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Mixed q-deformed Araki–Woods von Neumann algebras

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Abstract. Given a strongly continuous orthogonal representation $(U_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, a decomposition $\mathcal{H}_{\mathbb{R}} := \bigoplus_{i\in N} \mathcal{H}_{\mathbb{R}}^{(i)}$ consisting of invariant subspaces of $(U_t)_{t\in\mathbb{R}}$ and an appropriate matrix $((q_{ij}))_{N\times N}$ of real parameters, we associate representations of the mixed commutation relations on twisted Fock spaces. The associated von Neumann algebras are (usually) non-tracial and are generalizations of those constructed by Bożejko–Speicher and Hiai. We investigate the factoriality of these von Neumann algebras. Along the process, we show that the generating abelian subalgebras associated with the blocks of the aforesaid decomposition are strongly mixing masas when they admit appropriate conditional expectations. On the contrary, the generating abelian algebras which fail to admit appropriate conditional expectations are quasi-split. We also discuss non-injectivity and the Haagerup approximation property.

1. Introduction

In free probability, Voiculescu's C^*-free Gaussian functor associates a canonical C^*-algebra denoted by $\Gamma(\mathcal{H}_{\mathbb{R}})$ with a real Hilbert space $\mathcal{H}_{\mathbb{R}}$. The C^*-algebra $\Gamma(\mathcal{H}_{\mathbb{R}})$ is generated by the sum of canonical creation and annihilation operators on $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$, the full Fock space of the complexification of $\mathcal{H}_{\mathbb{R}}$. It is well known that the associated von Neumann algebra $\Gamma(\mathcal{H}_{\mathbb{R}})''$ is isomorphic to the free group factor $L(\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})})$ if $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ (see [31]) and is the central object of study in free probability.

There are natural deformations of Voiculescu's functor in the literature. The most prominent ones are the q-Gaussian functor due to Bożejko and Speicher [7] and the free CAR functor due to Shlyakhtenko [26]. The associated von Neumann algebras are called q-Gaussian von Neumann algebras and free Araki–Woods factors. These von Neumann algebras are very well studied.

Free Araki–Woods factors are type III counterparts of the free group factors. They are full factors when $(U_t)_{t\in\mathbb{R}}$ is non-trivial and $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. They have many more interesting properties; for example, they lack Cartan subalgebras, satisfy the complete metric approximation property and are strongly solid (see [5, 17]).

There is also a generalization of the q-Gaussian functor, namely, the mixed q-Gaussian functor introduced in [8]. The associated von Neumann algebras, namely, the *mixed q-Gaussian* von Neumann algebras, are tracial.

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In [16], Hiai introduced a new functor by combining the ones considered in [7, 26]. The associated von Neumann algebras are called q-deformed Araki–Woods von Neumann algebras and are known to be very complicated objects. The factoriality of these algebras is not even known in the fullest generality.

In this paper, we extend Hiai's construction by combining Shlyakhtenko's construction in [26] and the mixed q-Gaussian functor due to Bożejko and Speicher in [8]. Thus, in our context, the associated von Neumann algebras depend on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, a strongly continuous group of orthogonal representation $(U_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$ and parameters $q_{ij} \in (-1, 1)$ such that $q_{ij} = q_{ji}$ for all i, j, and $\sup_{i,j} |q_{ij}| < 1$, $i, j \in N$ where $N = \{1, 2, \dots, r\}$, $r \in \mathbb{N}$, or $N = \mathbb{N}$. We call these algebras the mixed q-deformed Araki–Woods von Neumann algebras. Our construction is also functorial. As expected, these algebras act in standard form on “twisted” Fock spaces, and the associated $(U_t)_{t\in\mathbb{R}}$ encodes the data of the modular automorphisms associated with the canonical vacuum state.

Like the q-commutation relations associated with the q-deformed Araki–Woods von Neumann algebras, here too, we obtain the mixed q_{ij}-commutation relations. Thus, our construction provides Fock-type representations of the following relations:

$$l^*(\xi)l(\eta) - q_{ij}l(\eta)l^*(\xi) = \langle \xi, \eta\rangle_U 1 \quad \text{for } \xi \in \mathcal{H}_{\mathbb{R}}^{(i)},\ \eta \in \mathcal{H}_{\mathbb{R}}^{(j)},$$

where $\mathcal{H}_{\mathbb{R}}^{(i)}$ and $\mathcal{H}_{\mathbb{R}}^{(j)}$ are appropriate subspaces of $\mathcal{H}_{\mathbb{R}}$ and $\langle\cdot,\cdot\rangle_U$ is the inner product on a complexification of $\mathcal{H}_{\mathbb{R}}$ that has been twisted by the representation $(U_t)_{t\in\mathbb{R}}$. The above commutation relations clearly generalize the commutation relations considered in [8, 16].

Now, we discuss the overview of this paper. This paper relies heavily on the techniques developed in [1, 16]. In Section 2, we construct the mixed q-deformed Araki–Woods von Neumann algebras combining the construction in [8, 16]. Subsequently, we study the standard representation of these algebras with respect to the canonical vacuum state and the modular theory associated with the vacuum state in Section 3.

Section 4 is devoted to the study of generator masas in the mixed q-deformed Araki–Woods von Neumann algebras. Following [1], we show that a canonical self-adjoint generator of the aforesaid von Neumann algebra corresponding to a unit vector $\xi \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, generates a masa with appropriate conditional expectation if and only if $U_t\xi = \xi$ for all $t \in \mathbb{R}$. In fact, such a masa is strongly mixing (with respect to the vacuum state) (Theorems 4.5 and 4.11). Similar ideas can be found in [25, 27].

Section 5 is concerned with the factoriality of the mixed q-deformed Araki–Woods von Neumann algebras, which is a hard problem. Hiai proved the factoriality of the q-deformed Araki–Woods von Neumann algebras, when the almost periodic part of $(U_t)_{t\in\mathbb{R}}$ is infinite-dimensional [16, Thm. 3.2]. The factoriality of the same was proved in [1] under the assumption that $(U_t)_{t\in\mathbb{R}}$ is non-ergodic and $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, or $(U_t)_{t\in\mathbb{R}}$ has a non-trivial weakly mixing component for all $q \in (-1, 1)$. Unfortunately, there is a gap in the proof of [16, Thm. 3.2]. To be precise, Hiai's proof holds only in the case when the set of eigenvalues of the analytic generator of $(U_t)_{t\in\mathbb{R}}$ has a limit point in $\mathbb{R}$ other than 0. Without this assumption, the conclusion “$\varphi(y^*x) = 0$” in the last equation in [16, Thm. 3.2] would fail, and hence, the final statement cannot be concluded. Thus, in Section 5, assuming the same hypothesis that fixes [16, Thm. 3.2] and adapting the techniques in [16] and also the techniques in [1], we prove the factoriality and type classification of the mixed q-deformed Araki–Woods von Neumann algebras. As the results in [1], our results on factoriality are partial.

With the results in Section 5, namely, Theorems 5.4, 5.1 and 5.2, the factoriality of the mixed q-deformed Araki–Woods von Neumann algebras remain open only in the cases when:

(1) $\dim(\mathcal{H}_{\mathbb{R}})$ is even and $(U_t)_{t\in\mathbb{R}}$ is ergodic;

(2) $(U_t)_{t\in\mathbb{R}}$ is ergodic, almost periodic, and 0 is the only limit point of the set of eigenvalues of the analytic generator of $(U_t)_{t\in\mathbb{R}}$.

Further, in Theorems 5.7, 5.8 and 5.9, we show that in the cases the mixed q-deformed Araki–Woods von Neumann algebras are factors, their type is completely determined by the spectral data of $(U_t)_{t\in\mathbb{R}}$.

In Section 6, we show that the construction of the mixed q-deformed Araki–Woods von Neumann algebras is functorial. Using this statement, in the same section, we show that these von Neumann algebras have the Haagerup property.

In Section 7, we show that, in contrast to the results proved in Section 4, if $\xi \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, is a unit vector *not* fixed by $(U_t)_{t\in\mathbb{R}}$, then the inclusion of the associated abelian subalgebra in the mixed q-deformed Araki–Woods von Neumann algebra is *split* whenever the latter is a type III factor, forcing such abelian subalgebras to have huge relative commutants. Thus, when $(U_t)_{t\in\mathbb{R}}$ is ergodic, there is no obvious way to construct masas in such von Neumann algebras.

Finally, in Section 8, by adapting the techniques in [16], we show that the mixed q-deformed Araki–Woods von Neumann algebras are non-injective in many cases.

The last section is an appendix. In Appendix A, we have included some known results concerning Hilbert spaces which are inevitable for our purpose but for which we lack an appropriate reference.

2. Mixed q-deformed Araki–Woods algebras: Construction

In this section, we describe the construction of the mixed q-deformed Araki–Woods von Neumann algebras. Our construction generalizes the constructions considered in [8, 16]. Following [16], we begin with a real Hilbert space and a strongly continuous one-parameter orthogonal group on it. As in [8], our construction also involves an operator T which is a self-adjoint contraction and satisfies the Yang–Baxter relation. As a convention, all Hilbert spaces in this paper are assumed to be separable, all von Neumann algebras have separable preduals and inner products are linear in the *second variable*. There is some overlap of materials in this section with [26] to keep the paper self-contained. We proceed to describe the construction below.

Let $\mathcal{H}_{\mathbb{R}}$ be a separable real Hilbert space and let $t \mapsto U_t$, $t \in \mathbb{R}$, be a strongly continuous orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$. Let $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$ denote the complexification of $\mathcal{H}_{\mathbb{R}}$. We denote the inner product and norm on $\mathcal{H}_{\mathbb{C}}$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$ and $\| \cdot \|_{\mathcal{H}_{\mathbb{C}}}$, respectively. Identify $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}_{\mathbb{C}}$ by $\mathcal{H}_{\mathbb{R}} \otimes 1$. Since $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$, as a real Hilbert space, the inner product of $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}_{\mathbb{C}}$ is given by $\Re\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$. Consider the bounded antilinear operator (complex conjugation) $\mathcal{J} : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ defined as $\mathcal{J}(\xi + i\eta) = \xi - i\eta$ for $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$. Note that $\mathcal{J}\xi = \xi$ for all $\xi \in \mathcal{H}_{\mathbb{R}}$. Moreover,

$$\langle \xi, \eta \rangle_{\mathcal{H}_{\mathbb{C}}} = \overline{\langle \eta, \xi \rangle}_{\mathcal{H}_{\mathbb{C}}} = \langle \eta, \mathcal{J}\xi \rangle_{\mathcal{H}_{\mathbb{C}}} \quad \text{for all } \xi \in \mathcal{H}_{\mathbb{C}} \text{ and } \eta \in \mathcal{H}_{\mathbb{R}}. \tag{1}$$

Linearly extend the flow $t \mapsto U_t$ from $\mathcal{H}_{\mathbb{R}}$ to a strongly continuous one-parameter group of unitaries on $\mathcal{H}_{\mathbb{C}}$. We denote the extension again by U_t for each $t \in \mathbb{R}$ with slight abuse of notation. Let A denote the analytic generator and H the associated Hamiltonian of the strongly continuous one-parameter group $\{U_t : t \in \mathbb{R}\}$ acting on $\mathcal{H}_{\mathbb{C}}$. Then, A is positive, nonsingular and self-adjoint, while H is self-adjoint. Since $\mathcal{H}_{\mathbb{R}}$ is invariant under U_t for all $t \in \mathbb{R}$, $\mathcal{H}_{\mathbb{R}}$ is invariant for iH as well. Let us denote $\mathfrak{D}(\cdot)$ to be the domain of an (unbounded) operator. One notes that $\mathfrak{D}(H) = \mathfrak{D}(iH)$ and H maps $\mathfrak{D}(H) \cap \mathcal{H}_{\mathbb{R}}$ into $i\mathcal{H}_{\mathbb{R}}$. It follows that $\mathcal{J}H = -H\mathcal{J}$ and $\mathcal{J}A = A^{-1}\mathcal{J}$.

Define a new inner product $\langle \cdot, \cdot \rangle_U$ on $\mathcal{H}_{\mathbb{C}}$ as follows:

$$\langle \xi, \eta \rangle_U = \left\langle \frac{2}{1 + A^{-1}}\xi, \eta \right\rangle_{\mathcal{H}_{\mathbb{C}}} \quad \text{for } \xi, \eta \in \mathcal{H}_{\mathbb{C}}.$$

Denote the completion of $\mathcal{H}_{\mathbb{C}}$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_U$ by $\mathcal{H}$. We denote the inner product and norm on $\mathcal{H}$ by $\langle \cdot, \cdot \rangle_U$ and $\| \cdot \|_U$, respectively. Since A is affiliated to $vN(U_t : t \in \mathbb{R})$, one has

$$\langle U_t\xi, U_t\eta \rangle_U = \langle \xi, \eta \rangle_U \quad \text{for } \xi, \eta \in \mathcal{H}_{\mathbb{C}}.$$

Consequently, $(U_t)_{t\in\mathbb{R}}$ extends to a strongly continuous unitary representation $(\widetilde{U_t})_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{H}$. Let $\widetilde{A}$ be the analytic generator associated with $(\widetilde{U_t})_{t\in\mathbb{R}}$, which is clearly an extension of A. From [1, Prop. 2.1] and the discussion prior to it, one notes that the spectral data of A and $\widetilde{A}$ are essentially the same. Therefore, we denote the extensions $(\widetilde{U_t})_{t\in\mathbb{R}}$ and $\widetilde{A}$ again by $(U_t)_{t\in\mathbb{R}}$ and A, respectively, with slight abuse of notation.

A vector $\xi \in \mathcal{H}$ is said to be analytic with respect to the strongly continuous one-parameter group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ if the mapping $\mathbb{R} \ni t \mapsto U_t\xi \in \mathcal{H}$ has a weakly entire extension on $\mathcal{H}$. The value of the extended function at $z \in \mathbb{C}$ is denoted by $U_z\xi$. Further, it is easy to check that $U_z = A^{iz}$ for all $z \in \mathbb{C}$.

The next few computations are statutory and follow from [26]. We present them for the sake of completeness. Let $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$. Then,

$$\begin{aligned}\langle \xi, \eta \rangle_U &= \left\langle \frac{2}{1 + A^{-1}}\xi, \eta \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \left\langle \eta, \mathcal{J}\frac{2}{1 + A^{-1}}\xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \left\langle \eta, \frac{2}{1 + A}\xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} \\ &= \left\langle \eta, \frac{2A^{-1}}{1 + A^{-1}}\xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \left\langle \frac{2}{1 + A^{-1}}\eta, A^{-1}\xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \langle \eta, A^{-1}\xi \rangle_U.\end{aligned} \tag{2}$$

Also,

$$\langle \xi, \xi \rangle_U = \left\langle \frac{2}{1 + A^{-1}} \xi, \xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \langle \xi, \xi \rangle_{\mathcal{H}_{\mathbb{C}}} + \left\langle \frac{1 - A^{-1}}{1 + A^{-1}} \xi, \xi \right\rangle_{\mathcal{H}_{\mathbb{C}}}. \tag{3}$$

But

$$\left\langle \frac{1 - A^{-1}}{1 + A^{-1}} \xi, \xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \left\langle \xi, \frac{1 - A}{1 + A} \xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = -\left\langle \xi, \frac{1 - A^{-1}}{1 + A^{-1}} \xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = 0. \tag{4}$$

Thus, it follows from (3) and (4) that $\|\xi\|_U = \|\xi\|_{\mathcal{H}_{\mathbb{C}}}$ for all $\xi \in \mathcal{H}_{\mathbb{R}}$. Hence, $\mathcal{H}_{\mathbb{R}}$ embeds in $\mathcal{H}$ isometrically as a real Hilbert space.

Let N denote $\{1, 2, \ldots, r\}$, $r \in \mathbb{N}$, or $\mathbb{N}$. Fix a decomposition of $\mathcal{H}_{\mathbb{R}}$ as follows:

$$\mathcal{H}_{\mathbb{R}} := \bigoplus_{i \in N} \mathcal{H}_{\mathbb{R}}^{(i)}, \tag{5}$$

where $\mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, are non-trivial invariant subspaces of $(U_t)_{t \in \mathbb{R}}$ (direct sum taken with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$). Choose $-1 < q_{ij} = q_{ji} < 1$ for $i, j \in N$ with $\sup_{i,j \in N} |q_{ij}| < 1$. In this paper, we will often denote the scalar q_{ij} also by $q(i, j)$ for $i, j \in N$.

Note that $\mathcal{H}_{\mathbb{C}} = \bigoplus_{i \in N} \mathcal{H}_{\mathbb{C}}^{(i)}$, where $\mathcal{H}_{\mathbb{C}}^{(i)}$ is the complexification of $\mathcal{H}_{\mathbb{R}}^{(i)}$ for $i \in N$. Also, since $\mathcal{H}_{\mathbb{C}}^{(i)}$, $i \in N$, are invariant for $(U_t)_{t \in \mathbb{R}}$, it follows that $\mathcal{H} = \bigoplus_{i \in N} \mathcal{H}^{(i)}$, where $\mathcal{H}^{(i)}$, $i \in N$, are, respectively, the completions of $\mathcal{H}_{\mathbb{C}}^{(i)}$, $i \in N$, with respect to $\langle \cdot, \cdot \rangle_U$. For $\xi \in \mathcal{H}$, the associated unique decomposition will be denoted by $\xi := \bigoplus_{i \in N} \xi^{(i)}$.

Fix $i, j \in N$. Define $T_{i,j} : \mathcal{H}_{\mathbb{R}}^{(i)} \otimes \mathcal{H}_{\mathbb{R}}^{(j)} \to \mathcal{H}_{\mathbb{R}}^{(j)} \otimes \mathcal{H}_{\mathbb{R}}^{(i)}$ to be the bounded extension of

$$\xi \otimes \eta \mapsto q_{ij} (\eta \otimes \xi) \quad \text{for } \xi \in \mathcal{H}_{\mathbb{R}}^{(i)},\ \eta \in \mathcal{H}_{\mathbb{R}}^{(j)}.$$

Then, $T_{\mathbb{R}} := \bigoplus_{i,j \in N} T_{i,j} \in \mathbf{B}(\mathcal{H}_{\mathbb{R}} \otimes \mathcal{H}_{\mathbb{R}})$. Linearly extend $T_{\mathbb{R}}$ to $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$ and denote the extension by $T_{\mathbb{C}}$.

By a simple density argument, it follows that $T_{\mathbb{C}}$ admits a unique bounded extension T to $\mathcal{H} \otimes \mathcal{H}$. It is easy to verify that $T := \bigoplus_{i,j \in N} T_{ij}$, where $T_{ij} : \mathcal{H}^{(i)} \otimes \mathcal{H}^{(j)} \to \mathcal{H}^{(j)} \otimes \mathcal{H}^{(i)}$ is defined as the bounded extension of the map

$$\xi \otimes \eta \mapsto q_{ij} (\eta \otimes \xi) \quad \text{for } \xi \in \mathcal{H}^{(i)},\ \eta \in \mathcal{H}^{(j)}. \tag{6}$$

Moreover, T has the following properties:

$$\begin{aligned} &T^* = T, && (\text{since } q_{ij} = q_{ji} \text{ for } i, j \in N), \\ &\|T\|_{\mathcal{H} \otimes \mathcal{H}} < 1, && \big(\text{since } \sup_{i,j \in N} |q_{ij}| < 1\big), \\ &(1 \otimes T)(T \otimes 1)(1 \otimes T) = (T \otimes 1)(1 \otimes T)(T \otimes 1), && \end{aligned} \tag{7}$$

where $1 \otimes T$ and $T \otimes 1$ are the natural amplifications of T to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. The third relation listed in (7) is referred to as the Yang–Baxter equation (see [18, 20, 34]).

Let $\mathcal{F}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ be the full Fock space of $\mathcal{H}$, where Ω is a distinguished unit vector (vacuum vector) in $\mathbb{C}$. By convention, $\mathcal{H}^{\otimes 0} := \mathbb{C}\Omega$. The canonical inner product and norm on $\mathcal{F}(\mathcal{H})$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{F}(\mathcal{H})}$ and $\| \cdot \|_{\mathcal{F}(\mathcal{H})}$, respectively.

For $\xi \in \mathcal{H}$, let $a(\xi)$ and $a^*(\xi)$ denote the canonical left creation and annihilation operators acting on $\mathcal{F}(\mathcal{H})$ which are defined as follows:

$$\begin{aligned} a(\xi)\Omega &= \xi, \quad a(\xi)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n, \\ a^*(\xi)\Omega &= 0, \quad a^*(\xi)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \langle \xi, \xi_1 \rangle_U \xi_2 \otimes \cdots \otimes \xi_n, \end{aligned} \tag{8}$$

where $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$ ($\mathcal{H}^{\odot n}$ denoting the n-fold algebraic tensor product of $\mathcal{H}$) for $n \geq 1$. The operators $a(\xi)$ and $a^*(\xi)$ are bounded and are adjoints of each other on $\mathcal{F}(\mathcal{H})$.

Let T_i be the operator acting on $\mathcal{H}^{\otimes(i+1)}$ for $i \in \mathbb{N}$ as follows:

$$T_i := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes T. \tag{9}$$

Extend T_i to $\mathcal{H}^{\otimes n}$ for all $n > i+1$ by $T_i \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1}$ and denote the extension again by T_i with slight abuse of notation.

Let S_n denote the symmetric group of n elements. Note that S_1 is trivial. For $n \geq 2$, let τ_i be the transposition between i and $i+1$. It is well known that $\{\tau_i\}_{i=1}^{n-1}$ is a generating set of S_n.

For $n \in \mathbb{N}$, let $\pi : S_n \to \mathbf{B}(\mathcal{H}^{\otimes n})$ be the quasi-multiplicative extension of the map given by $\pi(1) = 1$ and $\pi(\tau_i) = T_i$ ($i = 1, \ldots, n-1$). The extension is well defined and unique, and this follows from the proof of [4, Prop. 5].

Consider $P^{(n)} \in \mathbf{B}(\mathcal{H}^{\otimes n})$, defined as follows:

$$P^{(n)} := \sum_{\sigma \in S_n} \pi(\sigma). \tag{10}$$

By convention, $P^{(0)}$ on $\mathcal{H}^{\otimes 0}$ is identity.

From the properties of T in (7) and [8, Thm. 2.3], it follows that $P^{(n)}$ is a strictly positive operator for every $n \in \mathbb{N}$. Following [8], the association

$$\langle \xi, \eta \rangle_T = \delta_{n,m} \langle \xi, P^{(n)} \eta \rangle_{\mathcal{F}(\mathcal{H})} \quad \text{for } \xi \in \mathcal{H}^{\otimes m},\ \eta \in \mathcal{H}^{\otimes n}, \tag{11}$$

defines a definite sesquilinear form on $\mathcal{F}(\mathcal{H})$, and let $\mathcal{F}_T(\mathcal{H})$ denote the completion of $\mathcal{F}(\mathcal{H})$ with respect to the norm on $\mathcal{F}(\mathcal{H})$ induced by $\langle \cdot, \cdot \rangle_T$. We denote the inner product and the norm on $\mathcal{F}_T(\mathcal{H})$ by $\langle \cdot, \cdot \rangle_T$ and $\| \cdot \|_T$, respectively. We also denote $\mathcal{F}_T^{\text{finite}}(\mathcal{H}) := \text{span}_{\mathbb{C}}\{\mathcal{H}^{\otimes n},\ n \geq 0\}$ and $\mathcal{H}^{\otimes_T^n} = \overline{\mathcal{H}^{\otimes n}}^{\|\cdot\|_T}$ for $n \in \mathbb{N}$.

Lemma 2.1. *For $n, m \in \mathbb{N}$, let $\eta_{i_k} \in \mathcal{H}^{(i_k)}$ for $i_k \in N$, $1 \leq k \leq n$, and $\xi_{j_l} \in \mathcal{H}^{(j_l)}$ for $j_l \in N$, $1 \leq l \leq m$. Let $\boldsymbol{\eta} = \eta_{i_1} \otimes \cdots \otimes \eta_{i_n}$ and $\boldsymbol{\xi} = \xi_{j_1} \otimes \cdots \otimes \xi_{j_m}$. Then,*

$$\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle_T = \delta_{n,m} \sum_{\sigma \in S_n} a(\sigma, \boldsymbol{\xi}) \langle \eta_{i_1}, \xi_{j_{\sigma(1)}} \rangle_U \cdots \langle \eta_{i_n}, \xi_{j_{\sigma(m)}} \rangle_U, \tag{12}$$

where $a(\sigma, \boldsymbol{\xi})$ is given by

$$a(\sigma, \boldsymbol{\xi}) = \begin{cases} 1, & \text{if } \sigma = id, \\ \prod_{t=1}^{k-1} q(j_{\sigma_t(u_{k-t})}, j_{\sigma_t(u_{k-t}+1)}) q(j_{u_k}, j_{u_k+1}), & \text{if } \sigma := \tau_{u_1} \cdots \tau_{u_k}, \end{cases}$$

where $\sigma := \tau_{u_1}\cdots\tau_{u_k}$ is the reduced product of transpositions and σ_t denotes the permutation $\tau_{u_{k-t+1}}\cdots\tau_{u_k}$ for $1 \le t \le k-1$.

Proof. Fix $\sigma \in S_m$. Let σ be written as the reduced product of transpositions as $\sigma := \tau_{u_1}\cdots\tau_{u_k}$. First, we show that

$$T_{u_1}\cdots T_{u_k}(\xi_{j_1}\otimes\cdots\otimes\xi_{j_m}) = a(\sigma,\boldsymbol{\xi})(\xi_{j_{\sigma(1)}}\otimes\cdots\otimes\xi_{j_{\sigma(m)}}).$$

Note that

$$\begin{aligned}
&T_{u_k}(\xi_{j_1}\otimes\cdots\otimes\xi_{j_m})\\
&= q(j_{u_k}, j_{u_k+1})(\xi_{j_1}\otimes\cdots\otimes\xi_{j_{u_k-1}}\otimes\xi_{j_{u_k+1}}\otimes\xi_{j_{u_k}}\otimes\cdots\otimes\xi_{j_m}) \quad \big(\text{by } (9)\big)\\
&= q(j_{u_k}, j_{u_k+1})(\xi_{j_{\tau_{u_k}(1)}}\otimes\cdots\otimes\xi_{j_{\tau_{u_k}(m)}}).
\end{aligned} \tag{13}$$

Again,

$$\begin{aligned}
&T_{u_{k-1}}T_{u_k}(\xi_{j_1}\otimes\cdots\otimes\xi_{j_m})\\
&= T_{u_{k-1}}\big(q(j_{u_k}, j_{u_k+1})(\xi_{j_{\tau_{u_k}(1)}}\otimes\cdots\otimes\xi_{j_{\tau_{u_k}(m)}})\big) \quad \big(\text{by } (13)\big)\\
&= q(j_{u_k}, j_{u_k+1})T_{u_{k-1}}(\xi_{j_{\tau_{u_k}(1)}}\otimes\cdots\otimes\xi_{j_{\tau_{u_k}(m)}})\\
&= q(j_{u_k}, j_{u_k+1})q(j_{\tau_{u_k}(u_{k-1})}, j_{\tau_{u_k}(u_{k-1}+1)})(\xi_{j_{\tau_{u_{k-1}}\tau_{u_k}(1)}}\otimes\cdots\otimes\xi_{j_{\tau_{u_{k-1}}\tau_{u_k}(m)}}) \quad \big(\text{by } (9)\big)\\
&= q(j_{u_k}, j_{u_k+1})q(j_{\sigma_1(u_{k-1})}, j_{\sigma_1(u_{k-1}+1)})(\xi_{j_{\tau_{u_{k-1}}\tau_{u_k}(1)}}\otimes\cdots\otimes\xi_{j_{\tau_{u_{k-1}}\tau_{u_k}(m)}}).
\end{aligned}$$

Iterating as above, one gets

$$\begin{aligned}
&T_{u_1}\cdots T_{u_k}(\xi_{j_1}\otimes\cdots\otimes\xi_{j_m})\\
&\quad= \prod_{t=1}^{k-1} q(j_{\sigma_t(u_{k-t})}, j_{\sigma_t(u_{k-t}+1)})q(j_{u_k}, j_{u_k+1})(\xi_{j_{\sigma(1)}}\otimes\cdots\otimes\xi_{j_{\sigma(m)}})\\
&\quad= a(\sigma,\boldsymbol{\xi})(\xi_{j_{\sigma(1)}}\otimes\cdots\otimes\xi_{j_{\sigma(m)}}).
\end{aligned} \tag{14}$$

Then,

$$\begin{aligned}
\langle\boldsymbol{\eta},\boldsymbol{\xi}\rangle_T &= \delta_{n,m}\big\langle\eta_{i_1}\otimes\cdots\otimes\eta_{i_n}, P^{(m)}(\xi_{j_1}\otimes\cdots\otimes\xi_{j_m})\big\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{by } (11)\big)\\
&= \delta_{n,m}\Big\langle\eta_{i_1}\otimes\cdots\otimes\eta_{i_n}, \sum_{\sigma\in S_m}\pi(\sigma)(\xi_{j_1}\otimes\cdots\otimes\xi_{j_m})\Big\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{by } (10)\big)\\
&= \delta_{n,m}\Big\langle\eta_{i_1}\otimes\cdots\otimes\eta_{i_n}, \sum_{\sigma\in S_m}a(\sigma,\boldsymbol{\xi})\xi_{j_{\sigma(1)}}\otimes\cdots\otimes\xi_{j_{\sigma(m)}}\Big\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{by } (14)\big)\\
&= \delta_{n,m}\sum_{\sigma\in S_n}a(\sigma,\boldsymbol{\xi})\langle\eta_{i_1}\otimes\cdots\otimes\eta_{i_n}, \xi_{j_{\sigma(1)}}\otimes\cdots\otimes\xi_{j_{\sigma(m)}}\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{as } a(\sigma,\boldsymbol{\xi})\in\mathbb{R}\big)\\
&= \delta_{n,m}\sum_{\sigma\in S_n}a(\sigma,\boldsymbol{\xi})\langle\eta_{i_1},\xi_{j_{\sigma(1)}}\rangle_U\cdots\langle\eta_{i_n},\xi_{j_{\sigma(m)}}\rangle_U.
\end{aligned}$$

This completes the proof. ∎

Note that $\bar{\mathcal{H}}^{\|\cdot\|_T} = \mathcal{H}$. Following [8], for $\xi \in \mathcal{H}$, consider the T-deformed left creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$ defined as follows:

$$l(\xi) := a(\xi),$$
$$l^*(\xi) := \begin{cases} a^*(\xi)(1 + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1}), & \text{on } \mathcal{H}^{\otimes n}, \\ 0, & \text{on } \mathbb{C}\Omega. \end{cases} \tag{15}$$

Then, $l(\xi)$ and $l^*(\xi)$ admit bounded extensions to $\mathcal{F}_T(\mathcal{H})$, and we have the following.

Proposition 2.2. *Let $\xi \in \mathcal{H}$. Then, the following hold:*

(i) *if $\|T\|_{\mathcal{H}\otimes\mathcal{H}} = q < 1$, then $\|l(\xi)\| \le \|\xi\|_U(1-q)^{-\frac{1}{2}}$;*

(ii) *$l(\xi)$ and $l^*(\xi)$ are adjoints of each other on $\mathcal{F}_T(\mathcal{H})$.*

Proof. The proof follows exactly along the same lines of [8, Thm. 3.1]. We omit the details. ∎

Following (15), the definition of $l^*(\xi)$ involves the operator T. From (6), it follows that the action of $l^*(\xi)$ on $\mathcal{F}_T(\mathcal{H})$ is as follows. Fix $n \in \mathbb{N}$, and $\xi_{i_k} \in \mathcal{H}^{(i_k)}$ for $i_k \in N$ and $1 \le k \le n$. Then,

$$\begin{aligned} &l^*(\xi)\Omega = 0, \\ &l^*(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \sum_{k=1}^{n} \langle \xi, \xi_{i_k}\rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} (\xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}). \end{aligned} \tag{16}$$

In the following lemma, we show that the T-deformed creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$ satisfy the q_{ij}-commutation relations, which generalize the commutation relations considered in [8, 16].

Lemma 2.3. *Fix $i, j \in N$ and let $\xi \in \mathcal{H}^{(i)}, \eta \in \mathcal{H}^{(j)}$. Then,*

$$l^*(\xi)l(\eta) - q_{ij}l(\eta)l^*(\xi) = \langle \xi, \eta\rangle_U 1, \tag{17}$$

where 1 is the identity operator on $\mathcal{F}_T(\mathcal{H})$.

Proof. By (8) and (15), it follows that

$$l^*(\xi)l(\eta)\Omega - q_{ij}l(\eta)l^*(\xi)\Omega = l^*(\xi)l(\eta)\Omega = \langle \xi, \eta\rangle_U \Omega.$$

Thus, it remains to check (17) on the n-particle spaces. Fix $n \in \mathbb{N}$, and let $\xi_{i_k} \in \mathcal{H}^{(i_k)}$ for $i_k \in N$ and $1 \le k \le n$. By (15) and (16), it follows that

$$\begin{aligned} &l^*(\xi)l(\eta)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = l^*(\xi)(\eta \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \langle \xi, \eta\rangle_U \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \\ &\qquad + \sum_{k=1}^{n} \langle \xi, \xi_{i_k}\rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} q_{i_k j} (\eta \otimes \xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
&l(\eta)l^*(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\
&\quad = \sum_{k=1}^{n} \langle \xi, \xi_{i_k} \rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} (\eta \otimes \xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}).
\end{aligned}
$$

Therefore, $l^*(\xi)l(\eta) - q_{ij} l(\eta)l^*(\xi) = \langle \xi, \eta \rangle_U 1$ on simple tensors if and only if

$$
q_{kj} \langle \xi, \zeta \rangle_U = q_{ij} \langle \xi, \zeta \rangle_U \quad \forall \zeta \in \mathcal{H}^{(k)},\ k \in N. \tag{18}
$$

To verify (18), fix $\zeta \in \mathcal{H}^{(k)}$ for $k \in N$. Indeed, if $i \neq k$, then both sides of (18) are equal to 0. Now, if $i = k$, then $q_{kj} = q_{ij}$; so both sides of (18) are equal to $q_{ij} \langle \xi, \zeta \rangle_U$. Hence, (17) holds on simple tensors. Therefore, it follows by a simple density argument that (17) holds on n-particle spaces. This completes the proof. ■

Define

$$
s(\xi) := l(\xi) + l^*(\xi) \quad \text{for } \xi \in \mathcal{H}_{\mathbb{R}}. \tag{19}
$$

Consider the C^*-algebra $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)$ and the associated von Neumann algebra

$$
\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)'' \subseteq \mathbf{B}\big(\mathcal{F}_T(\mathcal{H})\big)
$$

generated by the self-adjoint operators $\{s(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}$. In this paper, we are interested in $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$, and we call it the mixed q-deformed Araki–Woods von Neumann algebra.

