

Ideal structure and pure infiniteness of inverse semigroup crossed products

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Abstract. We give efficient conditions under which a C^* -subalgebra $A \subseteq B$ separates ideals in a C^* -algebra B , and B is purely infinite if every positive element in A is properly infinite in B . We specialise to the case when B is a crossed product for an inverse semigroup action by Hilbert bimodules or a section C^* -algebra of a Fell bundle over an étale, possibly non-Hausdorff, groupoid. Then our theory works provided B is the recently introduced essential crossed product and the action is essentially exact and residually aperiodic or residually topologically free. These last notions are developed in the article.

1. Introduction

Many authors have given sufficient criteria for crossed products by discrete group actions or for C^* -algebras associated to étale locally compact groupoids to be purely infinite (see, for instance, [2, 6, 19, 22, 34–36, 38, 41, 43]). These articles mostly deal with the case when the bigger C^* -algebra B is simple or when the C^* -subalgebra $A \subseteq B$ on which the action takes place is commutative and has totally disconnected spectrum. In addition, étale groupoids are required to be Hausdorff. These pure infiniteness criteria also imply that A separates ideals in B . Then the ideal lattice of B is isomorphic to the lattice of invariant ideals in A . Here we formulate sufficient conditions for A to separate ideals in B and for B to be purely infinite, which allow A to be noncommutative and which impose no Hausdorffness restrictions. In this generality, it is natural to study actions of inverse semigroups by Hilbert bimodules (see [11]) or, equivalently, section algebras of Fell bundles over inverse semigroups. This contains Fell bundles over discrete groups and over étale groupoids – possibly non-Hausdorff – as special cases. Another special case are Exel’s noncommutative Cartan C^* -inclusions (see [15, 29]), which generalise Renault’s (commutative) Cartan subalgebras.

This article is based on our recent papers, [28, 30–32], where two key concepts are developed. The first one is the *essential crossed product* introduced in [31], which is a variation on the reduced crossed product that “always” has the expected ideal structure – even

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for general actions of inverse semigroups and for actions of non-Hausdorff groupoids. The second concept is *aperiodicity*. It is a strong regularity property, abstracted from the work of Kishimoto, Olesen–Pedersen, and others, defined for a general C^* -inclusion $A \subseteq B$ in [31]. As shown in [32], aperiodicity implies that there is a *unique pseudo-expectation* – a unique generalised conditional expectation for $A \subseteq B$ taking values in Hamana’s injective hull of A . If in addition this pseudo-expectation is almost faithful, then A *supports* B in the sense that every element in $B^+ \setminus \{0\}$ is supported by an element in $A^+ \setminus \{0\}$ in the Cuntz preorder of [12]. This, in turn, implies that A *detects ideals* in B , that is, $J \cap A \neq 0$ for any ideal $0 \neq J \triangleleft B$. All these properties are closely related. In fact, when B is an essential crossed product and A is separable or of Type I, then the following conditions are equivalent: aperiodicity, unique pseudo-expectation, supporting positive elements in all intermediate C^* -algebras, detection of ideals in all intermediate C^* -algebras, and *topological freeness* of the dual groupoid (see [32]).

In the present paper, we study “residual” versions of these conditions, that is, when they hold for quotient inclusions $A/I \subseteq B/BIB$ for all ideals $I \triangleleft A$ that are restricted from B . The relationship between ideals in B and *restricted ideals* in A is thoroughly studied in [30]. For a crossed product inclusion, the restricted ideals in A are exactly those that are *invariant* under the action that produces B from A . The residual version of detection of ideals is separation of ideals. We say that A *separates ideals* in B if $I \cap A = J \cap A$ for ideals $I, J \triangleleft B$ implies $I = J$. This identifies the ideal lattice of B with the lattice of restricted ideals in A . Under some extra assumptions, we may also identify the primitive ideal space of B with the *quasi-orbit space* of the induced action on the primitive ideal space of A (see [30]). A residual version of supporting is closely related to the formally stronger condition called *filling*, which was used in [22, 23] to prove strong pure infiniteness. For a large class of C^* -inclusion $A \subseteq B$, we show that A residually supports B if and only if A fills B . Namely, this holds for the *symmetric* inclusions defined in [30], which include all sorts of C^* -inclusions coming from crossed products. For general residually supporting C^* -inclusions $A \subseteq B$ and a family $\mathcal{F} \subseteq A^+$ of elements in A that are properly infinite in B , we give sufficient conditions for B to be purely infinite (see Theorem 2.37 below).

To ensure that A^+ residually supports B we assume that the inclusion $A \subseteq B$ is *residually aperiodic*, in the sense that for any restricted ideal $I \triangleleft A$, the inclusion $A/I \rightarrow B/BIB$ is aperiodic. We also need to assume that the pseudo-expectations for these quotient inclusions are almost faithful. For actions by discrete groups this “residual faithfulness of conditional expectations” is also called *exactness* (see [1, 34, 44]) and for groupoids inner exactness (see [3, 6]). We generalise this concept to inverse semigroup actions by Hilbert bimodules and Fell bundles over étale groupoids. In particular, we prove that the full crossed product is an exact functor and the reduced crossed product is an injective functor, but only when restricted to special homomorphisms between actions (Propositions 4.2 and 4.15). The essential crossed product is not functorial. Therefore, exactness for these crossed products, which we call *essential exactness*, is more subtle.

As an appetizer, we formulate here a theorem that summarises and illustrates some of our results. An action of a unital inverse semigroup S by Hilbert bimodules on a C^* -algebra A is a semigroup $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$, where each fibre \mathcal{E}_t is a Hilbert A -bimodule and the semigroup product is compatible with the internal tensor product (see Definition 3.1 below). This induces an S -action by partial homeomorphisms on the spectrum \widehat{A} and the primitive ideal space \check{A} . The corresponding transformation groupoids $\widehat{A} \rtimes S$ and $\check{A} \rtimes S$ are called *dual groupoids* of \mathcal{E} . If the unit spaces in these groupoids are closed, we call \mathcal{E} a *closed action*. Then the essential crossed product $A \rtimes_{\text{ess}} S$ coincides with the reduced crossed product $A \rtimes_r S$. The following theorem combines Theorems 5.8, 5.9 and 5.15, and Corollary 5.22.

Theorem 1. *Let $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ be an inverse semigroup action by Hilbert bimodules on a C^* -algebra A . Assume that \mathcal{E} is essentially exact (or exact when the action is closed) and residually aperiodic (this holds when $\widehat{A} \rtimes S$ is residually topologically free). Then*

- (1) *A fills the essential crossed product $A \rtimes_{\text{ess}} S$. So $A \rtimes_{\text{ess}} S$ is strongly purely infinite if and only if every pair of elements in A^+ satisfies the matrix factorisation property of [23];*
- (2) *the ideal lattice of $A \rtimes_{\text{ess}} S$ is isomorphic to the lattice $\mathbb{I}^{\mathcal{E}}(A)$ of \mathcal{E} -invariant ideals in A . If \check{A} is second countable, the primitive ideal space of $A \rtimes_{\text{ess}} S$ is homeomorphic to the quasi-orbit space \check{A}/\sim of the dual groupoid $\check{A} \rtimes S$;*
- (3) *if $\mathcal{F} \subseteq A^+$ residually supports A and consists of residually \mathcal{E} -infinite elements (see Definition 5.18), and $\mathbb{I}^{\mathcal{E}}(A)$ is finite or the projections in \mathcal{F} separate the ideals in $\mathbb{I}^{\mathcal{E}}(A)$, then $A \rtimes_{\text{ess}} S$ is purely infinite and has the ideal property.*

The above theorem directly applies to (twisted) crossed products by discrete groups. As we explain in Section 5.2, it covers all the pure infiniteness results in [19, 22, 34, 35, 38, 43]. Theorem 1 has an analogue for Fell bundles over étale groupoids (Corollary 5.10). In particular, it can be used to generalise the results in [6, 36, 41] to twisted (not necessarily Hausdorff) étale groupoids. The twisted version covers all Cartan inclusions of Renault [42] (see Corollary 5.29 and Remark 5.30). In fact, Theorem 1 may be applied to all regular residually aperiodic C^* -inclusions $A \subseteq B$ with a residually faithful conditional expectation (see Proposition 5.12). This includes a large class of noncommutative Cartan inclusions in the sense of Exel [15]. See [29, Theorem 4.3] for a number of equivalent characterisations of such inclusions.

The article is organised as follows. Section 2 reviews general results about restriction and induction of ideals from [30], discusses residually supporting and filling families and their relationship, and presents our general pure infiniteness criteria for C^* -algebras. In Section 3, we recall actions of inverse semigroups by Hilbert bimodules and their crossed products. In Section 4, we discuss restrictions, functoriality and exactness of inverse semigroup crossed products. Section 5 discusses our general pure infiniteness criteria for crossed products by inverse semigroup actions.

2. Separation of ideals and pure infiniteness criteria for C^* -inclusions

2.1. Separation and detection of ideals

Let $A \subseteq B$ be a C^* -inclusion. We recall some results from [30] that relate the ideal structure of the two C^* -algebras A and B . (In [30], we also consider the more general situation of an inclusion $A \rightarrow \mathcal{M}(B)$ into the multiplier algebra of B .)

Definition 2.2. Let $\mathbb{I}(A)$ and $\mathbb{I}(B)$ be the complete lattices of (closed, two-sided) ideals in A and B , respectively. For $I \in \mathbb{I}(A)$, let $BIB \in \mathbb{I}(B)$ be the ideal in B generated by I . We define the *restriction map* r and the *induction map* i by

$$\begin{aligned} r: \mathbb{I}(B) &\rightarrow \mathbb{I}(A), & J &\mapsto J \cap A, \\ i: \mathbb{I}(A) &\rightarrow \mathbb{I}(B), & I &\mapsto BIB. \end{aligned}$$

We call $I \in \mathbb{I}(A)$ *restricted* if $I = r(J)$ for some $J \in \mathbb{I}(B)$. We call $J \in \mathbb{I}(B)$ *induced* if $J = i(I)$ for some $I \in \mathbb{I}(A)$. Let $\mathbb{I}^B(A) \subseteq \mathbb{I}(A)$ and $\mathbb{I}^A(B) \subseteq \mathbb{I}(B)$ be the subsets of restricted and induced ideals, respectively.

The maps r and i form a (monotone) *Galois connection*, that is, if $I \in \mathbb{I}(A)$, $J \in \mathbb{I}(B)$, then $I \subseteq r(J)$ if and only if $i(I) \subseteq J$. This observation goes back to Green [17]. It has a number of consequences. For instance, the maps i and r are monotone and satisfy $r \circ i(I) \supseteq I$ and $i \circ r \circ i(I) = i(I)$ for all $I \in \mathbb{I}(A)$, and $i \circ r(J) \subseteq J$ and $r \circ i \circ r(J) = r(J)$ for all $J \in \mathbb{I}(B)$. The map i preserves joins and r preserves meets. The maps $r: \mathbb{I}(B) \rightarrow \mathbb{I}^B(A) \subseteq \mathbb{I}(A)$ and $i: \mathbb{I}(A) \rightarrow \mathbb{I}^A(B) \subseteq \mathbb{I}(B)$ restrict to mutually inverse isomorphisms of partially ordered sets

$$\mathbb{I}^A(B) \cong \mathbb{I}^B(A).$$

Thus $\mathbb{I}^A(B) \cong \mathbb{I}^B(A)$ are complete lattices with inclusion as the partial order. The map r is injective if and only if i is surjective, if and only if $\mathbb{I}(B) = \mathbb{I}^A(B)$. Subalgebras with this property are said to separate ideals in the sense of the following definition.

Definition 2.3. We say that A *separates ideals* in B if $J_1 \cap A \neq J_2 \cap A$ for all $J_1, J_2 \triangleleft B$ with $J_1 \neq J_2$ or, equivalently, r is injective. We say that A *detects ideals* in B if $J \cap A \neq 0$ for all $J \triangleleft B$ with $J \neq 0$ or, equivalently, $r^{-1}(0) = \{0\}$.

Detection of ideals is sometimes called the intersection property. Separation of ideals is a residual version of detection of ideals, as the following lemma shows.

Lemma 2.4 ([30, Proposition 2.12]). *A C^* -subalgebra $A \subseteq B$ separates ideals if and only if $A/I \subseteq B/BIB$ detects ideals for all restricted ideals $I \in \mathbb{I}^B(A)$.*

2.2. Symmetric and regular C^* -inclusions

We will be mainly interested in regular inclusions, and these are symmetric as in the following definition.

Definition 2.5 ([30, Definition 5.2]). A C^* -inclusion $A \subseteq B$ is *nondegenerate* if $AB = B$. It is *symmetric* if for all $I \in \mathbb{I}^B(A)$ the inclusion $I \rightarrow BIB$ is nondegenerate. (This is equivalent to $IB = BI$ by [30, Lemma 5.1].)

Let \check{A} and \check{B} denote the primitive ideal spaces of A and B , respectively. Let $\text{Prime}^B(A)$ denote the space of prime ideals in the lattice $\mathbb{I}^B(A)$ of restricted ideals. For $I \in \mathbb{I}^B(A)$, let $U_I := \{\mathfrak{p} \in \text{Prime}^B(A) : I \not\subseteq \mathfrak{p}\}$. We equip $\text{Prime}^B(A)$ with the topology $\{U_I\}_{I \in \mathbb{I}^B(A)}$.

The next theorem summarises several desirable properties of symmetric inclusions.

Theorem 2.6 ([30]). *Let $A \subseteq B$ be a symmetric C^* -inclusion.*

- (1) *If \mathfrak{p} is a primitive ideal in B , then $r(\mathfrak{p}) \in \mathbb{I}^B(A)$ is prime, and this defines a continuous map $r: \check{B} \rightarrow \text{Prime}^B(A)$.*
- (2) *If \mathfrak{p} is a primitive ideal in A , then there is a largest restricted ideal in A that is contained in \mathfrak{p} , which we denote by $\pi(\mathfrak{p})$. This element of $\mathbb{I}^B(A)$ is prime, and the resulting map $\pi: \check{A} \rightarrow \text{Prime}^B(A)$ is continuous. Define an equivalence relation on \check{A} by $\mathfrak{p} \sim \mathfrak{q}$ if and only if $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$. So π descends to a continuous map $\tilde{\pi}: \check{A}/\sim \rightarrow \text{Prime}^B(A)$.*
- (3) *If $\text{Prime}^B(A)$ is first countable – this always holds when \check{A} is second countable – then π is open and surjective and $\tilde{\pi}$ is a homeomorphism. Then there is a continuous map*

$$\varrho: \check{B} \rightarrow \check{A}/\sim, \quad \mathfrak{p} \mapsto \tilde{\pi}^{-1}(r(\mathfrak{p})).$$

It is a homeomorphism if and only if A separates ideals in B .

Proof. By [30, Corollary 5.5] we may apply [30, Lemmas 4.8 and 4.1] to get (1) and (2), and [30, Theorem 4.5] gives (3), see also [30, Corollary 4.7]. ■

Definition 2.7 ([30, Definitions 4.4, 4.10]). The space \check{A}/\sim in Theorem 2.6 is called the *quasi-orbit space* and the map $\varrho: \check{B} \rightarrow \check{A}/\sim$ is called the *quasi-orbit map* of $A \subseteq B$.

Remark 2.8. If B is the full or reduced crossed product – or an exotic crossed product, for an action of a discrete group G on a C^* -algebra A , then \check{A}/\sim coincides with the usual quasi-orbit space of the dual action of G on \check{A} . In [30], the quasi-orbit space is also described in several other cases.

Next we turn to regular inclusions. To link them to crossed products for inverse semigroup actions, we describe them through gradings by inverse semigroups.

Definition 2.9 ([30, Definition 6.15]). Let S be an inverse semigroup with unit $1 \in S$. An S -graded C^* -algebra is a C^* -algebra B with a family of closed linear subspaces $(B_t)_{t \in S}$ such that $B_g^* = B_{g^*}$, $B_g \cdot B_h \subseteq B_{gh}$ for all $g, h \in S$ and $B_g \subseteq B_h$ if $g \leq h$ in S (that is, $g = hg^*g$), and $\sum B_t$ is dense in B . We call $A := B_1 \subseteq B$ the *unit fibre* of the S -grading. The grading is *saturated* if $B_g \cdot B_h = B_{gh}$ for all $g, h \in S$.

Definition 2.10 ([24, 42]). Let $A \subseteq B$ be a C^* -subalgebra. We call $b \in B$ a *normaliser* of A in B if $bAb^* \subseteq A$ and $b^*Ab \subseteq A$. The inclusion $A \subseteq B$ is *regular* if it is nondegenerate and B is the closed linear span of the normalisers of A in B .

Proposition 2.11. *Let $A \subseteq B$ be a C^* -inclusion. The following are equivalent:*

- (1) A is a regular subalgebra of B ;
- (2) A is the unit fibre for some S -grading on B ;
- (3) A is the unit fibre for some saturated S -grading on B .

If the inclusion is regular, then it is symmetric, and the set $\mathcal{S}(A, B)$ of all closed linear A -subbimodules $M \subseteq B$ that consist entirely of normalisers is an inverse semigroup with the operations $M \cdot N := \overline{\text{span}}\{mn : m \in M, n \in N\}$ and $M^ := \{m^* : m \in M\}$, and it gives a saturated grading on B .*

Proof. Combine [30, Lemma 6.25 and Proposition 6.26]. ■

Definition 2.12. Let $(B_t)_{t \in S}$ be an S -grading on B with $A = B_1$. An ideal $I \in \mathbb{I}(A)$ is called *$(B_t)_{t \in S}$ -invariant* if $IB_t = B_tI$ for all $t \in S$. An ideal $J \in \mathbb{I}(B)$ is called *S -graded* if $J = \overline{\sum_{t \in S} (J \cap B_t)}$.

Proposition 2.13 ([30, Propositions 6.19 and 6.20]). *Let $(B_t)_{t \in S}$ be an S -grading on B with $A = B_1$. An ideal $I \in \mathbb{I}(A)$ is restricted, $I \in \mathbb{I}^B(A)$, if and only if it is $(B_t)_{t \in S}$ -invariant. An ideal $J \in \mathbb{I}(B)$ is induced, $J \in \mathbb{I}^A(B)$, if and only if J is S -graded.*

The following lemma on quotients of C^* -inclusions is not yet considered in [30].

Lemma 2.14. *Let $A \subseteq B$ be a C^* -subalgebra and $J \in \mathbb{I}(B)$. Let $I := J \cap A$. Let $q: B \rightarrow B/J$ denote the quotient map. View A/I as a C^* -subalgebra of B/J .*

- (1) *If $(B_t)_{t \in S}$ is an S -grading of B with A as unit fibre, then $(q(B_t))_{t \in S}$ is an S -grading on B/J with unit fibre A/I . And $q(B_t) \cong B_t/B_tI$ as Banach spaces – and even as Hilbert A/I -bimodules – for all $t \in S$.*
- (2) *If $A \subseteq B$ is regular, then $A/I \subseteq B/J$ is regular.*

Proof. The canonical map from A/I to B/J is injective because $J \cap A = I$. Thus we may view A/I as a C^* -subalgebra of B/J . We prove (1). It is easy to see that $(q(B_t))_{t \in S}$ is an S -grading on B/J . Each B_t is naturally a right Hilbert A -module with inner product $\langle a | b \rangle := a^*b \in A$ for $a, b \in B_t$ and the right multiplication in B . The proof of the Rieffel correspondence between ideals in A and $\mathbb{K}(B_t)$ shows that $B_tI = \{b \in B_t : \langle b | b \rangle \in I\}$. The quotient Banach space B_t/B_tI is a right Hilbert A/I -module with the induced multiplication and the inner product

$$\langle a + B_tI | b + B_tI \rangle := \langle a | b \rangle + I = q(a^*b) \in A/I$$

for $a, b \in B_t$. We claim that the norm defined by this inner product is equal to the quotient norm on B_t/B_tI . To show this, let $(u_n)_{n \in \mathbb{N}}$ be an approximate unit for I . Then

$$\begin{aligned} \|q(b)\|^2 &= \lim \|b - bu_n\|^2 = \lim \|(1 - u_n)b^*b(1 - u_n)\| = \|q(b^*b)\| \\ &= \|\langle b + B_tI \mid b + B_tI \rangle\|. \end{aligned}$$

This finishes the proof of (1). Assertion (2) follows from (1) and Proposition 2.11. ■

2.3. Generalised expectations

Definition 2.15. A *generalised expectation* for a C^* -inclusion $A \subseteq B$ consists of another C^* -inclusion $A \subseteq \tilde{A}$ and a completely positive, contractive map $B \rightarrow \tilde{A}$ that restricts to the identity map on A . If $\tilde{A} = A$, $\tilde{A} = A''$, or $\tilde{A} = I(A)$ is Hamana’s injective envelope of A , then we speak of a *conditional expectation*, a *weak expectation*, or a *pseudo-expectation*, respectively.

Let $E: B \rightarrow \tilde{A} \supseteq A$ be a generalised expectation. It is called *faithful* if $E(b^*b) = 0$ for some $b \in B$ implies $b = 0$, *almost faithful* if $E((bc)^*bc) = 0$ for all $c \in B$ and some $b \in B$ implies $b = 0$, and *symmetric* if $E(b^*b) = 0$ for some $b \in B$ implies $E(bb^*) = 0$. The largest two-sided ideal in B contained in $\ker E$ is equal to

$$\begin{aligned} \mathcal{N}_E &:= \{b \in B : E((bc)^*bc) = 0 \text{ for all } c \in B\} \\ &= \{b \in B : E(xby) = 0 \text{ for all } x, y \in B\} \end{aligned}$$

(see [31, Proposition 3.6]), and $\mathcal{N}_E = 0$ if and only if E is almost faithful.

