

Conormal homology of manifolds with corners

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Abstract. Given a manifold with corners X , we associate to it the corner structure simplicial complex Σ_X . Its reduced K-homology is isomorphic to the K-theory of the C^* -algebra $\mathcal{K}_b(X)$ of b-compact operators on X . Moreover, the homology of Σ_X is isomorphic to the conormal homology of X .

In this note, we construct for an arbitrary abstract finite simplicial complex Σ a manifold with corners X such that $\Sigma_X \cong \Sigma$. As a consequence, the homology and K-homology which occur for finite simplicial complexes also occur as conormal homology of manifolds with corners and as K-theory of their b-compact operators. In particular, these groups can contain torsion.

1. Introduction

In this note, we contribute to the index theory and homology of compact manifolds with corners. More specifically, a fundamental question in this field asks for obstructions to the Fredholm property for boundary value problems, initiated in [3]. Let X be such a manifold with corners. The geometrically relevant boundary value problems (and their inverses) are contained in the algebra of b-pseudodifferential operators as introduced by [7]. Given such an operator which is *elliptic* (has an elliptic principal symbol), it is in general not true that the operator is Fredholm.

In this situation, one asks if one can find a smoothing perturbation (possibly after stabilization) to render the given operator Fredholm. If this is possible, the operator satisfies the stable Fredholm perturbation (SFP) property. As one can guess, not all operators have the SFP: it is proven in [8] that the obstruction to this is precisely the boundary analytic index which takes values in $K_*(\mathcal{K}_b(\partial X))$, where $\mathcal{K}_b(\partial X)$ is the C^* -algebra of b-compact operators on the boundary of X (the C^* -algebra $\mathcal{K}_b(\partial X)$ is defined as the quotient $\mathcal{K}_b(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the C^* -algebra of compact operators on $L^2(X)$; for details see [3]). A modern proof of the same fact, using deformation groupoids, is given in [10]. By [3, Proposition 5.6], the restriction to the boundary induces an isomorphism $K_*(\mathcal{K}_b(X)) \rightarrow K_*(\mathcal{K}_b(\partial X))$ provided that $\partial X \neq \emptyset$.

It turns out that the relevant group $K_*(\mathcal{K}_b(X))$ depends only on the combinatorics of the faces of the manifold with corners X and how they intersect. Indeed, one essentially can compute these K-groups combinatorially. To this end, Bunke [2] introduced the *conor-*

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mal homology of X (called differently in [2]), computed from a chain complex generated abstractly by the faces of X . In [3], it is shown that this conormal homology contains the obstructions to SFP if the corners in X are of codimension ≤ 3 .

More systematically, in [4] a natural Chern character

$$K_*(\mathcal{K}_b(X)) \rightarrow H_{*+2\mathbb{Z}}^{\text{cn}}(X) \otimes \mathbb{Q} \tag{1}$$

is constructed and proven to be rationally an isomorphism.

In the situation studied in [3], it is useful if the conormal homology is torsion free. There and in [4], the authors therefore ask if the conormal homology always is torsion free, and whether perhaps the Chern character (1) can be improved to an integral isomorphism.

The main goal of the present paper is to provide examples of manifolds with corners which show that the conormal homology does not have any of these nice properties. Instead, it can be as rich as the homology of an arbitrary finite simplicial complex. The same applies to the K-theory of $\mathcal{K}_b(X)$.

More specifically, in the present paper, we associate to a manifold with embedded corners X its *corner structure complex*, a simplicial complex Σ_X encoding the *corner structure* in such a way that the reduced K-homology of Σ_X is isomorphic to the topological K-theory of the C^* -algebra $\mathcal{K}_b(X)$, and the conormal homology of X is isomorphic to the homology of Σ_X . We then prove that every finite simplicial complex can be realized as Σ_X of some manifold with embedded corners X .

In particular, this implies that the conormal homology defined in [3,4] in general does contain torsion.

2. The corner structure complex of a manifold with corners

Let us recall the definition of a smooth manifold with embedded corners. We adopt the approach where a manifold with corners is defined as a suitable subset of a smooth ordinary manifold (without boundary).

