

Pontrjagin duality on multiplicative gerbes

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Abstract. We use Segal–Mitchison’s cohomology of topological groups to define a convenient model for topological gerbes. We introduce multiplicative gerbes over topological groups in this setup and define their representations. For a specific choice of representation, we construct its category of endomorphisms, and we show that it induces a new multiplicative gerbe over another topological group. This new induced group is fiberwise Pontrjagin dual of the original one, and therefore we called the pair of multiplicative gerbes “Pontrjagin dual”. We show that Pontrjagin dual multiplicative gerbes have equivalent categories of representations. In addition, we show that their monoidal centers are equivalent. Examples of Pontrjagin dual multiplicative gerbes over finite and discrete, as well as compact and non-compact, Lie groups are provided.

1. Introduction

Gerbes with structure group $U(1)$ over a manifold are geometrical objects whose isomorphism classes are classified by degree three integer cohomology classes [4]. They could be thought of as groupoid extensions of the underlying manifold by the groupoid $U(1)[1]$ [11], as $U(1)$ -bundle gerbes [20], or as principal $U(1)[1]$ -2-bundles [22]. In each of these versions, gerbes over manifolds define a monoidal 2-category. Thus, one may consider weak monoid objects [2] in the 2-category of gerbes. These weak monoid objects in gerbes are what are known in the literature as multiplicative gerbes [5, 6, 32]. The study of multiplicative gerbes over compact, simple, and simply connected Lie groups [6, 8, 9, 13, 32] has been of particular interest due, among others, to its relation with Chern–Simons theories and Wess–Zumino–Witten models. The classification of isomorphism classes of multiplicative gerbes over compact Lie groups is well understood; isomorphic multiplicative gerbes over a fixed compact Lie group are classified by degree four integer cohomology classes of its classifying space, and every known equivalence relation between multiplicative gerbes over compact Lie groups involves isomorphic underlying Lie groups. In the paper, we address the question of how to classify multiplicative gerbes up to Morita equivalence. Namely, can multiplicative gerbes over non-isomorphic Lie groups have equivalent categories of representations? Let us elaborate.

Recall that multiplicative gerbes are weak monoid objects in the 2-category of gerbes, and monoid objects in a 2-category may act on objects of the same 2-category. Therefore,

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we may define the 2-category of representations of a multiplicative gerbe as the 2-category of its weak module objects [2]. Now, the previous questions make sense; in particular, we will call multiplicative gerbes *Morita equivalent* if they have equivalent 2-categories of representations. Clearly, isomorphic multiplicative gerbes over isomorphic Lie groups will be Morita equivalent. In this work, we are looking for Morita equivalent multiplicative gerbes defined over non-isomorphic Lie groups.

When considering multiplicative gerbes over finite groups, the previous question has been completely solved [21, 31]. A multiplicative gerbe over a finite group defines a pointed fusion category [7], and its module categories (representations) are well understood [21, 23, 24]. The endomorphism category of particular module categories induces again pointed fusion categories of the same Frobenius–Perron dimension, and in these cases, the pointed fusion categories become Morita equivalent [21]. An explicit description of the Morita equivalence classes in terms of groups and cocycles has been obtained by the second author in [31].

This work generalizes the ideas and procedures carried out to classify Morita equivalence classes of multiplicative gerbes over finite groups to the more general case of topological groups. To make this generalization possible, we construct a model for gerbes over topological spaces similar to the one done for the finite group case. We draw on the cohomology theory for topological groups constructed by Segal and Mitchison [30], and we define our model for gerbes using spaces and degree 2 cocycles over the spaces. We call these gerbes *Segal–Mitchison gerbes*.

We emphasize here that the cohomology theory of topological groups defined by Segal and Mitchison incorporates the information about the topological invariants of these groups, as well as the cohomological information of the complex of continuous cochains of the group. This cohomology theory permits treating on an equal footing finite, discrete, compact, and non-compact Lie groups.

We define the monoidal 2-category of Segal–Mitchison gerbes, we consider weak monoid objects in this category as our model for multiplicative gerbes, and we define their 2-categories of representations. We take a representation of a multiplicative gerbe satisfying a cohomological condition expressed with the help of the Lyndon–Hochschild–Serre spectral sequence, and we consider the monoidal category of its endomorphism. We show that this monoidal category of endomorphisms defines a new multiplicative Segal–Mitchison gerbe. This new multiplicative Segal–Mitchison gerbe is called the “fiberwise Pontrjagin dual” of the original one because it is constructed throughout a dualization procedure over a normal and abelian subgroup of the original multiplicative gerbe. This dualization produces a new group with a normal and abelian group, which is the Pontrjagin dual of the normal and abelian group of the original group.

We note here that the monoidal category of endomorphisms is a gerbe of a particular kind called *lifting bundle gerbe* [20]. We include an appendix where we summarize the properties of lifting bundle gerbes, and we show that Segal–Mitchison gerbes and lifting bundle gerbes form canonically equivalent monoidal 2-categories.

We present an array of examples of Pontrjagin dual multiplicative gerbes, including finite, discrete, compact, and non-compact Lie groups, to demonstrate that the model of Segal–Mitchison gerbes is ample enough to encompass them all. It is important to notice that the classical Pontrjagin duality on abelian topological groups could be recovered as the fiberwise Pontrjagin dual of the trivial multiplicative Segal–Mitchison gerbe defined by the abelian groups once they fiber over a trivial group.

Then, we proceed to show that the 2-categories of representations of fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes are equivalent; and therefore, this justifies calling them “Morita equivalent”. We show furthermore that this equivalence induces an equivalence of the categorical centers of the fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes.

We have thus succeeded in finding sufficient conditions for two multiplicative Segal–Mitchison gerbes to be Morita equivalent. We conjecture that these conditions are also necessary, as it is the case for multiplicative gerbes over finite groups. Unfortunately, the foundations underlying the categorical constructions performed in this work with topological groups are not as developed as the ones over fusion categories. These foundations would have to be first developed in order to be able to generalize the procedures and proofs that have been performed using pointed fusion categories. Thus, the classification of Morita equivalence classes of multiplicative Segal–Mitchison gerbes remains open. Nevertheless, the results of this work offer a clear path on how to carry out the complete classification.

We start this work in Section 2 with an introduction to Segal–Mitchison’s cohomology of topological groups; we summarize its most important features, we show its relation to the cohomology of continuous cochains and to the cohomology of the classifying space of the group, and we introduce the Lyndon–Hochschild–Serre spectral sequence for this cohomology theory. In Section 3 we define the monoidal 2-category of Segal–Mitchison gerbes, in Section 4, we define multiplicative Segal–Mitchison gerbes as weak monoid objects in Segal–Mitchison gerbes, and in Section 5, we define the 2-category of representations of multiplicative Segal–Mitchison gerbes as weak module objects over weak monoid objects in Segal–Mitchison gerbes. In Section 6, we introduce the fiberwise Pontrjagin duality of multiplicative Segal–Mitchison gerbes and we present several examples of this duality. Finally, in Section 7, we show that fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes have equivalent 2-categories of representations, thus becoming Morita equivalent.

The final two sections include a summary on crossed modules and 2-groups in Appendix A and a summary on lifting bundle gerbes and their relation to Segal–Mitchison gerbes in Appendix B.

2. Segal–Mitchison cohomology

G. Segal in [30] defined cohomology groups suited to understanding the cohomology of a topological group with coefficients in a topological abelian group. The description of

this cohomology theory is closely related to the cohomology of a discrete group with coefficients in an abelian group. In what follows, we will review the definition of Segal that appears in [30].

2.1. Coefficients

The coefficients of the Segal–Mitchison cohomology are in the category \mathfrak{Topab} of compactly generated and locally contractible Hausdorff topological abelian groups and continuous homomorphisms. A sequence

$$A' \xrightarrow{i} A \xrightarrow{p} A''$$

in \mathfrak{Topab} is called a short exact sequence if i embeds A' as a closed subgroup in A , p induces an isomorphism of topological groups $A/A' \cong A''$, and $p : A \rightarrow A''$ is a principal A' -bundle. If $A \in \mathfrak{Topab}$, then EA and $BA := EA/A$ are also in \mathfrak{Topab} and

$$A \rightarrow EA \rightarrow BA$$

is a short exact sequence. Recall that EA is a contractible topological space obtained from the nerve of the contractible category whose space of objects is A and whose space of morphisms is $A \times A$, one morphism for each pair of elements in A . Since A is abelian, the contractible space EA may be endowed with the structure of a topological abelian group [30, Appendix A].

The previous construction may be iterated, thus giving the short exact sequences in \mathfrak{Topab} :

$$B^l A \xrightarrow{i_l} EB^l A \xrightarrow{p_l} B^{l+1} A$$

for every $l \geq 0$. Composing the projections $EB^l A \xrightarrow{p_l} B^{l+1} A$ with the embeddings

$$B^{l+1} A \xrightarrow{i_{l+1}} EB^{l+1} A,$$

we obtain continuous homomorphisms $EB^l A \xrightarrow{\partial_l} EB^{l+1} A$ which can be assembled to produce a resolution of A by contractible groups in \mathfrak{Topab} :

$$A \hookrightarrow EA \xrightarrow{\partial_0} EBA \xrightarrow{\partial_1} EB^2 A \xrightarrow{\partial_2} EB^3 A \xrightarrow{\partial_3} \dots$$

For simplicity, the subindex in ∂_i will be omitted in what follows.

2.2. Cohomology of spaces

Let X be a paracompact space and consider the complex

$$C^l(X, \underline{A}) := \text{Map}(X, EB^l A)$$

with differential $\partial f := \partial \circ f$. The space of maps is endowed with the compact-open topology and has a canonical group structure. Hence, $C^l(X, \underline{A})$ belongs to \mathfrak{Topab} .

Definition 2.1. Let X be a topological paracompact space. The cohomology of the complex $(C^*(X, \underline{A}), \partial)$ is called the *Segal–Mitchison cohomology* of X with coefficients in A and will be denoted by

$$H^*(X, \underline{A}) := H^*(C^*(X, \underline{A}), \partial).$$

Note that the cocycles $Z^l(X, \underline{A})$ consist of maps $X \xrightarrow{f} EB^l A$ such that the composition $X \xrightarrow{f} EB^l A \xrightarrow{p_l} B^{l+1} A$ is the identity in $B^{l+1} A$. Since the fiber of p_l at the identity element is $B^l A$, we have an isomorphism of topological groups:

$$Z^l(X, \underline{A}) \cong \text{Map}(X, B^l A).$$

By the classification of principal bundles over paracompact spaces, we know that a map $X \rightarrow B^l A$ is null-homotopic if and only if the map lifts to $EB^{l-1} A$. Therefore, we obtain the isomorphism

$$H^l(X, \underline{A}) \cong [X, B^l A]$$

whenever $l \geq 1$. For $l = 0$, we obtain the isomorphism

$$H^0(X, \underline{A}) \cong \text{Map}(X, A),$$

and we may say that the Segal–Mitchison cohomology calculates the cohomology of X with coefficients in the sheaf \underline{A} of A -valued functions, i.e., it is the derived functor of $\text{Map}(X, A)$.

Whenever we have a short exact sequence in \mathfrak{Topab}

$$A' \xrightarrow{i} A \xrightarrow{p} A'',$$

a standard diagram chase argument shows that there is an induced long exact sequence in cohomology:

$$\rightarrow H^{k-1}(X, \underline{A}'') \rightarrow H^k(X, \underline{A}') \rightarrow H^k(X, \underline{A}) \rightarrow H^k(X, \underline{A}'') \rightarrow H^{k+1}(X, \underline{A}') \rightarrow .$$

2.3. Cohomology of simplicial spaces

Whenever X_\bullet is a simplicial paracompact space, we may define the double complex

$$C^{p,q}(X_\bullet, \underline{A}) := \text{Map}(X_p, EB^q A)$$

whose horizontal differential

$$\delta : C^{p,q}(X_\bullet, \underline{A}) \rightarrow C^{p+1,q}(X_\bullet, \underline{A})$$

is the one induced by the structural maps $X_{p+1} \rightarrow X_p$ of the simplicial space and whose vertical differential is ∂ . The total complex of the double complex will be denoted by

$$C^*(X_\bullet, \underline{A}) := \text{Tot}(C^{*,*}(X_\bullet, \underline{A})),$$

and its differential is $d := \delta + (-1)^p \partial$.

Definition 2.2. Let X_\bullet be a simplicial paracompact space. The cohomology of the complex $C^*(X_\bullet, \underline{A})$ with differential d is denoted by

$$H^*(X_\bullet, \underline{A}) := H^*(C^*(X_\bullet, \underline{A}), d)$$

and is called the Segal–Mitchison cohomology of the simplicial space X_\bullet with coefficients in the topological abelian group A .

We may filter the total complex by the subcomplexes $\text{Tot}(C^{*\geq p,*}(X_\bullet, \underline{A}))$, thus obtaining a spectral sequence whose first page is isomorphic to

$$E_1^{p,q} = H^q(X_p, \underline{A}).$$

Note that whenever the simplicial space X_\bullet is homotopy equivalent to a simplicial set, or when A is contractible, we know that $E_1^{p,q} = 0$ for $q > 0$. For $q = 0$, we obtain $E_1^{p,0} = \text{Map}(X_p, A)$, and therefore in this case, $H^*(X_\bullet, \underline{A})$ is simply the cohomology of the complex $(\text{Map}(X_*, A), \delta)$. The cohomology of this complex will be denoted as

$$H_{\text{cont}}^*(X_\bullet, A) := H^*(\text{Map}(X_*, A), \delta)$$

and will be called the *cohomology of continuous cochains*.

Lemma 2.3. *Suppose that X_\bullet is a simplicial paracompact space and A is a contractible topological group, or alternatively suppose that the simplicial space X_\bullet is homotopy equivalent to a simplicial set. Then, in either case, the forgetful homomorphism induces an isomorphism of cohomologies:*

$$H^*(X_\bullet, \underline{A}) \xrightarrow{\cong} H_{\text{cont}}^*(X_\bullet, A).$$

2.4. Cohomology of topological groups

Let G be a paracompact and compactly generated topological group. Denote by G_\bullet the simplicial space defined by the nerve of the topological category with one object, and denote by G the space of morphisms. In this case, we have that $G_p = G^p$ and the differential

$$\delta_G : \text{Map}(G^p, EB^q A) \rightarrow \text{Map}(G^{p+1}, EB^q A)$$

is defined by the formula

$$\begin{aligned} \delta_G(f)(h_0, \dots, h_p) &= f(h_1, \dots, h_p) + \sum_{i=1}^p (-1)^i f(h_1, \dots, h_{i-1}h_i, \dots, h_p) + (-1)^{p+1} f(h_0, \dots, h_{p-1}). \end{aligned}$$

Definition 2.4. Let G be a paracompact and compactly generated topological group and let A be a compactly generated and locally contractible Hausdorff topological abelian group. The Segal–Mitchison cohomology of the group G with coefficients in A is

$$H^*(G_\bullet, \underline{A}) = H^*(\text{Tot}(C^{*,*}(G_\bullet, \underline{A})), d_G = \delta_G + (-1)^p \partial).$$

Segal in [30, §4] shows that the cohomology groups $H^*(G_\bullet, \underline{A})$ for $* = 1, 2$ have their usual interpretations. Let us recall the proofs since they will be used in what follows.

Lemma 2.5. *The first Segal–Mitchison cohomology group is isomorphic to the topological group of continuous homomorphisms, i.e.,*

$$H^1(G_\bullet, \underline{A}) \cong \text{Hom}(G, A).$$

Proof. An element in $Z^1(G_\bullet, \underline{A})$ consists of a pair of maps $f_1 : G \rightarrow EA$ and $f_0 : \{*\} \rightarrow EBA$ such that $\delta_G f_1(g_1, g_2) = f(g_2) - f(g_1g_2) + f(g_1) = 0$, $\partial f_1(g) = -\delta f_0(*) = 0$, and $\partial f_0(*) = 0$. The last two equations imply that the image of f_1 lies in A , the kernel of $\partial : EA \rightarrow EBA$, and the image of f_0 lies in BA , the kernel of $\partial : EBA \rightarrow EB^2A$. The first equation implies that $f_1 : G \rightarrow A \subset EA$ is a homomorphism of groups. Therefore, we obtain the isomorphism of groups as follows:

$$Z^1(G_\bullet, \underline{A}) \cong \text{Hom}(G, A) \oplus BA.$$

Now, the image of $C^0(G_\bullet, \underline{A}) = \text{Map}(*, EA)$ under $d = \delta + \partial$ is simply $\{0\} \oplus BA$, and therefore, we obtain the desired isomorphism: $H^1(G_\bullet, \underline{A}) \cong \text{Hom}(G, A)$. ■

Lemma 2.6. *The second Segal–Mitchison cohomology group is isomorphic to the group of isomorphism classes of A -central extensions of G , i.e.,*

$$H^2(G_\bullet, \underline{A}) \cong \text{Ext}(G, A).$$

Proof. We will consider A -central extensions of groups, $A \rightarrow P \xrightarrow{p} G$, i.e., short exact sequences of groups where A is central in P . We recall that short exact sequence means, in particular, that P is a principal A -bundle over G . Two of these A -central extensions $P \xrightarrow{p} G$ and $P' \xrightarrow{p'} G$ are isomorphic if there is a homomorphism of groups $\phi : P \rightarrow P'$ with $p' \circ \phi = p$ such that ϕ is a morphism of principal A -bundles. Denote by $\text{Ext}(G, A)$ the set of isomorphism classes of A -central extensions of G .

The group of 2-cocycles $Z^2(G_\bullet, \underline{A})$ consist of three maps: $F_2 : G \times G \rightarrow EA$, $F_1 : G \rightarrow EBA$, and $F_0 : \{*\} \rightarrow EB^2A$, satisfying the following equations:

$$\begin{aligned} F_2(g_2, g_3) + F_2(g_1, g_2g_3) &= F_2(g_1g_2, g_3) + F_2(g_1, g_2), \\ \partial F_2(g_1, g_2) &= F_1(g_1g_2) - F_1(g_1) - F_1(g_2), \\ \partial F_1(g) &= F_0(*) - F_0(*) = 0, \\ \partial F_0(*) &= 0. \end{aligned}$$

Hence, F_1 takes values in $BA \subset EBA$, and we may take the pullback bundle F_1^*EA over G as $F_1^*EA := \{(g, b) \in G \times EA \mid F_1(g) = p_0(b)\}$. We may define the group structure on F_1^*EA by the equation

$$(g_1, b_1) \cdot (g_2, b_2) := (g_1g_2, b_1 + b_2 + F_2(g_1, g_2))$$

since we know by the second equation that

$$\partial(b_1 + b_2 + F_2(g_1, g_2)) = \partial(b_1) + \partial(b_2) + F_1(g_1g_2) - F_1(g_1) - F_1(g_2) = F_1(g_1g_2).$$

The associativity of the product structure in F_1^*EA follows from the first equation, thus making F_1^*EA an A -central extension of G . Therefore, we have constructed an A -central extension F_1^*EA over G with the information of the maps F_1 and F_2 (the information of F_0 is not important).

