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Bordism, rho-invariants and the Baum–Connes conjecture

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Abstract. Let Γ be a finitely generated discrete group. In this paper we establish vanishing results for rho-invariants associated to

(i) the spin Dirac operator of a spin manifold with positive scalar curvature and fundamental group Γ;

(ii) the signature operator of the disjoint union of a pair of homotopy equivalent oriented manifolds with fundamental group Γ.

The invariants we consider are more precisely

- the Atiyah–Patodi–Singer ($\equiv$ APS) rho-invariant associated to a pair of finite dimensional unitary representations $\lambda_1, \lambda_2 \colon \Gamma \to \mathrm{U}(d)$,
- the L^2-rho-invariant of Cheeger–Gromov,
- the delocalized eta-invariant of Lott for a non-trivial conjugacy class of Γ which is finite.

We prove that all these rho-invariants vanish if the group Γ is *torsion-free* and the Baum–Connes map for the maximal group C*-algebra is bijective. This condition is satisfied, for example, by torsion-free amenable groups or by torsion-free discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. For the delocalized invariant we only assume the validity of the Baum–Connes conjecture for the *reduced* C*-algebra. In addition to the examples above, this condition is satisfied e.g. by Gromov hyperbolic groups or by cocompact discrete subgroups of $\mathrm{SL}(3,\mathbb{C})$.

In particular, the three rho-invariants associated to the signature operator are, for such groups, *homotopy invariant*. For the APS and the Cheeger–Gromov rho-invariants the latter result had been established by Navin Keswani. Our proof reestablishes this result and also extends it to the delocalized eta-invariant of Lott. The proof exploits in a fundamental way results from bordism theory as well as various generalizations of the APS-index theorem; it also embeds these results in general vanishing phenomena for degree zero higher rho-invariants (taking values in $A/\overline{[A,A]}$ for suitable C*-algebras A). We also obtain precise information about the eta-invariants in question themselves, which are usually much more subtle objects than the rho-invariants.

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Contents

1. Introduction and main results

Throughout this paper, M is an oriented closed manifold of odd dimension $2n-1$. Let $u\colon M \to B\Gamma$ be a continuous map, classifying a normal Γ-covering $\widetilde{M}$ of M. Let $(M', u'\colon M' \to B\Gamma)$ be a different oriented Γ-covering. We shall say that $(M', u'\colon M' \to B\Gamma)$ and $(M, u\colon M \to B\Gamma)$ are oriented Γ-*homotopy equivalent* if there exists an oriented homotopy equivalence $h\colon M \to M'$ such that $u' \circ h$ is homotopic to u.

1.1. Atiyah–Patodi–Singer rho-invariant. Let $\lambda_1, \lambda_2\colon \Gamma \to \mathrm{U}(d)$ be two finite dimensional unitary representations of Γ of the same dimension. Equivalently, $\lambda_1 - \lambda_2$ is a virtual representation in the representation ring of Γ of dimension 0. The representations λ_1, λ_2 induce flat bundles L_1 and L_2 on M endowed with flat unitary connections: if V_j denotes the representation space of λ_j, then $L_j := \widetilde{M} \times_\Gamma V_j$ and the unitary connection is induced from the trivial connection on the product $\widetilde{M} \times L_j$.

Let D be a Dirac-type operator acting on the sections of a Clifford module E. We shall always assume that the Clifford module is unitary and endowed with a unitary Clifford connection ∇^E. The operator D can be twisted with the flat bundles L_1

and L_2. We use the notation D_{L_j}, but also D_{λ_j}, for the twisted operator, acting on sections of $E \otimes L_j$.

The usual integral defines the eta-invariant of the twisted operator

$$\eta(D_{L_j}) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2}\,\mathrm{Tr}(D_{L_j} \exp(-(sD_{L_j}^2))\, ds. \tag{1.1}$$

Convergence near $s = 0$ is ensured by our assumptions on E and ∇^E, see [5]. In particular, if M is equipped with a spin structure and $D := \not{D}$ is the associated Dirac operator, then $\eta(\not{D}_{L_j})$ is defined. Similarly, if $D := D^{\mathrm{sign}}$ is the signature operator of an orientation and Riemannian metric on M, the eta-invariant $\eta(D_{L_j}^{\mathrm{sign}})$ is defined by (1.1).

1.2 Definition. The Atiyah–Patodi–Singer rho-invariant associated to the Dirac-type operator D and the virtual representation $\lambda_1 - \lambda_2$ is the difference

$$\rho(D)_{\lambda_1-\lambda_2} := \eta(D_{L_1}) - \eta(D_{L_2}). \tag{1.3}$$

The first important result we shall prove in this paper is the following.

1.4 Theorem. *Assume that Γ is a torsion-free discrete group and that the Baum–Connes assembly map is an isomorphism*

$$\mu_{\max}\colon K_*(B\Gamma) \to K^*(C^*\Gamma),$$

where $C^*\Gamma$ is the maximal C-algebra of Γ.*

(1) *Let D be the Dirac operator of a spin structure, $D = \not{D}$. If the metric on M has positive scalar curvature then the Atiyah–Patodi–Singer rho-invariant is zero.*

(2) *Let D be the signature operator, $D = D^{\mathrm{sign}}$, then the Atiyah–Patodi–Singer rho-invariant $\rho(D^{\mathrm{sign}})_{\lambda_1-\lambda_2}$ only depends on the oriented Γ-homotopy type of $(M, u\colon M \to B\Gamma)$.*

1.5 Remark. The result about the APS-rho-invariant of the *signature operator* is due to Navin Keswani [29]. We give an independent proof of Keswani's result. The first special case of this result is due to Mathai [47], who proved it for Bieberbach groups. We shall also try to present a "philosophical" reason for the theorem. It should be added Keswani's argument is most likely to be adapted so as to cover the result on the spin Dirac operator as well. Finally, Shmuel Weinberger [66] proves the homotopy invariance of the APS-rho-invariants for the signature operator under the assumption that Γ is torsion free and that the L-theory isomorphism conjecture holds for Γ. Chang [9] has used the same assumption to prove the corresponding result for the L^2-rho-invariant.

1.6 Remark. Note that for *finite* groups the rho-invariants of the spin Dirac operator are effectively used to distinguish different metrics with positive scalar curvature, compare e.g. [6]. We see that this is not possible under our assumption on the group Γ. This might be particularly surprising in view of the genuinely non-local definition of the eta-invariants.

We shall use different C*-algebras associated to the group Γ.

1.7 Definition. Let Γ be a discrete group. The *maximal C*-algebra* $C^*\Gamma$ is the completion of the group ring $\mathbb{C}[\Gamma]$ with respect to the maximal possible C^*-norm on this ring [13].

The maximal C*-algebra has the universal property that for each unitary representation $\phi\colon \Gamma \to \mathrm{U}(H)$ on a Hilbert space H, there is a unique C*-algebra homomorphism $C^*\Gamma \to \mathcal{B}(H)$ extending ϕ.

More frequently used is the *reduced C*-algebra* of Γ. It is by definition the closure of $\mathbb{C}[\Gamma]$, considered as subalgebra of $\mathcal{B}(l^2(\Gamma))$ with respect to the right regular representation $((\sum_{\gamma\in\Gamma}\lambda_\gamma\gamma)\cdot g := \sum_{\gamma\in\Gamma}\lambda_\gamma(\gamma g))$. Note that by the universal property of $C^*\Gamma$, we have a canonical homomorphism $C^*\Gamma \to C^*_{\mathrm{red}}\Gamma$ extending the identity on Γ.

We will also make use of the *group von Neumann algebra* $\mathcal{N}\Gamma$. This is by definition the weak closure of the right regular representation of $\mathbb{C}[\Gamma]$ in $\mathcal{B}(l^2\Gamma)$. In particular it is also the weak closure of $C^*_{\mathrm{red}}\Gamma$, i.e., we have an inclusion $C^*_{\mathrm{red}}\Gamma \hookrightarrow \mathcal{N}\Gamma$. By the bicommutant theorem, it is also equal to $\mathcal{B}(l^2(\Gamma))^\Gamma$, i.e. all operators which commute with the right regular representation of Γ on $l^2(\Gamma)$. Note that $\mathcal{N}\Gamma$ is also a C*-algebra.

1.8 Remark. The Baum–Connes conjecture for a *torsion-free* group Γ asserts that the assembly map

$$K_*(B\Gamma) \to K^*(C^*_{\mathrm{red}}\Gamma)$$

is an isomorphism, where the right-hand side is the *reduced* C*-algebra of the group Γ. It is a fact that $K_*(C^*\Gamma)$ and $K_*(C^*_{\mathrm{red}}\Gamma)$ are identical for large classes of groups, e.g. amenable groups or discrete subgroups of $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. However, for groups with Kazhdan's property T, they definitely differ, and $K_*(B\Gamma) \to K^*(C^*\Gamma)$ is known not to be surjective in this case.

1.2. L^2-rho-invariant. If we look at the Γ-covering $\widetilde{M}$ directly, we can lift any *differential* operator D from M to a Γ-invariant differential operator $\widetilde{D}$ on $\widetilde{M}$. In particular, this is possible for a Dirac-type operator D. Moreover, by using Schwartz kernels, but integrating only over a fundamental domain for the covering $\widetilde{M} \to M$, we get an L^2-trace $\mathrm{Tr}_{(2)}$, see [1]. This yields the L^2-eta-invariant defined using essentially formula (1.1) by

$$\eta_{(2)}(\widetilde{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2}\, \mathrm{Tr}_{(2)}(\widetilde{D}\exp(-(s\widetilde{D}^2)))\, ds,$$

where the L^2-trace is defined by

$$\mathrm{Tr}_{(2)}(\widetilde{D}\exp(-t\widetilde{D}^2)) = \int_{\mathcal{F}} \mathrm{tr}_x\, k_t(x,x). \tag{1.9}$$

Here tr_x is the fiberwise trace and $\mathcal{F}$ is a fundamental domain.

In order to ensure the convergence of the integral near $t = 0$ we still require the operator D to be associated to a unitary Clifford module endowed with a unitary Clifford connection. The convergence of the integral for large t is discussed in [10] and also in [54].

1.10 Definition. We define the L^2-rho-invariant as the difference

$$\rho_{(2)}(D) := \eta_{(2)}(\widetilde{D}) - \eta(D).$$

1.11 Theorem. *If Γ is torsion-free and the Baum–Connes assembly map for the maximal C^*-algebra of Γ, $K_*(B\Gamma) \to K_*(C^*\Gamma)$, is an isomorphism, then*

(1) *the L^2-rho-invariant of the Dirac operator of a spin manifold with positive scalar curvature vanishes*;

(2) *the L^2-rho-invariant of the signature operator depends only on the oriented Γ-homotopy type of $(M, u\colon M \to B\Gamma)$.*

1.12 Remark. For the signature operator, this result is proved by Keswani in [30] using methods different from ours.

The result for the spin Dirac operator was originally obtained (with a similar, but slightly more complicated method as the one presented here) in unpublished work of Nigel Higson and the second author. It should be possible to adapt Keswani's arguments so as to cover the result on the spin Dirac operator.

Our proof also provides some rather delicate information about the eta-invariants directly (and not just of the less subtle rho-invariants).

1.3. Delocalized eta-invariants. We continue with the geometric setup of Section 1.2, with a Galois Γ-covering $\Gamma \longrightarrow \widetilde{M} \longrightarrow M$ classified by a map $u\colon M \to B\Gamma$. To construct the L^2-eta-invariant, we used the integral kernel $k_t(x,y)$ of $\widetilde{D}\exp(-t\widetilde{D}^2)$ on $\widetilde{M}$. Fix now a non-trivial conjugacy class $\langle g\rangle$ of Γ. Define the *delocalized* trace

$$\mathrm{Tr}_{\langle g\rangle}(\widetilde{D}\exp(-t\widetilde{D}^2)) := \sum_{g\in\langle g\rangle} \int_{\mathcal{F}} \mathrm{tr}_x\, k_t(x, gx). \tag{1.13}$$

Note that the fibers at x and gx of the pull back vector bundle on which $\widetilde{D}$ acts are canonically identified, so that $k_t(x, gx)$ can be considered as an endomorphism of this fiber, and its fiberwise trace tr is defined.

Convergence of this (possibly infinite) sum follows from exponential decay, compare [41].

Finite conjugacy classes. If the conjugacy class $\langle g\rangle$ contains only finitely many elements, Lott proves in [41] that the following integral, defining the *delocalized eta-invariant*, converges.

$$\eta_{\langle g\rangle}(\widetilde{D}) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\,\mathrm{Tr}_{\langle g\rangle}(\widetilde{D}\exp(-t\widetilde{D}^2))\,dt. \tag{1.14}$$

We prove:

1.15 Theorem. *Assume that Γ is torsion-free and that the assembly map $\mu_{\mathrm{red}}\colon K_*(B\Gamma) \to K_*(C^*_{\mathrm{red}}\Gamma)$ for the reduced group C*-algebra is an isomorphism. Let $\langle g\rangle$ be a non-trivial finite conjugacy class in Γ.*

(1) *If $D = \not{D}$ is the spin Dirac operator of a spin manifold with positive scalar curvature then*
$$\eta_{\langle g\rangle}(\widetilde{\not{D}}) = 0.$$

(2) *If $D = D^{\mathrm{sign}}$ is the signature operator of an oriented Riemannian manifold then $\eta_{\langle g\rangle}(\widetilde{D}^{\mathrm{sign}})$ depends only on the oriented Γ-homotopy type of $(M, u\colon M \to B\Gamma)$.*

1.16 Remark. Notice that our assumption here involves the *reduced* group C*-algebra. There are substantially more groups which satisfy the Baum–Connes conjecture for the reduced C*-algebra, even some with property T, e.g. all Gromov hyperbolic groups [52] or cocompact discrete subgroups of $\mathrm{SL}(3,\mathbb{C})$ [31].

1.17 Remark. The stated result is only interesting if the group contains elements with finite conjugacy class. Note that this does by no means imply that the corresponding element has finite order.

Evidently, each central element has a finite conjugacy class (consisting only of itself). This means that, starting with an arbitrary group Γ, each central extension with kernel $\mathbb{Z}$ will be a group with large center, and if $H^2(\Gamma,\mathbb{Z}) \neq \{0\}$, there are non-trivial such central extensions.

Infinite conjugacy classes. Lott establishes the convergence of the delocalized eta-invariant under more general assumptions: it suffices to assume that the Dirac-type operator on the covering has a gap at 0 (with 0 allowed in the spectrum) and that the conjugacy class is of polynomial growth with respect to a word-metric on Γ. Thus, for the spin Dirac operator of a manifold with positive scalar curvature, the delocalized eta-invariant $\eta_{\langle g\rangle}(\widetilde{\not{D}})$ is well defined, provided $\langle g\rangle$ is of polynomial growth. If, in addition, Γ is torsion-free and the reduced Baum–Connes map is bijective, then we shall prove that, as in the case of finite conjugacy classes stated above, $\eta_{\langle g\rangle}(\widetilde{\not{D}}) = 0$.

1.4. General principle of the proofs of vanishing results. In order to simplify the exposition we shall concentrate on the case where Γ is the fundamental group of our manifold and the Γ-covering is the universal covering.

First observe that the *homotopy invariance* of the rho-invariants for the signature operator reduces to the *vanishing* of the rho-invariant for the disjoint union $X \amalg -X'$, if X and X' are homotopy equivalent, since all rho-invariants are certainly additive under disjoint union.

For this reason we shall be only concerned with vanishing results. To establish these results, we apply the following *general principle*. To avoid undue repetitions, let ρ for the moment stand for any of the rho-invariants we want to investigate.

(1) We first define a *stable* variant ρ^s of ρ. This will be defined as the invariant of a perturbation of our generalized Dirac operator. Such perturbations do not always exist, we need the vanishing of the index class of the generalized Dirac operator. This very strong assumption is satisfied for geometric reasons if one looks at the Dirac operator of a spin manifold with positive scalar curvature, as well as for the signature operator on the disjoint union $X \amalg -X'$ of two homotopy equivalent manifolds.

    We study the main properties of ρ^s. Most important is that it appears as the correction term in an index theorem for manifolds with boundary for suitably perturbed Dirac operators. We use this fact and the assumed *surjectivity* of the Baum–Connes map in order to show that the stable rho-invariant is *well defined*, independent of the chosen perturbation (we are always under the assumption that the index class of our operator is zero). Under these assumptions we also prove the fundamental fact that ρ^s is a *bordism invariant*: suitable index theorems on manifolds with boundary will again play a crucial role here.

(2) Then we use our injectivity assumption on the Baum–Connes map, fundamental results in bordism theory and the bordism invariance established in (1) in order to show that the stable invariant ρ^s, whenever it is defined, is equal to the stable invariant of a particularly nice manifold. For this nice manifold we compute the stable invariant and show that *it vanishes*.

    To this point, we have therefore shown that in certain special situations one can define an invariant ρ^s which turns out to be zero.

(3) As a last step we show that in the two geometric situations we are studying, the stable invariant ρ^s coincides with the unstable invariant ρ. This will be done by constructing very special perturbations (used in the definition of the stable invariant) which make the direct comparison of the stable and unstable invariant possible. For the signature operator on $X \amalg -X'$ we use perturbations that are inspired by the work of Hilsum–Skandalis.

    In fact, our results here are much more precise: they give information directly about the unstable eta-invariants. This is quite remarkable because of the nonstable nature of the eta-invariants under perturbations and might lead to future applications or calculations of eta-invariants.

In the case of a spin manifold with positive scalar curvature, we do not have to perturb at all, so the last step is trivial.

1.5. Examples. Let Γ be a torsion-free discrete group satisfying our basic assumption, the bijectivity of the Baum–Connes map. One might very well wonder whether the signature rho-invariants considered in this paper are non-zero and whether they can be effectively used in order to distinguish manifolds that are not homotopy equivalent. In the last part of the paper we give a careful treatment of some non-trivial examples. In particular, we construct manifolds with the same cohomology but which are not in the same homotopy class and we distinguish their homotopy type through their rho-invariants. This can be done for all non-trivial groups which satisfy our basic examples. The results might also be obtainable using Blanchfield forms of classical algebraic topology; however, it seems that one has to use some advanced version to cover all the cases covered by our invariant. The advantage of our approach is that it is very easy to carry out the calculations.

We also show, along the way, that the vanishing of the signature index class is not sufficient for establishing the vanishing of the signature-rho-invariants. The mere vanishing of the signature index class does not imply the vanishing of our rho-invariants.

1.6. Plan of the paper. In Section 2 we gather the index theoretic results that will be needed throughout the paper. We treat the general case of a Dirac operator twisted by a bundle $\mathcal{L}$ of finitely generated projective A-modules, with A a C*-algebra. We state $A/\overline{[A,A]}$-valued index theorems on closed manifolds and on manifolds with boundary. Proofs are given in Appendix A.

In Section 3 we specialize to the C*-algebras defined by a discrete group Γ and to the corresponding Mishchenko–Fomenko bundle; we also discuss the corresponding notions when we twist with the group von Neumann algebra $\mathcal{N}\Gamma$.

In Section 4 we use the *surjectivity* of the assembly map in order to define the *stable* rho-invariants and establish their bordism invariance. The stable rho-invariants are C*-algebraic objects and they are defined under the additional assumption that the index class in $K_1(C^*\Gamma)$ is equal to zero.

In Section 5 we employ the *injectivity* of the assembly map in order to construct a suitable bordism between d copies of the manifold $X = M \cup (-M')$, with M and M' homotopy equivalent, and a manifold of a special type. A similar result is proved if X is a manifold with positive scalar curvature. These bordisms take the classifying maps into account.

In Section 6 we show that the stable rho-invariants of these special manifolds are zero. This fact and the bordism invariance of Section 4 are then used in Section 7 to prove that under our assumptions the stable rho-invariants for $D = D^{\text{sign}}$ are zero if $X = M \cup (-M')$, with M and M' homotopy equivalent. The corresponding result for X a manifold with positive scalar curvature and $D = \not{D}$ is established in Section 12.

In Section 8 we introduce *unstable* rho-invariants; these are von Neumann objects producing the three rho-invariants defined in Subsections 1.1, 1.2, 1.3 as special cases. The unstable rho-invariants are always defined.

In Section 9, Section 10 and Section 11 we show that if $X = M \cup (-M')$, with M and M' homotopy equivalent, then the unstable rho-invariants are suitable limits of stable rho-invariants. This step completes the proof of our theorems for the signature operator, since we know that stable rho-invariants are zero under our assumptions. The proof for the spin Dirac operator, which is much easier since there is no need for this limit-argument, is given in Section 12

In Section 13 we gather several remarks on delocalized eta-invariant for infinite conjugacy classes. We also prove vanishing results for *higher* rho-invariants.

In Section 14 we state a general von Neumann signature formula for manifolds with boundary.

In Section 15 we give examples of closed manifolds with torsion-free fundamental group satisfying the Baum–Connes assumption and for which the relevant rho-invariants are non-zero. We also construct manifolds with isomorphic homology but with different homotopy type and we distinguish them by using the L^2-rho-invariant or the delocalized eta-invariant.

In four additional appendices we recall signatures and the signature operator of Hilbert A-module chain complexes with symmetry, and we give a detailed account of the relationship between spectral invariants on Γ-coverings and certain algebra-valued invariants; we also discuss naturality properties of these algebra-valued invariants. All appendices either recall known results or immediate extensions of known results; they are included for the reader's convenience.

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2. Index theory: statement of results

Many of the results of this paper are based on suitable generalizations of the Atiyah–Singer and Atiyah–Patodi–Singer index theorem. In this section we gather the index theoretic results that we shall need in the rest of the paper. Most of these index theorems are described (sometimes implicitly) in the literature; they are due to Mishchenko–Fomenko in the closed case and to Leichtnam and the first author in the boundary case. Nevertheless, we shall give a direct and simplified account of these results, showing in particular that once the index class in $K_*(C_r^*\Gamma)$ is carefully described, it is possible to give heat kernel proofs of all needed results following the same steps as for the numeric case (see, for example, [49, Introduction]).

This section is devoted to the statements; Appendix A will contain the proofs. We start with the closed case.

2.1. Atiyah–Singer index theory for arbitrary C*-algebras

2.1.1. Geometric setup. Let M be a closed manifold. We assume that D is a Dirac-type operator acting on a finite dimensional Clifford module E on M, see [5] for the definitions. Our main interest will be the spin Dirac operator $\not{D}$ of a spin structure, and the signature operator D^{sign} of an orientation and Riemannian metric on M. Let A be a C*-algebra, and $\mathcal{L}$ a bundle of finitely generated projective Hilbert A-modules,[1] with an A-connection $\nabla_{\mathcal{L}}$. We define $D_{\mathcal{L}}$ to be the operator D twisted with the bundle with connection $\mathcal{L}$, acting on sections of $E \otimes \mathcal{L}$. Then $D_{\mathcal{L}}$ is an A-linear differential operator,

$$D_{\mathcal{L}} \in \mathrm{Diff}^1_A(M; E \otimes \mathcal{L}, E \otimes \mathcal{L}) \subset \Psi^*_A(M; E \otimes \mathcal{L}, E \otimes \mathcal{L}).$$

On the right-hand side the Mishchenko–Fomenko pseudodifferential calculus appears. $D_{\mathcal{L}}$ is elliptic in the sense of Mishchenko–Fomenko with a well-defined index class $\mathrm{Ind}(D_{\mathcal{L}}) \in K_{\dim(M)}(A)$. We shall recall below the definition of $\mathrm{Ind}(D_{\mathcal{L}}) \in K_{\dim(M)}(A)$ when $\dim(M)$ is even.

In many situations, instead of working with the differential operator $D_{\mathcal{L}}$ itself, we will have to perturb the operator slightly (thereby leaving the world of differential operators). We do this because on the one hand we will have to improve some technical properties of the operator, such as the large time behavior of the associated heat kernel; on the other hand, when the operator arises as a boundary operator, we shall need invertibility in order to define a Fredholm problem on the manifold with boundary. Of course, each time such a perturbation is introduced, we shall have then to control how invariants such as the index or the eta-invariant behave.

We start with some generalities about perturbations; this material will be needed throughout the paper.

[1] A comment on notations: objects in the C*-algebraic context will always be denoted by script letters.

2.1 Definition. Let $\mathcal{C} \in \Psi_A^{-\infty}$ be a smoothing operator in the Mishchenko–Fomenko calculus, acting on sections of $E \otimes \mathcal{L}$. Set

$$D_{\mathcal{L},\mathcal{C}} := D_{\mathcal{L}} + \mathcal{C}.$$

This is a (smoothing) *perturbation* of $D_{\mathcal{L}}$.

The following lemma is a direct consequence of the Mishchenko–Fomenko ($\equiv$ MF) pseudodifferential calculus.

2.2 Lemma. *$D_{\mathcal{L},\mathcal{C}}$ is an elliptic element in the MF-calculus*

$$D_{\mathcal{L},\mathcal{C}} \in \Psi_A^1(M; E \otimes \mathcal{L}, E \otimes \mathcal{L}) \subset \Psi_A^*(M; E \otimes \mathcal{L}, E \otimes \mathcal{L}). \tag{2.3}$$

Moreover, the operators $D_{\mathcal{L}}$ and $D_{\mathcal{L},\mathcal{C}}$ have the same index

$$\operatorname{Ind}(D_{\mathcal{L}}) = \operatorname{Ind}(D_{\mathcal{L},\mathcal{C}}) \in K_{\dim(M)}(A).$$

Fundamental in what follows is the following Mishchenko–Fomenko decomposition theorem.

2.4 Theorem. *Let M be even dimensional so that $E = E^+ \oplus E^-$. Let $(E \otimes \mathcal{L})^{\pm} := E^{\pm} \otimes \mathcal{L}$. There is a Mishchenko–Fomenko decomposition of the space of sections of $E \otimes \mathcal{L}$ with respect to $D_{\mathcal{L}}$, i.e.,*

$$C^\infty(M, (E \otimes \mathcal{L})^+) = \mathcal{J}_+ \oplus \mathcal{J}_+^{\perp}, \quad C^\infty(M, (E \otimes \mathcal{L})^-) = \mathcal{J}_- \oplus D_{\mathcal{L}}(\mathcal{J}_+^{\perp}). \tag{2.5}$$

By completion, we obtain a decomposition of the Sobolev Hilbert A-modules $H^m(M, (E \otimes \mathcal{L})^{\pm})$ for any $m \in \mathbb{N}$.

The second decomposition is not, a priori, orthogonal. However, $D_{\mathcal{L}}$ induces an isomorphism (in the Fréchet topology) between $\mathcal{J}_+^{\perp}$ and $D_{\mathcal{L}}(\mathcal{J}_+^{\perp})$, and $\mathcal{J}_+$ and $\mathcal{J}_-$ are finitely generated projective Hilbert A-modules (consisting of smooth section of $L^2(M, E \otimes \mathcal{L})$). The projections $\Pi_{\mathcal{J}_+}$ onto $\mathcal{J}_+$ (orthogonal) and $\Pi_{\mathcal{J}_-}$ onto $\mathcal{J}_-$ (along $D_{\mathcal{L}}(\mathcal{J}_+^{\perp})$) are smoothing operators in the Mishchenko–Fomenko calculus. Because $\mathcal{J}_{\pm}$ are already complete finitely generated projective Hilbert A-modules they are unchanged under the completions. The Mishchenko–Fomenko index class is given by

$$\operatorname{Ind}(D_L) = [\mathcal{J}_+] - [\mathcal{J}_-] \in K_0(A). \tag{2.6}$$

2.7 Remark. The theorem is ultimately a consequence of the ellipticity of $D_{\mathcal{L}}$ and more precisely of the existence of an inverse for $D_{\mathcal{L}}$ modulo smoothing operators. Thus, an identical statement remains true for the more general (elliptic) operator $D_{\mathcal{L},\mathcal{C}}$. The existence of the decomposition is implicitly proved in [53]; the structure of the two projections is analyzed in [33, Appendix A].

2.1.2. The $A/\overline{[A,A]}$-valued index. We want to give a heat kernel proof of a suitable Atiyah–Singer ($\equiv$ AS) index theorem for the operator $D_{\mathcal{L}}$ of 2.1.1. First of all, we introduce the index we want to compute. Consider the index class $\mathrm{Ind}(D_{\mathcal{L}})$ expressed through the MF-decomposition theorem (see 2.6): $\mathrm{Ind}(D_{\mathcal{L}}) = [\mathcal{I}_+] - [\mathcal{I}_-] \in K_0(A)$. Expressing the finitely generated projective modules $\mathcal{I}_\pm$ as the images of idempotent matrices and taking the difference of the traces, we obtain a well-defined element in $A/\overline{[A,A]}$, with $[A,A]$ equal to the $\mathbb{C}$-subspace generated by the commutators $[a,b] := ab - ba$ for $a, b \in A$.