Note that $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ (and hence $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)$) is equipped with a canonical vacuum state φ given by $\varphi(\cdot) = \langle \Omega, \cdot\Omega \rangle_T$.

3. Standard form of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$

This section is concerned with the faithful representation of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ on $\mathcal{F}_T(\mathcal{H})$. We show that the GNS space of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ associated with the canonical vacuum state φ is $\mathcal{F}_T(\mathcal{H})$, and hence, the identity representation of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ on $\mathcal{F}_T(\mathcal{H})$ is in standard form. The modular theory of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ associated with the vacuum state φ is an inevitable component for the further analysis of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$. In this section, we also discuss the same, and subsequently we identify the commutant of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$. We proceed as follows.

For $\xi \in \mathcal{H}$, let $b(\xi)$ and $b^*(\xi)$ denote the canonical right creation and annihilation operators acting on $\mathcal{F}(\mathcal{H})$ which are defined as follows:

$$
\begin{aligned}
&b(\xi)\Omega = \xi, \quad && b(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi, \\
&b^*(\xi)\Omega = 0, \quad && b^*(\xi)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \langle \xi, \xi_n \rangle_U \xi_1 \otimes \cdots \otimes \xi_{n-1},
\end{aligned} \tag{20}
$$

where $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$ for $n \geq 1$. The operators $b(\xi)$ and $b^*(\xi)$ are bounded and adjoints of each other on $\mathcal{F}(\mathcal{H})$.

Consider the unitary operator $j : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ defined as follows:

$$\begin{aligned} j\Omega &= \Omega, \\ j(\xi_1 \otimes \cdots \otimes \xi_n) &= \xi_n \otimes \cdots \otimes \xi_1, \end{aligned} \tag{21}$$

where $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$ for $n \geq 1$.

Lemma 3.1. *The operator j extends to a unitary on $\mathcal{F}_T(\mathcal{H})$.*

Proof. Fix $n \in \mathbb{N}$, and let $\xi_{i_m} \in \mathcal{H}^{(i_m)}$ for $i_m \in N, 1 \leq m \leq n$. For $k < n$, one has

$$\begin{aligned} & jT_k j(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &= jT_k(\xi_{i_n} \otimes \cdots \otimes \xi_{i_1}) \quad \big(\text{by } (21)\big) \\ &= j\big(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes T \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-k-1}\big)(\xi_{i_n} \otimes \cdots \otimes \xi_{i_1}) \quad \big(\text{by } (9)\big) \\ &= jq_{i_{(n-k)}i_{(n-k+1)}}(\xi_{i_n} \otimes \cdots \otimes \xi_{i_{(n-k+2)}} \otimes \xi_{i_{(n-k)}} \otimes \xi_{i_{(n-k+1)}} \otimes \xi_{i_{(n-k-1)}} \otimes \cdots \otimes \xi_{i_2} \otimes \xi_{i_1}) \\ &= q_{i_{(n-k)}i_{(n-k+1)}}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_{(n-k-1)}} \otimes \xi_{i_{(n-k+1)}} \otimes \xi_{i_{(n-k)}} \otimes \xi_{i_{(n-k+2)}} \otimes \cdots \otimes \xi_{i_{(n-1)}} \otimes \xi_{i_n}) \\ &= T_{n-k}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}). \end{aligned} \tag{22}$$

Therefore, by a simple density argument, it follows that $jT_k j = T_{n-k}$ on $\mathcal{H}^{\otimes n}$ for all $k < n$.

Consequently, it follows that there is an injection (and hence a bijection) $S_n \ni \sigma \to \sigma' \in S_n$ such that

$$j\pi(\sigma)j = \pi(\sigma'), \quad \sigma \in S_n.$$

Therefore,

$$\begin{aligned} jP^{(n)}j &= \sum_{\sigma \in S_n} j\pi(\sigma)j \quad \big(\text{by } (10)\big) \\ &= \sum_{\sigma' \in S_n} \pi(\sigma') \\ &= P^{(n)}. \end{aligned} \tag{23}$$

Then, one has

$$\begin{aligned} & \big\langle j(\xi_1 \otimes \cdots \otimes \xi_n), j(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_T \\ &= \delta_{n,m}\big\langle j(\xi_1 \otimes \cdots \otimes \xi_n), P^{(n)} j(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_{\mathcal{F}(\mathcal{H})} \\ &= \delta_{n,m}\big\langle j(\xi_1 \otimes \cdots \otimes \xi_n), (jP^{(n)}j)j(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{by } (23)\big) \\ &= \delta_{n,m}\big\langle \xi_1 \otimes \cdots \otimes \xi_n, P^{(n)}(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{since } j^2 = jj^* = j^*j = 1 \text{ on } \mathcal{F}(\mathcal{H})\big) \\ &= \langle \xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_m\rangle_T \end{aligned}$$

for $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$, $\eta_1 \otimes \cdots \otimes \eta_m \in \mathcal{H}^{\odot m}$.

Hence, by a simple density argument, it follows that j extends to a unitary on $\mathcal{F}_T(\mathcal{H})$. ∎

The extension of j in Lemma 3.1 will be denoted by j again with slight abuse of notation.

For $\xi \in \mathcal{H}$, consider the densely defined T-deformed right creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$ defined as follows:

$$
\begin{aligned}
&r(\xi)=b(\xi),\\
&r^*(\xi)=\begin{cases} b^*(\xi)(1+T_{n-1}+T_{n-1}T_{n-2}+\cdots+T_{n-1}T_{n-2}\cdots T_1), & \text{on } \mathcal{H}^{\otimes n} \text{ for } n\geq 1,\\ 0, & \text{on } \mathbb{C}\Omega. \end{cases}
\end{aligned} \tag{24}
$$

Proposition 3.2. *Let $\xi \in \mathcal{H}$. Then, $r(\xi), r^*(\xi)$ extend as bounded operators in $\mathcal{F}_T(\mathcal{H})$. Denoting the extensions by the same symbols, the following hold:*

(i) *if $\|T\|_{\mathcal{H}\otimes\mathcal{H}} = q < 1$, then $\|r(\xi)\| \leq \|\xi\|_U(1-q)^{-\frac{1}{2}}$;*

(ii) *$r(\xi)$ and $r^*(\xi)$ are adjoints of each other on $\mathcal{F}_T(\mathcal{H})$.*

Proof. For $n \geq 1$, on $\mathcal{H}^{\otimes n}$, define

$$
\begin{aligned}
R_n &:= 1 + T_1 + T_1T_2 + \cdots + T_1T_2\cdots T_{n-1},\\
R'_n &:= 1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1}T_{n-2}\cdots T_1.
\end{aligned}
$$

By (22), it follows that $jR_nj = R'_n$ on $\mathcal{H}^{\otimes n}$.

Note that, for $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$, one has

$$
\begin{aligned}
jl(\xi)j(\xi_1 \otimes \cdots \otimes \xi_n) &= jl(\xi)(\xi_n \otimes \cdots \otimes \xi_1) = j(\xi \otimes \xi_n \otimes \cdots \otimes \xi_1)\\
&= \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi = r(\xi)(\xi_1 \otimes \cdots \otimes \xi_n).
\end{aligned}
$$

Therefore, it follows directly from Proposition 2.2 and Lemma 3.1 that $r(\xi)$ extends to $\mathcal{F}_T(\mathcal{H})$ as a bounded operator and $jl(\xi)j = r(\xi)$ on $\mathcal{F}_T(\mathcal{H})$.

Fix $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$. Again, as j keeps $\mathcal{F}(\mathcal{H})$ invariant, one has

$$
\begin{aligned}
ja^*(\xi)j(\xi_1 \otimes \cdots \otimes \xi_n) &= ja^*(\xi)(\xi_n \otimes \cdots \otimes \xi_1)\\
&= j\langle \xi, \xi_n\rangle_U(\xi_{n-1} \otimes \cdots \otimes \xi_1) \quad \big(\text{see } (8)\big)\\
&= \langle \xi, \xi_n\rangle_U(\xi_1 \otimes \cdots \otimes \xi_{n-1})\\
&= b^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) \quad \big(\text{see } (20)\big).
\end{aligned} \tag{25}
$$

Therefore, by (15),

$$
\begin{aligned}
jl^*(\xi)j(\xi_1 \otimes \cdots \otimes \xi_n) &= ja^*(\xi)R_nj(\xi_1 \otimes \cdots \otimes \xi_n)\\
&= ja^*(\xi)jjR_nj(\xi_1 \otimes \cdots \otimes \xi_n)\\
&= b^*(\xi)R'_n(\xi_1 \otimes \cdots \otimes \xi_n) \quad \big(\text{see } (25)\big)\\
&= r^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) \quad \big(\text{see } (24)\big).
\end{aligned}
$$

Arguing as in the previous case, it follows that $jl^*(\xi)j = r^*(\xi)$ on $\mathcal{F}_T(\mathcal{H})$.

The rest follows from Proposition 2.2. This completes the proof. ∎

Equation (24) entails that the definition of the right annihilation operator $r^*(\xi)$ involves the operator T. Following (6), note that for $\xi \in \mathcal{H}$, one has

$$\begin{aligned} &r^*(\xi)\Omega = 0, \\ &r^*(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \sum_{k=1}^{n} \langle \xi, \xi_{i_k} \rangle_U q_{i_k i_{k+1}} \cdots q_{i_k i_n} \xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}, \end{aligned} \tag{26}$$

where $n \in \mathbb{N}$ and $\xi_{i_k} \in \mathcal{H}^{(i_k)}$ for $i_k \in N$, $1 \leq k \leq n$.

Define

$$d(\xi) := r(\xi) + r^*(\xi) \quad \text{for } \xi \in \mathcal{H}. \tag{27}$$

Let $\xi \in \mathcal{H}^{(i)}$, $\eta \in \mathcal{H}^{(j)}$ for $i, j \in N$. Then, from Lemmas 2.3 and 3.1, one has

$$\begin{aligned} r^*(\xi)r(\eta) - q_{ij} r(\eta) r^*(\xi) &= j l^*(\xi) j j l(\eta) j - q_{ij} j l(\eta) j j l^*(\xi) j \\ &= j l^*(\xi) l(\eta) j - q_{ij} j l(\eta) l^*(\xi) j \\ &= j \big(l^*(\xi) l(\eta) - q_{ij} l(\eta) l^*(\xi) \big) j \\ &= \langle \xi, \eta \rangle_U 1. \end{aligned} \tag{28}$$

For $j \in N$, let

$$(\mathcal{H}_{\mathbb{R}}^{(j)})' = \{\xi \in \mathcal{H}^{(j)} : \langle \xi, \eta \rangle_U \in \mathbb{R} \text{ for all } \eta \in \mathcal{H}_{\mathbb{R}}\}.$$

Note that $(\mathcal{H}_{\mathbb{R}}^{(j)})'$ is a real subspace of $\mathcal{H}$, and by the Hahn–Hellinger theorem, it follows that

$$\overline{(\mathcal{H}_{\mathbb{R}}^{(j)})' + i(\mathcal{H}_{\mathbb{R}}^{(j)})'}^{\|\cdot\|_U} = \mathcal{H}^{(j)}.$$

Define

$$\mathcal{H}'_{\mathbb{R}} = \bigoplus_{j \in N} (\mathcal{H}_{\mathbb{R}}^{(j)})'. \tag{29}$$

Lemma 3.3. *Let $B = \{d(\xi) : \xi \in \mathcal{H}'_{\mathbb{R}}\}''$. Then, $B \subseteq \Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)'$.*

Proof. Let $\xi \in \mathcal{H}_{\mathbb{R}}$ and $\eta \in \mathcal{H}'_{\mathbb{R}}$. Let $\xi = \bigoplus_{i \in N} \xi^{(i)}$ for $\xi^{(i)} \in \mathcal{H}_{\mathbb{R}}^{(i)}$ and $\eta = \bigoplus_{i \in N} \eta^{(i)}$ for $\eta^{(i)} \in (\mathcal{H}_{\mathbb{R}}^{(i)})'$, $i \in N$, be the unique decompositions of ξ, η, respectively. We show that $[d(\eta), s(\xi)] = 0$ on $\mathcal{F}_T(\mathcal{H})$.

If $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$, from (19) and (27), it follows that $s(\xi) = \sum_{i \in N} s(\xi^{(i)})$ and $d(\eta) = \sum_{i \in N} d(\eta^{(i)})$. From Proposition 2.2 (resp., Proposition 3.2), it follows that $\mathcal{H}_{\mathbb{R}} \ni \zeta \mapsto s(\zeta) \in \Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ (resp., $\mathcal{H}'_{\mathbb{R}} \ni \zeta \mapsto d(\zeta) \in B$) is $\|\cdot\|_U$ to $\|\cdot\|$ continuous. Hence, if $N = \mathbb{N}$, then $s(\xi) = \sum_{i \in N} s(\xi^{(i)})$ and $d(\eta) = \sum_{i \in N} d(\eta^{(i)})$ in strong operator topology (s.o.t.) as well.

Therefore, we may assume without loss of generality that $\xi \in \mathcal{H}_{\mathbb{R}}^{(t)}$ and $\eta \in (\mathcal{H}_{\mathbb{R}}^{(r)})'$ for $t, r \in N$. Note that $\langle \xi, \eta \rangle_U \in \mathbb{R}$. From (19) and (27), it follows that

$$s(\xi) d(\eta) \Omega = \langle \xi, \eta \rangle_U \Omega + \xi \otimes \eta,$$

while

$$d(\eta)s(\xi)\Omega = \langle \eta, \xi\rangle_U \Omega + \xi \otimes \eta.$$

Since $\langle \xi, \eta\rangle_U \in \mathbb{R}$, one has $\langle \xi, \eta\rangle_U = \langle \eta, \xi\rangle_U$. Therefore, $[d(\eta), s(\xi)]\Omega = 0$.

Thus, it remains only to show that $[d(\eta), s(\xi)] = 0$ on the n-particle spaces of $\mathcal{F}_T(\mathcal{H})$. Now, from (19) and (27), it follows that

$$\begin{aligned} d(\eta)s(\xi) &= r(\eta)l(\xi) + r(\eta)l^*(\xi) + r^*(\eta)l(\xi) + r^*(\eta)l^*(\xi), \\ s(\xi)d(\eta) &= l(\xi)r(\eta) + l(\xi)r^*(\eta) + l^*(\xi)r(\eta) + l^*(\xi)r^*(\eta). \end{aligned} \tag{30}$$

Fix $n \in \mathbb{N}$. Let $\xi_{i_k} \in \mathcal{H}^{(i_k)}$ for $i_k \in N$, $1 \le k \le n$. Let $\gamma := \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}$. From (15) and (24), it follows that $l(\xi)r(\eta)\gamma = r(\eta)l(\xi)\gamma$. Therefore, by a simple density argument, it follows that $l(\xi)r(\eta) = r(\eta)l(\xi)$ on $\mathcal{F}_T(\mathcal{H})$. Hence, $l^*(\xi)r^*(\eta) = r^*(\eta)l^*(\xi)$ on $\mathcal{F}_T(\mathcal{H})$. Therefore, from (30), it follows that to show $[d(\eta), s(\xi)]\gamma = 0$, it is enough to show that

$$\big(l(\xi)r^*(\eta) + l^*(\xi)r(\eta)\big)\gamma = \big(r(\eta)l^*(\xi) + r^*(\eta)l(\xi)\big)\gamma.$$

We proceed to show the same. First, let us note the following computations that follow from (15), (16), (24) and (26). We have

$$\begin{aligned} &l(\xi)r^*(\eta)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \sum_{k=1}^{n} \langle \eta, \xi_{i_k}\rangle_U q_{i_k i_{k+1}} \cdots q_{i_k i_n} (\xi \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}). \end{aligned} \tag{31}$$

Again,

$$\begin{aligned} &l^*(\xi)r(\eta)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = l^*(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \eta) = \langle \xi, \eta\rangle_U q_{r i_n} \cdots q_{r i_1} (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\qquad + \sum_{k=1}^{n} \langle \xi, \xi_{i_k}\rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} (\xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n} \otimes \eta), \\ &r(\eta)l^*(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \sum_{k=1}^{n} \langle \xi, \xi_{i_k}\rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} (\xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n} \otimes \eta), \\ &r^*(\eta)l(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = r^*(\eta)(\xi \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \langle \eta, \xi\rangle_U q_{t i_1} \cdots q_{t i_n} (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\qquad + \sum_{k=1}^{n} \langle \eta, \xi_{i_k}\rangle_U q_{i_k i_{k+1}} \cdots q_{i_k i_n} (\xi \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}). \end{aligned}$$

From (31), it follows that $(l(\xi)r^*(\eta) + l^*(\xi)r(\eta))\gamma = (r(\eta)l^*(\xi) + r^*(\eta)l(\xi))\gamma$ if and only if

$$\langle \eta, \xi\rangle_U q_{t i_1} \cdots q_{t i_n} = \langle \xi, \eta\rangle_U q_{r i_n} \cdots q_{r i_1}. \tag{32}$$

We verify (32) below. First, assume that $t \neq r$. Then, both sides of (32) are equal to 0. Otherwise, since $\langle \xi, \eta \rangle_U \in \mathbb{R}$, both sides of (32) are equal to $\langle \xi, \eta \rangle_U q_{ri_n} \cdots q_{ri_1}$.

Therefore, we have $[d(\eta), s(\xi)]\gamma = 0$ for $\xi \in \mathcal{H}_{\mathbb{R}}$, $\eta \in \mathcal{H}'_{\mathbb{R}}$. This completes the proof. ■

Lemma 3.4. (i) $\mathrm{span}_{\mathbb{C}}\{\Omega, s(\xi_{t_1}) \cdots s(\xi_{t_n})\Omega : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)}$ *for* $t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\}$ *is dense in* $\mathcal{F}_T(\mathcal{H})$.

(ii) $\mathrm{span}_{\mathbb{C}}\{\Omega, d(\xi_{t_1}) \cdots d(\xi_{t_n})\Omega : \xi_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})'$ *for* $t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\}$ *is dense in* $\mathcal{F}_T(\mathcal{H})$.

Proof. (i) First, we show that

$$\begin{aligned}&\left\{\xi_{t_1} \otimes \cdots \otimes \xi_{t_n} : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)},\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}\\ &\quad \subseteq \mathrm{span}_{\mathbb{C}} \left\{\Omega, s(\xi_{t_1}) \cdots s(\xi_{t_n})\Omega : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)} \text{ for } t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}.\end{aligned}$$

The proof is by induction. Let $\xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)}$ for $1 \leq k \leq n$. The case $n = 1$ is trivial since by (15) and (16), one has $s(\xi_{t_1})\Omega = \xi_{t_1}$.

Suppose that the result is true for $(n-1)$ with $n \geq 2$. Thus,

$$\xi_{t_2} \otimes \cdots \otimes \xi_{t_n} \in \mathrm{span}_{\mathbb{C}} \left\{\Omega, s(\xi_{t_1}) \cdots s(\xi_{t_n})\Omega : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)} \text{ for } t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}.$$

Note that

$$\begin{aligned}&s(\xi_{t_1})(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\ &= \big(l(\xi_{t_1}) + l^*(\xi_{t_1})\big)(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\ &= \xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_n} + l^*(\xi_{t_1})(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n}) \quad \big(\text{by (15)}\big)\\ &= \xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_n}\\ &\quad + \sum_{k=2}^{n} \langle \xi_{t_1}, \xi_{t_k} \rangle_U q_{t_k t_{k-1}} \cdots q_{t_k t_1} (\xi_{t_2} \otimes \cdots \otimes \xi_{t_{k-1}} \otimes \xi_{t_{k+1}} \otimes \cdots \otimes \xi_{t_n}) \quad \big(\text{by (16)}\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_n}\\ &\quad = s(\xi_{t_1})(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\ &\qquad - \sum_{k=2}^{n} \langle \xi_{t_1}, \xi_{t_k} \rangle_U q_{t_k t_{k-1}} \cdots q_{t_k t_1} (\xi_{t_2} \otimes \cdots \otimes \xi_{t_{k-1}} \otimes \xi_{t_{k+1}} \otimes \cdots \otimes \xi_{t_n}).\end{aligned} \tag{33}$$

Note that the simple tensors in the right-hand side of (33) contain only $(n-1)$ vectors. Therefore, by the induction hypothesis, it follows that

$$\begin{aligned}&\xi_{t_1} \otimes \xi_{t_2} \otimes \cdots \otimes \xi_{t_n}\\ &\quad \in \mathrm{span}_{\mathbb{C}} \left\{\Omega, s(\xi_{t_1}) \cdots s(\xi_{t_n})\Omega : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)} \text{ for } t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}.\end{aligned}$$

Hence, the containment of the sets as claimed is established.

Therefore, $\mathcal{H}_{\mathbb{C}}^{\odot m} \subseteq \text{span}_{\mathbb{C}}\{\Omega, s(\xi_{t_1})\cdots s(\xi_{t_n})\Omega : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)}\}$ for all $m \geq 0$. The rest follows by a straightforward density argument.

(ii) Proceeding analogously as in (i), it follows that

$$\begin{aligned}&\left\{\xi_{t_1} \otimes \cdots \otimes \xi_{t_n} : \xi_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})',\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}\\ &\quad \subseteq \text{span}_{\mathbb{C}}\left\{\Omega,\ d(\xi_{t_1})\cdots d(\xi_{t_n})\Omega : \xi_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})' \text{ for } t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}.\end{aligned}$$

Since $\overline{(\mathcal{H}_{\mathbb{R}}^{(j)})' + i(\mathcal{H}_{\mathbb{R}}^{(j)})'}^{\|\cdot\|_U} = \mathcal{H}^{(j)}$ for $j \in N$, by a simple density argument, it follows that

$$\text{span}_{\mathbb{C}}\left\{\xi_{t_1} \otimes \cdots \otimes \xi_{t_n} : \xi_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})',\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}$$

is dense in $\text{span}_{\mathbb{C}}\{\eta_{t_1} \otimes \cdots \otimes \eta_{t_n} : \eta_{t_k} \in \mathcal{H}^{(t_k)},\ 1 \leq k \leq n,\ n \in \mathbb{N}\}$ with respect to $\|\cdot\|_{\mathcal{F}(\mathcal{H})}$. Further, by an approximation argument, one has that

$$\text{span}_{\mathbb{C}}\left\{\eta_{t_1} \otimes \cdots \otimes \eta_{t_n} : \eta_{t_k} \in \mathcal{H}^{(t_k)},\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\} \cup \{\mathbb{C}\Omega\}$$

is dense in $\mathcal{F}(\mathcal{H})$. The rest is clear. ∎

Proposition 3.5. *The vacuum vector Ω is cyclic and separating for $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$.*

Proof. By Lemma 3.4 (i), one has that

$$\text{span}_{\mathbb{C}}\left\{\Omega, s(\xi_{t_1})\cdots s(\xi_{t_n})\Omega : \xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)},\ t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}$$

is dense in $\mathcal{F}_T^{\text{finite}}(\mathcal{H})$. Therefore, it follows that Ω is cyclic for $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$.

By Lemma 3.4 (ii), one has that

$$\text{span}_{\mathbb{C}}\left\{\Omega, d(\eta_{t_1})\cdots d(\eta_{t_n})\Omega : \eta_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})',\ t_k \in N,\ 1 \leq k \leq n,\ n \in \mathbb{N}\right\}$$

is dense in $\mathcal{F}_T^{\text{finite}}(\mathcal{H})$. Therefore, from Lemma 3.3, it follows that Ω is cyclic for the commutant of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ as well. Hence, Ω is cyclic and separating for $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$. This completes the proof. ∎

Remark 3.6. The vacuum state φ is faithful on $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$, and the identity representation of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ on $\mathcal{F}_T(\mathcal{H})$ is the GNS representation with respect to φ.

Remark 3.7. (1) If $U_t = I$ for all $t \in \mathbb{R}$, and $\mathcal{H}_{\mathbb{R}} = \bigoplus_{i \in N} \mathbb{R}$, then $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ is the mixed q-Gaussian von Neumann algebra constructed in [8].

(2) Consider the decomposition $\mathcal{H}_{\mathbb{R}} := \bigoplus_{i \in N} \mathcal{H}_{\mathbb{R}}^{(i)}$ in (5), and let $-1 < q_{ij} = q = q_{ji} < 1$ for $i, j \in N$. Then, the above construction reduces to the construction in [16], and if $q = 0$, then the construction reduces to the construction in [26].

The remaining part of this section is devoted for describing the modular theory of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ associated with the vacuum state φ. Before discussing the same, we prepare ourselves with some useful lemmas. We proceed as follows.

Fix $t \in \mathbb{R}$. Note that the second quantization $\mathcal{F}(U_t)$, defined on the full Fock space $\mathcal{F}(\mathcal{H})$ by

$$\mathcal{F}(U_t)\Omega = \Omega,$$
$$\mathcal{F}(U_t)(\eta_1 \otimes \cdots \otimes \eta_n) = (U_t\eta_1) \otimes \cdots \otimes (U_t\eta_n) \quad \text{for } \eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{H}^{\odot n},\ n \in \mathbb{N},$$

is unitary on $\mathcal{F}(\mathcal{H})$.

Lemma 3.8. *For $t \in \mathbb{R}$, the operator $\mathcal{F}(U_t)$ extends to a unitary on $\mathcal{F}_T(\mathcal{H})$.*

Proof. By the definition of T in (6), it is easy to verify that $(U_t \otimes U_t)T = T(U_t \otimes U_t)$ on $\mathcal{H}^{(i)} \otimes \mathcal{H}^{(j)}$ for $i, j \in N$, $t \in \mathbb{R}$. This implies that $(U_t \otimes U_t)T = T(U_t \otimes U_t)$ on $\mathcal{H} \otimes \mathcal{H}$ for all $t \in \mathbb{R}$. Thus, it follows that T_i commutes with

$$\underbrace{U_t \otimes U_t \otimes \cdots \otimes U_t}_{i+1}$$

on $\mathcal{H}^{\otimes(i+1)}$ for all $i \in \mathbb{N}$, $t \in \mathbb{R}$ (see (9)). Therefore, $P^{(n)}$ commutes with

$$\underbrace{U_t \otimes U_t \otimes \cdots \otimes U_t}_{n}$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ as $\pi(1) = 1$.

Fix $t \in \mathbb{R}$ and $n, m \in \mathbb{N}$. For $\zeta_1 \otimes \cdots \otimes \zeta_n \in \mathcal{H}^{\odot n}$, $\eta_1 \otimes \cdots \otimes \eta_m \in \mathcal{H}^{\odot m}$, one has

$$\begin{aligned}
&\big\langle \mathcal{F}(U_t)(\zeta_1 \otimes \cdots \otimes \zeta_n),\ \mathcal{F}(U_t)(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_T \\
&\quad = \delta_{n,m}\big\langle U_t\zeta_1 \otimes \cdots \otimes U_t\zeta_n, P^{(n)}(U_t\eta_1 \otimes \cdots \otimes U_t\eta_m)\big\rangle_{\mathcal{F}(\mathcal{H})} \quad \big(\text{by (11)}\big) \\
&\quad = \delta_{n,m}\big\langle U_t\zeta_1 \otimes \cdots \otimes U_t\zeta_n, (U_t \otimes \cdots \otimes U_t)P^{(n)}(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_{\mathcal{F}(\mathcal{H})} \\
&\quad = \delta_{n,m}\big\langle \zeta_1 \otimes \cdots \otimes \zeta_n, P^{(n)}(\eta_1 \otimes \cdots \otimes \eta_m)\big\rangle_{\mathcal{F}(\mathcal{H})} \\
&\quad = \langle \zeta_1 \otimes \cdots \otimes \zeta_n, \eta_1 \otimes \cdots \otimes \eta_m\rangle_T.
\end{aligned}$$

Therefore, $\mathcal{F}(U_t)$ extends to a unitary on $\mathcal{F}_T(\mathcal{H})$ for all $t \in \mathbb{R}$. This completes the proof. ∎

We will denote the unitary extension of $\mathcal{F}(U_t)$ on $\mathcal{F}_T(\mathcal{H})$ by $\mathcal{F}_T(U_t)$, $t \in \mathbb{R}$.

Lemma 3.9. *For $n \geq 1$, the following assertions hold:*

(1) *the operator $A^{\odot n} : (\mathfrak{D}(A))^{\odot n} \subseteq \mathcal{H}^{\otimes_T^n} \to \mathcal{H}^{\otimes_T^n}$ defined by*

$$A^{\odot n}(\eta_1 \otimes \cdots \otimes \eta_n) = A(\eta_1) \otimes \cdots \otimes A(\eta_n),$$

$\eta_i \in \mathfrak{D}(A)$ for $1 \leq i \leq n$, is positive and symmetric;

(2) *let $A^{\otimes_T^n}$ be the closure of $A^{\odot n}$ on $\mathcal{H}^{\otimes_T^n}$. Then, $A^{\otimes_T^n}$ is positive and self-adjoint;*

(3) *for $k \in \mathbb{N}$, $(A^{\otimes_T^n})^{\frac{1}{k}} = (A^{\frac{1}{k}})^{\otimes_T^n}$.*

Proof. Firstly, for $n = 1$, there is nothing to prove. So, fix $n \geq 2$.

(i) Note that $P^{(n)}$ is a positive bounded operator on $\mathcal{H}^{\otimes n}$ (see [8, Thm. 2.3]). Since A leaves $\mathcal{H}^{(i)}$ invariant for $i \in N$, by the definition of T in (6), it follows that $T(A \odot A) \subseteq (A \odot A)T$ on $\mathfrak{D}(A) \odot \mathfrak{D}(A)$. Then, by (10), one has

$$P^{(n)} A^{\odot n} \subseteq A^{\odot n} P^{(n)} \quad \text{on } \big(\mathfrak{D}(A)\big)^{\odot n}.$$

Note that $A^{\odot n}$ is preclosed in $\mathcal{H}^{\otimes n}$. Denoting $A^{\otimes n}$ to be the closure of $A^{\odot n}$ on $\mathcal{H}^{\otimes n}$, it follows that $A^{\otimes n} P^{(n)}$ is a densely defined closed operator on $\mathcal{H}^{\otimes n}$. Since $(\mathfrak{D}(A))^{\odot n}$ is a core for $A^{\otimes n}$, it follows that $P^{(n)} A^{\otimes n} \subseteq A^{\otimes n} P^{(n)}$ on $\mathcal{H}^{\otimes n}$. Therefore, by [28, Exer. E.9.21], it follows that

$$\overline{P^{(n)} A^{\otimes n}} = A^{\otimes n} P^{(n)} \quad \text{on } \mathcal{H}^{\otimes n}.$$

Next, we show that $A^{\otimes n} P^{(n)}$ is positive and self-adjoint on $\mathcal{H}^{\otimes n}$. Since $P^{(n)} A^{\otimes n} \subseteq A^{\otimes n} P^{(n)}$, it is easy to verify that $p(P^{(n)}) A^{\otimes n} \subseteq A^{\otimes n} p(P^{(n)})$ for every polynomial p on $\sigma(P^{(n)})$. Let $\{p_m\}$ be a sequence of polynomials such that $p_m(0) = 0$ for all m and $p_m(\lambda) \to \lambda^{\frac{1}{2}}$ uniformly on $\sigma(P^{(n)})$ as $m \to \infty$. Let $\zeta \in \mathfrak{D}(A^{\otimes n})$. Then, one has

$$\begin{aligned}
(P^{(n)})^{\frac{1}{2}} A^{\otimes n} \zeta &= \lim_{m\to\infty} p_m(P^{(n)}) A^{\otimes n} \zeta \\
&= \lim_{m\to\infty} A^{\otimes n} p_m(P^{(n)}) \zeta \\
&= A^{\otimes n} (P^{(n)})^{\frac{1}{2}} \zeta
\end{aligned}$$

since $\lim_{m\to\infty} p_m(P^{(n)})\zeta = (P^{(n)})^{\frac{1}{2}}\zeta$ and $A^{\otimes n}$ is closed. Therefore,

$$(P^{(n)})^{\frac{1}{2}} A^{\otimes n} \subseteq A^{\otimes n} (P^{(n)})^{\frac{1}{2}}.$$

Since $A^{\otimes n}$ is positive, one has

$$\begin{aligned}
\langle \zeta, P^{(n)} A^{\otimes n} \zeta \rangle_{\mathcal{F}(\mathcal{H})} &= \big\langle (P^{(n)})^{\frac{1}{2}} \zeta, (P^{(n)})^{\frac{1}{2}} A^{\otimes n} \zeta \big\rangle_{\mathcal{F}(\mathcal{H})} \\
&= \big\langle (P^{(n)})^{\frac{1}{2}} \zeta, A^{\otimes n} (P^{(n)})^{\frac{1}{2}} \zeta \big\rangle_{\mathcal{F}(\mathcal{H})} \\
&\geq 0, \quad \zeta \in \mathfrak{D}(A^{\otimes n}).
\end{aligned}$$

Thus, $P^{(n)} A^{\otimes n}$ is positive. This implies that $\overline{P^{(n)} A^{\otimes n}} = A^{\otimes n} P^{(n)}$ is positive and hence symmetric.

Now, since $P^{(n)} A^{\otimes n} \subseteq A^{\otimes n} P^{(n)}$ and $A^{\otimes n} P^{(n)}$ is symmetric, one has

$$\begin{aligned}
(A^{\otimes n} P^{(n)})^* &\subseteq (P^{(n)} A^{\otimes n})^* && \big(\text{by [28, p. 191]}\big) \\
&= (A^{\otimes n})^* (P^{(n)})^* && \big(\text{by [28, §9.2]}\big) \\
&= A^{\otimes n} P^{(n)} \\
&\subseteq (A^{\otimes n} P^{(n)})^*.
\end{aligned}$$

Therefore, $A^{\otimes n} P^{(n)}$ is self-adjoint as well.

Let $C := \overline{P^{(n)} A^{\otimes n}}$ and $\zeta \in (\mathfrak{D}(A))^{\odot n}$. Then, one has

$$\begin{aligned}\langle \zeta, A^{\odot n}\zeta\rangle_T &= \langle \zeta, P^{(n)}A^{\otimes n}\zeta\rangle_{\mathcal{F}(\mathcal{H})}\\ &= \langle \zeta, C\zeta\rangle_{\mathcal{F}(\mathcal{H})}\\ &= \langle C^{\frac{1}{2}}\zeta, C^{\frac{1}{2}}\zeta\rangle_{\mathcal{F}(\mathcal{H})}\\ &\geq 0.\end{aligned}$$

Thus, (i) in the statement follows.