Now, for an A -central extension $P \rightarrow G$, which is moreover a principal A -bundle, we may define its classifying map $F_1 : G \rightarrow BA$ together with its lift $F'_1 : P \rightarrow EA$, a morphism of principal A -bundles. For any pair of elements $p_1, p_2 \in P$, we define

$$\bar{F}_2(p_1, p_2) := F'_1(p_1p_2) - F'_1(p_2) - F'_1(p_1),$$

and we note that for any $a_1, a_2 \in A$ we have $\bar{F}_2(p_1, p_2) = \bar{F}_2(a_1p_1, a_2p_2)$. Therefore, \bar{F}_2 is $A \times A$ -invariant and therefore it induces a map $F_2 : (A \setminus P) \times (A \setminus P) \cong G \times G \rightarrow EA$. The associativity of P implies that $\delta_G F_2 = 0$, and the definition of \bar{F}_2 implies that

$$\partial F_2(g_1, g_2) \cong F_1(g_1g_2) - F_1(g_1) - F_1(g_2).$$

Therefore, the maps F_2 and F_1 are the essential information that defines a 2-cocycle in $Z^2(G_\bullet, \underline{A})$.

The facts that cohomologous 2-cocycles induce isomorphic A -central extensions, and that isomorphic A -central extensions induce cohomologous 2-cocycles are straightforward. ■

By Lemma 2.3, whenever A is contractible, we obtain an isomorphism with the cohomology of continuous cochains of the group with coefficients in A :

$$H^*(G_\bullet, \underline{A}) \cong H^*_{\text{cont}}(G, A).$$

Whenever A is discrete, by [30, Prop. 3.3], there is an isomorphism with the cohomology of the complex of singular cochains with coefficients in A :

$$H^*(G_\bullet, \underline{A}) \cong H^*(BG, A).$$

The short exact sequence $A \rightarrow EA \rightarrow BA$ induces a long exact sequence in cohomology groups, and since EA is a contractible, we obtain the long exact sequence in cohomology:

$$\rightarrow H^n_c(G, EA) \rightarrow H^n(G_\bullet, \underline{BA}) \rightarrow H^{n+1}(G_\bullet, \underline{A}) \rightarrow H^{n+1}_c(G, EA) \rightarrow .$$

When $n = 2$, the homomorphism $H^2(G_\bullet, \underline{BA}) \rightarrow H^3(G_\bullet, \underline{A})$ could be understood as the assignment that takes the isomorphism class of a BA -central extension of G ,

$$BA \rightarrow \Gamma \rightarrow G,$$

to the isomorphism class of the crossed module $EA \rightarrow \Gamma$ which induces the exact sequence

$$A \rightarrow EA \rightarrow \Gamma \rightarrow G.$$

The 2-group $\Gamma \times EA \rightrightarrows \Gamma$ that defines the crossed module could be understood as a 2-group central extension of the 2-group $G \rightrightarrows G$ by the 2-group $A \rightrightarrows \{*\}$, and its extension class is an element in the cohomology group $H^3(G_\bullet, \underline{A})$. We refer to Appendix A.5 where we elaborate on the previous statement.

2.5. Segal–Mitchison cohomology as a derived functor

The cohomology groups $H^*(G_\bullet, \underline{A})$ could be understood as the derived functor of G -invariant sections of the trivial G -equivariant topological abelian group A . Whenever A is endowed with an action of G , we need to elaborate a little further.

Denote by $G\text{-}\mathfrak{Topab}$ the category of compactly generated locally contractible topological abelian groups on which G acts continuously [30, §2]. A sequence $A' \rightarrow A \rightarrow A''$ in $G\text{-}\mathfrak{Topab}$ is called a short exact sequence if it is one in \mathfrak{Topab} when the action of G is neglected. Call an object in $G\text{-}\mathfrak{Topab}$ *soft* if it is of the form $\text{Map}(G, A)$, where A is a contractible group and Map denotes space of continuous maps endowed with the compact open topology [30, p. 380]. Note that a soft group is contractible. Segal shows that every object A in $G\text{-}\mathfrak{Topab}$ has a soft resolution

$$A \rightarrow E_{(G)}A \rightarrow E_{(G)}B_{(G)}A \rightarrow E_{(G)}B_{(G)}^2A \rightarrow \dots,$$

where $E_{(G)}A := \text{Map}(G, EA)$ and $B_{(G)}A$ fits in the short exact sequence

$$A \rightarrow E_{(G)}A \rightarrow B_{(G)}A, \tag{2.1}$$

where A embeds in $E_{(G)}A$ as the constant maps taking values in A .

Alternatively, one may take the double complex $\text{Map}(G^{k+1}, EB^l A)$, $k, l \geq 0$, with differentials ∂ and δ , with δ defined by the following equation:

$$\delta(f)(h_1, \dots, h_n) = f(h_2, \dots, h_n) + \sum_{i=1}^{n-1} (-1)^i f(h_1, \dots, h_i h_{i+1}, \dots, h_n).$$

The total complex becomes a soft resolution of A in $G\text{-}\mathfrak{Topab}$. Then, the derived functor of the G -invariant sections $\Gamma^G(A)$ of A can be calculated by the cohomology of the G -invariant sections of the previous total complex. The G -invariant double complex is isomorphic to the double complex $\text{Map}(G^k, EB^l A)$, $k, l \geq 0$ whose differentials are ∂ and

$$\begin{aligned} &\delta_G(f)(h_0, \dots, h_n) \\ &= f(h_1, \dots, h_n) + \sum_{i=1}^n (-1)^i f(h_1, \dots, h_{i-1} h_i, \dots, h_n) + (-1)^{n+1} f(h_0, \dots, h_{n-1}) \cdot h_n. \end{aligned}$$

Denoting by $H^*(G_\bullet, \underline{A})$ the cohomology of

$$(\text{Tot}^*(\text{Map}(G^*, EB^*A)), d_G = \delta_G + (-1)^p \partial),$$

we see that

$$R^p \Gamma^G(A) = H^p(G_\bullet, \underline{A}).$$

Note that whenever the action of G on A is trivial, the cohomology groups previously defined agree with the cohomology of topological groups defined in Section 2.4.

2.6. Lyndon–Hochschild spectral sequence

Let us consider a short exact sequence of locally contractible and compactly generated groups

$$S \hookrightarrow G \rightarrow K,$$

where S is abelian and central. Denote by $K \rtimes G$ the action groupoid determined by the right action of G on K ; the space of objects is K , the space of morphisms is $K \times G$, $s(k, g) = k$, and $t(k, g) = kg$. Recall that the inclusion $S[1] \rightarrow K \rtimes G$, $s \mapsto (1_K, s)$ is a weak equivalence of topological groupoids.

Consider first the double complex

$$C^{l,m}((K \rtimes G)_\bullet, \underline{A}) := \text{Map}(K \times G^l, EB^m A)$$

with differentials $\delta_{K \rtimes G} : C^{l-1,m}((K \rtimes G)_\bullet, \underline{A}) \rightarrow C^{l,m}((K \rtimes G)_\bullet, \underline{A})$

$$\begin{aligned} (\delta_{K \rtimes G} f)(k, g_1, \dots, g_l) &= f(kg_1, g_2, \dots, g_l) \\ &\quad + \sum_{i=1}^{l-1} (-1)^i f(k, g_1, \dots, g_i g_{i+1}, \dots, g_l) \\ &\quad + (-1)^l f(k, g_1, \dots, g_{l-1}) \end{aligned}$$

and $\partial_2 : C^{l,m-1}((K \rtimes G)_\bullet, \underline{A}) \rightarrow C^{l,m}((K \rtimes G)_\bullet, \underline{A})$, $\partial_2 f = \partial \circ f$ (the reason for the sub-index 2 in the second differential will become clear in what follows). The cohomology of the total complex is $H^*((K \rtimes G)_\bullet, \underline{A})$.

Note that the groups $C^{l,m}((K \rtimes G)_\bullet, \underline{A})$ belong to \mathfrak{Topab} and they come endowed with a continuous right K action; for $f \in C^{l,m}((K \rtimes G)_\bullet, \underline{A})$ and $h \in K$, let

$$(f \cdot h)(k, g_1, \dots, g_l) := f(hk, g_1, \dots, g_l).$$

The groups $C^{l,m}((K \rtimes G)_\bullet, \underline{A})$ then belong to $K\text{-}\mathfrak{Topab}$. Moreover, they are *soft* since they are of the form $\text{Map}(K, \text{Map}(G^l, EB^m A))$ with $\text{Map}(G^l, EB^m A)$ contractible.

Let us now consider the quadruple complex

$$C^{i,j,l,m} := \text{Map}(K^i, EB^j(\text{Map}(K \times G^l, EB^m A)))$$

with differentials $\delta_K : C^{i-1,j,l,m} \rightarrow C^{i,j,l,m}$ defined by

$$\begin{aligned}
 &(\delta_K F)(k_1, \dots, k_i) \\
 &= F(k_2, \dots, k_i) + \sum_{r=1}^{i-1} (-1)^r F(k_1, \dots, k_r k_{r+1}, \dots, k_i) + (-1)^i F(k_1, \dots, k_{i-1}) \cdot k_i,
 \end{aligned}$$

$\partial_1 : C^{i,j-1,l,m} \rightarrow C^{i,j,l,m}$, $\partial_1 F = \partial \circ F$ with

$$\partial : EB^{j-1}(\text{Map}(K \times G^l, EB^m A)) \rightarrow EB^j(\text{Map}(K \times G^l, EB^m A)),$$

and the ones induced by $\delta_{K \rtimes G}$ and ∂_2 which will be denoted by the same symbols $\delta_{K \rtimes G} : C^{i,j,l-1,m} \rightarrow C^{i,j,l,m}$ and $\partial_2 : C^{i,j,l,m-1} \rightarrow C^{i,j,l,m}$.

At this point, it is important to notice that the functors E and B send exact sequences in $K\text{-}\mathfrak{Topab}$ to exact sequences in $K\text{-}\mathfrak{Topab}$ [30, Prop. A.3]. Therefore, if we have a complex of groups (V^*, d) in $K\text{-}\mathfrak{Topab}$, there is a canonical isomorphism

$$H^*(EB^l V^*, EB^l d) \cong EB^l H^*(V^*, d)$$

in $K\text{-}\mathfrak{Topab}$. This canonical isomorphism will be used in what follows.

Define the double complex

$$D^{p,q} = \bigoplus_{i+j=p, l+m=q} C^{i,j,l,m} \tag{2.2}$$

with differentials $d_K : D^{p-1,q} \rightarrow D^{p,q}$, $d_K := \delta_K + (-1)^i \partial_1$, and $d_{K \rtimes G} : D^{p,q-1} \rightarrow D^{p,q}$, $d_{K \rtimes G} := \delta_{K \rtimes G} + (-1)^l \partial_2$, and consider the total complex

$$\text{Tot}(C^{*,*,*,*}) = \text{Tot}(D^{*,*})$$

with total differential

$$d = d_K + (-1)^p d_{K \rtimes G} = \partial_K + (-1)^i \partial_1 + (-1)^{i+j} \delta_{K \rtimes G} + (-1)^{i+j+l} \partial_2.$$

We claim the following result.

Proposition 2.7. *Consider the short exact sequence of locally contractible and compactly generated groups*

$$S \hookrightarrow G \rightarrow K,$$

where $K = S \backslash G$, S is abelian and is central. Then, the inclusion of the K -invariant groups

$$C^*((K \rtimes G)_\bullet, \underline{A})^K \hookrightarrow \text{Tot}(C^{*,*,*,*})$$

is a quasi-isomorphism. Hence, the cohomology groups

$$H^*(G_\bullet, \underline{A}) \xrightarrow{\cong} H^*(\text{Tot}(C^{*,*,*,*}), d)$$

are canonically isomorphic.

Proof. Filter the total complex $\text{Tot}(C^{*,*,*,*})$ by the degree q in $D^{p,q}$, i.e.,

$$F_q = \text{Tot}(D^{*,*\geq q}).$$

The associated graded group $\bigoplus_{q \geq 0} F_q/F_{q+1}$ is isomorphic to the total complex, but here we only take into account the differential d_K . Since the groups $\text{Map}(K \times G^l, EB^m A)$ are soft in $K\text{-}\mathfrak{Topab}$, the first page of the spectral sequence becomes the K -invariant subgroup

$$E_1^{p,0} \cong C^p((K \rtimes G)_\bullet, \underline{A})^K$$

with $E_1^{*,q \geq 1} = 0$. Now, since $C^p((K \rtimes G)_\bullet, \underline{A})^K \cong C^p(G_\bullet, \underline{A})$, we have that $E_2^{p,0} \cong H^p(G_\bullet, \underline{A})$. The result follows. ■

Proposition 2.8 (Lyndon–Hochschild spectral sequence). *Under the same conditions of above, filtering the total complex $F'_p = \text{Tot}(D^{*\geq p,*})$ by the degree p in $D^{p,q}$, we obtain a spectral sequence whose second page is*

$$E_2^{p,q} \cong H^p(K_\bullet, \underline{H^q(S_\bullet, \underline{A})})$$

which converges to $H^*(G_\bullet, \underline{A})$.

Proof. The associated graded group $\bigoplus_{p \geq 0} F'_p/F'_{p+1}$ is isomorphic to the total complex, but where we only take into account the differential $d_{K \rtimes G}$. The first page of the spectral sequence is

$$E_1^{q,p} \cong \bigoplus_{i+j=p} \text{Map}(K^i, EB^j(H^q((K \rtimes G)_\bullet, \underline{A}))),$$

and since $H^q((K \rtimes G)_\bullet, \underline{A}) \cong H^q(S_\bullet, \underline{A})$, we have that the first page is

$$E_1^{q,p} \cong \bigoplus_{i+j=p} \text{Map}(K^i, EB^j(H^q(S_\bullet, \underline{A}))).$$

The differential of the first page of the spectral sequence coincides with the differential of the complex $C^*(K_\bullet, \underline{H^q(S_\bullet, \underline{A})})$, and therefore the second page is

$$E_2^{q,p} \cong H^p(K_\bullet, \underline{H^q(S_\bullet, \underline{A})}).$$

The spectral sequence converges to $H^*(G_\bullet, \underline{A})$ by the previous proposition. ■

3. Segal–Mitchison gerbes

In this section, we introduce a homological model for gerbes based on the cycles of the Segal–Mitchison cohomology. Let A be an abelian group in \mathfrak{Topab} . In what follows, we will define the strict 2-category $A\text{-}\mathfrak{SM}\text{-}\mathfrak{Grbs}$ of Segal–Mitchison gerbes with structure group A .

3.1. Objects

The objects are pairs (M, α) , where M is a paracompact and locally compact space and α belongs to $Z^2(M, \underline{A})$.

Note that the 2-cocycle is a map $\alpha : M \rightarrow EB^2A$ such that $\partial \circ \alpha$ is constant to the identity in EB^3A . Since $B^3A \rightarrow EB^3A$ is an inclusion, we have that the composition $M \xrightarrow{\alpha} EB^2A \rightarrow B^3A$ is constant to the identity. Therefore, the image of α lies on the fiber of the identity and this fiber is isomorphic to B^2A . Therefore, $Z^2(M, \underline{A}) \cong \text{Map}(M, B^2A)$, and we may consider α as a map $\alpha : M \rightarrow B^2A$.

Any Segal–Mitchison A -gerbe (M, α) defines a lifting bundle A -gerbe $[\alpha^*EBA/EA]$ as defined in Appendix B, where α^*EBA is the pullback of the canonical bundle $EBA \rightarrow B^2A$ along $\alpha : M \rightarrow B^2A$.

3.2. Morphisms

Let (M, α) and (N, β) be two Segal–Mitchison gerbes with structure group A . A morphism from (M, α) to (N, β) is a pair (F, c) where $F : M \rightarrow N$ is a continuous map and $c \in C^1(M, \underline{A})$ such that

$$\alpha - F^*\beta = \partial c.$$

This equation implies that the principal BA -bundles α^*EBA and $F^*\beta^*EBA$ are isomorphic, and that one can construct a bundle morphism $\alpha^*EBA \rightarrow \beta^*EBA$ covering $F : M \rightarrow N$. This bundle morphism determines a strict 1-morphism $[\alpha^*EBA/EA] \rightarrow [\beta^*EBA/EA]$ of lifting bundle A -gerbes in the following way. Let

$$\begin{aligned} \alpha^*EBA &:= \{(m, \lambda) \in M \times EBA \mid \alpha(m) = \partial(\lambda)\}, \\ \beta^*EBA &:= \{(n, \sigma) \in N \times EBA \mid \beta(n) = \partial(\sigma)\} \end{aligned}$$

for $p : EBA \rightarrow B^2A$ the projection map and define the following map:

$$\begin{aligned} [F, c] : \alpha^*EBA &\rightarrow \beta^*EBA \\ (m, \lambda) &\mapsto (F(m), \lambda - c(m)). \end{aligned}$$

The structural equation $\alpha - F^*\beta = \partial c$ implies that $\beta(F(m)) = \alpha(m) - \partial c(m)$ and therefore $[F, c]$ is well defined. Moreover, the map $[F, c]$ is EA -equivariant and therefore, it defines a strict morphism of lifting bundle A -gerbes:

$$[F, c] : [\alpha^*EBA/EA] \rightarrow [\beta^*EBA/EA].$$

If we have morphisms

$$(M, \alpha) \xrightarrow{(F,c)} (N, \beta) \xrightarrow{(H,d)} (O, \gamma),$$

the composition is

$$(H, d) \circ (F, c) = (H \circ F, F^*d + c).$$

This composition is associative, and it follows that the functor to lifting bundle A -gerbes strictly preserves this composition.

3.3. 2-Morphisms

If we have two morphisms between the same objects, $(F_1, c_1), (F_2, c_2) : (M, \alpha) \rightarrow (N, \beta)$, there are 2-morphisms between (F_1, c_1) and (F_2, c_2) whenever $F = F_1 = F_2$. If this is the case, a 2-morphism

$$(F, c_1) \xrightarrow{e} (F, c_2)$$

consists of an element $e \in C^0(M, \underline{A})$ such that

$$\partial e = c_2 - c_1.$$

In the realm of lifting bundle A -gerbes, the 2-morphism provides a natural transformation between the two morphisms of lifting bundle A -gerbes from $[(\alpha^*EBA)/EA]$ and $[(F^*\beta^*EBA)/EA]$. The natural transformation is given by the map

$$\begin{aligned} \alpha^*EBA &\rightarrow F^*\beta^*EBA \times EA, \\ (m, \lambda) &\mapsto ((F(m), \lambda - c_1(m)), -\partial e(m)), \end{aligned}$$

where $((F(m), \lambda - c_1(m)), -\partial e(m))$ is the morphism in the class $[(F^*\beta^*EBA)/EA]$ from $(F(m), \lambda - c_1(m))$ to $(F(m), \lambda - c_1(m) - \partial e(m)) = (F(m), \lambda - c_2(m))$.