2.8 Definition. Consider the closure $\overline{[A,A]}$ of the subspace $[A,A]$. We set

$$A_{\mathrm{ab}} := A/\overline{[A,A]}. \tag{2.9}$$

where the subscript ab stands for *abelianization*. It should be noticed that A_{ab} is a commutative Banach algebra.

In this way we define a homomorphism of abelian groups

$$\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}. \tag{2.10}$$

Since this is nothing but the zero-degree part of the Karoubi–Chern character [27], we denote $\mathrm{tr}^{\mathrm{alg}}([\mathcal{I}_+] - [\mathcal{I}_-])$ by

$$[\mathcal{I}_+]_{[0]} - [\mathcal{I}_-]_{[0]} \in A_{\mathrm{ab}}, \tag{2.11}$$

or, equivalently, by $\mathrm{Ind}_{[0]}(D_{\mathcal{L}}) \in A_{\mathrm{ab}}$.

Our aim is to give a formula for $\mathrm{Ind}_{[0]}(D_{\mathcal{L}}) \in A_{\mathrm{ab}}$ and to prove it via heat kernel techniques. First we need the existence of the heat kernel for the Dirac Laplacian $D^2_{\mathcal{L}}$ and its perturbation $D^2_{\mathcal{L},C}$. This is not completely obvious, given that we are in the C*-algebraic context. We shall give the precise statement in Lemma A.1 of Appendix A.1 For the time being we shall content ourselves with the statement that the heat semigroup $\{e^{-tD^2_{\mathcal{L},C}},\ t > 0\}$ exists, provides a fundamental solution of the heat equation and is such that $e^{-tD^2_{\mathcal{L},C}} \in \Psi_A^{-\infty}$.

Since the heat operator is a smoothing operator in the Mishchenko–Fomenko calculus, we can consider its supertrace

$$\mathrm{STR}(e^{-tD^2_{\mathcal{L}}}) \in A_{\mathrm{ab}}.$$

This is defined as follows: we restrict the heat kernel to the diagonal Δ in $M \times M$, $\Delta \leftrightarrow M$, thus obtaining an element in $C^\infty(M, \mathrm{End}(E \otimes \mathcal{L}))$, we take for each fiber over x the A_{ab}-valued supertrace on $\mathrm{End}(E \otimes \mathcal{L})_x$ and integrate

$$\mathrm{STR}(e^{-tD^2_{\mathcal{L}}}) := \int_M \mathrm{str}_x^{\mathrm{alg}}(e^{-tD^2_{\mathcal{L}}}(x,x))\,\mathrm{vol}_M \in A_{\mathrm{ab}}. \tag{2.12}$$

Our first result concerns the large time behavior of the heat kernel in the Mishchenko–Fomenko calculus.

2.13 Proposition.

$$\lim_{t\to\infty} \mathrm{STR}(e^{-tD_{\mathcal{L}}^2}) = [I_+]_{[0]} - [I_-]_{[0]} \equiv \mathrm{Ind}_{[0]}(D_{\mathcal{L}}) \in A_{\mathrm{ab}},$$

where part of the assertion is that the limit exists.

The proof of this proposition can be found in Appendix A.1. Next we want to connect $\mathrm{Ind}_{[0]}(D_{\mathcal{L}}) \in A_{\mathrm{ab}}$ defined using the index class and the algebraic trace $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$ to the integral-kernel-trace, TR, of the projection operators $P_+ := \Pi_{\mathcal{I}_+}$, $P_- := \Pi_{\mathcal{I}_-}$ given in the Mishchenko–Fomenko decomposition. We recall that these are smoothing operators; the trace TR is thus well defined. We state the result here and defer the proof to Appendix A.1.

2.14 Proposition. *The algebraic trace*

$$\mathrm{Ind}_{[0]}(D_{\mathcal{L}}) := [I_+]_{[0]} - [I_-]_{[0]} \quad \textit{of } \mathrm{Ind}(D_{\mathcal{L}}),$$

i.e. the image of $\mathrm{Ind}(D_L)$ *under the map* $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$, *can be calculated as*

$$\begin{aligned}[I_+]_{[0]} - [I_-]_{[0]} &= \mathrm{TR}(P_+) - \mathrm{TR}(P_-)\\ &\equiv \int_M \mathrm{tr}_x^{\mathrm{alg}} P_+(x,x) - \int_M \mathrm{tr}_x^{\mathrm{alg}} P_-(x,x) \in A_{\mathrm{ab}},\end{aligned}$$

where P_+ *and* P_- *are the projections onto* $\mathcal{I}_+$ *and* $\mathcal{I}_-$ *as given by the Mishchenko–Fomenko decomposition.*

Proceeding now as in the classical case, one proves the following A_{ab}-valued Atiyah–Singer index theorem:

2.15 Theorem. *We have*

$$\mathrm{Ind}_{[0]}(D_{\mathcal{L}}) = \mathrm{TR}(P_+) - \mathrm{TR}(P_-) = \int_M \mathrm{AS}(D)(x) \wedge \mathrm{ch}\,\mathcal{L}(x)_{[\dim M]} \in A_{\mathrm{ab}}$$

where $\mathrm{AS}(D)(x)$ *is the local integrand in the Atiyah–Singer formula for* D.

In the above formula the differential form

$$(x \mapsto \mathrm{AS}(D)(x) \wedge \mathrm{ch}(E)(x) \wedge \mathrm{ch}\,\mathcal{L}(x)_{[\dim M]}) \in \Omega^{\dim M}(M, A_{\mathrm{ab}})$$

can be calculated as usual using Chern–Weyl theory and the curvature of the connections, see [60].

2.2. APS index theory for arbitrary C*-algebras

2.2.1. Geometric setup. Let W be a compact manifold of even dimension with boundary $\partial W = M$. We assume that D is a Dirac-type operator acting on a finite dimensional Dirac-bundle E on W. Our main interest will be the spin Dirac operator of a spin structure, and the signature operator of an orientation and Riemannian metric on W. We always assume that all these structures are of product type near the boundary.

Let A be a C*-algebra, and $\mathcal{L}$ a bundle of finitely generated projective Hilbert A-modules, with an A-connection $\nabla_{\mathcal{L}}$ (again everything has a product structure near the boundary). We define $D_{\mathcal{L}}$ to be the operator D twisted with the bundle with connection $\mathcal{L}$. Associated to this are the boundary operators $D_M \equiv D_{\partial W}$ of D and $D_{M,\mathcal{L}} \equiv D_{\partial W,\mathcal{L}}$ of $D_{\mathcal{L}}$. We can attach an infinite cylinder $(-\infty, 0] \times M$ to W along its boundary $\partial W = M$, thus obtaining a manifold with cylindrical ends $\widehat{W}$ and with product metric $dt^2 + g_{\partial W}$ along the cylinder. The operators $D_{\mathcal{L}}$ extends in a natural way to the manifold $\widehat{W}$. The change of coordinates $t = \log x$ compactifies $\widehat{W}$ to a manifold with boundary and with product b-metric $dx^2/x^2 + g_{\partial M}$ near the boundary. The operator on $\widehat{W}$ then defines in a natural way a b-differential operator on the compactified manifold. We refer the reader to the book [49] of Melrose for basics about the b-calculus. As we shall mainly work in the framework of the b-calculus, we keep denoting the compactified manifold by W and the resulting b-differential operators by $D_{\mathcal{L}}$.

2.2.2. Trivializing perturbations. Let M be odd dimensional and without boundary, and let us assume that $\mathrm{Ind}(D_{M,\mathcal{L}}) = 0 \in K_1(A)$. This will be the case in the following examples:

- D_M is the signature operator on $M = X \sqcup (-X')$ with X and X' homotopy equivalent.
- D_M is the signature operator and $(M, \mathcal{L})$ bounds, i.e., there is a manifold W with $\partial W = M$ such that $\mathcal{L}$ extends to W.
- M is spin with positive scalar curvature and D_M is the spin Dirac operator.

For more on these examples we refer the reader to [22], [56], [34] and [57]. According to [68], [38, Theorem 3] we can find a non-commutative spectral section $\mathcal{P}$ for $D_{M,\mathcal{L}}$.[2] Using the projection $\mathcal{P}$ one can construct [34, Proposition 2.10] a smoothing operator $\mathcal{C}_{\mathcal{P}} \in \Psi_A^{-\infty}(M; E \otimes \mathcal{L})$ such that

$$D_{M,\mathcal{L}} + \mathcal{C}_{\mathcal{P}} \in \Psi_A^1(M; E \otimes \mathcal{L}) \text{ is invertible in } \Psi_A^*(M; E \otimes \mathcal{L}).$$

[2]Thus, by definition, $\mathcal{P}$ is a self-adjoint projection, $\mathcal{P} \in \Psi_A^0$ and there exists functions $\chi_1, \chi_2 \in C^\infty(\mathbb{R}, [0, 1])$ such that $\chi_i(t) = 0$ for $t \ll 0$, $\chi_i(t) = 1$ for $t \gg 0$, $\chi_2 \equiv 1$ on a neighborhood of the support of χ_1 and $\mathrm{im}\,\chi_1(D_{M,\mathcal{L}}) \subset \mathrm{im}\,\mathcal{P} \subset \mathrm{im}\,\chi_2(D_{M,\mathcal{L}})$.

Moreover, $\mathcal{C}_{\mathcal{P}}$ is symmetric with respect to the A-valued scalar product on $C^\infty(M, E \otimes \mathcal{L})$. We shall consider in what follows the set of allowable perturbations:

$$\begin{aligned}\mathfrak{P} := \mathfrak{P}_{D_{M,\mathcal{L}}} := \{\mathcal{C} \in \Psi_A^{-\infty}(M; E \otimes \mathcal{L}) \mid \mathcal{C} \text{ is symmetric} \\ \text{and } D_{M,\mathcal{L}} + \mathcal{C} \text{ is invertible in } \Psi_A^*(M; E \otimes \mathcal{L})\}.\end{aligned} \tag{2.16}$$

Summarizing, we have just seen that

$$\operatorname{Ind}(D_{M,\mathcal{L}}) = 0 \Rightarrow \mathfrak{P} \neq \emptyset. \tag{2.17}$$

Let us go back to the case where $M = \partial W$, with W even dimensional. As mentioned above, by cobordism invariance, the index class of $D_{M,\mathcal{L}}$ is zero in $K_1(A)$; thus $\mathfrak{P} \neq \emptyset$. Let $\mathcal{C} \in \mathfrak{P}$ and consider the invertible operator $D_{M,\mathcal{L}} + \mathcal{C}$. According to [34, Lemma 6.1] there exists a smoothing operator in the Mishchenko–Fomenko b-calculus

$$\mathcal{C}_W^+ \in \Psi_{b,A}^{-\infty}(W; E^+ \otimes \mathcal{L}, E^- \otimes \mathcal{L})$$

such that $D_{\mathcal{L}}^+ + \mathcal{C}_W^+$ has $D_{M,\mathcal{L}} + \mathcal{C}$ as boundary operator. Using the invertibility of $D_{M,\mathcal{L}} + \mathcal{C}$ one proves the invertibility of the *indicial family* associated to $D_{\mathcal{L}}^+ + \mathcal{C}_W^+$; this property can in turn be used to prove that $D_{\mathcal{L}}^+ + \mathcal{C}_W^+$ is A-Fredholm as an operator between suitable Sobolev Hilbert A-modules. Thus there is a well defined index class in $K_0(A)$, denoted $\operatorname{Ind}_b(D_{\mathcal{L}} + \mathcal{C}_W)$. More precisely

$$\operatorname{Ind}_b(D_{\mathcal{L}} + \mathcal{C}_W) = [\mathcal{J}_+] - [\mathcal{J}_-] \in K_0(A) \tag{2.18}$$

with $\mathcal{J}_\pm$ finitely generated projective A-modules and

$$\mathcal{J}_\pm \subset x^\varepsilon H_b^\infty(W, (E \otimes L)^\pm), \quad \varepsilon > 0.$$

The construction of $\mathcal{C}_W$ from $\mathcal{C}$ involves choices; the index class, on the other hand, does not depend on these choices. We refer the reader to [34] for the details.

2.19 Remark. The results in [34], [38] are noncommutative generalizations of the results first proved in [50], [51] for families of Dirac operators (i.e. for $C(B)$-linear operators, with B a compact manifold).

2.20 Notation. We shall often denote by $\operatorname{Ind}_b(D_L, \mathcal{C})$ the index class in (2.18).

2.21 Remark. Notice that the index class $\operatorname{Ind}_b(D_L, \mathcal{C})$ *does depend* on $\mathcal{C}$.[3]

[3]For example if $\mathcal{C}_{\mathcal{P}} \in \mathfrak{P}$ and $\mathcal{C}_{\mathcal{Q}} \in \mathfrak{P}$ are defined by spectral sections $\mathcal{P}$ and $\mathcal{Q}$ respectively, then (see [36], [39])

$$\operatorname{Ind}_b(D_{\mathcal{L}}, \mathcal{C}_{\mathcal{P}}) - \operatorname{Ind}_b(D_{\mathcal{L}}, \mathcal{C}_{\mathcal{Q}}) = [\mathcal{Q} - \mathcal{P}] \quad \text{in} \quad K_0(A),$$

where on the right-hand side the difference class defined by the two projections $\mathcal{Q}$, $\mathcal{P}$ appears. This is known to be non-zero in general.

2.22 Definition. Let $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$ be the trace introduced in Subsection 2.1.2. The A_{ab}-valued b-index is by definition $\mathrm{tr}^{\mathrm{alg}}(\mathrm{Ind}_b(D_{\mathcal{L}}, \mathcal{C})) \in A_{\mathrm{ab}}$. As in the closed case we use the notation

$$\mathrm{tr}^{\mathrm{alg}}(\mathrm{Ind}_b(D_{\mathcal{L}}, \mathcal{C})) =: \mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C}) = [\mathcal{J}_+]_{[0]} - [\mathcal{J}_-]_{[0]} \in A_{\mathrm{ab}}.$$

Our goal is to prove an index formula for $\mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C})$. First of all, we introduce the boundary correction term that will appear in the formula.

2.2.3. Eta-invariants for perturbed operators. Let $\mathcal{C} \in \mathfrak{P}_{D_{M,\mathcal{L}}}$. Consider the pseudo-differential operator

$$D_{M,\mathcal{L}} + \mathcal{C} \in \Psi^1_A(M; E \otimes \mathcal{L}).$$

We define the A_{ab}-valued eta-invariant $\eta_{[0]}(D_{M,\mathcal{L}} + \mathcal{C}) \in A_{\mathrm{ab}}$ by

$$\eta_{[0]}(D_{M,\mathcal{L}} + \mathcal{C}) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{TR}((D_{M,\mathcal{L}} + \mathcal{C}) e^{-t(D_{M,\mathcal{L}}+\mathcal{C})^2}) \frac{dt}{\sqrt{t}}. \tag{2.23}$$

The integral converges for large t because of the invertibility of $D_{M,\mathcal{L}} + \mathcal{C}$. The convergence for $t \downarrow 0$ follows from the local A_{ab}-valued index theorem on closed manifold and the observation that

$$e^{-t(D_{M,\mathcal{L}}+C)^2} = e^{-tD^2_{M,\mathcal{L}}} + tC^\infty([0,\infty), \Psi_A^{-\infty}(M, E \otimes \mathcal{L})),$$

a consequence of Duhamel's formula and the fact that $\mathcal{C} \in \Psi_A^{-\infty}(M, E \otimes \mathcal{L})$.

2.2.4. The $A/\overline{[A,A]}$-valued Atiyah–Patodi–Singer index formula. In Appendix A.2 we shall recall the precise form of the Mishchenko–Fomenko decomposition theorem in the b-context. From this result we get the two finitely generated projective modules $\mathcal{J}_\pm$ entering into the definition of the b-index class $\mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C})$. b-elliptic regularity implies that the projections $P_\pm$ onto these two modules have well defined traces $\mathrm{TR}(P_\pm) \in A_{\mathrm{ab}}$. Analyzing the large-time behavior of the b-supertrace of the heat kernel for $(D_{\mathcal{L}} + \mathcal{C}_W)^2$, computing the short-time limit and using the commutator formula for the b-trace, one finally proves the following theorem (see Appendix A.2 for proofs and relevant definitions).

2.24 Theorem.

$$\begin{aligned} \mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C}) &\equiv [\mathcal{J}_+]_{[0]} - [\mathcal{J}_-]_{[0]} \\ &= \lim_{t \to +\infty} b\,\mathrm{STR}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}) = \mathrm{TR}(P_+) - \mathrm{Tr}(P_-) \\ &= \int_W \mathrm{AS}(D)(x) \wedge \mathrm{ch}(E)(x) \wedge \mathrm{ch}\,\mathcal{L}(x)\,dx - \frac{1}{2}\eta_{[0]}(D_{M,\mathcal{L}} + \mathcal{C}) \\ &\in A_{\mathrm{ab}}. \end{aligned}$$

where $\mathrm{AS}(D)$ *is the local integrand in the APS index theorem for* D.

3. Discrete groups, twisting bundles and index theorems

In this section we show how we can derive the particular index theorems needed for the results stated in the introduction from the general ones presented above. We shall specialize to the maximal or the reduced C*-algebra of the group, and also to the group von Neumann algebra.

3.1. The group C*-algebra and the classical Mishchenko–Fomenko twisting bundle. We now specialize the C*-algebra A and the Hilbert A-module bundle $\mathcal{L}$ in 2.2.1 and 2.1.1 to the cases which lead to essentially all our applications.

Specifically, let Γ be a discrete finitely generated group. Let $\Gamma \to \tilde{M} \to M$ be a Galois covering and let $u\colon M \to B\Gamma$ be the associated classifying map, i.e., $\tilde{M}$ is the pull back of the universal Γ-covering $E\Gamma \to B\Gamma$ under u. We twist our Dirac type operator D with the flat $C^*\Gamma$-bundle $\mathcal{L}$ associated to the covering, i.e. with

$$\mathcal{L} := \tilde{M} \times_\Gamma C^*\Gamma.$$

3.1 Theorem. *For this operator*

$$\mathrm{Ind}_{[0]}(D_{M,\mathcal{L}}) = \left(\int_M \mathrm{AS}(D)\right) \cdot 1 \in (C^*\Gamma)_{\mathrm{ab}}. \tag{3.2}$$

Here AS *is the integrand in the classical Atiyah–Singer index theorem and we recall that* $(C^*\Gamma)_{\mathrm{ab}} := C^*\Gamma/\overline{[C^*\Gamma, C^*\Gamma]}$.

Proof. We apply Theorem 2.15. Since $\mathcal{L}$ is flat, and each fiber is isomorphic to the free $C^*\Gamma$-module of rank 1, $\mathrm{ch}(\mathcal{L})(x) = 1 \in (C^*\Gamma)_{\mathrm{ab}}$, so that we get the constant 1 in the formula. □

We now look at a compact manifold W with boundary M, again with a Γ-covering $\Gamma \to \tilde{W} \to W$, classified by a map (also called u) $u\colon W \to B\Gamma$. Note that this induces, by restriction, a Γ-covering $\tilde{M}$ of the boundary.

Let D be a Dirac type operator on W. Then let $\mathcal{C} \in \mathfrak{P}_{D_{M,L}}$ give rise to an allowable perturbation of the boundary operator $D_{M,\mathcal{L}}$.

3.3 Theorem.

$$\mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C}) = \left(\int_W \mathrm{AS}(D)\right) \cdot 1 - \frac{1}{2}\eta_{[0]}(D_{M,\mathcal{L}} + \mathcal{C}) \in (C^*\Gamma)_{\mathrm{ab}}. \tag{3.4}$$

Proof. This is a special case of Theorem 2.24. Note that, since the twisting bundle $\mathcal{L}$ is flat and fiberwise isomorphic to the free Hilbert $C^*\Gamma$-module of rank 1, again $\mathrm{ch}\,\mathcal{L}(x) = 1 \in (C^*\Gamma)_{\mathrm{ab}}$, so that it does only contribute the 1 in the formula. □

3.5 Remark. Consider the reduced group C*-algebra $C_r^*\Gamma$ of Definition 1.7. One can form a Mishchenko–Fomenko twisting bundle $\widetilde{M} \times_\Gamma C_r^*\Gamma$ and prove $(C_r^*\Gamma)_{\mathrm{ab}}$-valued index formulas completely analogous to formulas (3.2) and (3.4).

4. Surjectivity of the Baum–Connes map and stable rho-invariants

4.1. The ρ-index and its vanishing. Let Γ be a discrete group. Let $\Gamma \to \widetilde{W} \to W$ be an even-dimensional Galois covering of a compact manifold with boundary W, classified by $u\colon W \to B\Gamma$. Let $\Gamma \to \widetilde{M} \to M$ be the boundary covering. As before, we consider $\mathcal{D} := D_{\mathcal{L}}$, a Dirac-type operator on W twisted with the Mishchenko–Fomenko line bundle $\mathcal{L}$. Since the boundary operator $D_{M,\mathcal{L}}$ has trivial index in $K_1(C^*\Gamma)$ we can choose a trivializing perturbation $\mathcal{C} \in \mathfrak{P}_{D_{M,\mathcal{L}}}$. With respect to this perturbation, we have $\mathrm{Ind}_b(\mathcal{D},\mathcal{C}) \in K_0(C^*\Gamma)$. By applying $\mathrm{tr}^{\mathrm{alg}}\colon K_0(C^*\Gamma) \to (C^*\Gamma)_{\mathrm{ab}}$ we have finally defined the $(C^*\Gamma)_{\mathrm{ab}}$-valued b-index $\mathrm{Ind}_{b,[0]}(\mathcal{D},\mathcal{C}) \in (C^*\Gamma)_{\mathrm{ab}}$.

Since $(C^*\Gamma)_{\mathrm{ab}}$ is a vector space with one dimensional subspace generated by $[1] = 1 + \overline{[C^*\Gamma, C^*\Gamma]}$, we can project the degree zero part of the index onto the quotient $(C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$.

4.1 Definition. The *ρ-index* associated to $\mathcal{D}$ is the image of $\mathrm{Ind}_{b,[0]}(\mathcal{D},C)$ in the quotient $(C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$. We shall denote the ρ-index by

$$\mathrm{ind}^{\rho}_{b,[0]}(\mathcal{D},C) \in (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle. \tag{4.2}$$

4.3 Remark. Suppose now that W is *closed* and let $\mathrm{Ind}_{[0]}(\mathcal{D}) \in (C^*\Gamma)_{\mathrm{ab}}$ be the associated $(C^*\Gamma)_{\mathrm{ab}}$-valued index. Using the degree-zero Atiyah–Singer index formula 3.1, we see that the ρ-index of $\mathcal{D}$ vanishes:

$$\mathrm{ind}^{\rho}_{[0]}(\mathcal{D}) = 0 \in (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle. \tag{4.4}$$

This simple observation plays a fundamental role.

4.5 Lemma. *If Γ is torsion-free and the Baum–Connes map $\mu_{\max}\colon K_0(B\Gamma) \to K_0(C^*\Gamma)$ is surjective, then the ρ-index vanishes*:

$$\mathrm{ind}^{\rho}_{b}(\mathcal{D},C)_{[0]} = 0 \quad \text{in } (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle.$$

Proof. Observe that we can define a homomorphisms

$$\Phi^{\rho}\colon K_0(C^*\Gamma) \to (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle,$$

by simply composing $\mathrm{tr}^{\mathrm{alg}}\colon K_0(C^*\Gamma) \to (C^*\Gamma)_{\mathrm{ab}}$ with the quotient map $(C^*\Gamma)_{\mathrm{ab}} \to (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$. Using the above remark, we see that this homomorphism is zero on the image of $\mu_{\max}$, since the image of $\mu_{\max}$ consists precisely of index classes associated to *closed* manifolds. We use the Baum–Douglas [4] description of the K-homology of $B\Gamma$ here. By assumption, $\mu_{\max}$ is surjective, so that $\Phi^\rho = 0$ on all of $K_0(C^*\Gamma)$. Thus $0 = \Phi^\rho(\mathrm{Ind}_b(\mathcal{D}, C)) = \mathrm{ind}_b^\rho(\mathcal{D}, C)_{[0]}$ and the assertion follows. □

4.2. The stable rho-invariant. Assume that $\mathcal{D}$ is a Dirac type operator on a closed odd-dimensional manifold M, twisted with the Mishchenko–Fomenko line bundle $\mathcal{L}$ associated to some classifying map $u\colon M \to B\Gamma$. Assume that $\mathrm{Ind}(\mathcal{D}) = 0 \in K_1(C^*\Gamma)$. Pick a trivializing perturbation $C \in \mathfrak{P}_{\mathcal{D}}$.

4.6 Lemma. *If Γ is torsion-free and the Baum–Connes map $\mu_{\max}\colon K_0(B\Gamma) \to K_0(C^*\Gamma)$ is surjective, then*

$$[\eta_{[0]}(\mathcal{D} + C)] \in (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle, \tag{4.7}$$

i.e., the image of $\eta_{[0]}(\mathcal{D} + C) \in (C^*\Gamma)_{\mathrm{ab}}$ in the quotient $(C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$, does not depend on the particular perturbation $\mathcal{C}$ chosen.

Proof. Let $\mathcal{C}' \in \mathfrak{P}$ be a different perturbation. Consider the cylinder $[-1,1]\times M$. On the boundary of the cylinder we have an invertible operator, obtained by considering the operator $\mathcal{D} + \mathcal{C}$ on $\{-1\} \times M$ and the operator $\mathcal{D} + \mathcal{C}'$ on $\{1\} \times M$. As already explained, there is a well-defined b-index class in $K_0(C^*\Gamma)$, obtained by lifting the two perturbations to the cylinder and defining a b-pseudodifferential operator $\mathcal{D}_{[-1,1]\times M} + \mathcal{C}_{[-1,1]\times M}$ with *invertible* indicial family. We denote this index class by

$$\mathrm{Ind}_b(\mathcal{D}_{[-1,1]\times M} + \mathcal{C}_{[-1,1]\times M}).$$

Using the APS-index formula 3.3 we obtain

$$\begin{aligned}\mathrm{ind}_b^\rho&(\mathcal{D}_{[-1,1]\times M} + \mathcal{C}_{[-1,1]\times M})_{[0]}\\ &= -\tfrac{1}{2}(\rho_{[0]}(\mathcal{D} + \mathcal{C}) - \rho_{[0]}(\mathcal{D} + \mathcal{C}')) \in (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle.\end{aligned}$$

On the other hand, by the assumed surjectivity of the Baum–Connes map we know that $\mathrm{ind}_b^\rho(\mathcal{D}_{[-1,1]\times M} + \mathcal{C}_{[-1,1]\times M}) = 0$ and the lemma is proved. □

4.8 Definition. Assume that $\mathrm{Ind}(\mathcal{D}) = 0 \in K_1(C^*\Gamma)$ and that the max-Baum–Connes map $K_0(B\Gamma) \to K_0(C^*\Gamma)$ is surjective. Then the *stable degree-zero rho-invariant* is the class

$$\rho^s_{[0]}(\mathcal{D}) := [\eta_{[0]}(\mathcal{D} + C)] \in (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle \tag{4.9}$$

for any perturbation $C \in \mathfrak{P}_{\mathcal{D}}$, and it is well defined because of Lemma 4.6.

4.10 Notation. Let $r\colon M \to B\Gamma$ be a classifying map. We shall frequently use the notation $\rho^s_{[0]}(M,r)$ with the understanding that this rho-invariant is associated either to the signature operator on the oriented Riemannian manifold M, or to the spin Dirac operator on the spin manifold M (both twisted by the Mishchenko–Fomenko bundle), depending on the context. Sometimes we shall omit the classifying map from the notation thus writing $\rho^s_{[0]}(M)$.

4.3. Bordism invariance. Let (M,g) be an odd dimensional Riemannian manifold endowed with a bundle of Clifford modules defining a Dirac-type operator D_M. Let $(M, r\colon M \to B\Gamma)$ be a Galois covering and let $\mathcal{D} := D_{M,\mathcal{L}}$ be the twisted Dirac operator in the Mishchenko–Fomenko calculus. Let (M',g') be another odd dimensional Riemannian manifold endowed with a Clifford module and let $(M', r'\colon M' \to B\Gamma)$ be a Galois covering on M'; set $\mathcal{D}' := D_{M',\mathcal{L}'}$. Assume now the existence of a bordism $(W, R\colon W \to B\Gamma)$ between $(M, r\colon M \to B\Gamma)$ and $(M', r'\colon M' \to B\Gamma)$; we also assume the existence of a Riemannian metric and of a bundle of Clifford modules on W restricting to the given data on M and M'.