(ii) Since by (i), $A^{\odot n}$ is densely defined, positive and symmetric on $\mathcal{H}^{\otimes_T^n}$, it follows that $A^{\odot n}$ is closable (on $\mathcal{H}^{\otimes_T^n}$) and $A^{\otimes_T^n}$ is also densely defined. Hence, $A^{\otimes_T^n}$ is positive and symmetric.

From Lemma 3.8, it follows that

$$\mathbb{R} \ni t \mapsto \mathcal{F}_T(U_t)_{\restriction \mathcal{H}^{\otimes_T^n}}$$

is a strongly continuous one-parameter group of unitaries on $\mathcal{H}^{\otimes_T^n}$. Let B denote the analytic generator of $(\mathcal{F}_T(U_t)_{\restriction \mathcal{H}^{\otimes_T^n}})_{t\in\mathbb{R}}$. Note that $A^{\odot n} \subseteq A^{\otimes_T^n} \subseteq B$. Since $\mathfrak{D}(A^{\odot n})$ is a core for both $A^{\otimes_T^n}$ and B, and both $A^{\otimes_T^n}$ and B are closed, one has that $A^{\otimes_T^n} = B$. Since B is self-adjoint on $\mathcal{H}^{\otimes_T^n}$ (by Stone's Theorem), one has that $A^{\otimes_T^n}$ is also self-adjoint. This completes the proof.

(iii) Replacing the role of A by $A^{\frac{1}{k}}$ in (ii), one has that $(A^{\frac{1}{k}})^{\otimes_T^n}$ is positive and self-adjoint on $\mathcal{H}^{\otimes_T^n}$. Therefore, by functional calculus, it follows that $((A^{\frac{1}{k}})^{\otimes_T^n})^k$ is positive and self-adjoint. Note that

$$A^{\odot n} = \underbrace{(A^{\frac{1}{k}})^{\odot n}\cdots(A^{\frac{1}{k}})^{\odot n}}_{k \text{ times}} \subseteq \left((A^{\frac{1}{k}})^{\otimes_T^n}\right)^k.$$

Hence, one has $A^{\otimes_T^n} \subseteq ((A^{\frac{1}{k}})^{\otimes_T^n})^k$. Therefore, by (ii) and [28, Exer. E.9.28], it follows that $A^{\otimes_T^n} = ((A^{\frac{1}{k}})^{\otimes_T^n})^k$. Hence, it follows by the uniqueness of the k-th root of $A^{\otimes_T^n}$ that $(A^{\otimes_T^n})^{\frac{1}{k}} = (A^{\frac{1}{k}})^{\otimes_T^n}$. This completes the proof. ∎

Lemma 3.10. *Fix $n \in \mathbb{N}$, and let $\xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)}$ for $t_k \in N$, $1 \leq k \leq n$. Then,*

$$\begin{aligned}&s(\xi_{t_1})s(\xi_{t_2})\cdots s(\xi_{t_n})\Omega\\ &= \sum_{\nu=\{\{i(r),j(r)\}_{1\leq r\leq l},\{k(p)\}_{1\leq p\leq m}\}} f_\nu(q_{ij})\left(\prod_{r=1}^{l}\langle \xi_{t_{i(r)}}, \xi_{t_{j(r)}}\rangle_U\right)\left(\xi_{t_{k(1)}}\otimes\cdots\otimes\xi_{t_{k(m)}}\right),\end{aligned} \quad (34)$$

where the summation is over all partitions

$$\nu = \left\{\{i(r), j(r)\}_{1\leq r\leq l}, \{k(p)\}_{1\leq p\leq m}\right\}$$

of $\{1,\ldots,n\}$ having blocks of one or two elements such that

$$l, m \geq 0, \quad 2l + m = n, \quad i(r) < j(r) \quad \text{for } 1 \leq r \leq l,\ k(1) < \cdots < k(m),$$

and $f_\nu(q_{ij})$ *is given by*

$$f_\nu(q_{ij}) = \Bigg(\prod_{\nu_{(1)}=\{(r,s):1\le r,\ s\le l,\ i(r)<i(s)<j(r)<j(s)\}} q(t_{j(r)},\ t_{j(s)})\Bigg) \times \Bigg(\prod_{\nu_{(2)}=\{(r,p):1\le r\le l,\ 1\le p\le m,\ i(r)<k(p)<j(r)\}} q(t_{k(p)}, t_{j(r)})\Bigg).$$

Proof. The proof is by induction on n. The case $n = 1$ is trivial. Suppose the formula holds for $n - 1$. Therefore, we have

$$s(\xi_{t_2})\cdots s(\xi_{t_n})\Omega = \sum_{\nu=\{\{i(r),j(r)\}_{1\le r\le l},\{k(p)\}_{1\le p\le m}\}} f_\nu(q_{ij})\Bigg(\prod_{r=1}^{l}\langle \xi_{t_{i(r)}}, \xi_{t_{j(r)}}\rangle_U\Bigg)\big(\xi_{t_{k(1)}}\otimes\cdots\otimes\xi_{t_{k(m)}}\big),$$

where ν runs over the partitions of $\{2,\dots,n\}$ as specified in the statement (with $2l + m = n - 1$). Then, we compute

$$\begin{aligned} s(\xi_{t_1})\cdots s(\xi_{t_n})\Omega &= \sum_\nu f_\nu(q_{ij})\Bigg(\prod_{r=1}^{l}\langle \xi_{t_{i(r)}}, \xi_{t_{j(r)}}\rangle_U\Bigg)\xi_{t_1}\otimes\xi_{t_{k(1)}}\otimes\cdots\otimes\xi_{t_{k(m)}} \\ &\quad + \sum_\nu\sum_{u=1}^{m} f_\nu(q_{ij})q(t_{k(u)}, t_{k(u-1)})\cdots q(t_{k(u)}, t_{k(1)}) \\ &\quad \times\Bigg(\langle \xi_{t_1}, \xi_{t_{k(u)}}\rangle_U\prod_{r=1}^{l}\langle \xi_{t_{i(r)}}, \xi_{t_{j(r)}}\rangle_U\Bigg)\xi_{t_{k(1)}}\otimes\cdots\otimes\xi_{t_{k(u-1)}}\otimes\xi_{t_{k(u+1)}}\otimes\cdots\otimes\xi_{t_{k(m)}}. \end{aligned}$$

For each partition $\nu = \{\{i(r), j(r)\}_{1\le r\le l}, \{k(p)\}_{1\le p\le m}\}$ of $\{2,\dots,n\}$ in the above, we consider the following partitions of $\{1,\dots,n\}$:

$$\begin{aligned} \nu_0 &:= \big\{\{i(r), j(r)\}_{1\le r\le l}, \{1\}, \{k(p)\}_{1\le p\le m}\big\}, \\ \nu_u &:= \big\{\{1, k(u)\}, \{i(r), j(r)\}_{1\le r\le l}, \{k(p)\}_{1\le p\le m, p\ne u}\big\} \quad \text{for } 1\le u\le m. \end{aligned}$$

It is clear that the partitions of $\{1,\dots,n\}$ obtained above exhaust all partitions of $\{1,\dots,n\}$ as described in the statement of the lemma. Moreover, it is easy to see that

$$\begin{aligned} f_{\nu_0}(q_{ij}) &= f_\nu(q_{ij}), \\ f_{\nu_u}(q_{ij}) &= f_\nu(q_{ij})q(t_{k(u)}, t_{k(u-1)})\cdots q(t_{k(u)}, t_{k(1)}) \quad \text{for } 1\le u\le m. \end{aligned}$$

Hence, we obtain the desired formula for n. This completes the proof. ■

Using Lemma 3.10, for $n\in\mathbb{N}$, the value of φ at $s(\xi_{t_1})s(\xi_{t_2})\cdots s(\xi_{t_n})$, for $\xi_{t_k}\in\mathcal{H}_{\mathbb{R}}^{(t_k)}$, $t_k\in N$, $1\le k\le n$, is given by

$$\varphi\big(s(\xi_{t_1})s(\xi_{t_2})\cdots s(\xi_{t_n})\big) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \sum_\nu g_\nu(q_{ij})\prod_{r=1}^{n/2}\langle \xi_{t_{i(r)}}, \xi_{t_{j(r)}}\rangle_U, & \text{if } n \text{ is even}, \end{cases} \tag{35}$$

where the summation is over all pair partitions $\nu = \{i(r), j(r)\}_{1\leq r\leq n/2}$ of $\{1,\dots,n\}$ with $i(r) < j(r)$, and $g_\nu(q_{ij})$ is given by

$$g_\nu(q_{ij}) = \prod_{i(\nu)=\{(r,s):1\leq r,s\leq n/2,\ i(r)<i(s)<j(r)<j(s)\}} q(t_{j(r)}, t_{j(s)}).$$

From Lemma 3.8, it follows that $\mathbb{R} \ni t \mapsto \mathcal{F}_T(U_t)$ defines a strongly continuous unitary representation of $\mathbb{R}$ on $\mathcal{F}_T(\mathcal{H})$. Notice that

$$\mathcal{F}_T(U_t)s(\xi)\mathcal{F}_T(U_t)^* = s(U_t\xi) \quad \text{for } \xi \in \mathcal{H}_\mathbb{R} \text{ and } t \in \mathbb{R}. \tag{36}$$

Hence, $\rho_t = \mathrm{Ad}(\mathcal{F}_T(U_{-t}))$, $t \in \mathbb{R}$, defines a strongly continuous one-parameter group of automorphisms on $\Gamma_T(\mathcal{H}_\mathbb{R}, U_t)''$.

Definition 3.11 ([19, Def. 9.2.10]). A one-parameter group $\{\alpha_t : t \in \mathbb{R}\}$ of $*$-automorphisms of a von Neumann algebra M satisfies the KMS (modular) condition relative to a normal state ψ of M if, given any elements x and y of M, there is a complex-valued function f, bounded and continuous on the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq 1\}$, and analytic on the interior of that strip, such that

$$f(t) = \psi\big(\alpha_t(x)y\big), \quad f(t+i) = \psi\big(y\alpha_t(x)\big) \quad \text{for } t \in \mathbb{R}.$$

In fact, it is possible to drop the condition of "boundedness" of the function in Definition 3.11 (see [9, Prop. 5.3.7]).

Theorem 3.12. *The one-parameter group $\{\rho_t : t\in\mathbb{R}\}$ of $*$-automorphisms of $\Gamma_T(\mathcal{H}_\mathbb{R}, U_t)''$ is the modular automorphism group of φ, and therefore it satisfies the KMS condition with respect to φ.*

Proof. Note that if $\xi, \eta \in \mathcal{H}_\mathbb{R}$ are analytic for $(U_t)_{t\in\mathbb{R}}$, then

$$\begin{aligned}
\varphi\big(\rho_t\big(s(\xi)\big)s(\eta)\big) &= \big\langle \Omega, \rho_t\big(s(\xi)\big)s(\eta)\Omega\big\rangle_U \\
&= \big\langle \Omega, s(U_{-t}\xi)s(\eta)\Omega\big\rangle_U \\
&= \big\langle s(U_{-t}\xi)\Omega, s(\eta)\Omega\big\rangle_U \\
&= \langle U_{-t}\xi, \eta\rangle_U \\
&= \langle \xi, U_t\eta\rangle_U, \quad t \in \mathbb{R}.
\end{aligned}$$

On the other hand,

$$\varphi\big(s(\eta)\rho_t\big(s(\xi)\big)\big) = \langle U_t\eta, \xi\rangle_U, \quad t \in \mathbb{R}.$$

Let $\mathcal{I} = \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq 1\}$. Let $F_{\xi,\eta} : \mathcal{I} \to \mathbb{C}$ be the function defined as

$$F_{\xi,\eta}(z) = \langle \xi, U_z\eta\rangle_U, \quad z \in \mathcal{I}.$$

Then, $F_{\xi,\eta}$ is continuous in $\mathcal{I}$ and analytic in the interior of $\mathcal{I}$. Clearly,

$$F_{\xi,\eta}(t) = \varphi\big(\rho_t\big(s(\xi)\big)s(\eta)\big) \quad \text{for all } t \in \mathbb{R}.$$

Further,

$$\begin{aligned} F_{\xi,\eta}(t+i) &= \langle \xi, U_{t+i}\eta\rangle_U \\ &= \langle \xi, U_i U_t \eta\rangle_U \\ &= \langle \xi, A^{-1} U_t \eta\rangle_U \\ &= \langle U_t\eta, \xi\rangle_U \quad \big(\text{by } (2)\big) \\ &= \varphi\big(s(\eta)\rho_t\big(s(\xi)\big)\big), \quad t \in \mathbb{R}. \end{aligned}$$

Let $\xi_{t_k} \in \mathcal{H}_{\mathbb{R}}^{(t_k)}$ for $t_k \in N$, $1 \le k \le n$, $n \in \mathbb{N}$, be analytic vectors of $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$. Let $x = s(\xi_{t_1})s(\xi_{t_2})\cdots s(\xi_{t_m})$ and $y = s(\xi_{t_{m+1}})s(\xi_{t_{m+2}})\cdots s(\xi_{t_n})$ for $1 \le m < n$. From (35), it follows that

$$\begin{aligned} \varphi\big(\rho_t(x)y\big) &= \varphi\big(s(U_{-t}\xi_{t_1})s(U_{-t}\xi_{t_2})\cdots s(U_{-t}\xi_{t_m})s(\xi_{t_{m+1}})s(\xi_{t_{m+2}})\cdots s(\xi_{t_n})\big) \\ &= \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \sum_\nu g_\nu(q_{ij}) \prod_{r=1}^{n/2} \langle \eta_{t_{i(r)}}, \eta_{t_{j(r)}}\rangle_U, & \text{if } n \text{ is even}, \end{cases} \end{aligned}$$

where the summation is over all pair partitions $\nu = \{i(r), j(r)\}_{1\le r\le n/2}$ of $\{1,\dots,n\}$ with $i(r) < j(r)$ and $\eta_{t_p} = U_{-t}\xi_{t_p}$ for $p = 1,\dots,m$, and $\eta_{t_p} = \xi_{t_p}$ for $p = (m+1),\dots,n$. The summand corresponding to each ν in the summation above is non-zero only if $t_{i(r)} = t_{j(r)}$ for $r = 1,\dots,\frac{n}{2}$.

With each partition $\nu = \{i(r), j(r)\}_{1\le r\le n/2}$ of $\{1,\dots,n\}$ with $i(r) < j(r)$, we associate a new partition $\nu' = \{(i'(r), j'(r)) := (i(r)-(n-m), j(r)-(n-m))\}_{1\le r\le n/2}$, where the arithmetic is performed modulo n. Thus, again from (35), one has

$$\begin{aligned} \varphi(y\rho_t(x)) &= \varphi\big(s(\xi_{t_{m+1}})s(\xi_{t_{m+2}})\cdots s(\xi_{t_n})s(U_{-t}\xi_{t_1})s(U_{-t}\xi_{t_2})\cdots s(U_{-t}\xi_{t_m})\big) \\ &= \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \sum_{\nu'} g_{\nu'}(q_{ij}) \prod_{r=1}^{n/2} \langle \eta'_{t_{i'(r)}}, \eta'_{t_{j'(r)}}\rangle_U, & \text{if } n \text{ is even}, \end{cases} \end{aligned} \tag{37}$$

where $\eta'_{t_p} = U_{-t}\xi_{t_p}$ for $p = 1,\dots,m$, and $\eta'_{t_p} = \xi_{t_p}$ for $p = (m+1),\dots,n$, and $g_{\nu'}(q_{ij})$ is given by

$$g_{\nu'}(q_{ij}) = \prod_{i(\nu)=\{(r,s):1\le r,\ s\le n/2,\ i(r)<i(s)<j(r)<j(s)\}} q(t_{j'(r)}, t_{j'(s)}).$$

As before, the summand corresponding to each ν' in the summation above is non-zero only if $t_{i'(r)} = t_{j'(r)}$ for $r = 1,\dots,\frac{n}{2}$.

Let $f_{x,y} : \mathcal{I} \to \mathbb{C}$ be the function defined as

$$f_{x,y}(z) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \sum_\nu g_\nu(q_{ij}) \prod_{r=1}^{n/2} \langle \zeta_{t_{i(r)}}, \zeta_{t_{j(r)}}\rangle_U, & \text{if } n \text{ is even}, \end{cases} \tag{38}$$

where the summation is over all pair partitions $\nu = \{i(r), j(r)\}_{1\le r\le n/2}$ of $\{1,\dots,n\}$ with $i(r) < j(r)$ and $\zeta_{t_p} = U_{-\bar{z}}\xi_{t_p}$ for $p = 1,\dots,m$, and $\zeta_{t_p} = \xi_{t_p}$ for $p = (m+1),\dots,n$.

Since the term corresponding to each partition ν in (38) is non-zero only if $t_{i(r)} = t_{j(r)}$ for $r = 1, \ldots, \frac{n}{2}$, and $U_z^* = U_{-\bar{z}}$ for all $z \in \mathbb{C}$, from the first part of the argument above and (37), it follows that

$$f_{x,y}(t) = \varphi\big(\rho_t(x)y\big), \quad f_{x,y}(t+i) = \varphi\big(y\rho_t(x)\big), \quad t \in \mathbb{R}.$$

By applying an argument analogous to [1, Prop. 2.5] to each $\mathcal{H}_{\mathbb{R}}^{(i)}$ for $i \in N$, one can construct an orthonormal basis $\mathcal{O}_i := \{\xi_j^{(i)} : 1 \leq j \leq \dim(\mathcal{H}_{\mathbb{R}}^{(i)})\}$ of $\mathcal{H}_{\mathbb{R}}^{(i)}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}$ consisting of analytic vectors from $\mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, with respect to the strongly continuous one-parameter group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$. Consequently, $\mathcal{O} = \bigcup_{i\in N} \mathcal{O}_i$ is an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ consisting of analytic vectors and orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}$.

Proposition 2.2 yields that if $\mathcal{H}_{\mathbb{R}} \ni \xi_n \to \xi \ni \mathcal{H}_{\mathbb{R}}$ in $\|\cdot\|_U$ (equivalently in $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}}$), then $s(\xi_n) \to s(\xi)$ in $\|\cdot\|$. Therefore, $vN(s(\xi_i) : \xi_i \in \mathcal{O}) = \Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$. Hence, the $*$-algebra $\mathcal{A} = \mathrm{span}_{\mathbb{C}}\{s(\xi_{j_1})\cdots s(\xi_{j_n}) : \xi_{j_k} \in \mathcal{O},\ 1 \leq k \leq n,\ n \in \mathbb{N}\}$ is s.o.t. dense in $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$.

The above argument shows that for $x, y \in \mathcal{A}$, the KMS condition holds. Further, φ is faithful as Ω is separating for $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ (see Remark 3.6). Consequently, from [19, Lem. 9.2.17], the result follows. This completes the proof. ∎

3.1. Modular theory

Following the standard notations in modular theory, let S_φ be the closure of the operator $x\Omega \mapsto x^*\Omega$ for $x \in \Gamma_T(\mathcal{H}, U_t)''$, and let Δ_φ, J_φ be the associated modular operator and Tomita's modular conjugation, respectively.

From Theorem 3.12, it follows that $\sigma_t^\varphi = \rho_t = \mathrm{Ad}(\mathcal{F}_T(U_{-t}))_{\upharpoonright \Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''}$, $t \in \mathbb{R}$, where (σ_t^φ) denotes the modular automorphism group of $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ associated with the vacuum state φ. Further, from (36), one has

$$\sigma_t^\varphi\big(s(\xi)\big) = s(U_{-t}\xi) \quad \text{for all } \xi \in \mathcal{H}_{\mathbb{R}} \text{ and } t \in \mathbb{R}. \tag{39}$$

Let $x \in \Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$. Then,

$$\begin{aligned}
(\Delta_\varphi)^{it} x\Omega &= (\Delta_\varphi)^{it} x (\Delta_\varphi)^{-it}\Omega \\
&= \sigma_t^\varphi(x)\Omega \\
&= \mathcal{F}_T(U_{-t}) x \mathcal{F}_T(U_{-t})^*\Omega \\
&= \mathcal{F}_T(U_{-t}) x\Omega, \quad t \in \mathbb{R}.
\end{aligned}$$

Since Ω is cyclic for $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$, one has

$$(\Delta_\varphi)^{it} = \mathcal{F}_T(U_{-t}) \quad \text{for all } t \in \mathbb{R} \text{ on } \mathcal{F}_T(\mathcal{H}). \tag{40}$$

Theorem 3.13. *For each $n \geqslant 1$, the following assertions hold:*

(i) $S_\varphi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1$ *for* $\xi_1, \ldots, \xi_n \in \mathcal{H}_{\mathbb{R}}$;

(ii) $\Delta_\varphi(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1}\xi_1 \otimes \cdots \otimes A^{-1}\xi_n$ *for* $\xi_1, \ldots, \xi_n \in \mathcal{H}_{\mathbb{R}} \cap \mathfrak{D}(A^{-1})$;

(iii) *Δ_φ restricted on $\mathfrak{D}(\Delta_\varphi)\cap\mathcal{H}^{\otimes_T^n}$ is the closure $(A^{-1})^{\otimes_T^n}$ of $(A^{-1})^{\odot n}$ on $\mathcal{H}^{\otimes_T^n}$;*

(iv) *$J_\varphi(\xi_1\otimes\cdots\otimes\xi_n) = A^{-\frac{1}{2}}\xi_n\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_1$ for $\xi_1,\dots,\xi_n\in\mathcal{H}_\mathbb{R}\cap\mathfrak{D}(A^{-\frac{1}{2}})$.*

Proof. (i) Fix $n\in\mathbb{N}$. Let $\xi_{t_k}\in\mathcal{H}_\mathbb{R}^{(t_k)}$ for $t_k\in N$, $1\le k\le n$. First, we show that

$$S_\varphi(\xi_{t_1}\otimes\cdots\otimes\xi_{t_n}) = \xi_{t_n}\otimes\cdots\otimes\xi_{t_1}.$$

The proof is by induction on n. For $n=1$, the formula is true since

$$S_\varphi\xi_{t_1} = S_\varphi\big(s(\xi_{t_1})\Omega\big) = s(\xi_{t_1})\Omega = \xi_{t_1}.$$

Suppose that the formula is true up to $n-1$. From Lemma 3.10, it follows that $\xi_{t_1}\otimes\cdots\otimes\xi_{t_n}\in\mathfrak{D}(S_\varphi)$. This is because in (34), there is a single simple tensor of length n, and all other simple tensors are of order less than n.

By applying S_φ to (34), we get

$$\begin{aligned} &s(\xi_{t_n})\cdots s(\xi_{t_1})\Omega\\ &= S_\varphi(\xi_{t_1}\otimes\cdots\otimes\xi_{t_n})\\ &+\sum_{\nu=\{\{i(r),j(r)\}_{1\le r\le l},\{k(p)\}_{1\le p\le m}\}} f_\nu(q_{ij})\Big(\prod_{r=1}^{l}\langle\xi_{t_{j(r)}},\xi_{t_{i(r)}}\rangle_U\Big)\big(\xi_{t_{k(m)}}\otimes\cdots\otimes\xi_{t_{k(1)}}\big), \end{aligned}\tag{41}$$

where $2l+m=n$ and $l\geqslant 1$.

Next, applying Lemma 3.10 to the reverse sequence $\xi_{t_n},\dots,\xi_{t_1}$, we get

$$\begin{aligned} &s(\xi_{t_n})\cdots s(\xi_{t_1})\Omega\\ &= \xi_{t_n}\otimes\cdots\otimes\xi_{t_1}\\ &+\sum_{\nu=\{\{i(r),j(r)\}_{1\le r\le l},\{k(p)\}_{1\le p\le m}\}} f_\nu(q_{ij})\Big(\prod_{r=1}^{l}\langle\xi_{t_{j(r)}},\xi_{t_{i(r)}}\rangle_U\Big)\big(\xi_{t_{k(m)}}\otimes\cdots\otimes\xi_{t_{k(1)}}\big), \end{aligned}\tag{42}$$

where $2l+m=n$ and $l\geqslant 1$.

Comparing (41) and (42) yields the desired formula for n.

Now, since every $\xi\in\mathcal{H}_\mathbb{R}$ has a unique decomposition $\xi := \bigoplus_{i\in N}\xi^{(i)}$ for $\xi^{(i)}\in\mathcal{H}_\mathbb{R}^{(i)}$, S_φ is closed and j is unitary on $\mathcal{F}_T(\mathcal{H})$ by Lemma 3.1, (i) follows by a simple density argument.

(ii) Fix $n\in\mathbb{N}$. Let $\xi_1,\dots,\xi_n,\zeta_1,\dots,\zeta_n\in\mathcal{H}_\mathbb{R}\cap\mathfrak{D}(A^{-1})$. Following [26], we have

$$\begin{aligned} &\big\langle\zeta_1\otimes\cdots\otimes\zeta_n,\ S_\varphi(\xi_n\otimes\cdots\otimes\xi_1)\big\rangle_{\mathcal{F}(\mathcal{H})}\\ &\quad= \langle\zeta_1\otimes\cdots\otimes\zeta_n,\ \xi_1\otimes\cdots\otimes\xi_n\rangle_{\mathcal{F}(\mathcal{H})}\\ &\quad= \prod_{k=1}^{n}\langle\zeta_k,\xi_k\rangle_U = \prod_{k=1}^{n}\langle\xi_k,A^{-1}\zeta_k\rangle_U \quad (\text{by } (2))\\ &\quad= \langle\xi_n\otimes\cdots\otimes\xi_1,\ A^{-1}\zeta_n\otimes\cdots\otimes A^{-1}\zeta_1\rangle_{\mathcal{F}(\mathcal{H})}. \end{aligned}\tag{43}$$

Arguing as in Lemma 3.9, it is easy to show that $T(A^{-1}\odot A^{-1})\subseteq(A^{-1}\odot A^{-1})T$ on $\mathfrak{D}(A^{-1})\odot\mathfrak{D}(A^{-1})$. Then, for $k<n$, one has $T_k((A^{-1})^{\odot n})\subseteq((A^{-1})^{\odot n})T_k$ on $\mathfrak{D}(A^{-1})^{\odot n}$. Hence, from (10), it follows that $P^{(n)}((A^{-1})^{\odot n})\subseteq((A^{-1})^{\odot n})P^{(n)}$ on $\mathfrak{D}(A^{-1})^{\odot n}$. Therefore,

$$\begin{aligned}
&\langle\zeta_1\otimes\cdots\otimes\zeta_n,\ S_\varphi(\xi_n\otimes\cdots\otimes\xi_1)\rangle_T\\
&\quad=\langle\zeta_1\otimes\cdots\otimes\zeta_n,\ P^{(n)}S_\varphi(\xi_n\otimes\cdots\otimes\xi_1)\rangle_{\mathcal{F}(\mathcal{H})}\\
&\quad=\langle P^{(n)}(\zeta_1\otimes\cdots\otimes\zeta_n),\ S_\varphi(\xi_n\otimes\cdots\otimes\xi_1)\rangle_{\mathcal{F}(\mathcal{H})}\\
&\quad=\langle\xi_n\otimes\cdots\otimes\xi_1,\ (A^{-1})^{\odot n}jP^{(n)}(\zeta_1\otimes\cdots\otimes\zeta_n)\rangle_{\mathcal{F}(\mathcal{H})}\quad\text{(by (43))}\\
&\quad=\langle\xi_n\otimes\cdots\otimes\xi_1,\ (A^{-1})^{\odot n}jP^{(n)}j^2(\zeta_1\otimes\cdots\otimes\zeta_n)\rangle_{\mathcal{F}(\mathcal{H})}\quad(\text{since } j^2=1)\\
&\quad=\langle\xi_n\otimes\cdots\otimes\xi_1,\ (A^{-1})^{\odot n}P^{(n)}(\zeta_n\otimes\cdots\otimes\zeta_n)\rangle_{\mathcal{F}(\mathcal{H})}\quad\text{(by (23))}\\
&\quad=\langle\xi_n\otimes\cdots\otimes\xi_1,\ P^{(n)}(A^{-1}\zeta_n\otimes\cdots\otimes A^{-1}\zeta_1)\rangle_{\mathcal{F}(\mathcal{H})}\\
&\quad=\langle\xi_n\otimes\cdots\otimes\xi_1,\ A^{-1}\zeta_n\otimes\cdots\otimes A^{-1}\zeta_1\rangle_T.
\end{aligned}$$

Since $\mathrm{span}_{\mathbb{C}}\{\{\eta_1\otimes\cdots\otimes\eta_m:\eta_i\in\mathcal{H}_{\mathbb{R}}\cap\mathfrak{D}(A^{-1}),\ 1\le i\le m,\ m\in\mathbb{N}\}\cup\mathbb{C}\Omega\}$ is a core of S_φ, it follows that $\zeta_1\otimes\cdots\otimes\zeta_n\in\mathfrak{D}(S_\varphi^*)$ and

$$S_\varphi^*(\zeta_1\otimes\cdots\otimes\zeta_n)=A^{-1}\zeta_n\otimes\cdots\otimes A^{-1}\zeta_1.$$

Again, since $\Delta_\varphi=S_\varphi^*S_\varphi$, one has

$$\begin{aligned}
\Delta_\varphi(\xi_1\otimes\cdots\otimes\xi_n)&=S_\varphi^*S_\varphi(\xi_1\otimes\cdots\otimes\xi_n)=S_\varphi^*(\xi_n\otimes\cdots\otimes\xi_1)\quad\big(\text{by (i)}\big)\\
&=A^{-1}\xi_1\otimes\cdots\otimes A^{-1}\xi_n.
\end{aligned}$$

Therefore, (ii) holds.

(iii) Fix $n\in\mathbb{N}$. In (ii), we have noted that $P^{(n)}(A^{-1})^{\odot n}\subseteq(A^{-1})^{\odot n}P^{(n)}$ on $\mathfrak{D}(A^{-1})^{\odot n}$. Therefore, replacing the role of A with A^{-1} in the proof of Lemma 3.9, one has that $(A^{-1})^{\otimes_T^n}$ is a densely defined positive self-adjoint operator on $\mathcal{H}^{\otimes_T^n}$.

Let $P_n:\mathcal{F}_T(\mathcal{H})\to\mathcal{H}^{\otimes_T^n}$ denote the orthogonal projection. Since $\mathcal{H}^{\otimes_T^n}$ is invariant under $\mathcal{F}_T(U_{-t})$, $t\in\mathbb{R}$, from (40), it follows that $(\Delta_\varphi)^{it}P_n=\mathcal{F}_T(U_{-t})P_n$ for $t\in\mathbb{R}$ and $\Delta_\varphi P_n$ is the analytic generator of the strongly continuous group $\mathbb{R}\ni t\mapsto\mathcal{F}_T(U_{-t})$ on $\mathcal{H}^{\otimes_T^n}$.

From (ii), it follows that $(A^{-1})^{\otimes_T^n}\subseteq\Delta_\varphi P_n$. By [28, Exer. E.9.28], it follows that $(A^{-1})^{\otimes_T^n}=\Delta_\varphi P_n$.

(iv) First, note that replacing the role of A by A^{-1} in Lemma 3.9 (iii), one has

$$((A^{-1})^{\otimes_T^n})^{\frac{1}{k}}=(A^{-\frac{1}{k}})^{\otimes_T^n}\quad\text{for } k,n\in\mathbb{N}.$$

Let $\xi_1,\dots,\xi_n\in\mathcal{H}_{\mathbb{R}}\cap\mathfrak{D}(A^{-\frac{1}{2}})$. Then,

$$\begin{aligned}
J_\varphi(\xi_1\otimes\cdots\otimes\xi_n)&=J_\varphi S_\varphi(\xi_n\otimes\cdots\otimes\xi_1)&&\big(\text{by } (i)\big)\\
&=\Delta_\varphi^{\frac{1}{2}}(\xi_n\otimes\cdots\otimes\xi_1)&&(\text{since } J_\varphi S_\varphi=\Delta_\varphi^{\frac{1}{2}})\\
&=A^{-\frac{1}{2}}\xi_n\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_1,&&\big(\text{since } \big((A^{-1})^{\otimes_T^n}\big)^{\frac{1}{2}}=(A^{-\frac{1}{2}})^{\otimes_T^n}\big).
\end{aligned}$$

This completes the proof. ∎

3.2. Commutant

With the help of J_φ, we proceed to describe the commutant of $\Gamma_T(\mathcal{H}_\mathbb{R}, U_t)''$. Let $\zeta \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_\mathbb{R}$, and let $\zeta := \bigoplus_{i\in N} \zeta^{(i)}$ for $\zeta^{(i)} \in \mathcal{H}_\mathbb{R}^{(i)}$ be the unique decomposition of ζ. Since $\mathcal{H}_\mathbb{R}^{(i)}$ is invariant for $(U_t)_{t\in\mathbb{R}}$, by the Hahn–Hellinger theorem, it follows that $\zeta^{(i)} \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_\mathbb{R}$ for all $i \in N$, and

$$A^{-\frac{1}{2}}\zeta = \bigoplus_{i\in N} A^{-\frac{1}{2}}\zeta^{(i)},$$

where the direct sum is taken with respect to $\langle\cdot,\cdot\rangle_U$. Then, for $i \in N$, by using (1), one has

$$\begin{aligned}\langle A^{-\frac{1}{2}}\zeta^{(i)}, \eta\rangle_U &= \left\langle \frac{2A^{-\frac{1}{2}}}{1+A^{-1}}\zeta^{(i)}, \eta\right\rangle_{\mathcal{H}_\mathbb{C}} = \left\langle \eta, \mathcal{J}\frac{2A^{-\frac{1}{2}}}{1+A^{-1}}\zeta^{(i)}\right\rangle_{\mathcal{H}_\mathbb{C}} \\ &= \left\langle \eta, \frac{2A^{\frac{1}{2}}}{1+A}\zeta^{(i)}\right\rangle_{\mathcal{H}_\mathbb{C}} = \left\langle \frac{2}{1+A^{-1}}\eta, A^{-\frac{1}{2}}\zeta^{(i)}\right\rangle_{\mathcal{H}_\mathbb{C}} \\ &= \langle \eta, A^{-\frac{1}{2}}\zeta^{(i)}\rangle_U, \quad \eta \in \mathcal{H}_\mathbb{R}. \end{aligned} \tag{44}$$

From (29) and (44), it follows that

$$A^{-\frac{1}{2}}\zeta \in \mathcal{H}'_\mathbb{R} \quad \text{for all } \zeta \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_\mathbb{R}. \tag{45}$$

Also, for $\xi, \eta \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_\mathbb{R}$, it follows from (2) that

$$\langle \eta, \xi\rangle_U = \langle \xi, A^{-1}\eta\rangle_U = \langle A^{-\frac{1}{2}}\xi, A^{-\frac{1}{2}}\eta\rangle_U \quad \big(\text{as } \mathfrak{D}(A^{-1}) \subseteq \mathfrak{D}(A^{-\frac{1}{2}})\big). \tag{46}$$

The next result describes the commutant of $\Gamma_T(\mathcal{H}_\mathbb{R}, U_t)''$, and the proof is an adaptation of [1, Thm. 2.4] to the current setting.