Note that the category $\text{Hom}_{A\text{-}\mathfrak{M}\text{-}\mathfrak{Grb}}((M, \alpha), (N, \beta))$, whose objects are the morphisms of Segal–Mitchison gerbes and whose morphisms are the 2-morphisms of Segal–Mitchison gerbes, defines a lifting bundle gerbe with structure group $H^0(M, \underline{A})$. According to the notation in Section B.1, it can be written as follows:

$$C^0(M, \underline{A}) \overset{\partial}{\cup} \text{Hom}_{A\text{-}\mathfrak{M}\text{-}\mathfrak{Grb}}^0((M, \alpha), (N, \beta)).$$

3.4. Monoidal structure

The 2-category $A\text{-}\mathfrak{M}\text{-}\mathfrak{Grb}$ can be endowed with a symmetrical monoidal structure. Take two Segal–Mitchison gerbes (M, α) and (N, β) and define their product:

$$(M, \alpha) \times (N, \beta) := (M \times N, \pi_1^*\alpha + \pi_2^*\beta),$$

where $\pi_1 : M \times N \rightarrow M$ and $\pi_2 : M \times N \rightarrow N$ are the projections.

This monoidal structure is compatible with the monoidal structure of the associated lifting bundle A -gerbes in Appendix B.4. From the maps

$$\alpha : M \rightarrow B^2A \quad \text{and} \quad \beta : N \rightarrow B^2A,$$

we construct the map $M \times N \rightarrow B^2A$, $(m, n) \mapsto \alpha(m) + \beta(n)$ using the group structure of B^2A . This map encodes the Segal–Mitchison A -gerbe of the product of lifting bundle A -gerbes $[(\alpha^*EBA)/EA] \otimes [(\beta^*EBA)/EA]$.

In Appendix B.4, we give a summary of the properties of the 2-category of lifting bundle A -gerbes, and in Theorem B.2, we show its equivalence with the 2-category of Segal–Mitchison A -gerbes.

4. Multiplicative Segal–Mitchison gerbes

Multiplicative Segal–Mitchison gerbes will be nothing else but weak monoid objects in the monoidal 2-category $A\text{-}\mathcal{CM}\text{-}\mathcal{Grbs}$ of Segal–Mitchison gerbes. They also form a 2-category which will be defined in the following.

4.1. Objects

A multiplicative Segal–Mitchison gerbe consists of an object (M, α_1) , a morphism

$$(M, \alpha_1) \times (M, \alpha_1) \xrightarrow{(m, \alpha_2)} (M, \alpha_1),$$

a 2-morphism which controls the defect of the associativity, and a cocycle condition on the 2-morphism for the pentagon property.

Since at the level of spaces the maps are indeed associative, the space itself together with the product $M \times M \xrightarrow{m} M$ is an associative monoid in spaces. We may consider the double complex $C^{p,q}(M_\bullet, \underline{A})$ that the monoid M induces since the maps that define the differential δ_M only require the product of the monoid. In terms of that double complex, a multiplicative Segal–Mitchison gerbe over M consists of the following:

- A Segal–Mitchison gerbe (M, α_1) with $\partial\alpha_1 = 0$.
- A morphism lifting the product in M

$$(M, \alpha_1) \times (M, \alpha_1) \xrightarrow{(m, \alpha_2)} (M, \alpha_1)$$

whose structural equation is

$$\pi_1^*\alpha_1 - m^*\alpha_1 + \pi_2^*\alpha_1 = \partial\alpha_2$$

with $\alpha_2 \in C^1(M^2, \underline{A})$; this equation can be read as

$$-\delta_M\alpha_1 + \partial\alpha_2 = 0$$

in the double complex $C^{p,q}(M_\bullet, \underline{A})$ with $\alpha_1 \in C^{1,2}(M_\bullet, \underline{A})$ and $\alpha_2 \in C^{2,1}(M_\bullet, \underline{A})$.

- A 2-morphism $\alpha_3 \in C^0(M^3, \underline{A})$ controlling the defect of associativity

$$(m, \alpha_2) \circ (1_M \times m, \pi_{23}^*\alpha_2) \xrightarrow{\alpha_3} (m, \alpha_2) \circ (m \times 1_M, \pi_{12}^*\alpha_2),$$

where $\pi_{23} : M^3 \rightarrow M^2$ denotes the projection on the last two coordinates, $\pi_{12} : M^3 \rightarrow M^2$ denotes the projection on the first two, and 1_M denotes the identity map on M . The structural equation becomes

$$\pi_{12}^*\alpha_2 - (1_M \times m)^*\alpha_2 + (m \times 1_M)^*\alpha_2 - \pi_{23}^*\alpha_2 = \partial\alpha_3,$$

which can be read as

$$\delta_M\alpha_2 + \partial\alpha_3 = 0$$

in the double complex $C^{p,q}(M_\bullet, \underline{A})$ with $\alpha_3 \in C^{3,0}(M_\bullet, \underline{A})$.

- The pentagon condition for the 2-morphism α_3 which can be simply written as

$$\delta_M \alpha_3 = 0$$

in the double complex $C^{p,q}(M_\bullet, \underline{A})$.

The element $\alpha := \alpha_1 \oplus -\alpha_2 \oplus \alpha_3$ becomes a 3-cocycle in $Z^3(M_\bullet, \underline{A})$ since

$$(\delta_M + (-1)^p \partial) \alpha = (-\partial \alpha_1) \oplus (\delta_M \alpha_1 - \partial \alpha_2) \oplus (-\delta_M \alpha_2 - \partial \alpha_3) \oplus (\delta_M \alpha_3) = 0,$$

and any 3 cocycle in $Z^3(M_\bullet, \underline{A})$ whose component in $C^{0,3}(M_\bullet, \underline{A}) = \text{Map}(pt, EB^3 A)$ is zero defines a 2-monoid structure on Segal–Mitchison A -gerbes over the monoid M .

Note that the 0-th column $C^{0,*}(M_\bullet, \underline{A}) = \text{Map}(pt, EB^q A)$ in the double complex is acyclic, and moreover, the differential δ_M is zero on all elements of this column. Therefore, we remove the 0-th column from the complex, just leaving the coefficients A at the $(0, 0)$ coordinate, thus obtaining a better suited complex for understanding Segal–Mitchison multiplicative gerbes. Then, define a new double complex $\bar{C}^{p,q}(M_\bullet, \underline{A})$ as the subcomplex of $C^{p,q}(M_\bullet, \underline{A})$ as follows:

$$\bar{C}^{p,q}(M_\bullet, \underline{A}) = \begin{cases} C^{p,q}(M_\bullet, \underline{A}) & \text{if } q > 0, \\ A & \text{if } p = q = 0, \\ 0 & \text{if } q = 0, p \neq 0. \end{cases}$$

Denote $\bar{C}^*(M_\bullet, \underline{A}) := \text{Tot}(\bar{C}^{*,*}(M_\bullet, \underline{A}))$ and by $\bar{Z}^k(M_\bullet, \underline{A}) := Z^k(\bar{C}^*(M_\bullet, \underline{A}))$ its cocycles. Note that the inclusion

$$\bar{C}^*(M_\bullet, \underline{A}) \xrightarrow{\cong} C^*(M_\bullet, \underline{A})$$

is a quasi-isomorphism, and therefore, $H^*(\bar{C}^*(M_\bullet, \underline{A}), d) \cong H^*(M_\bullet, \underline{A})$. We now may claim the following result.

Proposition 4.1. *Multiplicative Segal–Mitchison gerbes structures over the monoid M are in one-to-one correspondence with elements in $\bar{Z}^3(M_\bullet, \underline{A})$.*

Proof. We have seen by the construction of the 2-monoid that $\alpha := \alpha_1 \oplus -\alpha_2 \oplus \alpha_3$ defines a 3-cocycle in $\bar{Z}^3(M_\bullet, \underline{A})$ since we removed the group corresponding to $C^{0,3}(M_\bullet, \underline{A})$. Any 3-cocycle is also of this form and therefore it defines a multiplicative Segal–Mitchison A -gerbe over the monoid M . ■

Notation. We will denote multiplicative Segal–Mitchison A -gerbes by pairs $\langle M, \alpha \rangle$ where M is a monoid and $\alpha \in \bar{Z}^3(M_\bullet, \underline{A})$. All information is encoded in this pair.

4.2. Morphisms

A morphism of multiplicative Segal–Mitchison gerbes,

$$\langle M, \alpha \rangle \xrightarrow{\langle F, \beta \rangle} \langle M', \alpha' \rangle,$$

consists of a pair $\langle F, \beta \rangle$ with $F : M \rightarrow M'$, a morphism of monoids (a continuous map preserving the monoidal structure), together with $\beta \in \bar{C}^2(M_\bullet, \underline{A})$ such that

$$\alpha - F^*\alpha' = d_M\beta = (\delta_M + (-1)^p\partial)\beta.$$

4.3. 2-Morphisms

Take two morphisms of multiplicative Segal–Mitchison gerbes:

$$\langle F_1, \beta_1 \rangle, \langle F_2, \beta_2 \rangle : \langle M, \alpha \rangle \xrightarrow{\langle F, \beta \rangle} \langle M', \alpha' \rangle.$$

There exist 2-morphisms only when

$$F_1 = F_2 = F,$$

and in this case the 2-morphisms

$$\langle F, \beta_1 \rangle \xrightarrow{\gamma} \langle F, \beta_2 \rangle$$

consist of an element $\gamma \in \bar{C}^1(M_\bullet, \underline{A})$ such that

$$\beta_2 - \beta_1 = d_M\gamma.$$

If we restrict the 2-category to a fixed monoid M and we only consider morphisms which are the identity map on the monoid M , we obtain the complex

$$\bar{C}^1(M_\bullet, \underline{A}) \xrightarrow{d_M} \bar{C}^2(M_\bullet, \underline{A}) \xrightarrow{d_M} \bar{Z}^3(M_\bullet, \underline{A}),$$

whose cohomology groups are

$$H^1(M_\bullet, \underline{A}), \quad H^2(M_\bullet, \underline{A}), \quad \text{and} \quad H^3(M_\bullet, \underline{A}).$$

Therefore, we can classify multiplicative Segal–Mitchison A -gerbes over a fixed monoid M by the following result.

Proposition 4.2. *The isomorphism classes of multiplicative Segal–Mitchison gerbes with structure group A over the monoid M are in one-to-one correspondence with the elements of the group $H^3(M_\bullet, \underline{A})$.*

5. Representations of multiplicative Segal–Mitchison gerbes

In what follows, we will represent multiplicative Segal–Mitchison gerbes on the 2-category of Segal–Mitchison gerbes. We will define the 2-category $\text{Rep}_{(M, \alpha)}$ of representations of a multiplicative Segal–Mitchison gerbe $\langle M, \alpha \rangle$ by using the cohomological nature that the Segal–Mitchison gerbes possess, and we will provide explanations for the specific choices.

5.1. Objects

A representation of the multiplicative Segal–Mitchison gerbe $\langle M, \alpha \rangle$ consists of a pair (N, β) , where N is a space endowed with an action of the monoid M (in this case, it is a right action but it may be a left action as well). Denote the action by the map $\mu : N \times M \rightarrow N$. The second part is a 2-cochain $\beta \in C^2((N \times M)_\bullet, \underline{A})$ such that

$$d_{N \times M} \beta = \pi^* \alpha,$$

where π denotes the projection map $(N \times M)_\bullet \rightarrow M_\bullet$ which forgets the N -coordinate. Let us call (N, β) a $\langle M, \alpha \rangle$ -representation.

Let us untangle the definition in order to see its structural pieces. Let $\beta = \beta_0 \oplus -\beta_1 \oplus \beta_2$ with $\beta_i \in C^{i, 2-i}((N \times M)_\bullet, \underline{A}) = \text{Map}(N \times M^i, EB^{2-i}A)$, and recall that $\pi^* \alpha := \pi^* \alpha_1 \oplus -\pi^* \alpha_2 \oplus \pi^* \alpha_3$ with $\pi^* \alpha_j \in C^{j, 3-j}((N \times M)_\bullet, \underline{A})$.

- The first thing to notice is that $\partial \beta_0 = 0$ and therefore (N, β_0) is a Segal–Mitchison gerbe.
- The second equation that one obtains is

$$\delta_{N \times M} \beta_0 + \partial \beta_1 = \pi^* \alpha_1.$$

But since $\delta_{N \times M} \beta_0 = \mu^* \beta_0 - \pi_1^* \beta_0$, we may rearrange the equation as

$$\pi_1^* \beta_0 + \pi^* \alpha_1 - \mu^* \beta_0 = \partial \beta_1,$$

which says that

$$(N, \beta_0) \times (M, \alpha_1) \xrightarrow{(\mu, \beta_1)} (N, \beta_0)$$

is a morphism of Segal–Mitchison gerbes.

- The third equation is

$$-\delta_{N \times M} \beta_1 + \partial \beta_2 = -\pi^* \alpha_2.$$

Unwinding it, we obtain the equation for the defect of associativity

$$(\mu \times 1_M)^* \beta_1 + \pi_{12}^* \beta_1 - (1_N \times m)^* \beta_1 - \pi^* \alpha_2 = \partial \beta_2$$

associated with the diagram

$$\begin{array}{ccc}
 (N \times \beta_0) \times (M, \alpha_1) \times (M, \alpha_1) & \xrightarrow{(\mu \times 1_M, (\mu \times 1_M)^* \beta_1)} & (N \times \beta_0) \times (M, \alpha_1) \\
 \downarrow (1_N \times m, \pi^* \alpha_2) & \nearrow \beta_2 & \downarrow (\mu, \beta_1) \\
 (N \times \beta_0) \times (M, \alpha_1) & \xrightarrow{(\mu, \beta_1)} & (N \times \beta_0).
 \end{array}$$

- The last equation

$$\delta_{N \times M} \beta_2 = \pi^* \alpha_3$$

is simply the pentagon identity for the associator.

It should be now clear how to define the morphisms and 2-morphisms using the cohomological structure. We will not untangle the information anymore.

5.2. Morphisms

A morphism of multiplicative Segal–Mitchison gerbe representations

$$(N, \beta) \xrightarrow{(F, \gamma)} (N', \beta')$$

consists of an M -equivariant map $F : N \rightarrow N'$ and a cochain $\gamma \in C^1((N \rtimes M)_\bullet, \underline{A})$ such that

$$\beta - F^* \beta' = d_{N \rtimes M} \gamma.$$

The space of morphisms $\text{Hom}_{\langle M, \alpha \rangle}((N, \beta), (N', \beta'))$ is endowed with the subspace topology of $\text{Map}_M(N, N') \times C^1((N \rtimes M)_\bullet, \underline{A})$ where the space of M -equivariant maps $\text{Map}_M(N, N')$ is endowed with the compact open topology. The composition of morphisms is continuous and is given by the formula

$$(F', \gamma') \circ (F, \gamma) = (F' \circ F, \gamma + F^* \gamma').$$

5.3. 2-Morphisms

For two morphisms

$$(F_1, \gamma_1), (F_2, \gamma_2) : (N, \beta) \rightarrow (N', \beta'),$$

there are 2-morphisms from one to the other only if

$$F_1 = F_2 = F.$$

If this is the case, a 2-morphism

$$(F, \gamma_1) \xRightarrow{\nu} (F, \gamma_2)$$

is a cochain $\nu \in C^0((N \rtimes M)_\bullet, \underline{A})$ such that

$$\gamma_2 - \gamma_1 = d_{N \rtimes M} \nu.$$

Fix a space N on which M acts and fix a multiplicative Segal–Mitchison gerbe $\langle M, \alpha \rangle$. We see from the definition of a multiplicative Segal–Mitchison gerbe representation that a module over the space N exists if and only if $[\pi^* \alpha] = 0$ in $H^3((N \rtimes M)_\bullet, \underline{A})$. If this is the case, the $\langle M, \alpha \rangle$ -representation structures over N are (non-canonically) isomorphic with $Z^2((N \rtimes M)_\bullet, \underline{A})$; hence, we have the following result.

Proposition 5.1. *Consider a multiplicative Segal–Mitchison gerbe $\langle M, \alpha \rangle$ with structure group A , and a space N with a right action of M . Then, the space N can be endowed with the structure of a representation of $\langle M, \alpha \rangle$ if and only if*

$$[\pi^* \alpha] = 0 \quad \text{in } H^3((N \rtimes M)_\bullet, \underline{A}).$$

Whenever $[\pi^*\alpha] = 0$, the isomorphism classes of $\langle M, \alpha \rangle$ -representations with underlying topological space N are in (non-canonical) one-to-one correspondence with the group $H^2((N \rtimes M)_\bullet, \underline{A})$.

Example 5.2 (Canonical representation). Let us describe the canonical representation of a multiplicative Segal–Mitchison gerbe $\langle M, \alpha \rangle$ with structure group A . Recall that $\alpha = \alpha_1 \oplus -\alpha_2 \oplus \alpha_3$ with

$$\alpha_j \in C^{j,3-j}(M_\bullet, \underline{A}) = \text{Map}(M^j, EB^{3-j}A).$$

Let $\bar{\alpha}_j := \alpha_{j+1}$ and consider them as elements in

$$\bar{\alpha}_j \in C^{j,2-j}((M \rtimes M)_\bullet, \underline{A}) = \text{Map}(M \times M^j, EB^{2-j}A),$$

where the right action of M over M is given by right multiplication. Then,

$$\bar{\alpha} = \bar{\alpha}_1 \oplus -\bar{\alpha}_2 \oplus \bar{\alpha}_3 \in C^2((M \rtimes M)_\bullet, \underline{A}),$$

and it satisfies the equation

$$d_{M \rtimes M} \bar{\alpha} = (\delta_{M \rtimes M} + (-1)^p \partial) \bar{\alpha} = \pi^* \alpha,$$

where $\pi : (M \rtimes M)_\bullet \rightarrow M_\bullet$ forgets the first coordinate. Notice that

$$\pi^* \alpha - \delta_{M \rtimes M} \bar{\alpha} = \delta_M \alpha, \quad \text{and} \quad (-1)^p \partial \bar{\alpha} = (-1)^{p+1} \partial \alpha,$$

and therefore we get the desired equation

$$d_{M \rtimes M} \bar{\alpha} = (\delta_{M \rtimes M} + (-1)^p \partial) \bar{\alpha} = \pi^* \alpha - (\delta_M \alpha + (-1)^p \partial) \alpha = \pi^* \alpha. \tag{5.1}$$

Therefore, $(M, \bar{\alpha})$ is a representation of $\langle M, \alpha \rangle$, and we call it the canonical representation.