Since $E|_A = \text{Id}_A$ and $E|_{\mathcal{N}_E} = 0$, it follows that $A \cap \mathcal{N}_E = 0$. Hence the composite map $A \rightarrow B \rightarrow B/\mathcal{N}_E$ is injective and we may identify A with its image in $B_r := B/\mathcal{N}_E$. The map E descends to a generalised expectation $E_r: B_r \rightarrow \tilde{A} \supseteq A$ that we call the *reduced generalised expectation* associated to E (see [31, Definition 3.5]). The reduced generalised expectation E_r is always almost faithful. It is faithful if and only if E is symmetric (see [31, Corollary 3.8]).

We will mainly work with pseudo-expectations below. The injectivity of $I(A)$ implies that any C^* -inclusion has a pseudo-expectation. The following lemma links detection of ideals to almost faithfulness of pseudo-expectations.

Lemma 2.16. *Let $A \subseteq B$ be a C^* -inclusion. The following are equivalent:*

- (1) *A detects ideals in B ;*
- (2) *every generalised expectation for the C^* -inclusion $A \subseteq B$ is almost faithful;*
- (3) *every pseudo-expectation for $A \subseteq B$ is almost faithful.*

Proof. Let $E: B \rightarrow \tilde{A} \supseteq A$ be a generalised expectation. Since $\mathcal{N}_E \cap A = 0$, we must have $\mathcal{N}_E = 0$ if A detects ideals in B . That is, (1) implies (2). That (2) implies (3) is obvious. We prove by contradiction that (3) implies (1). Assume that A does not detect ideals in B . Then there is a nonzero ideal \mathcal{N} in B with $\mathcal{N} \cap A = 0$. The inclusion $A \hookrightarrow B/\mathcal{N}$ has a pseudo-expectation $E: B/\mathcal{N} \rightarrow I(A)$. Let $q: B \rightarrow B/\mathcal{N}$ be the quotient map.

Then $E \circ q: B \rightarrow I(A)$ is a pseudo-expectation for the inclusion $A \subseteq B$. It is not almost faithful because $0 \neq \mathcal{N} \subseteq \mathcal{N}_{E \circ q}$. ■

Remark 2.17 ([40, Theorem 3.5]). Every pseudo-expectation for $A \subseteq B$ is faithful – not only almost faithful – if and only if A detects ideals in C for each intermediate C^* -algebra $A \subseteq C \subseteq B$.

Lemma 2.4 says that A separates ideals in B if and only if it “residually detects” ideals in B . The residual version of Lemma 2.16 says that A separates ideals in B if and only if the following happens: if $I \in \mathbb{I}^B(A)$ is a restricted ideal and $E^I: B/BIB \rightarrow I(A/I)$ is a pseudo-expectation for the inclusion $A/I \hookrightarrow B/BIB$, then E^I is almost faithful.

In general, it is not practical to check that all pseudo-expectations $B \rightarrow I(A)$ are faithful or almost faithful. And the residual version of this statement looks even more hopeless. There are, however, inclusions with a unique pseudo-expectation. This is the case for aperiodic inclusions by [32, Theorem 3.6]. When the inclusion is even “residually” aperiodic, then the inclusions $A/I \hookrightarrow B/BIB$ have a unique pseudo-expectation for all $I \in \mathbb{I}^B(A)$. And if we know pseudo-expectations $E^I: B/BIB \rightarrow I(A/I)$ for all $I \in \mathbb{I}^B(A)$, then it becomes possible to check whether they are all (almost) faithful (see Theorem 2.34 below).

The residual version of Lemma 2.16 discussed above uses pseudo-expectations for the inclusion $A/I \hookrightarrow B/BIB$ for $I \in \mathbb{I}^B(A)$. The following examples show that these need not be closely related to pseudo-expectations for the original inclusion $A \subseteq B$. In fact, for inclusions that are not symmetric, even a genuine conditional expectation $E: B \rightarrow A$ need not “induce” a conditional expectation $E: B/BIB \rightarrow A/I$.

Example 2.18. Let $B = \mathbb{B}(H)$ be the algebra of bounded operators on a separable Hilbert space H , and let $A = \mathbb{K}(H) + 1$ be the minimal unitisation of the compacts. The inclusion $A \subseteq B$ is symmetric, $\mathbb{I}^B(A) = \{0, \mathbb{K}(H), A\}$, and $I(A) = \mathbb{B}(H) = B$. We may take the identity map as the pseudo-expectation $E: B \rightarrow B$ for $A \subseteq B$. Let $I := \mathbb{K}(H)$. Then, on the one hand, E descends to the identity map $E^I: B/I \rightarrow B/I$ on the Calkin algebra. On the other hand, pseudo-expectations for $\mathbb{C} \cong A/I \subseteq B/I$ are just states on the Calkin algebra. It seems that there is no universal way how to produce a state from the identity map.

Example 2.19 (See [30, Examples 2.16 and 7.9]). Let $B := \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$ and consider the commutative C^* -subalgebra $A \subseteq B$ spanned by the orthogonal diagonal projections $(P_{00}, 0)$, $(0, P_{00})$, and (P_{11}, P_{11}) . Let $E: B \rightarrow A \subseteq B$ be any faithful conditional expectation. For instance,

$$E\left(\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \oplus \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}\right) := \frac{1}{2} \begin{pmatrix} 2a_{00} & 0 \\ 0 & a_{11} + b_{11} \end{pmatrix} \oplus \frac{1}{2} \begin{pmatrix} 2b_{00} & 0 \\ 0 & a_{11} + b_{11} \end{pmatrix}.$$

Let $J = \mathbb{M}_2(\mathbb{C}) \oplus 0$. Then $I := J \cap A = \mathbb{C} \cdot (P_{00}, 0)$ is a restricted ideal in $A \cong \mathbb{C}^3$. We have $J = BIB$ and $E(J) = \{\lambda_1(P_{00}, 0) + \lambda_2(P_{11}, P_{11}) : \lambda_1, \lambda_2 \in \mathbb{C}\} \not\subseteq I$. Hence $E: B \rightarrow A$ does not factor through a map $B/J \rightarrow A/I$. Note that the inclusion $A \subseteq B$ is not symmetric.

Lemma 2.20. *Let $E: B \rightarrow A$ be a conditional expectation for a symmetric C^* -inclusion $A \subseteq B$ and let $I \in \mathbb{I}^B(A)$. Then E descends to a conditional expectation $E^I: B/BIB \rightarrow A/I$, $b + J \mapsto E(b) + I$, for the inclusion $A/I \rightarrow B/BIB$.*

Proof. Let $J \in \mathbb{I}^A(B)$ and put $I = J \cap A \in \mathbb{I}^B(A)$. Since $A \subseteq B$ is symmetric we have $J = IB I$. Thus $E(J) = E(IB I) \subseteq IE(B)I = I$ because E is A -bilinear. Since $I \subseteq E(J)$ always holds, this is equivalent to $I = E(J)$. Then E^I is well defined. ■

Many C^* -algebras, including section algebras for Fell bundles over Hausdorff étale groupoids, are naturally equipped with a conditional expectation, which is residually symmetric in the sense described in the following proposition. Then the residual faithfulness of E is called exactness of the corresponding action [1, 3, 6, 34, 44].

Proposition 2.21. *Let $A \subseteq B$ be a symmetric C^* -inclusion with a conditional expectation $E: B \rightarrow A$ which is residually symmetric, that is, for each $I \in \mathbb{I}^B(A)$, the conditional expectation E^I in Lemma 2.20 is symmetric. Then A separates ideals in B if and only if E is residually faithful – that is, E^I is faithful for each $I \in \mathbb{I}^B(A)$ – and E preserves ideals – that is, $E(J) \subseteq J$ for each $J \in \mathbb{I}(B)$.*

Proof. Assume first that A separates ideals in B . Lemmas 2.4 and 2.16 imply that E^I is faithful for all $I \in \mathbb{I}^B(A)$. Since we assume that $\mathbb{I}(B) = \mathbb{I}^A(B)$ and $A \subseteq B$ is symmetric, this implies $E(J) \subseteq J$ by Lemma 2.20. Conversely, assume that E is residually faithful and preserves ideals.

Let $J \in \mathbb{I}(B)$ and $I \in \mathbb{I}(A)$ satisfy $BIB \not\subseteq J$. Then $E(J) \neq I$ because E^I is faithful. Since E preserves ideals, $E(J) = J \cap A$. Thus $I \not\subseteq J \cap A$. This shows that $A/I \subseteq B/BIB$ detects ideals for any $I \in \mathbb{I}^B(A)$. This implies that A separates ideals in B by Lemma 2.4. ■

2.4. Supporting and filling families

Let B^+ be the set of positive elements in a C^* -algebra B . We equip B^+ with the *Cuntz preorder* \lesssim introduced in [12]: for $a, b \in B^+$, we write $a \lesssim b$ and say that a supports b if, for every $\varepsilon > 0$, there is $x \in B$ with $\|a - x^*bx\| < \varepsilon$. We call $a, b \in B^+$ *Cuntz equivalent* and write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$. We call $a, b \in B^+$ *Murray–von Neumann equivalent* and write $a \sim b$ if there is $z \in B$ with $a = z^*z$ and $b = zz^*$. Both \sim and \approx are equivalence relations, and $a \sim b$ implies $a \approx b$. In the converse direction, only a weaker result is true (see [21, Lemma 2.3 (iv)]). Namely, for $\varepsilon > 0$ and $a \in B^+ \setminus \{0\}$, let $(a - \varepsilon)_+ \in B$ be the positive part of $a - \varepsilon \cdot 1 \in \mathcal{M}(B)$. Let bBb be the hereditary subalgebra generated by b , that is, the closure of $\{bxb : x \in A\}$. Then $a \lesssim b$ if and only if every ε -cut-down of a is Murray–von Neumann equivalent to an element in bBb , that is

$$a \lesssim b \iff \forall_{\varepsilon>0} \exists_{z \in B} (a - \varepsilon)_+ = z^*z \text{ and } zz^* \in bBb.$$

In particular, $a \in bBb$ implies $a \lesssim b$, $aBa = bBb$ implies $a \approx b$, and $a \lesssim b$ implies $a \in BbB$. A C^* -algebra B is *purely infinite* [20] if it admits no characters and $a, b \in B^+ \setminus \{0\}$ satisfy $a \leq b$ if and only if $a \in BbB$.

Definition 2.22 (Compare [27, Definition 2.39]). A subset $\mathcal{F} \subseteq B^+$ supports B if, for each $b \in B^+ \setminus \{0\}$, there is $a \in \mathcal{F} \setminus \{0\}$ with $a \lesssim b$. It residually supports B if, for each $J \in \mathbb{I}(B)$, the image of \mathcal{F} in the quotient B/J supports B/J .

Lemma 2.23. Let B be a C^* -algebra and let $\mathcal{F} \subseteq B^+$.

- (1) Let \mathcal{F} support B . If $J \in \mathbb{I}(B)$, then $J \cap \mathcal{F}$ supports J . So \mathcal{F} detects ideals in B .
- (2) Let \mathcal{F} residually support B . If $J \in \mathbb{I}(B)$, then $J \cap \mathcal{F}$ residually supports J . In addition, \mathcal{F} separates ideals in B .

Proof. We first prove (1). Let $J \in \mathbb{I}(B)$ and $b \in J^+ \setminus \{0\}$. Then $x^*bx \in J$ for each $x \in B$. If $a \lesssim b$ for some $a \in B^+ \setminus \{0\}$, then $a = \lim x_k^*bx_k$ for some sequence $(x_k)_{k \in \mathbb{N}}$ in B . Therefore, any $a \in \mathcal{F}^+ \setminus \{0\}$ with $a \leq b$ and $b \in J^+ \setminus \{0\}$ belongs to J . So if \mathcal{F} supports B and $J \neq \{0\}$, then $J \cap \mathcal{F} \neq \{0\}$.

Next we prove (2). Let $J \in \mathbb{I}(B)$ and $I \in \mathbb{I}(J)$. Let $q: B \rightarrow B/I$ be the quotient map. For any $b \in q(J)^+ \setminus \{0\}$, there is $a \in \mathcal{F}$ with $0 \neq q(a) \lesssim b$. The proof above shows that $q(a) \in q(J)$. Since $q(a) \neq 0$, we have $a \in J \setminus I$. Thus $J \cap \mathcal{F}$ residually supports J . If $I, J \in \mathbb{I}(B)$ and $I \neq J$, then $I \cap J$ is a proper ideal of I or J . Assume, say, that $I \cap J \subsetneq J$. As we have seen above, there is $a \in (J \cap \mathcal{F}) \setminus I$. Hence $I \cap \mathcal{F} \neq J \cap \mathcal{F}$. ■

Corollary 2.24. Let $A \subseteq B$ be a C^* -inclusion. A family $\mathcal{F} \subseteq A^+$ residually supports B if and only if the image of \mathcal{F} supports B/J for each $J \in \mathbb{I}^A(B)$.

Proof. If the image of \mathcal{F} supports the quotient B/J for every $J \in \mathbb{I}^A(B)$, then by Lemma 2.23, $A/(J \cap A)$ detects ideals in B/J for each $J \in \mathbb{I}^A(B)$. Then A separates ideals in B by Lemma 2.4. Thus $\mathbb{I}(B) = \mathbb{I}^A(B)$. So our assumption already says that \mathcal{F} residually supports B . ■

A C^* -algebra B has the ideal property if projections in B separate ideals in B .

Corollary 2.25. If the set of projections in B residually supports B , then B has the ideal property. Conversely, a purely infinite C^* -algebra with the ideal property is residually supported by its projections.

Proof. Lemma 2.23 (2) gives the first statement. For the second statement, let $J \in \mathbb{I}(B)$ and $b \in B^+ \setminus J^+$. Let $q: B \rightarrow B/J$ be the quotient map. Since B has the ideal property, the ideal $BbB + J \supsetneq J$ contains a projection $p \notin J$. Then $0 \neq q(p) \in q(BbB) = q(B)q(b)q(B)$. The quotient B/J is purely infinite by [20, Proposition 4.3]. Hence we get $q(p) \leq q(b)$. ■

Let $\mathbb{H}(B)$ be the set of all nonzero hereditary C^* -subalgebras of B .

Lemma 2.26. Let $\mathcal{F} \subseteq B^+$. The following conditions are equivalent:

- (1) \mathcal{F} supports B ;
- (2) for each $b \in B^+ \setminus \{0\}$, there is $x \in B$ with $x^*bx \in \mathcal{F} \setminus \{0\}$;
- (3) for each $D \in \mathbb{H}(B)$, there is $z \in B$ with $zz^* \in D$ and $z^*z \in \mathcal{F} \setminus \{0\}$.

Proof. (1) \Rightarrow (2): Let $b \in B^+ \setminus \{0\}$. Let $\delta \in (0, \|b\|)$. Then (1) gives $a \in \mathcal{F} \setminus \{0\}$ with $a \lesssim (b - \delta)_+$. And [21, Lemma 2.4 (ii)] gives $x \in B$ with $x^*bx = a \in \mathcal{F} \setminus \{0\}$.

(2) \Rightarrow (3): Let $b \in D^+ \setminus \{0\}$ and choose $x \in B$ with $x^*bx \in \mathcal{F} \setminus \{0\}$ as in (2). Then $z := \sqrt{b} \cdot x$ satisfies $zz^* \in D$ and $z^*z \in \mathcal{F} \setminus \{0\}$, as required in (3).

(3) \Rightarrow (1): Let $b \in B^+ \setminus \{0\}$. Then $D := bBb$ is a nonzero hereditary C^* -subalgebra of B . Condition (3) gives $z \in B$ with $zz^* \in D$ and $a := z^*z \in \mathcal{F} \setminus \{0\}$. Then $a \sim zz^*$ and hence $a \approx zz^*$. And $zz^* \lesssim b$ by [20, Proposition 2.7 (i)]. So $a \lesssim b$. ■

Definition 2.27 ([23, Definition 4.2]). Let B be a C^* -algebra. A set $\mathcal{F} \subseteq B^+$ fills B (is a filling family for B) if, for each $D \in \mathbb{H}(B)$ and each $J \in \mathbb{I}(B)$ with $D \not\subseteq J$, there is $z \in B \setminus J$ with $zz^* \in D$ and $z^*z \in \mathcal{F}$.

In this definition, we may also require $z \in B$ with $zz^* \in D$ and $z^*z \in \mathcal{F} \setminus J$ because $z \in J$ if and only if $z^*z \in J$. For $J = 0$, this is the condition in Lemma 2.26 (3). This suggests that filling and residually supporting families are closely related. We are going to prove some results to this effect.

Proposition 2.28. *If $\mathcal{F} \subseteq B^+$ fills B , then it residually supports B .*

Proof. Let $J \in \mathbb{I}(B)$. Let $q: B \rightarrow B/J$ denote the quotient map. Let $b \in q(B)^+ \setminus \{0\}$. There is $d \in B^+ \setminus J$ with $q(d) = b$. For $D := dBd$ we have $q(D) = bq(B)b$ and $D \not\subseteq J$. Since \mathcal{F} fills B , there is $z \in B$ with $z^*z \in D$ and $zz^* \in \mathcal{F} \setminus J$. Then $q(z)^*q(z) \in bq(B)b$ and $q(z)q(z)^* \in q(\mathcal{F}) \setminus \{0\}$. Hence $q(z)q(z)^* \approx q(z)^*q(z) \lesssim b$. So $q(\mathcal{F})$ supports $q(B)$. ■

Proposition 2.29. *Let $A \subseteq B$ be a symmetric C^* -inclusion. Then A^+ residually supports B if and only if A^+ fills B .*

Proof. If A^+ fills B , then it residually supports B by Proposition 2.28. Conversely, assume that A^+ residually supports B . We are going to prove that A^+ fills B . Pick $D \in \mathbb{H}(B)$ and $J \in \mathbb{I}(B)$ with $D \not\subseteq J$. We need $z \in A$ with $z^*z \in D$ and $zz^* \in A^+ \setminus J$. By Lemma 2.23 (2), A separates ideals in B . Hence $\mathbb{I}^A(B) = \mathbb{I}(B)$. So $J = BIB$ with $I := A \cap J$. Let $q: B \rightarrow B/J$ be the quotient map. There is $d \in D^+ \setminus J$. Let $b := q(d) \in (B/J)^+ \setminus \{0\}$. Lemma 2.26 gives $x \in B/J$ with $a := x^*bx \in (A/I)^+ \setminus \{0\}$. There are $c \in A^+$ with $q(c) = a$ and $w \in B$ with $q(w) = x$. Then $q(c) = x^*bx = q(w^*dw)$. So $c = w^*dw + v$ for some $v \in J$.

Let $\varepsilon := \|a\|/2$. By assumption, an approximate unit in I is also one for J . So there is $f \in I^+$ with $\|f\| \leq 1$ and $\|v - fv\| < \varepsilon$. Let 1 denote the formal unit in the unitisation of B and let $g := 1 - f \in \mathcal{M}(A)^+$. Then $\|g\| \leq 1$ and

$$\|gw^*dwg - gcg\| = \|gvg\| \leq \|v - fv\| < \varepsilon.$$

Now [21, Lemma 2.2] gives $h \in B$ with

$$h^*(gw^*dwg)h = (gcv - \varepsilon)_+ \in A^+.$$

Let $z := d^{1/2}wgh$. Then $zz^* \in D$ because $d \in D$, and $z^*z = (gcg - \varepsilon)_+ \in A^+$. Since $q(gcg) = q(c - cf - fc + fcf) = q(c) = a$, we get

$$\|q(z^*z)\| = \|q((gcg - \varepsilon)_+)\| = \|(a - \varepsilon)_+\| \geq \|a\| - \varepsilon = \|a\|/2 > 0.$$

Hence $z^*z \notin J$, which is equivalent to $zz^* \notin J$. ■

Example 2.30. Let $B = C_0(\Omega)$ be commutative and let $\mathcal{F} \subseteq B^+$. The following conditions are equivalent:

- (1) \mathcal{F} fills B ;
- (2) \mathcal{F} residually supports B ;
- (3) the open supports of elements of \mathcal{F} form a basis of the topology of Ω .

Proposition 2.28 implies (1) \Rightarrow (2), and (3) \Rightarrow (1) is straightforward. We show (2) \Rightarrow (3). Fix an open subset $U \subseteq \Omega$ and a point $x_0 \in U$. There is a function $b \in C_0(\Omega)$ with $b(x_0) = 1$ and $b|_{\Omega \setminus U} \equiv 0$. Let J be the ideal in B consisting of functions vanishing on $\Omega \setminus U \cup \{x_0\}$. There is $a \in \mathcal{F} \setminus J$ with $a + J \lesssim b + J$. Then $V := \{x \in \Omega : a(x) > 0\}$ is an open subset of U that contains x_0 . This implies (3).

Example 2.30 suggests to view filling families and residually supporting subsets as noncommutative analogues of bases for topologies.

2.5. Residual aperiodicity and criteria for pure infiniteness

We introduce the residual version of aperiodicity and use it to characterise when a C^* -inclusion $A \subseteq B$ separates ideals. We also formulate some criteria for B to be purely infinite.

Definition 2.31 ([31, Definition 5.14]). A C^* -inclusion $A \subseteq B$ is *aperiodic* if the Banach A -bimodule B/A is aperiodic, that is, if for every $x \in B$, $D \in \mathbb{H}(A)$ and $\varepsilon > 0$, there are $a \in D^+$ and $y \in A$ with $\|axa - y\| < \varepsilon$ and $\|a\| = 1$.

Remark 2.32 ([32, Theorem 5.5]). If a C^* -inclusion $A \subseteq B$ has the almost extension property introduced in [37], then $A \subseteq B$ is aperiodic. The converse implication holds if B is separable.