Definition 2.1. A compact smooth manifold with embedded corners X is defined in the following way. Start with a compact smooth manifold \tilde{X} (without boundary) and with smooth maps $\rho_0, \dots, \rho_n: \tilde{X} \rightarrow \mathbb{R}$. Set

$$H_j = \rho_j^{-1}(0) \cap \tilde{X}, \quad j = 0, \dots, n, \quad H_{j_1, \dots, j_k} := H_{j_1} \cap \dots \cap H_{j_k}$$

such that

$$\{d\rho_{j_1}, \dots, d\rho_{j_k}\} \text{ has maximum rank at each } x \in H_{j_1, \dots, j_k} \quad \forall \{j_1, \dots, j_k\} \subset \{0, \dots, n\}.$$

This defines the manifold with corners X as follows:

- (1) $X := \bigcap_{j=1}^n \rho_j^{-1}([0, +\infty)) \subseteq \tilde{X}$;
- (2) each H_j is called a *boundary component of codimension 1*;

- (3) if $|\{j_1, \dots, j_k\}| = k$, then we call $H_{j_1} \cap \dots \cap H_{j_k}$ a *face of codimension k* ;
- (4) we denote by F_k the set of faces of codimension k .

Throughout the paper, we assume that each face $H_{j_1} \cap \dots \cap H_{j_k}$ of arbitrary codimension is connected, in particular each boundary component of codimension 1.

Definition 2.2. Let X be a manifold with embedded corners with $n + 1$ boundary components H_0, \dots, H_n of codimension 1. Define the *corner structure complex* as the abstract simplicial complex Σ_X associated to X with vertex set $\{H_0, \dots, H_n\}$ with the following simplices: for $A \subseteq \{H_0, \dots, H_n\}$

$$A \in \Sigma_X \quad \text{if and only if} \quad \bigcap_{H_i \in A} H_i \neq \emptyset.$$

It is clear that Σ_X is closed under inclusions and therefore is an abstract simplicial complex. Observe that an abstract simplicial complex as considered here is a set, the vertex set (here $\{H_0, \dots, H_n\}$), together with a collection of finite subsets (here Σ_X), called the simplices, which is closed under taking subsets (if $\tau \subset \sigma$, then τ is called a face of σ).

Every abstract simplicial complex Σ has a *geometric realization* denoted by

$$|\Sigma| := \left(\bigcup_{\sigma \in \Sigma} \sigma \times \Delta^{|\sigma|-1} \right) / \sim .$$

Here, Δ^k is the standard k -simplex and we glue according to the face relation in the abstract simplicial complex. For details, compare (3) or consult [9, Theorem 7.8].

- Example 2.3.**
- (1) If X is a smooth manifold without boundary, then $\Sigma_X = \emptyset$.
 - (2) Let $(Y, \partial Y)$ be a connected manifold with non-empty connected boundary. Then Σ_Y is a point.
 - (3) Let X be a connected manifold with boundary with n boundary components. Then Σ_X is the simplicial complex with n points and no edges.

More generally, we have the following result

Lemma 2.4. *Let Y_1 and Y_2 be manifolds with embedded corners, then $\Sigma_{Y_1 \times Y_2}$ is isomorphic to the join $\Sigma_{Y_1} * \Sigma_{Y_2}$.*

Proof. Recall that the join of two abstract simplicial complexes Σ_1, Σ_2 has vertex set the disjoint union $V(\Sigma_1) \amalg V(\Sigma_2)$, where $V(\Sigma_j)$ is the vertex set of Σ_j for $j = 1, 2$. A subset $A \subset V(\Sigma_1) \amalg V(\Sigma_2)$ is a simplex of $\Sigma_1 * \Sigma_2$ if and only if $A = A_1 \amalg A_2$ with $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$.

