mathsf{D}^\mathsf{b}(\mathsf{Y}_\mathsf{L}) \hookrightarrow \mathsf{D}^\mathsf{b}(\mathsf{X}_\mathsf{L}).$$

Let $\mathcal{A}$ be the exceptional collection $\{\mathsf{Sym}^3\, \mathsf{S}, \mathsf{Sym}^2\, \mathsf{S}, \mathsf{S}, \mathcal{O}\}$ of $\mathsf{D}^\mathsf{b}(\mathsf{Gr}(2, 10))$, where S is the tautological bundle on $\mathsf{Gr}(2, 10)$. It restricts to an exceptional collection in $\mathsf{D}^\mathsf{b}(\mathsf{X}_\mathsf{L})$ by techniques in [13]. Then, let $\langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(4)\rangle$ be an exceptional collection in $\mathsf{D}^\mathsf{b}(\mathsf{X}_\mathsf{L})$. It is right orthogonal to the above embedding of $\mathsf{D}^\mathsf{b}(\mathsf{Y}_\mathsf{L})$, see description in [28, Remark 3.8]. The Hochschild homology $\mathsf{HH}_0(\mathsf{X}_\mathsf{L}) \cong \mathbb{C}^{24}$ and $\mathsf{HH}_0(\mathsf{Y}_\mathsf{L}) \cong \mathbb{C}^4$. Thus, the 0-th Hochschild homology of the right orthogonal complement of $\langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(4), \mathsf{D}^\mathsf{b}(\mathsf{Y}_\mathsf{L})\rangle$ is trivial. Thus, the Hodge conjecture for X_L follows from the additive theory.

Inspired by the examples above, we expect that even though we do not have HPD, the duality of the Hodge conjecture between linear sections of the dual varieties can be proved by classical methods.

Conjecture 3.31. *Let $\mathsf{X} \subset \mathbb{P}(\mathsf{V})$ be a projective smooth variety. Suppose the Hodge conjecture is true for X. Let $\mathsf{Y} \subset \mathbb{P}(\mathsf{V}^*)$ be the projective dual of $\mathsf{X} \subset \mathbb{P}(V)$. Choose a linear subspace $\mathsf{L} \subset \mathsf{V}^*$. Suppose the linear sections $\mathsf{X}_\mathsf{L} = \mathsf{X} \cap \mathbb{P}(\mathsf{L}^\perp)$ and $\mathsf{Y}_\mathsf{L} = \mathsf{Y} \cap \mathbb{P}(\mathsf{L})$ are both of expected dimension and smooth. Then, the Hodge conjecture of X_L is equivalent to the Hodge conjecture of Y_L.*

3.3. Connective dg algebras

In this section, we prove that the noncommutative Hodge conjecture is true for the connective dg algebras.

Definition 3.32. An algebra A is called a connective dg algebra if $\mathsf{H}^i(\mathsf{A}) = 0$ for $i > 0$.

Theorem 3.33. *If A is a smooth and proper connective dg algebra, the noncommutative Hodge conjecture is true for A.*

Proof. According to recent work by Theo Raedschelders and Greg Stevenson [26, Corollary 4.3, Theorem 4.6], $\mathcal{U}(\mathsf{A})_{\mathbb{Q}} \cong \mathcal{U}(\mathsf{H}^0(\mathsf{A})/\mathsf{Jac}(\mathsf{H}^0(\mathsf{A})))_{\mathbb{Q}} \cong \bigoplus \mathcal{U}(\mathbb{C})_{\mathbb{Q}}$. Hence, the noncommutative Hodge conjecture is true for connective dg algebras. In particular, it is true for the proper smooth algebras (concentrated in degree 0). ■

We provide another proof which involves more calculation for smooth and proper algebras. Clearly, proper algebras are finite dimensional algebras. Due to R. Rouquier [27, Section 7], $\mathsf{Pdim}_{\mathsf{A}^e}(\mathsf{A}) = \mathsf{Pdim}(\mathsf{A})$, smooth algebras are finite global dimensional algebras. Consider the acyclic quiver Q with finitely many vertices. Let $\mathsf{A} := \mathsf{kQ}/\mathsf{I}$ be the quiver algebra with relations, where kQ is the path algebra of Q. Then, A is a smooth and proper algebra. The noncommutative Hodge conjecture is true for A.

Theorem 3.34. *Let* $\mathsf{A} = \mathsf{kQ}/\mathsf{I}$. *Consider the natural Chern character map*

$$\mathsf{Ch}\colon \mathsf{K}_0(\mathsf{A}) \to \mathsf{HH}_0(\mathsf{A}).$$

Then, $\mathsf{Im\,Ch}_{\mathbb{Q}} \otimes \mathbb{C} = \mathsf{HH}_0(\mathsf{A})$. *In particular, the noncommutative Hodge conjecture is true for* A.