Definition 2.33. A C^* -inclusion $A \subseteq B$ is *residually aperiodic* if, for all $I \in \mathbb{I}^B(A)$ and $J := B/B$, the C^* -inclusion $A/I \subseteq B/J$ is aperiodic.

Theorem 2.34. Let $A \subseteq B$ be a residually aperiodic C^* -inclusion. Then for each $I \in \mathbb{I}^B(A)$ there is a unique pseudo-expectation $E^I : B/B \rightarrow \mathbb{I}(A/I)$ for the C^* -inclusion $A/I \subseteq B/B$, and the following are equivalent:

- (1) the unique pseudo-expectation E^I is almost faithful for all $I \in \mathbb{I}^B(A)$;
- (2) A separates ideals in B ;
- (3) A^+ residually supports B .

If, in addition, the inclusion $A \subseteq B$ is symmetric, then (1)–(3) are equivalent to

- (4) A^+ fills B .

If $A \subseteq B$ is symmetric, \check{A} is second countable, and the above equivalent conditions hold, then $\check{B} \cong \check{A}/\sim$ via the quasi-orbit map.

Proof. It follows from [32, Theorem 3.6] that the expectation E^I is unique and that (1) implies (3). Lemma 2.23 shows that (3) implies (2). Next we prove that (2) implies (1). Indeed, if E^I is not almost faithful, then \mathcal{N}_{E^I} witnesses that A/I does not detect ideals in B/J . Then A does not separate ideals in B by Lemma 2.4. In the symmetric case, Proposition 2.29 shows that (3) is equivalent to (4). The claims in the last sentence follow from Theorem 2.6. ■

We end this section with pure infiniteness criteria that use filling and residually supporting families. Infinite and properly infinite elements in B^+ are defined in [20]. We recall their equivalent descriptions in [34, Lemma 2.1]. We also recall the notion of a separated pair of elements in B^+ from [28, Definition 5.1] and relate it to the matrix diagonalisation property. We write $a \approx_\varepsilon b$ if $\|a - b\| \leq \varepsilon$ for $a, b \in B$.

Definition 2.35. Let B be a C^* -algebra and $a \in B^+ \setminus \{0\}$.

- (1) We call $a \in B^+$ *infinite* in B if there is $b \in B^+ \setminus \{0\}$ such that for all $\varepsilon > 0$ there are $x, y \in aB$ with $x^*x \approx_\varepsilon a$, $y^*y \approx_\varepsilon b$ and $x^*y \approx_\varepsilon 0$.
- (2) We call $a \in B^+ \setminus \{0\}$ *properly infinite* if for all $\varepsilon > 0$ there are $x, y \in aB$ with $x^*x \approx_\varepsilon a$, $y^*y \approx_\varepsilon a$ and $x^*y \approx_\varepsilon 0$.
- (3) We call $a, b \in B^+$ *separated in B* if for all $\varepsilon > 0$ there are $x \in aB$ and $y \in bB$ with $x^*x \approx_\varepsilon a$, $y^*y \approx_\varepsilon b$ and $x^*y \approx_\varepsilon 0$.

By [20, Theorem 4.16], a C^* -algebra is purely infinite if and only if each element $a \in B^+ \setminus \{0\}$ is properly infinite. By [21, Remark 5.10], B is *strongly purely infinite* if and only if each pair of elements $a, b \in B^+ \setminus \{0\}$ is separated in B .

We say that a pair of elements $a, b \in B^+$ has the *matrix diagonalisation property in B* , if for each $x \in B$ with $\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(B)^+$ and each $\varepsilon > 0$ there are $d_1 \in B$ and $d_2 \in B$ such that

$$d_1^*ad_1 \approx_\varepsilon a, \quad d_2^*bd_2 \approx_\varepsilon b, \quad d_1^*xd_2 \approx_\varepsilon 0.$$

We say that a subset $\mathcal{F} \subseteq B^+$ is *invariant under ε -cut-downs* if $(a - \varepsilon)_+ \in \mathcal{F}$ for all $a \in \mathcal{F}$ and arbitrarily small $\varepsilon > 0$.

Theorem 2.36 ([23, Theorem 1.1]). *Let \mathcal{F} fill B and be invariant under ε -cut-downs. Then B is strongly purely infinite if and only if each pair of elements $a, b \in \mathcal{F}$ has the matrix diagonalisation property in B .*

The following theorem improves upon [28, Proposition 5.4].

Theorem 2.37. *Let $A \subseteq B$ be a C^* -subalgebra for which A^+ residually supports B ; this is the case, for instance, if $A \subseteq B$ is residually aperiodic and for each $I \in \mathbb{I}^B(A)$ the unique pseudo-expectation $E^I: B/BIB \rightarrow \mathbb{I}(A/I)$ is almost faithful. Let $\mathcal{F} \subseteq A^+$ residually support A . Assume that $\mathbb{I}^B(A)$ is finite or that the projections in \mathcal{F} separate*

the ideals in $\mathbb{I}^B(A)$ (this is automatic when \mathcal{F} consists of projections). Then the following statements are equivalent:

- (1) $\mathcal{F} \setminus \{0\}$ consists of elements that are properly infinite in B ;
- (2) B is purely infinite;
- (3) B is purely infinite and $\text{Prim } B$ has topological dimension zero;
- (4) B is purely infinite and has the ideal property;
- (5) B is strongly purely infinite.

Proof. The implications (3) \Leftarrow (4) \Rightarrow (5) are general facts (see [39, Propositions 2.11 and 2.14]). The implications (3) \Rightarrow (2) \Rightarrow (1) \Leftarrow (5) are clear. To close the cycles of implications, it suffices to show that (1) \Rightarrow (4). We have $\mathbb{I}(B) = \mathbb{I}^A(B) \cong \mathbb{I}^B(A)$ by Lemma 2.23. And \mathcal{F} residually supports B because \leq is transitive. In particular, \mathcal{F} separates ideals in $\mathbb{I}^B(A)$. Under our assumptions, the implication (1) \Rightarrow (4) follows from the proof of [28, Proposition 5.4]. We considered there the case $\mathcal{F} = A^+$. The proof still works, however, for any \mathcal{F} that residually supports A . ■

When A separates ideals in B , the assumption that A^+ residually supports B seems necessary for Theorem 2.37 to hold. Evidence for this is the following result abstracted from [43, Proposition 2.1] (see also [6, Proposition 4.1]).

Proposition 2.38. *Let $A \subseteq B$ be a symmetric C^* -inclusion with a residually symmetric conditional expectation $E: B \rightarrow A$ as in Proposition 2.21. If A separates ideals in B , then B is purely infinite if and only if every $a \in A^+ \setminus \{0\}$ is properly infinite in B and $E(b) \lesssim b$ for every $b \in B^+$. If B is purely infinite and A separates ideals in B , then A^+ fills B .*

Proof. Assume that A separates ideals in B . Suppose first that B is purely infinite. By Proposition 2.21, for any $b \in B^+ \setminus \{0\}$, $E(b)$ is in the ideal in B generated by b . Then $E(b) \lesssim b$ because b is properly infinite (see [20, Theorem 4.16]). Now suppose that every $a \in A^+ \setminus \{0\}$ is properly infinite in B and $E(b) \lesssim b$ for every $b \in B^+$. Let J be the ideal in B generated by $b \in B^+ \setminus \{0\}$. Proposition 2.21 implies $J \cap A = E(J)$. Since every ideal in B is induced, this implies $J = BE(J)B$. So b is in the ideal generated by $E(b)$. This implies that $b \lesssim E(b)$ using that $E(b)$ is properly infinite (see [20, Proposition 3.5 (ii)]). Since $b \oplus b \lesssim E(b) \oplus E(b) \lesssim E(b) \lesssim b$ in B , we conclude that b is properly infinite. This shows that B is purely infinite.

If A separates ideals in B and B is purely infinite, the same holds for all the quotient inclusions $A/I \subseteq B/J$, $I = J \cap A$, $J \in \mathbb{I}(B)$ (see [20, Proposition 4.3]). By Proposition 2.21, the conditional expectation $E^I: B/J \rightarrow A/I$ is faithful. Hence the first part of the assertion shows that $b \in (B/J)^+ \setminus \{0\}$ implies $0 \neq E^I(b) \lesssim b$. Thus A^+ residually supports B . Then A^+ fills B by Proposition 2.29. ■

3. Inverse semigroup actions and their crossed products

In this section, we briefly recall inverse semigroup actions by Hilbert bimodules and their crossed products, referring to [10, 31] for more details.

3.1. Inverse semigroup actions by Hilbert bimodules

Throughout this paper, S is an inverse semigroup with unit $1 \in S$.

Definition 3.1 ([11]). An *action* of S on a C^* -algebra A (by Hilbert bimodules) consists of Hilbert A -bimodules \mathcal{E}_t for $t \in S$ and Hilbert bimodule isomorphisms $\mu_{t,u}: \mathcal{E}_t \otimes_A \mathcal{E}_u \xrightarrow{\sim} \mathcal{E}_{tu}$ for $t, u \in S$, such that

(A1) for all $t, u, v \in S$, the following diagram commutes (associativity):

$$\begin{array}{ccc}
 (\mathcal{E}_t \otimes_A \mathcal{E}_u) \otimes_A \mathcal{E}_v & \xrightarrow{\mu_{t,u} \otimes_A \text{Id}_{\mathcal{E}_v}} & \mathcal{E}_{tu} \otimes_A \mathcal{E}_v & \xrightarrow{\mu_{tu,v}} & \mathcal{E}_{tuv}; \\
 \text{ass} \uparrow & & & & \\
 \mathcal{E}_t \otimes_A (\mathcal{E}_u \otimes_A \mathcal{E}_v) & \xrightarrow{\text{Id}_{\mathcal{E}_t} \otimes_A \mu_{u,v}} & \mathcal{E}_t \otimes_A \mathcal{E}_{uv} & \xrightarrow{\mu_{t,uv}} &
 \end{array}$$

(A2) \mathcal{E}_1 is the identity Hilbert A, A -bimodule A ;

(A3) $\mu_{t,1}: \mathcal{E}_t \otimes_A A \xrightarrow{\sim} \mathcal{E}_t$ and $\mu_{1,t}: A \otimes_A \mathcal{E}_t \xrightarrow{\sim} \mathcal{E}_t$ for $t \in S$ are the maps defined by $\mu_{1,t}(a \otimes \xi) = a \cdot \xi$ and $\mu_{t,1}(\xi \otimes a) = \xi \cdot a$ for $a \in A, \xi \in \mathcal{E}_t$.

Any S -action by Hilbert bimodules comes with canonical involutions $J_t: \mathcal{E}_t^* \rightarrow \mathcal{E}_{t^*}$ and inclusion maps $j_{u,t}: \mathcal{E}_t \rightarrow \mathcal{E}_u$ for $t \leq u$ that satisfy the conditions required for a saturated Fell bundle in [15] (see [11, Theorem 4.8]). Thus S -actions by Hilbert bimodules are equivalent to saturated Fell bundles over S . A nonsaturated Fell bundle over S is turned into a saturated Fell bundle over another inverse semigroup in [9], such that the full and reduced section C^* -algebras stay the same. Therefore, we usually restrict attention to saturated Fell bundles, which we may replace by inverse semigroup actions as in Definition 3.1. Definition 3.1 contains (twisted) actions by partial automorphisms, providing the following example.

Example 3.2 (Twisted actions of inverse semigroups, see [7, Definition 4.1]). A *twisted action of an inverse semigroup S by partial automorphisms* on a C^* -algebra A consists of partial automorphisms $\alpha_t: D_{t^*} \rightarrow D_t$ of A for $t \in S$ – that is, D_t is an ideal in A and α_t is a $*$ -isomorphism – and unitary multipliers $\omega(t, u) \in \mathcal{UM}(D_{tu})$ for $t, u \in S$, such that $D_1 = A$ and the following conditions hold for $r, t, u \in S$ and $e, f \in E(S) := \{s \in S : s^2 = s\}$:

- (1) $\alpha_r \circ \alpha_t = \text{Ad}_{\omega(r,t)} \alpha_{rt}$;
- (2) $\alpha_r(a\omega(t, u))\omega(r, tu) = \alpha_r(a)\omega(r, t)\omega(rt, u)$ for $a \in D_{r^*} \cap D_{tu}$;
- (3) $\omega(e, f) = 1_{ef}$ and $\omega(r, r^*r) = \omega(rr^*, r) = 1_r$, where 1_r is the unit of $\mathcal{M}(D_r)$;
- (4) $\omega(t^*, e)\omega(t^*e, t)a = \omega(t^*, t)a$ for all $a \in D_{t^*et}$.

Let $((\alpha_t)_{t \in S}, \omega(t, u))_{t, u \in S}$ be a twisted action as above. For $t \in S$, let \mathcal{E}_t be the Hilbert A -bimodule associated to the partial homeomorphism α_t ; this is D_t as a Banach space, and we denote elements by $b\delta_t$ to highlight the $t \in S$ in which we view $b \in D_t \subseteq A$ as an element; the Hilbert A -bimodule structure is defined by

$$\begin{aligned} a \cdot (b\delta_t) &:= (ab)\delta_t, & (b\delta_t) \cdot a &:= \alpha_t(\alpha_t^{-1}(b)a)\delta_t, \\ \langle\langle c\delta_t | b\delta_t \rangle\rangle &:= cb^*, & \langle\langle c\delta_t | b\delta_t \rangle\rangle &:= \alpha_t^{-1}(c^*b) \end{aligned}$$

for $a \in A, b, c \in D_t$. The formula

$$\mu_{t,u}(a\delta_t \otimes b\delta_u) = \alpha_t(\alpha_t^{-1}(b)a)\omega(t, u)\delta_t$$

well defines a Hilbert bimodule isomorphism $\mu_{t,u}: \mathcal{E}_t \otimes_A \mathcal{E}_u \xrightarrow{\sim} \mathcal{E}_{tu}$ for $t, u \in S$ by [7, Theorem 4.12]. And $(\mathcal{E}_t, \mu_{t,u})_{t, u \in S}$ is an action of S on A by Hilbert bimodules. Fell bundles over S that come from twisted partial actions are characterised in [7, Corollary 4.16], where they are called ‘‘regular’’.

Example 3.3 (Inverse semigroup gradings). An S -grading $(B_t)_{t \in S}$ of a C^* -algebra B as in Definition 2.9 gives a Fell bundle over S , using the multiplication and involution in B . This bundle is saturated if and only if the grading is saturated. Then it is an action of S by Hilbert bimodules on $A := B_1 \subseteq B$. Thus inverse semigroup actions by Hilbert bimodules are inevitable in the study of regular inclusions (see also Proposition 2.11). Any inverse semigroup action $\mathcal{E} = ((\mathcal{E}_t)_{t \in S}, (\mu_{t,u})_{t, u \in S})$ on a C^* -algebra A comes from an S -graded C^* -algebra. Namely, embed the spaces \mathcal{E}_t for $t \in S$ into, say, the full crossed product. They form an S -grading of the full crossed product.

Fell bundles over étale groupoids may be described through S -actions, as in the following example.

Example 3.4 (Fell bundles over groupoids). Let G be an étale groupoid with locally compact and Hausdorff unit space X . So the range and source maps $r, s: G \rightrightarrows X$ are local homeomorphisms. A Fell bundle over G is defined, for instance, in [8, Section 2]. It is an upper semicontinuous bundle $\mathcal{A} = (A_\gamma)_{\gamma \in G}$ of complex Banach spaces equipped with a continuous involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ and a continuous multiplication $\cdot: \{(a, b) \in \mathcal{A} \times \mathcal{A} : a \in A_{\gamma_1}, b \in A_{\gamma_2}, (\gamma_1, \gamma_2) \in G^{(2)}\} \rightarrow \mathcal{A}$, which satisfy some natural properties. The set of (open) bisections

$$\text{Bis}(G) := \{U \subseteq G : U \text{ is open and } s|_U, r|_U \text{ are injective}\}$$

is a unital inverse semigroup with $U \cdot V := \{\gamma \cdot \eta : \gamma \in U, \eta \in V\}$ for $U, V \in \text{Bis}(G)$. Namely, $X \in \text{Bis}(G)$ is the unit element and $U^* := \{\gamma^{-1} : \gamma \in U\}$ for $U \in \text{Bis}(G)$. Let A_U for $U \in \text{Bis}(G)$ be the space of continuous sections of $(A_\gamma)_{\gamma \in G}$ vanishing outside U . The spaces $A_{r(U)}$ and $A_{s(U)}$ are closed two-sided ideals in $A = A_X$, and A_U becomes a Hilbert $A_{r(U)}$ - $A_{s(U)}$ -bimodule with the bimodule structure $(a \cdot \xi \cdot b)(\gamma) := a(r(\gamma))\xi(\gamma)b(s(\gamma))$ and the right and left inner products

$$\langle\langle \xi | \eta \rangle\rangle(x) := \xi(s|_U^{-1}(x))^* \eta(s|_U^{-1}(x)), \quad \langle\langle \xi | \eta \rangle\rangle(x) := \xi(r|_U^{-1}(x)) \eta(r|_U^{-1}(x))^*.$$

For $U, V \in S$, the formula

$$\mu_{U,V}(\xi \otimes \eta)(\gamma) := \xi(r|_U^{-1}(r(\gamma)))\eta(s|_V^{-1}(s(\gamma))), \quad \gamma \in UV,$$

defines a Hilbert bimodule map $\mu_{U,V}: A_U \otimes_A A_V \rightarrow A_{UV}$. This data defines a Fell bundle over $\text{Bis}(G)$, which is saturated if \mathcal{A} is (see [8, 31] for details). If \mathcal{A} is not saturated, this may be naturally turned into a saturated Fell bundle over another inverse semigroup in a number of ways. For instance, for any inverse subsemigroup $S \subseteq \text{Bis}(G)$ we may let \tilde{S} be the family of all Hilbert subbimodules of A_U for $U \in S$. Equivalently, elements of \tilde{S} are of the form $A_U \cdot I$ for $U \in S$ and $I \triangleleft A$. Then \tilde{S} , with operations defined as above, forms an inverse semigroup that acts by Hilbert bimodules on A (see [31, Lemma 7.3]).

Let A be a C^* -algebra with an action \mathcal{E} of a unital inverse semigroup S . Let \hat{A} and $\check{A} = \text{Prim}(A)$ be the space of irreducible representations and the primitive ideal space of A , respectively. The action of S on A induces actions $\hat{\mathcal{E}} = (\hat{\mathcal{E}}_t)_{t \in S}$ and $\check{\mathcal{E}} = (\check{\mathcal{E}}_t)_{t \in S}$ of S by partial homeomorphisms on \hat{A} and \check{A} , respectively (see [11, Lemma 6.12], [31, Section 2.3]). The homeomorphisms

$$\check{\mathcal{E}}_t: s(\check{\mathcal{E}})_t \xrightarrow{\sim} r(\check{\mathcal{E}})_t, \quad \hat{\mathcal{E}}_t: s(\hat{\mathcal{E}})_t \xrightarrow{\sim} r(\hat{\mathcal{E}})_t$$

are given by Rieffel’s correspondence and induction of representations, respectively. Any action by partial homeomorphisms has a transformation groupoid, which is étale (see [14, Section 4] or [31, Section 2.1] for details).

Definition 3.5 ([30, 31]). We call $\hat{\mathcal{E}}$ and $\check{\mathcal{E}}$ *dual actions* to the action \mathcal{E} of S on A . The transformation groupoids $\hat{A} \rtimes S$ and $\check{A} \rtimes S$ are called *dual groupoids* of \mathcal{E} .

Example 3.6 (Dual groupoids to Fell bundles). Let \mathcal{A} be a Fell bundle over an étale groupoid G with locally compact Hausdorff object space X . Then G acts naturally both on the primitive ideal space \check{A} and the spectrum \hat{A} of the C^* -algebra $A := C_0(X, \mathcal{A})$. More specifically, every irreducible representation of A factors through the evaluation map $A \rightarrow A_x$ for some $x \in X$, and this defines a continuous map $\psi: \hat{A} \rightarrow X$, which is the anchor map of the G -action on \hat{A} given by the partial homeomorphisms $\psi_\gamma: \widehat{A_{s(\gamma)}} \rightarrow \widehat{A_{r(\gamma)}}$ induced by the Hilbert $A_{r(\gamma)}\text{-}A_{s(\gamma)}$ -bimodules A_γ for $\gamma \in G$ (see, for instance, [18, Section 2]). The corresponding transformation groupoid is

$$\hat{A} \rtimes G := \{([\pi], \gamma) \in \hat{A} \times G : \psi([\pi]) = s(\gamma)\}.$$

Two elements $([\rho], \eta)$ and $([\pi], \gamma)$ are composable if and only if $[\rho] = \psi_\gamma([\pi])$, and then their composite is $([\pi], \eta\gamma)$. The inverse is $([\pi], \gamma)^{-1} = (\psi_\gamma([\pi]), \gamma^{-1})$. The maps $\psi: \hat{A} \rightarrow X$ and $\psi_\gamma: \widehat{A_{s(\gamma)}} \rightarrow \widehat{A_{r(\gamma)}}$ for $\gamma \in G$, factor through to a G -action on \check{A} , which defines a transformation groupoid $\check{A} \rtimes G$ (see [18]). These actions give rise to transformation groupoids $\check{A} \rtimes G$ and $\hat{A} \rtimes G$. Let $S \subseteq \text{Bis}(G)$ be a unital, inverse subsemigroup of bisections of G which is *wide* in the sense that $\bigcup S = G$ and $U \cap V$ is a union of bisections

in S for all $U, V \in S$. Then turning this into the inverse semigroup action \tilde{S} on A described in Example 3.4, we have natural isomorphisms of groupoids (see [31, Remark 7.4])

$$\hat{A} \rtimes G \cong \hat{A} \rtimes \tilde{S}, \quad \hat{A} \rtimes G \cong \hat{A} \rtimes \tilde{S}.$$

We call $\check{A} \rtimes G$ and $\hat{A} \rtimes G$ *dual groupoids* for \mathcal{A} .