5.4. Category of morphisms between representations

Note that the category

$$\text{Hom}_{(M, \alpha)^{\text{op}}}((N, \beta), (N', \beta'))$$

of morphisms of multiplicative Segal–Mitchison gerbe representations is a lifting bundle gerbe with structure group

$$H^0(N \rtimes M, \underline{A}) = \text{Map}(N, A)^M,$$

the group of M -invariant maps from N to A . We have added the superscript op to recall that the action of the multiplicative Segal–Mitchison gerbe acts from the right. Using the notation in Section B.1, the category of morphisms of multiplicative Segal–Mitchison representations has the following form:

$$C^0(N \rtimes M, \underline{A}) \overset{d_{N \rtimes M}}{\curvearrowright} \text{Hom}_{(M, \alpha)^{\text{op}}}^0((N, \beta), (N', \beta')).$$

The underlying category is an action groupoid where the abelian group $C^0(N \rtimes M, \underline{A})$ acts on the morphisms $\text{Hom}_{(M, \alpha)^{\text{op}}}^0((N, \beta), (N', \beta'))$ by adding the image of the differential $d_{N \rtimes M}$.

5.5. Endomorphisms of representations

We will be interested in the category of endomorphisms of a right $\langle M, \alpha \rangle$ -representation (N, β) . Denote this category by

$$\text{End}_{\langle M, \alpha \rangle^{\text{op}}}(N, \beta) := \text{Hom}_{\langle M, \alpha \rangle^{\text{op}}}((N, \beta), (N, \beta)),$$

and note that its space of objects fits in the sequence of monoids

$$Z^1((N \rtimes M)_{\bullet}, \underline{A}) \rightarrow \text{End}_{\langle M, \alpha \rangle^{\text{op}}}^0(N, \beta) \rightarrow \text{End}_{M^{\text{op}}}(N, [\beta]),$$

where the monoid $\text{End}_{M^{\text{op}}}(N, [\beta])$ denotes the space of right M -equivariant maps $F : N \rightarrow N$ such that the cohomology class $[\beta - F^* \beta]$ is trivial. The space of morphisms of this category, $\text{End}_{\langle M, \alpha \rangle^{\text{op}}}^1(N, \beta)$, is parameterized by

$$\text{End}_{\langle M, \alpha \rangle^{\text{op}}}^0(N, \beta) \times C^0((N \rtimes M)_{\bullet}, \underline{A})$$

whose product structure is

$$((F_1, \gamma_1), v_1) \cdot ((F_2, \gamma_2), v_2) = ((F_1 \circ F_2, F_2^* \gamma_1 + \gamma_2), F_2^* v_1 + v_2).$$

Note furthermore that the space of isomorphism classes of objects fits in the sequence of monoids

$$H^1((N \rtimes M)_{\bullet}, \underline{A}) \rightarrow \pi_0(\text{End}_{\langle M, \alpha \rangle^{\text{op}}}(N, \beta)) \rightarrow \text{End}_{M^{\text{op}}}(N, [\beta])$$

and the group of endomorphisms of the identity map is the space of right M -invariant maps over N :

$$H^0((N \rtimes M)_{\bullet}, \underline{A}) = \text{Map}(N, A)^M.$$

The underlying four-term sequence of the monoid in lifting bundle gerbes is

$$\text{Map}(N, A)^M \rightarrow C^0((N \rtimes M)_{\bullet}, \underline{A}) \rightarrow \text{End}_{\langle M, \alpha \rangle^{\text{op}}}^0(N, \beta) \rightarrow \pi_0(\text{End}_{\langle M, \alpha \rangle^{\text{op}}}(N, \beta)),$$

and whenever $\text{End}_M(N, [\beta])$ is a group, it becomes the four-term sequence of the crossed module

$$C^0((N \rtimes M)_{\bullet}, \underline{A}) \rightarrow \text{End}_{\langle M, \alpha \rangle^{\text{op}}}^0(N, \beta)$$

associated with the 2-group. Note that $\text{End}_{\langle M, \alpha \rangle^{\text{op}}}(N, \beta)$ is also a group object in lifting bundle gerbes as it is explained in Appendix B.5. In this case, the structure group of the gerbe is the group $\text{Map}(N, A)^M$ of M -invariant maps.

Example 5.3 (Endomorphisms of the canonical representation). Consider a multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$, where the monoid is a topological group G , and $(G, \bar{\alpha})$, its canonical representation defined in Example 5.2. Here the group G acts by multiplication on the right on G .

The morphisms of the category of endomorphisms fit in the short exact sequence of groups

$$Z^1((G \rtimes G)_\bullet, \underline{A}) \rightarrow \text{End}_{(G,\alpha)^\text{op}}^0(G, \bar{\alpha}) \rightarrow G \tag{5.2}$$

since $\text{End}_{G^\text{op}}(G, G) \xrightarrow{\cong} G$, $F \mapsto F(1)$ and $H^*((G \rtimes G)_\bullet, \underline{A})$ is trivial for $* > 0$. The associated crossed module, or group object in lifting bundle gerbes (see Lemma B.1), produces a four-term exact sequence

$$A \rightarrow C^0((G \rtimes G)_\bullet, \underline{A}) \xrightarrow{d_{G \rtimes G}} \text{End}_{(G,\alpha)^\text{op}}^0(G, \bar{\alpha}) \rightarrow G,$$

where A sits inside $C^0((G \rtimes G)_\bullet, \underline{A}) = \text{Map}(G, EA)$ as the G invariant maps to A . Note that the crossed module, or group object in lifting bundle A -gerbes,

$$C^0((G \rtimes G)_\bullet, \underline{A}) \xrightarrow{d_{G \rtimes G}} \text{End}_{(G,\alpha)^\text{op}}^0(G, \bar{\alpha}),$$

encodes the information of the multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$.

Note that the complex $C^*((G \rtimes G)_\bullet, \underline{A})$ is a complex of G - \mathfrak{Topab} soft sheaves and moreover it is acyclic. Therefore, the short exact sequence

$$A \rightarrow C^0((G \rtimes G)_\bullet, \underline{A}) \xrightarrow{d_{G \rtimes G}} Z^1((G \rtimes G)_\bullet, \underline{A})$$

induces an isomorphism

$$H^2(G_\bullet, \underline{Z^1((G \rtimes G)_\bullet, \underline{A})}) \xrightarrow{\cong} H^3(G_\bullet, \underline{A}).$$

The inverse map of this isomorphism is given by the class in $H^2(G_\bullet, \underline{Z^1((G \rtimes G)_\bullet, \underline{A})})$ that encodes the group extension of (5.2).

Note that this 2-group of endomorphisms $\text{End}_{(G,\alpha)^\text{op}}(G, \bar{\alpha})$ is also group object in lifting bundle A -gerbes as it is explained in Lemma B.1. In this case, the structure group of the gerbe is A since the kernel of the differential $d_{G \rtimes G}$ consists of the G -invariant A -valued maps over the transitive space G .

The 2-group of endomorphisms $\text{End}_{(G,\alpha)^\text{op}}(G, \bar{\alpha})$ encodes the same topological information as the multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$. The first is a strict group object in lifting bundle A -gerbes, while the second is a weak monoid object in Segal–Mitchison gerbes. The former could be understood as a strictification of the latter.

6. Pontrjagin duality on multiplicative Segal–Mitchison $U(1)$ -gerbes

In this section, we will present a procedure by which we construct, from a specific choice of multiplicative Segal–Mitchison $U(1)$ -gerbe with a representation, another multiplicative Segal–Mitchison $U(1)$ -gerbe. This second multiplicative gerbe will be called fiberwise Pontrjagin dual because it generalizes Pontrjagin’s duality on locally compact topological abelian groups [16, 25].

Recall that for a topological abelian group S , its Pontrjagin dual group is

$$\widehat{S} := \text{Hom}(S, U(1)).$$

Whenever S is locally compact, \widehat{S} is also locally compact and the double Pontrjagin dual is canonically isomorphic to the original group $S \cong \widehat{\widehat{S}}$. In what follows, we will use the notation and results of Section 2.6.

Let us consider a central extension of locally contractible and compactly generated groups

$$S \rightarrow G \rightarrow K$$

with S locally compact. Let us take a cohomology class in $H^3(G_\bullet, \underline{U(1)})$ that vanishes once restricted to $H^3(S_\bullet, \underline{U(1)}) \cong E_2^{0,3} \supseteq E_\infty^{0,3}$ and once restricted to $\underline{E}_\infty^{1,2}$ in the Lyndon–Hochschild spectral sequence in Section 2.6. If we denote by $\Omega(G, S)$ those classes, the group is defined as the following subgroup of $H^3(G_\bullet, \underline{U(1)})$:

$$\Omega(G, S) := \text{Ker}(\text{Ker}(H^3(G_\bullet, \underline{U(1)}) \rightarrow E_\infty^{0,3}) \rightarrow E_\infty^{1,2}).$$

This group of cohomology classes fits into the short exact sequence

$$0 \rightarrow E_\infty^{3,0} \rightarrow \Omega(G, S) \rightarrow E_\infty^{2,1} \rightarrow 0,$$

where the left-hand side is the pullback of the classes from K

$$E_\infty^{3,0} = \text{Im}(H^3(K_\bullet, \underline{U(1)}) \rightarrow H^3(G_\bullet, \underline{U(1)})),$$

and the right-hand side is the cohomology of the middle term in the sequence

$$H^2(S_\bullet, \underline{U(1)}) \xrightarrow{d_2} H^2(K_\bullet, \widehat{S}) \xrightarrow{d_2} H^4(K_\bullet, \underline{U(1)})$$

with d_2 being the second differential of the spectral sequence. Here we have used the fact that $\widehat{S} \cong H^1(S_\bullet, \underline{U(1)})$.

Take a cocycle $\alpha \in Z^3(G_\bullet, \underline{U(1)})$ whose cohomology class lies in $\Omega(G, S)$. Consider the restriction homomorphism

$$H^3(G_\bullet, \underline{U(1)}) \xrightarrow{\pi^*} H^3((K \rtimes G)_\bullet, \underline{U(1)}) \cong H^3(S_\bullet, \underline{U(1)}),$$

and note that $[\alpha] \in \Omega(G, S)$ implies that $[\alpha|_S] = 0$. Therefore,

$$[\pi^* \alpha] = 0 \quad \text{in } H^3((K \rtimes G)_\bullet, \underline{U(1)}),$$

and thus there is a representation (K, β) with $\beta \in C^2(K \rtimes G, \underline{U(1)})$ such that $d_{K \rtimes G} \beta = \pi^* \alpha$. We should think of β as an element in $D^{0,2}$ in the double complex defined in the notation of (2.2), satisfying the equation

$$d_{K \rtimes G} \beta = \pi^* \alpha. \tag{6.1}$$

Recall that the double complex is

$$D^{p,q} = \bigoplus_{i+j=p, l+m=q} C^{i,j,l,m}$$

with differentials $d_K : D^{p-1,q} \rightarrow D^{p,q}$, $d_K := \delta_K + (-1)^i \partial_1$, and $d_{K \rtimes G} : D^{p,q-1} \rightarrow D^{p,q}$, $d_{K \rtimes G} := \delta_{K \rtimes G} + (-1)^l \partial_2$, and the total complex

$$\text{Tot}(C^{*,*,*,*}) = \text{Tot}(D^{*,*})$$

is endowed with the total differential $d = d_K + (-1)^p d_{K \rtimes G}$.

Since $[\alpha]$ belongs to $\Omega(G, S)$, the cohomology class $[\alpha]$ maps to zero in the cohomology group $E_{\infty}^{1,2}$, and therefore there must exist $\gamma \in D^{1,1}$ such that

$$d_K \beta - d_{K \rtimes G} \gamma = 0.$$

Therefore, $d_K \gamma \in D^{2,1}$ satisfies the equations

$$d_K(d_K \gamma) = 0 \quad \text{and} \quad d_{K \rtimes G}(d_K \gamma) = 0.$$

Hence, we have that

$$d_K \gamma \in Z^2(K_{\bullet}, \underline{Z^1((K \rtimes G)_{\bullet}, U(1))}),$$

and the cocycles $\pi^* \alpha$ and $d_K \gamma$ are cohomologous in $\text{Tot}^3(C^{*,*,*,*})$.

Recall that the inclusion $(pt \rtimes S) \rightarrow (K \rtimes G)$ of action groupoids induces an equivalence of groupoids. Therefore, the restriction map

$$Z^1((K \rtimes G)_{\bullet}, \underline{U(1)}) \rightarrow Z^1((pt \rtimes S)_{\bullet}, \underline{U(1)}) \cong \text{Hom}(S, U(1)) = \widehat{S}$$

induces a homomorphism

$$Z^2(K_{\bullet}, \underline{Z^1((K \rtimes G)_{\bullet}, U(1))}) \rightarrow Z^2(K_{\bullet}, \widehat{S}), \quad d_1 \gamma \mapsto \widehat{F}.$$

This cocycle \widehat{F} defines a central extension of the form

$$\widehat{S} \rightarrow \widehat{G} \rightarrow K.$$

We will see in what follows that by choosing $[\alpha] \in \Omega(G, S)$ we make sure that there is a representation (K, β) of the multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$ such that all the right G -equivariant homomorphisms $f_h \in \text{Map}_{G\text{-op}}(K, K)$ with $f_h(k) = hk$ and $h \in K$ may be lifted to morphisms of representations from (K, β) to $(K, f_h^* \beta)$.

Having set up the structure, we claim the following result.

Theorem 6.1. *Let $\langle G, \alpha \rangle$ be a multiplicative Segal–Mitchison $U(1)$ -gerbe such that $[\alpha] \in \Omega(G, S)$, with $S \rightarrow G \rightarrow K$ being a central extension by the locally compact abelian*

group S , and let (K, β) be a right representation of $\langle G, \alpha \rangle$ as above. Then, the endomorphism category $\text{End}_{(G, \alpha)^{\text{op}}}(K, \beta)$ is a crossed module, or group lifting bundle A -gerbe, whose four-term sequence fits into the middle column of the following diagram:

$$\begin{array}{ccccc}
 U(1) & \longrightarrow & U(1) & & \\
 \downarrow & & \downarrow & & \\
 C^0((K \rtimes G)_\bullet, \underline{U(1)}) & \longrightarrow & C^0((K \rtimes G)_\bullet, \underline{U(1)}) & & \\
 \downarrow & & \downarrow & & \\
 Z^1((K \rtimes G)_\bullet, \underline{U(1)}) & \longrightarrow & \text{End}_{(G, \alpha)^{\text{op}}}^0(K, \beta) & \longrightarrow & K \\
 \downarrow & & \downarrow & & \downarrow \\
 \widehat{S} & \longrightarrow & \widehat{G} & \longrightarrow & K.
 \end{array}$$

Proof. Since the class $[\alpha]$ belongs to $\Omega(G, S)$, we have shown above that there must exist $\gamma \in D^{1,1}$ such that $d_K \beta - d_{K \rtimes G} \gamma = 0$. Therefore, $d_K \gamma$ and $\pi^* \alpha$ are cohomologous and $d_K \gamma \in Z^2(K_\bullet, \underline{Z^1((K \rtimes G)_\bullet, \underline{U(1))})})$ defines the cocycle $\widehat{F} \in Z^2(K_\bullet, \widehat{S})$ by the canonical map induced by the restriction

$$r : Z^1((K \rtimes G)_\bullet, \underline{U(1)}) \rightarrow \text{Hom}(S, U(1)) = \widehat{S}.$$

Since we have the isomorphism of topological groups

$$\text{Map}_{G^{\text{op}}}(K, K) \cong K, \quad f_h \mapsto h \tag{6.2}$$

where $f_h(k) = hk$ for $h, k \in K$, we know that the existence of γ implies that for all $h \in K$

$$[d_K(\beta)](h) = \beta - f_h^* \beta = (d_{K \rtimes G} \gamma)(h). \tag{6.3}$$

Therefore, the projection map

$$\text{End}_{(G, \alpha)^{\text{op}}}^0(K, \beta) \rightarrow \text{Map}_{G^{\text{op}}}(K, K) \cong K \tag{6.4}$$

is surjective.

Now, γ lives in $\text{Map}(K, EC^1((K \rtimes G)_\bullet, \underline{U(1)}))$ and for every $h \in K$ it chooses an element in $C^1((K \rtimes G)_\bullet, \underline{U(1)})$ such that

$$\beta - f_h^* \beta = (d_{K \rtimes G} \gamma)(h).$$

The class $d_K \gamma \in Z^2(K_\bullet, \underline{Z^1((K \rtimes G)_\bullet, \underline{U(1))})})$ measures the obstruction for the assignment γ to be a section of the homomorphism $\text{End}_{(G, \alpha)^{\text{op}}}^0(K, \beta) \rightarrow K$. The cocycle \widehat{F} defined by $d_K \gamma$ through the restriction map

$$r_* : Z^2(K_\bullet, \underline{Z^1((K \rtimes G)_\bullet, \underline{U(1))})}) \rightarrow Z^2(K_\bullet, \widehat{S}), \quad d_K \gamma \mapsto r_*(d_K \gamma) = \widehat{F}$$

is equivalently defined as the cocycle in $Z^2(K_\bullet, H^1(S_\bullet, U(1)))$ that $d_K \gamma$ defines once one takes isomorphism classes of morphisms in $\text{End}_{(G,\alpha)}(K, \beta)$. Therefore, we have the isomorphism of groups

$$\widehat{G} \cong \pi_0(\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta))$$

and \widehat{F} defines the extension class of the exact sequence

$$\widehat{S} \rightarrow \pi_0(\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)) \rightarrow K.$$

The rest of the diagram follows from the four-term exact sequence

$$U(1) \rightarrow C^0((K \times G)_\bullet, U(1)) \rightarrow Z^1((K \times G)_\bullet, U(1)) \rightarrow \widehat{S}$$

since $H^1((K \times G)_\bullet, U(1)) = \widehat{S}$. ■

The crossed module, or group lifting bundle A -gerbe, and its four-term sequence

$$U(1) \rightarrow C^0((K \times G), U(1)) \xrightarrow{d_{K \times G}} \text{End}_{(G,\alpha)^{\text{op}}}^0(K, \beta) \rightarrow \widehat{G}$$

define a cohomology class in $H^3(\widehat{G}_\bullet, U(1))$, which is constructed as follows.

The central extension $\widehat{S} \rightarrow G \xrightarrow{\hat{p}} K$ defines an element $\widehat{F} = (-\widehat{F}_1, \widehat{F}_2) \in Z^2(K_\bullet, \widehat{S})$ whose components are $\widehat{F}_1 : K \rightarrow B\widehat{S}$ and $\widehat{F}_2 : K \times K \rightarrow E\widehat{S}$, satisfying the equations

$$\delta_K \widehat{F}_1(k_1, k_2) = \widehat{F}_1(k_2) - \widehat{F}_1(k_1 k_2) + \widehat{F}_1(k_1) = \partial \widehat{F}_2(k_1, k_2), \quad \delta_K \widehat{F}_2 = 0.$$

Let $\widehat{F}'_1 : \widehat{G} \rightarrow E\widehat{S}$ be the map of \widehat{S} -principal bundles making the diagram commutative

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\widehat{F}'_1} & E\widehat{S} \\ \hat{p} \downarrow & & \downarrow \\ K & \xrightarrow{\widehat{F}_1} & B\widehat{S}. \end{array}$$

Then, the map \widehat{F}'_1 could be seen as living in $C^1(\widehat{G}, \widehat{S})$, and it satisfies the equations

$$\delta_{\widehat{G}} \widehat{F}'_1 = \hat{p}^* \widehat{F}_2, \quad \partial \widehat{F}'_1 = \hat{p}^* \widehat{F}_1.$$

Hence, $(\delta_{\widehat{G}} + (-1)^l \partial) \widehat{F}'_1 = d_{\widehat{G}} \widehat{F}'_1 = \hat{p}^* \widehat{F}$.