Theorem 3.14. *Let $\xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_\mathbb{R}$. Then, $J_\varphi s(\xi) J_\varphi = d(A^{-\frac{1}{2}}\xi)$. Moreover,*

$$\Gamma_T(\mathcal{H}_\mathbb{R}, U_t)' = \{d(\xi) : \xi \in \mathcal{H}'_\mathbb{R}\}''.$$

Proof. Fix $n \in \mathbb{N}$. Let

$$\xi_{i_k} \in \mathcal{H}_\mathbb{R}^{(i_k)} \cap \mathfrak{D}(A^{-1}) \quad \text{for } i_k \in N,\ 1 \le k \le n.$$

Let $\gamma_k = q_{i_k i_{k-1}} \cdots q_{i_k i_1}$ for $1 \le k \le n$. By Theorem 3.13, one has

$$\begin{aligned} &J_\varphi s(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &= J_\varphi\left(\sum_{k=1}^n \langle \xi, \xi_{i_k}\rangle_U \gamma_k\, \xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n}\right) \\ &\quad + J_\varphi(\xi \otimes \xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_n}) \\ &= \sum_{k=1}^n \langle \xi_{i_k}, \xi\rangle_U \gamma_k\, A^{-\frac{1}{2}}\xi_{i_n} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_{k+1}} \otimes A^{-\frac{1}{2}}\xi_{i_{k-1}} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_1} \\ &\quad + A^{-\frac{1}{2}}\xi_{i_n} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_1} \otimes A^{-\frac{1}{2}}\xi \quad \big(\text{since } \mathfrak{D}(A^{-1}) \subseteq \mathfrak{D}(A^{-\frac{1}{2}})\big) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{n} \langle \xi, A^{-1}\xi_{i_k}\rangle_U \gamma_k \; A^{-\frac{1}{2}}\xi_{i_n} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_{k+1}} \otimes A^{-\frac{1}{2}}\xi_{i_{k-1}} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_1} \\
&\quad + A^{-\frac{1}{2}}\xi_{i_n} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_1} \otimes A^{-\frac{1}{2}}\xi \quad \big(\text{by } (46)\big) \\
&= \sum_{k=1}^{n} \langle A^{-\frac{1}{2}}\xi, A^{-\frac{1}{2}}\xi_{i_k}\rangle_U \gamma_k \; A^{-\frac{1}{2}}\xi_{i_n} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_{k+1}} \otimes A^{-\frac{1}{2}}\xi_{i_{k-1}} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_1} \\
&\quad + A^{-\frac{1}{2}}\xi_{i_n} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_1} \otimes A^{-\frac{1}{2}}\xi \quad \big(\text{since } \mathfrak{D}(A^{-1}) \subseteq \mathfrak{D}(A^{-\frac{1}{2}})\big) \\
&= d(A^{-\frac{1}{2}}\xi) J_\varphi(\xi_{i_1} \otimes \xi_{i_2} \otimes \cdots \otimes \xi_{i_n}).
\end{aligned}$$

Since $\mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}^{(i)}$ is dense in $\mathcal{H}_{\mathbb{R}}^{(i)}$ for $i \in N$, it follows that $J_\varphi s(\xi) J_\varphi = d(A^{-\frac{1}{2}}\xi)$.

By the fundamental theorem of the Tomita–Takesaki theory, we have

$$\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)' = J_\varphi \Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)'' J_\varphi.$$

From (45), it follows that $A^{-\frac{1}{2}}\xi \in \mathcal{H}_{\mathbb{R}}'$ for all $\xi \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_{\mathbb{R}}$. By what we have proved so far, it follows that $\{J_\varphi s(\xi) J_\varphi : \xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}\} \subseteq \{d(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}'\}''$. From Proposition 2.2, it follows that if $\mathcal{H}_{\mathbb{R}} \ni \xi_n \to \xi \in \mathcal{H}_{\mathbb{R}}$ in $\|\cdot\|_U$ (equivalently in $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}}$), then $s(\xi_n) \to s(\xi)$ in $\|\cdot\|$. Since $\mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}$ is dense in $\mathcal{H}_{\mathbb{R}}$, it follows that

$$\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)' \subseteq \{d(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}'\}''.$$

The reverse inclusion follows from Lemma 3.3. This completes the proof. ∎

3.3. Notations and facts

In order to reduce notations, in the remaining sections, we will denote $\Gamma_T(\mathcal{H}_{\mathbb{R}}, U_t)''$ by M_T, J_φ by J and Δ_φ by Δ, respectively. Also, we will denote the center of M_T by $\mathcal{Z}(M_T)$ and the centralizer of M_T associated with the state φ by M_T^φ, respectively; i.e., $\mathcal{Z}(M_T) = M_T \cap M_T'$, and $M_T^\varphi = \{x \in M_T : \sigma_t^\varphi(x) = x \ \forall t \in \mathbb{R}\}$.

Since Ω is separating for both M_T and M_T', for $\zeta \in M_T\Omega$ and $\eta \in M_T'\Omega$, there exist unique $x_\zeta \in M_T$ and $x_\eta' \in M_T'$ such that $\zeta = x_\zeta\Omega$ and $\eta = x_\eta'\Omega$. In this case, we will write

$$s(\zeta) = x_\zeta \quad \text{and} \quad d(\eta) = x_\eta'.$$

For example, as $\xi \in M_T\Omega$ for every $\xi \in \mathcal{H}_{\mathbb{R}}$, one has $s(\xi + i\eta) = s(\xi) + is(\eta)$ for all $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$.

4. Generator algebras M_ξ

In this section, we investigate the subalgebra M_ξ of M_T generated by the single self-adjoint variable $s(\xi)$ for $\xi \in \mathcal{H}_{\mathbb{R}}$. We show that for $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$, $i_0 \in N$, with $\|\xi_0\|_U = 1$ and $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, the associated subalgebra M_{ξ_0} of M_T is a φ-strongly mixing masa in M_T. Needless to say, such a masa is singular.

Fix $i \in N$, $n \in \mathbb{N}$. Let $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, with $\|\xi_0\|_U = 1$. From (35), it follows that the moments of the operator $s(\xi_0)$ with respect to the vacuum state φ are

$$\varphi\big((s(\xi_0))^n\big) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \sum_{\nu=\{i(r),j(r)\}_{1\le r\le n/2}} (q_{ii})^{c(\nu)}, & \text{if } n \text{ is even,} \end{cases} \tag{47}$$

where the summation is over all pair partitions $\nu = \{i(r), j(r)\}_{1\le r\le n/2}$ of $\{1, \dots, n\}$ with $i(r) < j(r)$ and $c(\nu)$ is the number of crossings of ν; i.e.,

$$c(\nu) = \#\big\{(r,s) : i(r) < i(s) < j(r) < j(s)\big\}.$$

Note that (47) shows that the distribution of the single generator $s(\xi_0)$ does not depend on the one-parameter group $(U_t)_{t\in\mathbb{R}}$. Therefore, as in the tracial case in [8], the distribution of the self-adjoint operator $s(\xi_0)$ for $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, obeys the semi-circular law $\nu_{q_{ii}}$ which is absolutely continuous with respect to the uniform measure supported on the interval $[-\frac{2}{\sqrt{1-q_{ii}}}, \frac{2}{\sqrt{1-q_{ii}}}]$. Thus, $M_{\xi_0} \cong L^\infty([-\frac{2}{\sqrt{1-q_{ii}}}, \frac{2}{\sqrt{1-q_{ii}}}], \nu_{q_{ii}})$, and hence, M_{ξ_0} is diffuse. The associated orthogonal polynomials, namely, the q_{ii}-Hermite polynomials, will be denoted by $H_n^{q_{ii}}$, $n \ge 0$.

The next lemma provides the descriptions of $s(\xi_{t_1} \otimes \cdots \otimes \xi_{t_n})$ (resp., $d(\eta_{t_1} \otimes \cdots \otimes \eta_{t_n})$) for $\xi_{t_k} \in \mathcal{H}_{\mathbb{C}}^{(t_k)}$ (resp., $\eta_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})' + i(\mathcal{H}_{\mathbb{R}}^{(t_k)})'$), $t_k \in N$, $1 \le k \le n$, in terms of the operators $l(\xi_{t_k})$ and $l^*(\xi_{t_k})$ (resp., $r(\eta_{t_k})$ and $r^*(\eta_{t_k})$). Similar formulas in the literature are known as the Wick product formulas.

Lemma 4.1. *Fix $n \in \mathbb{N}$. Let $\xi_{t_k} \in \mathcal{H}_{\mathbb{C}}^{(t_k)}$, $\eta_{t_k} \in (\mathcal{H}_{\mathbb{R}}^{(t_k)})' + i(\mathcal{H}_{\mathbb{R}}^{(t_k)})'$ for $t_k \in N$, $1 \le k \le n$. Then,*

(i) *Then, one has*

$$\begin{aligned} &s(\xi_{t_1} \otimes \cdots \otimes \xi_{t_n}) \\ &\quad = \sum_{\substack{l,m\ge 0 \\ l+m=n}} \sum_{\substack{\tilde{I}=\{i(1),\dots,i(l)\},\, i(1)<\cdots<i(l) \\ \tilde{J}=\{j(1),\dots,j(m)\},\, j(1)<\cdots<j(m) \\ \tilde{I}\cup\tilde{J}=\{1,\dots,n\} \\ \tilde{I}\cap\tilde{J}=\emptyset}} \begin{aligned}[t] &f_{(\tilde{I},\tilde{J})}(q_{ij}) l(\xi_{t_{i(1)}}) \cdots l(\xi_{t_{i(l)}}) \\ &\times l^*(\mathcal{J}\xi_{t_{j(1)}}) \cdots l^*(\mathcal{J}\xi_{t_{j(m)}}), \end{aligned} \end{aligned}$$

(ii) *also*

$$\begin{aligned} &d(\eta_{t_1} \otimes \cdots \otimes \eta_{t_n}) \\ &\quad = \sum_{\substack{l,m\ge 0 \\ l+m=n}} \sum_{\substack{\tilde{I}=\{i(1),\dots,i(l)\},\, i(1)<\cdots<i(l) \\ \tilde{J}=\{j(1),\dots,j(m)\},\, j(1)<\cdots<j(m) \\ \tilde{I}\cup\tilde{J}=\{1,\dots,n\} \\ \tilde{I}\cap\tilde{J}=\emptyset}} \begin{aligned}[t] &f_{(\tilde{I},\tilde{J})}(q_{ij}) r(\eta_{t_{i(1)}}) \cdots r(\eta_{t_{i(l)}}) \\ &\times r^*\big(\mathcal{J}_r(\eta_{t_{j(1)}})\big) \cdots r^*\big(\mathcal{J}_r(\eta_{t_{j(m)}})\big), \end{aligned} \end{aligned}$$

where $f_{(\tilde{I},\tilde{J})}(q_{ij}) = \prod_{\{(r,s):1\le r\le l, 1\le s\le m, i(r)>j(s)\}} q_{t_{i(r)} t_{j(s)}}$, and $\mathcal{J}$ and $\mathcal{J}_r$ are the complex conjugations defined on $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{H}'_{\mathbb{R}} + i\mathcal{H}'_{\mathbb{R}}$, respectively.

Proof. (i) The proof is by induction. For $n = 1$, the formula is trivial since

$$s(\xi_{t_1}) = l(\xi_{t_1}) + l^*(\mathcal{J}\xi_{t_1}) \quad \big(\text{see (15)}\big).$$

Suppose that the formula is true for $n-1$. Since Ω is separating for M_T, from (15) and (16), one has

$$\begin{aligned}
&s(\xi_{t_1} \otimes \cdots \otimes \xi_{t_n})\\
&\quad = s(\xi_{t_1})s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\
&\qquad - \sum_{i=2}^{n} q_{t_i t_{i-1}} \cdots q_{t_i t_2} \langle \mathcal{J}\xi_{t_1}, \xi_{t_i}\rangle_U\, s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_{i-1}} \otimes \xi_{t_{i+1}} \otimes \cdots \otimes \xi_{t_n})\\
&\quad = \big(l(\xi_{t_1}) + l^*(\mathcal{J}\xi_{t_1})\big)s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\
&\qquad - \sum_{i=2}^{n} q_{t_i t_{i-1}} \cdots q_{t_i t_2} \langle \mathcal{J}\xi_{t_1}, \xi_{t_i}\rangle_U\, s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_{i-1}} \otimes \xi_{t_{i+1}} \otimes \cdots \otimes \xi_{t_n})\\
&\quad = l(\xi_{t_1})s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n}) + l^*(\mathcal{J}\xi_{t_1})s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\
&\qquad - \sum_{i=2}^{n} q_{t_i t_{i-1}} \cdots q_{t_i t_2} \langle \mathcal{J}\xi_{t_1}, \xi_{t_i}\rangle_U\, s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_{i-1}} \otimes \xi_{t_{i+1}} \otimes \cdots \otimes \xi_{t_n})\\
&\quad = l(\xi_{t_1})\Bigg(\sum_{\substack{l,m\geq 0\\ l+m=n-1}} \sum_{\substack{\tilde{I}=\{i(1),\ldots,i(l)\},\, i(1)<\cdots<i(l)\\ \tilde{J}=\{j(1),\ldots,j(m)\},\, j(1)<\cdots<j(m)\\ \tilde{I}\cup\tilde{J}=\{2,\ldots,n\}\\ \tilde{I}\cap\tilde{J}=\emptyset}} f_{(\tilde{I},\tilde{J})}(q_{ij}) l(\xi_{t_{i(1)}}) l(\xi_{t_{i(2)}}) \cdots l(\xi_{t_{i(l)}})\\
&\qquad\qquad \times l^*(\mathcal{J}\xi_{t_{j(1)}}) \cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\Bigg) + l^*(\mathcal{J}\xi_{t_1})s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\
&\qquad - \sum_{i=2}^{n} q_{t_i t_{i-1}} \cdots q_{t_i, t_2} \langle \mathcal{J}\xi_{t_1}, \xi_{t_i}\rangle_U\, s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_{i-1}} \otimes \xi_{t_{i+1}} \otimes \cdots \otimes \xi_{t_n})\\
&\qquad\qquad\qquad \text{(by the induction hypothesis).}
\end{aligned} \tag{48}$$

Again, by the induction hypothesis, one has

$$\begin{aligned}
&l^*(\mathcal{J}\xi_{t_1})s(\xi_{t_2} \otimes \cdots \otimes \xi_{t_n})\\
&= \sum_{\substack{l,m\geq 0\\ l+m=n-1}} \sum_{\substack{\tilde{I}=\{i(1),\ldots,i(l)\},\, i(1)<\cdots<i(l)\\ \tilde{J}=\{j(1),\ldots,j(m)\},\, j(1)<\cdots<j(m)\\ \tilde{I}\cup\tilde{J}=\{2,\ldots,n\}\\ \tilde{I}\cap\tilde{J}=\emptyset}} \begin{aligned}[t] &f_{(\tilde{I},\tilde{J})}(q_{ij}) l^*(\mathcal{J}\xi_{t_1}) l(\xi_{t_{i(1)}})\\ &\times l(\xi_{t_{i(2)}}) \cdots l(\xi_{t_{i(l)}}) l^*(\mathcal{J}\xi_{t_{j(1)}}) \cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\end{aligned}\\
&= \Bigg(\sum_{\substack{l,m\geq 0\\ l+m=n-1}} \sum_{\substack{\tilde{I}=\{i(1),\ldots,i(l)\},\, i(1)<\cdots<i(l)\\ \tilde{J}=\{j(1),\ldots,j(m)\},\, j(1)<\cdots<j(m)\\ \tilde{I}\cup\tilde{J}=\{2,\ldots,n\}\\ \tilde{I}\cap\tilde{J}=\emptyset}} \begin{aligned}[t] &f_{(\tilde{I},\tilde{J})}(q_{ij}) q_{t_{i(1)},t_1} l(\xi_{t_{i(1)}}) l^*(\mathcal{J}\xi_{t_1})\\ &\times l(\xi_{t_{i(2)}}) \cdots l(\xi_{t_{i(l)}}) l^*(\mathcal{J}\xi_{t_{j(1)}}) \cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\Bigg)\end{aligned}
\end{aligned}$$

$$+\Bigg(\sum_{\substack{l,m\geq 0\\ l+m=n-1}}\sum_{\substack{\widetilde{I}=\{i(1),\ldots,i(l)\},\ i(1)<\cdots<i(l)\\ \widetilde{J}=\{j(1),\ldots,j(m)\},\ j(1)<\cdots<j(m)\\ \widetilde{I}\cup\widetilde{J}=\{2,\ldots,n\}\\ \widetilde{I}\cap\widetilde{J}=\emptyset}} \begin{aligned}&f_{(\widetilde{I},\widetilde{J})}(q_{ij})\langle \mathcal{J}\xi_{t_1},\xi_{t_{i(1)}}\rangle_U l(\xi_{t_{i(2)}})\cdots l(\xi_{t_{i(l)}})\\ &\times l^*(\mathcal{J}\xi_{t_{j(1)}})\cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\Bigg)\ \text{(by Lemma 2.3)}\end{aligned}$$

$$=\Bigg(\sum_{\substack{l,m\geq 0\\ l+m=n-1}}\sum_{\substack{\widetilde{I}=\{i(1),\ldots,i(l)\},\ i(1)<\cdots<i(l)\\ \widetilde{J}=\{j(1),\ldots,j(m)\},\ j(1)<\cdots<j(m)\\ \widetilde{I}\cup\widetilde{J}=\{2,\ldots,n\}\\ \widetilde{I}\cap\widetilde{J}=\emptyset}} \begin{aligned}&f_{(\widetilde{I},\widetilde{J})}(q_{ij})q_{t_{i(1)}t_1}l(\xi_{t_{i(1)}})l^*(\mathcal{J}\xi_{t_1})\\ &\times l(\xi_{t_{i(2)}})\cdots l(\xi_{t_{i(l)}})l^*(\mathcal{J}\xi_{t_{j(1)}})\cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\Bigg)\end{aligned}$$

$$+\,q_{t_{i(1)}t_{i(1)-1}}\cdots q_{t_{i(1)}t_2}\langle \mathcal{J}\xi_{t_1},\xi_{t_{i(1)}}\rangle_U s(\xi_{t_2}\otimes\cdots\otimes\xi_{t_{i(1)-1}}\otimes\xi_{t_{i(1)+1}}\otimes\cdots\otimes\xi_{t_n})$$

(by the induction hypothesis).

Recursively, one obtains

$$\begin{aligned}&l^*(\mathcal{J}\xi_{t_1})s(\xi_{t_2}\otimes\cdots\otimes\xi_{t_n})\\ &=\sum_{\substack{l,m\geq 0\\ l+m=n-1}}\sum_{\substack{\widetilde{I}=\{i(1),\ldots,i(l)\},\ i(1)<\cdots<i(l)\\ \widetilde{J}=\{j(1),\ldots,j(m)\},\ j(1)<\cdots<j(m)\\ \widetilde{I}\cup\widetilde{J}=\{2,\ldots,n\}\\ \widetilde{I}\cap\widetilde{J}=\emptyset}} \begin{aligned}&(q_{t_{i(1)}t_1}\cdots q_{t_{i(l)}t_1})f_{(\widetilde{I},\widetilde{J})}(q_{ij})l(\xi_{t_{i(1)}})\cdots l(\xi_{t_{i(l)}})\\ &\times l^*(\mathcal{J}\xi_{t_{i(1)}})l^*(\mathcal{J}\xi_{t_{j(1)}})\cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\end{aligned}\\ &\quad+\sum_{i=2}^{n}q_{t_it_{i-1}}\cdots q_{t_it_2}\langle \mathcal{J}\xi_{t_1},\xi_{t_i}\rangle_U s(\xi_{t_2}\otimes\cdots\otimes\xi_{t_{i-1}}\otimes\xi_{t_{i+1}}\otimes\cdots\otimes\xi_{t_n}).\end{aligned}\tag{49}$$

Combining (48) and (49), one has

$$\begin{aligned}&s(\xi_{t_1}\otimes\cdots\otimes\xi_{t_n})\\ &=\sum_{\substack{l,m\geq 0\\ l+m=n-1}}\sum_{\substack{\widetilde{I}=\{i(1),\ldots,i(l)\},\ i(1)<\cdots<i(l)\\ \widetilde{J}=\{j(1),\ldots,j(m)\},\ j(1)<\cdots<j(m)\\ \widetilde{I}\cup\widetilde{J}=\{2,\ldots,n\}\\ \widetilde{I}\cap\widetilde{J}=\emptyset}} \begin{aligned}&f_{(\widetilde{I},\widetilde{J})}(q_{ij})l(\xi_{t_1})l(\xi_{t_{i(1)}})l(\xi_{t_{i(2)}})\cdots l(\xi_{t_{i(l)}})\\ &\times l^*(\mathcal{J}\xi_{t_{j(1)}})\cdots l^*(\mathcal{J}\xi_{t_{j(m)}})\end{aligned}\\ &\quad+\sum_{\substack{l,m\geq 0\\ l+m=n-1}}\sum_{\substack{\widetilde{I}=\{i(1),\ldots,i(l)\},\ i(1)<\cdots<i(l)\\ \widetilde{J}=\{j(1),\ldots,j(m)\},\ j(1)<\cdots<j(m)\\ \widetilde{I}\cup\widetilde{J}=\{2,\ldots,n\}\\ \widetilde{I}\cap\widetilde{J}=\emptyset}} \begin{aligned}&(q_{t_{i(1)}t_1}\cdots q_{t_{i(l)}t_1})f_{(\widetilde{I},\widetilde{J})}(q_{ij})l(\xi_{t_{i(1)}})\cdots l(\xi_{t_{i(l)}})\\ &\times l^*(\mathcal{J}\xi_{t_1})l^*(\mathcal{J}\xi_{t_{j(1)}})\cdots l^*(\mathcal{J}\xi_{t_{j(m)}}).\end{aligned}\end{aligned}\tag{50}$$

Note that each partition $(\widetilde{I},\widetilde{J})$ of $\{2,\ldots,n\}$ considered in (50) corresponds to two partitions $(1\cup\widetilde{I},\widetilde{J})$ and $(\widetilde{I},1\cup\widetilde{J})$ of $\{1,\ldots,n\}$ as prescribed in the statement of the lemma. It is easy to see that

$$f_{(\widetilde{I},\widetilde{J})}(q_{ij})=f_{(1\cup\widetilde{I},\widetilde{J})}(q_{ij})\quad\text{and}\quad (q_{t_{i(1)}t_1}\cdots q_{t_{i(l)}t_1})f_{(\widetilde{I},\widetilde{J})}(q_{ij})=f_{(\widetilde{I},1\cup\widetilde{J})}(q_{ij}).$$

Hence, (i) follows from (50).

(ii) The proof is similar to that of (i), but one replaces the usage of (15) and (16) by (24) and (26), respectively, and also replaces the usage of the commutation relation in Lemma 2.3 by the commutation relation in (28). We omit the details. ■

The following lemma identifies vectors in the GNS space.

Lemma 4.2. *Let $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, with $\|\xi_0\|_U = 1$. Then, the following hold:*

(i) $H_n^{q_{ii}}(s(\xi_0))\Omega = \xi_0^{\otimes n}$ *for all* $n \geq 0$*;*

(ii) $\overline{M_{\xi_0}\Omega}^{\|\cdot\|_T} = \overline{\operatorname{span}\{\xi_0^{\otimes n} : n \geq 0\}}^{\|\cdot\|_T}$.

Proof. (i) The proof is by induction. Since

$$H_0^{q_{ii}}(x) = 1 \text{ on } \left[-\frac{2}{\sqrt{1-q_{ii}}}, \frac{2}{\sqrt{1-q_{ii}}}\right],$$

the result is trivial for $n = 0$. Since

$$H_1^{q_{ii}}(x) = x \text{ on } \left[-\frac{2}{\sqrt{1-q_{ii}}}, \frac{2}{\sqrt{1-q_{ii}}}\right] \quad \text{and} \quad s(\xi_0)\Omega = \xi_0,$$

the result is true for $n = 1$. Now, suppose the result is true for $n \in \mathbb{N}$. We have the following recurrence relation for q_{ii}-Hermite polynomials:

$$xH_n^{q_{ii}}(x) = H_{n+1}^{q_{ii}}(x) + [n]_{q_{ii}} H_{n-1}^{q_{ii}}(x), \quad x \in \left[-\frac{2}{\sqrt{1-q_{ii}}}, \frac{2}{\sqrt{1-q_{ii}}}\right], \ n \geq 1,$$

where $[n]_{q_{ii}} = 1 + q_{ii} + \cdots + (q_{ii})^{n-1}$ [6, Def. 1.9]. Therefore, by (15), (16) and functional calculus, one has

$$\begin{aligned} H_{n+1}^{q_{ii}}\big(s(\xi_0)\big)\Omega &= s(\xi_0)\big(H_n^{q_{ii}} s(\xi_0)\big)\Omega - [n]_{q_{ii}} H_{n-1}^{q_{ii}}\big(s(\xi_0)\big)\Omega \\ &= s(\xi_0)(\xi_0^{\otimes n}) - [n]_{q_{ii}}(\xi_0^{\otimes(n-1)}) \\ &= \big(l(\xi_0) + l^*(\xi_0)\big)(\xi_0^{\otimes n}) - [n]_{q_{ii}}(\xi_0^{\otimes(n-1)}) \\ &= \xi_0^{\otimes(n+1)} + l^*(\xi_0)(\xi_0^{\otimes n}) - [n]_{q_{ii}}(\xi_0^{\otimes(n-1)}) \\ &= \xi_0^{\otimes(n+1)}. \end{aligned}$$

This completes the proof.

(ii) We have $M_{\xi_0} \cong L^\infty([-\frac{2}{\sqrt{1-q_{ii}}}, \frac{2}{\sqrt{1-q_{ii}}}], \nu_{q_{ii}})$. Note that $\overline{M_{\xi_0}\Omega}^{\|\cdot\|_T}$ is canonically identified with $L^2(\nu_{q_{ii}})$. Since $\{H_n^{q_{ii}} : n \geq 0\}$ is a total orthogonal set in $L^2(\nu_{q_{ii}})$, the statement in (ii) is a direct consequence of that in (i). ■

Lemma 4.3. *Let $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, be a unit vector such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$. Then, the following hold.*

(i) *Let $\xi_{i_k} \in \mathcal{H}_{\mathbb{R}}^{(i_k)}$ for $i_k \in N$, $1 \leq k \leq n, n \in \mathbb{N}$, be non-zero vectors. For $m \geq 1$, if $n \neq m$ or $\langle \xi_0, \xi_{i_k}\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$ for at least one k, then*

$$\langle \xi_0^{\otimes m}, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}\rangle_T = 0.$$

(ii) *Let* $\xi_{i_k} \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_{\mathbb{R}}^{(i_k)}$ *for* $1 \leq i_k \leq N$, $1 \leq k \leq n, n \in \mathbb{N}$, *be non-zero vectors. For* $m \geq 1$, *if* $n \neq m$ *or* $\langle \xi_0, \xi_{i_k} \rangle_{\mathcal{H}_\mathbb{C}} = 0$ *for at least one* k, *then*

$$\langle \xi_0^{\otimes k}, A^{-\frac{1}{2}}\xi_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_n} \rangle_T = 0.$$

Proof. (i) Since $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, one has $\frac{2}{1+A^{-1}}\xi_0 = \xi_0$. From (12), it follows that

$$\begin{aligned}
\langle \xi_0^{\otimes m}, \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \rangle_T &= \delta_{m,n} \sum_{\sigma \in S_n} a(\sigma, \boldsymbol{\xi}) \langle \xi_0, \xi_{i_{\sigma(1)}} \rangle_U \cdots \langle \xi_0, \xi_{i_{\sigma(n)}} \rangle_U \\
&= \delta_{m,n} \sum_{\sigma \in S_n} a(\sigma, \boldsymbol{\xi}) \prod_{k=1}^{n} \left\langle \frac{2}{1+A^{-1}}\xi_0, \xi_{i_{\sigma(k)}} \right\rangle_{\mathcal{H}_\mathbb{C}} \\
&= \delta_{m,n} \sum_{\sigma \in S_n} a(\sigma, \boldsymbol{\xi}) \prod_{k=1}^{n} \langle \xi_0, \xi_{i_{\sigma(k)}} \rangle_{\mathcal{H}_\mathbb{C}} \\
&= \delta_{m,n} \prod_{k=1}^{n} \langle \xi_0, \xi_{i_k} \rangle_{\mathcal{H}_\mathbb{C}} \sum_{\sigma \in S_n} a(\sigma, \boldsymbol{\xi}).
\end{aligned}$$

Hence, the conclusion is immediate.

(ii) The proof follows along the same lines of the proof of (i). We omit the details. ■

For our purposes, we need to know the action of certain operators in M_T on simple tensors precisely. We note it down in the form of a lemma below. The proof is similar to [1, Lem. 3.2].

Lemma 4.4. *Let* $\xi_0 \in \mathcal{H}_\mathbb{R}$ *and* $\xi_{i_k} \in \mathcal{H}_\mathbb{R}^{(i_k)}$ *for* $i_k \in N$, $1 \leq k \leq n$, $n \in \mathbb{N}$, *be such that* $\langle \xi_{i_k}, \xi_0 \rangle_U = 0$ *for* $1 \leq k \leq n$. *Then,*

$$s(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n})(\xi_0^{\otimes m}) = \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \xi_0^{\otimes m} \quad \textit{for all } m \geq 0.$$

Proof. By definition, $s(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n})\Omega = \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}$. Hence, the result is true for $m = 0$. We prove the result only for $m = 1$ since the argument is similar for $m \geq 2$. The proof is by induction.

If $n = 1$, and $\langle \xi_{i_1}, \xi_0 \rangle_U = 0$, then

$$\begin{aligned}
s(\xi_{i_1})\xi_0 &= \xi_{i_1} \otimes \xi_0 + \langle \xi_{i_1}, \xi_0 \rangle_U \Omega \quad \big(\text{by (15) and (16)}\big) \\
&= \xi_{i_1} \otimes \xi_0.
\end{aligned}$$

Now suppose that the result is true for all $1 \leq p \leq n$. Let $\xi_{i_l} \in \mathcal{H}_\mathbb{R}^{(i_l)}$ be such that $\langle \xi_{i_l}, \xi_0 \rangle_U = 0$, $1 \leq l \leq n+1$. Since Ω is separating for M_T, from (15) and (16), one has

$$\begin{aligned}
&s(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \xi_{i_{n+1}}) \\
&\quad = s(\xi_{i_1}) s(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{n+1}}) - l^*(\xi_{i_1})(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{n+1}}) \\
&\quad = s(\xi_{i_1}) s(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{n+1}}) - \sum_{k=2}^{n+1} \langle \xi_{i_1}, \xi_{i_k} \rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_2} \\
&\qquad\qquad \times s(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_{n+1}}).
\end{aligned}$$

By the induction hypothesis, one has

$$
\begin{aligned}
&s(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \xi_{i_{n+1}})\xi_0 \\
&= s(\xi_{i_1})s(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{n+1}})\xi_0 - \sum_{k=2}^{n+1} \langle \xi_{i_1}, \xi_{i_k} \rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_2} \\
&\qquad\qquad \times s(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_{n+1}})\xi_0 \\
&= s(\xi_{i_1})(\xi_{i_2} \otimes \cdots \otimes \xi_{i_{n+1}} \otimes \xi_0) - \sum_{k=2}^{n+1} \langle \xi_{i_1}, \xi_{i_k} \rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_2} \\
&\qquad\qquad \times (\xi_{i_2} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_{n+1}} \otimes \xi_0) \\
&= \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \xi_{i_{n+1}} \otimes \xi_0.
\end{aligned}
$$

In the last step of the above equation, one uses (15) and (16). This completes the proof. ■

The following theorem is known in the case $q_{ij} = q$ for all $i, j \in N$. The proof follows along the same lines of the proof of [1, Thm. 4.2]. Thus, we only state the theorem below.

Theorem 4.5. *Let $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$ for some $i \in N$ be a unit vector. There exists a unique φ-preserving faithful normal conditional expectation $\mathbb{E}_{\xi_0} : M_T \to M_{\xi_0}$ if and only if $s(\xi_0) \in M_T^{\varphi}$; equivalently, $U_t \xi_0 = \xi_0$ for all $t \in \mathbb{R}$.*

Let M be a von Neumann algebra equipped with a faithful normal state ψ. Let M act on the GNS Hilbert space $L^2(M, \psi)$ via left multiplication, and let $\| \cdot \|_{2,\psi}$ denote the norm on $L^2(M, \psi)$. Let Ω_ψ, J_ψ and $(\sigma_t^\psi)_{t\in\mathbb{R}}$, respectively, denote the vacuum vector, modular conjugation operator and the modular automorphism group associated with ψ. Let $A \subseteq M$ be a diffuse abelian von Neumann subalgebra contained in $M^\psi = \{x \in M : \sigma_t^\psi(x) = x \ \forall t \in \mathbb{R}\}$. Then, there exists a unique faithful, normal, ψ-preserving conditional expectation $\mathbb{E}_A$ from M onto A (see [29, Thm. IX. 4.2]). Let M_a denote the $*$-subalgebra of all analytic elements of M with respect to $(\sigma_t^\psi)_{t\in\mathbb{R}}$. For $x \in M$ and $y \in M_a$, consider the densely defined operator

$$T_{x,y} : L^2(A, \psi) \to L^2(A, \psi) \quad \text{defined by } T_{x,y}(a\Omega_\psi) = \mathbb{E}_A(xay)\Omega_\psi, \ a \in A. \tag{51}$$

Since $y \in M_a$, it follows that $y^* \in \mathfrak{D}(\sigma_z^\psi)$ for all $z \in \mathbb{C}$. Hence,

$$J_\psi \sigma_{-\frac{i}{2}}^\psi(y^*) J_\psi a\Omega_\psi = ay\Omega_\psi$$

for all $a \in A$ [14]. Note that $T_{x,y}$ admits a bounded extension to $L^2(A, \psi)$. For

$$
\begin{aligned}
\left\| \mathbb{E}_A(xay)\Omega_\psi \right\|_{2,\psi} &\le \|xay\Omega_\psi\|_{2,\psi} \\
&\le \|x\| \|ay\Omega_\psi\|_{2,\psi} \\
&\le \|x\| \left\| J_\psi \sigma_{-\frac{i}{2}}^\psi(y^*) J_\psi \right\| \|a\Omega_\psi\|_{2,\psi} \\
&= \|x\| \left\| \sigma_{-\frac{i}{2}}^\psi(y^*) \right\| \|a\Omega_\psi\|_{2,\psi} \quad \text{for all } a \in A.
\end{aligned}
$$

The bounded extension of $T_{x,y}$ to $L^2(A, \psi)$ will also be denoted by $T_{x,y}$.