Let \tilde{Y}_1 be a smooth manifold and let $\rho_1, \dots, \rho_n: \tilde{Y}_1 \rightarrow \mathbb{R}$ be smooth maps defining the manifold with corners Y_1 . In the same way, let \tilde{Y}_2 and $\rho_{n+1}, \dots, \rho_{n+m}: \tilde{Y}_2 \rightarrow \mathbb{R}$ define Y_2 . Then the smooth manifold $\tilde{Y}_1 \times \tilde{Y}_2$ with the smooth maps $\rho_1 \circ \pi_1, \dots, \rho_n \circ \pi_1, \rho_{n+1} \circ \pi_2, \dots, \rho_{n+m} \circ \pi_2: \tilde{Y}_1 \times \tilde{Y}_2 \rightarrow \mathbb{R}$ define $Y_1 \times Y_2$ as a manifold with embedded cor-

ners. Here $\pi_j: Y_1 \times Y_2 \rightarrow Y_j, j = 1, 2$, are the projections. We denote the boundary components of Y_1 by $H_j, j = 1, \dots, n$, and the boundary components of Y_2 by $H_j, j = n + 1, \dots, n + m$. Then the set of boundary components of $Y_1 \times Y_2$ is $\{H_1 \times Y_2, \dots, H_n \times Y_2, Y_1 \times H_{n+1}, \dots, Y_1 \times H_{n+m}\}$ and clearly $\Sigma_{Y_1 \times Y_2}$ satisfies the conditions to be the join $\Sigma_{Y_1} * \Sigma_{Y_2}$. ■

Note that the geometric realization of the join of simplicial complexes is the topological join of the geometric realizations of the individual simplicial complexes. The above lemma gives more examples as follows:

- Example 2.5.** (1) Let $(Y, \partial Y)$ be a connected manifold with non-empty connected boundary and set $X = Y^n$. This is a manifold with embedded corners and $\Sigma_X = \Delta_n$ is the n -simplex.
- (2) We can also directly construct an $(n + 2)$ -dimensional manifold A with embedded corners (all of whose faces are connected) such that $\Sigma_A \cong \Delta_n$. For this aim, start with $\tilde{A} := S^{n+2}$, but decompose

$$S^{n+2} = \partial(D^{n+3}) = \partial(D^2 \times D^{n+1}) = S^1 \times D^{n+1} \cup_{S^1 \times S^n} D^2 \times S^n.$$

This way, we have an obvious projection map to the second factor $\pi: S^{n+2} \rightarrow D^{n+1}$, with $n + 1$ component functions $\rho_1, \dots, \rho_{n+1}: \tilde{A} \rightarrow \mathbb{R}$. Then we set

$$A := \bigcup_{j=1}^{n+1} \{\rho_j \geq 0\} = S^1 \times Q^{n+1} \cup_{S^1 \times (S^n \cap Q^{n+1})} D^2 \times (S^n \cap Q^{n+1}),$$

where we write $Q^{n+1} := \{(x_1, \dots, x_{n+1}) \in D^{n+1} \mid x_j \geq 0, \forall j\}$, the positive hyperquadrant sector. Then A is a manifold with embedded corners, where clearly all the faces are connected and as $\bigcap_{j=1}^{n+1} \{\rho_j = 0\} = S^1 \times \{(0, \dots, 0)\} \neq \emptyset$, according to Definition 2.2 we have $\Sigma_A \cong \Delta_n$, as desired.

Our first main result is constructive: for every finite simplicial complex we can construct a manifold with corners. A more general result will be proved in Theorem 2.11.

Theorem 2.6. *Let Σ be an arbitrary finite simplicial complex. Then there exists a compact manifold with embedded corners X such that Σ_X is isomorphic to Σ .*

As a preparation for the proof, we introduce further definitions.

Definition 2.7. Let X be a manifold with embedded corners, let $x_0 \in X$, and let V_0 be a coordinate neighborhood around x_0 . Then V_0 is a manifold with embedded corners and its corner structure complex Σ_{V_0} is called the simplicial complex of X around x_0 . Given points $x_0 \in X$ and $y_0 \in Y$, where X and Y are manifolds with embedded corners of the same dimension, we say that x_0 and y_0 have the same local corner structure if the corner structure complexes of X around x_0 and of Y around y_0 are isomorphic.

Now we will define the connected sum around 0-dimensional submanifolds. This will be a key ingredient in the constructions required to prove Theorem 2.6.