4.11 Proposition. *Assume Γ is torsion-free and such that the Baum–Connes map μ_{max} is surjective. Assume that* $\mathrm{Ind}(\mathcal{D}) = 0$ *in* $K_1(C^*\Gamma)$ *so that the stable rho-invariant* $\rho^s_{[0]}(\mathcal{D})$ *is well defined. By the bordism invariance of the index class we know that* $\mathrm{Ind}(\mathcal{D}') = 0$ *so that* $\rho^s_{[0]}(\mathcal{D}')$ *is also well defined. We have*

$$\rho^s_{[0]}(\mathcal{D}) = \rho^s_{[0]}(\mathcal{D}').$$

Proof. The argument establishing Lemma 4.6 can be repeated here, provided we substitute the cylinder $[-1,1] \times M$ there with the bordism W here. □

5. Injectivity of the Baum–Connes map and bordism

5.1. Statement of results. In contrast to the previous section we now consider only the spin Dirac operator $\not{D}$ and the signature operator D^{sign} and not arbitrary generalized Dirac type operators.

5.1 Definition. Consider $(M, u\colon M \to B\Gamma)$. If $d \in \mathbb{N} \setminus \{0\}$, then we denote by $d(M, u\colon M \to B\Gamma)$ the disjoint union of d copies of M with obvious induced map $du\colon dM \to B\Gamma$, and with obvious induced structure (orientation, etc.).

5.2 Proposition. *Assume that M is a closed oriented manifold, $u\colon M \to B\Gamma$ a classifying map. Let $\mathcal{D}^{\mathrm{sign}}$ be the associated Mishchenko–Fomenko signature operator. Assume that $\mu_{\max}\colon K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C^*\Gamma) \otimes \mathbb{Q}$ is injective and that*

$\mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}) = 0$ *in* $K_{\dim M}(C^*\Gamma) \otimes \mathbb{Q}$. *Then there exists* $d \in \mathbb{N} \setminus \{0\}$ *such that* $d \cdot [M, u\colon M \to B\Gamma]$ *is bordant to*

$$\bigcup_{j=1}^{k} (A_j \times B_j, r_j \times 1\colon A_j \times B_j \to B\Gamma)$$

in $\Omega^{\mathrm{SO}}_{\dim M}(B\Gamma)$, *with* $\dim B_j = 4b_j$, $\pi_1(B_j) = 1$ *and* $\langle L(B_j), [B_j]\rangle = \mathrm{sgn}(B_j) = 0$. *Here* $r_j\colon A_j \to B\Gamma$ *is a continuous map and* $(r_j \times 1)(a, b) := r_j(a)$.

5.3 Proposition. *Assume that* M *is a closed spin manifold,* $u\colon M \to B\Gamma$ *a classifying map. Let* $\mathcal{D} := D\!\!\!/_{\mathcal{L}}$ *be the associated Mishchenko–Fomenko spin Dirac operator. Assume that* $\mu_{\max}\colon K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C^*\Gamma) \otimes \mathbb{Q}$ *is injective and that* $\mathrm{Ind}(\mathcal{D}) = 0 \in K_*(C^*\Gamma) \otimes \mathbb{Q}$. *Then there is* $d \in \mathbb{N} \setminus \{0\}$ *such that* $d[M, u\colon M \to B\Gamma]$ *is bordant to*

$$\bigcup_{j=1}^{k} (A_j \times B_j, r_j \times 1\colon A_j \times B_j \to B\Gamma)$$

in $\Omega^{\mathrm{spin}}_{\dim M}(B\Gamma)$, *with* $\dim B_j = 4b_j$, $\pi_1(B_j) = 1$ *and* $\langle \hat{A}(B_j), [B_j]\rangle = 0$. *Here* $r_j\colon A_j \to B\Gamma$ *is a continuous map and* $(r_j \times 1)(a, b) := r_j(a)$.

5.4 Remark. In Propositions 5.2 and 5.3, the condition that the index of the Mishchenko–Fomenko operator in the K-theory of the maximal C*-algebra vanishes can be replaced by the more familiar condition that the index in the K-theory of the reduced C*-algebra vanishes, if we replace the assumption on the rational injectivity of the maximal Baum–Connes map by the analogous one for the reduced C*-algebra. Moreover, it should be remarked that the rational injectivity of the *complex* Baum–Connes map is equivalent to the rational injectivity of the *real* Baum–Connes map. This is a well-known result, a proof can be found in [61].

We prove Propositions 5.2 and 5.3 with the same method from algebraic topology, which characterizes rationalized homology theories.

5.2. Rational homology theories. Let h_* and k_* be two generalized homology theories. We will be interested in the examples Ω^{spin}_* of spin bordism, Ω^{SO}_* of oriented bordism, K_* and KO_* of complex or real K-homology (the homology theory dual to K-theory).

Let $\check{h}_* := h_*(pt)$ and $\check{k}_* := k_*(pt)$ be the coefficients.

It is easy to see that $h_* \otimes \mathbb{Q}$ and $H_*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} (\check{h}_* \otimes \mathbb{Q})$ (graded tensor product, the n-th group is $\bigoplus_{p+q=n} H_p(X) \otimes \check{h}_q$) are again homology theories, compare e.g. [23, 3.18 ff].[4]

[4]One remark on the reference [23]: here only cohomology is considered, but everything works exactly in the same way if cohomology is replaced by homology.

5.5 Proposition. (1) *Every natural transformation* $T\colon h_* \otimes \mathbb{Q} \to k_* \otimes \mathbb{Q}$ *is determined by* $\check{T}\colon \check{h}_* \otimes \mathbb{Q} \to \check{k}_* \otimes \mathbb{Q}$.

(2) *Every homomorphism* $\check{T}\colon \check{h}_* \otimes \mathbb{Q} \to \check{k}_* \otimes \mathbb{Q}$ *has a unique extension to a natural transformation* $T\colon h_* \otimes \mathbb{Q} \to k_* \otimes \mathbb{Q}$.

(3) *There is a unique natural transformation* $H_*(\,\cdot\,;\mathbb{Q}) \otimes (\check{h}_* \otimes \mathbb{Q}) \to h_*(\,\cdot\,) \otimes \mathbb{Q}$ *which is the identity on the point. It is a natural isomorphism of* $\check{h}_* \otimes \mathbb{Q}$*-modules if* h_* *is a multiplicative homology theory.*

Proof. Compare [23, (3.20), (3.21), (3.22)]. □

Now, we apply this to the examples we want to study. By definition, elements in $K_*(X)$ are represented by triples (M, E, ϕ), where M is a closed oriented Riemannian manifold of dimension congruent to $* \pmod 2$, E is a bundle of Clifford modules on M and $\phi\colon M \to X$ is continuous. Similarly, $KO_*(X)$ is defined by triples (M, E, ϕ) as above, with the exception that E is a bundle of real Clifford modules. Of course, a suitable equivalence relation has to be factored out. See [4] and [28].

In particular, we get natural transformations

$$\begin{aligned} T_{\mathrm{SO}}\colon \Omega_*^{\mathrm{SO}}(X) &\to K_*(X), \qquad [M \xrightarrow{\phi} X] \mapsto [M, \Lambda^{\mathrm{sign}}, \phi]; \\ T_{\mathrm{spin}}\colon \Omega_*^{\mathrm{spin}}(X) &\to KO_*(X), \quad [M \xrightarrow{\phi} X] \mapsto [M, \$, \phi]. \end{aligned}$$

Here Λ^{sign} stands for the complex Clifford module defining the signature operator on the oriented Riemannian manifold M, whereas $\$$ is the spin Clifford module defining the real Dirac operator of the given spin structure.[5] If X is a point, T_{SO} is just given by the signature (using the canonical isomorphism $K_{2n}(pt) = \mathbb{Z}$). Also if $X = \{pt\}$, T_{spin} is the α-invariant, which is (up to a multiple) equal to $\hat{A}(M)$ for dimensions divisible by 4 (again use the canonical isomorphism $KO_{4n}(pt) = \mathbb{Z}$). Note that, in other dimensions, $KO_k(pt)$ is finite, and $K_{2n+1}(pt) = 0$.

Using Proposition 5.5 we get commutative diagrams:

$$\begin{array}{ccc} H_*(X;\mathbb{Q}) \otimes (\Omega_*^{\mathrm{SO}}(pt) \otimes \mathbb{Q}) & \xrightarrow{\mathrm{id} \otimes T_{\mathrm{SO}}^{\mathbb{Q}}} & H_*(X;\mathbb{Q}) \otimes (K_*(pt) \otimes \mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ \Omega_*^{\mathrm{SO}}(X) \otimes \mathbb{Q} & \xrightarrow{T_{\mathrm{SO}}^{\mathbb{Q}}} & K_*(X) \otimes \mathbb{Q} \end{array} \tag{5.6}$$

[5]Topologists refer to these transformations as the *natural K-orientation associated to an* (*ordinary*) *orientation after inversion of* 2 and *natural KO-orientation associated to a spin structure* on M, respectively.

$$\begin{array}{ccc} H_*(X;\mathbb{Q}) \otimes (\Omega_*^{\mathrm{spin}}(pt) \otimes \mathbb{Q}) & \xrightarrow{\mathrm{id} \otimes T^{\mathbb{Q}}_{\mathrm{spin}}} & H_*(X;\mathbb{Q}) \otimes (KO_*(pt) \otimes \mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ \Omega_*^{\mathrm{spin}}(X) \otimes \mathbb{Q} & \xrightarrow{T^{\mathbb{Q}}_{\mathrm{spin}}} & KO_*(X) \otimes \mathbb{Q}. \end{array} \tag{5.7}$$

It is not really necessary for us to understand completely the vertical isomorphisms. The important point is that $\Omega_*^{\mathrm{spin}}(X) \otimes \mathbb{Q}$ is a *free* module over $\Omega_*^{\mathrm{spin}}(pt) \otimes \mathbb{Q}$ (free generators given by a basis of $H_*(X;\mathbb{Q})$ under the vertical isomorphism). The corresponding statements hold for $\Omega_*^{\mathrm{SO}}(X) \otimes \mathbb{Q}$ and $K_*(X) \otimes \mathbb{Q}$, $KO_*(X) \otimes \mathbb{Q}$. Moreover, we also see that the basis can be chosen in a natural way and such that the transformations $T^{\mathbb{Q}}_{\mathrm{SO}}$ and $T^{\mathbb{Q}}_{\mathrm{spin}}$ are diagonal with respect to them. In particular, $\ker(T^{\mathbb{Q}}_{\mathrm{SO}}(X))$ is a free module over $\ker(T^{\mathbb{Q}}_{\mathrm{SO}}(pt))$ and $\ker(T^{\mathbb{Q}}_{\mathrm{spin}}(X))$ is a free module over $\ker(T^{\mathbb{Q}}_{\mathrm{spin}}(pt))$.

This means that every element in the kernel of $T^{\mathbb{Q}}_{\mathrm{SO}}$ or $T^{\mathbb{Q}}_{\mathrm{spin}}$, respectively, will after multiplication with a suitable non-zero integer (to clear the denominators) have the form described in Proposition 5.2 or Proposition 5.3, respectively. Here we use the fact that the $\check{\Omega}_*$-module structure of $\Omega_*(X)$ is given by the Cartesian product.

Finally, recall that

$$\mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}) = \mu_{\max}(T_{\mathrm{SO}}([M \xrightarrow{u} B\Gamma])) \in K_{\dim M}(C^*\Gamma))$$

in the situation of Proposition 5.2, and that

$$\mathrm{Ind}(\not{\mathcal{D}}_{\mathbb{R}}) = \mu_{\mathbb{R},\max}(T_{\mathrm{spin}}([M \xrightarrow{u} B\Gamma])) \in KO_{\dim M}(C^*_{\mathbb{R}}\Gamma)$$

in the situation of Proposition 5.3. Because we assume that $\mu_{\max} \otimes \mathbb{Q}$ is injective, Proposition 5.2 on the signature operator follows from the descriptions of $\ker(T^{\mathbb{Q}}_{\mathrm{SO}})$ we have just given. Concerning Proposition 5.3, we point out that the rational vanishing of the complex index class associated to $\not{\mathcal{D}}$ is equivalent to the rational vanishing of the corresponding real index class for the real spin Dirac operator $\not{\mathcal{D}}_{\mathbb{R}}$. Recalling that the rational injectivity of the real Baum–Connes map $\mu_{\mathbb{R},\max}$ is equivalent to the rational injectivity of the complex Baum–Connes map $\mu_{\max}$ [61], we see that the result stated in Proposition 5.3 is again a consequence of the description of $\ker(T^{\mathbb{Q}}_{\mathrm{spin}})$ we have just given.

6. Vanishing of stable rho-invariants for certain product manifolds

6.1 Theorem. *Assume that $M = U \times V$, where V is a simply connected oriented Riemannian manifold of dimension divisible by* 4 *with vanishing signature, and U is* *any oriented Riemannian manifold of odd dimension. Let A be a C^*-algebra and $\mathcal{L}$ a flat finitely generated projective Hilbert A-module bundle on U (defining a bundle, also called $\mathcal{L}$, on M by pullback). Let $D_{\mathcal{L}}$ be the corresponding signature operator.*

Then $\operatorname{Ind} D_{\mathcal{L}} = 0 \in K_{\dim M}(A)$ *and there exists a perturbation $\mathcal{C} \in \mathfrak{P}_{D_{\mathcal{L}}}$ such that*

$$\eta_{[0]}(D_{\mathcal{L}} + \mathcal{C}) = 0 \in A_{\mathrm{ab}}.$$

Proof. For the differential forms on $U \times V$ we have the following decomposition:

$$L^2\Omega(U \times V) = L^2\Omega(U) \otimes L^2\Omega^+(V) \oplus L^2\Omega(U) \otimes L^2\Omega^-(V)$$

where $\Omega^{\pm}(V)$ refers to the grading associated to the signature operator on V. Since the bundle $\mathcal{L}$ pulls back from V, we get a corresponding decomposition of the Hilbert A-module of forms with coefficients in $\mathcal{L}$:

$$L^2\Omega(U \times V; \mathcal{L}) = L^2\Omega(U; \mathcal{L}) \otimes L^2\Omega^+(V) \oplus L^2\Omega(U; \mathcal{L}) \otimes L^2\Omega^-(V).$$

With respect to this decomposition, the signature operator splits as

$$D_{U\times V} = \begin{pmatrix} D_U \otimes 1_V & 1_U \otimes D_V^- \\ 1_U \otimes D_V^+ & -D_U \otimes 1_V \end{pmatrix} = D_U \otimes \tau + 1_U \otimes D_V,$$

where τ is the grading operator on $L^2\Omega(V)$. Since $\mathcal{L}$ is flat, we get a corresponding splitting of the signature operator twisted with $\mathcal{L}$:

$$D_{\mathcal{L},U\times V} = \begin{pmatrix} D_{\mathcal{L},U} \otimes 1_V & 1_U \otimes D_V^- \\ 1_U \otimes D_V^+ & -D_{\mathcal{L},U} \otimes 1_V \end{pmatrix} = D_{\mathcal{L},U} \otimes \tau + 1_U \otimes D_V. \quad (6.2)$$

Let us now define the perturbed operator. Choose an isometry $\Psi\colon \ker(D_V^+) \to \ker(D_V^-)$. Since V is compact with vanishing signature, such an isomorphism of finite dimensional vector spaces exists. Then

$$C := \begin{pmatrix} 0 & \Psi^* \\ \Psi & 0 \end{pmatrix} : L^2\Omega^+(V) \oplus L^2\Omega^-(V) \to L^2\Omega^+(V) \oplus L^2\Omega^-(V)$$

is a smoothing operator on V (here "Ψ" stands for the projection onto $\ker(D_V^+)$ follows by the operator Ψ).

The operator $D_{\mathcal{L},U\times V} + 1_U \otimes C$ is invertible. Let $f(D_{\mathcal{L},U})$ be any smoothing function of $D_{\mathcal{L},U}$ which is the identity on a sufficiently large neighborhood of zero in the spectrum of $D_{\mathcal{L},U}$ so that

$$\mathcal{D}_C := D_{\mathcal{L},U\times V} + f(D_{\mathcal{L},U}) \otimes C$$

becomes invertible. This will be our perturbation; notice that since we have found a smoothing perturbation of $D_{\mathcal{L},U\times V}$ which is invertible, we conclude that $\operatorname{Ind}(D_{\mathcal{L},U\times V}) = 0$ in $K_{\dim M}(A)$, as required.

We want to show that the degree 0 eta-invariant of this perturbation $\mathcal{D}_C$ vanishes. Since $L^2\Omega(V)$ is an ordinary Hilbert space where orthogonal complements exist, we get another orthogonal decomposition into projective Hilbert A-modules:

$$L^2\Omega(U \times V; \mathcal{L}) = (L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V)) \oplus (L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V)^{\perp}).$$

The operator $D_{\mathcal{L},U\times V}$ as well as $f(D_{\mathcal{L},U}) \otimes C$ preserves the splitting

$$L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V) \oplus L^2\Omega(U;\mathcal{L}) \otimes (\ker(D_V)^{\perp}).$$

Moreover, the second operator is zero on the second summand. Therefore, the integrand in the definition of $\eta_{[0]}(\mathcal{D}_C)$ splits into two summands. We continue by investigating the two summands separately. We will use freely from equation (6.2): $D_{\mathcal{L},U\times V} = D_{\mathcal{L},U} \otimes \tau + 1_U \otimes D_V$. We restrict to $L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V)^{\perp}$. We denote the restriction of $\mathcal{D}_C$ to this space by $(\mathcal{D}_C)_r$ and the restrictions of τ and D_V to $\ker(D_V)^{\perp}$ by τ_r and $(D_V)_r$. Then

$$\begin{aligned}(\mathcal{D}_C)_r\, e^{-t(\mathcal{D}_C)_r^2} &= D_{\mathcal{L},U\times V} e^{-tD^2_{\mathcal{L},U}} \otimes e^{-t(D_V)_r^2} \\ &= D_{\mathcal{L},U} e^{-tD^2_{\mathcal{L},U}} \otimes (\tau)_r e^{-t(D_V)_r^2} + e^{-tD^2_{\mathcal{L},U}} \otimes (D_V)_r e^{-t(D_V)_r^2}.\end{aligned} \tag{6.3}$$

We claim that the degree zero trace with values in A_{ab} of both summands vanishes. Note that TR is multiplicative for tensor products. Here, we get TR of the first factor multiplied with the ordinary complex-valued trace of the second factor.

For the first summand

$$\mathrm{Tr}((\tau)_r e^{-t(D_V)_r^2}) = 0$$

by the McKean–Singer formula (we restrict to the orthogonal complement of the kernel, so the index is zero); therefore

$$\mathrm{TR}(D_{\mathcal{L},U} e^{-tD^2_{\mathcal{L},U}} \otimes (\tau)_r e^{-t(D_V)_r^2}) = \mathrm{TR}(D_{\mathcal{L},U} e^{-tD^2_{\mathcal{L},U}})\, \mathrm{Tr}((\tau)_r e^{-t(D_V)_r^2}) = 0.$$

For the second summand, the same argument applies since $\mathrm{TR}(e^{-tD^2_{\mathcal{L},U}}) = 0$ on the odd dimensional manifold U by the usual symmetry argument.

Now we restrict $\mathcal{D}_C$ to $L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V)$ (and we again denote the relevant restriction with $(\ )_r$).

We will finish by showing that this summand also is identically equal to zero, which then implies immediately that $\eta_{[0]}(\mathcal{D}_C) = 0$. To see this, consider the operator

$$B := 1_U \otimes (\tau \circ C) \colon L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V) \to L^2\Omega(U;\mathcal{L}) \otimes \ker(D_V).$$

This is an isometry which anticommutes with $D_{\mathcal{L},U} \otimes \tau$ (since C anticommutes and τ commutes with τ) and which anticommutes with $1_U \otimes C$ (since C commutes and τ anticommutes with C). Similarly, B anticommutes with $f(D_{\mathcal{L},U}) \otimes C$. Consequently,

B anticommutes with the restriction of $\mathcal{D}_C$ to $L^2\Omega(U;\mathcal{L})\otimes\ker(D_V)$, which is just given by $D_{\mathcal{L},U}\otimes\tau+f(D_{\mathcal{L},U})\otimes C$. Of course, this implies also that B commutes with $e^{-t(D_{\mathcal{L},U\times V})_r^2}$. Therefore (everything restricted to $L^2\Omega(U;\mathcal{L})\otimes\ker(D_V)$)

$$\begin{aligned}\mathrm{TR}((\mathcal{D}_C)_r e^{-t(\mathcal{D}_C)_r^2}) &= \mathrm{TR}(B^{-1}B(\mathcal{D}_C)_r e^{-t(\mathcal{D}_C)_r^2})\\ &= -\mathrm{TR}(B^{-1}(\mathcal{D}_C)_r e^{-t(\mathcal{D}_C)_r^2}B)\\ &= -\mathrm{TR}((\mathcal{D}_C)_r e^{-t(\mathcal{D}_C)_r^2}) \in A_{\mathrm{ab}}.\end{aligned}$$

The last equality follows from the trace property. Consequently, the integrand also vanishes identically.

Putting everything together, we get

$$\eta_{[0]}(\mathcal{D}_C)=\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{TR}(\mathcal{D}_C e^{-t\mathcal{D}_C^2})=\frac{1}{\sqrt{\pi}}\int_0^\infty 0=0. \qquad \square$$

As a special case of the theorem, we obtain the following corollary.

6.4 Corollary. *Assume that $M=U\times V$, where V is a simply connected oriented $4k$-dimensional Riemannian manifold such that $\langle L(V),[V]\rangle=0$, and U is any oriented Riemannian manifold of odd dimension. Let $r\colon U\to B\Gamma$ be a continuous map. Then there exists a smoothing perturbation $\mathcal{C}$ such that $\mathcal{D}+\mathcal{C}$ is invertible and $\eta_{[0]}(\mathcal{D}+\mathcal{C})$ vanishes, where $\mathcal{D}$ is the Mishchenko–Fomenko signature operator associated to $r\times 1$. In particular, if the Baum–Connes map is surjective, then the stable rho-invariant is defined in $C^*\Gamma_{\mathrm{ab}}/\langle[1]\rangle$ and is equal to zero*: $\rho^s_{[0]}(U\times V, r\times 1)=0$.

There are similar results in the spin context:

6.5 Theorem. *Assume that $M=U\times V$, where V is a simply connected spin manifold of dimension divisible by 4 with $\widehat{A}(V)=0$, and U is any spin manifold of odd dimension. Let A be a real C^*-algebra and let $\mathcal{L}$ be a flat finitely generated projective Hilbert A-module bundle on U (defining a bundle, also called $\mathcal{L}$, on M by pullback). There is a product metric on $U\times V$ such that for the resulting twisted spin Dirac operator $D_{\mathcal{L}}$ we get*

$$\eta_{[0]}(D_{\mathcal{L}})=0\in A_{\mathrm{ab}}.$$

Proof. By Stephan Stolz's solution [62] of the Gromov–Lawson–Rosenberg conjecture for simply connected manifolds, V admits a metric with positive scalar curvature. If we shrink this metric appropriately, the corresponding product metric on $U\times V$ will still have positive scalar curvature. The Lichnerowicz formula then implies that its spin Dirac operator D_L is invertible (here we use that the bundle is flat). In particular, we do not need a perturbation and can define $\eta_{[0]}(D_{\mathcal{L}})$.

Secondly, we can use the product structure (of the Dirac operator) and argue in a way similar to the proof of Theorem 6.4 (but with the simplification that the operator on V has no kernel) to conclude that

$$\eta_{[0]}(D_L) = 0 \in (C^*\Gamma)_{\mathrm{ab}}. \qquad \square$$

7. Vanishing of stable rho-invariants

7.1 Theorem. *Let M be an odd dimensional oriented closed compact manifold and let $u\colon M \to B\Gamma$ be a continuous map, classifying a Galois Γ-covering $\widetilde{M} \to M$. If the max-Baum–Connes map $\mu_{\max}\colon K_*(B\Gamma) \to K_*(C^*\Gamma)$ is bijective and the signature index class of $(M, u\colon M \to B\Gamma)$ vanishes in $K_{\dim M}(C^*\Gamma)$ then*

$$\rho^s_{[0]}(M,u) = 0. \tag{7.2}$$

Proof. We use the injectivity of the max-Baum–Connes map, the assumption $\mathrm{Ind}(\mathcal{D}) = 0$ and Proposition 5.2 in order to conclude that there exists a bordism $(W, F\colon W \to B\Gamma)$ between $(dM, du\colon M \to B\Gamma)$ and

$$\bigcup_{j=1}^{k} (A_j \times B_j, r_j \times 1\colon A_j \times B_j \to B\Gamma)$$

with $\pi_1(B_j) = 1$ and $\langle L(B_j), [B_j]\rangle = 0$. Denote briefly

$$\bigcup_{j=1}^{k} (A_j \times B_j, r_j \times 1\colon A_j \times B_j \to B\Gamma)$$

by $(N, v\colon N \to B\Gamma)$. By cobordism invariance, the index of the signature operator associated to $(N, v\colon N \to B\Gamma)$ is zero in $K_{\dim M}(C^*\Gamma)$. From the surjectivity of the max-Baum–Connes map we know that there are well-defined stable rho-invariants $\rho^s_{[0]}(M,u)$, $\rho^s_{[0]}(N,v)$. Fix allowable perturbations $\mathcal{C}_M$, $\mathcal{C}_N$. Then there exists a well-defined signature b-index class, in $K_{\dim W}(C^*\Gamma)$, associated to the bordism $(W, F\colon W \to B\Gamma)$ and to the perturbations $d\mathcal{C}_M$, $\mathcal{C}_N$, which give rise to a Mishchenko–Fomenko b-smoothing operator $\mathcal{A}_W$ on W. We denote this b-index class by $\mathrm{Ind}_b(\mathcal{D}_W + \mathcal{A}_W) \in K_{\dim W}(C^*\Gamma)$. Now we proceed as in the proof of Lemma 4.6. On the one hand, by the surjectivity of $\mu_{\max}$ we have

$$\mathrm{Ind}^{\rho}_b(\mathcal{D}_W + \mathcal{A}_W) = 0 \in (C^*\Gamma)_{\mathrm{ab}}/\langle 1\rangle;$$

on the other hand by applying the APS index Theorem 3.3 we get

$$\mathrm{Ind}^{\rho}_b(\mathcal{D}_W + \mathcal{A}_W)) = -\frac{1}{2}(d\rho^s_{[0]}(M,u) - \rho^s_{[0]}(N,v)) \in (C^*\Gamma)_{\mathrm{ab}}/\langle 1\rangle,$$

from which we deduce that

$$\rho^s_{[0]}(M,u) = \frac{1}{d}\rho^s_{[0]}(N,v) = \frac{1}{d}\sum_{j=1}^{k}\rho^s_{[0]}(A_j \times B_j, r_j \times 1).$$

We finish the proof by applying Corollary 6.4. □

7.3 Theorem. *Let M be an odd dimensional closed compact spin manifold and let $u\colon M \to B\Gamma$ be a continuous classifying map. Let $\mathcal{D} := \not{D}_{\mathcal{L}}$ be the associated Mishchenko–Fomenko spin Dirac operator. If the max-Baum–Connes map $\mu_{\max}\colon K_*(B\Gamma) \to K_*(C^*\Gamma)$ is bijective and the index class* $\mathrm{Ind}(\mathcal{D}) = 0 \in K_{\dim M}(C^*\Gamma)$ *vanishes, then*

$$\rho^s_{[0]}(M,u) = 0. \tag{7.4}$$

Proof. The proof follows exactly the same pattern of the proof of Theorem 7.1. More details in Subsection 13.2 (see in particular the proof of Theorem 13.9). □

8. Unstable rho-invariants

In this section, let A be a von Neumann algebra. Let Z be a commutative von Neumann algebra, and $\tau\colon A \to Z$ a positive and normal trace on A with values in Z.

In Section 4 we managed to define, if the index class is zero and the Baum–Connes map is surjective, the stable rho-invariant $\rho_{[0]}$. In Section 7 we showed that, under the additional assumption of injectivity of the Baum–Connes map, it is zero.

We now introduce unstable rho-invariants which are potentially more interesting, since they are defined under much more general hypothesis.