Definition 4.6 ([11]). The diffuse abelian algebra $A \subseteq M$ is said to be ψ-strongly mixing in M if $\|\mathbb{E}_A(xa_ny)\|_{2,\psi} \to 0$ for all $x, y \in M$ with $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$, whenever $\{a_n\}$ is a bounded sequence in A that goes to 0 in the weak operator topology (w.o.t.).

One can identify $A = L^\infty(X, \nu)$, where X is a compact metric space and ν is a non-atomic probability measure on X such that $\varphi_{\restriction A} = \int_X \cdot d\nu$. The *left-right* measure of A is the measure (strictly speaking, the measure class) on $X \times X$ obtained from the direct integral decomposition of $L^2(M, \psi) \ominus L^2(A, \psi)$ over $X \times X$, so that $A \vee J_\psi A J_\psi$ is the algebra of diagonalizable operators with respect to the decomposition. For details, see [21].

Define $N(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ to be the normalizer of A. If A is a masa in M, then A is said to be *singular* if $N(A) = \mathcal{U}(A)$. In view of [1, Thm. 5.2], to show that A is a singular masa in M, it is enough to show that the *left-right* measure of A is Lebesgue absolutely continuous. For the sake of completeness, we state the theorem below.

Theorem 4.7 ([1, Thm. 5.2]). *Let $A \subseteq M$ be a diffuse abelian algebra such that $A \subseteq M^\psi$ and the left-right measure of A is Lebesgue absolutely continuous. Then, A is a ψ-strongly mixing masa in M. In particular, A is a singular masa in M.*

From the results of [22, §2], it follows that if A is identified with $L^\infty([a, b], \nu)$ as above, where $\lambda \ll \nu \ll \lambda$ and λ is the normalized Lebesgue measure on $[a, b]$, then the *left-right measure* of A is *Lebesgue absolutely continuous* when T_{x,y^*} is Hilbert–Schmidt for x, y varying over a set S such that $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$ for all $x, y \in S$ and the span of $S\Omega$ is dense in $L^2(M, \psi) \ominus L^2(A, \psi)$.

Note that $x \in M_T$ is analytic with respect to $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ if the function $\mathbb{R} \ni t \mapsto \sigma_t^\varphi(x) \in M_T$ extends to an M_T-valued weakly entire function. Note that this is equivalent to the extension being an entire function in the norm topology of M_T (see [28, §9.24]). Note that if $\xi \in \mathcal{H}_\mathbb{R}$ is analytic for $(U_t)_{t\in\mathbb{R}}$, then $s(\xi)$ is analytic for $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ (see [26, Rem. 2.5]).

Fix $i_0 \in N$ and a vector $\xi_0 \in \mathcal{H}_\mathbb{R}^{(i_0)}$ such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$ and $\|\xi_0\|_U = 1$. We proceed to show that the diffuse abelian subalgebra M_{ξ_0} is a φ-strongly mixing masa in M_T. Let $\mathbb{E}_{\xi_0}$ denote the unique φ-preserving, faithful, normal conditional expectation from M_T onto M_{ξ_0} as guaranteed by Theorem 4.5.

As prescribed in [1, Prop. 2.5], one can extend ξ_0 to an orthonormal basis of $\mathcal{H}_\mathbb{R}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathbb{C}}$ consisting of analytic vectors in $\mathcal{H}_\mathbb{R}^{(i)}$ for all $i \in N$. Let $\widetilde{\mathcal{O}}$ be such an extension; i.e.,

$$\widetilde{\mathcal{O}} = \left\{\eta_k^{(i)} : \eta_k^{(i)} \text{ analytic},\ k \in \Lambda_i,\ i \in N\right\},$$

where Λ_i is an index set of cardinality $\dim(\mathcal{H}_\mathbb{R}^{(i)})$ for $i \in N$. Let $\eta_{k_0}^{(i_0)} = \xi_0$. For simplicity of notation, we rename the elements of $\widetilde{\mathcal{O}}$ as

$$\widetilde{\mathcal{O}} := \{\xi_0\} \cup \left\{\eta_k : \eta_k \text{ analytic},\ k \in \bigcup_{i\neq i_0} \Lambda_i\right\} \cup \{\eta_l : l \in \Lambda_{i_0},\ \eta_l \neq \xi_0\}.$$

Fix $n \in \mathbb{N}$. If $\eta_{i_1}, \dots, \eta_{i_n} \in \widetilde{\mathcal{O}}$, then $s(\eta_{i_1} \otimes \cdots \otimes \eta_{i_n})$ lies in the $*$-algebra generated by $\{s(\eta_{i_j}) : 1 \leq j \leq n\}$ (see Lemma 4.1). Thus, $s(\eta_{i_1} \otimes \cdots \otimes \eta_{i_n})$ is analytic with respect to $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$. Further, $s(A^{-\frac{1}{2}}\eta)$ is analytic with respect to $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$ for each $\eta \in \widetilde{O}$, and this follows by arguing along the same lines of [1, Rem. 2.6]. Again, by a direct application of Lemma 4.1 (or the discussion prior to [1, Thm. 5.3]), it follows that $A^{-\frac{1}{2}}\eta_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i_n} \in M_T\Omega$ and $s(A^{-\frac{1}{2}}\eta_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i_n})$ is also analytic with respect to $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$. Therefore, $A^{-\frac{1}{2}}\eta_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i_n} \in \mathfrak{T}_{\varphi}$, where $\mathfrak{T}_{\varphi}$ is the Tomita algebra associated with φ. Consequently, $A^{-\frac{1}{2}}\eta_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i_n} \in M_T'\Omega$.

Note that from Lemma 4.2 (ii) and Lemma 4.3, it follows that if at least one letter η_{i_k} for $k \in \{1, \ldots, n\}$ is different from ξ_0, then $\mathbb{E}_{\xi_0}(s(\eta_{i_1} \otimes \cdots \otimes \eta_{i_n})) = 0$ and $\mathbb{E}_{\xi_0}(s(A^{-\frac{1}{2}}\eta_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i_n})) = 0$.

Lemma 4.8. *Let $\xi \in \mathcal{H}_{\mathbb{C}}^{(i)}, \eta \in \mathcal{H}_{\mathbb{C}}^{(j)}$, $i, j \in N$. Then, for all $n \geq 0$, the following holds:*

$$\left\| \left(l^*(\xi)r(\eta) - r(\eta)l^*(\xi)\right)_{1_{\mathcal{H}^{\otimes_T^n}}} \right\| \leq \left|\langle \xi, \eta\rangle_U\right| q^n, \quad \text{where } q = \|T\| < 1.$$

Proof. From (16) and (24), it follows that

$$\left(l^*(\xi)r(\eta) - r(\eta)l^*(\xi)\right)\Omega = \langle \xi, \eta\rangle_U.$$

Thus, the result holds for $n = 0$.

Fix $n \geq 1$. Let $\xi_{i_k} \in \mathcal{H}^{(i_k)}$ for $i_k \in N$, $1 \leq k \leq n$. Again, by (16) and (24), one has

$$\begin{aligned} &l^*(\xi)r(\eta)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \sum_{k=1}^{n} \langle \xi, \xi_{i_k}\rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} \xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n} \otimes \eta \\ &\qquad\quad + \langle \xi, \eta\rangle_U (q_{j i_n} \cdots q_{j i_1}) \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}, \qquad (52) \\ &r(\eta)l^*(\xi)(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) \\ &\quad = \sum_{k=1}^{n} \langle \xi, \xi_{i_k}\rangle_U q_{i_k i_{k-1}} \cdots q_{i_k i_1} \xi_{i_1} \otimes \cdots \otimes \xi_{i_{k-1}} \otimes \xi_{i_{k+1}} \otimes \cdots \otimes \xi_{i_n} \otimes \eta. \end{aligned}$$

Fix $j \in N$. For $i_k \in N$, with $1 \leq k \leq n$, define $V_{i_1,\ldots,i_n} : \mathcal{H}^{(i_1)} \otimes \cdots \otimes \mathcal{H}^{(i_n)} \to \mathcal{H}^{(i_1)} \otimes \cdots \otimes \mathcal{H}^{(i_n)}$ by $V_{i_1,\ldots,i_n} = (q_{j i_1} \cdots q_{j i_n})I$. Then, $\|V_{i_1,\ldots,i_n}\| \leq q^n$. Let $V : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ be the linear operator defined as follows:

$$V = \bigoplus_{(i_1,\ldots,i_n)\in N^n} V_{i_1,\ldots,i_n}.$$

Then, V is bounded and $\|V\| \leq q^n$.

By the definition of $P^{(n)}$ in (10), it is easy to verify that $VP^{(n)} = P^{(n)}V$ for all $n \in \mathbb{N}$. Therefore, by Proposition A.3, V admits a unique extension $\widetilde{V}$ to $\mathcal{H}^{\otimes_T^n}$ such that $\|\widetilde{V}\| = \|V\| \leq q^n$.

From (52), observe that

$$\langle \xi, \eta\rangle_U V = l^*(\xi)r(\eta) - r(\eta)l^*(\xi).$$

The rest is immediate. This completes the proof. ∎

We also need [32, Lem. 3] in this section. To keep the paper self-contained, we state it below.

Lemma 4.9 ([32, Lem. 3]). *Let $(H_n)_{n\geq 1}$ be a sequence of Hilbert spaces, and let $H = \bigoplus_{n\geq 1} H_n$. Let $r, s \in \mathbb{N}$, and let $(a_i)_{1\leq i\leq r}$, $(b_j)_{1\leq j\leq s}$ be two families of operators on H which send each H_n into H_{n+1} or H_{n-1} ($H_0 = 0$ by convention) such that there exists $0 < q < 1$ with*

$$\big\|(a_i b_j - b_j a_i)_{\restriction H_n}\big\| \leq q^n \quad \textit{for all } n \geq 1 \textit{ and for all } i, j.$$

For $n \geq 1$, let $K_n \subseteq H_n$ be a finite-dimensional subspace, and let $K = \bigoplus_{n\geq 1} K_n$. Suppose that

$$a_i(K) \subseteq K, \quad 1 \leq i \leq r-1,\ a_{r\restriction K} = 0.$$

Then, there exists a constant $C > 0$ independent of n such that

$$\big\|(a_r \cdots a_1 b_1 \cdots b_s)_{\restriction K_n}\big\| \leq C q^n \quad \textit{for all } n \geq 0.$$

Theorem 4.10. *Suppose there exists a unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$, $i_0 \in N$, such that*

$$U_t \xi_0 = \xi_0 \quad \textit{for all } t \in \mathbb{R}.$$

Let $x = s(\eta_{i_1} \otimes \cdots \otimes \eta_{i_m})$ and $y = s(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})$, where $\eta_{i_u}, \eta_{j_v} \in \widetilde{\mathcal{O}}$ for $1 \leq i \leq m$ and $1 \leq j \leq k$, be such that for at least one pair $\{\eta_{i_u}, \eta_{j_v}\}$, $\eta_{i_u} \neq \xi_0$ and $\eta_{j_v} \neq \xi_0$. Then, $T_{x,y}$ is a Hilbert–Schmidt operator.

Proof. The hypothesis along with Lemma 4.3 guarantees that $\mathbb{E}_{\xi_0}(x) = 0 = \mathbb{E}_{\xi_0}(y)$.

Since $U_t\xi_0 = \xi_0$ for $t \in \mathbb{R}$, from (39), it follows that the diffuse abelian subalgebra M_{ξ_0} of M_T is contained in M_T^{φ}. Since x and y are analytic with respect to $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$ (as discussed before Lemma 4.8), from (51), it follows that

$$T_{x,y} \in \mathbf{B}\big(L^2(M_{\xi_0}, \varphi)\big).$$

From Lemma 4.2 (ii), it follows that

$$\left\{\frac{1}{\|\xi_0^{\otimes n}\|_T}\,\xi_0^{\otimes n} : n \geq 0\right\}$$

forms an orthonormal basis of $L^2(M_{\xi_0}, \varphi)$. Thus, to show that $T_{x,y}$ is a Hilbert–Schmidt operator, it is enough to show that

$$\sum_{n=0}^{\infty} \big\|T_{x,y}(\xi_0^{\otimes n})\big\|_T^2 / \|\xi_0^{\otimes n}\|_T^2 < \infty.$$

Let $\mathcal{E}_{\xi_0} : L^2(M_T, \varphi) \to L^2(M_{\xi_0}, \varphi)$ denote the Jones projection associated with M_{ξ_0}. From Lemma 4.2, it follows that

$$\xi_0^{\otimes n} = H_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)\Omega \quad \text{for all } n \geq 0.$$

Therefore, from (51) and the fact that $A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k} \in M_T'\Omega$, one has

$$\begin{aligned}
&T_{x,y}\big(H_n^{(q_{i_0 i_0})}(s(\xi_0))\Omega\big) \\
&\quad = \mathcal{E}_{\xi_0}\big(xH_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)s(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})\Omega\big) \\
&\quad = \mathcal{E}_{\xi_0}\big(xH_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})\big) \\
&\quad = \mathcal{E}_{\xi_0}\big(xH_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)d(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})\Omega\big) \quad \text{(see Section 3.3)} \\
&\quad = \mathcal{E}_{\xi_0}\big(xd(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})H_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)\Omega\big) \\
&\quad = \mathcal{E}_{\xi_0}\big(xd(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})\xi_0^{\otimes n}\big) \\
&\quad = \mathcal{E}_{\xi_0}\big(s(\eta_{i_1} \otimes \cdots \otimes \eta_{i_m})d(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})\xi_0^{\otimes n}\big), \quad n \geq 0.
\end{aligned}$$

By the assumption, there exists at least one pair of vectors $\{\eta_{i_\ell}, \eta_{j_{\ell'}}\}$ such that $\eta_{i_\ell} \neq \xi_0$ and $\eta_{j_{\ell'}} \neq \xi_0$.

By the Wick product formula in Lemma 4.1,

$$s(\eta_{i_1} \otimes \cdots \otimes \eta_{i_m})d(A^{-\frac{1}{2}}\eta_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{j_k})$$

split as finite sums. A generic term in the aforesaid sum without the coefficient is of the form:

$$l(\eta_{i_1})\cdots l(\eta_{i_a})l^*(\eta_{i_{a+1}})\cdots l^*(\eta_{i_b})r(A^{-\frac{1}{2}}\eta_{j_1})\cdots r(A^{-\frac{1}{2}}\eta_{j_e})r^*(A^{-\frac{1}{2}}\eta_{j_{e+1}})\cdots r^*(A^{-\frac{1}{2}}\eta_{j_f}), \tag{53}$$

where the indices of the variables are constrained by Lemma 4.1 and with a pair of vectors $\{\eta_{i_u}, \eta_{j_v}\}$ such that $\eta_{i_u} \neq \xi_0$ and $\eta_{j_v} \neq \xi_0$.

Then, by Lemma 4.1, it suffices to show that for a generic term in the Wick product expansion as in (53), if

$$\begin{aligned}
\zeta_n = \mathcal{E}_{\xi_0}(l(\eta_{i_1})\cdots l(\eta_{i_a})l^*(\eta_{i_{a+1}})\cdots l^*(\eta_{i_b})r(A^{-\frac{1}{2}}\eta_{j_1})\cdots \\
\cdots r(A^{-\frac{1}{2}}\eta_{j_e})r^*(A^{-\frac{1}{2}}\eta_{j_{e+1}})\cdots r^*(A^{-\frac{1}{2}}\eta_{j_f})\xi_0^{\otimes n}), \quad n \geq 0,
\end{aligned}$$

then $\sum_{n=0}^{\infty} \|\zeta_n\|_T^2 / \|\xi_0^{\otimes n}\|_T^2 < \infty$.

We will use only the fact that one of $\{\eta_{i_1}, \ldots, \eta_{i_b}\}$ is different from ξ_0. Let $m_0 = \max\{w : 1 \leq w \leq b,\ \eta_{i_w} \neq \xi_0\}$. Note that if $m_0 < a+1$, then ζ_n is 0 from (16) and (26). Therefore, we will consider only the case $m_0 \geq a+1$.

In this step, we apply Lemma 4.9 to the following two sets of operators:

$$\begin{aligned}
A &= \big\{a_h = l^*(\eta_{i_{b-h+m_0}}),\ m_0 \leq h \leq b\big\}, \\
B &= \big\{r(A^{-\frac{1}{2}}\eta_{j_1}), \ldots, r(A^{-\frac{1}{2}}\eta_{j_e}), r^*(A^{-\frac{1}{2}}\eta_{j_{e+1}}), \ldots, r^*(A^{-\frac{1}{2}}\eta_{j_f})\big\}.
\end{aligned}$$

The finite-dimensional subspace K_n in Lemma 4.9 is replaced by $\mathbb{C}\xi_0^{\otimes n}$ for $n \geq 0$. Thus, $K = \bigoplus_{n\geq 0} K_n = L^2(M_{\xi_0}, \varphi)$.

Note that, by the choice of m_0, it follows that

$$l^*(\eta_{i_{m_0}})_{\upharpoonright L^2(M_{\xi_0},\varphi)} = 0.$$

Also, we have $l^*(\eta_{i_h})(L^2(M_{\xi_0},\varphi)) \subseteq L^2(M_{\xi_0},\varphi)$, for $m_0 < h \leq b$.

From Lemma 4.8, note that for all $m_0 \leq h \leq b$ and $1 \leq p \leq e$, one has

$$\left\| \left(l^*(\eta_{i_h}) r(A^{-\frac{1}{2}}\eta_{j_p}) - r(A^{-\frac{1}{2}}\eta_{j_p}) l^*(\eta_{i_h}) \right) \right\|_{\upharpoonright \mathcal{H}^{\otimes_T^n}} \leq Kq^n,$$

where

$$K = \max_{\substack{m_0 \leq h \leq b,\\ 1 \leq p \leq e}} \left| \langle \eta_{i_h}, A^{-\frac{1}{2}}\eta_{j_p} \rangle_U \right|.$$

From (16) and (26) again, it follows that

$$l^*(\eta_{i_h}) r^*(A^{-\frac{1}{2}}\eta_{j_{p'}}) - r^*(A^{-\frac{1}{2}}\eta_{j_{p'}}) l^*(\eta_{i_h}) = 0 \text{ for all } m_0 \leq h \leq b \text{ and } e+1 \leq p' \leq f.$$

Now applying Lemma 4.9 to

$$l^*(\eta_{i_{m_0}}) \cdots l^*(\eta_b) r(A^{-\frac{1}{2}}\eta_{j_1}) \cdots r(A^{-\frac{1}{2}}\eta_{j_e}) r^*(A^{-\frac{1}{2}}\eta_{j_{e+1}}) \cdots r^*(A^{-\frac{1}{2}}\eta_{j_f}),$$

it follows that there exists a constant $C > 0$ independent of n such that

$$\|\zeta_n\|_T \leq Cq^n \|\xi_0^{\otimes n}\|_T \quad \text{for all } n \geq 0.$$

Therefore,

$$\sum_{n=0}^{\infty} \|\zeta_n\|_T^2 / \|\xi_0^{\otimes n}\|_T^2 < \infty,$$

as desired. This completes the proof. ∎

The next result is an adaptation of [1, Thm. 5.4] to the current situation.

Theorem 4.11. *Let* $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$*, and suppose there exists a unit vector* $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$*,* $i_0 \in N$*, such that* $U_t\xi_0 = \xi_0$ *for all* $t \in \mathbb{R}$*. Then,* M_{ξ_0} *is a* φ*-strongly mixing masa in* M_T *whose left-right measure is Lebesgue absolutely continuous.*

Proof. The proof is similar to [1, Thm. 5.4]. The only change required is the replacement of the usage of [1, Thm. 5.3] by Theorem 4.10. ∎

We conclude this section by jotting the main results obtained so far.

Corollary 4.12. *Let* $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$*, and let* $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$*,* $i_0 \in N$*, be a unit vector. Then, the following are equivalent:*

(1) $s(\xi_0) \in M_T^{\varphi}$;

(2) $U_t\xi_0 = \xi_0$ *for all* $t \in \mathbb{R}$;

(3) *there exists a faithful normal conditional expectation* $\mathbb{E}_{\xi_0} : M_T \to M_{\xi_0}$ *such that* $\varphi(\mathbb{E}_{\xi_0}(x)) = \varphi(x)$ *for all* $x \in M_T$;

(4) M_{ξ_0} *is a* φ*-strongly mixing masa in* M_T.

5. Factoriality and type classification

In this section, we discuss the factoriality and type classification of M_T under different constraints.

First, we show that if there exists $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$, $i_0 \in N$, such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, then M_T is a factor. Its proof is a generalization of [1, Thm. 6.3].

Theorem 5.1. *Let* $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. *Suppose there exists* $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$, $i_0 \in N$, *such that*

$$U_t\xi_0 = \xi_0 \quad \textit{for all } t \in \mathbb{R}.$$

Then, M_T *is a factor.*

Proof. Let $x \in \mathcal{Z}(M_T)$. We show that x is a scalar multiple of identity. From Theorem 4.11, it follows that M_{ξ_0} is a diffuse masa in M_T. Hence, $\mathcal{Z}(M_T) \subseteq M_{\xi_0}$, and thus by Lemma 4.2, one has

$$x\Omega = \sum_{n=0}^{\infty} b_n \xi_0^{\otimes n},$$

where the summation converges in $\|\cdot\|_T$.

Since $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, there exist $\eta \in \mathcal{H}_{\mathbb{R}}$ such that $\langle \xi_0, \eta \rangle_{\mathcal{H}_{\mathbb{C}}} = 0$. Since $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, one has $\langle \xi_0, \eta \rangle_U = 0$. From (15) and (16) and by the continuity of $l(\eta)$, it follows that

$$s(\eta)x\Omega = \sum_{n=0}^{\infty} b_n (\eta \otimes \xi_0^{\otimes n}).$$

We claim that $xs(\eta)\Omega \in \overline{\operatorname{span}\{\xi_0^{\otimes n} \otimes \eta : n \geq 0\}}^{\|\cdot\|_T}$ as well. Note that x is a limit in s.o.t. of a sequence of operators from the linear span of $\{H_n^{q_{i_0 i_0}}(s(\xi_0)) : n \geq 0\}$. Proceeding along the same lines of (i) of Lemma 4.2, it follows that

$$H_n^{q_{i_0 i_0}}\big(s(\xi_0)\big)s(\eta)\Omega = \xi_0^{\otimes n} \otimes \eta \quad \text{for all } n \geq 1.$$

Therefore, the claim follows.

Since $xs(\eta) = s(\eta)x$, it follows that $b_n = 0$ for all $n \geq 1$. Therefore, $x\Omega = b_0\Omega$. Since Ω is separating for M_T, it follows that $x = b_0 1$. Hence, M_T is a factor. ∎

Theorem 5.2. *Suppose that the invariant subspace of weakly mixing vectors in* $\mathcal{H}_{\mathbb{R}}$ *is non-trivial. Then,* M_T *is a factor.*

Proof. The proof follows exactly along the same lines of [1, Thm. 6.2]. ∎

Now, we deal with the ergodic and almost periodic component of the representation $t \mapsto U_t, t \in \mathbb{R}$.

First, note that there is a unique decomposition of $\mathcal{H}_{\mathbb{R}}$ as follows [26]:

$$(\mathcal{H}_{\mathbb{R}}, U_t) := \left(\bigoplus_{h=1}^{N_1} (\mathbb{R}_h, \mathrm{id})\right) \oplus \left(\bigoplus_{k=1}^{N_2} \big(\mathcal{H}_{\mathbb{R}}(k), U_t(k)\big)\right) \oplus (\widetilde{\mathcal{H}_{\mathbb{R}}}, U_t), \quad t \in \mathbb{R}, \quad (54)$$

where $0 \leq N_1, N_2 \leq \aleph_0$,

$$\mathbb{R}_h = \mathbb{R}, \quad \mathcal{H}_{\mathbb{R}}(k) = \mathbb{R}^2, \quad U_t(k) = \begin{pmatrix} \cos(t \log \lambda_k) & -\sin(t \log \lambda_k) \\ \sin(t \log \lambda_k) & \cos(t \log \lambda_k) \end{pmatrix}, \quad \lambda_k > 1,$$

where $(\widetilde{\mathcal{H}_{\mathbb{R}}}, U_t)$ denotes the weakly mixing component of $(U_t)_{t\in\mathbb{R}}$.

In accordance with (5) concerned with the decomposition of $\mathcal{H}_{\mathbb{R}}$ into invariant subspaces of $(U_t)_{t\in\mathbb{R}}$, we will treat each component of the direct sum in (54) as a single block. Then, in this case, N will be regarded as a subset of $\mathbb{N}$ of cardinality at most $N_1 + N_2 + 1$.

If $N_1 \neq 0$, let $e_h^0 = 0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \cdots \oplus 0 \in \bigoplus_{h=1}^{N_1} \mathbb{R}_h$, where 1 appears at the h-th place for $1 \leq h \leq N_1$ (or $1 \leq h < N_1$, if $N_1 = \aleph_0$). Also, if $N_2 \neq 0$, let

$$f_k^1 = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{k=1}^{N_2} \mathcal{H}_{\mathbb{R}}(k),$$
$$f_k^2 = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{k=1}^{N_2} \mathcal{H}_{\mathbb{R}}(k)$$

be vectors with non-zero entries in the k-th position for $1 \leq k \leq N_2$ (or $1 \leq k < N_2$, if $N_2 = \aleph_0$).

Fix k. Denote

$$e_k^1 = \frac{\sqrt{\lambda_k + 1}}{2}(f_k^1 + i f_k^2) \quad \text{and} \quad e_k^2 = \frac{\sqrt{\lambda_k^{-1} + 1}}{2}(f_k^1 - i f_k^2).$$

Then, e_k^1 and e_k^2 are orthonormal vectors in $\mathcal{H}_{\mathbb{R}}(k) + i\,\mathcal{H}_{\mathbb{R}}(k)$ with respect to $\langle \cdot, \cdot \rangle_U$. The analytic generator $A(k)$ of $(U_t(k))_{t\in\mathbb{R}}$ is given by

$$A(k) = \begin{pmatrix} \lambda_k + \frac{1}{\lambda_k} & i(\lambda_k - \frac{1}{\lambda_k}) \\ -i(\lambda_k - \frac{1}{\lambda_k}) & \lambda_k + \frac{1}{\lambda_k} \end{pmatrix}.$$

Also, we have

$$A(k)e_k^1 = \frac{1}{\lambda_k} e_k^1 \quad \text{and} \quad A(k)e_k^2 = \lambda_k e_k^2. \tag{55}$$

When the almost periodic component of $(\mathcal{H}_{\mathbb{R}}, U_t)$ is infinite-dimensional and $q_{ij} = q$ for all $i, j \in N$, Hiai established the factoriality of M_T [16, Thm. 3.2]. Unfortunately, there is a gap in the proof, and the proof is valid only in the case when $\sigma(A)$ has a limit point in $\mathbb{R}$ away from 0 (assuming the vacuum state is almost periodic) [15].

Remark 5.3. It is proved in [23, Thm. 4.5] that for finite-dimensional $\mathcal{H}_{\mathbb{R}}$, the q-deformed Araki–Woods von Neumann algebras are isomorphic to free Araki–Woods factors for sufficiently small $|q|$; in particular, they are factors in this case. However, for arbitrary $q \in (-1, 1)$, the factoriality of the q-deformed Araki–Woods von Neumann algebras remained open in the following cases:

(1) $\dim(\mathcal{H}_{\mathbb{R}})$ is even and $(U_t)_{t\in\mathbb{R}}$ is ergodic;

(2) $(U_t)_{t\in\mathbb{R}}$ is ergodic, almost periodic, and 0 is the only limit point of the set of eigenvalues of the analytic generator of $(U_t)_{t\in\mathbb{R}}$.

In a very recent work [3], the factoriality of q-deformed Araki–Woods von Neumann algebras has been further investigated. It is proved in [3] that when $\mathcal{H}_{\mathbb{R}}$ has a two-dimensional ergodic sub-representation, the q-deformed Araki–Woods von Neumann algebras are factors in the following cases:

(1) $\dim(\mathcal{H}_{\mathbb{R}})=2$ and the parameter $\lambda\in(0,1)$ defining the aforesaid sub-representation is small;

(2) $\dim(\mathcal{H}_{\mathbb{R}})\geq 3$.

The next theorem discusses the factoriality of M_T under Hiai's framework by imposing the hypothesis necessary for Hiai's theorem to successfully pass through. Its proof is by adapting the techniques of Hiai to the current setup. Thus, this is essentially Hiai's proof and we do not claim originality of it.

If $N_2 \neq 0$, then there exists an injection Φ from a set of cardinality N_2 into N. For $1 \leq k \leq N_2$ (or $1 \leq k < N_2$, as the case may be), we will denote $\Phi(k)$ by k to reduce notation. This abuse of notation will cause no confusion.

Again, if $\zeta_l \in \mathcal{H}^{(l)}$ for $l \in N$, then the action of T on vectors of the form $(f_k^i \otimes \zeta_l) \in \mathcal{H}_{\mathbb{R}} \odot \mathcal{H}_{\mathbb{R}}$ will be given by

$$T(f_k^i \otimes \zeta_l) = q_{kl}(\zeta_l \otimes f_k^i), \quad i = 1,2.$$

Therefore, by linearity, one has $T(e_k^i \otimes \zeta_l) = q_{kl}(\zeta_l \otimes e_k^i)$, $i = 1,2$, and for all k.

Theorem 5.4. *Assume that $N_1 = 0$, the almost periodic component of $(\mathcal{H}_{\mathbb{R}}, U_t)$ is infinite-dimensional and the set of eigenvalues of the analytic generator A has a limit point other than 0 in $\mathbb{R}$. Then, $(M_T^{\varphi})' \cap M_T = \mathbb{C}1$. In particular, M_T is a factor.*

Proof. Note that $N_2 = \aleph_0$. Then, for $1 \leq k < N_2$, let $z_k := \frac{1}{2}(s(f_k^1) + is(f_k^2)) \in M_T$. By (19), it is easy to verify that

$$z_k = \frac{1}{\sqrt{\lambda_k + 1}} l(e_k^1) + \frac{1}{\sqrt{\lambda_k^{-1} + 1}} l^*(e_k^2). \tag{56}$$

Also, from (15) and (39), one has

$$\begin{aligned}
\sigma_t^{\varphi}(z_k) &= \mathcal{F}_T(U_{-t})\left(\frac{1}{\sqrt{\lambda_k + 1}} l(e_k^1) + \frac{1}{\sqrt{\lambda_k^{-1} + 1}} l^*(e_k^2)\right)\mathcal{F}_T(U_{-t})^* \\
&= \frac{1}{\sqrt{\lambda_k + 1}} l(U_{-t}e_k^1) + \frac{1}{\sqrt{\lambda_k^{-1} + 1}} l^*(U_{-t}e_k^2) \\
&= \frac{1}{\sqrt{\lambda_k + 1}} l(\lambda_k^{it}e_k^1) + \frac{1}{\sqrt{\lambda_k^{-1} + 1}} l^*(\lambda_k^{-it}e_k^2) \quad \big(\text{by (55)}\big) \\
&= \lambda_k^{it} z_k.
\end{aligned}$$

Consequently, $\sigma_t^{\varphi}(z_k^* z_k) = z_k^* z_k$, and hence, one has $y_k := \sqrt{1+\lambda_k}\, z_k^* z_k \in M_T^{\varphi}$.