Let X and Y be manifolds with embedded corners of dimension n and let

$$\{x_1, \dots, x_m\} \subseteq X \quad \text{and} \quad \{y_1, \dots, y_m\} \subseteq Y$$

be finite subsets of X and Y , respectively, such that for $i = 1, \dots, m$, x_i and y_i have the same local corner structure. By results in [6], there are tubular neighborhoods V and W with

$$\{x_1, \dots, x_m\} \subseteq V \subseteq X \quad \text{and} \quad \{y_1, \dots, y_m\} \subseteq W \subseteq Y.$$

We have

$$V \cong V_1 \sqcup \dots \sqcup V_m \quad \text{and} \quad W \cong W_1 \sqcup \dots \sqcup W_m,$$

where V_i is a coordinate neighborhood of x_i and W_i is a coordinate neighborhood of y_i . Then we can choose identifications

$$V_i = (-1, 1)^{n_i} \times [0, 1)^{n-n_i} \subset (-1, 1)^{n_i} \times (-1, 1)^{n-n_i} =: \widetilde{V}_i$$

and

$$W_i = (-1, 1)^{m_i} \times [0, 1)^{n-m_i} \subset \widetilde{W}_i.$$

In these coordinates, the face defining functions ρ_j of Definition 2.1 are just the coordinate functions for the closed intervals.

Moreover, as x_i and y_i have the same local corner structure, we have $n_i = m_i$.

We now follow the description of the ordinary connected sum in [5] to define our connected sum of manifolds with corners.

Definition 2.8. Define the *connected sum of X and Y along the subsets $\{x_0, \dots, x_m\}$ and $\{y_0, \dots, y_m\}$* as follows:

$$\begin{aligned} & (X, \{x_0, \dots, x_m\}) \sharp (Y, \{y_0, \dots, y_m\}) \\ & := \frac{(X - \{x_0, \dots, x_m\}) \sqcup (Y - \{y_0, \dots, y_m\})}{tz \in V_i \sim (1-t)z \in W_i} \end{aligned} \tag{2}$$

for every $z \in S^{n-1}$. When $\{x_0, \dots, x_m\}$ and $\{y_0, \dots, y_m\}$ are clear from the context, we denote the connected sum by $X \sharp Y$.

For the sake of completeness, let us give the details of how it is a straightforward verification that $X \sharp Y$ is a manifold with embedded corners. To do so, we have to define the ambient smooth ordinary manifold and the boundary defining functions.

Of course, the ambient manifold for the connected sum is defined as the connected sum of the ambient manifolds \widetilde{X} of X and \widetilde{Y} of Y , defined precisely with the same formula as (2).

The boundary defining function ρ_j remains unchanged outside the set $V_i \sqcup W_i$ but has to be modified in the coordinate regions where the identifications are carried out. We redefine ρ_j on $V_i \sqcup W_i$ by

$$\rho_j(x) := \phi_x(|x|) \cdot x_j,$$

with $\phi_x: (0, 1) \rightarrow (0, 1)$ a smooth monotonously decreasing function with

$$\phi_x(t) = \begin{cases} 1, & t > 0.6, \\ \frac{1-t}{t}, & t < 0.4. \end{cases}$$

In the coordinates of W_i , we have for $y \in W_i$ that $y \sim x := \frac{1-|y|}{|y|}y \in V_i$ and therefore

$$\rho_j(y) = \begin{cases} \frac{1-|y|}{|y|}y_j, & 1 - |y| > 0.6 \iff |y| < 0.4, \\ \frac{1-(1-|y|)}{|y|} \frac{1-|y|}{|y|}y_j = y_j, & |y| > 0.6. \end{cases}$$

We observe that we indeed defined a smooth function on the connected sum of \tilde{X} and \tilde{Y} as the expressions are unchanged on the “outer” part of U_i or V_i , where the norm is > 0.6 .

Remark 2.9. The connected sum of two manifolds with corners X and Y will again be a manifold with corners. However, even if all faces of arbitrary codimension of X and of Y are connected, in general this will not be the case for $X \# Y$.