3.2. Crossed products

Fix an action $\mathcal{E} = ((\mathcal{E}_t)_{t \in S}, (\mu_{t,u})_{t,u \in S})$ of S on A . For any $t \in S$, let $r(\mathcal{E}_t)$ and $s(\mathcal{E}_t)$ be the ideals in A generated by the left and right inner products of vectors in \mathcal{E}_t , respectively. Thus \mathcal{E}_t is an $r(\mathcal{E}_t)$ - $s(\mathcal{E}_t)$ -imprimitivity bimodule. If $v \leq t$, then the inclusion map $j_{t,v}$ restricts to a Hilbert bimodule isomorphism $\mathcal{E}_v \xrightarrow{\cong} r(\mathcal{E}_v) \cdot \mathcal{E}_t = \mathcal{E}_t \cdot s(\mathcal{E}_v)$. For $t, u \in S$ and for all $v \leq t, u$ this gives Hilbert bimodule isomorphisms

$$\vartheta_{u,t}^v : \mathcal{E}_t \cdot s(\mathcal{E}_v) \xleftarrow[\cong]{j_{t,v}} \mathcal{E}_v \xrightarrow[\cong]{j_{u,v}} \mathcal{E}_u \cdot s(\mathcal{E}_v).$$

Let

$$I_{t,u} := \overline{\sum_{v \leq t,u} s(\mathcal{E}_v)} \tag{3.7}$$

be the closed ideal generated by $s(\mathcal{E}_v)$ for $v \leq t, u$. It is contained in $s(\mathcal{E}_t) \cap s(\mathcal{E}_u)$, and the inclusion may be strict. There is a unique Hilbert bimodule isomorphism

$$\vartheta_{u,t} : \mathcal{E}_t \cdot I_{t,u} \xrightarrow{\cong} \mathcal{E}_u \cdot I_{t,u} \tag{3.8}$$

that restricts to $\vartheta_{u,t}^v$ on $\mathcal{E}_t \cdot s(\mathcal{E}_v)$ for all $v \leq t, u$ by [10, Lemma 2.4]. The $*$ -algebra $A \rtimes_{\text{alg}} S$ of the action \mathcal{E} is defined as the quotient vector space of $\bigoplus_{t \in S} \mathcal{E}_t$ by the linear span of $\vartheta_{u,t}(\xi)\delta_u - \xi\delta_t$ for all $t, u \in S$ and $\xi \in \mathcal{E}_t \cdot I_{t,u}$. The algebraic structure on $A \rtimes_{\text{alg}} S$ is given by the multiplication maps $\mu_{t,u}$ and the involutions J_t .

Definition 3.9. The (full) *crossed product* $A \rtimes S$ of the action \mathcal{E} is the maximal C^* -completion of the $*$ -algebra $A \rtimes_{\text{alg}} S$ described above.

Definition 3.10. A *representation* of \mathcal{E} in a C^* -algebra B is a family of linear maps $\pi_t : \mathcal{E}_t \rightarrow B$ for $t \in S$ such that $\pi_{tu}(\mu_{t,u}(\xi \otimes \eta)) = \pi_t(\xi)\pi_u(\eta)$, $\pi_t(\xi_1)^* \pi_t(\xi_2) = \pi_1(\langle \xi_1 | \xi_2 \rangle)$, and $\pi_t(\xi_1)\pi_t(\xi_2)^* = \pi_1(\langle \xi_1 | \xi_2 \rangle)$ for all $t, u \in S$, $\xi, \xi_1, \xi_2 \in \mathcal{E}_t$, $\eta \in \mathcal{E}_u$. The representation is called *injective* if π_1 is injective; then all the maps π_t for $t \in S$ are isometric.

Any representation π of \mathcal{E} in B induces a $*$ -homomorphism $\pi \rtimes S : A \rtimes S \rightarrow B$. Conversely, every $*$ -homomorphism $A \rtimes S \rightarrow B$ is equal to $\pi \rtimes S$ for a unique representation π (compare [10, Proposition 2.9]). This universal property determines $A \rtimes S$ uniquely up to isomorphism.

The C^* -algebra $A \rtimes S$ is canonically isomorphic to the full section C^* -algebra of the Fell bundle over S corresponding to \mathcal{E} . The reduced section C^* -algebra of a Fell bundle

over S was first defined using inducing pure states, see [15]. An equivalent definition appears in [10], where this is called the reduced crossed product $A \rtimes_r S$ of the action \mathcal{E} (the reduced C^* -algebra obtained in [5], using a regular representation, is in general different). The main ingredient in the construction in [10] is the weak conditional expectation for $A \rtimes S \rightarrow A''$ described in [10, Lemma 4.5] through the formula

$$E(\xi\delta_t) = \text{s-lim}_i \vartheta_{1,t}(\xi \cdot u_i), \tag{3.11}$$

where $\xi \in \mathcal{E}_t$, $t \in S$, (u_i) is an approximate unit for $I_{1,t}$ and s-lim denotes the limit in the strict topology on $\mathcal{M}(I_{1,t}) \subseteq A''$. This weak expectation is symmetric by [31, Theorem 3.22].

Definition 3.12. The *reduced crossed product* is the quotient C^* -algebra $A \rtimes_r S := (A \rtimes S) / \mathcal{N}_E$. Hence there is a canonical surjection $\Lambda: A \rtimes S \rightarrow A \rtimes_r S$, and

$$\ker \Lambda = \mathcal{N}_E = \{b \in A \rtimes S : E(b^*b) = 0\}. \tag{3.13}$$

So the induced weak expectation E_r on $A \rtimes_r S$ is faithful.

Remark 3.14. The canonical maps from $A \rtimes_{\text{alg}} S$ to $A \rtimes S$ and to $A \rtimes_r S$ are injective by [10, Proposition 4.3]. In particular, both $A \rtimes S$ and $A \rtimes_r S$ are naturally S -graded with the same Fell bundle $(\mathcal{E}_t)_{t \in S}$ over S .

Definition 3.15. The action $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ is called *closed* if the weak expectation $E: A \rtimes S \rightarrow A''$ given by (3.11) is A -valued. So it is a genuine conditional expectation $A \rtimes S \rightarrow A \subseteq A \rtimes S$.

Remark 3.16. The action \mathcal{E} is closed if and only if the unit space \hat{A} is closed in the dual groupoid $\hat{A} \rtimes S$ or, equivalently, \check{A} is closed in $\check{A} \rtimes S$ (see [10, Theorem 6.5] and [31, Proposition 3.20]). This explains the name. By [10, Proposition 6.3], \mathcal{E} is closed if and only if the ideal $I_{t,1}$ defined in (3.7) is complemented in $s(\mathcal{E}_t)$ for each $t \in S$.

Example 3.17 (C^* -algebras of Fell bundles over groupoids). Retain the notation from Example 3.4. The $*$ -algebra associated to the Fell bundle \mathcal{A} over the étale groupoid G is denoted by $\mathfrak{S}(G, \mathcal{A})$. It is the linear span of compactly supported continuous sections $A_U = C_c(U, \mathcal{A})$ for all bisections $U \in \text{Bis}(G)$ with a convolution and involution given by

$$(f * g)(\gamma) := \sum_{r(\eta)=r(\gamma)} f(\eta) \cdot g(\eta^{-1} \cdot \gamma), \quad (f^*)(\gamma) := f(\gamma^{-1})^*$$

for all $f, g \in \mathfrak{S}(G, \mathcal{A})$, $\gamma \in G$. The *full section C^* -algebra* $C^*(G, \mathcal{A})$ is defined as the maximal C^* -completion of the $*$ -algebra $\mathfrak{S}(G, \mathcal{A})$. This C^* -algebra contains $A := C^*(X, \mathcal{A})$ as a C^* -subalgebra. It is equipped with a generalised expectation $E: C^*(G, \mathcal{A}) \rightarrow \mathfrak{B}(X, \mathcal{A})$, where $\mathfrak{B}(X, \mathcal{A})$ is the C^* -algebra of bounded Borel sections of the C^* -bundle $\mathcal{A}|_X$ and E on $\mathfrak{S}(G, \mathcal{A})$ restricts sections to X . The *reduced section C^* -algebra* can be defined as the quotient

$$C_r^*(G, \mathcal{A}) := C^*(G, \mathcal{A}) / \mathcal{N}_E.$$

All this follows from [31, Proposition 7.10]. Let $S \subseteq \text{Bis}(G)$ be a unital, inverse sub-semigroup of bisections of G which is *wide* in the sense that $\bigcup S = G$ and $U \cap V$ is a union of bisections in S for all $U, V \in S$. If \mathcal{A} is saturated, then the spaces $(A_U)_{U \in S}$, with operations inherited from $\mathfrak{S}(G, \mathcal{A})$, form an action of S on A by Hilbert bimodules, and the associated crossed products are isomorphic to the corresponding section C^* -algebras. In general, we modify the construction as follows. The convolution and multiplication from $\mathfrak{S}(G, \mathcal{A})$ make $(A_U \cdot I)_{U \in S, I \triangleleft A}$ an action by Hilbert bimodules on A , and there are natural isomorphisms

$$C^*(G, \mathcal{A}) \cong A \rtimes \tilde{S}, \quad C_r^*(G, \mathcal{A}) \cong A \rtimes_r \tilde{S}$$

(see [31, Propositions 7.6 and 7.9]). If the groupoid G is Hausdorff, that is, the unit space X is closed in G , then the inverse semigroup action \tilde{S} is closed. The converse implication holds if \mathcal{A} is a Fell line bundle. Fell line bundles over G are equivalent to twists of G . If (G, Σ) is a twisted groupoid and \mathcal{L} the corresponding line bundle, then $C^*(G, \Sigma) \cong C^*(G, \mathcal{L}) \cong A \rtimes S$ and $C_r^*(G, \Sigma) \cong C_r^*(G, \mathcal{L}) \cong A \rtimes_r S$. That is, twisted groupoid C^* -algebras are also modelled by inverse semigroup actions.

3.3. Essential crossed products

The *local multiplier algebra* $\mathcal{M}_{\text{loc}}(A)$ of A is the inductive limit of the multiplier algebras $\mathcal{M}(J)$, where J runs through the directed set of essential ideals in A (see [4]). A key idea in [31] is a natural generalised expectation $EL: A \rtimes S \rightarrow \mathcal{M}_{\text{loc}}(A)$ with values in $\mathcal{M}_{\text{loc}}(A)$. It is defined as follows: for each $\xi \in \mathcal{E}_t, t \in S$, the element $EL(\xi) \in \mathcal{M}(I_{1,t} \oplus I_{1,t}^\perp) \subseteq \mathcal{M}_{\text{loc}}(A)$ is given by

$$EL(\xi)(u + v) := \vartheta_{1,t}(\xi u) \in A$$

for $u \in I_{1,t}, v \in I_{1,t}^\perp$. This generalised expectation is symmetric by [31, Theorem 4.11]. Hence EL factors through a faithful pseudo-expectation on the quotient

$$A \rtimes_{\text{ess}} S := (A \rtimes S) / \mathcal{N}_{EL}.$$

There is a canonical embedding $\iota: \mathcal{M}_{\text{loc}}(A) \hookrightarrow I(A)$ compatible with the inclusions $A \subseteq \mathcal{M}_{\text{loc}}(A)$ and $A \subseteq I(A)$ (see [16, Theorem 1]). Thus the *canonical \mathcal{M}_{loc} -expectation EL* may be viewed as a pseudo-expectation.

Definition 3.18 ([31, Definition 4.4]). We call $A \rtimes_{\text{ess}} S$ the *essential crossed product*.

For a closed action, both E and EL take values in A and then $A \rtimes_{\text{ess}} S = A \rtimes_r S$. In general, $\mathcal{N}_E \subseteq \mathcal{N}_{EL}$ and there are surjective maps

$$A \rtimes S \twoheadrightarrow A \rtimes_r S \twoheadrightarrow A \rtimes_{\text{ess}} S.$$

A C^* -algebra B with $*$ -epimorphisms $A \rtimes S \twoheadrightarrow B \twoheadrightarrow A \rtimes_{\text{ess}} S$ that compose to the canonical quotient map $A \rtimes S \twoheadrightarrow A \rtimes_{\text{ess}} S$ is called an *exotic crossed product* (see [31]). The following proposition characterises when the reduced and essential crossed products coincide.

Proposition 3.19 ([31, Corollary 4.17]). *We have $A \rtimes_{\text{ess}} S = A \rtimes_r S$ if and only if for every $b \in (A \rtimes S)^+ \setminus \{0\}$ there is $\varepsilon > 0$ such that $\{\pi \in \widehat{A} : \|\pi''(E(b))\| > \varepsilon\}$ has nonempty interior, if and only if for every $b \in (A \rtimes S)^+ \setminus \{0\}$ the set $\{\pi \in \widehat{A} : \|\pi''(E(b))\| \neq 0\}$ is not meagre in \widehat{A} .*

Example 3.20 (Essential section C^* -algebras). Let \mathcal{A} be a Fell bundle over an étale groupoid G with locally compact Hausdorff unit space X . There is a canonical generalised expectation $EL: C^*(G, \mathcal{A}) \rightarrow \mathcal{M}_{\text{loc}}(A)$ (see [31, Section 7.4]). Let

$$\mathfrak{M}(X, \mathcal{A}) := \{f \in \mathfrak{B}(X, \mathcal{A}) : f \text{ vanishes on a comeagre set}\}.$$

If the bundle $\mathcal{A}|_X$ is continuous, then there is a natural embedding

$$\mathcal{M}_{\text{loc}}(A) \hookrightarrow \mathfrak{B}(X, \mathcal{A})/\mathfrak{M}(X, \mathcal{A}),$$

and EL is the composite of E and the quotient map $\mathfrak{B}(X, \mathcal{A}) \twoheadrightarrow \mathfrak{B}(X, \mathcal{A})/\mathfrak{M}(X, \mathcal{A})$. If the bundle is discontinuous, we define EL using the isomorphism $C^*(G, \mathcal{A}) \cong A \rtimes \widetilde{S}$ from Example 3.17. The *essential section C^* -algebra* is defined in [31, Definition 7.12] as the quotient

$$C_{\text{ess}}^*(G, \mathcal{A}) := C^*(G, \mathcal{A})/\mathcal{N}_{EL}.$$

Example 3.21 (Essential twisted groupoid C^* -algebras). We define the *essential groupoid C^* -algebra* $C_{\text{ess}}^*(G, \Sigma)$ of a twisted groupoid (G, Σ) as $C_{\text{ess}}^*(G, \mathcal{L})$ for the corresponding Fell line bundle \mathcal{L} . Denoting by X the unit space of G , [31, Proposition 7.18] implies that the following are equivalent:

- (1) $C_r^*(G, \Sigma) = C_{\text{ess}}^*(G, \Sigma)$;
- (2) $\{x \in X : E_r(f)(x) \neq 0\}$ is not meagre for every $f \in C_r^*(G, \Sigma)^+ \setminus \{0\}$;
- (3) if $f \in C_r^*(G, \Sigma)^+ \setminus \{0\}$, then $\{x \in X : \|E_r(f)(x)\| > \varepsilon\}$ has nonempty interior for some $\varepsilon > 0$.

Here $E_r: C_r^*(G, \Sigma) \rightarrow \mathfrak{B}(X)$ is the canonical generalised expectation that restricts sections of the corresponding Fell line bundle to X .

Example 3.22 (Twisted crossed products by partial automorphisms). Let (α, ω) be a twisted action of an inverse semigroup S by partial automorphisms on a C^* -algebra A as in Example 3.2. By [7, Definition 6.2], a *covariant representation* of (α, ω) on a Hilbert space \mathcal{H} is a pair (ρ, v) consisting of a $*$ -homomorphism $\rho: A \rightarrow \mathbb{B}(\mathcal{H})$ and a family $v = (v_t)_{t \in S}$ of partial isometries in $\mathbb{B}(\mathcal{H})$ such that

$$\rho(\alpha_t(b)) = v_t \rho(b) v_t^*, \quad \bar{\rho}(\omega(t, u)) = v_t v_u v_{tu}^*, \quad v_t v_t^* = \bar{\rho}(1_{t^*t}), \quad v_t^* v_t = \bar{\rho}(1_{t^*t}),$$

for all $b \in D_t$, $t, u \in S$. Here $\bar{\rho}$ is the extension of ρ to the enveloping von Neumann algebra of A , so that $\bar{\rho}(\omega(t, u))$ and $\bar{\rho}(1_e)$ make sense. By definition, the *full crossed product* for (α, ω) is the universal C^* -algebra for covariant representations, and by [7, Theorem 6.3] it is naturally isomorphic to the full crossed product $A \rtimes S$ for the associated inverse semigroup action \mathcal{E} by Hilbert bimodules (see Example 3.2). The *reduced*

crossed product for (α, ω) may be identified with the reduced crossed product $A \rtimes_r S$ by [7, Definition 6.6]. We define the *essential crossed product* for (α, ω) as $A \rtimes_{\text{ess}} S$. By [7, Theorem 7.2], for any twisted groupoid (G, Σ) the bisections S of G that trivialise the twist Σ give rise to a twisted inverse semigroup action (α, ω) by partial automorphisms of $A := C_0(G^0)$ such that $C_r^*(G, \Sigma) \cong A \rtimes_r S$. By definition, this descends to an isomorphism $C_{\text{ess}}^*(G, \Sigma) \cong A \rtimes_{\text{ess}} S$.

The reduced section C^* -algebra $C_r^*(G, \mathcal{A})$ is usually defined through the regular representation, which is the direct sum of representations

$$\lambda_x: C^*(G, \mathcal{A}) \rightarrow \mathbb{B}(\ell^2(G_x, \mathcal{A})).$$

Here $\ell^2(G_x, \mathcal{A})$ is the Hilbert A_x -module completion of $\bigoplus_{s(\gamma)=x} A_\gamma$ with the obvious right multiplication and the standard inner product $\langle f | g \rangle := \sum_{s(\gamma)=x} f(\gamma)^* g(\gamma)$. For $f \in \mathfrak{S}(H, \mathcal{A})$ and $g \in \bigoplus_{s(\gamma)=x} A_\gamma$, define $\lambda_x(f)(g)(\gamma) := \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma)$. The kernel of $\bigoplus_{x \in X} \lambda_x$ is \mathcal{N}_E . So the reduced C^* -algebra $C_r^*(H, \mathcal{A})$ is isomorphic to the completion of $\mathfrak{S}(H, \mathcal{A})$ in the reduced norm $\|f\|_r := \sup_{x \in X} \|\lambda_x(f)\|$. We now describe essential algebras in a similar fashion.

Definition 3.23 ([31, Definition 7.14]). Call $x \in X$ *dangerous* if there is a net (γ_n) in G that converges towards two different points $\gamma \neq \gamma'$ in G with $s(\gamma) = s(\gamma') = x$.

Proposition 3.24. *Let \mathcal{A} be a continuous Fell bundle over an étale groupoid G with locally compact and Hausdorff unit space X . Assume G is covered by countably many bisections and let $D \subseteq X$ be the set of dangerous points. Then*

$$\ker \bigoplus_{x \in X \setminus D} \lambda_x = \mathcal{N}_{EL}.$$

That is, $C_{\text{ess}}^*(G, \mathcal{A})$ is isomorphic to the Hausdorff completion of $\mathfrak{S}(G, \mathcal{A})$ in the seminorm $\|f\|_{\text{ess}} := \sup_{x \in X \setminus D} \|\lambda_x(f)\|$.

Proof. For each $x \in X$, $\gamma \in G_x = s^{-1}(x)$ and $f \in C^*(G, \mathcal{A})$,

$$\|\lambda_\gamma(f)1_\gamma\|^2 = \langle 1_\gamma | \lambda_x(f^* f)1_\gamma \rangle = E(f^* f)(r(\gamma)).$$

Thus $\ker \lambda_x = \{f \in C^*(G, \mathcal{A}) : E(f^* f)(r(G_x)) = 0\}$. We claim that the set of dangerous points is G -invariant. Indeed, if $\eta \in s^{-1}(x)$ and there is a net (γ_n) that converges towards two different $\gamma \neq \gamma' \in H$ with $s(\gamma) = s(\gamma') = x$, then the net $(\gamma_n \eta^{-1})$ converges to $\gamma \eta^{-1} \neq \gamma' \eta^{-1} \in H$ with $s(\gamma \eta^{-1}) = s(\gamma' \eta^{-1}) = r(\eta)$. Hence $x \in D$ implies that $r(G_x) \subseteq D$. Then

$$\ker \bigoplus_{x \in X \setminus D} \lambda_x = \bigcap_{x \in X \setminus D} \ker \lambda_x = \{f \in C^*(G, \mathcal{A}) : E(f^* f)(x) = 0 \text{ for all } x \in X \setminus D\}.$$

The set on the right hand side is equal to \mathcal{N}_{EL} by [31, Proposition 7.18]. ■

4. Exactness of inverse semigroup actions

4.1. Functoriality

For group actions, the full and reduced crossed products are functors, and the reduced one preserves injective homomorphisms. We extend this to inverse semigroup actions by Hilbert bimodules.