Let us carry out the same construction for the central extension $S \rightarrow G \xrightarrow{p} K$. Define its extension class $F = (-F_1, F_2) \in Z^2(K_\bullet, S)$ and the map of S -principal bundles $F'_1 G \rightarrow ES$, satisfying $\delta_G F'_1 = p^* F_2$ and $\partial F'_1 = p^* F_1$.

Define the cocycle $E = (E_0, E_1) \in Z^1((K \times G)_\bullet, S)$ with $E_0 \in \text{Map}(K, BS)$ and $E_0 = F_1$ by the equations

$$E_0 = F_1, \quad E_1(k, g) = F'_1(g) - F_2(k, p(g)).$$

A simple calculation shows that the following equations are satisfied:

$$\begin{aligned} \delta_{K \rtimes G} E_0(k, g) &= F_1(kg) - F_1(k) = F_1(p(g)) - \partial F_2(k, p(g)) = \partial E_1(k, g) \\ \delta_{K \rtimes G} E_1(k, g_1, g_2) &= 0. \end{aligned}$$

Any element $\rho \in \hat{S}$ induces homomorphisms $ES \rightarrow EU(1)$ and $BS \rightarrow BU(1)$, and therefore, we may define the homomorphism

$$\psi : \hat{S} \rightarrow Z^1((K \rtimes G)_\bullet, \underline{U(1)}), \quad \rho \mapsto \psi(\rho) = \rho \circ (E_0, E_1),$$

which is an inverse to the restriction homomorphism $Z^1((K \rtimes G)_\bullet, \underline{U(1)}) \rightarrow \hat{S}$. Nevertheless, the map ψ does not preserve the structure of K -module and this deficiency is quite important.

Let us consider the homomorphism

$$C^1(\hat{G}_\bullet, \hat{S}) \xrightarrow{\psi_*} C^1(\hat{G}_\bullet, \underline{Z^1((K \rtimes G)_\bullet, \underline{U(1))})}$$

induced by the map ψ at the level of the coefficients and take $\psi_* \hat{F}'_1$ the image of \hat{F}'_1 . The element $\psi_* \hat{F}'_1 : \hat{G} \rightarrow EZ^1((K \rtimes G)_\bullet, \underline{U(1)})$ should be understood as the lift of the map $\hat{F}'_1 : \hat{G} \rightarrow E\hat{S}$ to $Z^1((K \rtimes G)_\bullet, \underline{U(1)})$ in the left vertical map of the following diagram:

$$\begin{array}{ccccc} Z^1((K \rtimes G)_\bullet, \underline{U(1)}) & \longrightarrow & \text{End}_{(G, \alpha)^{\text{op}}}^0(K, \beta) & \longrightarrow & K \\ \downarrow & & \downarrow & & \parallel \\ \hat{S} & \longrightarrow & \hat{G} & \longrightarrow & K. \end{array}$$

The failure of the map $\psi_* \hat{F}'_1$ to induce a section is measured by the cocycle

$$d_{\hat{G}}(\psi_* \hat{F}'_1) \in Z^2(\hat{G}_\bullet, \underline{Z^1((K \rtimes G)_\bullet, \underline{U(1))})},$$

and its restriction to $Z^2(\hat{G}_\bullet, \hat{S})$ is

$$r_*(d_{\hat{G}}(\psi_* \hat{F}'_1)) = \hat{p}^* \hat{F}.$$

The element

$$[d_{\hat{G}}(\psi_* \hat{F}'_1) - \hat{p}^*(d_K \gamma)] \in Z^2(\hat{G}_\bullet, \underline{Z^1((K \rtimes G)_\bullet, \underline{U(1))})}$$

restricts to zero in $Z^2(\hat{G}_\bullet, \hat{S})$ since $r_*(\hat{p}^*(d_K \gamma)) = \hat{p}^*(r_*(d_K \gamma)) = \hat{p}^* \hat{F}$. Therefore,

$$[d_{\hat{G}}(\psi_* \hat{F}'_1) - \hat{p}^*(d_K \gamma)] \in Z^2(\hat{G}_\bullet, \underline{B^1((K \rtimes G)_\bullet, \underline{U(1))})},$$

where $B^1((K \rtimes G)_\bullet, \underline{U(1)})$ are the coboundaries fitting in the short exact sequence

$$B^1((K \rtimes G)_\bullet, \underline{U(1)}) \hookrightarrow Z^1((K \rtimes G)_\bullet, \underline{U(1)}) \xrightarrow{r} \hat{S}.$$

Now, the coboundaries also fit in the short exact sequence of abelian groups in $\widehat{G}\text{-Topab}$

$$U(1) \rightarrow C^0((K \rtimes G)_\bullet, \underline{U(1)}) \rightarrow B^1((K \rtimes G)_\bullet, \underline{U(1)}),$$

thus inducing a map in cohomology

$$H^2(\widehat{G}_\bullet, \underline{B^1((K \rtimes G)_\bullet, \underline{U(1)})}) \rightarrow H^3(\widehat{G}_\bullet, \underline{U(1)})$$

$$[d_{\widehat{G}}(\psi_* \widehat{F}'_1) - \widehat{p}^*(d_K \gamma)] \mapsto [\widehat{\alpha}_{(K, \beta)}].$$

The explicit description of the cocycle $\widehat{\alpha}_{(K, \beta)} \in Z^3(\widehat{G}_\bullet, \underline{U(1)})$ depends on several choices, but its cohomology class only depends on the multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$ and the representation (K, β) . The cohomology class $[\widehat{\alpha}_{(K, \beta)}]$ characterizes the group lifting bundle A -gerbe $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$.

Bundling up, we obtain the following.

Theorem 6.2. *The crossed module, or group lifting bundle A -gerbe,*

$$C^0((K \rtimes G)_\bullet, \underline{U(1)}) \rightarrow \text{End}_{\langle G, \alpha \rangle^{\text{op}}}^0(K, \beta)$$

induced by the category $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$ of endomorphisms of the right $\langle G, \alpha \rangle$ -representation (K, β) has \widehat{G} for cokernel, $U(1)$ as kernel and it is classified by the cohomology class

$$[\widehat{\alpha}_{(K, \beta)}] \in H^3(\widehat{G}_\bullet, \underline{U(1)})$$

defined above.

Both groups G and \widehat{G} project to K and the fibers are S and $\widehat{S} = \text{Hom}(S, U(1))$, respectively. Since S is a locally compact abelian group, its Pontrjagin dual group \widehat{S} is also a locally compact abelian group [25, First Fundamental Theorem]. Therefore, we propose the following definition.

Definition 6.3. The multiplicative Segal–Mitchison gerbe $\langle \widehat{G}, \widehat{\alpha}_{(K, \beta)} \rangle$ is a fiberwise Pontrjagin dual of the multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$.

Let us now see some specific cases.

6.1. Classical Pontrjagin duality

Take S a locally compact abelian group and $\alpha = 0$. By the Second Fundamental Theorem of Pontrjagin [25, p. 377], cf. [16, Thm. 2], the group S is locally compact and locally contractible, and therefore it satisfies the conditions of Theorem 6.1. Consider the representation of $\langle S, 0 \rangle$ to be $(*, 0)$ where $*$ is a point.

The 2-group $\text{End}_{\langle S, 0 \rangle}(*, 0)$ is isomorphic to $U(1)[1] \times \widehat{S}$ since its four-term sequence is

$$U(1) \rightarrow C^0((\ast \rtimes S)_\bullet, \underline{U(1)}) \rightarrow Z^1((\ast \rtimes S)_\bullet, \underline{U(1)}) \rightarrow \widehat{S}$$

and its cohomology class in $H^3(\widehat{S}_\bullet, \underline{U(1)})$ is trivial because the last homomorphism splits through the inclusion of groups $\widehat{S} = \text{Hom}(S, U(1)) \subset Z^1((\ast \rtimes S)_\bullet, U(1))$.

We see that the fiberwise dual of the multiplicative Segal–Mitchison $U(1)$ -gerbe $\langle S, 0 \rangle$ is the multiplicative Segal–Mitchison $U(1)$ -gerbe $\langle \widehat{S}, 0 \rangle$ and vice versa, thus recovering Pontrjagin’s duality in the context of multiplicative Segal–Mitchison gerbes.

6.2. Multiplicative Segal–Mitchison gerbe with trivial cocycle

Consider a non-trivial central extension $S \rightarrow G \rightarrow K$ and $\alpha = 0$. Then, we may take the representation to be $(K, 0)$, and therefore, we get that

$$\text{End}_{(G,0)^{\text{op}}}^0(K, 0) \cong K \rtimes Z^1((K \rtimes G)_\bullet, \underline{U(1)}).$$

Hence, the fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbe has for underlying group $\widehat{G} = K \times \widehat{S}$. Now consider $\widehat{F}'_1 : K \times \widehat{S} \rightarrow E\widehat{S}$ to be defined by the inclusion $\widehat{S} \rightarrow E\widehat{S}$. Following the construction, we get $(\psi_* \widehat{F}'_1)(k, \rho) = \rho \circ (E_0, E_1)$ and

$$[d_{\widehat{G}}(\psi_* \widehat{F}'_1)]((k_1, \rho_1), (k_2, \rho_2)) = \rho_1(k_2^*(E_0, E_1) - (E_0, E_1)).$$

The cocycle thus defined $\widehat{\alpha}_{(K,0)} = (\widehat{\alpha}_{(K,0)}^3, \widehat{\alpha}_{(K,0)}^2)$ with $\widehat{\alpha}_{(K,0)}^3 : \widehat{G}^3 \rightarrow EU(1)$ and $\widehat{\alpha}_{(K,0)}^2 : \widehat{G}^2 \rightarrow BU(1)$ is defined in coordinates as follows:

$$\begin{aligned} \widehat{\alpha}_{(K,0)}^3((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) &= \rho_1(F_2(k_2, k_3)) \in EU(1), \\ \widehat{\alpha}_{(K,0)}^2((k_1, \rho_1), (k_2, \rho_2)) &= \rho_1(F_1(k_2)) \in BU(1). \end{aligned} \tag{6.5}$$

Therefore, the cohomology class $[\widehat{\alpha}_{(K,0)}] \in H^3(K \times \widehat{S}_\bullet, \underline{U(1)})$ is simply obtained by the composition of $[F] \in H^2(K_\bullet, \underline{S})$ with $\widehat{S} = \text{Hom}(S, U(1))$, and the Multiplicative Segal–Mitchison gerbe $\langle K \times \widehat{S}, \widehat{\alpha}_{(K,0)} \rangle$ is a fiberwise Pontrjagin dual of $\langle G, 0 \rangle$.

6.3. Multiplicative Segal–Mitchison gerbe over a trivial extension

Let us start now with the trivial extension $\widehat{G} = K \times \widehat{S}$ and the cocycle $\widehat{\alpha}_{(K,0)}$ defined by equations (6.5). In this case, the dual group is

$$\pi_0(\text{End}_{(K \times \widehat{S}, \widehat{\alpha}_{(K,0)})}^0(K, 0)) \cong G$$

with cocycle $F \in Z^2(K_\bullet, \underline{S})$, and the cohomology class of the crossed module is trivial because

$$d_G \psi_* F'_1 = p^* \psi_* F$$

and γ may be chosen so that $d_G \gamma = \psi_* F$. Therefore, we conclude the following.

Proposition 6.4. *The multiplicative Segal–Mitchison $U(1)$ -gerbes*

$$\langle G, 0 \rangle \quad \text{and} \quad \langle K \times \widehat{S}, \widehat{\alpha}_{(K,0)} \rangle$$

defined above are mutually fiberwise Pontrjagin dual.

6.4. Groups whose connected components of the identity are contractible

In the case that $G, \widehat{G}, S, \widehat{S}$, and K are groups whose connected components of the identity are contractible, we can provide an explicit description of the class $[\widehat{\alpha}_{(K,\beta)}]$ using the results of [31, §3].

For these specific choice of groups, we know by Lemma 2.3 that

$$H^*(G_\bullet, \underline{U(1)}) \cong H_{\text{cont}}^*(G, U(1)),$$

where $H_{\text{cont}}^*(G, U(1))$ denotes the cohomology of the complex of continuous cochains of the group G with values in $U(1)$.

Following the notation, the formulas, and the calculations of [31, §3], we may take $G \cong S \rtimes_F K$ where $F \in Z_c^2(K, S)$ and the product structure on $S \rtimes_F K$ is given by the formula

$$(a_1, k_1) \cdot (a_2, k_2) = (a_1 a_2 F(k_1, k_2), k_1 k_2).$$

From [31, Thm. 3.6], we know that the 3-cocycle $\alpha \in Z_c^3(G, U(1))$ can be defined as

$$\alpha((a_1, k_1), (a_2, k_2), (a_3, k_3)) = \widehat{F}(k_1, k_2)(a_3)\varepsilon(k_1, k_2, k_3)$$

where $\widehat{F} \in Z^2(K, \widehat{S})$ and

$$\delta_K \varepsilon(k_1, k_2, k_3, k_4) = \widehat{F}(k_1, k_2)(F(k_3, k_4))$$

with $\varepsilon \in C_c^3(K, U(1))$.

From this definition of α , we know that $\widehat{G} \cong K \rtimes_{\widehat{F}} \widehat{S}$ with product structure

$$(k_1, \rho_1) \cdot (k_2, \rho_2) = (k_1 k_2, \rho_1 \rho_2 \widehat{F}(k_1, k_2)).$$

From [31, Thm. 3.6], we know that $\beta \in C_c^2(K \rtimes G, U(1))$ can be written as follows:

$$\beta(k_1, (a_2, k_2), (a_3, k_3)) = (\widehat{F}(k_1, k_2)(a_3)\varepsilon(k_1, k_2, k_3))^{-1}$$

and $\gamma \in C_c^1(K, C_c^1(K \rtimes G, U(1)))$ by the equation

$$\gamma(k_1, (x_1, (a_2, x_2))) = \widehat{F}(k_1, x_1)(a_2)\varepsilon(k_1, x_1, x_2);$$

therefore, $\widehat{p}^* d_1 \gamma$ satisfies the following equation:

$$\begin{aligned} &(\widehat{p}^* d_1 \gamma)((k_1, \rho_1), (k_2, \rho_2))(x_1, (a_2, x_2)) \\ &= \widehat{F}(k_1, k_2)(a_2 F(x_1 x_2))\varepsilon(k_1, k_2, x_1 x_2)\varepsilon(k_1, k_2, x_1)^{-1}. \end{aligned}$$

The cochain \widehat{F}'_1 can be defined by the map $\widehat{F}'_1(k, \rho) = \rho^{-1}$ and therefore $\psi_* \widehat{F}'_1$ satisfies the following equation:

$$\psi_* \widehat{F}'_1(k, \rho)(x_1, (a_2, x_2)) = \rho(a_2 F(x_1, x_2))^{-1}.$$

Calculating the coboundary, we obtain

$$\begin{aligned} &\delta_{\widehat{G}}\psi_*\widehat{F}'_1((k_1, \rho_1), (k_2, \rho_2))(x_1, (a_2, x_2)) \\ &= \widehat{F}(k_1, k_2)(a_2F(x_1, x_2))\rho_1(F(k_2, x_1)F(k_2, x_1x_2)^{-1}), \end{aligned}$$

and therefore,

$$\begin{aligned} &[\delta_{\widehat{G}}(\psi_*\widehat{F}'_1) - \widehat{p}^*(d_1\gamma)]((k_1, \rho_1), (k_2, \rho_2))(x_1, (a_2, x_2)) \\ &= \varepsilon(k_1, k_2, x_1x_2)^{-1}\varepsilon(k_1, k_2, x_1)\rho_1(F(k_2, x_1x_2)^{-1}F(k_2, x_1)) \\ &= \delta_{K \rtimes G}\eta(((k_1, \rho_1), (k_2, \rho_2))(x_1, (a_2, x_2))), \end{aligned}$$

where $\eta((k_1, \rho_1), (k_2, \rho_2), x) = \varepsilon(k_1, k_2, x)^{-1}\rho_1(F(k_2, x))^{-1}$ with $\eta \in C_c^2(\widehat{G}, C_c^0(K \rtimes G, U(1)))$. The coboundary of η satisfies the equation

$$\delta_{\widehat{G}}\eta((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3), x) = \varepsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3))$$

and it is independent of the variable x . We obtain that the dual class $\widehat{\alpha}_{(K, \beta)}$ can be written explicitly by the following formula:

$$\widehat{\alpha}_{(K, \beta)}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) = \varepsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)).$$

From the previous calculations, we can conclude the following theorem which generalizes [31, Thm. 3.7].

Theorem 6.5. *Let K be a compactly generated topological group and S a locally compact and compactly generated topological abelian group such that the connected components of the identity on K , S , and \widehat{S} are contractible groups. Consider the groups $G = S \rtimes_F K$ and $\widehat{G} = K \rtimes_{\widehat{F}} \widehat{S}$ with \widehat{S} being the Pontrjagin dual of S , $F \in Z_c^2(K, S)$, and $\widehat{F} \in Z_c^2(K, \widehat{S})$ such that $\delta_K\varepsilon(k_1, k_2, k_3, k_4) = \widehat{F}(k_1, k_2)(F(k_3, k_4))$ for $\varepsilon \in C_c^3(K, U(1))$. Define the 3-cocycles $\alpha \in Z_c^3(G, U(1))$ and $\widehat{\alpha} \in Z_c^3(\widehat{G}, U(1))$ by the equations*

$$\begin{aligned} \alpha((a_1, k_1), (a_2, k_2), (a_3, k_3)) &= \widehat{F}(k_1, k_2)(a_3)\varepsilon(k_1, k_2, k_3), \\ \widehat{\alpha}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) &= \varepsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)). \end{aligned}$$

Then, the multiplicative Segal–Mitchison gerbes $\langle G, \alpha \rangle$ and $\langle \widehat{G}, \widehat{\alpha} \rangle$ are fiberwise Pontrjagin dual.