Lemma 2.10. *Assume that X is a compact manifold of dimension d with embedded corners (all faces connected). Assume that we have an embedding $\iota: \partial\Delta_n \hookrightarrow \Sigma_X$, but the simplex spanned by the vertices in the image of ι is not contained in Σ_X .*

Then, for each $d' \geq \max\{d, n + 2\}$, there is a compact smooth manifold Z with embedded corners (all faces connected) such that $\Sigma_Z \cong \Sigma_X \cup_\iota \Delta_n$; i.e. Σ_Z is isomorphic to the simplicial complex obtained from Σ_X by adding the simplex spanned by the image of ι .

Proof. We write $\Delta_n = \mathcal{P}(\{0, \dots, n\})$, the power set of $\{0, \dots, n\}$. Taking the product with a smooth connected manifold, a process which does not change the corner structure, we can assume that $\dim(X) = d'$. Let $H_k \subset X$ be the boundary component which corresponds to $\iota(k)$ for $k = 0, \dots, n$, and set

$$H_{k^c} := \bigcap_{j \neq k} H_j,$$

the face of X of codimension n corresponding to $\{0, \dots, n\} \setminus \{k\} \subset \partial\Delta_n$. By the assumption that $\iota(\partial\Delta_n) \subset \Sigma_X$, we have $H_{k^c} \neq \emptyset$ and $\dim(H_{k^c}) = d' - (n - 1) > 0$ for all k . Pick then $x_k \in (H_{k^c})^\circ$. The local corner structure around x_k is precisely the $(n - 1)$ -simplex spanned by $\{0, \dots, n\} \setminus \{k\}$.

Choose a d' -dimensional manifold M with embedded corners (all faces connected) with a fixed isomorphism $\Delta_n \xrightarrow{\cong} \Sigma_M$. For this, start with A of Example 2.3 (2) and take the Cartesian product with a connected smooth manifold to adjust the dimension. Given the fixed isomorphism $\Sigma_M \cong \Delta_n$, let H_k^M be the boundary face corresponding to the vertex k of Δ_n and set

$$H_{k^c}^M := \bigcap_{j \neq k} H_j^M.$$

Note that $\dim(H_{k^c}^M) = d' - n > 0$ and pick distinct $y_k \in (H_{k^c}^M)^\circ$, with local corner structure the $(n - 1)$ -simplex spanned by $\{0, \dots, n\} \setminus \{k\}$.

Define now $Z := (X; x_0, \dots, x_n) \# (M; y_0, \dots, y_n)$, using at each point the given identifications of the local corner structure with a determined $(n - 1)$ -dimensional sub-simplex of $\Delta_n = \mathcal{P}(\{0, \dots, n\})$. This way, the face H_k is glued near the n points x_j with $j \in \{0, \dots, n\} \setminus \{k\}$ to the face H_k^M . Moreover, for each $0 \leq k_0 < \dots < k_\alpha \leq n$ the face $H_{k_0} \cap \dots \cap H_{k_\alpha}$ is glued (at $n - \alpha$ points) to the intersection face $H_{k_0}^M \cap \dots \cap H_{k_\alpha}^M$. It follows that each face of arbitrary codimension of Z remains connected. The boundary faces of Z are in obvious bijection with those of X and the corner complex is a simplicial complex with vertex set the one of Σ_X . As $H_0^M \cap \dots \cap H_n^M \neq \emptyset$, taking everything together, we now get $\Sigma_Z = \Sigma_X \cup_{\iota(\partial\Delta_n)} \iota(\Delta_n)$. ■

Theorem 2.11. *Let K be a finite simplicial complex and let n be the maximal dimension of a simplex in K . If $d \geq n + 2$, then there is a compact d -dimensional manifold with embedded corners X (all faces connected) such that*

$$\Sigma_X \cong K.$$

Proof. The result follows by induction on the number of positive dimensional simplices in K .

By Example 2.3 (3) any 0-dimensional simplicial complex satisfies the result.

If K contains a positive dimensional simplex, let σ be one such of maximal dimension. Set $K' := K \setminus \{\sigma\}$. Then $K = K' \cup_{\partial\sigma} \sigma$. By induction, there exists the required X' with $\Sigma_{X'} \cong K'$ and by Lemma 2.10 we then can also construct the required X with $\Sigma_X \cong K$. ■

3. Groupoids and the space O_X

Let X be a manifold with embedded corners. In [4], for sufficiently large m a non-compact topological space O_X is introduced such that we have a Connes–Thom isomorphism

$$CT: K_*(C^*(\mathcal{K}_b(X))) \xrightarrow{\cong} K^{m+*}(O_X).$$

The space O_X is constructed as the orbit space of a free and proper groupoid. We recall this construction briefly.