Let $L := \{f_k^1, f_k^2\}_k \cup \widetilde{\mathcal{H}_{\mathbb{R}}}$. Fix $n \in \mathbb{N}$. Let $\zeta_1, \ldots, \zeta_n \in L$ be such that $\zeta_i \in \mathcal{H}_{\mathbb{R}}^{(m_i)}$ for $1 \leq i \leq n$. Let $x := s(\zeta_1 \otimes \cdots \otimes \zeta_n) \in M_T$ (see Lemma 4.1). Note that

$$|q_{km_1} q_{km_2} \cdots q_{km_n}| < q^n,$$

where $q = \sup_{i,j \in N} |q_{ij}|$ for all k. Using the hypothesis, replacing λ_k by $\frac{1}{\lambda_k}$ if necessary and further dropping to a subsequence if required, one can assume that

$$\begin{aligned}
&0 < \lambda_k \leq 1,\\
&\lambda_k \to \lambda \in (0, 1],\\
&q_{km_1} q_{km_2} \cdots q_{km_n} \to \tilde{q} \in [-q^n, q^n], \quad \text{as } k \to \infty.
\end{aligned}$$

First, we show that

$$\lim_{\mathcal{N} \to \infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} y_k x y_k = \frac{1 + \lambda \tilde{q}^2}{1 + \lambda} x, \quad \text{in w.o.t.}$$

It is enough to show that for $\xi, \eta \in \mathcal{F}_T(\mathcal{H})$,

$$\lim_{\mathcal{N} \to \infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \langle \xi, y_k x y_k \eta \rangle_T = \frac{1 + \lambda \tilde{q}^2}{1 + \lambda} \langle \xi, x \eta \rangle_T. \tag{57}$$

Fix $\xi = f_1 \otimes \cdots \otimes f_u$ and $\eta = g_1 \otimes \cdots \otimes g_v$ such that $f_i \in L$ for $1 \leq i \leq u$ and $g_j \in L$ for $1 \leq j \leq v$. By Proposition 2.2 and Lemma 4.1, to prove (57), it suffices to prove that if

$$\begin{aligned}
&l, r \geqslant 0,\ l + r = n,\ i(1) < \cdots < i(l),\ j(1) < \cdots < j(r),\\
&\{i(1), \ldots, i(l)\} \cup \{j(1), \ldots, j(r)\} = \{1, \ldots, n\},\\
&x_1 = l(\zeta_{i(1)}) \cdots l(\zeta_{i(l)}),\\
&x_2 = l^*(\zeta_{j(1)}) \cdots l^*(\zeta_{j(r)}),
\end{aligned}$$

then

$$\lim_{\mathcal{N} \to \infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \langle \xi, y_k x_1 x_2 y_k \eta \rangle_T = \frac{1 + \lambda \tilde{q}^2}{1 + \lambda} \langle \xi, x_1 x_2 \eta \rangle_T. \tag{58}$$

Consider the set $B = \{\zeta_1, \ldots, \zeta_n, f_1, \ldots, f_u, g_1, \ldots, g_v\}$, and let $\mathcal{N}_0 := \max\{k : f_k^1 \in B \text{ or } f_k^2 \in B\}$. We show that for $k > \mathcal{N}_0$,

$$\langle \xi, y_k x_1 x_2 y_k \eta \rangle_T = \frac{1 + \lambda_k (q_{km_1} q_{km_2} \cdots q_{km_n})^2}{1 + \lambda_k} \langle \xi, x_1 x_2 \eta \rangle_T. \tag{59}$$

Note that by the commutation relation in Lemma 2.3, one has

$$\begin{aligned}
&l^*(e_k^1) l(e_k^1) - q_{kk} l(e_k^1) l^*(e_k^1) = 1,\\
&l^*(e_k^2) l(e_k^2) - q_{kk} l(e_k^2) l^*(e_k^2) = 1,\\
&l^*(e_k^1) l(e_k^2) = q_{kk} l(e_k^2) l^*(e_k^1),\\
&l^*(e_k^2) l(e_k^1) = q_{kk} l(e_k^2) l^*(e_k^1) \quad \text{for all } k.
\end{aligned} \tag{60}$$

Therefore,

$$\begin{aligned}\sqrt{\lambda_k+1}\,y_k &= (\lambda_k+1)z_k^* z_k\\ &= l^*(e_k^1)l(e_k^1) + \sqrt{\lambda_k}\,l^*(e_k^1)l^*(e_k^2)\\ &\quad + \sqrt{\lambda_k}\,l(e_k^2)l(e_k^1) + \lambda_k l(e_k^2)l^*(e_k^2) \quad \big(\text{by } (56)\big)\\ &= 1 + q_{kk} l(e_k^1)l^*(e_k^1) + \sqrt{\lambda_k}\,l^*(e_k^1)l^*(e_k^2)\\ &\quad + \sqrt{\lambda_k}\,l(e_k^2)l(e_k^1) + \lambda_k l(e_k^2)l^*(e_k^2) \quad \big(\text{by } (60)\big).\end{aligned} \tag{61}$$

Note that for $k > \mathcal{N}_0$, $\langle e_k^1, g\rangle_U = \langle e_k^2, g\rangle_U = 0$ for all $g \in B$. Thus, for $k > \mathcal{N}_0$ and $j = 1, 2$, one has

$$\begin{aligned} l^*(e_k^j)\xi &= l^*(e_k^j)\eta = 0,\\ l^*(e_k^j)x_1^*\xi &= l^*(e_k^j)x_2\eta = 0. \end{aligned} \tag{62}$$

Therefore, for $k > \mathcal{N}_0$, one has

$$\begin{aligned}&(\lambda_k+1)\langle \xi, y_k x_1 x_2 y_k \eta\rangle_T\\ &= \big\langle \xi, \big(1+\sqrt{\lambda_k}\,l^*(e_k^1)l^*(e_k^2)\big)x_1x_2\big(1+\sqrt{\lambda_k}\,l(e_k^2)l(e_k^1)\big)\eta\big\rangle_T \quad \big(\text{by } (61), (62)\big)\\ &= \langle \xi, x_1x_2\eta\rangle_T + \big\langle \xi, \sqrt{\lambda_k}x_1x_2 l(e_k^2)l(e_k^1)\eta\big\rangle_T + \big\langle \xi, \sqrt{\lambda_k}\,l^*(e_k^1)l^*(e_k^2)x_1x_2\eta\big\rangle_T\\ &\quad + \lambda_k\big\langle \xi, l^*(e_k^1)l^*(e_k^2)x_1x_2 l(e_k^2)l(e_k^1)\eta\big\rangle_T\\ &= \langle \xi, x_1x_2\eta\rangle_T + \sqrt{\lambda_k}(q_{km_{j(1)}}\cdots q_{km_{j(r)}})^2\big\langle \xi, x_1 l(e_k^2)l(e_k^1)x_2\eta\big\rangle_T\\ &\quad + \sqrt{\lambda_k}(q_{km_{i(1)}}\cdots q_{km_{i(l)}})^2\big\langle \xi, x_1 l^*(e_k^1)l^*(e_k^2)x_2\eta\big\rangle_T\\ &\quad + \lambda_k(q_{km_1}q_{km_2}\cdots q_{km_n})^2\big\langle \xi, x_1 l^*(e_k^1)l^*(e_k^2)l(e_k^2)l(e_k^1)x_2\eta\big\rangle_T \quad (\text{by Lemma } 2.3)\\ &= \langle \xi, x_1x_2\eta\rangle_T + \lambda_k(q_{km_1}q_{km_2}\cdots q_{km_n})^2\big\langle \xi, x_1 l^*(e_k^1)l^*(e_k^2)l(e_k^2)l(e_k^1)x_2\eta\big\rangle_T \quad \big(\text{by } (62)\big).\end{aligned} \tag{63}$$

By (60), one has

$$\begin{aligned} l^*(e_k^1)l^*(e_k^2)l(e_k^2)l(e_k^1) &= 1 + q_{kk} l(e_k^1)l^*(e_k^1) + (q_{kk})^3 l(e_k^2)l^*(e_k^2)\\ &\quad + (q_{kk})^4 l(e_k^2)l(e_k^1)l^*(e_k^1)l^*(e_k^2). \end{aligned} \tag{64}$$

Hence, from (62), (63) and (64), it follows that

$$(\lambda_k+1)\langle \xi, y_k x_1 x_2 y_k \eta\rangle_T = \big(1+\lambda_k(q_{km_1}q_{km_2}\cdots q_{km_n})^2\big)\langle \xi, x_1x_2\eta\rangle_T.$$

This establishes (59).

Since

$$\lim_{k\to\infty}\frac{1+\lambda_k(q_{km_1}q_{km_2}\cdots q_{km_n})^2}{1+\lambda_k} = \frac{1+\lambda\tilde{q}^2}{1+\lambda},$$

Equation (58) is established by taking a Cesàro sum.

Working exactly as above and replacing $\mathcal{N}_0$ by

$$\widetilde{\mathcal{N}}_0 := \max\left\{k : f_k^1 \text{ or } f_k^2 \in \{f_1, \ldots, f_u, g_1, \ldots, g_v\}\right\},$$

it follows that $\langle \xi, y_k^2 \eta\rangle_T = \langle \xi, \eta\rangle_T$ for all $k > \widetilde{\mathcal{N}}_0$. Therefore, one has

$$\lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} y_k^2 = 1, \quad \text{in w.o.t.} \tag{65}$$

Next, we show that $\{y_k : k \geq 1\}' \cap M_T = \mathbb{C}1$. Let $y \in \{y_k : k \geq 1\}' \cap M_T$ and $x = s(\zeta_1 \otimes \cdots \otimes \zeta_n)$, for $\zeta_1, \ldots, \zeta_n \in L$ (as before).

Consequently,

$$\begin{aligned}
\frac{1+\lambda\tilde{q}^2}{1+\lambda}\varphi(y^*x) &= \frac{1+\lambda\tilde{q}^2}{1+\lambda}\langle y\Omega, x\Omega\rangle_T \\
&= \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \langle y\Omega, y_k x y_k \Omega\rangle_T \quad \big(\text{by (57)}\big) \\
&= \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \varphi(y^* y_k x y_k) \\
&= \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \varphi(y_k y^* x y_k) \\
&= \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} \varphi(y^* x y_k^2) \quad (\text{since } y_k \in M_T^{\varphi}) \\
&= \varphi(y^*x) \quad \big(\text{by (65)}\big).
\end{aligned}$$

This implies that $\varphi(y^*x) = 0$ since otherwise it forces that $1 + \lambda\tilde{q}^2 = 1 + \lambda$, which is impossible since $\lambda \neq 0$ and $\tilde{q} \in [-q^n, q^n]$, $0 \leq q < 1$. Therefore, one has $0 = \varphi(y^*x) = \langle y\Omega, x\Omega\rangle_T = \langle y\Omega, \zeta_1 \otimes \cdots \otimes \zeta_n\rangle_T$. Since $\mathrm{span}_{\mathbb{C}}\{\zeta_1 \otimes \cdots \otimes \zeta_n : \zeta_1, \ldots, \zeta_n \in L, n \geq 1\}$ is dense in $\mathcal{F}_T(\mathcal{H}) \ominus \mathbb{C}\Omega$, one has $y\Omega \in \mathbb{C}\Omega$. Since Ω is separating for M_T (Proposition 3.5), it follows that $y \in \mathbb{C}1$. This completes the proof. ∎

Remark 5.5. (i) Note that if 0 is the only limit point in $\mathbb{R}$ of the set of eigenvalues of A, then $\lambda = 0$ in the proof of Theorem 5.4. Thus, the equation $1 + \lambda\tilde{q}^2 = 1 + \lambda$ would be a tautology and Hiai's argument would be inconclusive.

(ii) Suppose that the hypothesis of Theorem 5.4 is true, and let $\mathcal{H}_{\mathbb{R}} := \bigoplus_{i\in N} \mathcal{H}_{\mathbb{R}}^{(i)}$ be an arbitrary but fixed decomposition of $\mathcal{H}_{\mathbb{R}}$ as in (5). Since the decomposition of $\mathcal{H}_{\mathbb{R}}$ in (54) is a refinement of the given decomposition, proceeding along the same lines of the proof of Theorem 5.4, one concludes that M_T is a factor as well, where T is the Yang–Baxter operator associated with the given decomposition. The only minor change in the proof will be the replacement of the sequence $\{q_{km_1} q_{km_2} \cdots q_{km_n}\}_k$ with a different sequence of parameters depending on the given decomposition of $\mathcal{H}_{\mathbb{R}}$. Thus, we have the following.

Theorem 5.6. *Assume that $N_1 = 0$, the almost periodic component of $(\mathcal{H}_{\mathbb{R}}, U_t)$ is infinite-dimensional and the set of eigenvalues of the analytic generator A has a limit point other than 0 in $\mathbb{R}$. Let $\mathcal{H}_{\mathbb{R}} := \bigoplus_{i \in N} \mathcal{H}_{\mathbb{R}}^{(i)}$ be an arbitrary but fixed decomposition of $\mathcal{H}_{\mathbb{R}}$ consisting of invariant subspaces of $(U_t)_{t \in \mathbb{R}}$. Let T be the Yang–Baxter operator corresponding to the decomposition and the parameters $-1 < q_{ij} = q_{ji} < 1$ for $i, j \in N$, with $\sup_{i,j \in N} |q_{ij}| < 1$. Then, $(M_T^{\varphi})' \cap M_T = \mathbb{C}1$. In particular, M_T is a factor.*

(iii) Note that, with the results obtained in this section, the factoriality of M_T remains open only for the following cases.

(1) $\dim(\mathcal{H}_{\mathbb{R}})$ is even and $(U_t)_{t \in \mathbb{R}}$ is ergodic;

(2) $(U_t)_{t \in \mathbb{R}}$ is ergodic, almost periodic, and 0 is the only limit point in $\mathbb{R}$ of the set of eigenvalues of the analytic generator of $(U_t)_{t \in \mathbb{R}}$.

In the remaining part of this section, we discuss the type classification of M_T.

The S invariant of a factor M is defined as the intersection over all faithful normal semifinite weights ϕ of the spectra of the associated modular operator Δ_ϕ. Further, if ϕ is a fixed faithful normal state on M, the S invariant can be written as

$$S(M) = \bigcap \big\{ \mathrm{Sp}(\Delta_{\phi_p}) : 0 \neq p \in \mathcal{P}\big(\mathcal{Z}(M^{\phi})\big) \big\},$$

where $\mathcal{P}(\mathcal{Z}(M^{\phi}))$ denotes the lattice of projections in the center of the centralizer M^{ϕ} and $\phi_p = \phi_{\restriction pMp}$ [13].

Connes classified type III factors using the S invariant as follows:

$$S(M) = \begin{cases} [0, \infty) & \text{if } M \text{ is type III}_1, \\ \{0, 1\} & \text{if } M \text{is type III}_0, \\ \{\lambda^n : n \in \mathbb{Z}\} \cup \{0\} & \text{if } M \text{ is type III}_\lambda,\ 0 < \lambda < 1. \end{cases}$$

Theorem 5.7. *Assume that A has infinitely many mutually orthogonal eigenvectors and the set of eigenvalues of A has a limit point in $\mathbb{R}$ other than 0. Let G be the closed subgroup of $\mathbb{R}_+^{\times}$ generated by the spectrum of A. Then,*

$$M_T \textit{ is of } \begin{cases} \textit{type } \mathrm{III}_1 & \textit{if } G = \mathbb{R}_+^{\times}, \\ \textit{type } \mathrm{III}_\lambda & \textit{if } G = \{\lambda^n : n \in \mathbb{Z}\},\ 0 < \lambda < 1, \\ \textit{type } \mathrm{II}_1 & \textit{if } G = \{1\}. \end{cases}$$

Proof. From Theorem 5.4, it follows that under the hypothesis of the theorem, M_T^{φ} is a factor. Hence, $S(M_T)$ is completely determined by $\mathrm{Sp}(\Delta)$. Therefore, the proof follows from Theorem 3.13 (iii) and [1, Prop. 3.3]. ∎

Theorem 5.8. *Suppose* $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, *the orthogonal representation* $\mathbb{R} \ni t \mapsto U_t$ *is almost periodic, and there exists* $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i_0)}$, $i_0 \in N$, *such that* $U_t \xi_0 = \xi_0$ *for all* $t \in \mathbb{R}$. *Let* G *be the closed subgroup of* $\mathbb{R}_+^{\times}$ *generated by the spectrum of* A. *Then,* $(M_T^{\varphi})' \cap M_T = \mathbb{C}1$, *and*

$$M_T \text{ is of } \begin{cases} \text{type III}_1 & \text{if } G = \mathbb{R}_+^{\times}, \\ \text{type III}_\lambda & \text{if } G = \{\lambda^n : n \in \mathbb{Z}\},\ 0 < \lambda < 1, \\ \text{type II}_1 & \text{if } G = \{1\}. \end{cases}$$

Proof. The hypothesis forces that if $\dim(\mathcal{H}_{\mathbb{R}}) = 2$, then $(U_t)_{t \in \mathbb{R}}$ is trivial. Thus, M_T coincides with Bożejko–Speicher's II_1 factor, and hence, there is nothing to prove.

Assume that $\dim(\mathcal{H}_{\mathbb{R}}) \geq 3$. First, we show that $(M_T^{\varphi})' \cap M_T = \mathbb{C}1$. Let $x \in (M_T^{\varphi})' \cap M_T$. From Corollary 4.12, note that $M_{\xi_0} = vN(s(\xi_0)) \subseteq M_T^{\varphi}$ is a masa in M_T. Therefore, $(M_T^{\varphi})' \cap M_T \subseteq M_{\xi}$. Choose $\zeta_{j_1}, \dots, \zeta_{j_m} \in \mathcal{H}_{\mathbb{C}}$ with real and imaginary part of ζ_{j_i} being orthogonal to ξ_0 such that $A\zeta_{j_i} = \beta_{j_i}\zeta_{j_i}$ for $1 \leq i \leq m$ and $\prod_{i=1}^m \beta_{j_i} = 1$. Let $y := s(\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m}) \in M_T$ (see Lemma 4.1). Since $\sigma_{-t}^{\varphi} = \mathrm{Ad}(\mathcal{F}_T(U_t))$ for all $t \in \mathbb{R}$ (see (39)), it follows that

$$\begin{aligned} \sigma_{-t}^{\varphi}(y)\Omega &= \mathcal{F}_T(U_t) y \mathcal{F}_T(U_t)^* \Omega \\ &= \mathcal{F}_T(U_t) y \Omega \\ &= \mathcal{F}_T(U_t)(\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m}) \\ &= U_t\zeta_{j_1} \otimes \cdots \otimes U_t\zeta_{j_m} \\ &= (\beta_{j_1} \cdots \beta_{j_m})^{it}(\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m}) \quad (\text{since } U_t = A^{it}) \\ &= s(\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m})\Omega \\ &= y\Omega \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Since Ω is separating for M_T, one has $y \in M_T^{\varphi}$.

From Lemma 4.2, one has

$$x\Omega = \sum_{n=0}^{\infty} a_n \xi_0^{\otimes n} = \sum_{n=0}^{\infty} a_n H_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)\Omega, \quad a_n \in \mathbb{C},$$

where the series converges in $\|\cdot\|_T$. From Lemma 4.4, it follows that

$$yx\Omega \in \overline{\mathrm{span}\{\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m} \otimes \xi_0^{\otimes n} : n \geq 0\}}^{\|\cdot\|_T}.$$

Since $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$ and $\langle \xi_0, \zeta_{j_i} \rangle_{\mathcal{H}_{\mathbb{C}}} = 0$ for $1 \leq i \leq m$, one has

$$\langle \xi_0, \zeta_{j_i} \rangle_U = 0 \quad \text{for } 1 \leq i \leq m.$$

Therefore, from (15) and (16), it follows that

$$s(\xi_0)(\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m}) = \xi_0 \otimes \xi_{j_1} \otimes \cdots \otimes \xi_{j_m}.$$

Also, proceeding along the same lines of Lemma 4.2, it follows that

$$H_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)(\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m}) = \xi_0^{\otimes n} \otimes \zeta_{j_1} \otimes \cdots \otimes \zeta_{j_m} \quad \text{for all } n \geq 0.$$

Therefore,

$$\begin{aligned} xy\Omega = Jy^*Jx\Omega &= \sum_{n=0}^{\infty} a_n Jy^*JH_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)\Omega \\ &= \sum_{n=0}^{\infty} a_n H_n^{(q_{i_0 i_0})}\big(s(\xi_0)\big)y\Omega \\ &= \sum_{n=0}^{\infty} a_n(\xi_0^{\otimes n} \otimes \xi_{j_1} \otimes \cdots \otimes \xi_{j_m}). \end{aligned}$$

Now, since $xy = yx$, one has $a_n = 0$ for all $n \geq 1$. Thus, x is a scalar multiple of 1. Hence, $(M_T^{\varphi})' \cap M_T = \mathbb{C}1$.

The rest of the statements follow similarly to Theorem 5.7. This completes the proof. ∎

Theorem 5.9. *Suppose that the invariant subspace of weakly mixing vectors in $\mathcal{H}_{\mathbb{R}}$ is non-trivial. Then, M_T is a type* III_1 *factor.*

Proof. The proof follows exactly along the same lines of [1, Thm. 8.1]. ∎

6. Second quantization and Haagerup approximation property

Second quantization is an indispensable tool for proving various approximation properties of the free Araki–Woods factors (see [17]) and q- deformed Araki–Woods von Neumann algebras (see [33]). In this section, we show that second quantization is also available for the mixed q-deformed Araki–Woods von Neumann algebras. In the same vein, we establish that M_T has the Haagerup approximation property.

Let $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{R}}$ be real Hilbert spaces with strongly continuous one-parameter groups of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$ and $(V_t)_{t\in\mathbb{R}}$, respectively. For $N_2 \geq N_1$, let $\mathcal{K}_{\mathbb{R}} := \bigoplus_{i\in N_1} \mathcal{K}_{\mathbb{R}}^{(i)}$ and $\mathcal{H}_{\mathbb{R}} := \bigoplus_{i\in N_2} \mathcal{H}_{\mathbb{R}}^{(i)}$ denote the decompositions of $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{R}}$ consisting of invariant subspaces of $(U_t)_{t\in\mathbb{R}}$ and $(V_t)_{t\in\mathbb{R}}$, respectively. Clearly, $\mathcal{K}_{\mathbb{C}} := \bigoplus_{i\in N_1} \mathcal{K}_{\mathbb{C}}^{(i)}$ and $\mathcal{H}_{\mathbb{C}} := \bigoplus_{i\in N_2} \mathcal{H}_{\mathbb{C}}^{(i)}$, where $\mathcal{K}_{\mathbb{C}}^{(i)}$ and $\mathcal{H}_{\mathbb{C}}^{(i)}$ denote the complexification of $\mathcal{K}_{\mathbb{R}}^{(i)}$ and $\mathcal{H}_{\mathbb{R}}^{(i)}$, respectively. Denote the complex conjugation on $\mathcal{K}_{\mathbb{C}}^{(i)}$ and $\mathcal{H}_{\mathbb{C}}^{(i)}$ by I_i and J_i, respectively. Let A denote the analytic generator of $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{K}_{\mathbb{C}}$ and B the analytic generator of $(V_t)_{t\in\mathbb{R}}$ on $\mathcal{H}_{\mathbb{C}}$. Since $\mathcal{K}_{\mathbb{C}}^{(i)}$, $i \in N_1$, are invariant for $(U_t)_{t\in\mathbb{R}}$, and $\mathcal{H}_{\mathbb{C}}^{(i)}$, $i \in N_2$, are invariant for $(V_t)_{t\in\mathbb{R}}$, it follows that $\mathcal{K} = \bigoplus_{i\in N_1} \mathcal{K}^{(i)}$ and $\mathcal{H} = \bigoplus_{i\in N_2} \mathcal{H}^{(i)}$, where $\mathcal{K}^{(i)}$, $i \in N_1$, are, respectively, the completions of $\mathcal{K}_{\mathbb{C}}^{(i)}$, $i \in N_1$, with respect to $\langle\cdot,\cdot\rangle_U := \langle \frac{2}{1+A^{-1}}\cdot,\cdot\rangle_{\mathcal{K}_{\mathbb{C}}}$, and $\mathcal{H}^{(i)}$, $i \in N_2$, are, respectively, the completions of $\mathcal{H}_{\mathbb{C}}^{(i)}$, $i \in N_2$, with respect to $\langle\cdot,\cdot\rangle_V := \langle \frac{2}{1+B^{-1}}\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}$.

Fix $-1 < q_{ij} < 1$ with $\sup_{i,j\in N_2} |q_{ij}| < 1$ and $q_{ij} = q_{ji}$ for $i, j \in N_2$. Consider the operators

$$T_{1,i_{\mathcal{K}},j_{\mathcal{K}}} : \mathcal{K}_{\mathbb{R}}^{(i_{\mathcal{K}})} \otimes \mathcal{K}_{\mathbb{R}}^{(j_{\mathcal{K}})} \to \mathcal{K}_{\mathbb{R}}^{(j_{\mathcal{K}})} \otimes \mathcal{K}_{\mathbb{R}}^{(i_{\mathcal{K}})},$$
$$T_{2,i_{\mathcal{H}},j_{\mathcal{H}}} : \mathcal{H}_{\mathbb{R}}^{(i_{\mathcal{H}})} \otimes \mathcal{H}_{\mathbb{R}}^{(j_{\mathcal{H}})} \to \mathcal{H}_{\mathbb{R}}^{(j_{\mathcal{H}})} \otimes \mathcal{H}_{\mathbb{R}}^{(i_{\mathcal{H}})},$$

defined by, respectively, extending the maps

$$\xi \otimes \eta \mapsto q_{i_{\mathcal{K}} j_{\mathcal{K}}}(\eta \otimes \xi), \quad i_{\mathcal{K}}, j_{\mathcal{K}} \in N_1,$$
$$\xi' \otimes \eta' \mapsto q_{i_{\mathcal{H}} j_{\mathcal{H}}}(\eta' \otimes \xi'), \quad i_{\mathcal{H}}, j_{\mathcal{H}} \in N_2.$$

Let $T_1 = \bigoplus_{i_{\mathcal{K}},j_{\mathcal{K}}} T_{1,i_{\mathcal{K}},j_{\mathcal{K}}}$ and $T_2 = \bigoplus_{i_{\mathcal{H}},j_{\mathcal{H}}} T_{2,i_{\mathcal{H}},j_{\mathcal{H}}}$.

Now, we are in a position to prove an appropriate second quantization theorem for mixed q-deformed Araki–Woods von Neumann algebras.

Proposition 6.1. *Suppose that $L_i : \mathcal{K}_{\mathbb{R}}^{(i)} \to \mathcal{H}_{\mathbb{R}}^{(i)}$ are contractions such that $L_i U_t = V_t L_i$ for all $t \in \mathbb{R}$ and $i \in N_1$. Let $L = \bigoplus_{i\in N_1} L_i$. Then, there exists a normal unital completely positive (u.c.p. in the sequel) map $\Gamma(L) : M_{T_1} \to M_{T_2}$ extending $s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) \mapsto s(L\xi_{j_1} \otimes \cdots \otimes L\xi_{j_n})$, $\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)}$, for $j_k \in N_1$, $1 \le k \le n$, $n \in \mathbb{N}$. Moreover, $\Gamma(L)$ is a Markov map; i.e., it preserves the vacuum state and intertwines the associated modular automorphism groups.*

Proof. Fix $i \in N_1$. Consider the dilation of L_i to an orthogonal operator U_{L_i} on $\mathcal{K}_{\mathbb{R}}^{(i)} \oplus \mathcal{H}_{\mathbb{R}}^{(i)}$ as follows:

$$U_{L_i} = \begin{pmatrix} (1_{\mathcal{K}_{\mathbb{R}}} - L_i^* L_i)^{\frac{1}{2}} & L_i^* \\ L_i & -(1_{\mathcal{H}_{\mathbb{R}}} - L_i L_i^*)^{\frac{1}{2}} \end{pmatrix}.$$

Let $\iota_i : \mathcal{K}_{\mathbb{R}}^{(i)} \to \mathcal{K}_{\mathbb{R}}^{(i)} \oplus \mathcal{H}_{\mathbb{R}}^{(i)}$ be the natural inclusion, and let $P_i : \mathcal{K}_{\mathbb{R}}^{(i)} \oplus \mathcal{H}_{\mathbb{R}}^{(i)} \to \mathcal{H}_{\mathbb{R}}^{(i)}$ denote the orthogonal projection. Then,

$$L_i = P_i U_{L_i} \iota_i.$$

Define $\iota = \bigoplus_{i\in N_1} \iota_i$, $U_L = \bigoplus_{i\in N_1} U_{L_i}$ and $P = \bigoplus_{i\in N_1} P_i$. It follows that

$$P U_L \iota = \Big(\bigoplus_{i\in N_1} P_i\Big)\Big(\bigoplus_{i\in N_1} U_{L_i}\Big)\Big(\bigoplus_{i\in N_1} \iota_i\Big) = \bigoplus_{i\in N_1} P_i U_{L_i} \iota_i = \bigoplus_{i\in N_1} L_i = L.$$

First, we intend to define appropriate maps $\Gamma(\iota)$, $\Gamma(U_L)$ and $\Gamma(P)$. We proceed as follows. Consider the orthogonal group $(U_t \oplus V_t)_{t\in\mathbb{R}}$ on $\mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}$. Note that $\iota \circ U_t = (U_t \oplus V_t) \circ \iota$ and $P \circ (U_t \oplus V_t) = (U_t \oplus V_t) \circ P$ for all $t \in \mathbb{R}$. Again, since $L_i U_t = V_t L_i$ for all $t \in \mathbb{R}$ and $i \in N_1$, one has $LU_t = V_t L$ for all $t \in \mathbb{R}$. This implies that U_L intertwines $U_t \oplus V_t$ for all $t \in \mathbb{R}$. Hence, the maps ι, U_L and P extend to contractions from $\mathcal{K}$ to $\mathcal{K} \oplus \mathcal{H}$, $\mathcal{K} \oplus \mathcal{H}$ to $\mathcal{K} \oplus \mathcal{H}$ and $\mathcal{K} \oplus \mathcal{H}$ to $\mathcal{H}$, respectively. We denote these extensions again by ι, U_L and P with slight abuse of notations.

Let $\xi_{i_{\mathcal{K}}} \in \mathcal{K}_{\mathbb{R}}^{(i_{\mathcal{K}})}$, $\xi_{j_{\mathcal{K}}} \in \mathcal{K}_{\mathbb{R}}^{(j_{\mathcal{K}})}$, for $i_{\mathcal{K}}, j_{\mathcal{K}} \in N_1$, and $\zeta_{i_{\mathcal{H}}} \in \mathcal{H}_{\mathbb{R}}^{(i_{\mathcal{H}})}$, $\zeta_{j_{\mathcal{H}}} \in \mathcal{H}_{\mathbb{R}}^{(j_{\mathcal{H}})}$ for $i_{\mathcal{H}}, j_{\mathcal{H}} \in N_2$. Define $\widetilde{T}$ on $(\mathcal{K} \oplus \mathcal{H}) \otimes (\mathcal{K} \oplus \mathcal{H})$ as the bounded linear extensions of the complexifications of the following:

$$\begin{aligned}
\widetilde{T}_{1_{\mathcal{K}_{\mathbb{R}}^{(i_{\mathcal{K}})} \otimes \mathcal{K}_{\mathbb{R}}^{(j_{\mathcal{K}})}}} &: \xi_{i_{\mathcal{K}}} \otimes \xi_{j_{\mathcal{K}}} \mapsto q_{i_{\mathcal{K}} j_{\mathcal{K}}} (\xi_{j_{\mathcal{K}}} \otimes \xi_{i_{\mathcal{K}}}),\\
\widetilde{T}_{1_{\mathcal{K}_{\mathbb{R}}^{(i_{\mathcal{K}})} \otimes \mathcal{H}_{\mathbb{R}}^{(j_{\mathcal{H}})}}} &: \xi_{i_{\mathcal{K}}} \otimes \zeta_{j_{\mathcal{H}}} \mapsto q_{i_{\mathcal{K}} j_{\mathcal{H}}} (\zeta_{j_{\mathcal{H}}} \otimes \xi_{i_{\mathcal{K}}}),\\
\widetilde{T}_{1_{\mathcal{H}_{\mathbb{R}}^{(i_{\mathcal{H}})} \otimes \mathcal{K}_{\mathbb{R}}^{(j_{\mathcal{K}})}}} &: \zeta_{i_{\mathcal{H}}} \otimes \xi_{j_{\mathcal{K}}} \mapsto q_{i_{\mathcal{H}} j_{\mathcal{K}}} (\xi_{j_{\mathcal{K}}} \otimes \zeta_{i_{\mathcal{H}}}),\\
\widetilde{T}_{1_{\mathcal{H}_{\mathbb{R}}^{(i_{\mathcal{H}})} \otimes \mathcal{H}_{\mathbb{R}}^{(j_{\mathcal{H}})}}} &: \zeta_{i_{\mathcal{H}}} \otimes \zeta_{j_{\mathcal{H}}} \mapsto q_{i_{\mathcal{H}} j_{\mathcal{H}}} (\zeta_{j_{\mathcal{H}}} \otimes \zeta_{i_{\mathcal{H}}}).
\end{aligned}$$

By construction, $\widetilde{T}$ satisfies the properties listed in (7). Also, it is easy to verify that $(P \otimes P)\widetilde{T} = T_2(P \otimes P)$, $(U_L \otimes U_L)\widetilde{T} = \widetilde{T}(U_L \otimes U_L)$ and $(\iota \otimes \iota)T_1 = \widetilde{T}(\iota \otimes \iota)$.

Define $\Gamma(U_L) : \mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H})) \to \mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H}))$ by

$$\Gamma(U_L)(x) := \mathcal{F}_{\widetilde{T}}(U_L) x \mathcal{F}_{\widetilde{T}}(U_L)^*, \quad x \in \mathbf{B}\big(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H})\big).$$

Since $(U_L \otimes U_L)\widetilde{T} = \widetilde{T}(U_L \otimes U_L)$, from Proposition A.4, it follows that $\Gamma(U_L)$ is a normal completely positive map from $\mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H}))$ to $\mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H}))$.

Now, we verify that

$$\Gamma(U_L)\big(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\big) = s(U_L\xi_{j_1} \otimes \cdots \otimes U_L\xi_{j_n}),$$

where $\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)} \oplus \mathcal{H}_{\mathbb{R}}^{(j_{k'})}$, $j_k \in N_1$, $j_{k'} \in N_2$ and $1 \leq k, k' \leq n$, $n \in \mathbb{N}$. By the Wick product formula in Lemma 4.1, it suffices to show that

$$\begin{aligned}
&\Gamma(U_L)\big(l(\xi_{j_{t_1}}) \cdots l(\xi_{j_{t_k}}) l^*(\xi_{j_{t_{(k+1)}}}) \cdots l^*(\xi_{j_{t_n}})\big)\\
&\quad = l(U_L\xi_{j_{t_1}}) \cdots l(U_L\xi_{j_{t_k}}) l^*(U_L\xi_{j_{t_{(k+1)}}}) \cdots l^*(U_L\xi_{j_{t_n}}),
\end{aligned}$$

where $\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)}$ for $j_k \in N_1$, $1 \leq k \leq n$, $n \in \mathbb{N}$.

It is easy to see that $\mathcal{F}_{\widetilde{T}}(U_L)$ is a unitary and $\mathcal{F}_{\widetilde{T}}(U_L) l(u) \mathcal{F}_{\widetilde{T}}(U_L)^* = l(U_L u)$ for all $u \in \mathcal{K}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}}$. Therefore, we have

$$\begin{aligned}
&\mathcal{F}_{\widetilde{T}}(U_L) l(\xi_{j_{t_1}}) \cdots l(\xi_{j_{t_k}}) l^*(\xi_{j_{t_{(k+1)}}}) \cdots l^*(\xi_{j_{t_n}}) \mathcal{F}_{\widetilde{T}}(U_L)^*\\
&\quad = l(U_L\xi_{j_{t_1}}) \cdots l(U_L\xi_{j_{t_k}}) l^*(U_L\xi_{j_{t_{(k+1)}}}) \cdots l^*(U_L\xi_{j_{t_n}}).
\end{aligned}$$

Since $U_L\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)} \oplus \mathcal{H}_{\mathbb{R}}^{(j_{k'})}$, $j_k \in N_1$, $j_{k'} \in N_2$ and $1 \leq k, k' \leq n$, $n \in \mathbb{N}$, we have

$$\Gamma(U_L)\big(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\big) = s(U_L\xi_{j_1} \otimes \cdots \otimes U_L\xi_{j_n}).$$

By an application of the Kaplansky density theorem, it follows that $\Gamma(U_L)$ maps $M_{\widetilde{T}}$ into $M_{\widetilde{T}}$.