Definition 4.1. Let $\mathcal{E} = ((\mathcal{E}_t)_{t \in S}, (\mu_{t,u})_{t,u \in S})$ and $\mathcal{F} = ((\mathcal{F}_t)_{t \in S}, (\nu_{t,u})_{t,u \in S})$ be two actions of S by Hilbert bimodules on A and B , respectively. A *homomorphism* ψ from \mathcal{E} to \mathcal{F} is a family of linear maps $\psi_t: \mathcal{E}_t \rightarrow \mathcal{F}_t$ for $t \in S$ such that for all $t, u \in S, \xi \in \mathcal{E}_t, \eta \in \mathcal{E}_u$ we have

$$\begin{aligned} \psi_{tu}(\mu_{t,u}(\xi \otimes \eta)) &= \nu_{t,u}(\psi_t(\xi) \otimes \psi_u(\eta)), \\ \langle \psi_t(\xi) \mid \psi_t(\eta) \rangle &= \psi_1(\langle \xi \mid \eta \rangle), \\ \langle\langle \psi_t(\xi) \mid \psi_t(\eta) \rangle\rangle &= \psi_1(\langle\langle \xi \mid \eta \rangle\rangle). \end{aligned}$$

The maps ψ_t are always contractive. We call ψ *injective* if ψ_1 is injective; then the maps ψ_t are isometric for all $t \in S$. We call ψ an *isomorphism* if all ψ_t are isomorphisms.

We use the superscripts \mathcal{E} and \mathcal{F} to distinguish between objects defined for the actions \mathcal{E} and \mathcal{F} .

Proposition 4.2. *Let ψ be a homomorphism from an action $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ on A to an action $\mathcal{F} = (\mathcal{F}_t)_{t \in S}$ on B . It induces a $*$ -homomorphism $\psi \rtimes S: A \rtimes S \rightarrow B \rtimes S$ where $(\psi \rtimes S)(\xi) = \psi_t(\xi)$ for $\xi \in \mathcal{E}_t, t \in S$. In particular, ψ respects the involution and inclusions maps on \mathcal{E} and \mathcal{F} , and $\psi \rtimes S$ restricts to a $*$ -homomorphism $\psi \rtimes_{\text{alg}} S: A \rtimes_{\text{alg}} S \rightarrow B \rtimes_{\text{alg}} S$. Moreover, the following conditions are equivalent:*

- (1) $\psi \rtimes S$ descends to a $*$ -homomorphism $\psi \rtimes_r S: A \rtimes_r S \rightarrow B \rtimes_r S$ that respects the canonical weak expectations, that is, the following diagram commutes:

$$\begin{array}{ccccccc} A \rtimes_{\text{alg}} S & \xrightarrow{\subseteq} & A \rtimes S & \xrightarrow{\Lambda^{\mathcal{E}}} & A \rtimes_r S & \xrightarrow{E^{\mathcal{E}}} & A'' \\ \downarrow \psi \rtimes_{\text{alg}} S & & \downarrow \psi \rtimes S & & \downarrow \psi \rtimes_r S & & \downarrow \psi''_1 \\ B \rtimes_{\text{alg}} S & \xrightarrow{\subseteq} & B \rtimes S & \xrightarrow{\Lambda^{\mathcal{F}}} & B \rtimes_r S & \xrightarrow{E^{\mathcal{F}}} & B'' \end{array}$$

where $\Lambda^{\mathcal{E}}$ (resp. $\Lambda^{\mathcal{F}}$) is the regular representation and $E^{\mathcal{E}}$ (resp. $E^{\mathcal{F}}$) is the canonical weak conditional expectation associated to the action \mathcal{E} (resp. \mathcal{F}).

- (2) $\psi''_1([I_{1,t}^{\mathcal{E}}]) = \psi''_1([s(\mathcal{E}_t)]) [I_{1,t}^{\mathcal{F}}]$, for all $t \in S$, where $[I_{1,t}^{\mathcal{E}}], [s(\mathcal{E}_t)] \in A''$ and $[I_{1,t}^{\mathcal{F}}] \in B''$ are the support projections of the ideals $I_{1,t}^{\mathcal{E}}, s(\mathcal{E}_t) \subseteq A$ and $I_{1,t}^{\mathcal{F}} \subseteq B$, respectively.

If the above equivalent conditions hold and ψ is injective, then so are $\psi \rtimes_r S, \psi \rtimes_{\text{alg}} S$, and ψ''_1 .

Proof. The Hilbert bimodules \mathcal{F}_t for $t \in S$ embed into $B \rtimes S$. Hence we may treat the maps $\psi_t: \mathcal{E}_t \rightarrow \mathcal{F}_t$ as taking values in $B \rtimes S$. Then ψ is a representation of \mathcal{E} in $B \rtimes S$. It integrates to a $*$ -homomorphism $\psi \rtimes S: A \rtimes S \rightarrow B \rtimes S$ by [10, Proposition 2.9]. This restricts to a $*$ -homomorphism $\psi \rtimes_{\text{alg}} S: A \rtimes_{\text{alg}} S \rightarrow B \rtimes_{\text{alg}} S$ and therefore ψ respects the induced involutions and inclusions maps on \mathcal{E} and \mathcal{F} . Our first goal is to show that (1) is equivalent to

$$E^{\mathcal{F}} \circ \Lambda^{\mathcal{F}} \circ (\psi \rtimes S) = \psi_1'' \circ E^{\mathcal{E}} \circ \Lambda^{\mathcal{E}}. \tag{4.3}$$

It is clear that (1) implies (4.3). Conversely, assume (4.3). For any $a \in (A \rtimes S)^+$, we get

$$\begin{aligned} a \in \ker \Lambda^{\mathcal{E}} &\iff E(\Lambda^{\mathcal{E}}(a)) = 0 \implies \psi_1''(E^{\mathcal{E}}(\Lambda^{\mathcal{E}}(a))) = 0 \\ &\iff E^{\mathcal{F}}(\Lambda^{\mathcal{F}}((\psi \rtimes S)(a))) = 0 \iff (\psi \rtimes S)(a) \in \ker \Lambda^{\mathcal{F}}. \end{aligned} \tag{4.4}$$

Hence $(\psi \rtimes S)(\ker \Lambda^{\mathcal{E}}) \subseteq \ker(\Lambda^{\mathcal{F}})$. So $\psi \rtimes S$ descends to a $*$ -homomorphism $\psi \rtimes_r S$ as in (1). If, in addition, ψ is injective, then so is ψ_1'' , and then the only one-sided implication in (4.4) may be reversed. Thus $(\psi \rtimes S)(\ker \Lambda^{\mathcal{E}}) = \ker(\Lambda^{\mathcal{F}})$, so that $\psi \rtimes_r S$ is injective. Since the canonical map $A \rtimes_{\text{alg}} S \rightarrow A \rtimes_r S$ is still injective, it also follows that $\psi \rtimes_{\text{alg}} S$ is injective. So we get all assertions in (1).

Next, we prove that (2) is equivalent to (4.3). By passing to biduals, we get a weakly continuous $*$ -homomorphism $(\psi \rtimes S)'' : (A \rtimes S)'' \rightarrow (B \rtimes S)''$. The C^* -algebra A'' and the Hilbert bimodules \mathcal{E}_t'' for $t \in S$ embed naturally into $(A \rtimes S)''$. Similarly, \mathcal{F}_t'' for $t \in S$ are embedded into $(B \rtimes S)''$. Then $(\psi \rtimes S)''|_{\mathcal{E}_t''} = \psi_t''$ for $t \in S$. Moreover, if $\xi \in \mathcal{E}_t \subseteq A \rtimes S$, then $E^{\mathcal{E}}(\Lambda^{\mathcal{E}}(\xi)) = \xi \cdot [I_{1,t}] \in A''$, where the product is taken in $(A \rtimes S)''$ (see (3.11) or the proof of [10, Lemma 4.5]). Hence (4.3) is equivalent to

$$\psi_t(\xi)\psi_1''([I_{1,t}^{\mathcal{E}}]) = \psi_t(\xi)[I_{1,t}^{\mathcal{F}}], \quad \text{for every } \xi \in \mathcal{E}_t, t \in S, \tag{4.5}$$

Since $\xi = \xi \cdot [s(\mathcal{E}_t)]$ holds inside $(A \rtimes S)''$, (2) implies (4.5),

$$\begin{aligned} \psi_t(\xi)\psi_1''([I_{1,t}^{\mathcal{E}}]) &= \psi_t(\xi)\psi_1''([s(\mathcal{E}_t)])\psi_1''([I_{1,t}^{\mathcal{E}}]) = \psi_t(\xi)\psi_1''([s(\mathcal{E}_t)])[I_{1,t}^{\mathcal{F}}] \\ &= \psi_t(\xi)[I_{1,t}^{\mathcal{F}}]. \end{aligned}$$

Conversely, (4.5) implies $\psi_t(\xi_1)^*\psi_t(\xi_2)\psi_1''([I_{1,t}^{\mathcal{E}}]) = \psi_t(\xi_1)^*\psi_t(\xi_2)[I_{1,t}^{\mathcal{F}}]$ for all $\xi_1, \xi_2 \in \mathcal{E}_t$. Taking linear combinations, we may then replace $\psi_t(\xi_1)^*\psi_t(\xi_2)$ by an approximate unit for the ideal $s(\mathcal{E}_t)$. And then we may take a strong limit over this approximate unit to arrive at $\psi_1''([s(\mathcal{E}_t)])\psi_1''([I_{1,t}^{\mathcal{E}}]) = \psi_1''([s(\mathcal{E}_t)])[I_{1,t}^{\mathcal{F}}]$. The left hand side then simplifies to $\psi_1''([I_{1,t}^{\mathcal{E}}])$ because $I_{1,t}^{\mathcal{E}} \subseteq s(\mathcal{E}_t)$ implies $\psi_1''([I_{1,t}^{\mathcal{E}}]) \leq \psi_1''([s(\mathcal{E}_t)])$. Hence (4.5) implies (2). ■

Remark 4.6. Retain the notation from Proposition 4.2. Since ψ respects inclusions, we have $\psi_1(I_{1,t}^{\mathcal{E}}) \subseteq I_{1,t}^{\mathcal{F}}$ and therefore $\psi_1''([I_{1,t}^{\mathcal{E}}]) \leq [I_{1,t}^{\mathcal{F}}]$. We also have $\psi_1''([I_{1,t}^{\mathcal{E}}]) \leq \psi_1''([s(\mathcal{E}_t)])$ because $I_{1,t}^{\mathcal{E}} \subseteq s(\mathcal{E}_t)$. Thus every representation satisfies

$$\psi_1''([I_{1,t}^{\mathcal{E}}]) \leq \psi_1''([s(\mathcal{E}_t)])[I_{1,t}^{\mathcal{F}}]$$

for all $t \in S$. So condition (2) only asserts the inverse inequality. This condition is always satisfied when $S = G$ is a group, as then $[I_{1,t}^\mathcal{E}] = 0$ for $t \neq 1$ (and $[I_{1,1}^\mathcal{E}] = 1$). For general actions, (2) holds whenever $\psi_1(I_{1,t}^\mathcal{E}) = \psi_1(s(\mathcal{E}_t))I_{1,t}^\mathcal{F}$ for all $t \in S$. The latter equality is automatic for inclusion and quotient homomorphisms in Proposition 4.15 below.

Condition (2) in Proposition 4.2 may fail (and perhaps for some purposes one might want to include it in the definition of a homomorphism). We thank Alcides Buss, Diego Martínez and Jonathan Taylor for point this to us.

Example 4.7. Consider the actions whose crossed products are described in [10, Proposition 8.5]. Namely, let $S = \{-1, 0, 1\}$ be the inverse semigroup with the usual number multiplication. Take any C^* -algebra A and any ideal I in A different from A . Let $\mathcal{E}_1 = \mathcal{E}_{-1} := A$ and $\mathcal{E}_0 = I$ be trivial Hilbert bimodules over A , and let $\mu_{t,u}(a \otimes b) = a \cdot b$ for $t, u \in S$ be just the multiplication in A . Then $A \rtimes^\mathcal{E} S = A \rtimes_r^\mathcal{E} S = A \rtimes_{\text{alg}}^\mathcal{E} S \cong A \oplus A/I$ (see [10, (8.6)]). We let \mathcal{F} be the similar action with I replaced by A . Then $A \rtimes^\mathcal{F} S = A \rtimes_r^\mathcal{F} S = A \rtimes_{\text{alg}}^\mathcal{F} S \cong A$. The inclusion maps yield a homomorphism ψ from \mathcal{E} to \mathcal{F} where $\psi \rtimes S = \psi \rtimes_{\text{alg}} S: A \oplus A/I \rightarrow A$ is given by $(a \oplus b + I) \mapsto a$. This homomorphism is not injective, although ψ is. So the conditions in Proposition 4.2 are not satisfied. Indeed, note that $\psi \rtimes_r S = \psi \rtimes S$ exists in this example, but it does not intertwine the canonical weak expectations, which are given by $E^\mathcal{E}(a \oplus (b + I)) = \frac{a+b}{2} + \frac{a-b}{2}[I]$ and $E^\mathcal{F}(a) = a$, for $a, b \in A$, where $[I] \in A''$ is the support projection of I .

4.2. Restrictions of actions

We fix an action $\mathcal{E} = ((\mathcal{E}_t)_{t \in S}, (\mu_{t,u})_{t,u \in S})$ of a unital inverse semigroup S on a C^* -algebra A by Hilbert bimodules.

Definition 4.8. Let $\mathbb{I}^\mathcal{E}(A) := \{I \in \mathbb{I}(A) : I\mathcal{E}_t = \mathcal{E}_t I \text{ for every } t \in S\}$ be the set of \mathcal{E} -invariant ideals in A .

Lemma 4.9. Let $I \in \mathbb{I}(A)$ and let B be any S -graded C^* -algebra with grading $(\mathcal{E}_t)_{t \in S}$. The following are equivalent:

- (1) I is \mathcal{E} -invariant, that is, $I \in \mathbb{I}^\mathcal{E}(A)$;
- (2) I is restricted, that is, $I \in \mathbb{I}^B(A)$;
- (3) the open subset $\check{I} \subseteq \check{A}$ is invariant in the dual groupoid $\check{A} \rtimes S$;
- (4) the open subset $\hat{I} \subseteq \hat{A}$ is invariant in the dual groupoid $\hat{A} \rtimes S$.

Proof. Proposition 2.13 implies that (1) and (2) are equivalent. It is easy to see that (3) and (4) are equivalent. If $t \in S$, then $I \cdot \mathcal{E}_t = \mathcal{E}_t \cdot I$ is equivalent to $\check{\mathcal{E}}_t(\check{I} \cap \check{D}_t) = \check{I} \cap \check{D}_t^*$ (see [26, page 645] or the proof of [1, Proposition 3.10]). That is, (1) and (3) are equivalent. ■

Let $I \in \mathbb{I}^\mathcal{E}(A)$ be an \mathcal{E} -invariant ideal in A . There are natural induced actions of S on I and A/I . Namely, the family $\mathcal{E}|_I := (\mathcal{E}_t I)_{t \in S}$ of Hilbert I -bimodules, with the restrictions of the isomorphisms $\mu_{t,u}$ to $\mathcal{E}_t I \otimes_A \mathcal{E}_u I = \mathcal{E}_t \otimes_A \mathcal{E}_u I \xrightarrow{\sim} \mathcal{E}_{tu} I$ for $t, u \in S$,

forms an inverse semigroup action by Hilbert bimodules on the C^* -algebra I . For $t \in S$, the quotient Banach space $\mathcal{E}_t/\mathcal{E}_t I$ is a Hilbert A/I -bimodule in a natural way because $\mathcal{E}_t I = I \mathcal{E}_t$. If $t, u \in S$, then the isomorphism $\mu_{t,u}: \mathcal{E}_t \otimes_A \mathcal{E}_u \xrightarrow{\sim} \mathcal{E}_{tu}$ induces an isomorphism

$$\tilde{\mu}_{t,u}: (\mathcal{E}_t \otimes_A \mathcal{E}_u) / (\mathcal{E}_t \otimes_A \mathcal{E}_u I) \xrightarrow{\sim} \mathcal{E}_{tu} / \mathcal{E}_{tu} I.$$

There are natural isomorphisms of Hilbert bimodules

$$q_{t,u}: (\mathcal{E}_t/\mathcal{E}_t I) \otimes_{A/I} (\mathcal{E}_u/\mathcal{E}_u I) \xrightarrow{\sim} (\mathcal{E}_t \otimes_A \mathcal{E}_u) / (\mathcal{E}_t \otimes_A \mathcal{E}_u I)$$

because $I \mathcal{E}_u = \mathcal{E}_u I$ (see, for instance, [25, Lemma 1.8]). Now $\mathcal{E}|_{A/I} := (\mathcal{E}_t/\mathcal{E}_t I)_{t \in S}$ with the isomorphisms $\tilde{\mu}_{t,u} \circ q_{t,u}: (\mathcal{E}_t/\mathcal{E}_t I) \otimes_{A/I} (\mathcal{E}_u/\mathcal{E}_u I) \xrightarrow{\sim} (\mathcal{E}_{tu}/\mathcal{E}_{tu} I)$ for $t, u \in S$ is an action of S by Hilbert bimodules on A/I .

Definition 4.10. We call the actions $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$ above the *restrictions* of \mathcal{E} to I and A/I , respectively.

Remark 4.11. Let $I \in \mathbb{I}^{\mathcal{E}}(A)$. The inclusions $\mathcal{E}_t I \subseteq \mathcal{E}_t$ for $t \in S$ yield an injective homomorphism from $\mathcal{E}|_I$ to \mathcal{E} . The quotient maps $\mathcal{E}_t \rightarrow \mathcal{E}_t/\mathcal{E}_t I$ for $t \in S$ yield a homomorphism from \mathcal{E} to $\mathcal{E}|_{A/I}$.

Remark 4.12. Let B be an S -graded C^* -algebra with grading $(B_t)_{t \in S}$ and let $I \in \mathbb{I}^B(A)$. The induced ideal BIB carries the S -grading $BIB \cap B_t$ by Proposition 2.13. The quotient B/BIB is S -graded by the images of B_t by Lemma 2.14.

Remark 4.13. If I is \mathcal{E} -invariant, then the dual actions for the restrictions $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$ are equal to the restrictions of the action $(\hat{\mathcal{E}}_t)_{t \in S}$ to \hat{I} and $\hat{A} \setminus \hat{I}$, respectively. Let $\hat{I} \rtimes S$ and $\hat{A} \setminus \hat{I} \rtimes S$ denote the transformation groupoids dual to $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$, respectively. Then

$$\hat{I} \rtimes S = (\hat{A} \rtimes S)|_{\hat{I}}, \quad \hat{A} \setminus \hat{I} \rtimes S = (\hat{A} \rtimes S)|_{\hat{A} \setminus \hat{I}},$$

where the right hand sides in the above equalities mean the restrictions of the transformation groupoid $\hat{A} \rtimes S$ to the invariant subsets \hat{I} and $\hat{A} \setminus \hat{I}$ of the unit space \hat{A} of $\hat{A} \rtimes S$. In particular, if the space of units in $\hat{A} \rtimes S$ is closed, then the same holds for the transformation groupoids dual to the restrictions $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$.

Proposition 4.14. *If the action \mathcal{E} on A is closed, then so are the restrictions $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$ for each \mathcal{E} -invariant ideal $I \triangleleft A$.*

Proof. This follows from Remarks 3.16 and 4.13. ■

4.3. Exact actions

Proposition 4.15. *Let \mathcal{E} be an action of S by Hilbert bimodules on a C^* -algebra A and let $I \in \mathbb{I}^{\mathcal{E}}(A)$. Let ι be the injective homomorphism from $\mathcal{E}|_I$ into \mathcal{E} and let κ be the quotient homomorphism from \mathcal{E} onto $\mathcal{E}|_{A/I}$ in Remark 4.11. They induce an exact sequence*

$$0 \rightarrow I \rtimes S \xrightarrow{\iota \rtimes S} A \rtimes S \xrightarrow{\kappa \rtimes S} A/I \rtimes S \rightarrow 0.$$

It descends to a sequence

$$0 \rightarrow I \rtimes_r S \xrightarrow{\iota \rtimes_r S} A \rtimes_r S \xrightarrow{\kappa \rtimes_r S} A/I \rtimes_r S \rightarrow 0, \tag{4.16}$$

which may fail to be exact only in the middle: $\iota \rtimes_r S$ is injective, $\kappa \rtimes_r S$ is surjective and the range of $\iota \rtimes_r S$ is contained in the kernel of $\kappa \rtimes_r S$.

Proof. Note that $\iota(I_{1,t}^{\mathcal{E}I}) = I_{1,t}^{\mathcal{E}} \cdot I = \iota(s(\mathcal{E}_t I)) \cdot I_{1,t}^{\mathcal{E}}$ and $\kappa(I_{1,t}^{\mathcal{E}}) = I_{1,t}^{\mathcal{E}/\mathcal{E}I}$, for all $t \in S$. Hence ι and κ satisfy the equivalent conditions in Proposition 4.2 by the last part of Remark 4.6. Thus, by Proposition 4.2, not only the $*$ -homomorphisms $\iota \rtimes S, \kappa \rtimes S$ but also $\iota \rtimes_r S, \kappa \rtimes_r S$ exist, and $\iota \rtimes_r S$ is injective. The maps $\kappa \rtimes S$ and $\kappa \rtimes_r S$ are surjective because their images contain the dense $*$ -subalgebra $A/I \rtimes_{\text{alg}} S$.