Let X be defined by the smooth manifold \tilde{X} and the defining functions ρ_1, \dots, ρ_n ; i.e.

$$\bigcap_{j=1}^n \{\rho_j \geq 0\} =: X \subseteq \tilde{X}.$$

The puff groupoid is defined as a subgroupoid of $\tilde{X} \times \tilde{X} \times \mathbb{R}^n$, where $\tilde{X} \times \tilde{X}$ is the arrow space of the pair groupoid and where \mathbb{R}^n is the additive group (groupoid with one object). The arrow space of the puff groupoid is then defined as

$$G(\tilde{X}, (\rho_i)) := \{(x, y, \lambda_1, \dots, \lambda_n) \in \tilde{X} \times \tilde{X} \times \mathbb{R}^n \mid \rho_i(x) = e^{\lambda_i} \rho_i(y)\}.$$

Denote by $G_c(\tilde{X}, (\rho_i))$ the s-connected component of $G(\tilde{X}, (\rho_i))$.

Choose for sufficiently large m an embedding

$$\iota: \tilde{X} \rightarrow \mathbb{R}^{m-n}.$$

Using this embedding, in [4, Section 3] the authors construct a new groupoid

$$\mathbb{R}^m \rtimes_t G_c(\tilde{X}, (\rho_i)_i) \rightrightarrows \mathbb{R}^m \times \tilde{X}.$$

By [4, Proposition 3.1], this groupoid is a free and proper Lie groupoid (this uses that all faces of each codimension are connected). Hence the orbit space

$$O_{\tilde{X}} = \text{Orb}(G_c(\tilde{X}, (\rho_i)_i) \rtimes_h \mathbb{R}^M) = \mathbb{R}^m \times \tilde{X} / \sim$$

has a natural structure of smooth manifold. Decomposing $v \in \mathbb{R}^m$ as $v = (v', v'') \in \mathbb{R}^{m-n} \times \mathbb{R}^n$, set

$$\begin{aligned} \tilde{\rho}_i: O_{\tilde{X}} &\rightarrow \mathbb{R}, \\ [(v, x)] &\rightarrow \rho_i(x)e^{v''}. \end{aligned}$$

By [4, Section 3], these maps are indeed well defined and determine a manifold with corners O_X .

Definition 3.1. We denote

$$O_X = \bigcap_{i=1}^n \{\tilde{\rho}_i \geq 0\}.$$

In [4], it is verified that O_X is a manifold with embedded corners with defining functions $\tilde{\rho}_1, \dots, \tilde{\rho}_n$.

The main result in [4] is the following.

Theorem 3.2. *There is a Connes–Thom isomorphism*

$$CT_h: K_*(C^*(\mathcal{K}_b(X))) \rightarrow K^{m+*}(O_X).$$

Here $K^*(O_X)$ denotes topological K -theory with compact support.

4. Relation between Σ_X and O_X

In [4, Section 3C], the authors construct a filtration of the space O_X ,

$$Y_0 \subseteq Y_1 \subseteq \dots \subseteq Y_m = O_X.$$

Moreover, in [4, Proposition 3.27] they prove for each q that

$$Y_q \setminus Y_{q-1} \cong \bigcup_{\substack{f \in \text{faces of } X \\ N-q \leq \text{codim}(f) \leq N}} \mathbb{R}_f^m.$$

Here \mathbb{R}_f^m is a certain subspace of \mathbb{R}^m , and the construction implies that there is a homeomorphism from O_X to a subspace of \mathbb{R}^m which is entirely determined by the combinatorics of the corner structure of X . We will now describe in detail this homeomorphism.