Proof. Firstly, for the algebra A, $\mathsf{HH}_0(\mathsf{A}) \cong \mathsf{A}/[\mathsf{A},\mathsf{A}] \cong \mathsf{k}\langle \mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_n\rangle$ where e_i is the vertex of the quiver Q. We write $\mathsf{S}_i = \mathsf{A}\cdot\mathsf{e}_i$ which is considered as a left A module, $[\mathsf{S}_i] \in \mathsf{K}_0(\mathsf{A})$. We prove that $\mathsf{Ch}([\mathsf{S}_i]) = \mathsf{e}_i$. According to the paper of McCarthy, "Cyclic homology of an exact category" [21, Section 2], there is a natural identification of Hochschild homologies

$$\bigoplus_n \mathsf{Hom}_{\mathsf{A}}(\mathsf{A},\mathsf{A}) \otimes \cdots \otimes \mathsf{Hom}_{\mathsf{A}}(\mathsf{A},\mathsf{A}) \to \bigoplus_{\mathsf{X},\mathsf{Y},n} \mathsf{Hom}_{\mathsf{A}}(\mathsf{X},\mathsf{E}_1) \otimes \cdots \otimes \mathsf{Hom}_{\mathsf{A}}(\mathsf{E}_n,\mathsf{Y}).$$

It is a natural quasi-isomorphism, the left-hand side is exactly the bar complex of A. X and Y are both projective left A modules. Under this identification, the image of the Chern character of the object $[\mathcal{P}]$ that is a projective A module is the homology class of $\mathsf{id}_{\mathcal{P}}$ in the right-hand side complex. Consider the local picture

$$\mathsf{Bar}\colon \mathsf{Hom}_{\mathsf{A}}(\mathsf{S}_i,\mathsf{A}) \otimes \mathsf{Hom}_{\mathsf{A}}(\mathsf{A},\mathsf{S}_i) \to \mathsf{Hom}_{\mathsf{A}}(\mathsf{S}_i,\mathsf{S}_i) \oplus \mathsf{Hom}_{\mathsf{A}}(\mathsf{A},\mathsf{A}).$$

Let $\mathsf{f} \in \mathsf{Hom}(\mathsf{S}_i,\mathsf{A})$ be the natural inclusion, $\mathsf{e}_i \in \mathsf{Hom}_{\mathsf{A}}(\mathsf{A},\mathsf{S}_i)$ be the multiplication by e_i. Then $\mathsf{Bar}(\mathsf{f}\otimes\mathsf{e}_i) = \mathsf{id}_{\mathsf{S}_i} - \mathsf{e}_i$. Therefore, $[\mathsf{e}_i] = [\mathsf{id}_{\mathsf{S}_i}]$ in $\mathsf{HH}_0(\mathsf{Proj\,A})$. Hence $\mathsf{Ch}([\mathsf{S}_i]) = [\mathsf{e}_i]$. Finally, $\mathsf{Im\,Ch}_{\mathbb{Q}} \otimes \mathbb{C} = \mathsf{HH}_0(\mathsf{A})$. Since $\mathsf{Im\,Ch}_{\mathbb{Q}} \subset \mathsf{HH}_{0,\mathbb{Q}}(\mathsf{A})$, we have $\mathsf{Im\,Ch}_{\mathbb{Q}} = \mathsf{HH}_{0,\mathbb{Q}}(\mathsf{A})$. ■

A finite dimensional algebra A is (derived) Morita equivalent to an elementary algebra which is isomorphic to kQ/I for some quiver Q. Clearly, kQ/I is smooth and proper if A is smooth and proper. Then, according to Theorem 3.34, the Hodge conjecture is true for any smooth and finite dimensional algebra A.

Remark 3.35. A. Perry pointed out to the author that if A is a smooth and proper algebra, $\mathsf{Perf}(\mathsf{A})$ can be an admissible subcategory of the $\mathsf{Perf}(\mathsf{X})$ which admits full exceptional collections for some smooth and projective varieties X by Orlov [23, Section 5.1]. Therefore, the noncommutative Hodge conjecture of A is true.