We prove that $\iota \rtimes S$ is injective. Let π be a faithful, nondegenerate representation of $I \rtimes S$ on a Hilbert space \mathcal{H} . Since I is nondegenerate in $I \rtimes S$, the representation $\pi|_I$ is also nondegenerate. Therefore, for each $\xi_t \in \mathcal{E}_t$, the formula

$$\tilde{\pi}(\xi_t)\pi(a)h := \pi(\xi_t a)h$$

for $a \in I$ and $h \in \mathcal{H}$ defines a bounded operator on \mathcal{H} . Alternatively, $\tilde{\pi}(\xi_t)$ could be defined using an approximate identity (μ_λ) for I , as the limit of the strongly convergent net $\pi(\xi_t \mu_\lambda)$. A standard proof shows that $(\tilde{\pi}_t)_{t \in S}$ is a representation $\tilde{\pi}$ of \mathcal{E} . The integrated representation $\tilde{\pi} \rtimes S: A \rtimes S \rightarrow \mathbb{B}(\mathcal{H})$ satisfies $(\tilde{\pi} \rtimes S) \circ (\iota \rtimes S) = \pi$. Hence $\iota \rtimes S$ is injective.

The composite maps $(\kappa \rtimes S) \circ (\iota \rtimes S)$ and $(\kappa \rtimes_r S) \circ (\iota \rtimes_r S)$ vanish. Hence the range of $\iota \rtimes S$ is contained in $\ker(\kappa \rtimes S)$ and the range of $\iota \rtimes_r S$ is contained in $\ker(\kappa \rtimes_r S)$. Conversely, we claim that the range of $\iota \rtimes S$ contains $\ker(\kappa \rtimes S)$. We identify $I \rtimes S$ with its image in $A \rtimes S$. Let $q: A \rtimes S \rightarrow (A \rtimes S) / (I \rtimes S)$ be the quotient map. For $t \in S$, the restriction of q to $\mathcal{E}_t I$ vanishes. Hence q induces maps

$$\psi_t: \mathcal{E}_t / \mathcal{E}_t I \rightarrow (A \rtimes S) / (I \rtimes S).$$

They form a representation of $\mathcal{E}_{A/I}$. It integrates to a homomorphism

$$\psi \rtimes S: (A/I) \rtimes S \rightarrow (A \rtimes S) / (I \rtimes S).$$

We have $(\psi \rtimes S) \circ (\kappa \rtimes S) = q$. Hence $\ker(\kappa \rtimes S) \subseteq \ker(q) = I \rtimes S$. ■

Definition 4.17. The action \mathcal{E} is *exact* if the sequence (4.16) is exact for each $I \in \mathbb{I}^{\mathcal{E}}(A)$.

Example 4.18 (Exactness of twisted groupoids). An action \mathcal{E} of an inverse semigroup S on a commutative C^* -algebra $A \cong C_0(X)$ corresponds to a twisted étale groupoid (Σ, G) with unit space X (see [8]). Here $G = X \rtimes S$ is the dual groupoid of \mathcal{E} . The action \mathcal{E} is exact if and only if the corresponding *twisted groupoid* (G, Σ) is *exact* in the sense that for any open invariant subset $U \subseteq X$, the sequence of reduced twisted groupoid C^* -algebras

$$C_r^*(G|_U, \Sigma|_U) \twoheadrightarrow C_r^*(G, \Sigma) \twoheadrightarrow C_r^*(G|_{X \setminus U}, \Sigma|_{X \setminus U})$$

is exact. If the twist is trivial, the name *inner exact* for such groupoids is introduced in [3]. This example generalises as follows.

Example 4.19 (Exactness of Fell bundles over groupoids). Let $\mathcal{A} = (A_\gamma)_{\gamma \in G}$ be a Fell bundle over an étale groupoid G with locally compact Hausdorff unit space X . Let $S \subseteq \text{Bis}(G)$ be a wide inverse semigroup of bisections and turn the Fell bundle \mathcal{A} into an action \mathcal{E} of \tilde{S} on the section C^* -algebra $A = A|_X$, such that the associated universal and reduced C^* -algebras remain the same (see Examples 3.4 and 3.17). This natural correspondence extends to invariant ideals and the associated algebras. Indeed, by Lemma 4.9 an ideal I in A is \mathcal{E} -invariant if and only if it is G -invariant in the sense of [18] (\tilde{I} is invariant under the dual groupoid $\check{A} \rtimes G \cong \check{A} \rtimes \tilde{S}$). The restricted actions $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$ of \tilde{S} correspond to *restricted Fell bundles* $\mathcal{A}|_I = (A_\gamma I_{s(\gamma)})_{\gamma \in G}$ and $\mathcal{A}_{A/I} = (A_\gamma / (A_\gamma I_{s(\gamma)}))_{\gamma \in G}$ over G as defined in [18, Propositions 3.3 and 3.4]. The authors of [18] consider separable Fell bundles over locally compact Hausdorff groupoids. However, neither separability nor Hausdorffness are used in the construction of $\mathcal{A}|_I$ and $\mathcal{A}_{A/I}$. Proposition 4.15 implies that the sequence

$$C^*(\mathcal{A}|_I) \twoheadrightarrow C^*(\mathcal{A}) \twoheadrightarrow C^*(\mathcal{A}_{A/I})$$

is exact. This extends the main result of [18] to étale groupoids that are not separable or not Hausdorff. We will say that the *Fell bundle \mathcal{A} is exact* if for every G -invariant ideal I in A , the sequence

$$C_r^*(\mathcal{A}|_I) \twoheadrightarrow C_r^*(\mathcal{A}) \twoheadrightarrow C_r^*(\mathcal{A}_{A/I})$$

is exact. Equivalently, the action \mathcal{E} corresponding to \mathcal{A} is exact.

4.4. Exactness for essential crossed products

The essential crossed product is not functorial (see [31, Remark 4.8]). This complicates the definition of “exactness” for essential crossed products. Only the quotient maps cause extra problems, as the following lemma shows.

Lemma 4.20. *Let \mathcal{E} be an action of S by Hilbert bimodules on a C^* -algebra A and let $I \in \mathbb{I}^\mathcal{E}(A)$. The injective homomorphism ι from $\mathcal{E}|_I$ into \mathcal{E} induces an injective $*$ -homomorphism $\iota \rtimes_{\text{ess}} S: I \rtimes_{\text{ess}} S \rightarrow A \rtimes_{\text{ess}} S$. Its image is the ideal in $A \rtimes_{\text{ess}} S$ generated by I .*

Proof. If J is an essential ideal in I , then $J \oplus I^\perp$ is an essential ideal in A . The obvious inclusions $\mathcal{M}(J) \rightarrow \mathcal{M}(J \oplus I^\perp)$ for the essential ideals $J \subseteq I$ induce a natural isomorphism from $\mathcal{M}_{\text{loc}}(I)$ onto an ideal in $\mathcal{M}_{\text{loc}}(A)$. Let $EL_I: I \rtimes S \rightarrow \mathcal{M}_{\text{loc}}(I)$ be the canonical essential expectation. We are going to prove below that the following diagram commutes:

$$\begin{array}{ccc}
 I \rtimes S & \xleftarrow{\iota \rtimes S} & A \rtimes S \\
 \downarrow EL_I & & \downarrow EL \\
 \mathcal{M}_{\text{loc}}(I) & \hookrightarrow & \mathcal{M}_{\text{loc}}(A)
 \end{array} \tag{4.21}$$

Then $\mathcal{N}_{EL} \cap \iota \rtimes S(I \rtimes S) = \iota \rtimes S(\mathcal{N}_{EL_I})$ because $\iota \rtimes S(I \rtimes S)$ is an ideal in $A \rtimes S$. This, in turn, implies that the injective homomorphism $\iota \rtimes S$ factors through an injective homomorphism $\iota \rtimes_{\text{ess}} S: I \rtimes_{\text{ess}} S = I \rtimes S/\mathcal{N}_{EL_I} \rightarrow A \rtimes_{\text{ess}} S = A \rtimes S/\mathcal{N}_{EL}$.

To check (4.21), let $t \in S$. If $I_{t,1} = \overline{\sum_{v \leq t, 1} s(\mathcal{E}_v)}$ then $I_{t,1}I = \overline{\sum_{v \leq t, 1} s(\mathcal{E}_v I)}$ and the restriction of the map $\vartheta_{t,1}: \mathcal{E}_t \cdot I_{t,1} \xrightarrow{\sim} I_{t,1}$ (defined in (3.8)) to $\mathcal{E}_t I \cdot I_{t,1}$ coincides with the corresponding map defined for the restricted action $(\mathcal{E}_s I)_{s \in S}$. Hence for each $\xi \in \mathcal{E}_t I$ the element $EL(\xi) \in \mathcal{M}(I_{1,t} \oplus I_{1,t}^\perp) \subseteq \mathcal{M}(I_{1,t}I \oplus I_{1,t}^\perp I \oplus I_{1,t}I^\perp \oplus I_{1,t}^\perp I^\perp)$ acts on $u \in I_{1,t}I$ in the same way as $EL_I(\xi)$,

$$EL(\xi)u = \vartheta_{1,t}(\xi u) = EL_I(\xi)u.$$

Since $EL(\xi)(I_{1,t}^\perp I \oplus I_{1,t}I^\perp \oplus I_{1,t}^\perp I^\perp) = 0$, the embedding $\mathcal{M}_{\text{loc}}(I) \subseteq \mathcal{M}_{\text{loc}}(A)$ maps $EL_I(\xi)$ to $EL(\xi)$. This proves (4.21). ■

Example 4.22 ([31, Example 4.7]). Let $S := G \cup \{0\}$ be the inverse semigroup obtained by adjoining a zero element to an amenable discrete group G . Let G act on $A = C[0, 1]$ by $\mathcal{E}_g = C[0, 1]$ for $g \in G$ and $\mathcal{E}_0 = C_0(0, 1]$, equipped with the usual involution and multiplication maps. Then $A \rtimes_{\text{ess}} S = A$. Every ideal I in A is \mathcal{E} -invariant and $I \rtimes_{\text{ess}} S = I$, and so $I \rtimes_{\text{ess}} S \subseteq A \rtimes_{\text{ess}} S$. However, if $I := C_0(0, 1]$, then $A/I \rtimes_{\text{ess}} S = \mathbb{C} \rtimes G = C^*(G)$ and the quotient homomorphism κ from \mathcal{E} onto $\mathcal{E}|_{A/I}$ does not induce a map from A to $C^*(G)$.

Definition 4.23. We call the action \mathcal{E} *essentially exact* if for each $I \in \mathbb{I}^\mathcal{E}(A)$ there is a $*$ -homomorphism $\kappa \rtimes_{\text{ess}} S: A \rtimes_{\text{ess}} S \rightarrow A/I \rtimes_{\text{ess}} S$ whose restriction to each fibre \mathcal{E}_t is the quotient map onto $\mathcal{E}_t/\mathcal{E}_t I$, $t \in S$, and the kernel of $\kappa \rtimes_{\text{ess}} S$ is $\iota \rtimes_{\text{ess}} S(I \rtimes_{\text{ess}} S)$.

Remark 4.24. When the action \mathcal{E} is closed, then the reduced and essential crossed products coincide for all restrictions of \mathcal{E} (see Proposition 4.14). Thus essential exactness is the same as exactness for closed actions of inverse semigroups and for Fell bundles over Hausdorff groupoids.

Example 4.25 (Essentially exact Fell bundles). Consistently with Example 4.19, we will call a Fell bundle $\mathcal{A} = (A_\gamma)_{\gamma \in G}$ over an étale groupoid essentially exact if the corresponding action \mathcal{E} is essentially exact. More specifically, by Lemma 4.20, for any G -invariant ideal I in A , the inclusion $C_c(\mathcal{A}|_I) \subseteq C_c(\mathcal{A})$ extends to an injective $*$ -homomorphism $C_{\text{ess}}^*(\mathcal{A}|_I) \hookrightarrow C_{\text{ess}}^*(\mathcal{A})$. So \mathcal{A} is *essentially exact* if and only if, for every G -invariant ideal I in A , restriction of sections gives a well-defined $*$ -homomorphism $C_{\text{ess}}^*(\mathcal{A}) \twoheadrightarrow C_{\text{ess}}^*(\mathcal{A}|_{A/I})$ and the following sequence is exact:

$$C_{\text{ess}}^*(\mathcal{A}|_I) \hookrightarrow C_{\text{ess}}^*(\mathcal{A}) \twoheadrightarrow C_{\text{ess}}^*(\mathcal{A}|_{A/I}).$$

If the S -action on A is residually aperiodic, then A separates ideals in $A \rtimes_{\text{ess}} S$ if and only if the S -action on A is essentially exact (see Theorem 5.9 below). In Example 4.22, however, A separates ideals in $A \rtimes_{\text{ess}} S$ although the S -action on A is not essentially exact. The following proposition shows that the S -action on A must be essentially exact if A separates ideals in $A \rtimes_r S$.

Proposition 4.26. *The following are equivalent:*

- (1) *A separates ideals in the reduced crossed product $A \rtimes_r S$;*
- (2) *the action is exact and for each $I \in \mathbb{I}^{\mathcal{E}}(A)$, A/I detects ideals in $A/I \rtimes_r S$.*

If these equivalent conditions hold, then $A/I \rtimes_r S = A/I \rtimes_{\text{ess}} S$ for every $I \in \mathbb{I}^{\mathcal{E}}(A)$ and the action is essentially exact.

Proof. Lemma 2.4 and Proposition 2.13 show that A separates ideals in $A \rtimes_r S$ if and only if A/I detects ideals in $A \rtimes_r S / \iota_r \rtimes S(I \rtimes_r S)$ for all $I \in \mathbb{I}^{\mathcal{E}}(A)$. Then the action is exact because the kernel of $\kappa \rtimes_r S$ has to be $\iota_r \rtimes S(I \rtimes_r S)$, and then

$$A \rtimes_r S / \iota_r \rtimes S(I \rtimes_r S) \cong A/I \rtimes_r S$$

for all $I \in \mathbb{I}^{\mathcal{E}}(A)$. Thus (1) and (2) are equivalent. Condition (2) implies $A/I \rtimes_r S = A/I \rtimes_{\text{ess}} S$ because otherwise $A/I \rtimes_{\text{ess}} S$ would be a quotient of $A/I \rtimes_r S$ by a nonzero ideal not detected by A/I . ■

We illustrate by an example what can go wrong with the exactness of essential crossed products. Our example is closely related to the Reeb foliation or, more precisely, to its restriction to a transversal.

Example 4.27. Let $\vartheta: \mathbb{R} \rightarrow \mathbb{R}$ be a homeomorphism with $\vartheta(t) = t$ for $t \leq 0$ and $\vartheta(t) > t$ for all $t > 0$. Let G be the germ groupoid of the transformation groupoid $\mathbb{R} \times_{\vartheta} \mathbb{Z}$. We claim that $C_r^*(G) \cong C_{\text{ess}}^*(G)$. To see this, we use that the restrictions of G to $[0, \infty)$ and $(-\infty, 0)$ are Hausdorff. Indeed, $G|_{(0, \infty)}$ is the Hausdorff transformation groupoid $(0, \infty) \rtimes_{\vartheta} \mathbb{Z}$ because \mathbb{Z} acts freely (and properly) on $(0, \infty)$. Similarly, $G|_{[0, \infty)}$ is the Hausdorff transformation groupoid $[0, \infty) \rtimes_{\vartheta} \mathbb{Z}$; the action of \mathbb{Z} on $[0, \infty)$ fixes 0, but the germ of any $n \in \mathbb{Z}$ at 0 is nontrivial because ϑ^n acts nontrivially on $(0, \infty)$. Therefore, the support of any nonzero element of $C_r^*(G)$ must intersect $[0, \infty)$ and $(-\infty, 0)$ in relatively open subsets. Hence the support cannot be meagre, and this proves our claim. This equality of reduced and essential groupoid C^* -algebras for G is not inherited by the restriction to the closed invariant subset $(-\infty, 0]$. Indeed, the restriction of G to $(-\infty, 0]$ is the non-Hausdorff group bundle with trivial fibre over $(-\infty, 0)$ and the fibre \mathbb{Z} at 0 (see [31, Example 4.7] and Example 4.22). In this case, the full and reduced groupoid C^* -algebras are obtained by gluing together $C_0((-\infty, 0))$ and the fibre $C^*(\mathbb{Z})$ at 0. However, $C_{\text{ess}}^*(G|_{(-\infty, 0]}) = C_0((-\infty, 0])$, so

$$C_r^*(G|_{(-\infty, 0]}) \not\cong C_{\text{ess}}^*(G|_{(-\infty, 0]}).$$

Since the essential and reduced crossed products coincide for G and $G|_{(0, \infty)}$, but not for $G|_{(-\infty, 0]}$, the following sequence of essential crossed products exists, but fails to be exact:

$$0 \rightarrow C_{\text{ess}}^*(G|_{(0, \infty)}) \rightarrow C_{\text{ess}}^*(G) \rightarrow C_{\text{ess}}^*(G|_{(-\infty, 0]}) \rightarrow 0.$$

This is one way how essential crossed products may fail to be exact. The restriction $G|_{\{0\}}$ is simply the group \mathbb{Z} . So $C_r^*(G|_{\{0\}}) = C_{\text{ess}}^*(G|_{\{0\}}) \cong C^*(\mathbb{Z})$. The restriction $*$ -homomorphism $C^*(G|_{(-\infty,0]}) \rightarrow C^*(G|_{\{0\}})$ does not descend to the essential crossed products. That is, there is no canonical map from $C_{\text{ess}}^*(G|_{(-\infty,0]})$ to $C_{\text{ess}}^*(G|_{\{0\}})$. This is the second way how essential crossed products may fail to be exact.

Since $G|_{[0,\infty)} \cong [0, \infty) \rtimes_{\partial} \mathbb{Z}$, it follows that $C^*(G|_{[0,\infty)}) \cong C_r^*(G|_{[0,\infty)})$. By a diagram, it follows that the sequence

$$0 \rightarrow C_r^*(G|_{(-\infty,0)}) \rightarrow C_r^*(G) \rightarrow C_r^*(G|_{[0,\infty)}) \rightarrow 0$$

is exact. Since also $C^*(G|_{(-\infty,0)}) \cong C_r^*(G|_{(-\infty,0)})$, the Five Lemma shows that $C^*(G) \cong C_r^*(G)$. A similar argument shows $C^*(G|_U) \cong C_r^*(G|_U)$ for all locally closed G -invariant subsets $U \subseteq \mathbb{R}$. Therefore, G is inner exact.

4.5. Amenability vs exactness

Definition 4.28. Let \mathcal{E} be an S -action by Hilbert bimodules on a C^* -algebra A . We call the action \mathcal{E} *amenable* if the regular representation $\Lambda: A \rtimes S \rightarrow A \rtimes_r S$ is an isomorphism.

Remark 4.29. By (3.13), the action \mathcal{E} is amenable if and only if the weak conditional expectation $E: A \rtimes S \rightarrow A'$ is faithful.

Lemma 4.30. *Let \mathcal{E} be an S -action by Hilbert bimodules on a C^* -algebra A and let $I \in \mathbb{I}^{\mathcal{E}}(A)$ be an invariant ideal. If $\mathcal{E}|_{A/I}$ is amenable, then*

$$I \rtimes_r S \twoheadrightarrow A \rtimes_r S \twoheadrightarrow A/I \rtimes_r S$$

is exact. If this sequence is exact, then \mathcal{E} is amenable if and only if $\mathcal{E}|_I$ and $\mathcal{E}|_{A/I}$ are amenable.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} I \rtimes S & \xrightarrow{\iota \rtimes S} & A \rtimes S & \xrightarrow{\kappa \rtimes S} & A/I \rtimes S \\ \downarrow \Lambda^I & & \downarrow \Lambda^A & & \downarrow \Lambda^{A/I} \\ I \rtimes_r S & \xrightarrow{\iota \rtimes_r S} & A \rtimes_r S & \xrightarrow{\kappa \rtimes_r S} & A/I \rtimes_r S, \end{array}$$

where the top horizontal sequence is exact, and $\Lambda^I, \Lambda^A, \Lambda^{A/I}$ denote the respective regular representations. If $\Lambda^{A/I}$ is injective, then

$$\ker(\kappa \rtimes_r S) = \Lambda^A(\ker(\kappa \rtimes S)) = \Lambda^A(\iota \rtimes S(I \rtimes S)) = \iota \rtimes_r S(I \rtimes_r S).$$

Hence $I \rtimes_r S \twoheadrightarrow A \rtimes_r S \twoheadrightarrow A/I \rtimes_r S$ is exact. In general, if this sequence is exact, the Snake Lemma from homological algebra yields a short exact sequence

$$\ker(\Lambda^I) \twoheadrightarrow \ker(\Lambda^A) \twoheadrightarrow \ker(\Lambda^{A/I}).$$

Hence $\ker(\Lambda^A) = 0$ if and only if $\ker(\Lambda^I) = 0$ and $\ker(\Lambda^{A/I}) = 0$. ■

Remark 4.31. It is unclear whether the quotient action $\mathcal{E}|_{A/I}$ for $I \in \mathbb{I}^\mathcal{E}(A)$ is amenable if \mathcal{E} is (the proof of [34, Lemma 3.9] is incorrect). If so, then by Lemma 4.30, the amenable action is also exact. Let $S = G$ be a group. If \mathcal{E} satisfies the approximation property in [13, Definition 4.4], then also every restriction $\mathcal{E}|_{A/I}$ satisfies the approximation property. Hence the approximation property of a Fell bundle over a group implies both amenability and exactness. In particular, if there is an amenable group action \mathcal{E} such that $\mathcal{E}|_{A/I}$ is not amenable for some $I \in \mathbb{I}^\mathcal{E}(A)$, then the approximation property is strictly stronger than amenability (see also [13, page 169]).

Example 4.32. Let $\mathcal{A} = (A_\gamma)_{\gamma \in G}$ be a Fell bundle over an étale, locally compact, Hausdorff groupoid G . Suppose that G is second countable and that each fibre A_γ is separable. If G is amenable, then the regular representation $C^*(\mathcal{A}) \twoheadrightarrow C_r^*(\mathcal{A})$ is an isomorphism by [45, Theorem 1]. This applies also to every Fell bundle $\mathcal{A}_{A/I}$ over G where I is a G -invariant ideal in A . Thus if G is amenable, then \mathcal{A} is exact in the sense described in Example 4.19.