Assume that M is a closed manifold of odd dimension and let D be a Dirac type operator on M, acting on sections of a bundle E. Let $\mathcal{L}$ be a bundle of finitely generated projective Hilbert A-modules on M, with a connection preserving all the structure. Let $D_{\mathcal{L}}$ be the corresponding twisted Dirac operator. Each fiber of $\mathcal{L}$ is a finitely generated projective module; $\mathcal{L}_x = \mathrm{im}(p_x)$, with p_x a projection in $\mathrm{M}_{k\times k}(A)$ for some k; let $\tau(\mathcal{L}_x) := \tau(p_x)$ where we extend the trace τ to $\mathrm{M}_{k\times k}(A)$ in the obvious way. We assume that $\tau(\mathcal{L}_x)$ is constant in x.

8.1 Definition. We define the τ-eta-invariant as

$$\eta_\tau(D_{\mathcal{L}}) := \frac{1}{\sqrt{\pi}}\int_0^\infty \tau(\mathrm{TR}(D_{\mathcal{L}}\exp(-tD^2_{\mathcal{L}})))\,\frac{dt}{\sqrt{t}} \in Z. \tag{8.2}$$

We have to check that this integral converges. For $t \to 0$, this follows from the usual local heat expansion. Since τ is positive and normal, the estimates of Cheeger and Gromov [10] can be used to obtain convergence for large times.

8.3 Definition. Let A_1 and A_2 be von Neumann algebras; let $\mathcal{L}_1, \mathcal{L}_2$ be two Hilbert A_j-module bundles over M; let $\tau_j : A_j \to Z_j$ be positive normal traces with values in commutative von Neumann algebras Z_j. Let $\beta_j : Z_j \to V$ be homomorphisms to a fixed target space V (a vector space). Assume that $\beta_1(\tau_1((\mathcal{L}_1)_x)) + \beta_2(\tau_2((\mathcal{L}_2)_x)) = 0 \in V$. The *unstable rho-invariant* of $D_{\mathcal{L}}$ with respect to β_j, τ_j and $\mathcal{L}_j$ is defined as

$$\rho_{(\beta_j,\tau_j,\mathcal{L}_j)}(D) := \beta_1(\eta_{\tau_1}(D_{\mathcal{L}_1})) + \beta_2(\eta_{\tau_2}(D_{\mathcal{L}_2})) \in V.$$

The definition can be extended to a finite number of summands in the obvious way.

We give examples showing the interest of such a definition.

General example. Let M be a closed oriented Riemannian manifold of odd dimension. Assume that Γ is a discrete group; let $u\colon M \to B\Gamma$ be a map classifying a covering $\widetilde{M} = u^*E\Gamma$. Let $\alpha_j : C^*\Gamma \to A_j$ be homomorphisms to unital von Neumann algebras A_j (with $j = 1, 2$), and $\tau_j : A_j \to Z_j$ positive normal traces with values in abelian von Neumann algebras Z_j. Let $\mathcal{L}_j := \widetilde{M} \times_\Gamma A_j$ be the associated Hilbert A_j-module bundle, where Γ acts on A_j via $\Gamma \to C^*\Gamma \xrightarrow{\alpha_j} A_j$. Then $\tau_j((\mathcal{L}_j)_x) = \tau_j(\alpha_j(1))$. If $\beta_j : Z_j \to V$ satisfy

$$\beta_1(\tau_1(\alpha_1(1))) + \beta_2(\tau_2(\alpha_2(1))) = 0 \in V$$

then the unstable rho-invariant $\rho_{(\beta_j,\tau_j,\mathcal{L}_j)}(D) \in V$ is well defined and equal to

$$\beta_1(\eta_{\tau_1}(D_{\mathcal{L}_1})) + \beta_2(\eta_{\tau_2}(D_{\mathcal{L}_2})) \in V. \tag{8.4}$$

8.5 Example (Atiyah–Patodi–Singer rho-invariant). We refer to the general example. The relevant von Neumann algebras here are $A_1 = \mathrm{M}_d(\mathbb{C}) = A_2$, with two homomorphisms $\alpha_j : C^*\Gamma \to \mathrm{M}_d(\mathbb{C})$ induced by two representations $\lambda_1, \lambda_2 \colon \Gamma \to \mathrm{U}(d)$. The relevant trace is (in both cases) the usual trace $\tau\colon \mathrm{M}_d(\mathbb{C}) \to \mathbb{C}$ on matrices. Then $\tau(\alpha_1(1)) = d = \tau(\alpha_2(1))$, so that, with $V = \mathbb{C}$, we can choose $\beta_1 = \mathrm{id}$ and $\beta_2 = -\mathrm{id}$. By equation (D.1), the eta-invariants appearing in the definition of the APS-rho-invariant and those appearing in formula (8.4) coincide. Thus with these choices $\rho_{(\beta_j,\tau_j,\mathcal{L}_j)}(D) = \rho_{\lambda_1-\lambda_2}(D)$.

8.6 Example (center-valued rho-invariant). We refer to the general example. Let $A_1 := \mathcal{N}\Gamma$ be the group von Neumann algebra of Γ and Z its center. Let $A_2 = \mathbb{C}$. Let α_1 be induced by the natural map $C^*\Gamma \to \mathcal{N}\Gamma$ and let $\alpha_2\colon C^*\Gamma \to \mathbb{C}$ be induced by the trivial representation. Then (by definition) $\mathcal{L}_1 = \mathcal{N} := \widetilde{M} \times_\Gamma \mathcal{N}\Gamma$ and $\mathcal{L}_2 = M \times \mathbb{C}$.

We take $V = Z$, $\tau_1 = \tau\colon \mathcal{N}\Gamma \to Z$ equal to the canonical center-valued trace (compare [26, Chapter 8]); $\tau_2\colon \mathbb{C} \to Z$ given by $\tau_2(z) := z \cdot 1$. Note that both are positive and normal traces. Here $\tau_1(\alpha_1(1)) = 1 = \tau_2(\alpha_2(1))$, so that we can choose again $\beta_1 = \mathrm{id}$, $\beta_2 = -\mathrm{id}$. By Lemma C.8,

$$\rho_{(\beta_j,\tau_j,\mathcal{L}_j)}(D) = \eta_\tau(D_{\mathcal{N}}) - \eta(D)\cdot 1 \in Z. \tag{8.7}$$

We define this element in Z to be the *center-valued rho-invariant* of M.

8.8 Example (L^2-rho- and delocalized eta-invariants). Let $\langle g\rangle$ be a finite conjugacy class in Γ. This defines a trace

$$\tau_{\langle g\rangle}\colon \mathcal{N}\Gamma \to \mathbb{C}, \quad \sum_{h\in\Gamma}\lambda_h h \mapsto \sum_{h\in\langle g\rangle}\lambda_h. \tag{8.9}$$

By the universal property of the central-valued trace τ, $\tau_{\langle g\rangle} = \tau_{\langle g\rangle}\circ\tau$, where we use the restriction of $\tau_{\langle g\rangle}$ to the center Z on the right-hand side. We now apply $\tau_{\langle g\rangle}$ to both sides of Equation (8.7). If $g \neq 1$, $\tau_{\langle g\rangle}(1) = 0$, and we obtain (using Proposition E.11)

$$\tau_{\langle g\rangle}(\rho_{(\beta_j,\tau_j,\mathcal{L}_j)}(D)) = \eta_{\langle g\rangle}(\widetilde{D}).$$

If $g = 1$, then $\tau_{\langle g\rangle} =: \tau_\Gamma$ is the canonical trace, and we obtain by Proposition E.11, using $\tau_\Gamma(1) = 1$,

$$\tau_\Gamma(\rho_{(\beta_j\tau_j\mathcal{L}_j)}(D)) = \rho_{(2)}(\widetilde{D}).$$

Our goal is to prove vanishing results for such generalized unstable rho-invariants and, due to the examples just given, derive the assertions of the Introduction as special cases.

9. Special perturbations of the signature operator

Let $f\colon M \to M'$ be a smooth orientation-preserving homotopy equivalence between two closed manifolds. Let A be a C*-algebra and $\mathcal{V}'$ be a be flat bundle of finitely generated Hilbert A-modules on M'. We consider the signature operator on M' with values in $\mathcal{V}'$. We denote this operator by $D'_{\mathcal{V}'}$. Next, we consider the signature operator on M with values in the flat bundle $\mathcal{V} := f^*\mathcal{V}'$ and we denote it by $D_{\mathcal{V}}$. These two operators come from the de Rham complexes and the Hodge star operator on M' and M with values in the flat bundles $\mathcal{V}'$ and $f^*\mathcal{V}'$, respectively. We denote these de Rham differentials by $d'_{\mathcal{V}'}$ and $d_{\mathcal{V}}$.

In this section we shall construct an *explicit* trivializing perturbation for the Mishchenko–Fomenko signature operator

$$\mathcal{D} := \begin{pmatrix} D_{\mathcal{V}} & 0 \\ 0 & -D'_{\mathcal{V}'} \end{pmatrix} \tag{9.1}$$

on $M \sqcup (-M')$.

Recall that, since f is an orientation-preserving homotopy equivalence, the index in $K_*(A)$ of $\mathcal{D}$ is zero.[6] Thus, we already know that there exists a smoothing perturbation of the operator which is invertible, see [68], [38, Theorem 3]. In this section we shall sharpen this result, showing that we can construct certain *special* perturbations which are *spectrally concentrated near zero* (definition below). We will use these special perturbations in Section 10 in order to show that, in this situation, the stable rho-invariant of $\mathcal{D}$ in (9.1) coincides with the unstable one, getting a more precise result about their eta-invariants in that case.

9.2 Theorem. *Let $f: M \to M'$ be a smooth orientation-preserving homotopy equivalence between two closed manifolds, with $\dim(M) = \dim(M')$ odd. Then, for each $\varepsilon > 0$ we can find a special self-adjoint smoothing perturbation $\mathcal{B}_\varepsilon$ such that $\mathcal{D} + \mathcal{B}_\varepsilon$ is invertible and with the additional property that $\mathcal{B}_\varepsilon$ is ε-spectrally concentrated, that is,*

$$\mathcal{B}_\varepsilon \circ \phi(\mathcal{D}) = 0 = \phi(\mathcal{D}) \circ \mathcal{B}_\varepsilon$$

for each function $\phi: \mathbb{R} \to \mathbb{R}$ with $\phi(t) = 0$ for $|t| < \varepsilon$.

9.3 Remark. We expect that a corresponding result holds if $\dim(M)$ is even. In this case we will also need that $\mathcal{B}_\varepsilon$ is an *odd* operator with respect to the signature grading (where we use on M' the reverse orientation). Since we do not need the even dimensional case in this paper, we leave this for further investigation.

Proof. In order to prove Theorem 9.2, we must carefully recall the definition of the signature operator for odd dimensional manifolds. This is done in Appendix B.

We denote the Hilbert A-module $L^2(M, \Lambda^* M \otimes f^*\mathcal{V}')$ by $\mathcal{L}^2(M)^*$. Similarly we set $L^2(M', \Lambda^* M' \otimes \mathcal{V}') =: \mathcal{L}^2(M')^*$. We would like to compare $\mathcal{L}^2(M)^*$ with $\mathcal{L}^2(M')^*$ using the pull-back map f^*; but this is in general not L^2-bounded. As in Hilsum–Skandalis [22, p. 90], we modify f^* in order to obtain a L^2-bounded cochain map between $\mathcal{L}^2(M')^*$ and $\mathcal{L}^2(M)^*$ as follows. From [22, p. 90], for suitably large N, there is a submersion $F: D^N \times M \to M'$ such that $F(0, m) = f(m)$. Here D^N is an open ball in an Euclidean space of dimension N. Fix $v \in \Omega_c^N(D^N)$ with $\int_{D^N} v = 1$. Define a *bounded* cochain map

$$T: \mathcal{L}^2(M')^* \longrightarrow \mathcal{L}^2(M)^*, \quad \omega \to \int_{D^N} v \wedge F^*(\omega).$$

[6]This result, the homotopy invariance of the signature index class, has been established by several people and with different techniques. Mishchenko and Kasparov prove it by showing its equality with the C*-algebraic Mishchenko symmetric signature, an a-priori homotopy invariant. Kaminker–Miller give a more analytical treatment, adapting to the noncommutative context the proof that Lusztig gives in the case $\Gamma = \mathbb{Z}^k$; Hilsum–Skandalis prove the homotopy invariance in a purely analytical fashion; we shall follow their approach. For a thorough treatment of the homotopy invariance of the signature index class and its connections with surgery theory we refer the reader to the recent papers by Higson and Roe [18], [19], [20].

Let $\mathcal{D}_{\mathcal{V}}$ and $\mathcal{D}'_{\mathcal{V}'}$ be the signature operators on M and M', as introduced above. There is a well-defined functional calculus associated to these regular operators on Hilbert modules. In particular, if $\phi\colon \mathbb{R} \to \mathbb{R}$ is a rapidly decreasing (i.e. Schwartz) function, then $\phi(\mathcal{D}_{\mathcal{V}}) \in \Psi_A^{-\infty}$.

9.4 Definition. We define

$$\mathcal{C}_{\phi,f} := \phi(D_{\mathcal{V}}) \circ T \circ \phi(\mathcal{D}'_{\mathcal{V}'})\colon \mathcal{L}^2(M')^* \to \mathcal{L}^2(M)^*.$$

If ϕ is an even function, $\phi(D_{\mathcal{V}})$ is a function of the Laplacian $\mathcal{D}_{\mathcal{V}}^2 = \Delta$ and therefore preserves the degree of the differential forms. Consequently, the same is true for $\mathcal{C}_{\phi,f}$.

9.5 Lemma. *The operator $\mathcal{C}_{\phi,f}$ is an integral A-linear operator with smooth kernel on $M \times M'$. Moreover, if ϕ is even then $d_{\mathcal{V}} \circ \mathcal{C}_{\phi,f} = \mathcal{C}_{\phi,f} \circ d'_{\mathcal{V}'}$.*

Proof. Since T is bounded and $\phi(\mathcal{D}_{\mathcal{V}}), \phi(\mathcal{D}'_{\mathcal{V}'}) \in \Psi_A^{-\infty}$ we immediately get that $\mathcal{C}_{\phi,f}$ is an A-linear smoothing operator. We know that $d_{\mathcal{V}} g^*\omega = g^*(d'_{\mathcal{V}'}\omega)$ for any smooth map g. Moreover, $d_{\mathcal{V}}$ commutes with $(d_{\mathcal{V}} + d_{\mathcal{V}}^*)^2 = \mathcal{D}_{\mathcal{V}}^2$ and therefore also with $\phi(\mathcal{D}_{\mathcal{V}})$ since ϕ is even; similarly $d'_{\mathcal{V}'}$ commutes with $\phi(\mathcal{D}'_{\mathcal{V}'})$. Since

$$\mathcal{C}_{\phi,f}(\omega) = \phi(\mathcal{D}_{\mathcal{V}}) \circ \left(\int_{D^N} v \wedge F^*\phi(\mathcal{D}'_{\mathcal{V}'})(\omega) \right)$$

we also get $d_{\mathcal{V}} \circ \mathcal{C}_{\phi,f} = \mathcal{C}_{\phi,f} \circ d'_{\mathcal{V}'}$ as required and explained in [22, Proof of Theorem 3.3 on p. 90], where we use that the form v is closed. □

9.6 Definition. We use the usual inner product (coming from a fixed Riemannian metric) on the space of differential forms. We use * for the adjoint with respect to this inner product.

The involution τ coming from the Hodge-$*$ operator (compare Appendix B) and the above inner product define the signature quadratic form on the space of differential forms, we use † for the adjoint with respect to this quadratic form. Recall that

$$A^* = \tau A^\dagger \tau, \quad A^\dagger = \tau A^* \tau.$$

Let now ϕ_ε be a smooth even function $\phi_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that $\phi_\varepsilon(t) = 1$ if $|t| \leq \varepsilon/4$, and $\phi_\varepsilon(t) = 0$ if $|t| \geq \varepsilon/2$. Recall that $f\colon M \to M'$ was an orientation-preserving homotopy equivalence and write

$$T_\varepsilon := \mathcal{C}_{\phi_\varepsilon,f}\colon \Omega^*(M', \mathcal{V}') \to \Omega^*(M, \mathcal{V}).$$

Clearly T_ε extends to an L^2-bounded A-linear operator. Let T_ε^* be the adjoint of T_ε and define $T_\varepsilon^\dagger := \tau' T_\varepsilon^* \tau$, with τ and τ' denoting the involutions defined by the Hodge $*$-operators on M and M' (see Appendix B). Introduce a new differential d by setting $d\alpha := i^{|\alpha|} d_{\mathcal{V}}\alpha$ and similarly for d'.

9.7 Lemma. *The bounded operator T_ε verifies the following properties*:

(a) $T_\varepsilon(\mathrm{dom}(d)) \subset \mathrm{dom}(d')$; $T_\varepsilon d' = d T_\varepsilon$.

(b) *T_ε induces an isomorphism in cohomology, with inverse induced by $T_\varepsilon^\dagger$.*

(c) *There exists a bounded operator y_ε of degree -1 on $\mathcal{L}^2(M')^*$, with $y_\varepsilon^\dagger = -y_\varepsilon$ and such that $y_\varepsilon(\mathrm{dom}(d')) \subset \mathrm{dom}(d')$ and $1 - T_\varepsilon^\dagger T_\varepsilon = d' y_\varepsilon + y_\varepsilon d'$.*

(d) *$y_\varepsilon = y_1 + y_2$ where y_1 is ε-spectrally concentrated and y_2 commutes with Δ'.*

(e) *T_ε is ε-spectrally concentrated.*

Proof. The results (a) to (c) for the operator T are proved by Hilsum–Skandalis. We need first of all to extend them to T_ε. Observe that Lemma 9.5 is precisely assertion (a).

Next, we observe that the T of Hilsum–Skandalis and our T_ε are chain homotopic. Indeed, recall that $T_\varepsilon = \phi_\varepsilon(d_{\mathcal{V}} + d_{\mathcal{V}}^*) \circ T \circ \phi_\varepsilon(d'_{\mathcal{V}'} + (d'_{\mathcal{V}'})^*)$. Here, we can replace $\mathcal{D}_{\mathcal{V}}$ by $d_{\mathcal{V}} + d_{\mathcal{V}}^*$, since $\mathcal{D}_{\mathcal{V}}^2 = (d_{\mathcal{V}} + d_{\mathcal{V}}^*)^2$, and ϕ is even. Choose a homotopy ϕ_ε^t of even functions with $\phi_\varepsilon^0 = \phi_\varepsilon$, $\phi_\varepsilon^1 = 1$ and such that $\phi_\varepsilon^t(x) = 1$ for all $t \in [0,1]$ and $|x| < \varepsilon/4$. Then $\frac{d}{dt}\phi_t(x) = x g_t(x) = g_t(x)x$ with a suitable smooth odd family of functions $g_t\colon \mathbb{R} \to \mathbb{R}$. Observe that

$$
\begin{aligned}
T - T_\varepsilon &= \int_0^1 \frac{d}{dt}(\phi_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) \circ T \circ \phi_t(d'_{\mathcal{V}} + (d'_{\mathcal{V}'})^*))\,dt \\
&= (d_{\mathcal{V}} + d_{\mathcal{V}}^*) \int_0^1 (g_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) \circ T \circ \phi_t(d'_{\mathcal{V}'} + (d'_{\mathcal{V}'})^*))\,dt \\
&\quad + \int_0^1 (\phi_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) \circ T \circ g_t(d'_{\mathcal{V}'} + (d'_{\mathcal{V}'})^*))\,dt \circ (d'_{\mathcal{V}'} + (d'_{\mathcal{V}'})^*) \\
&= d_{\mathcal{V}} z + z d'_{\mathcal{V}'} = dw + wd'
\end{aligned}
\tag{9.8}
$$

with

$$
\begin{aligned}
z &= \int_0^1 g_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) \circ T \circ \phi_t(d'_{\mathcal{V}'} + (d'_{\mathcal{V}'})^*)\,dt \\
&\quad + \int_0^1 \phi_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) \circ T \circ g_t(d'_{\mathcal{V}'} + (d'_{\mathcal{V}'})^*)\,dt, \\
w(\omega) &= i^{1-|\omega|} z(\omega).
\end{aligned}
$$

The last but one equality in (9.8) is true since $d_{\mathcal{V}}^* g_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) = g_t(d_{\mathcal{V}} + d_{\mathcal{V}}^*) d_{\mathcal{V}}$ and $d_{\mathcal{V}} T = T d'_{\mathcal{V}}$. Since by [22] T induces an isomorphism in homology, and T_ε is chain homotopic to T, so does T_ε, proving (b).

Apply now $\cdot^\dagger$ to equation (9.8) and use that $d^\dagger = -d$ to get

$$T^\dagger - T_\varepsilon^\dagger = -w^\dagger d' - dw^\dagger.$$

Let now y be an operator such that $1 - T^\dagger T = dy + yd'$, as constructed in [22]. Then

$$\begin{aligned} 1 - T_\varepsilon^\dagger T_\varepsilon &= 1 - T^\dagger T + (T^\dagger - T_\varepsilon^\dagger)T + T_\varepsilon^\dagger(T - T_\varepsilon) \\ &= dy + yd' + (-dw^\dagger - w^\dagger d')T + T_\varepsilon^\dagger(dw + wd') \\ &= d(y - w^\dagger T + T_\varepsilon^\dagger w) + (y - w^\dagger T + T_\varepsilon^\dagger w)d', \end{aligned} \tag{9.9}$$

using that T and T_ε are chain maps, and therefore in particular $T_\varepsilon^\dagger d' = -T_\varepsilon^\dagger(d')^\dagger = -d^\dagger T_\varepsilon^\dagger = dT_\varepsilon^\dagger$. Equation (9.9) means that we get (c) with $\tilde{y}_\varepsilon = y - w^\dagger T + T_\varepsilon^\dagger w$.

We will now modify $\tilde{y}_\varepsilon$ in such a way that it splits as required in (d). To do this, we make the following general observation. Assume that $\psi_1 \colon \mathbb{R} \to \mathbb{R}$ vanishes in a neighborhood of 0 and write $\psi_1(x) = x\psi_2(x)$ with a smooth function ψ_2. Define

$$u := (d'_{\mathcal{V}'})^* \psi_2(\Delta') = \psi_2(\Delta')(d'_{\mathcal{V}'})^*. \tag{9.10}$$

Then

$$d'_{\mathcal{V}'}u + ud'_{\mathcal{V}'} = d'_{\mathcal{V}}(d'_{\mathcal{V}'})^*\psi_2(\Delta') + (d'_{\mathcal{V}'})^* d'_{\mathcal{V}'}\psi_2(\Delta') = \psi_1(\Delta'). \tag{9.11}$$

Choose $\psi \colon \mathbb{R} \to \mathbb{R}$ with support contained in $[\varepsilon/2, \infty)$ and such that $\psi(x) = 1$ for $x \geq \varepsilon$. Define now

$$y_1 := (1 - \psi)(\Delta') \circ \tilde{y}_\varepsilon \circ (1 - \psi)(\Delta').$$

By construction, this operator is ε-spectrally concentrated. Set $y_2 := \psi(\Delta')\tilde{y}_\varepsilon(1 - \psi)(\Delta')$, $y_3 := \tilde{y}_\varepsilon \psi(\Delta')$. We compute (since $T_\varepsilon^\dagger$ and T_ε are $\varepsilon/2$-spectrally concentrated)

$$\begin{aligned} d'y_2 + y_2 d' &= d' \circ (\psi(\Delta')\tilde{y}_\varepsilon(1 - \psi)(\Delta')) + (\psi(\Delta')\tilde{y}_\varepsilon(1 - \psi)(\Delta')) \circ d' \\ &= \psi(\Delta')(d'\tilde{y}_\varepsilon + \tilde{y}_\varepsilon d')(1 - \psi)\Delta') \\ &= \psi(\Delta') \circ (1 - \psi)(\Delta') \\ &= d' \circ u_1 + u_1 d' \end{aligned}$$

with a suitable operator u_1 defined as in (9.10) and (9.11) which commutes with Δ'. Similarly,

$$d'y_3 + y_3 d' = d' \circ \tilde{y}_\varepsilon \psi(\Delta') + \tilde{y}_\varepsilon \psi(\Delta') \circ d' = d'u_2 + u_2 d'$$

with a u_2 which also commutes with Δ'. Consequently,

$$\begin{aligned} 1 - T_\varepsilon^\dagger T_\varepsilon &= d'\tilde{y}_\varepsilon + \tilde{y}_\varepsilon d' \\ &= d'(y_1 + y_2 + y_3) + (y_1 + y_2 + y_3)d' \\ &= d'(y_1 + u_1 + u_2) + (y_1 + u_1 + u_2)d'. \end{aligned}$$

We now set $y_\varepsilon := y_1 + u_1 + u_2$ and observe that $(u_1 + u_2)$ commutes with Δ'. This proves (d).

(e) is an immediate consequence of the construction of T_ε. □

Following closely [22], consider now the operators on $\mathcal{L}^2(M')^* \oplus \mathcal{L}^2(M)^*$:

$$R_\varepsilon = \begin{pmatrix} 1 & 0 \\ -T_\varepsilon\gamma & 1 \end{pmatrix}, \quad L_{\varepsilon,\alpha} = \begin{pmatrix} 1 - T_\varepsilon^\dagger T_\varepsilon & (\gamma + \alpha y_\varepsilon)T_\varepsilon^\dagger \\ T_\varepsilon(-\gamma - \alpha y_\varepsilon) & 1 \end{pmatrix}, \quad \delta_{\varepsilon,\alpha} = \begin{pmatrix} d' & \alpha T_\varepsilon^\dagger \\ 0 & -d \end{pmatrix} \tag{9.12}$$

with $\gamma\omega := (-1)^{|\omega|}\omega$ and α a real number.[7] Note that γ anticommutes with d and τ and commutes with T and $T^\dagger$, and that $\gamma^\dagger = -\gamma$. It is clear that R_ε and $L_{\varepsilon,\alpha}$ are bounded. The crucial relation is $L_{\varepsilon,\alpha}\delta_{\varepsilon,\alpha} = -\delta_{\varepsilon,\alpha}^\dagger L_{\varepsilon,\alpha}$.

We notice that R_ε is invertible and that $R_\varepsilon^\dagger R_\varepsilon = L_{\varepsilon,0}$. Thus $L_{\varepsilon,\alpha}$ is invertible for $|\alpha|$ small enough. Let $S_{\varepsilon,\alpha} := \frac{\tau\circ L_{\varepsilon,\alpha}}{|\tau\circ L_{\varepsilon,\alpha}|}$, with $\tau = \left(\begin{smallmatrix} \tau' & 0 \\ 0 & \tau \end{smallmatrix}\right)$. Then $S_{\varepsilon,\alpha}$ is an involution. We now endow $\Omega^*(M', \mathcal{V}') \oplus \Omega^*(M, \mathcal{V})$ with the new inner product

$$\langle \omega_1, \omega_2 \rangle_{\varepsilon,\alpha} := \langle \omega_1, |\tau \circ L_{\varepsilon,\alpha}|\omega_2 \rangle.$$

Notice that $|\tau \circ L_{\varepsilon,\alpha}|$ is positive and self-adjoint with respect to both scalar-products. Let $\widehat{\mathcal{D}}_{\varepsilon,\alpha} := -i(\delta_{\varepsilon,\alpha}S_{\varepsilon,\alpha} + S_{\varepsilon,\alpha}\delta_{\varepsilon,\alpha})$ be the signature operator associated to $\delta_{\varepsilon,\alpha}$ and to the grading $S_{\varepsilon,\alpha}$. Using [22, Lemme 2.1], $\widehat{\mathcal{D}}_{\varepsilon,\alpha}$ is *invertible* and self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\varepsilon,\alpha}$. For the adjoint with respect to the original inner product we therefore get

$$(\widehat{\mathcal{D}})^*_{\varepsilon,\alpha} = (|\tau \circ L_{\varepsilon,\alpha}|)^{-1} \circ \widehat{\mathcal{D}}_{\varepsilon,\alpha} \circ (|\tau \circ L_{\varepsilon,\alpha}|).$$

9.13 Definition. We define the special perturbed signature operator

$$\mathcal{D}_{\varepsilon,\alpha} := -i\left(\delta_{\varepsilon,\alpha}\tau L_{\varepsilon,\alpha} + \frac{\tau L_{\varepsilon,\alpha}}{|\tau L_{\varepsilon,\alpha}|}\delta_{\varepsilon,\alpha}\frac{\tau L_{\varepsilon,\alpha}}{|\tau L_{\varepsilon,\alpha}|}\tau L_{\varepsilon,\alpha}\right)$$

with

$$\delta_{\varepsilon,\alpha} = \begin{pmatrix} d' & \alpha T_\varepsilon^\dagger \\ 0 & -d \end{pmatrix}, \quad L_{\varepsilon,\alpha} = \begin{pmatrix} 1 - T_\varepsilon^\dagger T_\varepsilon & (\gamma + \alpha y_\varepsilon)T_\varepsilon^\dagger \\ T_\varepsilon(-\gamma - \alpha y_\varepsilon) & 1 \end{pmatrix}.$$

Then $\mathcal{D}_{\varepsilon,\alpha} := \widehat{\mathcal{D}_{\varepsilon,\alpha}} \circ |\tau \circ L_{\varepsilon,\alpha}|$ is self-adjoint with respect the *original* inner product and is invertible for $\alpha \neq 0$ sufficiently small (depending on ε). We complete the proof of the theorem with the following result.