Canonically, \tilde{O}_X is defined as a subset of $\mathbb{R}^{m-|V|} \times \mathbb{R}^V$, where V is the set of boundary faces of X , i.e. the vertex set of Σ_X . We denote the standard basis vectors $\delta_H \in (\mathbb{R}_{\geq 0})^V$ for $H \in V$. Note that then $|\Delta(V)| \subset \mathbb{R}^V$ is the convex hull of the $\delta_H, H \in V$.

To define \tilde{O}_X , for every face F of X set

$$B_F := \{x \in \mathbb{R}^V \mid x_H = 0 \text{ if } F \subset H, x_H > 0 \text{ if } F \cap H = \emptyset \iff F \not\subset H\}.$$

Recall that we have a correspondence between the faces F and the subsets $\sigma \in \Sigma_X \subset \mathcal{P}(V)$, namely $F = \bigcap_{H \in \sigma} H$, and B_F is the open positive quadrant “spanned” by all δ_H with $H \notin \sigma$.

Define then

$$\tilde{O}_X := \mathbb{R}^{m-|V|} \times \bigcup_{F \text{ face of } X} B_F \subset \mathbb{R}^{m-|V|} \times \mathbb{R}^V$$

equipped with the subspace topology. Note that, as sets, the union is disjoint. The computation in [4, Section 3] then gives a homeomorphism between O_X and \tilde{O}_X .

Note that $\Sigma_X = \Delta(V)$ is the full simplex spanned by V if and only if $\tilde{O}_X = \mathbb{R}^{m-|V|} \times (\mathbb{R}_{\geq 0})^V$. Otherwise, $0 \notin \tilde{O}_X$ and we have a homeomorphism

$$\tilde{O}_X \cong \mathbb{R}^{m-|V|} \times \mathbb{R}_{>0} \times P_X,$$

where the factor $\mathbb{R}_{>0}$ is the norm (radial variable) and

$$P_X = |\Delta(V)| \cap \tilde{O}_X \subset (\mathbb{R}_{\geq 0})^V.$$

Observe that, if $F = \bigcap_{H \in \sigma} H$ then $B_F \cap |\Delta(V)|$ is the interior of the convex hull $F_{V \setminus \sigma}$ of the δ_H for $H \notin \sigma$:

$$B_F \cap |\Delta(V)| = (F_{V \setminus \sigma})^\circ.$$

Now

$$|\Delta(V)|^\circ \subset P_X \subset |\Delta(V)|;$$

i.e. $|\Delta(V)|$ is a compactification of P_X . This implies that we get for the one-point compactification $(P_X)^+$

$$(P_X)^+ \cong |\Delta(V)| / (|\Delta(V)| \setminus P_X);$$

i.e. we identify all the missing points in the compactification $|\Delta(V)|$ of P_X to one point.

We next show that $|\Delta(V)| \setminus P_X$ is the geometric realization of the dual of the simplicial complex Σ_X . First recall that the geometric realization of a simplicial complex K with vertex set V can be defined as follows.

If $I \subseteq V$, set

$$F_I := \text{closed convex hull of } \delta_H, \quad H \in I.$$

Then

$$|K| = \bigcup_{I \in K} F_I = \coprod_{I \in K} (F_I)^\circ \subset \mathbb{R}^K. \tag{3}$$

Here we define the open face $F_I^\circ := F_I \setminus \partial F_I$, in particular $F_{\{H\}} = \{\delta_H\}$ is the singleton (not the empty set).

Let us now also recall the definition of the dual of a simplicial complex.

Definition 4.1. Let K be a simplicial complex with vertex set V . Then the (Alexander) dual of K is

$$K^\vee = \{A \subset V \mid V \setminus A \notin K\}.$$

Note that, in particular,

$$|\Delta(V)| = \coprod_{\sigma \subset V} (F_\sigma)^\circ. \tag{4}$$

Lemma 4.2. *We have*

$$|\Sigma_X^\vee| = |\Delta(V)| \setminus P_X.$$

Proof. By definition,

$$|\Sigma_X^\vee| = \coprod_{\sigma \in \Sigma_X^\vee} (F_\sigma)^\circ = \coprod_{V \setminus \sigma \notin \Sigma_X} (F_\sigma)^\circ. \tag{5}$$

On the other hand, as observed above,

$$P_X = \coprod_{\sigma \in \Sigma_X} (F_{V \setminus \sigma})^\circ = \coprod_{V \setminus \sigma \in \Sigma_X} (F_\sigma)^\circ. \tag{6}$$

The three decompositions (4), (5), and (6) prove the claim. ■

Then we have proved the following result.