Example 4.33. If \mathcal{E} is a closed action of a countable inverse semigroup S on a separable, commutative C^* -algebra $A = C_0(X)$ such that $A \rtimes_r S$ is nuclear, then the action \mathcal{E} is amenable and exact. Indeed, the action \mathcal{E} corresponds to a twisted étale, locally compact, Hausdorff groupoid (G, Σ) , where $G = X \rtimes S$ is the dual groupoid of \mathcal{E} . Equivalently, this corresponds to a Fell line bundle over G . Then G is amenable by [46, Theorem 5.4]. Hence our claim follows from Example 4.32. In particular, a twisted étale, locally compact, second countable, Hausdorff groupoid (G, Σ) with nuclear $C_r^*(G, \Sigma)$ is exact in the sense of Example 4.18.

5. Ideals and pure infiniteness for inverse semigroup crossed products

In this section, we present efficient criteria for separation of ideals and pure infiniteness in essential crossed products.

5.1. Residual aperiodicity and ideal structure

Let $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ be an action of a unital inverse semigroup S by Hilbert bimodules on a C^* -algebra A .

Definition 5.1 ([31, Definition 6.1]). The action $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ is *aperiodic* if the Hilbert A -bimodules $\mathcal{E}_t \cdot I_{1,t}^\perp$ are aperiodic for all $t \in S$, where $I_{1,t}$ is defined in (3.7).

Proposition 5.2 ([31, Proposition 6.3]). *Let B be an S -graded C^* -algebra with a grading $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$. Let $A := \mathcal{E}_1 \subseteq B$ and turn \mathcal{E} into an S -action on A . If this action is aperiodic, then the inclusion $A \subseteq B$ is aperiodic. The converse holds if the grading on B is topological, that is, the canonical quotient map $A \rtimes S \twoheadrightarrow A \rtimes_{\text{ess}} S$ factors through $A \twoheadrightarrow B$.*

Definition 5.3 ([31, Definition 2.20]). Let G be an étale groupoid and $X \subseteq G$ its unit space. The *isotropy group* of a point $x \in X$ is $G(x) := s^{-1}(x) \cap r^{-1}(x) \subseteq G$. We call G *topologically free* if, for every open $U \subseteq G \setminus X$, the set $\{x \in X : G(x) \cap U \neq \emptyset\}$ has empty interior.

Remark 5.4. A groupoid G is *effective* if any open subset $U \subseteq G$ with $r|_U = s|_U$ is contained in X . Effective groupoids are topologically free. The converse implication holds if X is closed in G , but not in general.

Aperiodicity is related in [28, 31, 32] to topological freeness and several other conditions. We now introduce residual versions of aperiodicity and topological freeness.

Definition 5.5. The action \mathcal{E} is *residually aperiodic* if, for each $I \in \mathbb{I}^{\mathcal{E}}(A)$, the restricted action $\mathcal{E}|_{A/I}$ is aperiodic.

Definition 5.6. An étale groupoid G with unit space $X \subseteq G$ is *residually topologically free* if, for each nonempty closed G -invariant subset $Y \subseteq X$, the restricted groupoid $r^{-1}(Y) = s^{-1}(Y)$ is topologically free.

The following lemma may help to show that a transformation groupoid is (residually) topologically free.

Lemma 5.7. *Let G be an étale groupoid. Let X and Y be topological spaces with G -actions and let $f : X \rightarrow Y$ be a continuous G -equivariant map.*

- (1) *If Y is closed in $G \times Y$, then X is closed in $G \times X$.*
- (2) *If $f : X \rightarrow Y$ is open and $G \times Y$ is topologically free, then so is $G \times X$.*
- (3) *If $f : X \rightarrow Y$ is open, maps closed G -invariant sets to closed sets, and $G \times Y$ is residually topologically free, then so is $G \times X$.*

Proof. The map f induces a continuous groupoid homomorphism $f_* : G \times X \rightarrow G \times Y$ such that $f_*^{-1}(Y) = X$. So (1) follows. The map f_* is open if and only if f is open. Assume this. To show (2) assume that $G \times Y$ is topologically free and let $U \subseteq (G \times X) \setminus X$. Then $f_*(U)$ is open and contained in $(G \times Y) \setminus Y$. Let $I_U := \{x \in X : G(x) \cap U \neq \emptyset\}$ and $I_{f_*(U)} := \{y \in Y : G(y) \cap f_*(U) \neq \emptyset\}$. We claim that $f(I_U) = I_{f_*(U)}$. To see this, let $x \in I_U$. Then there is an arrow $(x, g, x) \in U$; here $g \in G$ is such that $\varrho(x) = s(g)$ and $g \cdot x = x$. Since f is G -equivariant, then $(f(x), g, f(x)) \in f_*(U)$. This witnesses that $f(x) \in I_{f_*(U)}$. Since $G \times Y$ is topologically free, $I_{f_*(U)}$ has empty interior. Since f is open, the preimage of $I_{f_*(U)}$ in X has empty interior as well. It follows that I_U has empty interior. This witnesses that $G \times X$ is topologically free and proves (2).

Finally, for any G -invariant set $D \subseteq X$ the set $f(D)$ is G -invariant. If we assume $f(D)$ is closed in Y and $G \times Y$ is residually topologically free, then $G \times f(D)$ is topologically free. Since the restriction of f_* to $G \times D$ is a continuous open map onto $G \times f(D)$, (2) implies that $G \times D$ is topologically free. This proves (3). ■

Theorem 5.8. *Let \mathcal{E} be an action of a unital inverse semigroup S on a C^* -algebra A by Hilbert bimodules. If A is separable or of Type I, then the following are equivalent:*

- (1) *the dual groupoid $\widehat{A} \rtimes S$ is residually topologically free;*
- (2) *the action \mathcal{E} is residually aperiodic;*
- (3) *for any $I \in \mathbb{I}^{\mathcal{E}}(A)$, the full crossed product for the restricted action $\mathcal{E}|_{A/I}$ has a unique pseudo-expectation, namely, the canonical \mathcal{M}_{loc} -expectation;*
- (4) *for any $I \in \mathbb{I}^{\mathcal{E}}(A)$, $(A/I)^+$ supports C for each intermediate C^* -subalgebra $A/I \subseteq C \subseteq A/I \rtimes_{\text{ess}} S$ for the restricted action $\mathcal{E}|_{A/I}$;*
- (5) *for any $I \in \mathbb{I}^{\mathcal{E}}(A)$, A/I detects ideals in each intermediate C^* -subalgebra $A/I \subseteq C \subseteq A/I \rtimes_{\text{ess}} S$ for the restricted action $\mathcal{E}|_{A/I}$.*

For general A , (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) and (2) \Rightarrow (4) \Rightarrow (5).

Proof. A subset of \widehat{A} is closed and invariant if and only if it is of the form $X = \widehat{A/I}$ for an \mathcal{E} -invariant ideal I . The dual groupoid of the induced action on A/I is the restriction $X \rtimes S$ of the dual groupoid to X . Hence the implications (1) \Rightarrow (2) \Rightarrow (3), (4) follow from [32, Corollary 4.8 and Theorem 3.6], applied to the quotients A/I and intermediate C^* -algebras. Lemma 2.23 shows that (4) implies (5), and [40, Theorem 3.5] shows that (3) implies (5). If A is separable or of Type I, so are its quotients A/I , and then [32, Proposition 6.1] shows that (5) implies (1). ■

The following result justifies introducing the notion of essential exactness – it is an instance of condition (1) in Theorem 2.34.

Theorem 5.9. *Let B be an S -graded C^* -algebra with a grading $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ that forms a residually aperiodic action of S on $A := \mathcal{E}_1$ (this holds if the dual groupoid $\widehat{A} \rtimes S$ is residually topologically free). The following are equivalent:*

- (1) *$B \cong A \rtimes_{\text{ess}} S$ and \mathcal{E} is essentially exact;*
- (2) *A separates ideals in B , so $\mathbb{I}(B) \cong \mathbb{I}^{\mathcal{E}}(A)$;*
- (3) *A^+ residually supports B ;*
- (4) *A^+ fills B .*

If the above equivalent conditions hold and the primitive ideal space \check{A} is second countable, then the quasi-orbit map induces a homeomorphism $\check{B} \cong \check{A}/\sim$, where \check{A}/\sim is the quasi-orbit space of the dual groupoid $\check{A} \rtimes S$, that is, $p_1, p_2 \in \check{A}$ satisfy $p_1 \sim p_2$ if and only if

$$\overline{(\check{A} \rtimes S) \cdot p_1} = \overline{(\check{A} \rtimes S) \cdot p_2}.$$

Proof. The C^* -inclusion $A \subseteq B$ is symmetric and residually aperiodic by Propositions 2.13 and 5.2. Hence by Theorem 2.34, conditions (2)–(4) are equivalent to the condition that for each $I \in \mathbb{I}^B(A) = \mathbb{I}^{\mathcal{E}}(A)$ the unique pseudo-expectation $E^I: B/BIB \rightarrow A/I$ is almost

faithful. Let us assume this. Then there is a commutative diagram

$$\begin{array}{ccc}
 A \rtimes S/I \rtimes S \cong A/I \rtimes S & \xrightarrow{\Psi} & B/BIB \\
 \downarrow EL^I & & \downarrow E^I \\
 \mathcal{M}_{\text{loc}}(A/I) & \xrightarrow{\hookrightarrow} & I(A/I),
 \end{array}$$

where Ψ is the homomorphism that exists by universality of $A/I \rtimes S$ because B/BIB is graded by $\mathcal{E}|_{A/I}$ by Lemma 2.14, EL^I is the canonical essential expectation for $A/I \rtimes S$, and $\mathcal{M}_{\text{loc}}(A/I) \hookrightarrow I(A/I)$ is the canonical embedding. The diagram commutes because the inclusion $A/I \subseteq A/I \rtimes S$ is aperiodic and hence there is a unique pseudo-expectation by [32, Theorem 3.6]. As a consequence, $\ker \Psi = \Psi^{-1}(0) = \Psi^{-1}(\mathcal{N}_{E^I}) = \mathcal{N}_{EL}$ and thus Ψ factors through an isomorphism $A/I \rtimes_{\text{ess}} S \cong B/BIB$. Since this holds for every $I \in \mathbb{I}^{\mathcal{E}}(A)$ we get that $B \cong A \rtimes_{\text{ess}} S$ and that \mathcal{E} is essentially exact ($A \rtimes_{\text{ess}} S/I \rtimes_{\text{ess}} S \cong B/BIB \cong (A/I) \rtimes_{\text{ess}} S$ for $I \in \mathbb{I}^{\mathcal{E}}(A)$).

Theorem 2.34 implies easily that (1) implies (2). This finishes the proof that the four conditions are equivalent. The remaining claims follow mostly from the last part of Theorem 2.34. That the quasi-orbit space has the asserted form follows from [30, Theorem 6.22]. ■

Corollary 5.10. *Let $\mathcal{A} = (A_\gamma)_{\gamma \in G}$ be a Fell bundle over an étale groupoid G with locally compact, Hausdorff unit space X , and put $A = C_0(\mathcal{A}|_X)$. Define the dual groupoid $\check{A} \rtimes G$ as in Example 3.6. Assume that it is residually topologically free (this holds, for instance, if G is residually topologically free and the base map for the C^* -bundle \mathcal{A} is open and closed). Assume also one of the following:*

- (1) $B := C_{\text{ess}}^*(\mathcal{A})$ and \mathcal{A} is essentially exact;
- (2) $B := C_{\text{r}}^*(\mathcal{A})$, \mathcal{A} is exact, and the unit space in $\check{A} \rtimes G$ is closed (the latter is automatic if G is Hausdorff);
- (3) $B := C^*(\mathcal{A})$, \mathcal{A} is separable, and G is amenable and Hausdorff.

Then A separates ideals in B and, even more, A^+ fills B . The lattice $\mathbb{I}(B)$ is naturally isomorphic to the lattice of G -invariant ideals in A . If, in addition, \check{A} is second countable, then $\check{B} \cong \check{A}/\sim$, where \check{A}/\sim is the quasi-orbit space of the dual groupoid $\check{A} \rtimes G$, that is, $p_1, p_2 \in \check{A}$ satisfy $p_1 \sim p_2$ if and only if $(\check{A} \rtimes G) \cdot p_1 = (\check{A} \rtimes G) \cdot p_2$.

Proof. The claims in brackets follow from Lemma 5.7. The claim in case (1) follows from Theorem 5.9 (see also Example 4.25). Case (2) follows from (1) and Remarks 3.16 and 4.24. Example 4.32 explains why (3) is a special case of (2). ■

Theorem 5.9 allows us to describe the ideal structure of B in terms of A under the following assumptions:

$$\mathcal{E} \text{ is an essentially exact, residually aperiodic action and } B = A \rtimes_{\text{ess}} S. \tag{5.11}$$

We are going to study whether B is purely infinite using the same assumption. Before we do this, we simplify (5.11) in the presence of a conditional expectation.

Proposition 5.12. *If there is a genuine conditional expectation $E: B \rightarrow A \subseteq B$, then (5.11) is equivalent to*

$$\mathcal{E} \text{ is closed, exact, residually aperiodic and } B = A \rtimes_r S. \tag{5.13}$$

For any C^ -inclusion $A \subseteq B$, an action \mathcal{E} as in (5.13) exists if and only if $A \subseteq B$ is regular, residually aperiodic and there is a conditional expectation $E: B \rightarrow A$ which is residually faithful in the sense that E descends to a faithful conditional expectation $E^I: B/BIB \rightarrow A/I$ for any $I \in \mathbb{I}^B(A)$ (see Lemma 2.20).*

Proof. Assumption (5.13) implies (5.11) by Remark 4.24. Conversely, assume (5.11). Then $E: B \rightarrow A$ is the unique pseudo-expectation for $A \subseteq B$ by [32, Theorem 3.6]. Hence $E = EL$, \mathcal{E} is closed and $A \rtimes_r S = A \rtimes_{\text{ess}} S$. The same argument works for all the restrictions \mathcal{E}^I and quotient inclusions $A/I \subseteq B/BIB$ for $I \in \mathbb{I}^{\mathcal{E}}(A) = \mathbb{I}^B(A)$. This gives (5.13). Combining this reasoning with [29, Theorem 6.3] also gives the second part of the assertion. ■

Corollary 5.14. *If A is type I, then an action \mathcal{E} as in (5.13) exists if and only if $A \subseteq B$ is residually Cartan, that is, for each $I \in \mathbb{I}^B(A)$, $A/I \subseteq B/BIB$ is a noncommutative Cartan subalgebra in the sense of Exel [15].*

Proof. Combine the second part of Proposition 5.12 and [29, Theorem 6.3]. ■

5.2. Pure infiniteness criteria

In this section, we assume that B is an S -graded C^* -algebra with a grading $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ and A is the unit fibre of the grading. We give pure infiniteness criteria for B under the assumption (5.11). This covers (5.13) and the assumptions in Corollary 5.10 as special cases. In view of Theorem 5.9, the following two theorems are immediate corollaries of Theorems 2.36 and 2.37.

Theorem 5.15. *Assume (5.11). Let $\mathcal{F} \subseteq A^+$ fill A and be invariant under ε -cut-downs. Then B is strongly purely infinite if and only if each pair of elements $a, b \in \mathcal{F}$ has the matrix diagonalisation property in B .*

Theorem 5.16. *Assume (5.11). Let $\mathcal{F} \subseteq A^+$ residually support A . Suppose that $\mathbb{I}^{\mathcal{E}}(A)$ is finite or the projections in \mathcal{F} separate the ideals in $\mathbb{I}^{\mathcal{E}}(A)$. Then B is strongly purely infinite (with the ideal property) if and only if every element in $\mathcal{F} \setminus \{0\}$ is properly infinite in B .*

In order to use these results, we need conditions that suffice for $a, b \in A^+$ to have the matrix diagonalisation property in B or for $a \in A^+ \setminus \{0\}$ to be properly infinite in B . Checking the matrix diagonalisation property is usually difficult. Nevertheless, the following lemma may be useful (see [22, 28]).

Lemma 5.17. *Let $a, b \in A^+ \setminus \{0\}$. Suppose that for each $c \in \mathcal{E}_t$, $t \in S$, and each $\varepsilon > 0$ there are $n, m \in \mathbb{N}$ and $a_i \in a\mathcal{E}_{s_i}$, $s_i \in S$, for $i = 1, \dots, n$ and $b_j \in b\mathcal{E}_{t_j}$, $t_j \in S$, for $j = 1, \dots, m$ such that*

$$a \approx_\varepsilon \sum_{i=1}^n a_i^* a_i, \quad b \approx_\varepsilon \sum_{i=1}^m b_i^* b_i, \quad \sum_{\substack{i,j=1, \\ i \neq j}}^n a_i^* a_j \approx_\varepsilon 0, \quad \sum_{\substack{i,j=1, \\ i \neq j}}^{n,m} b_i^* b_j \approx_\varepsilon 0$$

and $\sum_{i=1, j=1}^{n,m} a_i^* c b_j \approx_\varepsilon 0$. Then $a, b \in A^+ \setminus \{0\}$ have the matrix diagonalisation property in B .

Proof. Let $\mathcal{C} := \bigcup_{t \in S} \mathcal{E}_t$ and $\mathcal{S} := \bigcup_{t \in S} \mathcal{E}_t$. We claim that $a, b \in A^+ \setminus \{0\}$ have the matrix diagonalisation property with respect to \mathcal{C} and \mathcal{S} as introduced in [23, Definition 4.6]. Indeed, let $x \in \mathcal{E}_t$ be such that $\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(B)^+$ and let $\varepsilon > 0$. Let $a_i \in a\mathcal{E}_{s_i}$ and $b_j \in b\mathcal{E}_{t_j}$ satisfy the conditions described in the assertion with $c := a^{1/2} x b^{1/2}$. We may write $a_i = a^{1/2} x_i$ and $b_j = b^{1/2} y_j$ for some x_i, y_j . Let $d_1 := \sum_{i=1}^n x_i$ and $d_2 := \sum_{j=1}^m y_j$. The assumed estimates imply that

$$d_1^* a d_1 \approx_{2\varepsilon} a, \quad d_2^* b d_2 \approx_{2\varepsilon} b, \quad d_1^* x d_2 \approx_\varepsilon 0.$$

This proves our claim. Clearly, \mathcal{S} is a multiplicative subsemigroup of B , $\mathcal{S}^* \mathcal{C} \mathcal{S} \subseteq \mathcal{C}$, $A\mathcal{S}A \subseteq \mathcal{S}$, and the closed linear span of \mathcal{C} is B . Thus $a, b \in A^+ \setminus \{0\}$ have the matrix diagonalisation property in B by [23, Lemma 5.6]. ■

Now we will focus on ways to check whether $a \in A^+ \setminus \{0\}$ is properly infinite in B . The following definition generalises [34, Definition 5.1] and [28, Definition 5.5] from groups to inverse semigroups. A crucial point is that the properties depend only on the Fell bundle $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$, not on the norm in B .

Definition 5.18. An element $a \in A^+ \setminus \{0\}$ is called

- (1) \mathcal{E} -infinite if there is $b \in A^+ \setminus \{0\}$ such that for each $\varepsilon > 0$, there are $n, m \in \mathbb{N}$, $t_i \in S$ and $a_i \in a\mathcal{E}_{t_i}$ for $1 \leq i \leq n + m$, such that

$$a \approx_\varepsilon \sum_{i=1}^n a_i^* a_i, \quad b \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \sum_{\substack{i,j=1, \\ i \neq j}}^{n+m} \|a_i^* a_j\| \leq \varepsilon;$$

- (2) residually \mathcal{E} -infinite if $a + I$ is $\mathcal{E}|_{A/I}$ -infinite for all $I \in \mathbb{I}^\mathcal{E}(A)$ with $a \notin I$;
- (3) properly \mathcal{E} -infinite if for all $\varepsilon > 0$ there are $n, m \in \mathbb{N}$, $t_i \in S$ and $a_i \in a\mathcal{E}_{t_i}$ for $1 \leq i \leq n + m$, such that

$$a \approx_\varepsilon \sum_{i=1}^n a_i^* a_i, \quad a \approx_\varepsilon \sum_{j=n+1}^m a_j^* a_j, \quad \sum_{\substack{i,j=1, \\ i \neq j}}^{n+m} \|a_i^* a_j\| \leq \varepsilon;$$

- (4) \mathcal{E} -paradoxical if the condition in (3) holds with $\varepsilon = 0$, that is, there are $n, m \in \mathbb{N}$, $t_i \in S$, and $a_i \in a\mathcal{E}_{t_i}$ for $1 \leq i \leq n + m$, such that

$$a = \sum_{i=1}^n a_i^* a_i, \quad a = \sum_{j=n+1}^{n+m} a_j^* a_j, \quad a_i^* a_j = 0 \quad \text{for } i \neq j.$$

Lemma 5.19. *If $a \in A^+ \setminus \{0\}$ is \mathcal{E} -infinite, then it is infinite in B . If $a \in A^+ \setminus \{0\}$ is properly \mathcal{E} -infinite, then it is properly infinite in B .*

Proof. First, let $a \in A^+ \setminus \{0\}$ be \mathcal{E} -infinite. Let $b \in A^+ \setminus \{0\}$ be as in Definition 5.18 (1). For $\varepsilon > 0$, there are $n, m \in \mathbb{N}$ and $t_i \in S$ and $a_i \in a\mathcal{E}_{t_i}$ for $i = 1, \dots, n + m$, $n, m \in \mathbb{N}$ as in Definition 5.18 (1). Let $x := \sum_{i=1}^n a_i$ and $y := \sum_{i=n+1}^m a_i$. Then $x, y \in aB$. Simple estimates such as

$$\|x^*x - a\| = \left\| \sum_{j=1}^n a_j^* a_j - a + \sum_{i,j=1, i \neq j}^n a_i^* a_j \right\| \leq \varepsilon + \varepsilon$$

show that $x^*x \approx_{2\varepsilon} a$, $y^*y \approx_{2\varepsilon} b$ and $y^*x \approx_\varepsilon 0$. Hence a is infinite in B (see Definition 2.35). The proof when a is properly \mathcal{E} -infinite is the same with $b = a$. ■

Let us compare the definitions of infinite and properly infinite elements in Definition 2.35 with the definitions of \mathcal{E} -infinite and properly \mathcal{E} -infinite elements in Definition 5.18. There are two differences. First, we now choose the elements $x, y \in aB$ in the subalgebra $A \rtimes_{\text{alg}} S$, so that we may write them as a finite sum $\sum a_i$ with $a_i \in a\mathcal{E}_{t_i}$. Secondly, we estimate each product $a_i^* a_j$ for $i \neq j$ separately. The first change does not achieve much because $A \rtimes_{\text{alg}} S$ is dense in B and we only aim for approximate equalities anyway. The second change simplifies the estimates a lot because $\|a_i^* a_j\|$ is computed in the Hilbert A -bimodule $\mathcal{E}_{t_i^* t_j}$, whereas the norm estimates in Definition 2.35 involve the C^* -norm of B . For an \mathcal{E} -paradoxical element, we even assume the products $a_i^* a_j$ for $i \neq j$ to vanish exactly. This is once again much easier to check. Paradoxical elements are also important because they are related to paradoxical decompositions, which were studied already by Banach and Tarski. In the setting of purely infinite crossed products, their importance was highlighted by Rørdam and Sierakowski [43]. The implications among our infiniteness conditions hinted at above are summarised in the following proposition.