[7] The differences in signs between these operators and the ones appearing in [22] are due to the fact that we take τ' for the grading on M', whereas Hilsum–Skandalis take $-\tau'$.

9.14 Lemma. *Consider the operators of Definition* 9.13. *Then* $\tilde{\delta}_{\varepsilon,\alpha} := \delta_{\varepsilon,\alpha} - \left(\begin{smallmatrix} d & 0 \\ 0 & -d' \end{smallmatrix}\right)$ *and* $\tau L_{\varepsilon,\alpha} - \tau$ *are ε-spectrally concentrated. Consequently,* $\frac{\tau L_{\varepsilon,\alpha}}{|\tau L_{\varepsilon,\alpha}|} - \tau$ *and* $\mathcal{D}_{\varepsilon,\alpha} - \mathcal{D}$ *are ε-spectrally concentrated (recall that* $|\tau| = 1$*).*

Moreover, all these differences belong to $\Psi_A^{-\infty}$*, in particular* $\mathcal{D}_{\varepsilon,\alpha} - \mathcal{D} \in \Psi_A^{-\infty}$.

Proof. First of all, using the definition of $L_{\varepsilon,\alpha}$ and Lemma 9.7 we claim that $L_{\varepsilon,\alpha} = 1 + \Theta_{\varepsilon,\alpha}$ with $\Theta_{\varepsilon,\alpha} \in \Psi_A^{-\infty}$ and ε-spectrally concentrated. Indeed

$$\begin{aligned} L_{\varepsilon,\alpha} &= 1 + \begin{pmatrix} -T_\varepsilon^\dagger T_\varepsilon & \gamma T_\varepsilon^\dagger \\ -T_\varepsilon \gamma & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha y_\varepsilon T_\varepsilon^\dagger \\ -T_\varepsilon \alpha y_\varepsilon & 0 \end{pmatrix} \\ &= 1 + \Theta^1_{\varepsilon,\alpha} + \begin{pmatrix} 0 & \alpha y_1 T_\varepsilon^\dagger \\ -T_\varepsilon \alpha y_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha y_2 T_\varepsilon^\dagger \\ -T_\varepsilon \alpha y_2 & 0 \end{pmatrix} \\ &= 1 + \Theta^1_{\varepsilon,\alpha} + \Theta^2_{\varepsilon,\alpha} + + \begin{pmatrix} 0 & \alpha y_2 T_\varepsilon^\dagger \\ -T_\varepsilon \alpha y_2 & 0 \end{pmatrix} \end{aligned}$$

with $\Theta^j_{\varepsilon,\alpha}$, $j = 1, 2$ smoothing and ε-spectrally concentrated because of Lemma 9.7. It remains to be checked that the last term appearing in the above formula is also smoothing and ε-spectrally concentrated: but since y_2 commutes with Δ', it also commutes with $\phi(\mathcal{D}'_{\mathcal{V}'})$ if ϕ is even; using the very definition of T_ε we then get that

$$\Theta^3_{\varepsilon,\alpha} = \begin{pmatrix} 0 & \alpha\phi_\varepsilon(D_{\mathcal{V}}) \circ y_{2,\varepsilon} \circ T^\dagger \circ \phi_\varepsilon(D'_{\mathcal{V}'}) \\ -\alpha\phi_\varepsilon(D'_{\mathcal{V}'}) T \circ \circ y_{2,\varepsilon} \circ \phi_\varepsilon(D_{\mathcal{V}}) & 0 \end{pmatrix}$$

is smoothing and ε-spectrally concentrated. Our claim now follows with $\Theta_{\varepsilon,\alpha} = \Theta^1_{\varepsilon,\alpha} + \Theta^2_{\varepsilon,\alpha} + \Theta^3_{\varepsilon,\alpha}$.

Thus $\tau \circ L_{\varepsilon,\alpha} = \tau + \Phi_{\varepsilon,\alpha}$, with $\Phi_{\varepsilon,\alpha} \in \Psi_A^{-\infty}$ and ε-spectrally concentrated. Let us now consider $|\tau \circ L_{\varepsilon,\alpha}| = \sqrt{(\tau \circ L_{\varepsilon,\alpha})^2}$. We can write

$$|\tau \circ L_{\varepsilon,\alpha}| = \frac{i}{2\pi} \int_{\mathcal{C}} \lambda^{\frac{1}{2}} ((\tau \circ L_{\varepsilon,\alpha})^2 - \lambda)^{-1} d\lambda$$

with $\mathcal{C}$ equal to a circle of radius larger than the norm of $(\tau \circ L_{\varepsilon,\alpha})^2$. From this integral representation it follows that $|\tau \circ L_{\varepsilon,\alpha}| = 1 + \Lambda_{\varepsilon,\alpha}$ with $\Lambda_{\varepsilon,\alpha} \in \Psi_A^{-\infty}$ and ε-spectrally concentrated; indeed, $(\tau \circ L_{\varepsilon,\alpha})^2 - \lambda$ is equal to $(1-\lambda)$ plus a ε-spectrally concentrated smoothing operator. Thus by Lemma A.12 its inverse will be of type $(1-\lambda)^{-1}(1 + \Xi_{\varepsilon,\alpha}(\lambda))$ with $\Xi_{\varepsilon,\alpha}(\lambda) \in \Psi_A^{-\infty}$ and ε-spectrally concentrated; carrying out the integral we obtain immediately that $|\tau \circ L_{\varepsilon,\alpha}| = 1 + \mathcal{F}_{\varepsilon,\alpha}$ with $\mathcal{F}_{\varepsilon,\alpha} \in \Psi_A^{-\infty}$ and ε-spectrally concentrated. It then follows that the same is true for $|\tau \circ L_{\varepsilon,\alpha}|^{-1}$. Thus for $S_{\varepsilon,\alpha}$ we obtain $S_{\varepsilon,\alpha} = \tau + \mathcal{G}_{\varepsilon,\alpha}$ with $\mathcal{G}_{\varepsilon,\alpha}$ smoothing and ε-spectrally concentrated. Consider now $\widetilde{\mathcal{D}}$; this equals $-i(\delta_{\varepsilon,\alpha} \circ S_{\varepsilon,\alpha} + S_{\varepsilon,\alpha} \circ \delta_{\varepsilon,\alpha}) |\tau \circ L_{\varepsilon,\alpha}|$. By its very definition and Lemma 9.7, $\delta_{\varepsilon,\alpha} = d + \mathcal{E}_{\varepsilon,\alpha}$ with $\mathcal{E}_{\varepsilon,\alpha}$ smoothing and ε-spectrally concentrated.

Thus

$$\mathcal{D}_{\varepsilon,\alpha} = -i((d + \mathcal{E}_{\varepsilon,\alpha}) \circ (\tau + \mathcal{G}_{\varepsilon,\alpha}) + (\tau + \mathcal{G}_{\varepsilon,\alpha}) \circ (d + \mathcal{E}_{\varepsilon,\alpha})) \circ (1 + \mathcal{F}_{\varepsilon,\alpha}),$$

which is equal to $\mathcal{D}$ plus a smoothing operator ε-spectrally concentrated. □

Theorem 9.2 is proved. □

10. From stable to unstable eta-invariants

We continue with the setting of Section 9. In this section, we analyze the τ-eta-invariant of the special perturbations $\mathcal{D}_{\varepsilon,\alpha}$ of $\mathcal{D}$, where we use the notation and conventions of Section 8. In particular, A is a von Neumann algebra and $\tau\colon A \to Z$ a positive normal trace with values in the commutative von Neumann algebra Z.

10.1. Limits of eta-invariants

10.1 Theorem. *Let $f: M' \to M$ be a smooth oriented homotopy equivalence between closed Riemannian oriented manifolds as in Section* 9. *This gives rise to twisted signature operators $D_{\mathcal{L}}$ on M and $D'_{\mathcal{L}'}$ on M', with $\mathcal{L}' := f^*\mathcal{L}$. Consider the* (*unperturbed*) *τ-eta-invariants $\eta_\tau(D_{\mathcal{L}})$, $\eta_\tau(D'_{\mathcal{L}'}) \in Z$, see* 8.1.

Assume, in addition to the above, that the von Neumann algebra A admits a positive faithful normal trace $\tau_A\colon A \to \mathbb{C}$.

There are sequences $\varepsilon_k > 0$ and $\alpha_k > 0$ such that α_k is small enough for $\mathcal{D}_{\varepsilon_k,\pm\alpha_k}$ to be an invertible perturbation of $\mathcal{D}$ and such that

$$\lim_{k\to\infty} \frac{\eta_\tau(\mathcal{D}_{\varepsilon_k,\alpha_k}) + \eta_\tau(\mathcal{D}_{\varepsilon_k,-\alpha_k})}{2} = \eta_\tau(\mathcal{D}).$$

Here $\mathcal{D}_{\varepsilon,\alpha}$ is the special smoothing perturbation of the signature operator $\mathcal{D}$ on the manifold $M \amalg (-M')$ as defined in Definition 9.13. *Note that on the left-hand side we have a sequence of averaged perturbed eta-invariants, converging to the unperturbed eta-invariant on the right-hand side.*

Proof. We shall prove that there are sequences $\varepsilon_k > 0$ and $\alpha_k > 0$ such that

$$\lim_{k\to\infty} \eta_\tau(\mathcal{D}_{\varepsilon_k,\alpha_k}) = \eta_\tau(\mathcal{D}) + \sigma, \quad \lim_{k\to\infty} \eta_\tau(\mathcal{D}_{\varepsilon_k,-\alpha_k}) = \eta_\tau(\mathcal{D}) - \sigma, \quad \sigma \in \mathbb{R}.$$

Let $p_\varepsilon := \chi_{[-\varepsilon,\varepsilon]}(\Delta)$, where χ_Y is the characteristic function of a set Y.

Since $\mathcal{D}_{\varepsilon,\alpha}$ is an ε-perturbation of $\mathcal{D}$, we get

$$\mathcal{D}_{\varepsilon,\alpha} = p_\varepsilon \mathcal{D}_{\varepsilon,\alpha} p_\varepsilon + (1-p_\varepsilon)\mathcal{D}_{\varepsilon,\alpha}(1-p_\varepsilon), \ (1-p_\varepsilon)\mathcal{D}_{\varepsilon,\alpha}(1-p_\varepsilon) = (1-p_\varepsilon)\mathcal{D}(1-p_\varepsilon).$$

Consequently, by the spectral theorem and the definition of η_τ,

$$\eta_\tau(\mathcal{D}_{\varepsilon,\alpha}) = \eta_\tau(p_\varepsilon \mathcal{D}_{\varepsilon,\alpha} p_\varepsilon) + \eta_\tau((1-p_\varepsilon)\mathcal{D}(1-p_\varepsilon)). \tag{10.2}$$

To analyze η_τ of the compressed operators, we will make use of the following translation into simple functional calculus.

10.3 Lemma. *Let p and B be self-adjoint bounded Hilbert A-module morphisms on a Hilbert A-module H, p a projection such that $\tau(p) < \infty$ and $B = pBp$. Let $q\colon \mathbb{R} \to \mathbb{R}$ be defined by $q(x) = 1$ for $x > 0$, $q(0) = 0$ and $q(x) = -1$ for $x < 0$. We define* $\operatorname{sgn}_\tau(B) := \tau(q(B))$. *Then*

$$\eta_\tau(B) = \operatorname{sgn}_\tau(B). \tag{10.4}$$

Proof. We simply observe that

$$q(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty x \exp(-tx^2)\, \frac{dt}{\sqrt{t}} = \frac{2}{\pi} \int_0^\infty x \exp(-t^2x^2)\, dt,$$

and $|q(x)| \le 1$, therefore $q(B) = \frac{1}{\pi}\int_0^\infty B \exp(-tB^2)\, dt/\sqrt{t}$. Here we observe that $\int_0^T B\exp(-t^2B^2)\,dt$ is defined for each $T \ge 0$ since B is bounded and is equal to the functional calculus of B applied to $\int_0^T x\exp(-t^2x^2)\,dt$. The latter family of functions, depending on the parameter T, converges pointwise to q, therefore the limit $\int_0^\infty B\exp(-t^2B^2)\,dt$ exists in the von Neumann algebra of Hilbert A-module morphisms on H and equals $q(B)$. Finally observe that we can throughout write pBp instead of B. By assumption p is of τ-trace class. Since those operators form an ideal, it follows that both pBp and $pBp\exp(-t^2B^2)$ are τ-trace class. Normality implies that for a strongly convergent sequence X_n of operators, $\tau(pX_n) \xrightarrow{n\to\infty} \tau(pX)$. We can now interchange in the above argument integration, passage to limit and application of τ, and the statement follows. □

10.5 Lemma.

$$\lim_{\varepsilon\to 0} \eta_\tau((1-p_\varepsilon)\mathcal{D}(1-p_\varepsilon)) = \eta_\tau(\mathcal{D}),$$

where existence of the limit is part of the statement.

Proof. We have discussed in Section 8 that each of the integrals defining the eta-invariants exists in the above statement. Moreover, for $0 < \varepsilon < 1$,

$$\eta_\tau((1-p_\varepsilon)\mathcal{D}(1-p_\varepsilon)) = \eta_\tau((1-p_1)\mathcal{D}(1-p_1)) + \eta_\tau((p_1-p_\varepsilon)\mathcal{D}(p_1-p_\varepsilon)).$$

Observe that $\tau(p_1) \le \tau(5\exp(-\mathcal{D})^2) < \infty$ and $p_1(p_1-p_\varepsilon)\mathcal{D}(p_1-p_\varepsilon)p_1 = (p_1-p_\varepsilon)\mathcal{D}(p_1-p_\varepsilon)$. Then Lemma 10.3 implies that $\eta_\tau((p_1-p_\varepsilon)\mathcal{D}(p_1-p_\varepsilon)) =$

$\operatorname{sgn}_\tau((p_1 - p_\varepsilon)\mathcal{D}(p_1 - p_\varepsilon))$. Finally, p_ε strongly converges to p_0 so that $q((p_1-p_\varepsilon)\mathcal{D}(p_1-p_\varepsilon))$ strongly converges to $q((p_1-p_0)\mathcal{D}(p_1-p_0))$. By normality of τ, therefore

$$\begin{aligned}&\operatorname{sgn}_\tau((p_1 - p_\varepsilon)\mathcal{D}(p_1 - p_\varepsilon))\\ &\quad = \tau(q((p_1 - p_\varepsilon)\mathcal{D}(p_1 - p_\varepsilon))) \xrightarrow{\varepsilon\to 0} \operatorname{sgn}_\tau((p_1 - p_0)\mathcal{D}(p_1 - p_0)).\end{aligned}$$

Finally, by definition and Lemma 10.3,

$$\eta_\tau(\mathcal{D}) = \eta_\tau((1-p_1)\mathcal{D}(1-p_1)) + \operatorname{sgn}_\tau((p_1 - p_0)\mathcal{D}(p_1 - p_0)). \qquad \square$$

10.6 Lemma. *There are sequences $\varepsilon_k > 0$ and $\alpha_k > 0$ such that α_k is small enough for $\mathcal{D}_{\varepsilon_k,\alpha_k}$ to be smoothing invertible perturbations of $\mathcal{D}$ and such that*

$$\operatorname{sgn}_\tau(p_{\varepsilon_k}\mathcal{D}_{\varepsilon_k,\pm\alpha_k}p_{\varepsilon_k}) \xrightarrow{k\to\infty} \operatorname{sgn}_\tau(\mathcal{D}_{0,\pm 1}).$$

Here

$$\mathcal{D}_{0,\alpha} := -ip_0\Big(\delta_{0,\alpha}\tau L_0 + \frac{\tau L_0}{|\tau L_0|}\delta_{0,\alpha}\frac{\tau L_0}{|\tau L_0|}\tau L_0\Big)p_0$$

with

$$\delta_{0,\alpha} = p_0\begin{pmatrix}0 & \alpha T_0^\dagger\\ 0 & 0\end{pmatrix}p_0, \quad L_0 = \begin{pmatrix}1 - T_0^\dagger T_0 & \gamma T_0^\dagger\\ -T_0\gamma & 1\end{pmatrix}, \quad T_0 = p_0 T p_0.$$

In particular, $\mathcal{D}_{0,\alpha} = \alpha\mathcal{D}_{0,1}$ and therefore

$$\operatorname{sgn}_\tau(\mathcal{D}_{0,1}) = -\operatorname{sgn}_\tau(\mathcal{D}_{0,-1}). \tag{10.7}$$

Assuming Lemma 10.6, we can now finish the proof of Theorem 10.1. Observe that

$$\eta_\tau(\mathcal{D}_{\varepsilon_k,\pm\alpha_k}) = \underbrace{\eta_\tau((1-p_{\varepsilon_k})\mathcal{D}_{\varepsilon_k,\pm\alpha_k}(1-p_{\varepsilon_k}))}_{=\eta_\tau(1-p_{\varepsilon_k})\mathcal{D}(1-p_{\varepsilon_k})\xrightarrow{k\to\infty}\eta_\tau(\mathcal{D})} + \underbrace{\eta_\tau(p_{\varepsilon_k}\mathcal{D}_{\varepsilon_k,\pm\alpha_k}p_{\varepsilon_k})}_{\xrightarrow{k\to\infty}\eta_\tau(\mathcal{D}_{0,\pm 1})}$$

where we use Lemma 10.5 for the first convergence statement, and Lemma 10.6 for the second. Averaging the two resulting equations for α_k and $-\alpha_k$ and using equation (10.7) gives the statement of the theorem. $\square$

10.8 Remark. We leave it to the reader to check directly that $\eta_\tau(\mathcal{D}_{0,1}) = 0$. Using the same proof as above, this gives the following stronger version of Theorem 10.1:

$$\lim_{k\to\infty}\eta_\tau(\mathcal{D}_{\varepsilon_k,\alpha_k}) = \eta_\tau(\mathcal{D}).$$

10.2. Proof of Lemma 10.6. Define for $\alpha \neq 0$

$$\mathcal{C}_{\varepsilon,\alpha} := |\alpha|^{-1} p_\varepsilon \mathcal{D}_{\varepsilon,\alpha} p_\varepsilon$$

and observe that $\operatorname{sgn}_\tau(\mathcal{C}_{\varepsilon,\alpha}) = \operatorname{sgn}_\tau(p_\varepsilon \mathcal{D}_{\varepsilon,\alpha} p_\varepsilon)$. Set for $\alpha \neq 0$

$$\widetilde{\mathcal{C}}_{\varepsilon,\alpha} := -i\,|\alpha|^{-1}\; p_\varepsilon\Big(\tilde{\delta}_{\varepsilon,\alpha}\tau L_{\varepsilon,\alpha} + \frac{\tau L_{\varepsilon,\alpha}}{|\tau L_{\varepsilon,\alpha}|}\tilde{\delta}_{\varepsilon,\alpha}\frac{\tau L_{\varepsilon,\alpha}}{|\tau L_{\varepsilon,\alpha}|}\tau L_{\varepsilon,\alpha}\Big) p_\varepsilon$$

with

$$\tilde{\delta}_{\varepsilon,\alpha} = \begin{pmatrix} 0 & \alpha T_\varepsilon^\dagger \\ 0 & 0 \end{pmatrix}, \quad L_{\varepsilon,\alpha} = \begin{pmatrix} 1 - T_\varepsilon^\dagger T_\varepsilon & (\gamma + \alpha y_\varepsilon) T_\varepsilon^\dagger \\ T_\varepsilon(-\gamma - \alpha y_\varepsilon) & 1 \end{pmatrix}.$$

Note that $\widetilde{\mathcal{C}}_{\varepsilon,\alpha}$ is in general neither self-adjoint nor invertible.

10.9 Lemma. *There is a constant $K > 0$ such that the following holds. For each $\varepsilon > 0$ there exists $\alpha_\varepsilon > 0$ such that*

$$\|\alpha_\varepsilon y_\varepsilon\| < \varepsilon, \quad \|L^{-1}_{\varepsilon,\pm\alpha_\varepsilon}\| \le K.$$

Proof. We first observe that there is a uniform bound for $\|T_\varepsilon\|$. This implies a uniform bound for $\|L_{\varepsilon,0}\|$ and for $\|L^{-1}_{\varepsilon,0}\|$, using $L_{\varepsilon,0} = \left(\begin{smallmatrix} 1 & \gamma T_\varepsilon^\dagger \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -T_\varepsilon\gamma & 1 \end{smallmatrix}\right)$, so that $(L_{\varepsilon,0})^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ T_\varepsilon\gamma & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\gamma T_\varepsilon^\dagger \\ 0 & 1 \end{smallmatrix}\right)$. Since each y_ε is bounded and $L_{\varepsilon,\alpha} = L_{\varepsilon,0} + \alpha\left(\begin{smallmatrix} 0 & y_\varepsilon T_\varepsilon^\dagger \\ -T_\varepsilon y_\varepsilon & 0 \end{smallmatrix}\right)$, we can now choose α_ε such that all the assertions are fulfilled. □

10.10 Lemma. *Choose a sequence $\varepsilon_k > 0$ with $\varepsilon_k \xrightarrow{k\to\infty} 0$, and choose $\alpha_k := \alpha_{\varepsilon_k} > 0$ as given by Sublemma* 10.9. *Then $\widetilde{\mathcal{C}}_{\varepsilon_k,\pm\alpha_k}$ strongly converges to $\mathcal{D}_{0,\pm 1}$.*

Proof. For $\varepsilon \to 0$, p_ε strongly converges to p_0, T_ε strongly converges to T_0, and $T_\varepsilon^\dagger$ strongly converges to $T_0^\dagger$; $\|\alpha_\varepsilon y_\varepsilon\| \xrightarrow{\varepsilon\to 0} 0$. Since products of strongly convergent sequences strongly converge to the product of the limits, by its construction $L_{\varepsilon_k,\alpha_k}$ strongly converges to L_0. Moreover, since $\|(\tau L_{\varepsilon_k,\alpha_k})^{-1}\| \le K$ independent of $k \in \mathbb{N}$, $\frac{\tau L_{\varepsilon_k,\alpha_k}}{|\tau L_{\varepsilon_k,\alpha_k}|} = f(\tau L_{\varepsilon_k,\alpha_k})$ (and $\frac{\tau L_{0,0}}{|\tau L_{0,0}|} = f(\tau L_{0,0})$) for any continuous function f with $f(x) = -1$ for $x \le -1/K$, $f(x) = 1$ for $x \ge 1/K$. Since bounded continuous functions of strongly convergent sequences strongly converge by [55, Theorem VIII.20], $\frac{\tau L_{\varepsilon_k,\alpha_k}}{|L_{\varepsilon_k,\alpha_k}|}$ strongly converges to $\frac{\tau L_{0,0}}{|\tau L_{0,0}|}$. Putting everything together, and using in particular that $|\alpha|^{-1}\tilde{\delta}_{\varepsilon,\alpha} = |\alpha|^{-1}\left(\begin{smallmatrix} 0 & \alpha T_\varepsilon^\dagger \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \pm T_\varepsilon^\dagger \\ 0 & 0 \end{smallmatrix}\right)$, the result follows. □

10.11 Lemma. *There is a sequence $\varepsilon_k > 0$, with corresponding α_k as in Sublemma* 10.10, *such that $\mathcal{C}_{\varepsilon_k,\pm\alpha_k}$ converges in the strong resolvent sense to $\mathcal{D}_{0,\pm 1}$. Here we will use the positive finite faithful normal trace $\tau_A\colon A \to \mathbb{C}$.*

Proof. We will construct a sequence of monotonously increasing projections $Q_1 \leq Q_2 \leq \cdots$ with $\sup_k Q_k = p_1$. The latter property will be ensured by showing that for all k, $Q_k \leq p_1$ and that $\tau_A(Q_k) \xrightarrow{k\to\infty} \tau_A(p_1) < \infty$, hence $\tau_A(p_1 - (\sup_k Q_k)) = 0$ so that $p_1 = \sup_k Q_k$, using faithfulness and normality of τ_A.

The main property of the Q_k will be that

$$\tilde{\mathcal{C}}_{\varepsilon_n,\pm\alpha_n} Q_k = \mathcal{C}_{\varepsilon_n,\pm\alpha_n} Q_k \quad \text{for all } n \geq k. \tag{10.12}$$

The assertion then follows from Sublemma 10.10. To see this, we can represent A as bounded operators on a Hilbert space H. Since $\tilde{\mathcal{C}}_{\varepsilon_k,\pm\alpha_k}(1-p_1) = \mathcal{C}_{\varepsilon_k,\pm\alpha_k}(1-p_1) = \mathcal{D}_{0,\pm 1}(1-p_1) = 0$, it suffices to study the restriction to $\mathrm{im}(p_1)$. Now all operators in question are bounded, therefore $\bigcup_{k\in\mathbb{N}}(\mathrm{im}(Q_k))$ will be a common core for all of them (since $\sup_k Q_k = p_1$, $\bigcup_{k\in\mathbb{N}} \mathrm{im}(Q_k)$ is dense in $\mathrm{im}(p_1)$). A reference for the lattice of projections and properties of it we are using here is [25], [26], in particular Section 2.5. Finally, for each $v \in \bigcup_k \mathrm{im}(Q_k)$ there is a $k \in \mathbb{N}$ such that $v \in \mathrm{im}(Q_k)$. Then, for $n \geq k$, $\mathcal{C}_{\varepsilon_n,\pm\alpha_n} v = \tilde{\mathcal{C}}_{\varepsilon_n,\pm\alpha_n} v \xrightarrow{n\to\infty} \mathcal{D}_{0,\pm 1} v$. By [55, Theorem VIII.25], $\mathcal{C}_{\varepsilon_n,\pm\alpha_n}$ converges in the strong resolvent sense to $\mathcal{D}_{0,\pm 1}$.

We now tackle the construction of the projections Q_k. Choose ε_k such that $\tau_A(p_{\varepsilon_k} - p_0) < 10^{-k}$.

Restrict now all operators in question to $\mathrm{im}(p_{\varepsilon_k})$. Note that $A_k := \tau L_{\varepsilon_k,\alpha_{\varepsilon_k}}$ now is an invertible operator, mapping $\mathrm{im}(p_{\varepsilon_k})$ to itself, and the same is true for $B_k := \frac{L_{\varepsilon_k,\alpha_{\varepsilon_k}}}{|L_{\varepsilon_k,\alpha_{\varepsilon_k}}|} L_{\varepsilon_k,\alpha_{\varepsilon_k}}$. Choose maximal projections Q_{A_k} and Q_{B_k} such that

$$p_0 A_k Q_{A_k} = A_k Q_{A_k} \quad \text{and} \quad p_0 B_k Q_{B_k} = B_k Q_{B_k}. \tag{10.13}$$

In other words, Q_{A_k} is the projection onto the inverse image under A_k of $\mathrm{im}(p_0)$. Set $Q_k^0 := \inf\{p_0, Q_{A_k}, Q_{B_k}\}$.