Classically, given any projective smooth variety X, there is a compact generator E of $\mathsf{D}_{\mathsf{Qch}}(\mathsf{X})$. Write E again after the resolution to an injective complex. Denote $\mathsf{A} = \mathsf{Hom}_{\mathsf{dg}}(\mathsf{E}, \mathsf{E})$, then there is an equivalence $\mathsf{D}^{\mathsf{per}}(\mathsf{A}) \cong \mathsf{Perf}(\mathsf{X})$ and a chain of derived Morita equivalences between $\mathsf{Per}_{\mathsf{dg}}(\mathsf{A})$ and $\mathsf{Per}_{\mathsf{dg}}(\mathsf{X})$. Thus, the commutative Hodge conjecture for $\mathsf{X} \Leftrightarrow$ the noncommutative Hodge conjecture for a dg algebra A. By the results above, suppose A is a smooth and finite dimensional algebra, then the Hodge conjecture of A is true.

Definition 3.36. Let X be a projective smooth variety. An object T is a called tilting sheaf if the following property holds:

(1) T classically generates $\mathsf{D}^{\mathsf{b}}(\mathsf{X})$;

(2) $\mathsf{A} := \mathsf{Hom}(\mathsf{T}, \mathsf{T})$ is of finite global dimension;

(3) $\mathsf{Ext}^k(\mathsf{T}, \mathsf{T}) = 0$ for $k > 0$.

The reader can refer to Alastair Craw's note, "Explicit methods for derived categories of sheaves" [9] for more discussions.

Due to Van den Bergh, there are many examples of varieties which admit a tilting bundle.

Example 3.37 (Van den Bergh [34, Theorem A]). Suppose there is a projective morphism $\mathsf{f}\colon \mathsf{X} \to \mathsf{Y} = \mathsf{Spec}\,\mathsf{R}$ between noetherian schemes. Furthermore, $\mathsf{Rf}_*(\mathcal{O}_{\mathsf{X}}) \cong \mathcal{O}_{\mathsf{Y}}$ and the fibers are at most one-dimensional. Then there is a tilting bundle $\mathcal{E}$ of X.

Corollary 3.38. *Suppose X admits a tilting sheaf, then the Hodge conjecture for X is true.*

Proof. Let T be a tilting sheaf of X. We write T again after resolution to an injective complex. Define $\mathsf{A} := \mathsf{Hom}_{\mathsf{dg}}(\mathsf{T}, \mathsf{T})$, which is quasi-isomorphic (hence derived Morita equivalent) to a smooth and finite dimensional algebra. Thus, the Hodge conjecture for X is true. ∎

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References

[1] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk, Torification and factorization of birational maps. *J. Amer. Math. Soc.* **15** (2002), no. 3, 531–572 Zbl 1032.14003 MR 1896232

[2] D. Arapura, The Hodge conjecture for rationally connected fivefolds. 2005, arXiv:math/0502257

[3] M. F. Atiyah and F. Hirzebruch, Analytic cycles on complex manifolds. *Topology* **1** (1962), 25–45 Zbl 0108.36401 MR 145560

[4] M. Bernardara and G. Tabuada, From semi-orthogonal decompositions to polarized intermediate Jacobians via Jacobians of noncommutative motives. *Mosc. Math. J.* **16** (2016), no. 2, 205–235 Zbl 1386.14012 MR 3480702

[5] P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des intersections et théorème de Riemann–Roch. Séminaire de Géométrie Algébrique du Bois Marie 1966 /67 (SGA 6).* Lecture Notes in Math. 225, Springer, Berlin, 1971

[6] A. Blanc, Topological K-theory of complex noncommutative spaces. *Compos. Math.* **152** (2016), no. 3, 489–555 Zbl 1343.14003 MR 3477639

[7] A. Bondal and D. Orlov, Derived categories of coherent sheaves. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pp. 47–56, Higher Education Press, Beijing, 2002 Zbl 0996.18007 MR 1957019

[8] A. Conte and J. P. Murre, The Hodge conjecture for fourfolds admitting a covering by rational curves. *Math. Ann.* **238** (1978), no. 1, 79–88 Zbl 0373.14006 MR 510310

[9] A. Craw, Explicit methods for derived categories of sheaves. 2007, https://www.math.utah.edu/dc/tilting.pdf, visited on Aug 24, 2023

[10] D. Kaledin, Spectral sequences for cyclic homology. In *Algebra, geometry, and physics in the 21st century*, pp. 99–129, Progr. Math. 324, Birkhäuser/Springer, Cham, 2017 Zbl 1390.14011 MR 3702384

[11] B. Keller, On the cyclic homology of ringed spaces and schemes. *Doc. Math.* **3** (1998), 231–259 Zbl 0917.19002 MR 1647519

[12] B. Keller, On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pp. 151–190, European Mathematical Society, Zürich, 2006 Zbl 1140.18008 MR 2275593

[13] A. Kuznetsov, Homological projective duality for Grassmannians of lines. 2006, arXiv:math/0610957

[14] A. Kuznetsov, Homological projective duality. *Publ. Math. Inst. Hautes Études Sci.* (2007), no. 105, 157–220 Zbl 1131.14017 MR 2354207