Since $(P \otimes P)\widetilde{T} = T_2(P \otimes P)$, from Proposition A.4, it follows that the map $\Gamma(P) : \mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H})) \to \mathbf{B}(\mathcal{F}_{T_2}(\mathcal{H}))$ by $\Gamma(P)(x) := \mathcal{F}(P) x \mathcal{F}(P)^*$, $x \in \mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H}))$, is normal and completely positive.

Let $\Omega_{\widetilde{T}}$ and Ω_{T_2} denote the standard vacuum vectors in $\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H})$ and $\mathcal{F}_{T_2}(\mathcal{H})$, respectively. Since $\Omega_{\widetilde{T}}$ and Ω_{T_2} are, respectively, cyclic and separating for $M_{\widetilde{T}}$ and M_{T_2} (see Proposition 3.5), it follows that

$$\begin{aligned}\Gamma(P)\big(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\big)(\Omega_{T_2}) &= \mathcal{F}(P)s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\Omega_{\widetilde{T}} \\ &= \mathcal{F}(P)(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) \\ &= P\xi_{j_1} \otimes \cdots \otimes P\xi_{j_n} \\ &= s(P\xi_{j_1} \otimes \cdots \otimes P\xi_{j_n})\Omega_{T_2}.\end{aligned}$$

Thus, $\Gamma(P)(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})) = s(P\xi_{j_1} \otimes \cdots \otimes P\xi_{j_n})$, where $\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)} \oplus \mathcal{H}_{\mathbb{R}}^{(j_{k'})}$, $j_k \in N_1$, $j_{k'} \in N_2$ and $1 \leq k, k' \leq n$, $n \in \mathbb{N}$. As before, it follows that $\Gamma(P)$ maps $M_{\widetilde{T}}$ into M_{T_2}.

Now, we proceed to define $\Gamma(\iota)$. Let $M_{\mathcal{K}} = \{s(\xi) : \xi \in \mathcal{K}_{\mathbb{R}} \oplus 0 \subseteq \mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}\}'' \subseteq M_{\widetilde{T}}$. Since $(\iota \otimes \iota)T_1 = \widetilde{T}(\iota \otimes \iota)$, from Proposition A.4, it follows that $\mathcal{F}(\iota)$ is a contraction.

Note that $\mathcal{F}(\iota) : \mathcal{F}_{T_1}(\mathcal{K}) \to \mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H})$ is an isometry whose final space is the range of the Jones projection associated with $M_{\mathcal{K}}$. Consequently, $\Gamma(\iota) : \mathbf{B}(\mathcal{F}_{T_1}(\mathcal{K})) \to \mathbf{B}(\mathcal{F}_{\widetilde{T}}(\mathcal{K} \oplus \mathcal{H}))$ defined as $\Gamma(\iota)(x) = \mathcal{F}(\iota)x\mathcal{F}(\iota)^*$, $x \in \mathbf{B}(\mathcal{F}_{T_1}(\mathcal{K}))$, is a unital injective $*$-homomorphism. Arguing as before, it is clear that

$$\Gamma(\iota)\big(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\big) = s(\iota\xi_{j_1} \otimes \cdots \otimes \iota\xi_{j_n}),$$

where $\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)}$, $j_k \in N_1$, $1 \leq k \leq n$, $n \in \mathbb{N}$. Further, $\Gamma(\iota)$ maps M_{T_1} into $M_{\widetilde{T}}$, and the range of $\Gamma(\iota)$ is easily identified with $M_{\mathcal{K}}$.

Define $\Gamma(L) : M_{T_1} \to M_{T_2}$ by

$$\Gamma(L)(x) = \Gamma(P)\big(\Gamma(U_L)_{\restriction M_{\widetilde{T}}}\big)\big(\Gamma(\iota)_{\restriction M_{T_1}}\big)(x), \quad x \in M_{T_1}.$$

Clearly, $\Gamma(L)$ is a normal u.c.p. map. Further,

$$\Gamma(L)\big(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\big) = s(L\xi_{j_1} \otimes \cdots \otimes L\xi_{j_n})$$

for $\xi_{j_k} \in \mathcal{K}_{\mathbb{R}}^{(j_k)}$, $j_k \in N_1$, $1 \leq k \leq n$, $n \in \mathbb{N}$.

The rest of the statements are routine. This completes the proof. ∎

Corollary 6.2. *Let $\iota : \mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}$ be the inclusion map. Then, there exists an injective, normal unital $*$-homomorphism $\Gamma(\iota) : M_{T_1} \to M_{\widetilde{T}}$. Moreover, $\Gamma(\iota)$ is a Markov map.*

Proof. We have noted the result in the proof of Proposition 6.1. Hence, we omit the details. ∎

Next, we show that second quantization for mixed q-deformed Araki–Woods von Neumann algebras can be defined for a contraction between $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{R}}$ which respects the invariant blocks of the one-parameter groups $(U_t)_{t\in\mathbb{R}}$ and $(V_t)_{t\in\mathbb{R}}$ and extends to a contraction between $\mathcal{K}$ and $\mathcal{H}$. The proof is similar to [33, Thm. 3.4].

Theorem 6.3. *Let $L_i : \mathcal{K}^{(i)} \to \mathcal{H}^{(i)}$ be contractions such that $L_i = J_i L_i I_i$ for $i \in N_1$. Let $L = \bigoplus_{i\in N_1} L_i$. Then, there exists a normal u.c.p. map $\Gamma(L) : M_{T_1} \to M_{T_2}$ extending $s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) \mapsto s(L\xi_{j_1} \otimes \cdots \otimes L\xi_{j_n})$, $\xi_{j_k} \in \mathcal{K}_{\mathbb{C}}^{(j_k)}$ for $j_k \in N_1$, $1 \le k \le n$, $n \in \mathbb{N}$. Moreover, $\Gamma(L)$ preserves the vacuum state.*

Proof. The proof follows along the same lines of [33, Thm. 3.4]. Therefore, we provide only an outline of the proof.

As in the proof of Proposition 6.1, for $i \in N_1$, consider the dilation of L_i to a unitary operator U_{L_i} on $\mathcal{K}^{(i)} \oplus \mathcal{H}^{(i)}$ as follows:

$$U_{L_i} = \begin{pmatrix} (1_{\mathcal{K}^{(i)}} - L_i^* L_i)^{\frac{1}{2}} & L_i^* \\ L_i & -(1_{\mathcal{H}^{(i)}} - L_i L_i^*)^{\frac{1}{2}} \end{pmatrix}.$$

Note that $L_i = P_i U_{L_i} \iota_i$, where $\iota_i : \mathcal{K}^{(i)} \to \mathcal{K}^{(i)} \oplus \mathcal{H}^{(i)}$ is the natural inclusion and $P_i : \mathcal{K}^{(i)} \oplus \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}$ is the orthogonal projection. Define $\iota = \bigoplus_{i\in N_1} \iota_i, U_L = \bigoplus_{i\in N_1} U_{L_i}$ and $P = \bigoplus_{i\in N_1} P_i$. It follows that

$$P U_L \iota = \Big(\bigoplus_{i\in N_1} P_i\Big)\Big(\bigoplus_{i\in N_1} U_{L_i}\Big)\Big(\bigoplus_{i\in N_1} \iota_i\Big) = \bigoplus_{i\in N_1} P_i U_{L_i} \iota_i = \bigoplus_{i\in N_1} L_i = L.$$

It is easy to see that ι maps $\mathcal{K}_{\mathbb{R}}$ inside $\mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}$ and $\iota \circ U_t = (U_t \oplus V_t) \circ \iota$ for $t \in \mathbb{R}$. Therefore, by the proof of Proposition 6.1, $\Gamma(\iota)_{\restriction M_{T_1}} : M_{T_1} \to M_{\tilde{T}}$ is the second quantization of ι. Let $\Gamma(U_L) : \mathbf{B}(\mathcal{F}_{\tilde{T}}(\mathcal{K} \oplus \mathcal{H})) \to \mathbf{B}(\mathcal{F}_{\tilde{T}}(\mathcal{K} \oplus \mathcal{H}))$ be the automorphism given by

$$\Gamma(U_L)(x) := \mathcal{F}_{\tilde{T}}(U_L) x \mathcal{F}_{\tilde{T}}(U_L)^*, \quad x \in \mathbf{B}\big(\mathcal{F}_{\tilde{T}}(\mathcal{K} \oplus \mathcal{H})\big).$$

Also, let $\Gamma(P) : \mathbf{B}(\mathcal{F}_{\tilde{T}}(\mathcal{K} \oplus \mathcal{H})) \to \mathbf{B}(\mathcal{F}_{T_2}(\mathcal{H}))$ be the automorphism given by

$$\Gamma(P)(x) = \mathcal{F}(P) x \mathcal{F}(P)^*, \quad x \in \mathbf{B}\big(\mathcal{F}_{\tilde{T}}(\mathcal{K} \oplus \mathcal{H})\big).$$

Define $\Gamma(L) : M_{T_1} \to \mathbf{B}(\mathcal{F}_{T_2}(\mathcal{H}))$ by $\Gamma(L) := \Gamma(P)\Gamma(U_L)\Gamma(\iota)_{\restriction M_{T_1}}$. Using the hypothesis and arguing along the same lines of [33, Thm. 3.4], it follows that

$$\Gamma(L)\big(s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n})\big) = s(L\xi_{j_1} \otimes \cdots \otimes L\xi_{j_n})$$

for $\xi_{j_k} \in \mathcal{K}_{\mathbb{C}}^{(j_k)}$ for $j_k \in N_1$, $1 \le k \le n$, $n \in \mathbb{N}$. By an obvious application of Kaplansky density theorem, it follows that $\Gamma(L)$ maps M_{T_1} into M_{T_2}.

The rest is clear. This completes the proof. ∎

Remark 6.4. Consider two different decompositions of $\mathcal{H}_{\mathbb{R}}$ consisting of invariant subspaces of $(U_t)_{t\in\mathbb{R}}$ as follows:

$$\mathcal{H}_{\mathbb{R}} := \bigoplus_{i\in N} \mathcal{H}_{\mathbb{R}}^{(i)} \quad \text{and} \quad \mathcal{H}_{\mathbb{R}} := \bigoplus_{i\in N} \mathcal{K}_{\mathbb{R}}^{(i)}.$$

For $i, j \in N$, define $T_{i,j} : \mathcal{H}_{\mathbb{R}}^{(i)} \otimes \mathcal{H}_{\mathbb{R}}^{(j)} \to \mathcal{H}_{\mathbb{R}}^{(j)} \otimes \mathcal{H}_{\mathbb{R}}^{(i)}$ to be the bounded extension of

$$\xi \otimes \eta \mapsto q_{ij}(\eta \otimes \xi) \quad \text{for } \xi \in \mathcal{H}_{\mathbb{R}}^{(i)},\ \eta \in \mathcal{H}_{\mathbb{R}}^{(j)}.$$

Also, define $T'_{i,j} : \mathcal{K}_{\mathbb{R}}^{(i)} \otimes \mathcal{K}_{\mathbb{R}}^{(j)} \to \mathcal{K}_{\mathbb{R}}^{(j)} \otimes \mathcal{K}_{\mathbb{R}}^{(i)}$ to be the bounded extension of

$$\xi' \otimes \eta' \mapsto q_{ij}(\eta' \otimes \xi') \quad \text{for } \xi' \in \mathcal{K}_{\mathbb{R}}^{(i)},\ \eta' \in \mathcal{K}_{\mathbb{R}}^{(j)}.$$

Let M_T and $M_{T'}$ be the associated von Neumann algebras represented in standard form on $\mathcal{F}_T(\mathcal{H})$ and $\mathcal{F}_{T'}(\mathcal{H})$, respectively. Suppose that there exist orthogonal operators $V_i : \mathcal{H}_{\mathbb{R}}^{(i)} \to \mathcal{K}_{\mathbb{R}}^{(i)}$ such that $V_i U_t = U_t V_i$ for $i \in N$, $t \in \mathbb{R}$. Let $V = \bigoplus_{i \in N} V_i$. Then, from Proposition 6.1, it follows that the map $\Gamma(V) : M_T \to M_{T'}$ defines an isomorphism.

We are prepared to discuss the Haagerup property of M_T.

Definition 6.5 ([12]). Let M be a von Neumann algebra equipped with a faithful normal semifinite weight ψ. Then, M has a *Haagerup approximation property* if there exists a sequence of normal u.c.p. maps $\Phi_k : M \to M$, $k \in \mathbb{N}$ such that

(1) $\psi \circ \Phi_k \leq \psi$ for all $k \in \mathbb{N}$;

(2) the GNS-implementation $T_{\Phi_k} : L^2(M, \psi) \to L^2(M, \psi)$ of Φ_k is compact for all $k \in \mathbb{N}$, and $T_{\Phi_k} \to 1_{L^2(M,\psi)}$ strongly.

Remark 6.6. From [12, Thm. 1.3], it follows that the Haagerup approximation property is an intrinsic property of the von Neumann algebra M; i.e., it does not depend on the choice of the faithful normal semifinite weight ψ.

Fix $k \in N$. Let $\mathcal{J}_k$ denote the complex conjugation on $\mathcal{H}_{\mathbb{C}}^{(k)}$. Then, the following statement [17] holds.

Proposition 6.7 (cf. [17, Prop. 3.17]). *For $i \in N$, there exists a sequence of finite rank contractions $\{T_k^{(i)}\}_{k \in \mathbb{N}}$ on $\mathcal{H}^{(i)}$ such that $\mathcal{J}_i T_k^{(i)} \mathcal{J}_i = T_k^{(i)}$ and $\lim_{k\to\infty} T_k^{(i)} = 1$ (on $\mathcal{H}^{(i)}$) pointwise.*

In the next theorem, we show that M_T has the Haagerup approximation property. It generalizes [33, Thm. 1.1] to the current setting.

Theorem 6.8. *M_T has the Haagerup approximation property.*

Proof. For $i \in N$, by Proposition 6.7, we get a sequence of finite rank contractions

$$\{T_k^{(i)}\}_{k \in \mathbb{N}} \quad \text{on } \mathcal{H}^{(i)}$$

such that $\mathcal{J}_i T_k^{(i)} \mathcal{J}_i = T_k^{(i)}$ for all $k \in \mathbb{N}$ and $T_k^{(i)} \to 1$ (on $\mathcal{H}^{(i)}$) in s.o.t. as $k \to \infty$.

If $|N| < \infty$, define $L_k : \mathcal{H} \to \mathcal{H}$ by $L_k := \sum_{i \in N} T_k^{(i)}$. Again, if $N = \mathbb{N}$, then define $L_k : \mathcal{H} \to \mathcal{H}$ as

$$L_k := T_k^{(1)} \oplus T_k^{(2)} \oplus \cdots \oplus T_k^{(k-1)} \oplus T_k^{(k)} \oplus 0.$$

Note that L_k is a finite rank contraction for each $k \in \mathbb{N}$. Also, note that $L_k(\mathcal{H}^{(i)}) \subseteq \mathcal{H}^{(i)}$, and $\mathcal{J}_i L_k \mathcal{J}_i = L_k$ for all $i \in N$, $k \in \mathbb{N}$. If $|N| < \infty$, it is obvious that $L_k \to 1$ in s.o.t. Again, if $N = \mathbb{N}$, let $P_{(l)} : \mathcal{H} \to \bigoplus_{m=1}^{l} \mathcal{H}^{(m)}$ denote the orthogonal projection. Then,

$L_k\xi \to \xi$ for all $\xi \in \mathrm{Ran}(P_{(l)})$ for all l. By a simple density argument, it follows that $L_k \to 1$ in s.o.t. on $\mathcal{H}$ as $k \to \infty$.

By replacing the role of J_i and I_i for $i \in N_1$ in Theorem 6.3 by $\mathcal{J}_i$, $i \in N$, and proceeding along the same lines of the proof of Theorem 6.3, we get a second quantization $\Gamma(L_k)$ for M_T such that $\Gamma(L_k)s(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) = s(L_k\xi_{j_1} \otimes \cdots \otimes L_k\xi_{j_n})$ for $\xi_{j_k} \in \mathcal{H}_{\mathbb{C}}^{(j_k)}$, $j_k \in N$, $1 \le k \le n, n \in \mathbb{N}$.

Let $V_{k,t} := \Gamma(e^{-t}L_k) : M_T \to M_T$ for $k \in \mathbb{N}$ and $t > 0$ (see Theorem 6.3). Then, $V_{k,t}$ is a normal u.c.p. map, and it preserves the vacuum state φ for $k \in \mathbb{N}$ and $t > 0$. We show that the GNS-implementations of these maps are compact and are converging pointwise to 1 as $k \to \infty$ and $t \to 0$.

Since $\Gamma(e^{-t}L_k)(x) = \mathcal{F}_T(e^{-t}L_k)x\mathcal{F}_T(e^{-t}L_k)^*$ for $x \in M_T$, $\mathcal{F}_T(e^{-t}L_k)^*\Omega = \Omega$ and Ω is generating for M_T, it follows that the GNS implementation $T_{V_{k,t}}$ of $V_{k,t}$ is $\mathcal{F}_T(e^{-t}L_k)$, $k \in \mathbb{N}$ and $t > 0$.

Let $P_n : \mathcal{F}(\mathcal{H}) \to \bigoplus_{m=0}^{n} \mathcal{H}^{\otimes m}$ denote the orthogonal projection, $n \in \mathbb{N}$ ($\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$). Note that $P^{(n)}$ maps $\mathcal{H}^{\otimes n}$ into $\mathcal{H}^{\otimes n}$ for $n \in \mathbb{N}$. Therefore, one has

$$\Big(\bigoplus_{i=n+1}^{\infty} P^{(i)}\Big)P_n^{\perp} = P_n^{\perp}\Big(\bigoplus_{i=n+1}^{\infty} P^{(i)}\Big), \quad n \in \mathbb{N}$$

(note that $\bigoplus_{i=n+1}^{\infty} P^{(i)}$ is unbounded, densely defined, closed, positive and self-adjoint). It is easy to verify that $T_{1_{\mathcal{H}^{(i)} \otimes \mathcal{H}^{(j)}}}$ commutes with $T_k^{(i)} \otimes T_k^{(j)}$ for $k \in \mathbb{N}$, $i, j \in N$. Therefore, T commutes with $L_k \otimes L_k$ for all $k \in \mathbb{N}$. Therefore, $\mathcal{F}(e^{-t}L_k)_{1_{\mathcal{H}^{\otimes n}}}$ also commutes with $P^{(n)}$ for $n \in \mathbb{N}$ (cf. Proposition A.4). Note that $\mathcal{F}_T(L_k)_{1_{\mathcal{F}(\mathcal{H})}} = \mathcal{F}(L_k)$, $k \in \mathbb{N}$. Hence, by Proposition A.1, one has

$$\|P_n^{\perp}\mathcal{F}(e^{-t}L_k)\| = \|\widetilde{P_n}^{\perp}\mathcal{F}_T(e^{-t}L_k)\|, \tag{66}$$

where $\widetilde{P_n} : \mathcal{F}_T(\mathcal{H}) \to \bigoplus_{m=0}^{n} \mathcal{H}^{\otimes_T^m}$ is the orthogonal projection, which is an extension of P_n, $n \in \mathbb{N}$.

Now, we check the compactness of $\mathcal{F}_T(e^{-t}L_k)$ for $k \in \mathbb{N}$ and $t > 0$. Fix $k \in \mathbb{N}$ and $t > 0$. Since L_k is of finite rank, it follows that $\widetilde{P_n}\mathcal{F}_T(e^{-t}L_k)$ is also of finite rank. Therefore, it is enough to show that

$$\|\widetilde{P_n}^{\perp}\mathcal{F}_T(e^{-t}L_k)\| \to 0 \quad \text{as } n \to \infty.$$

By (66), it is enough to show that

$$\|P_n^{\perp}\mathcal{F}(e^{-t}L_k)\| \to 0 \quad \text{as } n \to \infty.$$

Note that

$$\|P_n^{\perp}\mathcal{F}(e^{-t}L_k)\|^2 = \|P_n^{\perp}\mathcal{F}(e^{-2t}L_kL_k^*)P_n^{\perp}\|.$$

Now, since $L_kL_k^*$ is a finite rank positive contraction, there exists an orthonormal basis $\{e_i^{(k)}\}_{i\in\Lambda}$ of $\mathcal{H}$ such that $L_kL_k^*e_i^{(k)} = \lambda_i^{(k)}e_i^{(k)}$ for $\lambda_i^{(k)} \in [0,1]$, $i \in \Lambda$. The tensor products of the elements of the orthonormal basis $\{e_i^{(k)}\}_{i\in\Lambda}$ form an orthonormal basis of the free Fock space $\mathcal{F}(\mathcal{H})$. For a multi-index $\mathfrak{J} = \{i_1, \dots, i_n\}$, let us denote $e_{\mathfrak{J}}^{(k)} = e_{i_1}^{(k)} \otimes \cdots \otimes e_{i_n}^{(k)}$ and $\lambda_{\mathfrak{J}}^{(k)} = \lambda_{i_1}^{(k)} \cdots \lambda_{i_n}^{(k)}$.

Let $\zeta \in \mathcal{F}(\mathcal{H})$ be written as $\zeta = \sum_{\mathfrak{J}} a_{\mathfrak{J}}^{(k)} e_{\mathfrak{J}}^{(k)}$. Then, one has

$$\begin{aligned}\|P_n^{\perp}\mathcal{F}(e^{-2t}L_k L_k^*)P_n^{\perp}\zeta\|_{\mathcal{F}(\mathcal{H})}^2 &= \Big\| \sum_{|\mathfrak{J}|>n} e^{-2|\mathfrak{J}|t} a_{\mathfrak{J}}^{(k)} \lambda_{\mathfrak{J}}^{(k)} e_{\mathfrak{J}}^{(k)} \Big\|_{\mathcal{F}(\mathcal{H})}^2 \\ &= \sum_{|\mathfrak{J}|>n} e^{-4|\mathfrak{J}|t} |a_{\mathfrak{J}}^{(k)}|^2 |\lambda_{\mathfrak{J}}^{(k)}|^2 \\ &\le e^{-4(n+1)t}\|\zeta\|^2,\end{aligned}$$

where the last inequality follows since $|\lambda_{\mathfrak{J}}^{(k)}| \le 1$. Therefore, $\|P_n^{\perp}\mathcal{F}(e^{-t}L_k)\| \to 0$ as $n \to \infty$.

From Proposition A.4, note that $\|\mathcal{F}_T(e^{-t}L_k)\| \le 1$ for $k \in \mathbb{N}$ and $t > 0$. Let $\zeta := \zeta_1 \otimes \cdots \otimes \zeta_n \in \mathcal{H}^{\odot n}$. It is easy to verify that $\|\mathcal{F}_T(e^{-t}L_k)\zeta - \zeta\|_T \to 0$ as $k \to \infty$ and $t \to 0$. Hence, by a simple density argument, it follows that $\mathcal{F}_T(e^{-t}L_k) \to 1_{\mathcal{F}_T(\mathcal{H})}$ in s.o.t. as $k \to \infty$ and $t \to 0$. This completes the proof. ∎

7. The relative commutant of M_ξ

In Section 4, we proved that for $\xi_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, the generating subalgebra M_{ξ_0} is a masa in M_T. In this section, we show that if $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ is *not* fixed by $(U_t)_{t\in\mathbb{R}}$, then the inclusion $M_{\xi_0} \subseteq M_T$ is *quasi-split*, and hence, whenever M_T is of type III, then $M_{\xi_0}' \cap M_T$ is large. Therefore, there is no easy way to construct a masa in M_T other than the ones constructed in Section 4. The results obtained in this section are analogous to the results obtained for the q-deformed Araki–Woods von Neumann algebras in [2].

Let M be a von Neumann algebra represented in standard form on the GNS Hilbert space $\mathcal{H}_\phi := L^2(M, \phi)$ with respect to a faithful normal state ϕ. Let $B \subseteq M$ be a unital von Neumann subalgebra of M. Also, let J_ϕ, Δ_ϕ and Ω_ϕ denote Tomita's conjugation operator, modular operator and the standard vacuum vector, respectively. The inner product and norm on $\mathcal{H}_\phi$ are denoted by $\langle\cdot,\cdot\rangle_\phi$ and $\|\cdot\|_{2,\phi}$, respectively. We have the following natural embeddings of M:

$$\begin{aligned}\Phi_1 &: M \to L^1(M) \text{ by } \Phi_1(x) = \langle J_\phi x\Omega_\varphi, \cdot\Omega_\phi\rangle_\phi, \quad && x \in M,\\ \Phi_2 &: M \to L^2(M) \text{ by } \Phi_2(x) = \Delta_\phi^{\frac{1}{4}} x\Omega_\phi, && x \in M.\end{aligned}$$

Definition 7.1 ([2, Def. 3.2]). (1) The inclusion $B \subseteq M$ is said to be *split* if there exists a type I factor F such that $B \subseteq F \subseteq M$.

(2) The inclusion $B \subseteq M$ is said to be *quasi-split* if the map

$$B \otimes_{\mathrm{alg}} M^{\mathrm{op}} \ni a \otimes y^{\mathrm{op}} \mapsto aJ_\phi y^* J_\phi \in \mathbf{B}(\mathcal{H}_\phi)$$

extends to a normal $*$-homomorphism η of $B\overline{\otimes}M^{\mathrm{op}}$ (acting on $\mathcal{H}_\phi \otimes \mathcal{H}_\phi$) onto $B \vee M'$.

It follows from [2, Lem. 3.9] that when M is type III and $B \subseteq M$ is split, then $B' \cap M$ is of type III. We state it for the sake of completeness.

Lemma 7.2 ([2, Lem. 3.9]). *Let $B \subseteq M \subseteq \mathbf{B}(\mathcal{H}_\phi)$ be von Neumann algebras. Then, the following are equivalent.*

(1) *The inclusion $B \subseteq M$ is split.*

(2) *There exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and faithful normal representations $\pi_B : B \to \mathbf{B}(\mathcal{H}_1)$ and $\pi_{M'} : M' \to \mathbf{B}(\mathcal{H}_2)$ such that*

$$xy' \mapsto \pi_B(x) \otimes \pi_{M'}(y'), \quad x \in B \text{ and } y' \in M',$$

extends to a spatial *isomorphism between $B \vee M'$ and $\pi_B(B)\overline{\otimes}\pi_{M'}(M')$. Moreover, $B' \cap M \cong \pi_B(B)'\overline{\otimes}(\pi_{M'}(M'))'$.*

Further, if M is of type III *and $B \subseteq M$ is a split inclusion, then $B' \cap M$ is of type* III.

Definition 7.3 ([10]). Let N and M be von Neumann algebras, and let $p = 1, 2$. A normal c.p. map $\Phi_p : N \to L^p(M)$ is said to be extendable if for any von Neumann algebra $\widetilde{N}$ with separable predual containing N, there exists a normal c.p. map $\widetilde{\Phi}_p : \widetilde{N} \to L^p(M)$, which extends Φ_p.

Definition 7.4 ([2]). Let X and Y be two Banach spaces, and let $\Psi : X \to Y$ be a bounded linear map. Then, Ψ is called a *nuclear* map if and only if there exist sequences $x_n^* \in X^*$ and $y_n \in Y$ such that $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ and

$$\Psi(x) = \sum_{n=1}^\infty x_n^*(x) y_n \quad \text{for all } x \in X.$$

The following proposition in [2] is crucial for our purpose. For the sake of completeness, we state it below.

Proposition 7.5 ([2, Prop. 3.7]). *Let $B \subseteq M$ be an inclusion of von Neumann algebras, where M is represented in standard form on the GNS Hilbert space $\mathcal{H}_\phi$ with respect to a faithful normal state ϕ. Then, $\Phi_{p_{1B}}$ is nuclear $\Rightarrow$ $\Phi_{p_{1B}}$ is extendable $\Leftrightarrow$ $B \subseteq M$ is* quasi-split*, $p = 1, 2$.*

In this section, we follow notations from Section 5. First, suppose that $N_2 \neq 0$. Fix $\tilde{k}$, with $1 \leq \tilde{k} \leq N_2$. In order to make the notations simple, we denote the pair $(f_{\tilde{k}}^1, f_{\tilde{k}}^2)$ by (f_0, f_0'). Also, we denote $\mathcal{H}_\mathbb{R}(\tilde{k})$, $A(\tilde{k})$, $\lambda_{\tilde{k}}$ and the pair $(e_{\tilde{k}}^1, e_{\tilde{k}}^2)$ by $\mathcal{H}_\mathbb{R}(0)$, $A(0)$, λ_0 and (e_0, e_0'), respectively. In this section, we show that $M_{f_0} \subseteq M_T$ (and hence by symmetry $M_{f_0'} \subseteq M_T$) is a *quasi-split* inclusion. We only work with M_{f_0}, as the analysis for $M_{f_0'}$ is analogous.

Denote $\tilde{f}_0 = \frac{1}{\sqrt{2}}(f_0 + if_0')$ and $\tilde{f}_0' = \frac{1}{\sqrt{2}}(f_0 - if_0')$. Note that $\tilde{f}_0, \tilde{f}_0'$ are scalar multiplies of e_0 and e_0', respectively, and are orthonormal vectors with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathbb{C}}$. Also, $A(0)\tilde{f}_0 = \frac{1}{\lambda_0}\tilde{f}_0$ and $A(0)\tilde{f}_0' = \lambda_0 \tilde{f}_0'$. Further, note that

$$f_0 = \frac{1}{\sqrt{2}}(\tilde{f}_0 + \tilde{f}_0') \quad \text{and} \quad f_0' = \frac{i}{\sqrt{2}}(\tilde{f}_0' - \tilde{f}_0).$$

The following lemma is crucial for our purpose, and the proof follows along the same lines as [2, Lem. 4.1].

Lemma 7.6. *Let* $\alpha, \beta \in \mathbb{R}$ *and* $z = \alpha + i\beta$. *Then,* $\|\Delta^z f_0\|_T = \sqrt{\frac{\lambda_0^{2\alpha}+\lambda_0^{1-2\alpha}}{1+\lambda_0}}$. *In particular,*

$$\|\Delta^{\frac{1}{4}} f_0\|_T = \sqrt{\frac{2\lambda_0^{\frac{1}{2}}}{1+\lambda_0}}.$$

Proof. The proof follows exactly along the same lines of [2, Lem. 4.1]. ∎

Theorem 7.7. $\Delta^{\frac{1}{4}}{\restriction}_{L^2(M_{f_0},\varphi)} : L^2(M_{f_0},\varphi) \to \mathcal{F}_T(\mathcal{H})$ *is a Hilbert–Schmidt operator of norm* 1. *In particular,* $\Phi_2{\restriction}_{M_{f_0}} : M_{f_0} \to \mathcal{F}_T(\mathcal{H})$ *is compact.*

Proof. Let $\mu = \frac{2\lambda_0^{\frac{1}{2}}}{1+\lambda_0}$. Note that $\mu < 1$ since $\lambda_0 > 1$. From Lemma 4.2 (ii), it follows that

$$\left\{\frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T} : n \geq 0\right\}$$

forms an orthonormal basis of $L^2(M_{f_0},\varphi)$. Let $x \in M_{f_0}$. Expand

$$x\Omega = \sum_{n=0}^{\infty} a_n \frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T},$$

where $a_n \in \mathbb{C}$ for all n and $\sum_{n=0}^{\infty} |a_n|^2 = \|x\Omega\|_T^2$.

From Lemma 3.9 (iii) and Theorem 3.13 (iii), it follows that if $f_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, then

$$f_0 \in \mathfrak{D}(\Delta^{\frac{1}{4}}), \quad \Delta^{\frac{1}{4}} f_0 \in \mathcal{H}_{\mathbb{C}}^{(i)},\ i \in N.$$

Therefore, it is easy to verify that

$$P^{(n)}(\Delta^{\frac{1}{4}} f_0)^{\otimes n} = \|f_0^{\otimes n}\|_T^2 (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \quad \text{for } n \in \mathbb{N}.$$

Note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} |a_n|^2 \left\|\Delta^{\frac{1}{4}}\left(\frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T}\right)\right\|_T^2 \\
&\quad = \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{\|f_0^{\otimes n}\|_T^2} \|\Delta^{\frac{1}{4}}(f_0^{\otimes n})\|_T^2 \\
&\quad = \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{\|f_0^{\otimes n}\|_T^2} \|(\Delta^{\frac{1}{4}} f_0)^{\otimes n}\|_T^2 \quad \text{(by Theorem 3.13 (ii))} \\
&\quad = \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{\|f_0^{\otimes n}\|_T^2} \langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, P^{(n)}(\Delta^{\frac{1}{4}} f_0)^{\otimes n}\rangle_{\mathcal{F}(\mathcal{H})}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} |a_n|^2 \frac{1}{\| f_0^{\otimes n} \|_T^2} \langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, \| f_0^{\otimes n} \|_T^2 (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \rangle_{\mathcal{F}(\mathcal{H})} \\
&= \sum_{n=0}^{\infty} |a_n|^2 \langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \rangle_{\mathcal{F}(\mathcal{H})} \\
&= \sum_{n=0}^{\infty} |a_n|^2 \big(\| \Delta^{\frac{1}{4}} f_0 \|_U \big)^{2n} \\
&= \sum_{n=0}^{\infty} |a_n|^2 \big(\| \Delta^{\frac{1}{4}} f_0 \|_T \big)^{2n} \\
&= \sum_{n=0}^{\infty} |a_n|^2 \mu^n \quad \text{(by Lemma 7.6)} \\
&\leq \| x\Omega \|_T^2 .
\end{aligned} \tag{67}
$$

Consequently, the series

$$
\sum_{n=0}^{\infty} a_n \left(\Delta^{\frac{1}{4}} \frac{f_0^{\otimes n}}{\| f_0^{\otimes n} \|_T} \right)
$$

defines a unique element in $\mathcal{F}_T(\mathcal{H})$. Now, approximating $x\Omega$ with

$$
\left\{ \sum_{n=0}^{l} a_n \frac{f_0^{\otimes n}}{\| f_0^{\otimes n} \|_T} \right\}_l ,
$$

noting $\Delta^{\frac{1}{4}}$ is a closed operator, using (iii) of Lemma 3.9 and (iii) of Theorem 3.13, one has

$$
\Delta^{\frac{1}{4}} x\Omega = \sum_{n=0}^{\infty} a_n \left(\Delta^{\frac{1}{4}} \frac{f_0^{\otimes n}}{\| f_0^{\otimes n} \|_T} \right).
$$

From (67), it follows that

$$
\| \Delta^{\frac{1}{4}} x\Omega \|_T \leq \| x\Omega \|_T .
$$

Therefore, $\Delta^{\frac{1}{4}}_{\restriction M_{f_0}\Omega}$ admits a bounded extension to $L^2(M_{f_0}, \varphi)$. Further,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left\| \Delta^{\frac{1}{4}} \left(\frac{f_0^{\otimes n}}{\| f_0^{\otimes n} \|_T} \right) \right\|_T^2 \\
&\qquad = \sum_{n=0}^{\infty} \frac{1}{\| f_0^{\otimes n} \|_T^2} \| (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \|_T^2 \quad \text{(by Theorem 3.13 (ii))} \\
&\qquad = \sum_{n=0}^{\infty} \frac{1}{\| f_0^{\otimes n} \|_T^2} \langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, P^{(n)} (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \rangle_{\mathcal{F}(\mathcal{H})} \\
&\qquad = \sum_{n=0}^{\infty} \frac{1}{\| f_0^{\otimes n} \|_T^2} \langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, \| f_0^{\otimes n} \|_T^2 (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \rangle_{\mathcal{F}(\mathcal{H})}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \rangle_{\mathcal{F}(\mathcal{H})} \\
&= \sum_{n=0}^{\infty} \big(\|\Delta^{\frac{1}{4}} f_0\|_U\big)^{2n} \\
&= \sum_{n=0}^{\infty} \big(\|\Delta^{\frac{1}{4}} f_0\|_T\big)^{2n} \\
&= \sum_{n=0}^{\infty} \mu^n \quad \text{(by Lemma 7.6)} \\
&= \frac{1}{1-\mu} \\
&< \infty.
\end{aligned}
$$

Hence, it follows that

$$
\Delta^{\frac{1}{4}}_{\restriction L^2(M_{f_0},\varphi)} : L^2(M_{f_0},\varphi) \to \mathcal{F}_T(\mathcal{H})
$$

is a Hilbert–Schmidt operator of norm 1, as $\Delta^{\frac{1}{4}}\Omega = \Omega$.