Observe that, by construction, $\tilde{\delta}_{\varepsilon,\alpha} p_0 = \delta_{\varepsilon,\alpha} p_0$. Consequently,

$$\tilde{\mathcal{C}}_{\varepsilon_k,\alpha_k} Q_k^0 = \mathcal{C}_{\varepsilon_k,\alpha_k} Q_k^0 \quad \text{for all } k \in \mathbb{N}. \tag{10.14}$$

Finally, we set $Q_k := (\inf_{l\geq k} Q_l^0) + p_1 - p_{\varepsilon_k}$. Observe that (10.12) immediately follows from (10.14) and the fact that $\tilde{\mathcal{C}}_{\varepsilon_k,\alpha_k}(p_1 - p_{\varepsilon_k}) = \mathcal{C}_{\varepsilon_k,\alpha_k}(p_1 - p_{\varepsilon_k}) = 0$. Note that $\tau_A(Q_{A_k}) = \tau_A(p_0) = \tau_A(Q_{B_k})$. This follows since instead of Q_{A_k} we can first construct the idempotent $e_k := A_k^{-1} p_0 A_k$; it satisfies the relation (10.13) and $\tau_A(e_k) = \tau_A(p_0)$. Use then the standard construction of a self-adjoint projection Q_{A_k} with $e_k Q_{A_k} = Q_{A_k}$ and $Q_{A_k} e_k = e_k$ (compare e.g. [7, Theorem 2.1]), then $\tau_A(Q_{A_k}) = \tau_A(e_k Q_{A_k}) = \tau_A(Q_{A_k} e_k) = \tau_A(e_k)$, and $p_0 A_k Q_{A_k} = p_0 A_k e_k Q_{A_k} = A_k e_k Q_{A_k} = A_k Q_{A_k}$.

Note that each of the projections Q_{A_k}, Q_{B_k}, p_0 are bounded above by p_{ε_k}, with $\tau_A(Q_{A_k}) = \tau_A(Q_{B_k}) = \tau_A(p_0)$, $\tau_A(p_{\varepsilon_k}) < \tau(p_0) + 10^{-k}$. The Kaplansky formula

[26, Theorem 6.1.7]

$$p_0 - \inf\{Q_{A_k}, p_0\} = \sup\{p_0, Q_{A_k}\} - Q_{A_k} \le p_{\varepsilon_k} - Q_{A_k},$$

the linearity and positivity of τ_A imply $\tau_A(Q_k^0) \ge \tau_A(p_0) - 2\cdot 10^{-k}$, and similarly,

$$\tau_A(\inf_{l\ge k} Q_l^0) \ge \tau_A(p_0) - 2\cdot 10^{-k}\sum_{l=0}^{\infty} 10^{-l} \ge \tau_A(p_0) - 4\cdot 10^{-k}.$$

It follows that

$$\tau_A(Q_k) \ge \tau_A(p_0) - 4\cdot 10^{-k} + \tau_A(p_1 - p_{\varepsilon_k}) \xrightarrow{k\to\infty} \tau_A(p_0) + \tau_A(p_1 - p_0) = \tau_A(p_1).$$

Since $Q_k \le p_1$, the properties of Q_k and therefore Sublemma 10.11 follow. □

10.15 Sublemma. *Assume that self-adjoint operators A_n converge in the strong resolvent sense to the self-adjoint operator A. Assume that $\chi_{\{0\}}(A_n)$ strongly converges to $\chi_{\{0\}}(A)$. Then $q(A_n)$ strongly converges to $q(A)$, where $q = \chi_{(0,\infty)} - \chi_{(-\infty,0)}$.*

Proof. By assumption, $\chi_{\{0\}}(A_n) + A_n$ converges in the strong resolvent sense to $\chi_{\{0\}}(A) + A$ (this is best seen using the Trotter criterion [55, Theorem VIII.21]). Thus by [55, Theorem VIII.24] we know that $\chi_{[0,\infty)}(\chi_{\{0\}}(A_n) + A_n)$ strongly converges to $\chi_{[0,\infty)}(\chi_{\{0\}}(A) + A)$ and similarly for $\chi_{(-\infty,0]}$. Hence $\chi_{(0,\infty)}(A_n) + \chi_{\{0\}}(A_n)$ converges strongly to $\chi_{(0,\infty)}(A) + \chi_{\{0\}}(A)$, which implies that $\chi_{(0,\infty)}(A_n)$ converges strongly to $\chi_{(0,\infty)}(A)$. Similarly, $\chi_{(-\infty,0)}(A_n)$ converges strongly to $\chi_{(-\infty,0)}(A)$. The proof of the sublemma is complete. □

With all the sublemmas in place, it is now easy to finish the proof of Lemma 10.6. We can restrict to $\mathrm{im}(p_1)$ (by multiplication with p_1 from the left) since all operators in question commute with p_1 and vanish on the complement of $\mathrm{im}(p_1)$ (i.e. when multiplied with $(1 - p_1)$). Observe that τ, restricted to $\mathrm{im}(p_1)$, is strongly continuous (because $\tau(p_1) < \infty$).

Since $\mathcal{D}_{\varepsilon,\alpha}$ is invertible and ε-spectrally concentrated near zero, $\chi_{\{0\}}(\mathcal{C}_{\varepsilon,\pm\alpha} p_1) = p_1 - p_\varepsilon$. In the same way, when restricted to $\mathrm{im}(p_0)$, $\mathcal{D}_{0,\alpha}$ is invertible and therefore $\chi_{\{0\}}(\mathcal{D}_{0,\alpha} p_1) = p_1 - p_0$. Consequently, $\chi_{\{0\}}(\mathcal{C}_{\varepsilon_k,\pm\alpha_k})$ converges strongly to $\chi_{\{0\}}(\mathcal{D}_{0,\pm 1})$. Sublemmas 10.11 and 10.15 imply that $q(\mathcal{C}_{\varepsilon_k,\pm\alpha_k})$ converge strongly to $q(\mathcal{D}_{0,1})$ so that finally, using Lemma 10.3,

$$\eta_\tau(p_{\varepsilon_k}\mathcal{D}_{\varepsilon_k,\pm\alpha_k} p_{\varepsilon_k}) = \tau(q(\mathcal{C}_{\varepsilon_k,\pm\alpha_k})) \xrightarrow{k\to\infty} \tau(q(\mathcal{D}_{0,\pm 1})) = \eta_\tau(\mathcal{D}_{0,\pm 1}).$$

10.16 Remark. We want to observe that if 0 is isolated in the spectrum of $\mathcal{D}$, the proof of Lemma 10.6 is almost trivial. By construction, for sufficiently small ε in this case $\mathcal{D}_{\varepsilon,\alpha}$ is equal to $\mathcal{D}_{0,\alpha}$.

11. Homotopy invariance of unstable rho-invariants

We are now in the position to prove the statements concerning homotopy invariance of the introduction, along with some more general results.

11.1 Theorem. *Let M be a closed oriented Riemannian manifold of odd dimension. Assume that Γ is a torsion-free discrete group such that the maximal Baum–Connes map*

$$\mu_{\max}\colon K_*(B\Gamma) \to K_*(C^*\Gamma)$$

is an isomorphism. Let $u\colon M \to B\Gamma$ be a map classifying a covering $\widetilde{M} = u^*E\Gamma$. Let A_j be unital von Neumann algebras admitting positive finite faithful normal traces $\tau_j\colon A_j \to \mathbb{C}$. Let $\alpha_j\colon C^*\Gamma \to A_j$ be homomorphisms (with $j = 1, \dots, r$), and $\tau_1, \dots, \tau_r\colon A_j \to Z_j$ positive normal traces with values in abelian von Neumann algebras Z_j. Assume that $\beta_1, \dots, \beta_r\colon Z_j \to V$ are continuous homomorphisms to a fixed topological vector space V such that

$$\sum_{j=1}^{r} \beta_j \tau_j(\alpha_j(1)) = 0 \in V. \tag{11.2}$$

Let $\mathcal{L}_j := \widetilde{M} \times_\Gamma A_j$ be the associated Hilbert A-module bundle, where Γ acts on A_j via $\Gamma \to C^*\Gamma \xrightarrow{\alpha_j} A_j$.

Then $\rho_{(\tau_j,\beta_j,L_j)}(M) \in V$ is a homotopy invariant.

11.3 Remark. We leave it to the reader to remove the hypothesis that the A_j admit a finite faithful trace in Theorem 11.1.

Proof. Let $f\colon M' \to M$ be a homotopy equivalence, $\mathcal{L}'_j := f^*\mathcal{L}_j$. Let $\mathcal{V} := \widetilde{M} \times_\Gamma C^*\Gamma$ be the Mishchenko–Fomenko line bundle associated to u on M. Then $f^*\mathcal{V} =: \mathcal{V}'$ is the Mishchenko–Fomenko line bundle associated to $f \circ u$ on M'.

Let $\mathcal{D}_{\varepsilon_k,\pm\alpha_k}$ be the smoothing perturbation of $D_{\mathcal{V}} \amalg -\mathcal{D}'_{\mathcal{V}'}$ on $M \amalg (-M')$ as in Theorem 10.1. Associated to these perturbation we define the stable rho-invariant

$$\rho^s_{[0]}(M \amalg -M') = [\eta_{[0]}(\mathcal{D}_{\varepsilon_k,\pm\alpha_k})] \in (C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle,$$

where we recall that $(C^*\Gamma)_{\mathrm{ab}} := (C^*\Gamma)/\overline{[C^*\Gamma, C^*\Gamma]}$. By Theorem 7.1, ρ^s does not depend on the particular perturbation and vanishes identically.

From Lemma C.8 we conclude that

$$\alpha_j(\eta_{[0]}(\mathcal{D}_{\varepsilon_k,\pm\alpha_k})) = \eta_{[0]}(\mathcal{D}^{A_j}_{\varepsilon_k,\pm\alpha_k}) \in A_j/\overline{[A_j, A_j]}, \tag{11.4}$$

where $\mathcal{D}^{A_j}_{\varepsilon_k,\pm\alpha_k}$ is the perturbation of the A_j-twisted signature operator $D_{\mathcal{L}_j} \amalg D'_{\mathcal{L}'_j}$ on $M \amalg (-M')$ given by the corresponding recipe. On the other hand, by Theorem 10.1,

$$\frac{\eta_{\tau_j}(\mathcal{D}^{A_j}_{\varepsilon_k,\alpha_k}) + \eta_{\tau_j}(\mathcal{D}^{A_j}_{\varepsilon_k,-\alpha_k})}{2} \xrightarrow{k\to\infty} \eta_{\tau_j}(\mathcal{D}^{A_j}) = \eta_{\tau_j}(D_{\mathcal{L}_j}) - \eta_{\tau_j}(D'_{\mathcal{L}'_j}).$$

Therefore

$$\sum_{j=1}^{r} \beta_j \frac{\eta_{\tau_j}(\mathcal{D}^{A_j}_{\varepsilon_k,\alpha_k}) + \eta_{\tau_j}(\mathcal{D}^{A_j}_{\varepsilon_k,-\alpha_k})}{2} \xrightarrow{k\to\infty} \sum_{j=1}^{r} \beta_j \eta_\tau(D_{\mathcal{L}_j}) - \sum_{j=1}^{r} \beta_j \eta_\tau(D'_{\mathcal{L}'_j}). \tag{11.5}$$

By Equation (11.4), for the left-hand side we can write

$$\sum_{j=1}^{r} \beta_j \eta_{\tau_j}(\mathcal{D}^{A_j}_{\varepsilon_k,\pm\alpha_k}) = \Big(\sum_{j=1}^{r} \beta_j \tau_j \alpha_j\Big)(\eta_{[0]}(\mathcal{D}_{\varepsilon_k,\pm\alpha_k})).$$

The maps $\tau_j\alpha_j : C^*\Gamma \to Z_j$ are traces. Therefore, they factorize through $(C^*\Gamma)_{\mathrm{ab}}$. After composing with β_j and summing up, we get that $\sum_{j=1}^r \beta_j \tau_j \alpha_j : C^*\Gamma \to V$ maps 1 to 0, therefore this map factorizes through $C^*_{\mathrm{ab}}/\langle[1]\rangle$.

But the projection of $\eta_{[0]}(\mathcal{D}_{\varepsilon_k,\pm\alpha_k})$ to $(C^*\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$ is the stable rho-invariant of $M \amalg (-M')$ which is identically zero.

The assertion of the theorem follows now from (11.5). □

11.6 Corollary. *The statements about the signature operator in Theorem* 1.4*, Theorem* 1.11 *and Theorem* 1.15 *are true.*

Proof. It suffices to specialize the result of our main theorem to the Examples 8.5 and 8.8. Notice that in this way we have established the result about the delocalized eta-invariant only under the Baum–Connes assumption for the maximal C*-algebra. In order to sharpen this result to the reduced C*-algebra we simply observe that if in the statement of Theorem 11.1 we take $j = 1$, $A = \mathcal{N}\Gamma$, $\alpha\colon C^*_r\Gamma \to \mathcal{N}\Gamma$ the natural map, $Z = \mathbb{C}$, $\tau = \tau_{\langle g\rangle}$ and $\beta_1 = 1$, then the basic condition (11.2) is satisfied, $\rho_{(\tau,\beta,\mathcal{L})}(M) = \eta_{\langle g\rangle}(\widetilde{D}^{\mathrm{sign}})$ and the proof carries over. □

11.7 Remark. Let M be a closed odd-dimensional manifold. We want to stress here that the mere vanishing of the signature index class in $K_1(C^*\Gamma)$ does not imply the vanishing of the corresponding L^2-rho-invariant: we wish to clarify this point. There certainly exist allowable perturbations to define the corresponding stable rho-invariant (by [38])); under our standard assumptions on the fundamental group and the Baum–Connes map this stable rho-invariant will be equal to zero. However, in this generality it cannot be guaranteed that *stable* = *unstable*. Indeed, there are examples where the index class is trivial but where the rho-invariants are non-trivial (even for a fundamental groups as simple as $\mathbb{Z}$); we shall construct them in Section 15. From this point of view, the case $M = X \sqcup (-X')$, with X and X' homotopy equivalent, is indeed very special.

12. Vanishing results on spin manifolds with positive scalar curvature

Let (M, g) be a Riemannian manifold. We assume that M is spin and that $\dim M = m = 2k - 1$, $k \geq 1$. We fix a spin structure and we let $\not{D}$ be the associated Dirac operator.

12.1 Theorem. *Assume that M has positive scalar curvature. Under the same assumptions on Γ, A_j, α_j, τ_j and β_j as in Theorem* 11.1 *we have*

$$\rho_{(\tau_j,\beta_j,L_j)}(M) = 0 \in Z. \tag{12.2}$$

Proof. The proof is parallel to the proof of Theorem 11.1, but much easier since under the assumption of positive scalar curvature the unstable rho-invariant coincides by definition with the stable one. □

12.3 Corollary. *The statements about the spin Dirac operator in Theorem* 1.4*, Theorem* 1.11 *and Theorem* 1.15 *are true.*

Proof. The proof is word by word parallel to the proof of Corollary 11.6, with the same remark applying in order to pass from $C^*\Gamma$ to $C_r^*\Gamma$ for the delocalized eta-invariant. □

12.4 Remark. Let M be a closed oriented odd dimensional manifold and let $\Gamma \to \widetilde{M} \to M$ be a Galois covering, with Γ torsion-free. Let us assume that the signature operator on $\widetilde{M}$ is L^2-invertible; thanks to the work of Farber–Weinberger [15] and Higson–Roe–Schick [21] we know that there are plenty of examples of such coverings. Then our arguments imply that

$$\rho_{\lambda_1-\lambda_2}(D^{\mathrm{sign}}) = 0 \quad \text{and} \quad \rho_{(2)}(\widetilde{D}^{\mathrm{sign}}) = 0,$$

provided that the max-Baum–Connes map is bijective, and

$$\eta_{\langle g\rangle}(\widetilde{D}^{\mathrm{sign}}) = 0$$

if $\langle g\rangle$ is of polynomial growth, and the reduced-Baum–Connes map is bijective. *Conclusion*: for such L^2-invisible coverings our signature-rho-invariants are zero, just as are the spin rho-invariants in the presence of positive scalar curvature.

13. Further remarks on delocalized invariants

13.1. Infinite conjugacy classes. Let $\langle g\rangle$ be a non-trivial conjugacy class in Γ, not necessarily of finite cardinality.

13.1 Definition. If the limit

$$\lim_{T\to+\infty}\frac{1}{\sqrt{\pi}}\int_0^T t^{-1/2}\,\mathrm{Tr}_{\langle g\rangle}(\widetilde{D}\exp(-t\widetilde{D}^2)dt \tag{13.2}$$

exists, then its value is, by definition, the delocalized eta-invariant $\eta_{\langle g\rangle}(\widetilde{D})$ associated to $\widetilde{D}$ and $\langle g\rangle$.

Lott [41], [43] establishes the convergence if the operator $\widetilde{D}$ on $\widetilde{M}$ has a gap near zero in its spectrum (i.e. zero is isolated in the spectrum) and if the trace $\tau_{\langle g\rangle}\colon \mathbb{C}\Gamma\to\mathbb{C}$ associated to $\langle g\rangle$ has a certain extension-property. Let us recall his results. Let $\mathcal{B}^\infty_\Gamma$, $\mathbb{C}\Gamma\subset\mathcal{B}^\infty_\Gamma\subset C^*_r\Gamma$, be the Connes–Moscovici algebra [12] and let $\mathcal{D}^\infty$ denote the Dirac operator on M twisted by the flat bundle $\widetilde{M}\times_\Gamma\mathcal{B}^\infty$. First of all, if $\widetilde{D}$ has a gap in its spectrum, then the integral defining the 0-degree eta-invariant of $\mathcal{D}^\infty$,

$$\begin{aligned}\eta_{[0]}(\mathcal{D}^\infty) &:= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{TR}(\mathcal{D}^\infty\exp(-t(\mathcal{D}^\infty)^2)dt\\ &\in \mathcal{B}^\infty_\Gamma/\overline{[\mathcal{B}^\infty_\Gamma,\mathcal{B}^\infty_\Gamma]},\end{aligned} \tag{13.3}$$

converges in the natural Fréchet topology induced by $\mathcal{B}^\infty_\Gamma$. We set as usual $(\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}} := \mathcal{B}^\infty_\Gamma/\overline{[\mathcal{B}^\infty_\Gamma,\mathcal{B}^\infty_\Gamma]}$. Next, if the trace $\tau_{\langle g\rangle}\colon\mathbb{C}\Gamma\to\mathbb{C}$ *extends* from $\mathbb{C}\Gamma$ to a continuous trace on $\mathcal{B}^\infty_\Gamma$, then one can prove (see Remark E.12 in Appendix E.2) that

$$\tau_{\langle g\rangle}(\mathrm{TR}(\mathcal{D}^\infty\exp(-t(\mathcal{D}^\infty)^2))) = \mathrm{Tr}_{\langle g\rangle}(\widetilde{D}\exp(-t\widetilde{D}^2)).$$

It is now clear that if $\widetilde{D}$ has a gap in its spectrum and if $\tau_{\langle g\rangle}$ extends, then the integral in (13.2) converges and the delocalized eta-invariant $\eta_{\langle g\rangle}(\widetilde{D})$ is well defined. Notice that under our two assumptions

$$\eta_{\langle g\rangle}(\widetilde{D}) = \tau_{\langle g\rangle}(\eta_{[0]}(\mathcal{D}^\infty)). \tag{13.4}$$

For example, groups of polynomial growth have the property that $\tau_{\langle g\rangle}$ extends for each conjugacy class $\langle g\rangle$. More generally:

13.5 Proposition. *If $\langle g\rangle$ is of polynomial growth with respect to a word metric, then $\tau_{\langle g\rangle}$ has the extension property.*

Proof. This follows from [12, Lemma 6.4] and the Hölder inequality. Compare also [24] □

13.6 Remark. It is in general very hard to check when a conjugacy class has polynomial growth. In particular, it is known that a group which has exponential growth will have a conjugacy class whose growth is not polynomial. On the other hand, the conjugacy classes of commutators in a metabelian group are in many cases of polynomial growth. For these results compare [67].

13.7 Theorem. *Assume that* Γ *is torsion-free,* $\mu_{\mathrm{red}}\colon K_*(B\Gamma) \to K_*(C^*_{\mathrm{red}}\Gamma)$ *is an isomorphism and that* $\langle g\rangle$ *is a non-trivial conjugacy class in* Γ *of polynomial growth. If* $D = \not{D}$ *is the spin Dirac operator on a spin manifold with positive scalar curvature, then* $\eta_{\langle g\rangle}(\widetilde{\not{D}})$ *is defined and* $\eta_{\langle g\rangle}(\widetilde{\not{D}}) = 0$.

Proof. We shall freely use a $(\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}}$-valued APS-index theory. This is the zero-degree part of the higher theory developed in [34].[8] Since we are in the presence of positive scalar curvature, the fact that $\eta_{\langle g\rangle}(\widetilde{\not{D}})$ is well defined follows from Lott's results and Proposition 13.5. The vanishing of $\eta_{\langle g\rangle}(\widetilde{\not{D}})$ follows from (13.4) and from the proof of Theorem 7.3 and Theorem 13.9 below, which show that $[\eta_{[0]}(\not{\mathcal{D}}^\infty)] \in (\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$ vanishes. Notice that $\tau_{\langle g\rangle}\colon (\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}} \to \mathbb{C}$ factors through $(\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$. □

13.8 Remark. For the signature operator one can prove the following statement:

If $D = D^{\mathrm{sign}}$ *is the signature operator of an oriented Riemannian manifold, and if* 0 *is isolated in the* L^2*-spectrum of* $\widetilde{D}^{\mathrm{sign}}$*, then* $\eta_{\langle g\rangle}(\widetilde{D}^{\mathrm{sign}})$ *is defined for each pair* $(M', u'\colon M' \to B\Gamma)$ Γ*-homotopy equivalent to* M*, and it is an oriented* Γ*-homotopy invariant.*

That the delocalized eta-invariants are well defined follows from Lott's results, from Proposition 13.5 and from the homotopy invariance of the Novikov–Shubin numbers, see Gromov–Shubin [17]. The proof of the homotopy invariance given in the previous sections can be modified so as to cover this case as well. We omit the details. In fact, under the gap assumption one can argue in the following alternative and more general way.

Because of the gap assumption the signature index class associated to $(M, u\colon M \to B\Gamma)$ is zero in $K_1(\mathcal{B}^\infty_\Gamma) = K_1(C^*_r\Gamma)$. Since the Baum–Connes assembly map is by assumption bijective, we conclude from Theorem 7.1 that the *stable* rho-invariant $\rho^s_{[0]}(M, u)$ is equal to zero in $(\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}}/\langle[1]\rangle$. However, in this case there is

[8]The $\mathcal{B}^\infty_\Gamma$ higher APS-index theory developed in [34] assumes the group to be of polynomial growth. Arbitrary groups are treated in [35] in the invertible case, based on results of Lott [42]. In order to extend the general $\mathcal{B}^\infty_\Gamma$-theory of [34] from groups of polynomial growth to arbitrary groups one simple needs to prove that if $\mathrm{Ind}(\mathcal{D}^\infty) = 0$ in $K_1(\mathcal{B}^\infty_\Gamma) = K_1(C^*_r\Gamma)$ then there exist trivializing perturbations in the $\mathcal{B}^\infty_\Gamma$-Mishchenko–Fomenko calculus. However, this is the consequence of a simple density argument. We thank Victor Nistor for pointing this out.

a very natural perturbation available for the definition of the stable rho-invariant, namely the projection $\Pi_{\ker(\mathcal{D}^\infty)}$ onto the null space of $\mathcal{D}^\infty$, which is a finitely generated projective $\mathcal{B}^\infty_\Gamma$-module because of the gap assumption. Thus $\rho^s_{[0]}(M,u) = [\eta_{[0]}(\mathcal{D}^\infty + \Pi)] = 0$. On the other hand, in general, it can be proved [34] that

$$\eta_{[0]}(\mathcal{D}^\infty + \Pi) = \eta_{[0]}(\mathcal{D}^\infty) + [\ker(\mathcal{D}^\infty)]_{[0]} \equiv \eta_{[0]}(\mathcal{D}^\infty) + \mathrm{TR}(\Pi_{\ker(\mathcal{D}^\infty)}) \text{ in } \mathcal{B}^\infty_\Gamma.$$

We conclude that under our three assumptions (gap + Baum–Connes + polynomial growth of $\langle g\rangle$) the following formula holds:

$$\eta_{\langle g\rangle}(\widetilde{D}) = -\tau_{\langle g\rangle}(\mathrm{TR}(\Pi_{\ker(\mathcal{D}^\infty)})).$$

On the other hand, because of the gap assumption, one can establish a Hodge theorem, proving that the whole null space $\ker(\mathcal{D}^\infty)$ is in this case a homotopy invariant, being isomorphic to the cohomology of M with values in the local system $\widetilde{M} \times_\Gamma \mathcal{B}^\infty_\Gamma$. Thus, under our assumption, the homotopy invariance of $\eta_{\langle g\rangle}(\widetilde{D}^{\mathrm{sign}})$ is a consequence of a much more general result.

13.2. Higher rho-invariants. Let us consider $(M, u\colon M \to \mathcal{B}\Gamma)$ with M spin and with a metric with positive scalar curvature. In this section M is not necessarily of odd dimension. Let $\not{\mathcal{D}}$ be the Dirac operator twisted by the Mishchenko–Fomenko bundle $u^*E\Gamma \times_\Gamma \mathcal{B}^\infty_\Gamma$. In this case, Lott's higher eta-invariant

$$\tilde{\eta}(\not{\mathcal{D}}) \in (\Omega_*(\mathcal{B}^\infty_\Gamma))_{\mathrm{ab}} := \Omega_*(\mathcal{B}^\infty_\Gamma)/\overline{[\Omega_*(\mathcal{B}^\infty_\Gamma), \Omega_*(\mathcal{B}^\infty_\Gamma)]}$$

is well defined, see [42], [34, Appendix]. Higher rho-invariants are obtained by pairing this noncommutative differential form with suitable closed graded traces on $\Omega_*(\mathcal{B}^\infty_\Gamma)$. Let us describe these traces. We start with a closed graded trace Φ on $\Omega_*(\mathbb{C}\Gamma)$; we assume that Φ is identically zero on the noncommutative differential forms concentrated at the identity conjugacy class, i.e. on elements of the form

$$\sum_{\gamma_0,\dots,\gamma_k;\gamma_0\cdots\gamma_k=1} \omega_{\gamma_0,\gamma_1,\dots,\gamma_k} \gamma_0 d\gamma_1 \dots d\gamma_k.$$

There are examples of such traces, see [40, p. 209]. We briefly refer to Φ as a *delocalized closed graded trace*. We *assume* that Φ *extends* to a closed graded trace Φ_∞ on $\Omega_*(\mathcal{B}^\infty_\Gamma)$. The *higher rho-invariant* associated to $\not{\mathcal{D}}$ and Φ is by definition the complex number

$$\tilde{\rho}_\Phi(\not{\mathcal{D}}) := \langle \tilde{\eta}(\not{\mathcal{D}}), \Phi_\infty\rangle.$$

It is clear that the delocalized eta-invariant of Lott is a special case of this construction. We shall also use the notation $\tilde{\rho}_\Phi(M,u) := \tilde{\rho}_\Phi(\not{\mathcal{D}})$.

It is proved in [37, Proposition 4.2] that the higher rho-invariants defines maps $\tilde{\rho}_\Phi\colon \mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \to \mathbb{C}$. If Γ has torsion, the latter information is used in [37] in order to distinguish metrics of positive scalar curvature (under suitable assumptions on Γ). In the torsion-free case, on the other hand, we can prove the following result.