Theorem 4.3. *Let X be a manifold with embedded corners (all faces connected) such that Σ_X is not the full simplex on the vertex set V with embedding $\tilde{X} \rightarrow \mathbb{R}^{m-|V|}$ as above. Then there is a homeomorphism*

$$O_X \cong \mathbb{R}^{m-|V|+1} \times (|\Delta(V)| \setminus |\Sigma_X^\vee|),$$

where $|\Sigma_X^\vee|$ denotes the geometric realization of the dual of Σ_X .

To prove the main result of this note we need to recall the Spanier–Whitehead duality theorem; for a proof see [1].

Theorem 4.4. *Let E^* be a generalized cohomology theory with dual homology theory E_* . Let $|K| \subsetneq |\partial\Delta(V)|$ be the geometric realization of a simplicial complex with vertex set V consisting of $|V|$ elements, so that, in particular,*

$$\dim(\partial|\Delta(V)|) = |V| - 2.$$

Then there is a canonical isomorphism

$$\tilde{E}^r(|K|) \cong \tilde{E}_{|V|-3-r}(|K^\vee|).$$

Applying this to K-theory and to $|\Sigma_X|$, we obtain the following theorem.

Theorem 4.5. *Let X be a manifold with embedded corners with associated simplicial complex $\Sigma_X \neq \Delta(V)$ on the vertex set V . Then we have a canonical isomorphism*

$$K_*(C^*(\mathcal{K}_b(X))) \rightarrow \tilde{K}_{-* - 1}(|\Sigma_X|).$$

Proof. We already know by Theorem 3.2 that there is an isomorphism

$$K_*(C^*(\mathcal{K}_b(X))) \rightarrow K^{m+*}(O_X).$$

On the other hand, by definition

$$K^*(O_X) \cong \tilde{K}^*(O_X^+),$$

where O_X^+ is the one-point compactification. But

$$\begin{aligned} O_X^+ &\cong S^{m-|V|+1} \wedge P_X^+ \\ &\cong S^{m-|V|+1} \wedge (|\Delta(V)| / (|\Delta(V)| \setminus P_X)) \\ &= S^{m-|V|+1} \wedge (|\Delta(V)| / (|\Sigma_X^\vee|)). \end{aligned}$$

We, therefore, get

$$\begin{aligned} K^{m+*}(\tilde{O}_X) &= \tilde{K}^{m+*}(\tilde{O}_X^+) \\ &\cong \tilde{K}^{*+|V|-1}(|\Delta(V)| / |\Sigma_X^\vee|) \\ &\cong \tilde{K}^{*+|V|-2}(|\Sigma_X^\vee|) \\ &\cong \tilde{K}_{-* - 1}(|\Sigma_X|). \end{aligned}$$

Here, the first isomorphism is the definition of compactly supported K-theory, the second is the suspension isomorphism, the third is the boundary map in the long exact pair sequence, using that $|\Delta(V)|$ is contractible and the last is Spanier–Whitehead duality. ■

We get the corresponding result for conormal homology $H_*^{\text{cn}}(-)$ defined in [3]. To prove this, one could construct a direct correspondence between the chain complex which defines conormal homology and the simplicial chain complex of Σ_X . We use the shortcut that in [4, Corollary 4.2] it is already established that $H_*^{\text{cn}}(X) \cong H^{m-*}(O_X)$. Combined with the argument above, applied to ordinary (co)homology instead of K-theory, we obtain the final result of this note.

Theorem 4.6. *Let X be a manifold with embedded corners (all faces connected) with associated simplicial complex Σ_X . Then we have an isomorphism*

$$H_*^{\text{cn}}(X) \rightarrow \tilde{H}_{* - 1}(|\Sigma_X|).$$

In particular, Theorem 2.11 implies that for every finite abelian group there are examples of conormal homology groups of manifolds with corner containing that torsion group as subgroup.

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