Proposition 5.20. *Assume that A separates ideals in B . Consider the following conditions which $a \in A^+ \setminus \{0\}$ may satisfy:*

- (1) a is properly infinite in B ;
- (2) for each $\varepsilon > 0$ there are $n, m \in \mathbb{N}$, $t_i \in S$, and $a_i \in a\mathcal{E}_{t_i}$ for $1 \leq i \leq n + m$, such that

$$a \approx_\varepsilon \sum_{i,j=1}^n a_i^* a_j, \quad a \approx_\varepsilon \sum_{i,j=n+1}^m a_i^* a_j, \quad \sum_{\substack{i,j=1, \\ i \neq j}}^{n+m} a_i^* a_j \approx_\varepsilon 0;$$

- (3) a is residually \mathcal{E} -infinite;
- (4) a is properly \mathcal{E} -infinite;
- (5) a is \mathcal{E} -paradoxical.

Then (1) \Leftrightarrow (2) \Leftarrow (3) \Leftarrow (4) \Leftarrow (5).

Proof. The implications (5) \Rightarrow (4) \Rightarrow (3) are straightforward. By [20, Proposition 3.14], a is properly infinite if and only if it is residually infinite. Since A separates ideals in B , any ideal in B comes from an invariant ideal in A , as in the definition that a is residually \mathcal{E} -infinite. Together with Lemma 5.19, this shows that (3) implies (1).

According to Definition 2.35, $a \in A^+ \setminus \{0\}$ is properly infinite in B if and only if, for all $\varepsilon > 0$, there are $x, y \in a \cdot B$ with $x^*x \approx_\varepsilon a$, $y^*y \approx_\varepsilon b$ and $x^*y \approx_\varepsilon 0$. Without loss of generality, we may pick $x, y \in a \cdot (\sum_{t \in S} \mathcal{E}_t)$ because $\sum_{t \in S} \mathcal{E}_t$ is dense in B . So $x = \sum_{j=1}^n a_j$ and $y = \sum_{j=n+1}^{n+m} a_j$ for some $n, m \in \mathbb{N}$, $t_i \in S$, and $a_i \in a\mathcal{E}_{t_i}$ for $1 \leq i \leq n + m$. The relations $x^*x \approx_\varepsilon a$, $y^*y \approx_\varepsilon b$ and $x^*y \approx_\varepsilon 0$ translate to those described in (2). This proves that (1) and (2) are equivalent. ■

It is unclear whether the implications in Proposition 5.20 may be reversed.

Remark 5.21. The example of graph C^* -algebras shows that it may be much easier to check that an element is residually \mathcal{E} -infinite than that it is properly \mathcal{E} -infinite (see also [34, Remark 7.10]).

Corollary 5.22. *Assume (5.11). Let $\mathcal{F} \subseteq A^+$ residually support A . Suppose that $\mathbb{I}^\mathcal{E}(A)$ is finite or that \mathcal{F} consists of projections, or that the projections in \mathcal{F} separate the ideals in $\mathbb{I}^\mathcal{E}(A)$. If every element in $\mathcal{F} \setminus \{0\}$ is residually \mathcal{E} -infinite, then $A \rtimes_{\text{ess}} S$ is purely infinite and has the ideal property.*

Proof. Combine Theorem 5.16 and Proposition 5.20. ■

We may simplify our conditions further if A is commutative. Then

$$A \rtimes_{\text{ess}} S \cong C_{\text{ess}}^*(G, \Sigma)$$

for a twisted étale groupoid G with object space \hat{A} . The twist Σ is always locally trivial. Therefore, the bisections that trivialise the twist Σ form a wide inverse subsemigroup S' among all bisections of G (see [7, Theorem 7.2]). Then $C_{\text{ess}}^*(G, \Sigma) \cong A \rtimes_{\text{ess}} S'$. The action of S' on A is equivalent to a twisted action as in [7, Definition 4.1], that is, each \mathcal{E}_t for $t \in S$ comes from an isomorphism between two ideals in A . We assume this because it allows us to identify elements of \mathcal{E}_t with C_0 -functions on $s(\mathcal{E}_t) \subseteq \hat{A}$. This discussion shows how to turn any inverse semigroup action on a commutative C^* -algebra into a twisted action by partial automorphisms.

Lemma 5.23. *Assume that A is commutative and that S acts on A by a twisted action by partial automorphisms as in Example 3.2. Equip \hat{A} with the dual action of S . Let $a \in A^+$ and $V := \{x \in \hat{A} : a(x) \neq 0\}$. Consider the following conditions:*

- (1) the condition in Definition 5.18 (1) holds with $\varepsilon = 0$;
- (2) there are $b \in (aAa)^+ \setminus \{0\}$, $n \in \mathbb{N}$ and $t_i \in S$, $a \in a\mathcal{E}_{t_i}$ for $1 \leq i \leq n$ such that

$$a = \sum_{i=1}^n a_i^* a_i, \quad \text{and} \quad a_i^* a_j = 0, \quad a_i^* b = 0 \quad \text{for all } i, j = 1, \dots, n, i \neq j;$$

- (3) there are $n \in \mathbb{N}$, $t_1, \dots, t_n \in S$, and open subsets $V_1, \dots, V_n \subseteq V$ such that
 - (a) V_i is contained in the domain of t_i for $1 \leq i \leq n$;
 - (b) $(t_i \cdot V_i) \cap (t_j \cdot V_j) = \emptyset$ if $1 \leq i < j \leq n$;
 - (c) $V = \bigcup_{i=1}^n V_i$ and $\overline{\bigcup_{i=1}^n t_i \cdot V_i} \subsetneq V$;
- (4) a is \mathcal{E} -infinite.

Then (1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4). The implications (1) \Leftrightarrow (2) \Rightarrow (4) hold in full generality.

Proof. If (2) holds, then taking $m := 1$, $t_{n+1} := 1$ and $a_{n+1} := \sqrt{b}$, we get (1). Conversely, if (1) holds, then there are $t_i \in S$, $a_i \in a\mathcal{E}_{t_i}$ for $i = 1, \dots, n + m$, such that $a = \sum_{i=1}^{n+m} a_i^* a_i$, $\sum_{i=n+1}^{n+m} a_i^* a_i \neq 0$ and $a_i^* a_j = 0$ for $i \neq j$. Thus putting $b := \sum_{i=n+1}^{n+m} a_i a_i^*$, gives (2). This shows that (1) and (2) are equivalent.

Now let $b \in (aAa)^+ \setminus \{0\}$, $n \in \mathbb{N}$, and $t_i \in S$, $a_i \in a\mathcal{E}_{t_i}$ for $i = 1, \dots, n$ be as in (2). We identify the fibres \mathcal{E}_t for $t \in S$ with spaces of sections of the associated line bundle over G . Put $U_i := \{\gamma \in G : \|a_i(\gamma)\| > 0\}$ for $i = 1, \dots, n$ and $W := \{x \in X : \|b(\gamma)\| > 0\}$. Then $V_i := s(U_i) = \{x \in X : (a_i^* a_i)(x) > 0\} \subseteq V$ is contained in the domain of t_i for $1 \leq i \leq n$. The equality $a = \sum_{i=1}^n a_i^* a_i$ implies that $V = \bigcup_{i=1}^n s(U_i) = \bigcup_{i=1}^n V_i$. Similarly, $a_i^* a_j = 0$ holds if and only if $(t_i \cdot V_i) \cap (t_j \cdot V_j) = r(U_i) \cap r(U_j) = \emptyset$ for all $i \neq j$ and $a_i^* b = 0$ holds if and only if $W \subseteq V \setminus \bigcup_{i=1}^n r(U_i) = V \setminus \bigcup_{i=1}^n (t_i \cdot V_i)$; here we identify functions in \mathcal{E}_t with C_0 -functions on bisections. Such a W exists if and only if $\overline{\bigcup_{i=1}^n t_i \cdot V_i} \subsetneq V$. Hence (2) implies (3).

Next we show that (3) implies (4). Let $t_1, \dots, t_n \in S$, and $V_1, \dots, V_n \subseteq V$ be as in (3). Let $b \in A^+ \setminus \{0\}$ be any function that vanishes outside the open set $V \setminus \overline{\bigcup_{i=1}^n t_i \cdot V_i}$. Fix $\varepsilon > 0$. Let $K := \{x \in \hat{A} : a(x) \geq \varepsilon\}$. Let $w_1, \dots, w_n \in A$ be a partition of unity subordinate to the open covering $K \subseteq \bigcup_{i=1}^n V_i$. Let $a_i := (a - \varepsilon)_+^{1/2} \cdot w_i^{1/2}$ for $i = 1, \dots, n$. These functions vanish outside K , and a_i belongs to the domain of t_i . Since \mathcal{E}_{t_i} comes from a partial automorphism, we may view a_i as an element of \mathcal{E}_{t_i} . It belongs to $a \cdot \mathcal{E}_{t_i}$ because the support of a_i is contained in V . The product $a_i^* a_j$ is defined using the Fell bundle structure. If $i \neq j$, then $a_i^* a_j = 0$ because $(t_i \cdot V_i) \cap (t_j \cdot V_j) = \emptyset$. Similarly, we get $a_i^* b = 0$. And

$$\sum_{i=1}^n a_i^* a_i = \sum_{i=1}^n (a - \varepsilon)_+ \cdot w_i = (a - \varepsilon)_+ \approx_\varepsilon a.$$

Hence a is \mathcal{E} -infinite. ■

Remark 5.24. For strongly boundary group actions (see [35]) and, more generally, for filling actions (see [19]) condition (3) in Lemma 5.23 holds for every nonempty open subset V . Thus if \mathcal{E} comes from such an action, then every element in $A^+ \setminus \{0\}$ is \mathcal{E} -infinite. This also holds when A is noncommutative (see [34, Lemma 5.12]).

Remark 5.25. An étale, Hausdorff, locally compact groupoid H is *locally contracting* if for each nonempty open set U in the unit space H^0 of H there is a bisection $B \subseteq H$ with $\overline{r(B)} \subsetneq s(B) \subseteq U$ (see [2]). Given a wide inverse subsemigroup $S \subseteq \text{Bis}(H)$, we may strengthen this criterion by requiring $B \subseteq t$ for some $t \in S$. Then we may rewrite $\overline{r(B)} \subsetneq s(B) \subseteq U$ as follows: there is $t \in S$ and $V \subseteq U$ contained in the domain of t with $\overline{t \cdot V} \subsetneq V$. This is the case $n = 1$ of condition (3) in Lemma 5.23. As a result, if the dual groupoid $\widehat{A} \rtimes S$ is locally contracting, then for any $a \in A^+ \setminus \{0\}$ there is $0 \neq a_2 \leq a$ that is \mathcal{E} -infinite; namely, choose $U = \text{supp } a$ and then a_2 with $\text{supp } a_2 = V$ and $a_2 \leq a$ for V as above.

Condition (3) in Lemma 5.23 could be relaxed so that it still implies \mathcal{E} -infiniteness, by using compact subsets of V . We formulate the relevant condition implying \mathcal{E} -proper infiniteness.

Lemma 5.26. *Retain the assumptions of Lemma 5.23. In particular, let $a \in A^+$ and $V := \{x \in \widehat{A} : a(x) \neq 0\}$. If for each compact subset $K \subseteq V$ there are $n, m \in \mathbb{N}$, $t_1, \dots, t_{n+m} \in S$, and open subsets $V_1, \dots, V_{n+m} \subseteq V$ such that $(t_i \cdot V_i) \cap (t_j \cdot V_j) = \emptyset$ if $1 \leq i < j \leq n + m$, $K \subseteq \bigcup_{i=1}^n V_i$ and $K \subseteq \bigcup_{i=n+1}^{n+m} V_i$, then a is \mathcal{E} -properly infinite.*

Proof. Fix $\varepsilon > 0$. Let $K := \{x \in \widehat{A} : a(x) \geq \varepsilon\}$. Choose n, m, t_i, V_i as in the assumption of the lemma. Let $w_1, \dots, w_n \in A$ and $w_{n+1}, \dots, w_{n+m} \in A$ be partitions of unity subordinate to the open coverings $K \subseteq \bigcup_{i=1}^n V_i$ and $K \subseteq \bigcup_{i=n+1}^{n+m} V_i$, respectively. Let $a_i := (a - \varepsilon)_+^{1/2} \cdot w_i^{1/2}$ for $i = 1, \dots, n + m$. As in the proof of the implication (3) \Rightarrow (4) in Lemma 5.23 one sees that treating a_i as an element of \mathcal{E}_{t_i} , the elements a_i satisfy the relations in Definition 5.18 (3). ■

Now we assume, in addition, that the spectrum \widehat{A} is totally disconnected. This implies that the compact open bisections form a basis for the topology and that A is spanned by projections. We are going to see that a projection is \mathcal{E} -paradoxical if and only if its support is (2, 1)-paradoxical as defined in [6]. Such open subsets give purely infinite elements in the type semigroup considered in [6, 36, 41].

Definition 5.27 ([6]). Let G be an ample groupoid. We say that a compact open set $V \subseteq G^0$ is (2, 1)-paradoxical if there are $n, m \in \mathbb{N}$ and compact open bisections $U_i \subseteq G$ for $1 \leq i \leq n + m$ such that $r(U_i) \subseteq V$ for $1 \leq i \leq n + m$ and

$$V = \bigsqcup_{i=1}^n s(U_i), \quad V = \bigsqcup_{i=n+1}^{n+m} s(U_i), \quad r(U_i) \cap r(U_j) = \emptyset \quad \text{for } i \neq j.$$

Proposition 5.28. *Let \mathcal{E} be an action of an inverse semigroup S by Hilbert bimodules on a commutative C^* -algebra A with totally disconnected spectrum \widehat{A} ; equivalently, the dual groupoid $G := \widehat{A} \rtimes S$ is ample. A projection $a \in A^+$ is \mathcal{E} -paradoxical if and only if its support $V := \{x \in \widehat{A} : a(x) \neq 0\}$ is $(2, 1)$ -paradoxical.*

Proof. Suppose first that $a \in A^+ \setminus \{0\}$ is \mathcal{E} -paradoxical. That is, there are $n, m \in \mathbb{N}$, $t_1, \dots, t_{n+m} \in S$, and $a_i \in a\mathcal{E}_{t_i}$ such that $a = \sum_{i=1}^n a_i^* a_i = \sum_{i=n+1}^{n+m} a_i^* a_i$ and $a_i^* a_j = 0$ for $i \neq j$. Let $1 \leq i \leq n + m$. Recall that we may treat \mathcal{E}_{t_i} as spaces of sections A_{U_i} of a line bundle over $G = \widehat{A} \rtimes S$ that are supported on open bisections $U_i \in \text{Bis}(G)$. Thus $U_i := \{\gamma \in G : \|a_i(\gamma)\| > 0\}$ is an open bisection of G contained in U_i . Since $a_i \in aA_{U_i}$, we have $r(U_i) = \{x \in \widehat{A} : (a_i a_i^*)(x) > 0\} \subseteq V$. And $a_i^* a_j = 0$ implies that $r(U_i) \cap r(U_j) = \emptyset$ for all $i \neq j$. Since $\{x \in X : (a_i^* a_i)(x) > 0\} = s(U_i)$, the equalities $a = \sum_{i=1}^n a_i^* a_i$ and $a = \sum_{i=n+1}^{n+m} a_i^* a_i$ imply $V = \bigsqcup_{i=1}^n s(U_i)$ and $V = \bigsqcup_{i=n+1}^{n+m} s(U_i)$. Hence the family $U_i \in \text{Bis}(G)$ for $1 \leq i \leq n + m$ has all the desired properties, except that U_i need not be compact. However, since G is ample, every U_i is a union of some compact open bisections. Since V is compact and $V = \bigsqcup_{i=1}^n s(U_i) = \bigsqcup_{i=n+1}^{n+m} s(U_i)$, we may, in fact, replace each U_i for $1 \leq i \leq n + m$ by a finite union of compact open bisections. This gives a compact open bisection.

Conversely, let $U_i \subseteq G$ for $1 \leq i \leq n + m$ be a family of bisection as in Definition 5.27. Let $S' \subseteq \text{Bis}(G)$ be the family of open compact bisections that trivialise the twist, that is, the restrictions of the associated line bundle over G to sets in S' are trivial. Note that S' forms an inverse semigroup and a basis for the topology of G ; this holds for the family of all open bisections that trivialise the twist, by the proof of [7, Theorem 7.2], and for the family of all compact open bisections because G is ample. Since

$$V = \bigsqcup_{i=1}^n s(U_i) = \bigsqcup_{j=n+1}^{n+m} s(U_j)$$

is compact, for each $i = 1, \dots, n$ we may find a finite family of sets $(U_{i,j})_{j=1}^{n_i} \subseteq S'$ such that $\bigcup_{j=1}^{n_i} U_{i,j} \subseteq U_i$ and $V = \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} s(U_{i,j})$. Since the bisections $(U_{i,j})_{j=1}^{n_i} \subseteq U_i$ are closed and open, we may arrange that the sets $(s(U_{i,j}))_{j=1}^{n_i}$ are pairwise disjoint. Then the sets $(U_{i,j})_{i=1, j=1}^{n, n_i}$ are pairwise disjoint, and since $\bigcup_{j=1}^{n_i} U_{i,j} \subseteq U_i$, for $i = 1, \dots, n$, also $(r(U_{i,j}))_{i=1, j=1}^{n, n_i}$ are pairwise disjoint. We put $a_{i,j} := 1_{U_{i,j}}$, for $i = 1, \dots, n, j = 1, \dots, n_i$. By the choice of bisections in S' , we may treat $a_{i,j}$ as an element of the space $C_c(U_{ij})$ of sections of the line bundle over G . By the construction of the Fell bundle over $\widehat{A} \rtimes S$, by passing if necessary to smaller sets, we may assume that each space $C_c(U_{i,j})$ is contained in $\mathcal{E}_{t_{ij}}$ for some $t_{ij} \in S$. Hence $a_{i,j} \in \mathcal{E}_{t_{ij}}$ for all i, j . Using the Fell bundle structure, we get

$$\sum_{i=1, j=1}^{n, n_i} a_{i,j}^* \cdot a_{i,j} = \sum_{i=1, j=1}^{n, n_i} 1_{s(U_{i,j})} = 1_V = a.$$

Similarly, we get $\sum_{i=n+1, j=1}^{n+m, n_i} a_{i,j}^* \cdot a_{i,j} = a$ and $a_{i,j}^* a_{i',j'} = 0$ for all $(i, j) \neq (i', j')$. Hence a is \mathcal{E} -paradoxical. ■

Corollary 5.29. *Let (G, Σ) be an essentially exact twisted groupoid where G is ample and residually topologically free with locally compact Hausdorff $X := G^0$. If every compact open subset of X is $(2, 1)$ -paradoxical, then the essential C^* -algebra $C_{\text{ess}}^*(G, \Sigma)$ is purely infinite (and has the ideal property).*

Proof. View $C_{\text{ess}}^*(G, \Sigma)$ as the essential crossed product by an inverse semigroup action \mathcal{E} on $C_0(X)$ as in Examples 3.21 and 3.22. The assertion follows from Proposition 5.28 and Corollary 5.22. ■

Remark 5.30. When G is Hausdorff, then $C_{\text{ess}}^*(G, \Sigma) = C_r^*(G, \Sigma)$ and (G, Σ) is essentially exact if and only if it is inner exact. Thus Corollary 5.29 generalises the pure infiniteness criteria in [6, 41], where the authors considered Hausdorff ample groupoids without a twist. They proved, in addition, that if the type semigroup associated to G is almost unperforated, then the implication in Corollary 5.29 may be reversed. We will generalise this and some other results of Ma [36] to étale twisted groupoids in the forthcoming paper [33].

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