13.9 Theorem. *Assume that Γ is torsion-free and that the assembly map $\mu_{\mathrm{red}}\colon K_*(B\Gamma) \to K_*(C^*_{\mathrm{red}}\Gamma)$ is an isomorphism. Let $\Phi\colon \Omega_*(\mathbb{C}\Gamma) \to \mathbb{C}$ be a delocalized closed graded trace. Assume that Φ extends to a closed graded trace on $\Omega_*(\mathcal{B}^\infty_\Gamma)$. If M is a spin manifold with positive scalar curvature and $u\colon M \to B\Gamma$ is a classifying map, then for the associated Dirac operator $\not{\mathcal{D}}$ the higher rho-invariant vanishes*:

$$\tilde{\rho}_\Phi(M,u) := \tilde{\rho}_\Phi(\not{\mathcal{D}}) = 0.$$

Proof. Since M has positive scalar curvature the index class of $\not{\mathcal{D}}$ is equal to zero. Thus, using Proposition 5.3 and the injectivity of μ_{red}, we conclude that for some $d \in \mathbb{N} \setminus \{0\}$, $d(M,u\colon M \to B\Gamma)$ is bordant to

$$\bigcup_{j=1}^{k} (A_j \times B_j, r_j \times 1\colon A_j \times B_j \to B\Gamma)$$

with $\dim B_j = 4b_j$, $\pi_1(B_j) = 1$ and $\langle \widehat{A}(B_j), [B_j]\rangle = 0$. We denote the latter manifold with classifying map by $(N, v\colon N \to B\Gamma)$. We can and we shall endow N with a metric of positive scalar curvature. Let $(W, F\colon W \to B\Gamma)$ be the manifold with boundary realizing the bordism. Since there is a metric of positive scalar curvature on ∂W, the b-index class $\mathrm{Ind}_b(\not{\mathcal{D}}_W)$ is well defined in $K_*(C^*_r\Gamma)$. By the surjectivity of μ_{red} we know that $\mathrm{Ind}_b(\not{\mathcal{D}}_W) = \mathrm{Ind}(\not{\mathcal{D}}_X)$ with X a *closed* spin manifold with classifying map $r\colon X \to B\Gamma$. In particular, using Lott's treatment of the Connes–Moscovici higher index theorem [12], [40], we see that the Karoubi–Chern character of $\mathrm{Ind}_b(\not{\mathcal{D}}_W)$ is concentrated in the trivial conjugacy class. This means that $\langle \mathrm{Ch}(\mathrm{Ind}_b(\not{\mathcal{D}}_W)), \Phi_\infty\rangle = 0$. On the other hand, by the higher APS-index theorem in [33] we know that

$$\langle \mathrm{Ch}(\mathrm{Ind}_b(\not{\mathcal{D}}_W)), \Phi_\infty\rangle = d\,\tilde{\rho}_\Phi(M,u) - \sum_{j=1}^{k} \tilde{\rho}_\Phi(A_j \times B_j, r_j \times 1),$$

since the local part in the index formula is concentrated in the trivial conjugacy class and it is thus sent to zero by Φ_∞. Hence using the bijectivity of μ_{red} we have proved that

$$\tilde{\rho}_\Phi(M,u) = \frac{1}{d} \sum_{j=1}^{k} \tilde{\rho}_\Phi(A_j \times B_j, r_j \times 1).$$

Using now the product formula for *higher* eta-invariants proved in [37, Proposition 2.1] we see that $\tilde{\rho}_\Phi(A_j \times B_j, r_j \times 1) = 0$ for all j. Thus $\tilde{\rho}_\Phi(M,u) = 0$ and the theorem is proved. □

14. The center-valued L^2-signature formula for manifolds with boundary

In this section we relate the eta- and rho-invariants of the signature operator, which show up in the APS-index theorem for the signature operator, to the signature of the manifold with boundary (the signature of some possibly degenerate intersection form).

Note that, even in the compact case, the ordinary signature formula for manifolds with boundary does not immediately follow from the Atiyah–Patodi–Singer index theorem for the signature operator. It is a non-trivial result of [2] to connect the APS-index of the signature operator to the signature of the manifold with boundary. This is much more complicated in the L^2-case, since in [2] eigenvalue decompositions of the space of L^2-sections on the boundary are used, which are not available in our setting. This is overcome in [46] together with [64] for the numerical L^2-signature. We now explain how this is done and how it generalizes to the situation we are considering in Section 15.

Therefore, let W be a compact oriented Riemannian manifold of dimension $4k$ with boundary $\partial W = M$. Let A be a von Neumann algebra and $\tau\colon A \to Z$ a finite positive normal trace with values in an abelian von Neumann algebra Z. We are thinking here in particular of the von Neumann algebra of a discrete group Γ with its canonical trace or with its center-valued trace.

Let $\mathcal{L}$ be a *flat* bundle of finitely generated projective Hilbert A-modules on W (giving rise to a local coefficient system of such modules). Recall that $\mathcal{L}$ is given by a representation of Γ in a finitely generated projective Hilbert A module (which we call $\mathcal{L}_x$ here). We assume that everything involved (Riemannian metric, bundle, connection) is of product type near the boundary.

We can now define three kinds of intersection forms on W, using the twisting bundle $\mathcal{L}$, and with a signature in $K_0(A)$.

The most computable one is obtained combinatorially: we consider a triangulation (or more generally a CW-decomposition) of W. This defines a cellular chain complex $C_*(W;\mathcal{L})$ of finitely generated free Hilbert A-modules, with coefficients in the local coefficient system $\mathcal{L}$. There is a Poincaré duality chain homotopy equivalence to the relative cochain complex $C^{4k-*}(W,\partial W;\mathcal{L})$. From there we can restrict to W to get a map to $C^{4k-*}(W;\mathcal{L})$. Since the second map is not a chain homotopy equivalence in general, neither is the composition. But it is self-dual (note that the cochain complex is dual to the chain complex).

Now one can pass to the projective part of the homology and cohomology of these Hilbert A-module (co)chain complexes. This passage to the projective part involves some additional consideration in homological algebra – special to finite von Neumann algebras –, developed in different languages and independently by Farber [14] and Lück [44]. We will later only look at the special example where the homology and cohomology in middle degree is itself a finitely generated free Hilbert A-module (in fact equal to the chain- and cochain-module). Then the projective part in middle degree is equal to the whole (co)homology. The Poincaré duality chain homotopy equivalence composed with restriction to the boundary will then define a self-dual map

$$H_{2k}(W;\mathcal{L}) \to H^{2k}(W;\mathcal{L}) \to \mathrm{Hom}_{\mathcal{A}}(H_{2k}(W;\mathcal{L}),\mathcal{A}).$$

In the special situation we are going to consider this will be given as follows: There is a free finitely generated $\mathbb{Z}\Gamma$-module $V \cong (\mathbb{Z}\Gamma)^l$ with (possibly singular) self-dual map $\sigma\colon V \to V^* := \mathrm{Hom}_{\mathbb{Z}\Gamma}(V,\mathbb{Z}\Gamma)$ of the form $\sigma = \psi + (-1)^k\psi^*$ (i.e., σ has a quadratic refinement). Identifying V^* with V using the given basis, σ is represented by a matrix $B = A + A^*$ with $A \in \mathrm{M}_n(\mathbb{Z}\Gamma)$. The self-dual map $H_{2k}(W;\mathcal{L}) \to H_{2k}(W;\mathcal{L})^*$ is then obtained by tensoring $\sigma\colon V \to V^*$ with the Γ-representation $\mathcal{L}_x$, i.e. $H_{2k}(W;\mathcal{L}) \cong V \otimes_{\mathbb{Z}\Gamma} \mathcal{L}_x \cong \mathcal{L}_x^l$, and the map is given as $B' := B \otimes \mathrm{id}_{\mathcal{L}_x}$.

The combinatorial signature $\mathrm{sgn}(W,\mathcal{L}) \in K_0(\mathcal{A})$ is then defined by

$$\mathrm{sgn}(W,\mathcal{L}) := \mathrm{sgn}_{\mathcal{A}}(B') := \chi_{(0,\infty)}(B') - \chi_{(-\infty,0)}(B') \in K_0(\mathcal{A}).$$

Note that, for manifolds with boundary, this can only be defined for a von Neumann algebra, where measurable functional calculus is available, since 0 might well be contained in the spectrum of B' (in contrast to the closed case, where the topology implies that an intersection form is necessarily invertible).

Using the trace $\tau\colon \mathcal{A} \to Z$, we can then define the Z-valued combinatorial signature

$$\mathrm{sgn}_\tau(W,\mathcal{L}) := \mathrm{sgn}_\tau(B') := \tau(\mathrm{sgn}_A(B')).$$

The second version of A-signature is obtained by replacing the combinatorial chain complex by the chain complex of differential forms with values in the flat bundle $\mathcal{L}$. Here the cup product of forms together with the A-valued inner product in the fibers of $\mathcal{L}$ induces an A-valued intersection pairing on the de Rham cohomology in middle degree with coefficients in $\mathcal{L}$. This intersection pairing is again defined by a self-adjoint operator B_{dR}, and then $\mathrm{sgn}_{\mathcal{A}}(B_{\mathrm{dR}}) := \chi_{(0,\infty)}(B_{\mathrm{dR}}) - \chi_{(-\infty,0)}(B_{\mathrm{dR}}) \in K_0(\mathcal{A})$ defines the de Rham signature of W with coefficients in $\mathcal{L}$.

Finally, we can attach infinite cylinders to the boundary of W, and then carry out the construction as above, but now with square integrable differential forms (with values in $\mathcal{L}$) on the enlarged non-compact manifold. We get a third signature invariant in $K_0(A)$.

Application of τ gives three signatures in Z.

One can now use the arguments of [45] in order to show that these three invariants in fact always coincide. In [45], the corresponding result is proved for the ordinary L^2-signature, i.e., where $\mathcal{L} = \mathcal{N}$ is the Mishchenko–Fomenko line bundle, and $\tau\colon \mathcal{N}\Gamma \to \mathbb{C}$ is the canonical trace. The techniques carry over. The relevant properties are that τ is a positive and normal trace. Note that in [45] one works with

$\mathcal{A}$-Hilbert spaces instead of the Hilbert $\mathcal{A}$-modules used here. However, one can translate between these two settings as explained in [60, Section 8.6].

Let $D_{\mathcal{L}}$ be the signature operator on W twisted by $\mathcal{L}$. It has boundary operator $D_{M,\mathcal{L}}$. Then we can express the three equal higher signature of the bordism W which were defined above using the signature operator.

14.1 Theorem.

$$\operatorname{sgn}_\tau(W;\mathcal{L}) = \left(\int_W \operatorname{AS}(D)\cdot\tau(\mathcal{L}_x)\right) - \frac{\eta_\tau(D_{M,\mathcal{L}})}{2} \in Z.$$

The Atiyah–Singer integrand $\operatorname{AS}(D)$ *for the signature operator is of course given by the Hirzebruch* L*-form of the Riemannian manifold* W*.* $\tau(\mathcal{L}_x)$ *is the trace of the projection onto the finitely generated projective fiber* $\mathcal{L}_x$*, this is a locally constant* A*-valued function on* W*.*

Proof. This is proved for the "ordinary" cylindrical end L^2-signature by Vaillant [64], using the L^2-Atiyah–Patodi–Singer index formula. His proof only uses the formal properties of the canonical L^2-trace of being positive and normal and therefore carries over to prove the asserted equality. As explained above, from the work in [45] it follows that the formula also holds for the combinatorially defined signature. □

15. Examples of non-trivial rho-invariants

In this section we show that there are many examples where the rho-invariants considered in this paper (known to be homotopy invariants) show that certain manifolds are not homotopy equivalent. The fundamental group can be as simple as $\mathbb{Z}$. Similar explicit calculations (without using the notion of rho-invariant) have been carried out in [11, Section 5]. In the latter paper these invariants are used to detect certain knots which do not have the slice property (which implies that certain types of bordisms can not exist).

We use the surgery construction of [46] employed there to construct manifolds with boundary with very general intersection form.

Recall from Section 14 that the combinatorially defined L^2-signatures of a triangulated manifold M (with or without boundary) is obtained as follows. Assume that $\Gamma = \pi_1(M)$. We have an "intersection form" on the combinatorial chain complex. This can be understood as a matrix B with entries in $\mathbb{Z}\Gamma$. Actually, because of the symmetry properties of the intersection form, $B = A + A^*$ for $A \in \mathrm{M}_n(\mathbb{Z}\Gamma)$. We now use $\mathcal{A} = \mathcal{N}\Gamma$ and $\mathcal{L} = \mathcal{N} = \widetilde{M} \times_\Gamma \mathcal{N}\Gamma$. In this situation, since $\mathbb{Z}\Gamma$ is a subset of $\mathcal{N}\Gamma$, we understand the matrices A and B to be matrices also over $\mathcal{N}\Gamma$ (i.e., we write B instead of B' in the notation of Section 14).

Then $\operatorname{sgn}(M,\mathcal{N}) = \operatorname{sgn}_{\mathcal{N}\Gamma}(B) = \chi_{(0,\infty)}(B) - \chi_{(-\infty,0)}(B)$.

Applying the center-valued trace $\tau\colon \mathcal{N}\Gamma \to Z$, we get then $\operatorname{sgn}(M, \mathcal{N}, \tau) = \tau(\operatorname{sgn}(M, \mathcal{N}))$. Set $\operatorname{sgn}_\tau(B) := \tau(\operatorname{sgn}_{\mathcal{N}\Gamma}(B))$. Applying the canonical trace, from this we get $\operatorname{sgn}_{(2)}(B) \in \mathbb{C}$, and applying the delocalized traces corresponding to a finite conjugacy class $\langle g\rangle$, we get $\operatorname{sgn}_{\langle g\rangle}(B) \in \mathbb{C}$.

Moreover, if $\lambda\colon \Gamma \to \mathrm{U}(d)$ is a finite dimensional representation of Γ, set $\lambda(B) \in \mathrm{M}_n(\mathrm{M}_d(\mathbb{C})) = \mathrm{M}_{nd}(\mathbb{C})$, the matrix obtained by applying λ entrywise. Then $\operatorname{sgn}(\lambda(B))$ is the λ-twisted signature of M. Recall that $\lambda(B)$ is a possibly indefinite self-adjoint matrix; $\operatorname{sgn}(\lambda(B))$ is the difference of the number of positive and negative eigenvalues, the eigenvalue 0 is ignored.

We will use the following Proposition 1.1 of [45].

15.1 Proposition. *Fix a dimension $4k \geq 6$ and a finitely presented group Γ. Let X be a closed $(4k-1)$-dimensional manifold with fundamental group Γ and with Morse decomposition without a k-handle. Choose $A \in \mathrm{M}_n(\mathbb{Z}\Gamma)$. Let $B = A + A^*$.*

Then there is a compact manifold with boundary $(W; X, Y)$ of dimension $4k$ with boundary $\partial W = X \amalg Y$ and fundamental group Γ such that the Morse chain complex $C_*(\widetilde{W})$ of the universal covering $\widetilde{W}$ is isomorphic to $C_*(\widetilde{X}) \oplus V$, where V is considered as trivial chain complex concentrated in the middle dimension k, and with inverse Poincaré duality homomorphism

$$C_{4k-*}(\widetilde{W}) \to C_{4k-*}(\widetilde{W}, \partial\widetilde{W}) \xrightarrow{PD^{-1}} C^*(\widetilde{W}),$$

which in the middle dimension is given by $B := A + A^*$, as explained in Section 14. Here PD^{-1} is a chain homotopy inverse to the cup product with $[W, \partial W]$.

In particular, $\operatorname{sgn}_\tau(W, \mathcal{N}) = \tau(\chi_{(0,\infty)}(B)) - \tau((-\infty, 0)(B))$, where $\tau\colon \mathcal{N}\Gamma \to Z$ is the center-valued trace.

Recall that, on the other hand, for the constructed manifold W the ordinary signature is the signature of $\rho(B)$, where $\rho\colon \Gamma \to \{1\}$ is the trivial representation.

Using the L^2-signature Theorem 14.1, we therefore get in principle a large number of examples of manifolds with interesting difference of rho-invariants.

15.2 Corollary. *Given any $A \in \mathrm{M}_n(\mathbb{Z}\Gamma)$, set $B := A + A^*$. Then there exist manifolds X, Y with $\pi_1(X) = \pi_1(Y) = \Gamma$ such that*

$$\rho_{(2)}(X) - \rho_{(2)}(Y) = \operatorname{sgn}_{(2)}(B) - \operatorname{sgn}(\rho(B)).$$

If $\lambda_1, \lambda_2\colon \Gamma \to U(d)$ are two representations, then

$$\rho_{\lambda_1-\lambda_2}(X) - \rho_{\lambda_1-\lambda_2}(Y) = \operatorname{sgn}(\lambda_1(B)) - \operatorname{sgn}(\lambda_2(B)).$$

If $g \in \Gamma$ has finite conjugacy class $\langle g\rangle$, then

$$\rho_{\langle g\rangle}(X) - \rho_{\langle g\rangle}(Y) = \operatorname{sgn}_{\langle g\rangle}(B).$$

15.3 Remark. Note that $X \amalg -Y$ with the reference map to $B\mathbb{Z}$ is a boundary and therefore has signature class $0 \in K_0(C^*\mathbb{Z})$. Nevertheless, our construction shows that the rho-invariants of $X \amalg -Y$ are non-zero. In particular, for such a manifold the stable and the unstable rho-invariant can differ (we know that the stable one vanishes in such an example).

This theorem implies that we have a great freedom of constructing manifolds X and Y such that the various rho-invariants differ. There is of course a problem in explicitly calculating these invariants for a given matrix B. Let us therefore recall how this can be done in the easiest case, i.e., if $\Gamma = \mathbb{Z}$.

In this case $\mathcal{N}\mathbb{Z} \cong L^\infty(S^1)$ using Fourier transform. The subring $\mathbb{Z}[\mathbb{Z}] \subset \mathcal{N}\mathbb{Z}$ corresponds to Laurent polynomials on S^1.

Therefore, $B = A + A^* \in \mathrm{M}_n(\mathbb{Z}\Gamma)$ can be understood as an $n \times n$-matrix with entries in Laurent polynomials on S^1, or alternatively as a function (a Laurent polynomial) $B(z)$ on S^1 with values in $\mathrm{M}_n(\mathbb{C})$. To compute $\operatorname{sgn}_{\mathcal{N}\mathbb{Z}}(B)$, one then computes pointwise $\operatorname{sgn}(B(z))$, getting an integer-valued function on S^1. Considered as an element of $L^\infty(S^1) = \mathcal{N}\mathbb{Z} = Z$, this is exactly $\operatorname{sgn}_\tau(B)$ (since $\mathcal{N}\mathbb{Z}$ is abelian the center-valued trace is the identity).

If $\lambda \colon \mathbb{Z} \to \mathrm{U}(1)$ is a 1-dimensional representation, sending the generator z to $\lambda \in \mathrm{U}(1) = S^1$, then $\operatorname{sgn}_\lambda(B) = \operatorname{sgn}(B(\lambda))$, i.e., we have to evaluate the function $\operatorname{sgn}(B(z))$ at the point λ. In particular, for the trivial representation, $\operatorname{sgn}_1(B) = \operatorname{sgn}(B(1))$.

If, on the other hand, $g = z^n \in \mathbb{Z}$, then $\operatorname{sgn}_{\langle g\rangle}(B)$ is the Fourier coefficient of g for the function $\operatorname{sgn}(B(z))$, i.e.,

$$\operatorname{sgn}_{\langle g\rangle}(B) = \int_{S^1} \operatorname{sgn}(B(z)) z^{-n}\, dz, \quad \text{in particular} \quad \operatorname{sgn}_{(2)}(B) = \int_{S^1} \operatorname{sgn}(B(z))\, dz.$$

Let us look at a particular example. If $A = z + z^{-1} + 1$ (a 1×1-matrix), then $B(z) = 2(z + z^{-1} + 1)$, and $\operatorname{sgn}(B(z)) = 0$ for $z = z_{1,2} = \exp(\pm 2\pi i/3)$, $\operatorname{sgn}(B(z)) = 1$ if z is contained in the connected component of 1 of $S^1 \setminus \{z_1, z_2\}$, and $\operatorname{sgn}(B(z)) = -1$ for the remaining points of z.

Observe that this is the general pattern: $\operatorname{sgn}(B(z))$ jumps (at most) at those points on S^1 where an eigenvalues of $B(z)$ crosses 0, i.e., where the rank of $B(z)$ is lower than the maximal rank. It follows e.g. that $\operatorname{sgn}_{(2)}(B) = 1/3$, but $\operatorname{sgn}_1(B) = 1$ and $\operatorname{sgn}_{\langle z\rangle}(B) = 2(\exp(2\pi i/3) - \exp(-2\pi i/3)) = 2i\sqrt{3}$. In particular, since $\operatorname{sgn}_{(2)}(B) - \operatorname{sgn}_1(B) \neq 0$ (or since $\operatorname{sgn}_{\langle z\rangle}(B) \neq 0$), if we use the matrix A in the construction of Proposition 15.1, the resulting closed manifolds X and Y satisfy

$$\rho_{(2)}(Y) - \rho_{(2)}(X) = \operatorname{sgn}_{(2)}(B) - \operatorname{sgn}_1(B) = -2/3,$$

and are therefore not homotopy equivalent.

Note that we can apply Theorems 1.4, 1.11 or 1.15 since $\mathbb{Z}$ is torsion-free and satisfies the Baum–Connes conjecture for the maximal C*-algebra.

Unfortunately, this conclusion has one flaw: by [65, Lemma 2.2] and the proof of [65, Theorem 5.8],

$$\begin{aligned} H_k(Y;\mathbb{Z}\Gamma) &\cong \ker(B) \oplus H_k(X;\mathbb{Z}\Gamma), \\ H_{k-1}(Y;\mathbb{Z}\Gamma) &\cong \operatorname{coker}(B) \oplus H_{k-1}(X;\mathbb{Z}\Gamma), \end{aligned} \tag{15.4}$$

whereas the homology of X and Y in all other degrees coincides. This implies that we might as well distinguish X and Y using their homology, which is of course much easier to compute that the rho-invariants.

One can easily construct more interesting examples as follows: consider a diagonal matrix A with entries $A_1(z),\dots,A_m(z)$, and a second diagonal matrix A' with entries $\varepsilon_1 A_1(a),\dots,\varepsilon_n A_n(z)$, with $\varepsilon_i \in \{-1,1\}$. Starting with a manifold X as in Proposition 15.1, we then get two manifolds Y and Y'. By [65, Lemma 2.2] and the proof of [65, Theorem 5.8], the homology of Y and Y' is isomorphic in all degree and with arbitrary coefficients.

However, the signatures and therefore the difference of rho-invariants changes sign if the matrix A changes sign. This means that we can easily arrange that certain rho-invariants of Y and Y' do not coincide, we have even enough freedom to make sure that there are examples where they neither coincide nor are negative of each other.

The conclusion is that although the homology of Y and Y' is isomorphic in all degrees and with arbitrary coefficients, there is no homotopy equivalence between Y and Y' (not even one which reverses the orientations). This result has been obtained using rho-invariants, it can be recast in terms of Blanchfield forms (also called linking forms), i.e. in terms of classical methods of advanced algebraic topology.

It is evident that with other matrices $A(z)$, we can get all kinds of further interesting examples.

This was all done for a group as simple as $\mathbb{Z}$. One is often interested to get examples for more complicated groups, compare also the problem of constructing knots; see e.g. [11]. We can "induce up" these examples by embedding $\mathbb{Z}$ in an arbitrary torsion-free group Γ and then use the fact that the signature calculations happen entirely in the subgroup $\mathbb{Z}$ and therefore are unchanged, as long as the L^2-signature and the ordinary signature are involved. The relevant result is stated and proved e.g. in [11, Proposition 5.13]. The following happens: when we have an inclusion $i\colon \mathbb{Z} \hookrightarrow \Gamma$, then this induces inclusions $l^2(\mathbb{Z}) \hookrightarrow l^2(\Gamma)$ and $i\colon \mathcal{N}\mathbb{Z} \hookrightarrow \mathcal{N}\Gamma$. The latter one being an inclusion of von Neumann algebras, it is compatible with functional calculus. If we therefore start with $B = A^* + A \in \mathrm{M}_n(\mathbb{Z}\Gamma)$, then $\chi_{(0,\infty)}(i(B)) = i(\chi_{(0,\infty)}(B))$. It is not clear to us how the classical methods of algebraic topology mentioned above could to be used in this general setting.

Finally, if we want to compute a delocalized trace of an element $i(x)$, we observe that by definition for $g \in \Gamma$ with $|\langle g\rangle| < \infty$,

$$\tau_{\langle g\rangle}(i(x)) = \sum_{h\in\langle g\rangle} \langle e \cdot i(x), h\rangle_{l^2(\Gamma)}.$$

But since $e \in l^2(\mathbb{Z})$, also $e \cdot i(x) \in i(l^2(\mathbb{Z}))$, and therefore the inner product is zero if $h \notin i(l^2(\mathbb{Z}))$, and

$$\tau_{\langle g\rangle}(i(x)) = \sum_{h\in\langle g\rangle\cap i(\mathbb{Z})} \langle e \cdot i(x), h\rangle_{l^2(\Gamma)} = \sum_{h\in\langle g\rangle\cap i(\mathbb{Z})} \tau_{i^{-1}(h)}(x).$$

In particular, if $g = e$ then $\operatorname{tr}_{(2)}(i(x)) = \operatorname{tr}_{(2)}(x)$, and if $g \neq e$, one has to analyze which conjugates of g lie in $i(\mathbb{Z})$. If $\langle g\rangle$ is finite, then g^{-1} can be the only power of g contained in $\langle g\rangle$. It follows that if g generates $i(\mathbb{Z})$, we understand $\langle g\rangle \cap i(\mathbb{Z})$ completely.

For example, if one embeds $\mathbb{Z}$ into the center of Γ, all delocalized invariants with respect to elements of this embedded $\mathbb{Z}$ will be the same, independent of the question whether the corresponding matrix is considered as a matrix over $\mathbb{Z}$ or over Γ. With a little extra care one can use the same "induction process" to obtain examples of non-trivial delocalized rho-invariants for finite conjugacy classes with more than one element.

A. Index theory: proofs

This first appendix is devoted to the proof of the two index theorems stated in Section 2.

A.1. Proof of the $A/\overline{[A, A]}$-valued Atiyah–Singer index formula. First of all we need to deal with the existence of the heat kernel for the Dirac Laplacian $D^2_{\mathcal{L}}$ and its perturbation $D^2_{\mathcal{L},\mathcal{C}} \equiv (D_{\mathcal{L}} + \mathcal{C})^2$.

A.1 Lemma. *For each $t > 0$ there exists a well-defined operator $e^{-tD^2_{\mathcal{L},\mathcal{C}}} \in \Psi^{-\infty}_A$. If $v \in C^\infty(M, E \otimes \mathcal{L})$ then $u(t,\cdot) := e^{-tD^2_{\mathcal{L},\mathcal{C}}} v$ is a solution of the heat equation $(\partial_t + D^2_{\mathcal{L},\mathcal{C}})u = 0$ with initial condition v at $t = 0$. The operators $e^{-tD^2_{\mathcal{L},\mathcal{C}}}$ form a semigroup.*

Proof. We begin with the Dirac Laplacian $D^2_{\mathcal{L}}$. The best way to prove the above lemma is by use of the *heat space* of Melrose; see [49], Chapter VII. First we make a general comment. Microlocal analysis can be viewed as a geometrization of operator theory, the operators of interest being certain distributions on $M \times M$ with precise singularities on the diagonal (conormal distributions). Smoothing operators, for example, are given as smooth functions on $M \times M$, or, more generally, as smooth sections of suitable homomorphism bundles on $M \times M$. It is not difficult to understand that the Mishchenko–Fomenko pseudodifferential calculus can be easily developed by simply considering A-valued conormal distributions. For example, smoothing operators in the Mishchenko–Fomenko calculus acting for simplicity on the trivial bundle $\mathbb{C}^\ell \otimes A$ are nothing but C^∞ functions on $M \times M$ with values in $\mathrm{M}_{\ell\times\ell}(A)$. From the microlocal point of view the presence of the C*-algebra A does not affect in a significant way all the usual arguments culminating in the existence of the pseudodifferential calculus. (Needless to say, operators in the Mishchenko–Fomenko calculus act on A-Hilbert modules and not on Hilbert spaces, it is at this point that much more care is needed.) This general philosophy will be applied here to the construction of the heat semigroup $e^{-tD^2_{\mathcal{L}}}$. The advantage of the treatment given by Melrose through the heat space (a certain blow-up of $M \times M \times [0,\infty)_t$) is that it is as geometric as it can be, thus generalizing without any difficulty to operators acting on sections of bundles of A-modules (such as $E \otimes \mathcal{L}$). This general principle has been already applied in [33] in the case $A = C^*_r\Gamma$ but it is clear that it extends readily to an arbitrary unital C*-algebra A.

Summarizing: By following closely the treatment given by Melrose for ordinary Dirac operator we can prove the lemma for the heat kernel associated to $D^2_{\mathcal{L}}$. A standard argument involving a Volterra series can be applied in order to obtain the lemma for the perturbed operator $D^2_{\mathcal{L},\mathcal{C}}$. See [5]. □

The problem we encounter with the heat kernel in the C*-algebraic context is that it does not behave well for $t \to \infty$, since our operators are in general not invertible in the Mishchenko–Fomenko calculus. For this reason, we introduce suitable perturbation which make them invertible.

A.2 Definition. Instead of M consider M_+, the disjoint union of the manifold with an additional point $*$. The Dirac type operator on the additional point is by definition the 0 operator on $\mathbb{C}$. Recall the finitely generated projective modules $\mathcal{I}_+$, $\mathcal{I}_-$ appearing in the Mishchenko–Fomenko decomposition theorem. We define $\mathcal{L}_+$, the twisting bundle on M_+ to be $\mathcal{L} \amalg (\mathcal{I}_+ \oplus \mathcal{I}_-)$, where we view $\mathcal{I}_\pm$ as an abstract finitely generated projective Hilbert A-module (therefore a bundle over the point).