Example 6.6. Consider $K = \mathbb{R}^2$, $S = \mathbb{R}$, and $G = K \times S = \mathbb{R}^3$. The Pontrjagin dual group $\widehat{S} = \text{Hom}(S, U(1))$ is isomorphic to the reals, and we may consider $\alpha \in Z_c^3(\mathbb{R}^3, U(1))$ by the formula

$$\alpha((r_1, s_1, t_1), (r_2, s_2, t_2), (r_3, s_3, t_3)) = e^{2\pi i r_1 s_2 t_3}.$$

Applying Proposition 6.4 and Theorem 6.5, we see that the fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbe is $\langle \mathcal{H}, 0 \rangle$, where \mathcal{H} is the Heisenberg group of upper

triangular 3×3 matrices with ones on the diagonal and trivial 3-cocycle. In this case, $\widehat{F}((r_1, s_1), (r_2, s_2)) = r_1 s_2$ with $\widehat{F} \in Z_c^2(\mathbb{R}^2, \mathbb{R})$ and $F = 0$. We conclude that

$$\langle \mathbb{R}^3, \alpha \rangle \quad \text{and} \quad \langle \mathcal{H}, 0 \rangle$$

are fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes.

6.5. Finite abelian fiber

Consider the case on which S is a finite abelian group. In what follows, we will provide an explicit construction of the crossed modules associated with the fiberwise Pontrjagin dual multiplicative Segal–Mitchison $U(1)$ -gerbes of Proposition 6.4.

Let $\widehat{S} := \text{Hom}(S, U(1))$ be the Pontrjagin dual finite group and take $b : S \times S \rightarrow U(1)$ a bilinear map inducing an isomorphism of groups $b^\# : S \xrightarrow{\cong} \widehat{S}, s \mapsto b(s, -)$.

Consider a topological group G on which S is a central subgroup fitting into the non-trivial group extension $S \rightarrow G \rightarrow K$.

We are going to define two non-equivalent crossed modules with this information which are fiberwise Pontrjagin dual. They will have the same structural groups N and E , same action of E on N , and same π_1 but different structural homomorphisms. Let us define the two 2-groups first.

Consider the groups

$$\mathcal{G}_0 := S \times G \quad \text{and} \quad \mathcal{G}_1 := (S \times G \times S) \ltimes U(1),$$

where the product in \mathcal{G}_1 is given by the formula

$$(s_1, g_1, \bar{s}_1, \lambda_1) \cdot (s_2, g_2, \bar{s}_2, \lambda_2) = (s_1 s_2, g_1 g_2, \bar{s}_1 \bar{s}_2, \lambda_1 + \lambda_2 + b(\bar{s}_1, s_2)).$$

In both cases, the underlying groupoid comes from the action of $S \times U(1)$ on $S \times G$. In one case, we let $S \times U(1)$ act on $S \times G$ by multiplication on the first coordinate, and in the other case by multiplication on the second coordinate. In both cases, the source map is the same $s(s, g, \bar{s}, \lambda) = (s, g)$ and the identity map also $e(s, g) = (s, g, 0, 0)$. Therefore, the induced right action of $S \times G$ on $S \times U(1)$ is given by the following formula:

$$(\bar{s}, \lambda)^{(s, g)} = (\bar{s}, \lambda + b(\bar{s}, s)).$$

The target maps are defined as follows:

$$t_1(s, g, \bar{s}, \lambda) = (s\bar{s}, g), \quad t_2(s, g, \bar{s}, \lambda) = (s, g\bar{s}),$$

and therefore they define crossed modules

$$S \times U(1) \xrightarrow{t_1} S \times G \quad \text{and} \quad S \times U(1) \xrightarrow{t_2} S \times G$$

such that in both cases $\text{Ker}(t_1) \cong U(1) \cong \text{Ker}(t_2)$.

Nevertheless, the crossed modules are non-equivalent since

$$\text{Coker}(t_1) = G \quad \text{and} \quad \text{Coker}(t_2) = S \times K.$$

Moreover, the cohomology class $[\alpha] \in H^3((S \times K)_\bullet, \underline{U(1)})$ that induces the second crossed module is not trivial.

These two crossed modules defined above are the fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes of Proposition 6.4.

Example 6.7. Consider $G = SU(2)$ and $S = \mathbb{Z}/2 \subset G$ its center. Then, $K = SO(3)$ and we may take the bilinear form $b : \mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow U(1)$, $(x, y) \mapsto \frac{xy}{2}$ with $x, y \in \{0, 1\}$ and $U(1) = \mathbb{R}/\mathbb{Z}$. The cohomology class of the crossed module that induces the four-term sequence

$$U(1) \rightarrow \mathbb{Z}/2 \times U(1) \rightarrow \mathbb{Z}/2 \times SU(2) \rightarrow \mathbb{Z}/2 \times SO(3)$$

is the only non-trivial class that comes from the map

$$H^1(H^2(SO(3)_\bullet, \underline{\mathbb{Z}/2}), U(1)) \rightarrow H^3((\mathbb{Z}/2 \times SO(3))_\bullet, \underline{U(1)}).$$

The two crossed modules previously defined induce multiplicative Segal–Mitchison $U(1)$ -gerbes. On the one hand, we have the group $SU(2)$ with trivial cohomology class and on the other hand, we have the group $\mathbb{Z}/2 \times SO(3)$ with the cohomology class in $H^3((\mathbb{Z}/2 \times SO(3))_\bullet, \underline{U(1)})$ defined by composition of the Stiefel–Whitney class $w_2 \in H^2(BSO(3), \mathbb{Z}/2)$ with the non-trivial homomorphism in $H^1((\mathbb{Z}/2)_\bullet, \underline{U(1)})$.

6.6. Finite groups

Multiplicative Segal–Mitchison gerbes over finite groups and pointed fusion categories are very related entities. A multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$ with G being a finite group and $\alpha \in Z^3(G, U(1))$ encodes the information of a pointed fusion category. A pointed fusion category is a rigid tensor category with finitely many isomorphism classes of simple objects which moreover are invertible [7]. The relevant pointed fusion categories are $\text{Vect}(G, \alpha)$ of G -graded complex vector spaces with associativity constraint α . Module categories over the pointed fusion categories $\text{Vect}(G, \alpha)$ play the role of representations of the multiplicative Segal–Mitchison gerbe $\langle G, \alpha \rangle$ and vice versa.

The realm of fusion categories and their module categories has been extensively studied [7, 17, 18, 21, 23, 24], and there are many properties and constructions which do not necessarily generalize to the continuous case. Nevertheless, in the finite case, we can compare our Theorem 6.5 with [31, Thm. 3.9], and we can conclude that Morita equivalent pointed fusion categories (namely such that their module categories are equivalent) induce fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes and that pointed fusion categories of fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes are Morita equivalent. Therefore we can conclude that fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes over finite groups have equivalent categories of representations. The

generalization to multiplicative Segal–Mitchison gerbes over topological groups will be approached in the next section.

Concrete examples of fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes can thus be obtained by looking at the examples that have been constructed out of Morita equivalent pointed fusion categories. A complete classification of all fiberwise Pontrjagin dual Segal–Mitchison gerbes over groups of dimension 8 appears in [19], over groups of order p^3 for p odd prime in [12], and over groups of order less than 32 in [14]. Several other examples appear in [17, 21, 23, 24] and the references therein.

7. Morita equivalence for fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes

In this section, we will focus on the relation between the categories of representations of fiberwise Pontrjagin dual multiplicative Segal–Mitchison gerbes. For this, let us take a multiplicative Segal–Mitchison $U(1)$ -gerbe $\langle G, \alpha \rangle$ and its representation (K, β) as it appears in Theorem 6.1. In Theorem 6.2, we have shown that the 2-group of endomorphisms $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$, which is also a group object in lifting bundle A -gerbes, has \widehat{G} for its underlying group and its classifying class is $[\widehat{\alpha}_{(K, \beta)}] \in H^3(\widehat{G}_\bullet, \underline{U(1)})$. The multiplicative Segal–Mitchison $U(1)$ -gerbe $\langle \widehat{G}, \widehat{\alpha}_{(K, \beta)} \rangle$ was called a fiberwise Pontrjagin dual of $\langle G, \alpha \rangle$ in Definition 6.3.

In order to define the functors relating the categories of representations of the Segal–Mitchison $U(1)$ -gerbes, we need to recall first the properties of the groupoids of morphisms of two representations from Section 5.4. Let us consider two right representations (V, χ) and (W, ν) of the Segal–Mitchison $U(1)$ -gerbe $\langle G, \alpha \rangle$. The groupoid of morphisms

$$\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((V, \chi), (W, \nu)) = [\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}^0((V, \chi), (W, \nu)) / C^0((V \rtimes G)_\bullet, \underline{U(1)})]$$

is an action groupoid where the group $C^0((V \rtimes G)_\bullet, \underline{U(1)})$ acts on the space of objects $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}^0((V, \chi), (W, \nu))$ by the image of the differential $d_{V \rtimes G}$. The kernel of the differential is

$$H^0((V \rtimes G)_\bullet, \underline{U(1)}) \cong \text{Map}(V, U(1))^G;$$

the space of G -invariant maps from V to $U(1)$, and therefore the groupoid of morphisms becomes a lifting bundle gerbe with structure group $\text{Map}(V, U(1))^G$ as described in Appendix B.

The endomorphisms 2-group $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})$ together with its Pontrjagin dual 2-group $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$, defined in Example 5.3 and Theorem 6.1, respectively, are both also group objects in lifting bundle $U(1)$ -gerbes. As such, they act on lifting bundle gerbes, and we will denote by

$$\text{Rep}_{\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})} \quad \text{and} \quad \text{Rep}_{\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)}$$

their 2-categories of representations.

The canonical covariant functor

$$\begin{aligned} \text{Rep}_{\langle G, \alpha \rangle^{\text{op}}} &\rightarrow \text{Rep}_{\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})^{\text{op}}} \\ (V, \chi) &\mapsto \text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (V, \chi)) \end{aligned}$$

induces an equivalence of 2-categories from the category of right representations of the Segal–Mitchison $U(1)$ -gerbe $\langle G, \alpha \rangle$ to the 2-category of lifting bundle $U(1)$ -gerbes right representations of $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})$. The endomorphisms act on the right by pre-composition of morphisms.

There is also the contravariant functor

$$\begin{aligned} \text{Rep}_{\langle G, \alpha \rangle^{\text{op}}} &\rightarrow \text{Rep}_{\text{End}_{\langle G, \alpha \rangle}(K, \beta)} \\ (V, \chi) &\mapsto \text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((V, \chi), (K, \beta)) \end{aligned}$$

that takes a right representation of the Segal–Mitchison $U(1)$ -gerbe $\langle G, \alpha \rangle$ to a left representation of the Pontrjagin dual group lifting bundle $U(1)$ -gerbe $\text{End}_{\langle G, \alpha \rangle}(K, \beta)$. The groupoid $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((V, \chi), (K, \beta))$ is a lifting bundle $H^0(V, \underline{U(1)})^G$ -gerbe and it becomes a left representation by composition of morphisms. Now, if we would want the image of the functor to produce only lifting bundle $U(1)$ -gerbes, we would need to restrict to G -spaces V which are homogeneous spaces themselves. In this way, we would have that $H^0(V, \underline{U(1)})^G = U(1)$.

The lifting bundle $U(1)$ -gerbe

$$\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$$

is both a right representation of $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})$ and a representation of $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$. This lifting bundle $U(1)$ -gerbe is the bi-representation that permits to relate the 2-group $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})$ with $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$; let us see more closely its structure.

Proposition 7.1. *The representation $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$ of $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$ defines a left representation $(K, \hat{\beta})$ of the multiplicative Segal–Mitchison $U(1)$ -gerbe $(\hat{G}, \hat{\alpha})$.*

Proof. Recall from (6.1) and (5.1) that

$$\begin{aligned} \beta &\in C^2(K \rtimes G, \underline{U(1)}), \quad d_{K \rtimes G} \beta = \pi^* \alpha, \\ \bar{\alpha} &\in C^2(G \rtimes G, \underline{U(1)}), \quad d_{G \rtimes G} \bar{\alpha} = \bar{\pi}^* \alpha, \end{aligned}$$

for $\pi : K \rightarrow *$ and $\bar{\pi} : G \rightarrow *$, respectively. Denote by $p : G \rightarrow K$ the projection homomorphism $g \mapsto [g]$ and recall from (6.2) the notation $f_h \in \text{Map}_{G^{\text{op}}}(K, K)$, $f_h(k) = hk$ for $h, k \in K$. Then, we have the isomorphism

$$\text{Map}_{G^{\text{op}}}(G, K) \cong K \quad f_h \circ p \mapsto h.$$

From the proof of Theorem 6.1 and (6.1), we know that there exists γ such that $\beta - f_h^* \beta = (d_{K \rtimes G} \gamma)(h)$ for all $h \in K$, and therefore,

$$d_{G \rtimes G}(\bar{\alpha} - p^* f_h^* \beta) = 0$$

for all $h \in K$. Since the complex $C^*(G \rtimes G, \underline{U(1)})$ is acyclic, we see that the four-term sequence for the lifting bundle $U(1)$ -gerbe $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$ is simply

$$U(1) \rightarrow C^0(G \rtimes G, \underline{U(1)}) \cup \text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}^0((G, \bar{\alpha}), (K, \beta)) \rightarrow K.$$

The action of $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$ on $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$ induces a left action of \widehat{G} on K by left multiplication. Moreover, this action implies that the Segal–Mitchison $U(1)$ -gerbe associated with $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$ can be made into a left representation of the multiplicative Segal–Mitchison $U(1)$ -gerbe $\langle \widehat{G}, \widehat{\alpha} \rangle$. Therefore, there the must exist

$$\widehat{\beta} \in C^2(\widehat{G} \ltimes K, \underline{U(1)}), \quad \text{with } d_{\widehat{G} \ltimes K} \widehat{\beta} = \widehat{\pi}^* \widehat{\alpha},$$

where $\widehat{\pi} : \widehat{G} \rightarrow K, \widehat{g} \mapsto [\widehat{g}]$.

Hence, $(K, \widehat{\beta})$ is the desired left representation of $\langle \widehat{G}, \widehat{\alpha} \rangle$ induced by

$$\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta)). \quad \blacksquare$$

Now we will argue that the representation $(K, \widehat{\beta})$ satisfies the hypothesis of Theorem 6.1, thus implying that $[\widehat{\alpha}] \in \Omega(\widehat{G}, \widehat{S})$. For this to happen, we just need to verify that the projection map

$$\text{End}_{\langle \widehat{G}, \widehat{\alpha} \rangle}(K, \widehat{\beta}) \rightarrow \text{Map}_{\widehat{G}}(K, K)$$

is surjective (see (6.4) and the argument of Theorem 6.1). This surjectivity is equivalent to the surjectivity of the projection map

$$\text{End}_{\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)}(\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))) \rightarrow \text{End}_{\text{End}_G(K)}(\text{Hom}_G(G, K))$$

induced by projecting on the components of the endomorphisms given by the maps among spaces.

This surjectivity property is part of a deeper result that relates the multiplicative Segal–Mitchison $U(1)$ -gerbe $\langle G, \alpha \rangle$ with its double dual constructed from the representation (K, β) . The setup is as follows.

Since left and right compositions commute, the composition on the right by elements in $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})$ induce endomorphisms of $\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$ as left $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$ representations. Therefore, we obtain the right composition functor

$$\begin{aligned} \Psi : \text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha}) &\rightarrow \text{End}_{\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)}(\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta))) \\ \mathcal{F} &\mapsto \Psi(\mathcal{F}) \quad (\Psi(\mathcal{F}) : \mathcal{Q} \mapsto \mathcal{Q} \circ \mathcal{F}) \end{aligned} \tag{7.1}$$

with $\mathcal{T} \circ \Psi(\mathcal{F})(\mathcal{Q}) = \mathcal{T} \circ \mathcal{Q} \circ \mathcal{F} = \Psi(\mathcal{F})(\mathcal{T} \circ \mathcal{Q})$ for all $\mathcal{T} \in \text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)$. We claim the following result.

Theorem 7.2. *Let $\langle G, \alpha \rangle$ be a multiplicative Segal–Mitchison $U(1)$ -gerbe with $[\alpha] \in \Omega(G, S)$ and (K, β) a representation as in Theorem 6.1. Then, the canonical right composition functor*

$$\Psi : \text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha}) \rightarrow \text{End}_{\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(K, \beta)}(\text{Hom}_{\langle G, \alpha \rangle^{\text{op}}}((G, \bar{\alpha}), (K, \beta)))$$

from $\text{End}_{\langle G, \alpha \rangle^{\text{op}}}(G, \bar{\alpha})$ to its double dual is an equivalence of 2-groups.

Proof. The outline of the proof is as follows. First, we will show that the projection map

$$\text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(K,\beta)}(\text{Hom}_{(G,\alpha)^{\text{op}}}((G, \bar{\alpha}), (K, \beta))) \rightarrow \text{End}_{\text{End}_{G^{\text{op}}}(K)}(\text{Hom}_{G^{\text{op}}}(G, K)) \quad (7.2)$$

is surjective. The surjectivity implies that $[\hat{\alpha}] \in \Omega(\hat{G}, \hat{S})$, and by Theorem 6.1, we obtain that the isomorphism classes of objects

$$\hat{G} = \pi_0(\text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(K,\beta)}(\text{Hom}_{(G,\alpha)^{\text{op}}}((G, \bar{\alpha}), (K, \beta))))$$

fits into the short exact sequence of groups

$$\hat{S} \rightarrow \hat{G} \rightarrow K.$$

This implies also that the induced homomorphism of groups $\Psi : G \rightarrow \hat{G}$ factors through the projection homomorphisms to K :

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & \hat{G} \\ & \searrow p & \swarrow \hat{p} \\ & & K. \end{array}$$

Second, we will show that the homomorphism, once restricted to the subgroup $S \subset G$, gives the fiber \hat{S} via the Pontrjagin dual isomorphism $S \xrightarrow{\cong} \hat{S}$. The two facts together imply that Ψ induces an isomorphism of groups $G \cong \hat{G}$, and together with the fact that Ψ is a functor of group lifting bundle $U(1)$ -gerbes, we have that Ψ induces an equivalence of 2-groups.

Let us start with the surjectivity of the projection map of (7.2). Let

$$(F_g, \eta_g) \in \text{End}_{(G,\alpha)^{\text{op}}}^0(G, \bar{\alpha})$$

with $F_g : G \rightarrow G$, $F_g(g') = gg'$ for all $g, g' \in G$, and $\bar{\alpha} - F_g^* \bar{\alpha} = d_{G \rtimes G} \eta_g$. Every morphism in $\text{Hom}_{(G,\alpha)^{\text{op}}}^0((G, \bar{\alpha}), (K, \beta))$ is of the form $(f_h \circ p, \varepsilon)$ with $f_h : K \rightarrow K$, $f_h(k) = hk$ for all $h, k \in K$, $p : G \rightarrow K$ the projection homomorphism and $\bar{\alpha} - p^* f_h^* \beta = d_{G \rtimes G} \varepsilon$.