[15] A. Kuznetsov, Derived categories of quadric fibrations and intersections of quadrics. *Adv. Math.* **218** (2008), no. 5, 1340–1369 Zbl 1168.14012 MR 2419925

[16] A. Kuznetsov, Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, pp. 219–243, Progr. Math. 282, Birkhäuser Boston, Boston, MA, 2010 Zbl 1202.14012 MR 2605171

[17] A. Kuznetsov, Semiorthogonal decompositions in algebraic geometry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pp. 635–660, Kyung Moon Sa, Seoul, 2014 Zbl 1373.18009 MR 3728631

[18] A. Kuznetsov, Calabi–Yau and fractional Calabi–Yau categories. *J. Reine Angew. Math.* **753** (2019), 239–267 Zbl 1440.14092 MR 3987870

[19] A. Kuznetsov and V. A. Lunts, Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN* (2015), no. 13, 4536–4625 Zbl 1338.14020 MR 3439086

[20] V. A. Lunts and D. O. Orlov, Uniqueness of enhancement for triangulated categories. *J. Amer. Math. Soc.* **23** (2010), no. 3, 853–908 Zbl 1197.14014 MR 2629991
[21] R. McCarthy, The cyclic homology of an exact category. *J. Pure Appl. Algebra* **93** (1994), no. 3, 251–296 Zbl 0807.19002 MR 1275967
[22] L. X. Meng, The Hodge conjecture for fiber bundles and blowups. 2019, arXiv:1907.10199v1
[23] D. Orlov, Smooth and proper noncommutative schemes and gluing of DG categories. *Adv. Math.* **302** (2016), 59–105 Zbl 1368.14031 MR 3545926
[24] A. Perry, Noncommutative homological projective duality. *Adv. Math.* **350** (2019), 877–972 Zbl 1504.14006 MR 3948688
[25] A. Perry, The integral Hodge conjecture for two-dimensional Calabi–Yau categories. *Compos. Math.* **158** (2022), no. 2, 287–333 Zbl 1492.14029 MR 4406785
[26] T. Raedschelders and G. Stevenson, Proper connective differential graded algebras and their geometric realizations. *Eur. J. Math.* **8** (2022), no. suppl. 2, S574–S598 Zbl 1505.14003 MR 4507255
[27] R. Rouquier, Dimensions of triangulated categories. *J. K-Theory* **1** (2008), no. 2, 193–256 Zbl 1165.18008 MR 2434186
[28] E. Segal and R. P. Thomas, Quintic threefolds and Fano elevenfolds. *J. Reine Angew. Math.* **743** (2018), 245–259 Zbl 1454.14051 MR 3859275
[29] G. Tabuada, Invariants additifs de DG-catégories. *Int. Math. Res. Not.* (2005), no. 53, 3309–3339 Zbl 1094.18006 MR 2196100
[30] G. Tabuada, Chow motives versus noncommutative motives. *J. Noncommut. Geom.* **7** (2013), no. 3, 767–786 Zbl 1296.14019 MR 3108695
[31] G. Tabuada, *Noncommutative motives*. Univ. Lecture Ser. 63, American Mathematical Society, Providence, RI, 2015 Zbl 1333.14002 MR 3379910
[32] G. Tabuada, Noncommutative counterparts of celebrated conjectures. In *K-theory in algebra, analysis and topology*, pp. 353–377, Contemp. Math. 749, American Mathematical Society, Providence, RI, 2020 Zbl 1456.46063 MR 4087645
[33] G. Tabuada and M. Van den Bergh, Noncommutative motives of Azumaya algebras. *J. Inst. Math. Jussieu* **14** (2015), no. 2, 379–403 Zbl 1356.14007 MR 3315059
[34] M. Van den Bergh, Three-dimensional flops and noncommutative rings. *Duke Math. J.* **122** (2004), no. 3, 423–455 Zbl 1074.14013 MR 2057015
[35] C. Voisin, *Hodge theory and complex algebraic geometry. II*. Cambridge Stud. Adv. Math. 77, Cambridge University Press, Cambridge, 2003 Zbl 1032.14002 MR 1997577
[36] C. Voisin, Some aspects of the Hodge conjecture. *Jpn. J. Math.* **2** (2007), no. 2, 261–296 Zbl 1159.14005 MR 2342587
[37] C. Weibel, The Hodge filtration and cyclic homology. *K-Theory* **12** (1997), no. 2, 145–164 Zbl 0881.19002 MR 1469140
[38] F. Xu, *On the smooth linear section of the Grassmannian Gr(2, n)*. Ph.D. thesis, Rice University, 2011

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