Consequently, $\Phi_{2\restriction M_{f_0}} : M_{f_0} \to \mathcal{F}_T(\mathcal{H})$ is compact. Indeed, if $M_{f_0} \ni x_n \to 0$ in the w^*-topology, then $x_n\Omega \to 0$ weakly in $L^2(M_{f_0},\varphi)$. By the compactness of $\Delta^{\frac{1}{4}}_{\restriction L^2(M_{f_0},\varphi)}$, it follows that $\Delta^{\frac{1}{4}} x_n\Omega \to 0$ in $\|\cdot\|_T$. This completes the proof. ∎

Theorem 7.8. *$\Phi_{2\restriction M_{f_0}} : M_{f_0} \to \mathcal{F}_T(\mathcal{H})$ is a nuclear map. Also, the inclusion $M_{f_0} \subseteq M_T$ is quasi-split.*

Proof. Following the proof of Theorem 7.7, for $x \in M_{f_0}$, one has

$$
\begin{aligned}
\Phi_{2\restriction M_{f_0}}(x) &= \Delta^{\frac{1}{4}} x\Omega \\
&= \sum_{n=0}^{\infty} \left\langle \frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T}, x\Omega \right\rangle_T \Delta^{\frac{1}{4}} \left(\frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T} \right) \\
&= \sum_{n=0}^{\infty} \psi_n(x) f_n,
\end{aligned}
$$

where $\psi_n \in (M_{f_0})_*$ is given by

$$
\psi_n(y) = \left\langle \frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T}, y\Omega \right\rangle_T \text{ for all } y \in M_{f_0}, \quad f_n = \Delta^{\frac{1}{4}} \left(\frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T} \right) \text{ for } n \in \mathbb{N} \cup \{0\}.
$$

By the Cauchy–Schwarz inequality, one has $\|\psi_n\| \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. From Lemma 3.9 (iii) and Theorem 3.13 (iii), it follows that if $f_0 \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, then

$$
f_0 \in \mathfrak{D}(\Delta^{\frac{1}{4}}), \quad \Delta^{\frac{1}{4}} f_0 \in \mathcal{H}_{\mathbb{C}}^{(i)}, \ i \in N.
$$

Therefore, it is easy to verify that $P^{(n)}(\Delta^{\frac{1}{4}} f_0)^{\otimes n} = \|f_0^{\otimes n}\|_T^2 (\Delta^{\frac{1}{4}} f_0)^{\otimes n}$. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} \|\psi_n\| \|f_n\|_T &\leq \sum_{n=0}^{\infty} \|f_n\|_T \\
&= \sum_{n=0}^{\infty} \left\| \Delta^{\frac{1}{4}} \left(\frac{f_0^{\otimes n}}{\|f_0^{\otimes n}\|_T} \right) \right\|_T \\
&= \sum_{n=0}^{\infty} \frac{1}{\|f_0^{\otimes n}\|_T} \left\| (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \right\|_T \quad \big(\text{by Theorem 3.13 (ii)}\big) \\
&= \sum_{n=0}^{\infty} \frac{1}{\|f_0^{\otimes n}\|_T} \left(\left\langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, P^{(n)} (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \right\rangle_{\mathcal{F}(\mathcal{H})} \right)^{\frac{1}{2}} \\
&= \sum_{n=0}^{\infty} \frac{1}{\|f_0^{\otimes n}\|_T} \left(\left\langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, \|f_0^{\otimes n}\|_T^2 (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \right\rangle_{\mathcal{F}(\mathcal{H})} \right)^{\frac{1}{2}} \\
&= \sum_{n=0}^{\infty} \left(\left\langle (\Delta^{\frac{1}{4}} f_0)^{\otimes n}, (\Delta^{\frac{1}{4}} f_0)^{\otimes n} \right\rangle_{\mathcal{F}(\mathcal{H})} \right)^{\frac{1}{2}} \\
&= \sum_{n=0}^{\infty} \left(\|\Delta^{\frac{1}{4}} f_0\|_U \right)^n \\
&= \sum_{n=0}^{\infty} \left(\|\Delta^{\frac{1}{4}} f_0\|_T \right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{2\lambda_0^{\frac{1}{2}}}{1+\lambda_0} \right)^{\frac{n}{2}} \quad (\text{by Lemma 7.6}) \\
&< \infty \quad \left(\text{since } \frac{2\lambda_0^{\frac{1}{2}}}{1+\lambda_0} < 1 \text{ as } \lambda_0 > 1 \right).
\end{aligned}$$

Hence, it follows that $\Phi_{2 \restriction M_{f_0}}$ is a nuclear map (see Definition 7.4). From Proposition 7.5, it follows that the inclusion $M_{f_0} \subseteq M_T$ is *quasi-split*. This completes the proof. ∎

Next, we extend the above investigations to general vectors in $\mathcal{H}_{\mathbb{R}}$. Let P_j^1, $1 \leq j \leq N_1$, P_k^2, $1 \leq k \leq N_2$, and P_{wm} be the orthogonal projections from $\mathcal{H}_{\mathbb{R}}$ onto $\mathbb{R}_j := \mathbb{R}$, $1 \leq j \leq N_1$, $\mathcal{H}_{\mathbb{R}}(k)$, $1 \leq k \leq N_2$, and $\widetilde{\mathcal{H}}_{\mathbb{R}}$, respectively. Let $\xi \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, be such that $\|\xi\|_{\mathcal{H}_{\mathbb{C}}} = \|\xi\|_U = 1$ and $P_k^2 \xi \neq 0$ for some k or $P_{wm}\xi \neq 0$. Then, following the same arguments as in [2], one has $\|\Delta^{\frac{1}{4}}\xi\|_T < 1$. We omit the details.

Therefore, assuming $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, $N_2 \neq 0$ or $\widetilde{\mathcal{H}}_{\mathbb{R}} \neq 0$, and from Theorems 7.7 and 7.8, one has the following.

Theorem 7.9. *Let $\xi \in \mathcal{H}_{\mathbb{R}}^{(i)}$, $i \in N$, be such that $\|\xi\|_U = 1$. Then, the following are equivalent.*

(1) *ξ is not fixed by $(U_t)_{t \in \mathbb{R}}$.*

(2) $\Delta^{\frac{1}{4}}{\restriction_{L^2(M_\xi,\varphi)}} : L^2(M_\xi,\varphi) \to \mathcal{F}_T(\mathcal{H})$ *is a Hilbert–Schmidt operator of norm* 1. *In particular,* $\Phi_2{\restriction_{M_\xi}} : M_\xi \to \mathcal{F}_T(\mathcal{H})$ *is compact.*

(3) $\Phi_2{\restriction_{M_\xi}} : M_\xi \to \mathcal{F}_T(\mathcal{H})$ *is a nuclear map.*

(4) $M_\xi \subseteq M_T$ *is a quasi-split inclusion.*

Proof. The proof of (1)$\Rightarrow$(2)$\Rightarrow$(3) follows from the above discussion and Theorems 7.7 and 7.8.

(3)$\Rightarrow$(1). Suppose to the contrary that $U_t\xi = \xi$ for all $t \in \mathbb{R}$. From Corollary 4.12, it follows that M_ξ is a masa in M_T possessing a φ-preserving normal conditional expectation $\mathbb{E}_\xi$. Since $\dim(\mathcal{H}_\mathbb{R}) \geq 2$, by Theorem 5.1, one has that M_T is a factor. Further, from Theorem 5.8, it follows that M_T is a factor of type II_1 or type III.

If M_T is a II_1 factor, then φ is a trace from Theorems 3.13 and 5.8. In this case, $\Delta = 1$, and thus, $\Phi_2{\restriction_{M_\xi}}$ cannot be nuclear. Thus, M_T is a type III factor.

Hence, from Proposition 7.5 and [2, Prop. 3.8], it follows that the inclusion $M_\xi \subseteq M_T$ is split. Let F be an intermediate type I factor between M_ξ and M_T. Then, $\mathbb{E}_\xi{\restriction_F} : F \to M_\xi$ is a faithful normal conditional expectation. This forces M_ξ to be completely atomic, which is a contradiction. Therefore, (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3).

(3)$\Leftrightarrow$(4) follows from Proposition 7.5 and [2, Prop. 3.8]. This completes the proof. ∎

In the general case, we have the following.

Corollary 7.10. *Let* $\xi \in \mathcal{H}_\mathbb{R}$ *be such that* $\|\xi\|_U = 1$. *Consider the following statements.*

(1) ξ *is not fixed by* $(U_t)_{t\in\mathbb{R}}$.

(2) $\Delta^{\frac{1}{4}}{\restriction_{L^2(M_\xi,\varphi)}} : L^2(M_\xi,\varphi) \to \mathcal{F}_T(\mathcal{H})$ *is a Hilbert–Schmidt operator of norm* 1. *In particular,* $\Phi_2{\restriction_{M_\xi}} : M_\xi \to \mathcal{F}_T(\mathcal{H})$ *is compact.*

(3) $\Phi_2{\restriction_{M_\xi}} : M_\xi \to \mathcal{F}_T(\mathcal{H})$ *is a nuclear map.*

(4) $M_\xi \subseteq M_T$ *is a quasi-split inclusion.*

Then, (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4). *Suppose* M_T *is of type* III *and* $\xi \in \mathcal{H}_\mathbb{R}$ *satisfies any one of the four conditions mentioned above. Then,* $M_\xi' \cap M_T$ *is of type* III. *If, in addition,* M_T *is a type* III *factor, then* $M_\xi \subseteq M_T$ *is a split inclusion.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) follows along the same lines of Theorem 7.9.

If M_T is of type III, then the conclusion follows directly from [2, Cor. 3.11]. If M_T is a type III factor, from [2, Prop. 3.8], it follows that the inclusion $M_\xi \subseteq M_T$ is split. This completes the proof. ∎

8. Non-injectivity of M_T

In this section, we show that M_T is non-injective in many cases. Our result on the non-injectivity of M_T is partial.

Definition 8.1. A von Neumann algebra $M \subseteq \mathbf{B}(H)$ is said to be *injective* if there exists a projection $\mathcal{E}$ of norm 1 from $\mathbf{B}(H)$ onto M.

Definition 8.2 ([30, Def. 3.5]). A von Neumann algebra $M \subseteq \mathbf{B}(H)$ is said to be *semi-discrete* if for any finite sequences $x_1, x_2, \ldots, x_n \in M$ and $y_1, y_2, \ldots, y_n \in M'$, the inequality

$$\left\| \sum_{i=1}^{n} x_i y_i \right\| \leq \left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_{\min}$$

holds.

Note in [30, Thm. 3.1] that a von Neumann algebra is injective if and only if it is semi-discrete.

Lemma 8.3. *Let* $\|T\| = q < 1$. *Also, let* $\eta_1, \ldots, \eta_n$ *be orthonormal vectors in* $(\mathcal{H}, \langle \cdot, \cdot \rangle_U)$ *for* $n \in \mathbb{N}$. *Then,*

$$\left\| \sum_{k=1}^{n} l(\eta_k) l^*(\eta_k) \right\| \leq \frac{1}{1-q}.$$

Proof. The proof follows along the same lines of [16, Lem. 2.1]. We omit the details. ■

In this section, we follow notations from Section 5. Let

$$\begin{aligned} x_h &= s(e_h) \quad \text{for } 1 \leq h \leq N_1 \text{ (or } 1 \leq h < N_1 \text{ if } N_1 = \aleph_0), \\ z_k &= \frac{1}{2}\big(s(f_k^1) + i s(f_k^2)\big) \quad \text{for } 1 \leq k \leq N_2 \text{ (or } 1 \leq k < N_2 \text{ if } N_2 = \aleph_0), \\ y_k &= z_k^*. \end{aligned}$$

The following theorem shows that M_T is non-injective in many cases. It adapts [16, Thms. 2.2, 2.3] to the current setting. We denote the spectral measure of A by E_A.

Theorem 8.4. *Let* $\|T\| = q < 1$. *Assume that any one of the following conditions hold.*

(i) *Suppose that* $\dim(E_A\{1\}) \geq 2$.

(ii) *Suppose that* $\dim(E_A\{1\}) = m_0 \leq 1$, $(\mathcal{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ *is almost periodic and for some* $K \in \mathbb{N}$, A *has eigenvalues* $\lambda_1, \ldots, \lambda_m \in (1, K)$, $m \in \mathbb{N}$, *counted with multiplicities such that*

$$\frac{1}{\sqrt{m_0 + m}} \left(m_0 + \sum_{k=1}^{m} \frac{2}{\sqrt{\lambda_k} + \sqrt{\lambda_k^{-1}}} \right) > \frac{4}{1-q}.$$

(iii) $(\mathcal{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ *is weak mixing.*

(iv) *Suppose that the weak mixing part of* $(\mathcal{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ *is non-trivial.*

Then, M_T *is not injective.*

Proof. (i) Following (54), let $\widetilde{\mathcal{K}_{\mathbb{R}}} := \bigoplus_{h=1}^{N_1} (\mathbb{R}_h, \mathrm{id})$. By assumption, $N_1 \geq 2$. Let $\mathcal{F}_T(\widetilde{\mathcal{K}})$ denote the twisted Fock space associated with $\widetilde{\mathcal{K}_{\mathbb{R}}}$ and the obvious compression of T. From [24, Thm. 2], it follows that $\Gamma_T(\widetilde{\mathcal{K}_{\mathbb{R}}})'' \subseteq \mathbf{B}(\mathcal{F}_T(\widetilde{\mathcal{K}}))$ is not injective.

Consider $\mathcal{H}_{\mathbb{R}} = \widetilde{\mathcal{K}_{\mathbb{R}}} \oplus (\widetilde{\mathcal{K}_{\mathbb{R}}})^{\perp}$ (direct sum taken with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$). Let $i : \widetilde{\mathcal{K}_{\mathbb{R}}} \to \widetilde{\mathcal{K}_{\mathbb{R}}} \oplus (\widetilde{\mathcal{K}_{\mathbb{R}}})^{\perp}$ be the inclusion map. By Corollary 6.2, there exists an injective, normal unital $*$-homomorphism $\Gamma(i) : \Gamma_T(\widetilde{\mathcal{K}_{\mathbb{R}}})'' \to M_T$. Hence, $M := \Gamma(i)(\Gamma_T(\widetilde{\mathcal{K}_{\mathbb{R}}})'')$ is not injective. In addition, $\Gamma(i)$ is a Markov map.

Suppose to the contrary that M_T is injective. Let $\mathcal{E}_1 : \mathbf{B}(\mathcal{F}_T(\mathcal{H})) \to M_T$ be a projection of norm 1 onto M_T. Since $\Gamma(i)$ is Markov, it follows that M is invariant under the modular automorphism group $(\sigma_t^{\varphi})_{t \in \mathbb{R}}$. By [29], there exists a φ-preserving conditional expectation $\mathcal{E}_2 : M_T \to M$ onto M. Hence, $\mathcal{E}_2 \circ \mathcal{E}_1 : \mathbf{B}(\mathcal{F}_T(\mathcal{H})) \to M$ is a projection of norm 1 onto M. This forces M to be injective, which is a contradiction.

(ii) Since $\mathcal{J} A \mathcal{J} = A^{-1}$, it follows that A has eigenvalues in $(1, \infty)$. Suppose to the contrary that M_T is injective. By the hypothesis, one has

$$m_0 + \sum_{k=1}^{m} \frac{2}{\sqrt{\lambda_k} + \sqrt{\lambda_k^{-1}}} > \frac{4}{1-q}(m_0 + m)^{\frac{1}{2}}. \tag{68}$$

Let

$$\begin{aligned} X &:= \sum_{h=1}^{m_0} x_h J x_h J + \sum_{k=1}^{m} z_k J z_k J + \sum_{k=1}^{m} y_k J y_k J \in C^*(M_T, M_T'), \\ \widetilde{X} &:= \sum_{h=1}^{m_0} x_h \otimes J x_h J + \sum_{k=1}^{m} z_k \otimes J z_k J + \sum_{k=1}^{m} y_k \otimes J y_k J \in M_T \otimes_{\min} M_T'. \end{aligned}$$

By the equivalence of injectivity and semi-discreteness, one has

$$\|X\| \le \|\widetilde{X}\|_{\min}.$$

Also, proceeding along the same lines of calculations in [16, Thm. 2.2], and by replacing the role of [16, Lem. 2.1] with Lemma 8.3, it follows that

$$\|X\| \ge m_0 + \sum_{k=1}^{m} \frac{2}{\sqrt{\lambda_k} + \sqrt{\lambda_k^{-1}}}$$

and

$$\|\widetilde{X}\|_{\min} \le \frac{4}{1-q}(m_0 + m)^{\frac{1}{2}}.$$

Therefore, one has

$$m_0 + \sum_{k=1}^{m} \frac{2}{\sqrt{\lambda_k} + \sqrt{\lambda_k^{-1}}} \le \frac{4}{1-q}(m_0 + m)^{\frac{1}{2}}.$$

This contradicts (68). Hence, M_T is not injective under the hypothesis.

(iii) Note that $\sigma(A)$ is continuous. Suppose to the contrary that M_T is injective. Since $\mathcal{J} A \mathcal{J} = A^{-1}$, it follows that $\sigma(A)$ has a continuous component in $(1, \infty)$. Let $F \subseteq (1, \infty) \cap \sigma(A)$ be a bounded set such that the spectral measure μ of A restricted to F

is continuous. Let $a = \inf\{x : x \in F\}$ and $b = \sup\{x : x \in F\}$. Then, $F \subseteq [a, b]$. Note that $L^\infty(F, \mu_{\restriction F})$ is diffuse. Therefore, there exists a disjoint partition $\{G_n\}_{n=1}^\infty$ of $[a, b)$ such that $G_n = [a_n, b_n)$ for $n \in \mathbb{N}$, with $a_1 = a$ and $\mu(G_n) > 0$ for all $n \in \mathbb{N}$. For $k \in \mathbb{N}$, let

$$\xi_k := \xi_k^1 + i\xi_k^2 \in E_A(G_k)(\mathcal{H}_\mathbb{C}),$$

such that $\xi_k^1, \xi_k^2 \in \mathcal{H}_\mathbb{R}$ and $\|\xi_k\|_{\mathcal{H}_\mathbb{C}} = 1$. Since $\mathcal{J} A \mathcal{J} = A^{-1}$, one has

$$\mathcal{J}\xi_k = \xi_k^1 - i\xi_k^2 \in E_A(G_k^{-1})(\mathcal{H}_\mathbb{C}), \quad \text{where } G_k^{-1} = \left\{\frac{1}{x} : x \in G_k\right\}.$$

Let $\zeta_k^1 = \frac{\xi_k}{\|\xi_k\|_U}$ and $\zeta_k^2 = \frac{\mathcal{J}\xi_k}{\|\mathcal{J}\xi_k\|_U}$ for $k \in \mathbb{N}$. Note that

$$\{\zeta_k^1\}_{k\in\mathbb{N}} \cup \{\zeta_k^2\}_{k\in\mathbb{N}}$$

is an orthonormal set in $\langle\cdot,\cdot\rangle_U$. Consider,

$$u_k = \frac{1}{\sqrt{a+1}\|\mathcal{J}\xi_k\|_U}\big(s(\xi_k^1) + is(\xi_k^2)\big), \quad v_k = u_k^*,\ k \in \mathbb{N}.$$

Fix $L \in \mathbb{N}$, and let

$$X := \sum_{k=1}^L u_k J u_k J + \sum_{k=1}^L v_k J v_k J \in C^*(M_T, M_T'),$$

$$\widetilde{X} := \sum_{k=1}^L u_k \otimes J u_k J + \sum_{k=1}^L v_k \otimes J v_k J \in M_T \otimes_{\min} M_T'.$$

As in the proof of (ii), by the equivalence of injectivity and semi-discreteness, one has

$$\|X\| \leq \|\widetilde{X}\|_{\min}.$$

Note that $\xi_k \in E_A([a, b_k))(\mathcal{H}_\mathbb{C})$ and $\mathcal{J}\xi_k \in E_A((b_k^{-1}, a^{-1}])(\mathcal{H}_\mathbb{C})$ for $k \in \mathbb{N}$. Since the function $f(x) = \frac{2x}{1+x}$ defined on $[0, \infty)$ is increasing, we get

$$\|\xi_k\|_U^2 \geq \frac{2a}{1+a} \quad \text{and} \quad \|\mathcal{J}\xi_k\|_U^2 \leq \frac{2}{1+a}.$$

Further, for $\xi \in \mathcal{H}$, one has $\|\xi\|_U = \|\xi\|_T$. Therefore, proceeding along the same lines of calculations in [16, Thm. 2.3] (as the vectors involved in the calculations are from $\mathcal{H}$), we get

$$\|X\| \geq \frac{a^{\frac{1}{2}}\big(1 + \big(\frac{a}{b}\big)^{\frac{1}{2}}\big)}{1+a} L.$$

Again, proceeding along the same lines of calculations in [16, Thm. 2.3] and by replacing the role of [16, Lem. 2.1] with Lemma 8.3, it follows that

$$\|\widetilde{X}\|_{\min} \leq \frac{4}{1-q}\left(\frac{1+b}{1+a}\right)^{\frac{1}{2}} L^{\frac{1}{2}}.$$

Therefore, one has

$$\frac{a^{\frac{1}{2}}\big(1+(\frac{a}{b})^{\frac{1}{2}}\big)}{1+a}L \leq \frac{4}{1-q}\Big(\frac{1+b}{1+a}\Big)^{\frac{1}{2}}L^{\frac{1}{2}},$$

which in turn implies

$$\frac{a^{\frac{1}{2}}\big(1+(\frac{a}{b})^{\frac{1}{2}}\big)}{\big((1+a)(1+b)\big)^{\frac{1}{2}}}L^{\frac{1}{2}} \leq \frac{4}{1-q}.$$

The last inequality does not hold for large $L \in \mathbb{N}$. This is a contradiction. Hence, M_T is not injective when $(\mathcal{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is weak mixing.

(iv) Let $(\widetilde{\mathcal{H}_{\mathbb{R}}}, (U_t)_{t\in\mathbb{R}})$ denote the weak mixing part of $(\mathcal{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$. Let $\mathcal{F}_T(\widetilde{\mathcal{H}})$ denote the twisted Fock space associated with $\widetilde{\mathcal{H}_{\mathbb{R}}}$ and the obvious compression of T associated with $\widetilde{\mathcal{H}_{\mathbb{R}}}$. By (iii), it follows that $\Gamma_T(\widetilde{\mathcal{H}_{\mathbb{R}}})'' \subseteq \mathbf{B}(\mathcal{F}_T(\widetilde{\mathcal{H}}))$ is not injective.

The rest of the arguments are similar to the arguments in (i) by considering the inclusion $\iota : \widetilde{\mathcal{H}_{\mathbb{R}}} \to \widetilde{\mathcal{H}_{\mathbb{R}}} \oplus (\widetilde{\mathcal{H}_{\mathbb{R}}})^{\perp}$ (direct sum taken with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}$). We omit the details. This completes the proof. ∎

Remark 8.5. The non-injectivity of the mixed q-deformed Araki–Woods von Neumann algebras is thus open only when $(\mathcal{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is almost periodic, $\dim(E_A\{1\}) \leq 1$, and the spectrum of the analytic generator A of $(U_t)_{t\in\mathbb{R}}$ is not thick in the sense as stated in (ii) of Theorem 8.4.

A. Some results on Hilbert spaces

In this section, we include some results concerning Hilbert spaces which are inevitable for our purpose. The results in this section are known, but we lack references. Therefore, we provide it for the sake of completeness.

Proposition A.1. *Let $\mathcal{H}_i$, $i = 1,2$, be Hilbert spaces, and let $P_i : \mathfrak{D}(P_i) \subseteq \mathcal{H}_i \to \mathcal{H}_i$ be densely defined strictly positive self-adjoint operators for $i = 1,2$. Let $B_i : \mathfrak{D}(P_i) \times \mathfrak{D}(P_i) \to \mathbb{C}$ be a sesquilinear form given by $B_i(\eta,\xi) = \langle \eta, P_i\xi\rangle_{\mathcal{H}_i}$, $\xi,\eta \in \mathfrak{D}(P_i)$, for $i = 1,2$. Suppose $\mathcal{H}_{P_i}$ denote the Hilbert space completion of $\mathfrak{D}(P_i)$ with respect to B_i, $i = 1,2$. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator such that $TP_1 \subseteq P_2T$. Then, T admits a unique extension $\widetilde{T} : \mathcal{H}_{P_1} \to \mathcal{H}_{P_2}$ such that $\|\widetilde{T}\| = \|T\|$.*

Proof. We denote the norm of $\mathcal{H}_i$ and $\mathcal{H}_{P_i}$ by $\|\cdot\|_i$ and $\|\cdot\|_{P_i}$, respectively, for $i = 1,2$. Let $\xi \in \mathfrak{D}(P_1)$. Then,

$$\begin{aligned}\|T\xi\|^2_{P_2} &= \langle T\xi, T\xi\rangle_{\mathcal{H}_{P_2}} = \langle T\xi, P_2T\xi\rangle_{\mathcal{H}_2} = \langle \xi, T^*P_2T\xi\rangle_{\mathcal{H}_1}\\ &= \langle \xi, P_1T^*T\xi\rangle_{\mathcal{H}_1} \quad \big(\text{since } TP_1 \subseteq P_2T \text{ and } (TP_1)^* = P_1T^* \text{ [28, §9.2]}\big)\\ &= \big\langle P_1^{\frac{1}{2}}\xi, (T^*T)P_1^{\frac{1}{2}}\xi\big\rangle_{\mathcal{H}_1} \quad \big(\mathfrak{D}(P_1^{\frac{1}{2}}) \supseteq \mathfrak{D}(P_1)\big)\\ &\leq \|T\|^2\langle P_1^{\frac{1}{2}}\xi, P_1^{\frac{1}{2}}\xi\rangle_{\mathcal{H}_1} = \|T\|^2\|\xi\|^2_{P_1}.\end{aligned}$$

From the density of $\mathfrak{D}(P_1)$ in $\mathcal{H}_{P_1}$, it follows that T has a unique extension $\widetilde{T}$ from $\mathcal{H}_{P_1}$ to $\mathcal{H}_{P_2}$, and $\|\widetilde{T}\| \leq \|T\|$.

Further, P_1 and P_2 are invertible and $(P_1)^{-1}$ and $(P_2)^{-1}$ admit bounded extensions. Then, $(P_2)^{-1}TP_1 \subseteq T$, and hence, $(P_2)^{-1}TP_1$ admits bounded extension as well. Thus, $(P_2)^{-1}TP_1 = T$, and hence, $(P_2)^{-1}T = T(P_1)^{-1}$. Further, $((P_2)^{-1})^k T = T((P_1)^{-1})^k$ for $k \in \mathbb{N}$. Thus, for $\xi \in \mathfrak{D}(P_1)$, by functional calculus, one has

$$\begin{aligned}
\|T\xi\|_2^2 &= \langle T\xi, T\xi\rangle_{\mathcal{H}_2} = \big\langle T\xi, (P_2)^{-1}TP_1\xi\big\rangle_{\mathcal{H}_2} \\
&= \big\langle (P_2)^{-\frac{1}{2}}T\xi, (P_2)^{-\frac{1}{2}}TP_1\xi\big\rangle_{\mathcal{H}_2} \\
&= \big\langle T(P_1)^{-\frac{1}{2}}\xi, (P_2)^{-\frac{1}{2}}TP_1\xi\big\rangle_{\mathcal{H}_2} \\
&= \big\langle T(P_1)^{-\frac{1}{2}}\xi, P_2(P_2)^{-\frac{3}{2}}TP_1\xi\big\rangle_{\mathcal{H}_2} \\
&= \big\langle T(P_1)^{-\frac{1}{2}}\xi, P_2T(P_1)^{-\frac{3}{2}}P_1\xi\big\rangle_{\mathcal{H}_2} \\
&= \big\langle T(P_1)^{-\frac{1}{2}}\xi, P_2T(P_1)^{-\frac{1}{2}}\xi\big\rangle_{\mathcal{H}_2} \\
&= \big\|\widetilde{T}(P_1)^{-\frac{1}{2}}\xi\big\|_{\mathcal{H}_{P_2}}^2 \leq \|\widetilde{T}\|^2\big\|(P_1)^{-\frac{1}{2}}\xi\big\|_{\mathcal{H}_{P_1}}^2 \quad (\text{as } \widetilde{T} \text{ is bounded}) \\
&\leq \|\widetilde{T}\|^2\|\xi\|_1^2 \quad \big(\mathfrak{D}(P_1^{\frac{1}{2}}) \supseteq \mathfrak{D}(P_1)\big).
\end{aligned}$$

This completes the proof. ∎

Proposition A.2. *Let $\mathcal{H}_i$, $i = 1, 2, \ldots,$ be Hilbert spaces, and let $P_i : \mathcal{H}_i \to \mathcal{H}_i$ be strictly positive bounded operators for $i = 1, 2, \ldots$. Then, $P = \bigoplus_i P_i$ is a densely defined closed strictly positive self-adjoint operator on $\bigoplus_i \mathcal{H}_i$.*

Proof. Note that $\mathcal{H}_i \subseteq \mathfrak{D}(P)$ for all i. Thus, P is densely defined. Clearly, $\bigoplus_i P_i$ is strictly positive. It is easy to see that P is closed and self-adjoint. ∎

For the next two results, we invoke the construction in [8] of twisted Fock spaces corresponding to arbitrary Yang–Baxter operators.

Let H, K be (complex) Hilbert spaces and T', T'' self-adjoint (strict) contractions satisfying the Yang–Baxter relation on H and K, respectively. For $n \in \mathbb{N}$, let $P_1^{(n)}$ and $P_2^{(n)}$ be positive operators on $H^{\otimes n}$ and $K^{\otimes n}$, respectively, defined similar to $P^{(n)}$ in (10). Let $H^{\otimes_{T'}^n}$ and $K^{\otimes_{T''}^n}$ denote the corresponding twisted tensor products as in Section 2.

Proposition A.3. *Fix $n \in \mathbb{N}$. Let $\mathcal{T} : H^{\otimes n} \to K^{\otimes n}$ be a bounded linear operator such that $\mathcal{T}P_1^{(n)} = P_2^{(n)}\mathcal{T}$. Then, $\mathcal{T}$ admits a unique bounded extension $\widetilde{\mathcal{T}} : H^{\otimes_{T'}^n} \to K^{\otimes_{T''}^n}$. In addition, $\|\mathcal{T}\| = \|\widetilde{\mathcal{T}}\|$.*

Proof. Recall that $P_1^{(n)}$ and $P_2^{(n)}$ are strictly positive on $H^{\otimes n}$ and $K^{\otimes n}$, respectively. The rest is immediate from Proposition A.1. ∎

Proposition A.4. *Let H and K be Hilbert spaces, and let $V : H \to K$ be a contraction. Let T_1 and T_2 be self-adjoint contractions satisfying the Yang–Baxter relations defined on*

$H \otimes H$ and $K \otimes K$, respectively. Suppose that $(V \otimes V)T_1 = T_2(V \otimes V)$. Then, the map $\mathcal{F}(V) : \mathcal{F}_{T_1}(H) \to \mathcal{F}_{T_2}(K)$ defined by extending

$$\mathcal{F}(V)(\Omega_H) = \Omega_K,$$
$$\mathcal{F}(V)(\eta_1 \otimes \cdots \otimes \eta_n) = V\eta_1 \otimes \cdots \otimes V\eta_n, \quad \eta_i \in H \text{ for } 1 \leq i \leq n,\ n \in \mathbb{N},$$

is a contraction, where Ω_H and Ω_K are the distinguished vacuum vectors in H and K, respectively. Moreover, $\|\mathcal{F}(V)\| \leq 1$.

Proof. Fix $n \in \mathbb{N}$. Let $P_1^{(n)}$ and $P_2^{(n)}$ denote the strictly positive operators on $H^{\otimes n}$ and $K^{\otimes n}$ associated with T_1 and T_2, respectively (as defined in (10)). Since $(V \otimes V)T_1 = T_2(V \otimes V)$, it follows that $(V^{\otimes n})P_1^{(n)} = P_2^{(n)}(V^{\otimes n})$.

Then, by Proposition A.3, it follows that $\mathcal{F}(V)_{\restriction H^{\otimes n}}$ extends uniquely to a bounded operator from $H^{\otimes_{T_1}^n}$ to $K^{\otimes_{T_2}^n}$ of norm $\|V\|^n$. Since V is a contraction, the rest is immediate. ∎

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