The right composition functor Ψ of (7.1) applied to (F_g, η_g) acts on $(f_h \circ p, \varepsilon)$ by right composition as follows:

$$(\Psi(F_g, \eta_g))(f_h \circ p, \varepsilon) = (f_h \circ p \circ F_g, \eta_g + F_g^* \varepsilon).$$

The composition map $f_h \circ p \circ F_g$ is equal to the map $f_{hp(g)} \circ p$. Therefore, the projection map of (7.2), applied to $\Psi(F_g, \eta_g)$, is equivalent to the assignment $f_h \circ p \mapsto f_{hp(g)} \circ p$ for all $f_h \circ p \in \text{Hom}_{G^{\text{op}}}(G, K)$ with $h \in K$. Since we know that

$$K \cong \text{End}_{\text{End}_{G^{\text{op}}}(K)}(\text{Hom}_{G^{\text{op}}}(G, K)), \quad l \mapsto (f_h \circ p \mapsto f_{hl} \circ p),$$

we see that the projection map of (7.2) is surjective.

Now let us understand the image under Ψ of the endomorphism functors $(F_s, \eta_s) \in \text{End}_{(G, \alpha)^{\text{op}}}^0(G, \bar{\alpha})$ whenever $s \in S \subset G$. Note that we can take a continuous homomorphism $S \rightarrow \text{End}_{(G, \alpha)^{\text{op}}}^0(G, \bar{\alpha})$, $s \mapsto (F_s, \eta_s)$ since we know that $[\alpha|_S] = 0$. Hence, S acts on the short exact sequence

$$U(1) \rightarrow C^0(G \rtimes G, \underline{U(1)}) \rightarrow Z^1(G \rtimes G, \underline{U(1)}) \tag{7.3}$$

via the map F_s^* , and the relevant part of $\text{End}_{(G, \alpha)^{\text{op}}}(K, \beta)$ acts via the pullback of the complex

$$U(1) \rightarrow C^0(K \rtimes G, \underline{U(1)}) \rightarrow Z^1(K \rtimes G, \underline{U(1)}); \tag{7.4}$$

both actions commute. The maps F_s^* act on the complex $C^*(G \rtimes G, U(1))$ via the pullback of the left translation action $F_s : G \rtimes G \rightarrow G \rtimes G$, $F_s(g, h) = (sg, \bar{h})$ for all $g, h \in G$. The map F_s^* provides an external automorphism of the short exact sequence of (7.3) which commutes with the action of the complex in (7.4). Therefore, the endomorphism functors $\Psi(F_s, \eta_s)$, once projected to their isomorphism classes, generate the group $S \cong \widehat{S} \subset \widehat{G}$. Hence, we have that the functor Ψ induces an equivalence of groups in lifting bundle $U(1)$ -gerbes as desired. ■

7.1. Morita equivalence

Let us now restrict our representation category $\text{Rep}_{(G, \alpha)}$ to objects (V, χ) such that

$$H^0(V \rtimes G, \underline{U(1)}) = U(1);$$

these are G -spaces V on which the action of G is transitive. Denote this subcategory by $\text{Rep}_{(G, \alpha)}^+$. We carry out these restrictions to ensure that the category of morphisms

$$\text{Hom}_{(G, \alpha)^{\text{op}}}((V, \chi), (K, \beta))$$

is a lifting bundle $U(1)$ -gerbe for all $(V, \chi) \in \text{Rep}_{(G, \alpha)}^+$.

Denote by

$$\begin{aligned} \mathcal{C} &:= \text{End}_{(G, \alpha)^{\text{op}}}(G, \bar{\alpha}), \\ \mathcal{D} &:= \text{End}_{(G, \alpha)^{\text{op}}}(K, \beta), \\ \mathcal{F} &:= \text{Hom}_{(G, \alpha)^{\text{op}}}((G, \bar{\alpha})(K, \beta)); \end{aligned}$$

note that Theorem 7.2 claims that

$$\Psi : \mathcal{C} \rightarrow \text{End}_{\mathcal{D}}(\mathcal{F})$$

is an equivalence of 2-groups, and denote also by Rep^+ the restriction of the categories of representations of the endomorphism groups in lifting bundle gerbes where the induced action of the groups G and \widehat{G} on the underlying spaces of the lifting bundle gerbes is transitive.

Consider the covariant functor

$$\text{Rep}_{\langle G, \alpha \rangle}^+ \rightarrow \text{Rep}_{\mathcal{C}^{\text{op}}}^+, \quad (V, \chi) \mapsto \text{Hom}_{\langle G, \alpha \rangle}((G, \bar{\alpha})(V, \chi))$$

and the contravariant functors

$$\begin{aligned} \text{Rep}_{\langle G, \alpha \rangle}^+ &\rightarrow \text{Rep}_{\mathcal{D}}^+ & \text{Rep}_{\mathcal{D}}^+ &\rightarrow \text{Rep}_{\text{End}_{\mathcal{D}}(\mathcal{F})}^+ \\ (V, \chi) &\mapsto \text{Hom}_{\langle G, \alpha \rangle}((V, \chi)(K, \beta)) & \mathcal{M} &\mapsto \text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{F}), \end{aligned} \tag{7.5}$$

where $\text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$ denotes strict morphisms of lifting bundle gerbes. The covariant functor compared with the composition of the contravariant functors gives us the following assignments:

$$\begin{aligned} \text{Rep}_{\langle G, \alpha \rangle}^+ &\rightarrow \text{Rep}_{\mathcal{C}^{\text{op}}}^+ & \text{Rep}_{\langle G, \alpha \rangle}^+ &\rightarrow \text{Rep}_{\text{End}_{\mathcal{D}}(\mathcal{F})}^+ \\ (G, \bar{\alpha}) &\mapsto \mathcal{C} & (G, \bar{\alpha}) &\mapsto \text{End}_{\mathcal{D}}(\mathcal{F}) \\ (K, \beta) &\mapsto \mathcal{F} & (K, \beta) &\mapsto \text{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{F}). \end{aligned}$$

From the previous assignments, we see that the composition of the contravariant functors provides, and equivalence of 2-categories, and therefore, we obtain as byproduct the desired Morita equivalence.

We have seen above in Theorem 6.2 that the group in lifting bundle $U(1)$ -gerbes $\text{End}_{\langle G, \alpha \rangle}(K, \beta)$ induces a multiplicative Segal–Mitchison $U(1)$ -gerbes $\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle$. Take the category of its left representations $\text{Rep}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}$ and restrict it to objects (W, η) such that

$$H^0(\hat{G} \ltimes W, \underline{U(1)}) = U(1).$$

Denote this subcategory by $\text{Rep}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}^+$.

Theorem 7.3. *Consider $\langle G, \alpha \rangle$ and $\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle$ a pair of Pontrjagin dual multiplicative Segal–Mitchison $U(1)$ -gerbes over the topological groups G and \hat{G} . Then, the 2-category $\text{Rep}_{\langle G, \alpha \rangle}^+$ of right representations of $\langle G, \alpha \rangle$ is equivalent to the 2-category $\text{Rep}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}^+$ of left representations of $\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle$.*

Proof. Our proof is based on the principles underlying a Morita context [28]. Nevertheless, we need to add extra equivalences of categories for the setup to work. The relevant bi-representations are

$$\text{Hom}_{\langle G, \alpha \rangle}((G, \bar{\alpha}), (K, \beta)) \quad \text{and} \quad \text{Hom}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}((\hat{G}, \bar{\alpha}_{(K, \beta)}), (K, \hat{\beta})),$$

where the former is an $\text{End}_{\langle G, \alpha \rangle}(K, \beta) - \text{End}_{\langle G, \alpha \rangle}(G, \bar{\alpha})$ representation and the former is an $\text{End}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}(K, \hat{\beta}) - \text{End}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}(\hat{G}, \bar{\alpha}_{(K, \beta)})$ representation. The necessary equivalences of 2-groups are

$$\text{End}_{\langle G, \alpha \rangle}(K, \beta) \simeq \text{End}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}(\hat{G}, \bar{\alpha}_{(K, \beta)}), \tag{7.6}$$

$$\text{End}_{\langle G, \alpha \rangle}(G, \bar{\alpha}) \simeq \text{End}_{\langle \hat{G}, \hat{\alpha}_{(K, \beta)} \rangle}(K, \hat{\beta}), \tag{7.7}$$

where the first equivalence of (7.6) follows from Theorem 6.2 and the second equivalence of (7.7) follows from Theorem 7.2.

The functors defined in (7.5) together with the equivalence of Theorem 7.2 provide the equivalence. ■

7.2. Equivalence of centers

The endomorphism categories of (7.6) and (7.7) are strict tensor categories and the equivalence of 2-groups of Theorem 7.2 implies that their centers are all tensor equivalent. Let us elaborate.

Let \mathcal{C} be a monoidal category and let its center be

$$\mathcal{Z}(\mathcal{C}) := \text{End}_{\mathcal{C}-\mathcal{C}}(\mathcal{C}),$$

the endomorphisms of \mathcal{C} which are compatible with the left and right monoidal structures.

Theorem 7.4. *The centers of $\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})$ and $\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)$ are equivalent:*

$$\mathcal{Z}(\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})) \simeq \mathcal{Z}(\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)).$$

Proof. Consider the following equivalences:

$$\begin{array}{c} \mathcal{Z}(\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})) \\ \simeq \downarrow \\ \text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta) - \text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})}(\text{Hom}_{(G,\alpha)^{\text{op}}}((G, \bar{\alpha}), (K, \beta))) \\ \simeq \uparrow \\ \mathcal{Z}(\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)). \end{array}$$

The upper one is an equivalence induced by the monoidal equivalence

$$\begin{array}{c} \text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})}(\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})) \\ \simeq \downarrow \Psi \\ \text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)}(\text{Hom}_{(G,\alpha)^{\text{op}}}((G, \bar{\alpha}), (K, \beta))) \end{array}$$

defined by Ψ of Theorem 7.2. The bottom is an equivalence which is induced by the following natural equivalences:

$$\begin{array}{c} \text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})}(\text{Hom}_{(G,\alpha)^{\text{op}}}((G, \bar{\alpha}), (K, \beta))) \\ \simeq \uparrow \\ \text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)}(\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)) \\ \simeq \uparrow \\ \text{End}_{(G,\alpha)^{\text{op}}}(K, \beta). \end{array}$$

■

From the results presented above we see that the Pontrjagin dual multiplicative Segal–Mitchison gerbes will have equivalent centers. This result will imply interesting consequences on the topological invariants associated with Pontrjagin dual multiplicative Segal–Mitchison gerbes. We postpone its study to a forthcoming publication.

A. Crossed modules and 2-groups

Here we summarize the equivalent concepts of crossed module and 2-group, together with their morphisms and classification.

A.1. Crossed modules

A topological crossed module $\phi : N \rightarrow E$ consists of topological groups N and E , of a continuous homomorphism ϕ , and of a right action of E on N by group homomorphisms, denoted by n^e for $n \in N$ and $e \in E$, such that ϕ is E -equivariant

$$\phi(n^e) = e^{-1}\phi(n)e$$

and ϕ satisfies the Peiffer identity

$$n^{\phi(m)} = m^{-1}nm.$$

The kernel of ϕ is always abelian and the image of ϕ is normal in E . Therefore, the crossed module induces an exact sequence

$$A \rightarrow N \rightarrow E \rightarrow G$$

where $A = \text{Ker}(\phi)$, $G = \text{Coker}(\phi)$, and there is an induced action of G on A . The groups A and G are sometimes denoted by $\pi_1(N \xrightarrow{\phi} E)$ and $\pi_0(N \xrightarrow{\phi} E)$, respectively.

Whenever the group N is abelian, the Pfeiffer identity implies that the subgroup $\phi(N)$ of E acts trivially on N . Therefore, a crossed module $\phi : N \rightarrow E$ with N abelian is nothing else but a homomorphism of groups ϕ and an action of G on N such that ϕ is E -equivariant. We will denote the crossed module by the homomorphism $\phi : N \rightarrow E$.

A.2. 2-groups

A topological 2-group is a group object in the category of topological groupoids, or equivalently a groupoid object in topological groups. Let \mathcal{G} be the groupoid in topological groups and denote by \mathcal{G}_1 the group of morphisms, \mathcal{G}_0 the group of objects, $s, t : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ the source and target homomorphisms, and $e : \mathcal{G}_0 \rightarrow \mathcal{G}_1$ the identity homomorphism. To a 2-group, one may associate the groups $\pi_0(\mathcal{G})$ and $\pi_1(\mathcal{G})$, the first being the isomorphism classes of objects of \mathcal{G} , which becomes a group by the group operation of \mathcal{G} , and the second the group of automorphism of the identity object $1_{\mathcal{G}_0} \in \mathcal{G}_0$. The group $\pi_1(\mathcal{G})$ is

abelian and is endowed with a right action of $\pi_0(\mathcal{G})$. Strict morphisms of 2-groups are continuous functors that are group homomorphisms on the level of objects and of morphisms. A strict morphism of 2-groups is an equivalence if it induces isomorphisms at the level of π_1 and π_0 [2, Thm. 8.3.7 and Cor. 8.3.8].

A.3. Equivalence of 2-groups and crossed modules

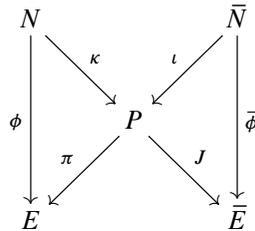
2-groups and crossed modules are equivalent concepts. If we have a 2-group \mathcal{G} , let $N = \text{Ker}(s) \subset \mathcal{G}_1$, $E = \mathcal{G}_0$, $\phi = t|_N$, and the right action of E on N is obtained by the equation $n^e := e(e)^{-1}ne(e)$, then $t : N \rightarrow E$ is a crossed module. If we have a crossed module $\phi : N \rightarrow E$, define the action groupoid $[E/N]$ whose group of morphisms is $\mathcal{G}_1 := E \times N$, whose group of objects is $\mathcal{G}_0 = E$, and the structural maps are $s(e, n) = e$, $t(e, n) = e\phi(n)$, and $e(e) = (e, 1_N)$. The product structure on $E \times N$ is given by the semidirect product, $(e, n) \cdot (e', n') = (ee', n^e n')$.

Two extremal cases are worth highlighting, i.e., whenever one of the two groups N or E on the crossed module $N \rightarrow E$ is trivial. For a group G , the crossed module $\{0\} \rightarrow G$ induces the 2-group $G \rightrightarrows G$ where the source and the target maps are the identity; this 2-group is simply denoted by G , sometimes by $G[0]$. For an abelian group A , the crossed module $A \rightarrow \{0\}$ induces the 2-group $A \rightrightarrows \{0\}$, which is sometimes denoted by $A[1]$. A 2-group of the form $A[1] \times G$ defines the crossed module $A \rightarrow G$ with trivial homomorphism and trivial action of G on A .

A.4. Morphisms of crossed modules

A strict morphism of crossed modules consists of group homomorphisms $\psi_2 : N \rightarrow \bar{N}$ and $\psi_1 : E \rightarrow \bar{E}$ commuting with ϕ and $\bar{\phi}$, which respect the actions. A strict morphism is an equivalence of crossed modules if the induced homomorphisms on π_2 and π_1 are isomorphisms [2, Thm. 8.3.7 and Cor. 8.3.8].

Non-strict morphisms between crossed modules were coined *butterflies* by Noohi [1, Def. 4.1.3], and they are the following. Take two crossed modules $\phi : N \rightarrow E$ and $\bar{\phi} : \bar{N} \rightarrow \bar{E}$; then, a butterfly from the first to the second one is defined as a diagram



of topological groups and group homomorphisms such that

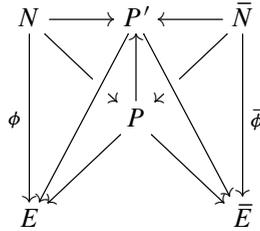
$$\bar{N} \xrightarrow{\iota} P \xrightarrow{\pi} E$$

is a group extension, and $N \xrightarrow{\kappa} P \xrightarrow{J} \bar{E}$ is a complex. The maps must satisfy the equivariance conditions

$$\iota(\bar{n}^{J(p)}) = p^{-1}\iota(\bar{n})p, \quad \kappa(n^{\pi(p)}) = p^{-1}\kappa(n)p,$$

where $\bar{n} \in \bar{N}$, $p \in P$, and $n \in N$.

A morphism of butterflies P and P' (a 2-morphisms) consists of a group homomorphism $\psi : P \rightarrow P'$ such that the diagram



commutes and is compatible with all structural maps.

A strict morphism between the crossed modules, consisting of homomorphisms $f_1 : N \rightarrow \bar{N}$ and $f_0 : E \rightarrow \bar{E}$ commuting with ϕ and ϕ' and preserving the actions, does define a butterfly in which $P = E \rtimes \bar{N}$, $\pi(e, \bar{n}) = e$, $J(e, \bar{n}) = f_0(e)\bar{\phi}(\bar{n})$, $\iota(\bar{n}) = (1, \bar{n})$ and $\kappa(n) = (\phi(n), f_1(n^{-1}))$. The product in $E \rtimes \bar{N}$ is

$$(e_1, \bar{n}_1) \cdot (e_2, \bar{n}_2) = (e_1 e_2, \bar{n}_1^{f_0(e_2)} \bar{n}_2).$$

A.5. Classification of crossed modules

Let us consider the category of crossed modules whose underlying groups are locally contractible, compactly generated, and paracompact. Then, the isomorphism classes of crossed modules in this setup are classified by a group G , an abelian group A , which is moreover a G -module, and a Segal–Mitchison cohomology class in $H^3(G_\bullet, \underline{A})$. This result follows from [3, Prop. 2.3.4], cf. [27, 29], when considering the isomorphism classes of crosses module extensions of the crossed module $G[0]$ by the crossed module $A[1]$.

Whenever the action of G on A is trivial, we may say that the crossed module defines a multiplicative Segal–Mitchison gerbe with structure group A in the sense of this article.

From the short exact sequence $A \rightarrow E_{(G)}A \rightarrow B_{(G)}A$ defined in (2.1), we obtain the isomorphism

$$H^3(G_\bullet, \underline{A}) \cong H^2(G_\bullet, \underline{B_{(G)}A})$$

since $E_{(G)}A = \text{Map}(G, A)$ is G -soft. A class in $H^3(G_\bullet, \underline{A})$ defines a short exact sequence of groups

$$B_{(G)}A \rightarrow W \rightarrow G,$$

which induces a crossed module whose four-term exact sequence is

$$A \rightarrow E_{(G)}A \rightarrow W \rightarrow G.$$

This procedure allow us to construct a crossed module extension of $G[0]$ by $A[1]$ for any element in $H^3(G_\bullet, A)$ in a canonical way.

B. Lifting bundle gerbes

The bundle gerbes that appear in this work are of specific type and deserve some explanation. They were denoted *lifting bundle gerbes* by Murray [20], and they are constructed using principal bundles. For a comprehensive study of gerbes and their properties, we refer to [22] and the references therein.

Consider the category \mathfrak{Topab} of compactly generated and locally contractible Hausdorff topological abelian groups and continuous homomorphisms. Let us fix a topological group A in \mathfrak{Topab} which will be the structure group of the gerbes [4].

B.1. Objects

A *lifting bundle A -gerbe* is a pair (N, E) consisting of a central extension $A \rightarrow N \rightarrow N/A$ and of a principal N/A -bundle E . A lifting bundle A -gerbe can be understood as a representative for the problem to lift the structure group of the principal bundle E from N/A to N . In this article, another perspective will be useful.

Associated with any lifting bundle A -gerbe is a topological action groupoid $[E/N]$, where N acts on E through the projection $N \rightarrow N/A$. We will use the notation $[E/N]$ for lifting bundle A -gerbes below. Note that the isotropy group of each object is A . The base space of a lifting bundle A -gerbe $[E/N]$ is the base space E/N of the principal bundle E ; it is the space of isomorphism classes of objects of the topological groupoid $[E/N]$. We will denote the lifting bundle A -gerbes also by the four-term sequence

$$A \rightarrow N \cup E \rightarrow E/N.$$

Given the abelian group A , there are two canonical examples:

- (1) The *canonical lifting A -bundle gerbe* is $[EBA/EA]$; here, the central extension is $A \rightarrow EA \rightarrow BA$, and the principal BA -bundle is $EBA \rightarrow B^2A$. As a four-term sequence, it looks as follows:

$$A \rightarrow EA \cup EBA \rightarrow B^2A.$$

- (2) The *trivial lifting A -gerbe* is $[*/A]$, i.e., $N = A$ and $E = *$.

The lifting bundle gerbes appearing in this work are ones with N abelian, which we then call abelian lifting A -gerbes. In most of the cases, it is the degree 0 group C^0 of a complex, and the group A is the zeroth cohomology group $H^0 \subset C^0$. The space E consists of pairs of points on a fixed space together with elements in the degree 1 group C^1 of the complex. The action of N on E is simply obtained by adding the image of the differential $d : C^0 \rightarrow C^1$.

B.2. Morphisms

Take two lifting bundle A -gerbes: $[E/N]$ and $[D/M]$. A strict morphism is a continuous functor $\psi : [E/N] \rightarrow [D/M]$ that is A -equivariant at the level of morphisms.

More explicitly, the functor ψ is at the level of morphisms given by a continuous map $\psi_1 : E \times N \rightarrow D \times M$, and A acts on both sides by multiplication on N and M , respectively. The requirement is that ψ_1 is equivariant for these actions. Denoting for one moment the 1-category of lifting A -gerbes by $A\text{-}\mathcal{Lft}\mathcal{G}\text{rb}\mathfrak{s}$, we have then

$$\begin{aligned} &\text{Hom}_{A\text{-}\mathcal{Lft}\mathcal{G}\text{rb}\mathfrak{s}}([E/N], [D/M]) \\ &= \{ \psi \in \text{Hom}_{\mathcal{T}\text{op}\mathcal{G}\text{p}\mathfrak{b}\mathfrak{s}}([E/N], [D/M]) \mid \psi_1 \text{ is } A\text{-equivariant} \}. \end{aligned}$$

It is important to notice that not all continuous functors are morphisms of lifting A -gerbes. The most simple case on which this could be seen is the trivial gerbe $[*/A]$. On the one hand, $\text{Hom}_{\mathcal{T}\text{op}\mathcal{G}\text{p}\mathfrak{b}\mathfrak{s}}([*/A], [*/A]) = \text{Hom}(A, A)$ consists of all endomorphism of groups of A . On the other hand, $\text{Hom}_{A\text{-}\mathcal{Lft}\mathcal{G}\text{rb}\mathfrak{s}}([*/A], [*/A])$ consists of only the identity map. Notice also that the natural transformations from the identity functor on $[A/*]$ to itself are parameterized by elements in A .

A special example of strict morphisms comes from principal bundle morphisms. Let E be a principal N/A -bundle, and let D be a principal M/A -bundle, let $\phi : N \rightarrow M$ be an A -equivariant group homomorphism, and let $\psi_0 : E \rightarrow D$ be a ϕ -equivariant map, i.e., $\psi_0(e[n]) = \psi_0(e)[\phi(n)]$. Then, ψ_0 and $\psi_1 := \psi_0 \times \phi$ form a strict morphism $\psi : [E/N] \rightarrow [D/M]$. Note that this works, in particular, when $M = N$ and $\phi = \text{id}_N$.

B.3. 2-Morphisms

Continuous natural transformations provide the 2-morphisms that turn the 1-category of lifting bundle A -gerbes into a 2-category, which from now on will be denoted by $A\text{-}\mathcal{Lft}\mathcal{G}\text{rb}\mathfrak{s}$.

B.4. Monoidal structure

The 2-category $A\text{-}\mathcal{Lft}\mathcal{G}\text{rb}\mathfrak{s}$ of lifting bundle A -gerbes can be moreover be endowed with a monoidal structure. Take two lifting bundle A -gerbes $[E/N]$ and $[D/M]$ and define their product

$$[E/N] \otimes [D/M] := [(E \times D)/(N \otimes M)]$$

where the group $N \otimes M$ is the quotient of $N \times M$ by its diagonal subgroup $\Delta := \{(a, a^{-1}) \mid a \in A\}$. The trivial lifting A -gerbe $[*/A]$ is neutral with respect to this tensor product.

We remark that the underlying groupoid of $[E/N] \otimes [D/M]$ is not the direct product of the groupoids $[E/N]$ and $[D/M]$, but that there is a continuous functor

$$[E/N] \times [D/M] \rightarrow [E/N] \otimes [D/M]$$

that is the identity on the level of objects. A continuous functor $\psi : [E/N] \times [E'/N'] \rightarrow [D/M]$ induces a strict morphism $[E/N] \otimes [E'/N'] \rightarrow [D/M]$ of lifting bundle A -gerbes if and only if its restriction to isotropy groups is the product of the group A . In terms of the trivial gerbe $[*/A]$, the only functor $[*/A] \times [*/A] \rightarrow [*/A]$ that induces a morphism of A -gerbes $[*/A] \otimes [*/A] \rightarrow [*/A]$ is the one induced by the product structure on A .

B.5. Strict group objects

A *strictly multiplicative lifting bundle A -gerbe* is a strict group object in lifting bundle A -gerbes [2]. Thus, it is a lifting bundle A -gerbe $[E/N]$ together with strict morphisms $[E/N] \otimes [E/N] \rightarrow [E/N]$ (multiplication), $[E/N] \rightarrow [E/N]$ (inversion), and $[*/A] \rightarrow [E/N]$ (unit), satisfying strictly the axioms of a group. Note that a strictly multiplicative lifting bundle A -gerbe is in particular a group in the category of topological groupoids, i.e., a strict topological 2-group. In fact, these two conditions are equivalent.

Lemma B.1. *A lifting bundle A -gerbe is strictly multiplicative if and only if the underlying groupoid is a strict topological 2-group.*

Proof. The reason for this phenomenon is the fact that for a groupoid $[*/A]$ to become a 2-group it is necessary that A is abelian and that the product structure $[*/A] \times [*/A] \rightarrow [*/A]$ coincides with the product structure of A . Hence, the structural properties of groups in groupoids impose the conditions necessary for the product to induce one on lifting bundle A -gerbes. ■

We remark that for a strictly multiplicative lifting bundle A -gerbe, the spaces E and E/N are in fact topological groups, and that there is a group homomorphism $N \rightarrow E$. Hence, we obtain an exact four-term sequence

$$A \rightarrow N \rightarrow E \rightarrow E/N.$$

Under the correspondence between strict 2-groups and crossed modules, the above four-term sequence is precisely the one associated with a crossed module; see Section A.1.

The trivial lifting bundle A -gerbe $[*/A]$ is strictly multiplicative in a unique way, and the canonical lifting bundle A -gerbe $[EBA/EA]$ is strictly multiplicative for the given group structure on EBA . Also, the 2-groups of endomorphisms $\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})$, $\text{End}_{(G,\alpha)}(K, \beta)$ and

$$\text{End}_{\text{End}_{(G,\alpha)^{\text{op}}}(K,\beta)}(\text{Hom}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha}), (K, \beta))$$

defined in Example 5.3, Theorem 6.1, and Theorem 7.2, respectively, are strictly multiplicative lifting bundle $U(1)$ -gerbes.

B.6. Representations of strictly multiplicative lifting bundle gerbes

Strict group objects may act on objects. In our case, lifting bundle A -gerbes representations of group objects in lifting bundle A -gerbes form a 2-category. This 2-category is

constructed in the usual way [2]. Objects are lifting bundle A -gerbes representations of the specific group object in lifting bundle A -gerbes, morphisms are morphisms of lifting bundle A -gerbes compatible with the action, and 2-morphisms are the natural transformations.

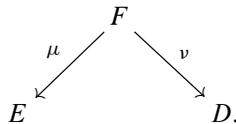
An example of a representation of a group in lifting bundle $U(1)$ -gerbes is the lifting bundle $U(1)$ -gerbe $\text{Hom}_{(G,\alpha)^{\text{op}}}((G, \bar{\alpha}), (K, \beta))$ of Proposition 7.1. It is a right representation of the group lifting bundle $U(1)$ -gerbe $\text{End}_{(G,\alpha)^{\text{op}}}(G, \bar{\alpha})$ and a left representation of the group lifting bundle $U(1)$ -gerbe $\text{End}_{(G,\alpha)^{\text{op}}}(K, \beta)$.

B.7. Weak morphisms

Strict morphisms as defined above are not enough to capture the essence of the bicategory of lifting bundle A -gerbes (though, for the purpose of this article, they suffice to discuss group objects). There is a well-established notion of weak morphisms between topological groupoids that correctly reflects the equivalence between topological stacks, known as anafunctors, bibundles, or Hilsum–Skandalis morphisms; see, e.g., [10, 15, 26]. An A -equivariant (smooth) version has been introduced in [22, App. A]. Adapted to our setting, we obtain the following. A *weak morphism* between lifting bundle A -gerbes $[E/N]$ and $[D/M]$ has the following structure:

- a space F together with maps $\mu : F \rightarrow E$ and $\nu : F \rightarrow D$,
- an action of N on F making μ N -equivariant,
- an action of M on F making ν M -equivariant and μ a principal M -bundle,

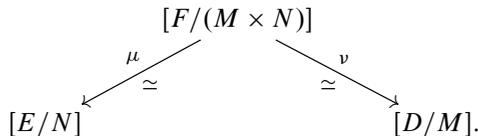
such that the actions of $A \subset N$ and $A \subset M$ agree on F . Weak morphisms can be pictured by the following diagram:



Whenever the map $\mu : F \rightarrow E$ is also an N -principal bundle, the weak morphism becomes a *weak isomorphism*, whose inverse is obtained by flipping the sides. In this case, we obtain homeomorphisms

$$F/D \cong E, \quad F/M \cong D, \quad E/N \cong D/M,$$

and the following diagram induce equivalence of topological groupoids



We will see now that a strict morphism induces a weak one. Indeed, suppose that

$$\Psi : [E/N] \rightarrow [D/M]$$

is a strict morphism. We obtain a map $\psi : E/N \rightarrow D/M$ of the underlying spaces and we may consider the space $\Delta^*(E \times \psi^*D)$ where $\Delta : E/N \rightarrow E/N \times E/N$ is the diagonal map. There is a canonical action of $N \times M$ on $\Delta^*(E \times \psi^*D)$ and canonical projection maps

$$\Delta^*(E \times \psi^*D) \rightarrow E \quad \text{and} \quad \Delta^*(E \times \psi^*D) \rightarrow D,$$

where the first one is an M -principal bundle. The space $\Delta^*(E \times \psi^*D)$ together with the action and the maps is the weak morphism induced by Ψ .

Weak morphisms are related through weak 2-morphisms, which are maps $F \rightarrow F'$ that exchange the maps to E and D , and are equivariant with respect to the actions of M and N . We refer the reader to the references above for a detailed study of the 2-categorical structure. Now we are ready to show the following result.

Theorem B.2. *Let us consider the 2-category of Segal–Mitchison gerbes with structure group A and the 2-category lifting bundle A -gerbes with weak morphisms and weak 2-morphisms. Then, the functor*

$$\begin{aligned} A\text{-}\mathcal{M}\text{-}\mathcal{G}\text{rbs} &\rightarrow A\text{-}\mathcal{L}\text{ft}\mathcal{G}\text{rbs} \\ (M, \alpha) &\mapsto [(\alpha^*EBA)/EA] \end{aligned}$$

is an equivalence of 2-categories.

Proof. The functor is defined in Section 3.1 at the level of objects, in Section 3.2 at the level of morphisms, and in Section 3.3 at the level of 2-morphisms.

Let us now show that the functor is essentially surjective. That is, for a lifting bundle A -gerbe $[E/N]$, we need to find a Segal–Mitchison A -gerbe $\alpha : E/N \rightarrow B^2A$ and a weak isomorphism between $[E/N]$ and $[\alpha^*EBA/EA]$. For this, we need some preparation.

Since the principal bundle

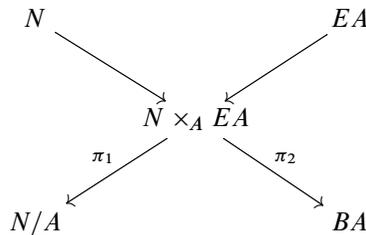
$$N/A \rightarrow E \rightarrow E/N$$

is obtained by pullback from the universal one $N/A \rightarrow E(N/A) \rightarrow B(N/A)$, we will focus our attention on finding a canonical Segal–Mitchison A -gerbe

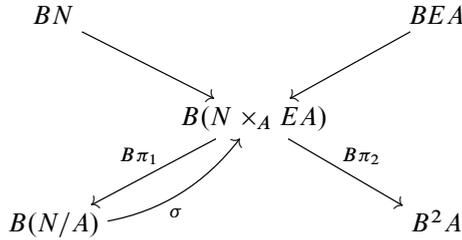
$$\alpha : B(N/A) \rightarrow B^2A$$

and a weak isomorphism from $[E(N/A)/N]$ to $[\alpha^*EBA/EA]$.

Consider the diagonal action of A on $N \times EA$ and denote by $N \times_A EA$ the quotient group. Consider the diagram of topological groups:



where the diagonals are central extensions of groups, and π_1 and π_2 are the homomorphisms induced by projections on the first and second coordinates, respectively. Applying the classifying space functor B , we obtain the diagram

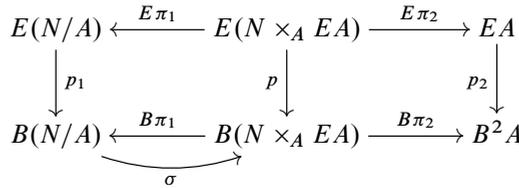


and note that there exists a section $\sigma : B(N/A) \rightarrow B(N \times_A EA)$ of the fiber map $B\pi_1$ since the fiber BEA is contractible. Take

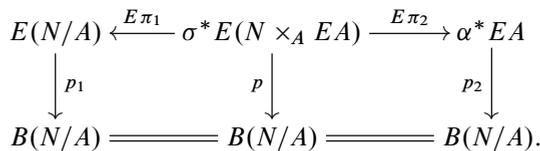
$$\alpha := B\pi_2 \circ \sigma : B(N/A) \rightarrow B^2A,$$

and let us show that the lifting bundle A -gerbes $[E(N/A)/N]$ and $[\alpha^*EBA/EA]$ are weakly isomorphic.

The diagram of principal bundles



induces the diagram of principal bundles over the same base $B(N/A)$:



The space $\sigma^* E(N \times_A EA)$ has free commuting actions of both N and EA and their actions agree on A . The bundle $E\pi_2 : \sigma^* E(N \times_A EA) \rightarrow \alpha^* EA$ is an N -principal bundle and the bundle $E\pi_1 : \sigma^* E(N \times_A EA) \rightarrow E(N/A)$ is an EA -principal bundle. Hence, we have a weak isomorphism.

Note that whenever $N = EA$, we may take $\sigma : B^2A \rightarrow B(EA \times_A EA)$ as $\sigma := B\Delta$ where $\Delta : BA \rightarrow EA \times_A EA$, $\Delta[\lambda] = [(\lambda, \lambda)]$. The composition becomes

$$\alpha = B\pi_2 \circ B\Delta = B(\pi_2 \circ \Delta) = B(id) = id$$

and therefore the map $\alpha : B^2A \rightarrow B^2A$ is the identity.

Let us now show that the functor is fully faithful. Consider two Segal–Mitchison gerbes (M, α) and (N, β) , and let F be a weak morphism between $[\alpha^*EBA/EA]$ and $[\beta^*EBA/EA]$. We have then maps

$$\begin{array}{ccc}
 & F & \\
 \mu \swarrow & & \searrow \nu \\
 \alpha^*EBA & & \beta^*EBA
 \end{array}$$

where F is endowed with an action of $EA \times EA$ and μ is a principal EA -bundle. The maps μ and ν induce maps at the level of quotient spaces

$$\begin{array}{ccc}
 & F/(EA \times EA) & \\
 \tilde{\mu} \swarrow & & \searrow \tilde{\nu} \\
 M = \alpha^*EBA/EA & & \beta^*EBA/EA = N
 \end{array}$$

\cong

where $\tilde{\mu}$ is a homeomorphism and $\tilde{\mu}^{-1}$ is its inverse. It is now straightforward to check that the maps α and $\beta \circ \tilde{\nu} \circ \tilde{\mu}^{-1}$ are homotopic, where $\tilde{\nu} \circ \tilde{\mu}^{-1} : M \rightarrow N$. This implies that the functor

$$\text{Hom}_{A\text{-}\mathfrak{Gm}\text{-}\mathfrak{Grb}_{\mathfrak{S}}}((M, \alpha), (N, \beta)) \rightarrow \text{Hom}_{A\text{-}\mathfrak{Lft}\mathfrak{Grb}_{\mathfrak{S}}}^{\text{weak}}([\alpha^*EBA/EA], [\beta^*EBA/EA])$$

is essentially surjective. In order to see that it is fully faithful, we recall that morphisms of Segal–Mitchison gerbes are mapped to *strict* morphisms of lifting bundle A -gerbes. It is well known from above references about weak morphisms that the functor

$$\text{Hom}_{A\text{-}\mathfrak{Lft}\mathfrak{Grb}_{\mathfrak{S}}}^{\text{strict}}([E/N], [D/M]) \rightarrow \text{Hom}_{A\text{-}\mathfrak{Lft}\mathfrak{Grb}_{\mathfrak{S}}}^{\text{weak}}([E/N], [D/M])$$

is fully faithful. Thus, it suffices to show that the 2-morphisms between Segal–Mitchison gerbes correspond one-to-one with 2-morphisms between strict morphisms of the corresponding lifting gerbes. This follows directly from the definitions. ■

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