Following [40], Section VI, we now define for $\alpha \in \mathbb{R}$ the perturbed operators (of $D_{\mathcal{L}}$, but we suppress the $\mathcal{L}$ in the notation)

$$D^+_\alpha := \begin{pmatrix} D^+_{\mathcal{L}}|_{\mathcal{I}^\perp_+} & 0 & 0 \\ 0 & D^+_{\mathcal{L}}|_{\mathcal{I}_+} & \alpha \\ 0 & \alpha & 0 \end{pmatrix},$$

and D_α^- as the (formal) adjoint of D_α^+. The description with respect to the Mishchenko–Fomenko decomposition of Theorem 2.4 is given by

$$C^\infty(M_+, (E\otimes\mathcal{L})_+^+) = \mathcal{I}_+^\perp \oplus \mathcal{I}_+ \oplus \mathcal{I}_-,$$
$$C^\infty(M_+, (E\otimes\mathcal{L})_+^-) = D_{\mathcal{L}}(\mathcal{I}_+^\perp) \oplus I_- \oplus \mathcal{I}_+.$$

Note that $\mathcal{I}_\pm$ has two roles here: first as subset of the section of $E\otimes\mathcal{L}$ on M, secondly as possible values of the sections at the additional point $*$, where the fiber is $\mathcal{I}_- \oplus \mathcal{I}_+$.

A.3 Lemma. *The operator $e^{-tD_\alpha^2}$ is defined for each $t > 0$ and each $\alpha \in \mathbb{R}$, and is a smoothing operator in the Mishchenko–Fomenko calculus on M_+. In particular* $\mathrm{STR}(e^{-tD_\alpha^2}) \in A_{\mathrm{ab}}$ *is defined.*

Proof. The heat operator $e^{-tD_\alpha^2}$ is defined by Duhamel's expansion, using the fact that $D_\alpha - D_0$ is finite rank as A-linear operator, see [40, Section VI]. Since the heat operator of D_0^2 is smoothing on M_+, the lemma follows. In fact, by using the information that the orthogonal projection onto $\mathcal{I}_+$ and the projection onto $\mathcal{I}_-$ along $D_{\mathcal{L}}(\mathcal{I}_+^\perp)$ are smoothing operators, it is possible to check that $D_\alpha - D_0$ is smoothing on M_+, i.e. an element in $\Psi_A^{-\infty}(M_+, (E\otimes\mathcal{L})_+, (E\otimes\mathcal{L})_+)$. □

The following lemma is clear.

A.4 Lemma. *Let $\alpha = 0$, then*

$$\mathrm{STR}(e^{-tD_0^2}) = \mathrm{STR}_M(e^{-tD_{\mathcal{L}}^2}) + [I_-]_{[0]} - [I_+]_{[0]} \in A_{\mathrm{ab}}.$$

A.5 Lemma. *For α sufficiently large, D_α is invertible in the MF-calculus and therefore*

$$\mathrm{STR}(e^{-tD_\alpha^2}) \xrightarrow{t\to\infty} 0 \in A_{\mathrm{ab}}. \tag{A.6}$$

Proof. The invertibility is explained in [40], Section VI, after formula (107). One can proceed as in [49] and show that the heat kernel defined through the heat space is indeed expressible in terms of the usual integral involving the resolvent. Using the invertibility of D_α^2 one then gets (A.6). □

A.7 Lemma. *For each $\alpha \in \mathbb{R}$ and $t > 0$*

$$\mathrm{STR}(e^{-tD_\alpha^2}) - \mathrm{STR}(e^{-tD_0^2}) = 0 \in A_{\mathrm{ab}}.$$

Proof. Use Duhamel's formula to compute the derivative with respect to α of $\mathrm{STR}(e^{-tD_\alpha^2})$. The usual calculations show that this is the supertrace of a supercommutator and therefore vanishes in A_{ab}. Details are as in [5], Chapter 3. □

As a corollary we obtain Proposition 2.13:

A.8 Corollary.

$$\lim_{t\to\infty} \mathrm{STR}_M(e^{-tD^2_{\mathcal{L}}}) = [I_+]_{[0]} - [I_-]_{[0]} \equiv \mathrm{Ind}_{[0]}(D_L) \in A_{\mathrm{ab}},$$

where part of the assertion is that the limit exists.

Proof. By Lemmas A.4 and A.7 we have in A_{ab}, for each $\alpha \in \mathbb{R}$ and $t > 0$:

$$\mathrm{STR}_M(e^{-tD^2_L}) = \mathrm{STR}(e^{-tD^2_0}) - [I_-]_{[0]} + [I_+]_{[0]} = \mathrm{STR}(e^{-tD^2_\alpha}) - [I_-]_{[0]} + [I_+]_{[0]}.$$

Taking α large enough we get the corollary by applying Lemma A.5. □

A.1.1. The integral operator index. Next we tackle the problem of connecting the index $\mathrm{Ind}_{[0]}(D_{\mathcal{L}}) \in A_{\mathrm{ab}}$ defined using the index class and the algebraic trace $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$ to the integral-kernel-trace, TR, of the projection operators given by the Mishchenko–Fomenko decomposition.

A.9 Definition. Define a second smoothing perturbation of D by

$$B^+_\alpha := D^+_{\mathcal{L}} - \alpha P_- D^+_{\mathcal{L}} P_+,$$

for $\alpha \in \mathbb{R}$, with B^-_α the (formal) adjoint of B^+_α and with $P_+ := \Pi_{\mathcal{I}_+}$, $P_- := \Pi_{\mathcal{I}_-}$.

A.10 Remark. Observe that

$$B_\alpha = \begin{pmatrix} 0 & B^-_\alpha \\ B^+_\alpha & 0 \end{pmatrix}$$

is a smoothing perturbation of $D_{\mathcal{L}}$. Note that P_+ is self-adjoint, but P_- is not necessarily. Thus we will also use its adjoint P^*_-. Note, finally, that $\mathrm{im}(1 - P_-) = D^+_{\mathcal{L}}(\mathcal{I}^\perp_+)$.

A.11 Lemma. *We have decompositions*

$$C^\infty(M, (E \otimes \mathcal{L})^-) = \mathrm{im}(P_-) \oplus \mathrm{im}(1 - P_-) = \mathrm{im}(P^*_-) \oplus \mathrm{im}(1 - P_-).$$

The second decomposition is an orthogonal decomposition since

$$\mathrm{im}(1 - P_-)^\perp = \ker(1 - P^*_-) = \mathrm{im}(P^*_-).$$

Let pr *be the orthogonal projection onto* $\mathrm{im}(P^*_-)$. *Then* pr *is also a smoothing operator.*

Proof. $P_-^* P_- + (1 - P_-) = 1 + (P_-^* P_- - P_-)$ is an isomorphism of $C^\infty(M, E \otimes \mathcal{L})^-)$ which is diagonal with respect to the two decompositions (it is an isomorphism, since P_-^* has kernel $\operatorname{im}(P_-)^\perp$ and is surjective, so that $P_-^* \colon \operatorname{im}(P_-) \to \operatorname{im}(P_-^*)$ is an isomorphism by the open mapping theorem). Then

$$\mathrm{pr} = (1 + (P_-^* P_- - P_-))^{-1} P_- (1 + (P_-^* P_- - P_-))$$

is smoothing by Lemma A.12 below. □

A.12 Lemma. *If P is smoothing and $1 + P$ is invertible in the sense of Hilbert-A-module morphisms (on the completed space of sections), then*

$$(1 + P)^{-1} = 1 + Q \quad \textit{with } Q \textit{ smoothing}.$$

Proof. Write $(1 + P)^{-1} = 1 + Q$; then Q satisfies $Q = -P - P^2 - PQP$ and it is therefore smoothing. Here we are using the fact that smoothing operators in L^2 are a semi-ideal (a subring R of a ring R is a semi-ideal if $i, j \in I,\ r \in R$, then $irj \in I$ for all $i, j \in I$ and for all $r \in R$). □

In the next lemma we shall suppress the $\mathcal{L}$ subscript in the notation of $D_{\mathcal{L}}$.

A.13 Lemma. *For $\alpha = 0$, $B_\alpha = D_{\mathcal{L}}$; for $\alpha = 1$ we have*

$$B_1^+ = \begin{pmatrix} D|_{I_+^\perp} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition

$$C^\infty(M, (E \otimes \mathcal{L})^+) = \mathcal{I}_+^\perp \oplus \mathcal{I}_+, \quad C^\infty(M, (E \otimes \mathcal{L})^-) = \operatorname{im}(1 - P_-) \oplus \operatorname{im}(P_-^*).$$

Observe that this operator decomposes as an invertible operator plus (direct sum) the zero operator between two finitely generated projective modules.

Since the decompositions of domain and range are both orthogonal, the adjoint B_1^* decomposes (with respect to the same decomposition) as

$$B_1^- = \begin{pmatrix} D^*|_{I_+^\perp} & 0 \\ 0 & 0 \end{pmatrix},$$

and again the left upper corner is an isomorphism. From this

$$B_1^2 = \begin{pmatrix} D_{I_+^\perp}^* D_{I_+^\perp} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_{I_+^\perp} D_{I_+^\perp}^* & \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where we use the same decomposition as before.

It follows that

$$e^{-tB_1^2} \xrightarrow{t\to\infty} P_+ \oplus \mathrm{pr},$$

where P_+ *is the even part of the operator and* pr *the odd part. For the supertraces this implies*

$$\mathrm{STR}(e^{-tB_1^2}) \xrightarrow{t\to\infty} \mathrm{TR}(P_+) - \mathrm{TR}(\mathrm{pr}) \in A_{\mathrm{ab}}.$$

Here the trace is always taken in the sense of integration over the diagonal. Note that pr *is a smoothing operator by Lemma* A.11.

A.14 Lemma.

$$\int_M \mathrm{tr}_x^{\mathrm{alg}}\, \mathrm{pr}(x,x) = \int_M \mathrm{tr}_x^{\mathrm{alg}}\, P_-(x,x) \in A_{\mathrm{ab}}.$$

Proof. By the proof of Lemma A.11 (using Lemma A.12),

$$\mathrm{pr} = (1-P)^{-1} P_-(1-P) = (1-Q)P_-(1-P),$$

where P and Q are smoothing operators. The assertion follows from the trace property for the integral trace for smoothing operators. □

A.15 Lemma. $\mathrm{STR}(e^{-tB_\alpha^2})$ *is independent of* α.

Proof. The independence follows, as before, from Duhamel's formula. □

We are now in the position to prove Proposition 2.14, which is stated once again here for the convenience of the reader.

A.16 Corollary. *The algebraic trace* $[I_+]_{[0]} - [I_-]_{[0]}$ *of* $[I_+] - [I_-] \equiv \mathrm{Ind}(D_{\mathcal{L}})$*, i.e. the image under the induced map* $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$ *of the index class* $\mathrm{Ind}(D_{\mathcal{L}})$*, can be calculated as*

$$[I_+]_{[0]} - [I_-]_{[0]} = \int_M \mathrm{tr}^{\mathrm{alg}}\, P_+(x,x) - \int_M \mathrm{tr}\, P_-(x,x) \equiv \mathrm{TR}\, P_+ - \mathrm{TR}\, P_- \in A_{\mathrm{ab}},$$

where P_+ *and* P_- *are the projections onto* I_+ *and* I_- *as given by the Mishchenko–Fomenko decomposition.*

Proof. Both expressions are limits for $t \to \infty$ of $\mathrm{STR}(e^{-tD_{\mathcal{L}}^2})$ by Lemma A.15 and Lemma A.7. □

A.17 Remark. Note that the proof shows that $[I_+]_{[0]} - [I_-]_{[0]}$ can also be expressed by means of many of the other integral operators we have used throughout the proof. In fact it might be useful to observe that the following general proposition holds.

A.18 Proposition. *Let $D_{\mathcal{L}}$ as above and let $\mathcal{Q} \in \Psi_A^{-1}(M, E^- \otimes \mathcal{L}, E^+ \otimes \mathcal{L})$ be a parametrix for $D_{\mathcal{L}}^+$ with remainders $S_\pm \in \Psi_A^{-\infty}$. Then*

$$\operatorname{ind}_{[0]}(D_{\mathcal{L}}) = \operatorname{TR}(S_+) - \operatorname{TR}(S_-).$$

The proof of the proposition is standard.

Having analyzed the large-time behavior of the A_{ab}-valued supertrace of the heat kernel, we now turn our attention to the short-time behavior.

A.19 Lemma. *The local supertrace*

$$\operatorname{str}_x^{\mathrm{alg}}(e^{-tD_{\mathcal{L}}^2}(x,x)) \operatorname{vol}(x)$$

has a limit for $t \to 0$ which is precisely the differential form

$$(x \mapsto \operatorname{AS}(D)(x) \wedge \operatorname{ch}(E)(x) \wedge \operatorname{ch} \mathcal{L}(x)_{[\dim M]}) \in \Omega^{\dim M}(M, A_{\mathrm{ab}}),$$

where the Chern forms are defined as usual using Chern–Weyl theory and the curvature of the connections.

Proof. We use Getzler's proof, as in the book by Berline, Getzler and Vergne [5]. □

A.20 Remark. In fact, the same statement holds for the smoothing perturbation $D_{\mathcal{L},\mathcal{C}}$ since, as already remarked, for the heat kernel

$$e^{-t(D_{\mathcal{L}}+\mathcal{C})^2} = e^{-tD_{M,\mathcal{L}}^2} + tC^\infty([0,\infty), \Psi_A^{-\infty}(M, E \otimes \mathcal{L}, E \otimes \mathcal{L}))$$

(a consequence of Duhamel's formula together with the fact that $\mathcal{C} \in \Psi_A^{-\infty}(M, E \otimes \mathcal{L}, E \otimes \mathcal{L})$).

A.21 Lemma.

$$\frac{d}{dt} \operatorname{STR}(e^{-tD_{\mathcal{L}}^2}) = 0 \quad \text{in } A_{\mathrm{ab}}.$$

Proof. This is, once again, a consequence of Duhamel's formula. □

We can finally give a complete proof of the A_{ab}-valued Atiyah–Singer index Theorem 2.15:

Proof. Only the index formula itself remains to be established. We integrate from $0 < \varepsilon < 1$ to $1/\varepsilon$ the derivative $\frac{d}{dt} \operatorname{STR}(e^{-tD_{\mathcal{L}}^2})$; we apply the fundamental theorem of calculus, we let $\varepsilon \downarrow 0$ and use the large and short time behavior of $\operatorname{STR}(e^{-tD_{\mathcal{L}}^2})$ as in A.19 and in A.8. □

A.2. Proof of the $A/\overline{[A,A]}$-valued APS index formula. In this subsection we shall recall and complement (a special case of) the Atiyah–Patodi–Singer index theory developed by Leichtnam and the first author in [34], [38]. In order to simplify the exposition we shall only consider even dimensional manifolds with boundary. Fundamental in our treatment is the extension of Melrose's b-calculus to the C*-algebraic setting. This is developed in [33], [34] when the C*-algebra A is equal to the reduced C*-algebra of a discrete group; exactly the same arguments work when A is an arbitrary (unital) C*-algebra. We shall not enter into the precise definition of the Mishchenko–Fomenko b-calculus with bounds $\Psi^{*,\varepsilon}_{b,A}$, $\varepsilon > 0$; we only recall that operators in $\Psi^{*,\varepsilon}_{b,A}$ are characterized by their behavior, as distributions, on $W \times W$ or, more precisely, by the behavior of their lifts on the so-called b-stretched product $W \times_b W$ (the manifold with corners obtained by blowing up $\partial W \times \partial W$ in $W \times W$). The b-calculus with bounds is the sum of three spaces of operators

$$\Psi^{*,\varepsilon}_{b,A} = \Psi^{*}_{b,A} + \Psi^{-\infty,\varepsilon}_{b,A} + \Psi^{-\infty,\varepsilon}_{A}.$$

The first space on the right-hand side is the *small* b-calculus; it is an algebra and contains as a subalgebra the space of b-differential operators. The elements in the second space are called (smoothing) boundary terms whereas the operators in the third space are called *residual* and are directly characterized on $W \times W$. The residual elements play the role of the smoothing operators in the closed case. There are natural composition rules for elements in the b-calculus with bounds. Finally, elements in the b-calculus are bounded on suitable b-Sobolev Hilbert modules:

$$P \in \Psi^{m,\varepsilon}_{b,A}(W, E\otimes\mathcal{L}, F\otimes\mathcal{L}) \Rightarrow P\colon H^{\ell}_b(W, E\otimes\mathcal{L}) \to H^{\ell-m}_b(W, F\otimes\mathcal{L}) \text{ is bounded,}$$

where the subscript b in the Sobolev modules indicates that these modules are defined using a b-metric and b-differential operators. We set $H^{\infty}_b := \bigcap_{k\in\mathbb{N}} H^k_b$. It is also important to consider weighted b-Sobolev modules $x^{\delta} H^m_b$, with x a boundary defining function; in fact the inclusion $x^{\delta} H^{\ell+\varepsilon}_b \hookrightarrow H^{\ell}_b$ is an A-compact operator for all $\varepsilon > 0$ and all $\delta > 0$.

A short guide to the literature. The basic reference for the b-calculus and its applications to index theory on manifolds with boundary is the book by Melrose [49]. Short but rather complete introductions to the theory are given in the surveys [48] and [16] and also in the appendix of [50]. The existence of spectral sections, and thus of trivializing perturbations, for an A-linear Dirac-type operator $D_{M,\mathcal{L}}$ on a closed manifold M with vanishing index class in $K_{\dim M}(A)$ (see 2.17) is established in [68], [38], building on work of Melrose–Piazza [50]. The b-calculus in the C*-algebraic context (including b-Sobolev modules) is studied in [33]. If $M = \partial W$ and $\mathcal{C}$ is a trivializing perturbation for $D_{\partial W,\mathcal{L}}$, then the lifted perturbation $\mathcal{C}_W \in \Psi^{-\infty}_{b,A}$ is defined in [50] for families and in [34] in the C*-context. Using the b-calculus and the invertibility of

$D_{\partial W,\mathcal{L}} + \mathcal{C}$ one proves that the operator $D_{W,\mathcal{L}} + \mathcal{C}_W$ is invertible modulo residual operators; since a residual operator extends to an A-compact operator on b-Sobolev modules we see that $D_{W,\mathcal{L}} + \mathcal{C}_W$ has a well-defined index class in $K_0(A)$ (see below for more on this point).

Our main interest is thus in the perturbed b-operator $D_{\mathcal{L}} + \mathcal{C}_W \in \Psi^1_{b,A}(M, E \otimes \mathcal{L}, E \otimes \mathcal{L})$. We begin with the existence of the heat kernel.

A.22 Lemma. *The operator $H(t) := e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}$ is a smoothing operator in the (small) Mishchenko–Fomenko b-calculus on W; $e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2} \in \Psi^{-\infty}_{b,A}$. The heat operator $H(\cdot)$ satisfies the heat equation with initial condition* $\lim_{t\downarrow 0} H(t) = \mathrm{id}$.

Proof. The result for the (unperturbed) b-differential operator $D^2_{\mathcal{L}}$ is obtained by employing the b-heat space as in [49] and applying the same reasoning as in the proof of Lemma A.1. For the perturbed operator $(D_{\mathcal{L}}+\mathcal{C}_W)^2$ we apply the same arguments used in the proof of Proposition 8 in [50]. □

The following result is fundamental in developing a higher APS-index theory.

A.23 Lemma. *There is a Mishchenko–Fomenko decomposition of the space of sections of $E \otimes \mathcal{L}$ with respect to $D^+_{\mathcal{L}} + \mathcal{C}^+_W$, i.e.,*

$$H^\infty_b(W, (E \otimes \mathcal{L})^+) = \mathcal{I}_+ \oplus \mathcal{I}^\perp_+, \quad H^\infty_b(W, (E \otimes \mathcal{L})^-) = \mathcal{I}_- \oplus (D^+_{\mathcal{L}} + \mathcal{C}^+_W)(\mathcal{I}^\perp_+).$$

In this decomposition, $\mathcal{I}_\pm \subset x^\varepsilon H^\infty_b$, with $\varepsilon > 0$. By completion, this decomposition gives a decomposition of the Hilbert A-modules $H^m_b(W, (E \otimes \mathcal{L})^\pm)$, $m \in \mathbb{N}$.

The second decomposition is not a priori orthogonal. However, $D^+_{\mathcal{L}}+\mathcal{C}^+_W$ induces an isomorphism (in the Fréchet topology) between $\mathcal{I}^\perp_+$ and $(D^+_{\mathcal{L}} + \mathcal{C}^+_W)(\mathcal{I}^\perp_+)$. Moreover, $\mathcal{I}_+$ and $\mathcal{I}_-$ are finitely generated projective Hilbert A-modules. The projections $\Pi_{\mathcal{I}_+}$ onto $\mathcal{I}_+$ (orthogonal) and $\Pi_{\mathcal{I}_-}$ onto $\mathcal{I}_-$ (along $(D^+_{\mathcal{L}} + \mathcal{C}^+_W)(\mathcal{I}^\perp_+)$) are residual, i.e. belong to $\Psi^{-\infty,\varepsilon}_A(W; E \otimes \mathcal{L}, E \otimes \mathcal{L})$. The $\mathcal{I}_\pm$ are already complete finitely generated projective Hilbert A-modules, i.e. unchanged when passing to any completion $H^m_b(W, E \otimes \mathcal{L})$. The index class associated to $D^+_{\mathcal{L}} + \mathcal{C}^+_W$, denoted $\mathrm{Ind}_b(D_{\mathcal{L}}, \mathcal{C})$*, is by definition*

$$\mathrm{Ind}_b(D_{\mathcal{L}}, \mathcal{C}) := [\mathcal{I}_+] - [\mathcal{I}_-] \in K_0(A). \tag{A.24}$$

Proof. The lemma is proved in [33, Appendix B] for an operator $D_{\mathcal{L}}$ with invertible boundary operator. As explained in [34], Theorems 6.2 and 6.5, the same proof applies to perturbed operators, such as $D_{\mathcal{L}} + \mathcal{C}_W$, with invertible indicial family. □

A.25 Remark. Suppose now that $\mathcal{C}$ is defined from a spectral section $\mathcal{P}$ associated to $D_{\partial W,\mathcal{L}}$: thus $\mathcal{C} \equiv \mathcal{C}_{\mathcal{P}}$ for some $\mathcal{P}$. Then

$$\mathrm{Ind}_b(D_{\mathcal{L}}, \mathcal{C}_{\mathcal{P}}) = \mathrm{Ind}_{\mathrm{APS}}(D_{\mathcal{L}}, \mathcal{P}) \quad \text{in } K_0(A)$$

where on the right-hand side a suitable generalization of the Atiyah–Patodi–Singer boundary condition appears. See [68], [38].

Recall that our goal is to prove an index formula for

$$\mathrm{tr}^{\mathrm{alg}}(\mathrm{Ind}_b(D_{\mathcal{L}},\mathcal{C})) := \mathrm{Ind}_{b,[0]}(D_L,\mathcal{C}) \equiv [\mathcal{I}_+]_{[0]} - [\mathcal{I}_-]_{[0]} \in A_{\mathrm{ab}}.$$

A.2.1. The algebraic index perturbation

A.26 Definition. Instead of W consider W_+, the disjoint union of the manifold with an additional point. The Dirac type operator on the additional point is by definition the 0 operator on $\mathbb{C}$. Note that the boundary of W_+ is still M. We define L_+, the twisting bundle on W_+ to be $L \amalg (\mathcal{I}_+ \oplus \mathcal{I}_-)$, where we view $\mathcal{I}_\pm$ as an abstract finitely generated projective Hilbert A-module (therefore a bundle over the point).

Now we define for $\alpha \in \mathbb{R}$ the perturbed operators (of $D^+_{\mathcal{L}} + \mathcal{C}^+_W$, but we suppress $\mathcal{L}$ and $\mathcal{C}_W$ in the notation of the perturbation)

$$D^+_\alpha := \begin{pmatrix} (D^+_{\mathcal{L}} + \mathcal{C}^+_W)|_{\mathcal{I}^\perp_+} & 0 & 0 \\ 0 & (D^+_{\mathcal{L}} + \mathcal{C}^+_W)|_{\mathcal{I}_+} & \alpha \\ 0 & \alpha & 0 \end{pmatrix},$$

and D^-_α as the (formal) adjoint of D^+_α. The description with respect to the b-Mishchenko–Fomenko decomposition of Lemma A.23 is given by

$$\begin{aligned} H^\infty_b(W_+,(E\otimes L)^+_+) &= \mathcal{I}^\perp_+ \oplus \mathcal{I}_+ \oplus \mathcal{I}_-, \\ H^\infty_b(W_+,(E\otimes L)^-_+) &= (D^+_{\mathcal{L}} + \mathcal{C}^+_W)(\mathcal{I}^\perp_+) \oplus \mathcal{I}_- \oplus \mathcal{I}_+. \end{aligned}$$

Note that $\mathcal{I}_\pm$ has two roles here: first as subset of the section of $E\otimes L$ on W, secondly as possible values of the sections at the additional point $*$, where the fiber is $\mathcal{I}_- \oplus \mathcal{I}_+$.

A.27 Lemma. *For each $\alpha \in \mathbb{R}$, D_α is a bounded perturbation of D_0. More precisely, $D_\alpha - D_0$ belongs to the residual space $\Psi^{-\infty,\varepsilon}_A(M_+,(E\otimes L)_+,(E\otimes L)_+)$ for some $\varepsilon > 0$.*

It follows by Duhamel expansion that $e^{-tD^2_\alpha}$ is defined for each $t > 0$ and each $\alpha \in \mathbb{R}$, and is an element in $\Psi^{-\infty,\varepsilon}_{b,A}(M_+,(E\otimes L)_+,(E\otimes L)_+)$. In particular the b-supertrace $\mathrm{bSTR}(e^{-tD^2_\alpha}) \in A_{\mathrm{ab}}$ is defined. Let us recall the definition of the b-supertrace from [49]. First we recall the definition of the regularized integral on the manifold W endowed with a product b-metric. Let $\mathrm{vol}_{b,W}$ be the associated volume form. Let us fix once and for all a trivialization $\nu \in C^\infty(\partial W, N_+\partial W)$ of the inward pointing normal bundle to the boundary and let $x \in C^\infty(W)$ be a boundary defining function for ∂W such that $dx\cdot\nu = 1$ on ∂W. For any function $f \in C^\infty(W)$ we set

$$ {}^{\nu}\!\!\int_W f := \lim_{\varepsilon\to 0^+}\Big[\int_{x>\varepsilon} f\,\mathrm{vol}_{b,W} + \log\varepsilon \int_{\partial W} f|_{\partial W}\,\mathrm{vol}_{\partial W}\Big].$$

Consider now an element $K \in \Psi_{b,A}^{-\infty,\varepsilon}$ and its restriction to the lifted diagonal $\Delta_b \subset M \times_b M$. Using the identification $\Delta_b \equiv W$ and taking the A_{ab}-valued supertrace of the endomorphism of $(E \otimes L)_x$, denoted str_x, we define

$$\mathrm{bSTR}_\nu(K) := {}^{\nu}\!\!\int_W \mathrm{str}_x^{\mathrm{alg}} K|_{\Delta_b}(x).$$

One can easily compute the effect of changing the trivialization ν; from now on we shall suppress the subscript ν in the notation of the b-supertrace. Needless to say, in the ungraded case we can similarly define the b-trace. The b-trace does not vanish on commutators; however, there is an explicit formula due to Melrose, see [49], for computing the defect. The formula involves the indicial families of the two operators:

A.28 Proposition. *For any K, K' belonging to $\Psi_{b,A}^{-\infty,\varepsilon}$ one has*

$$\mathrm{bSTR}[K, K'] = \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{R}} \mathrm{STR}\left(\frac{\partial|}{\partial\lambda} I(K,\lambda) \circ I(K',\lambda)\right) d\lambda. \tag{A.29}$$

If we replace K by a differential operator in $\mathrm{Diff}_{b,A}^*$ and K' by the composition of K' with an element of the calculus with bounds $\Psi_{b,A}^{*,\varepsilon}$ then the same commutator formula is valid.

A.30 Lemma. *The following formula is clear from the definition*:

$$\mathrm{bSTR}(e^{-tD_0^2}) = \mathrm{bSTR}_W(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}) + [I_-]_{[0]} - [I_+]_{[0]} \in A_{\mathrm{ab}}.$$

A.31 Lemma. *For α sufficiently large, D_α is invertible in the MF b-calculus with bounds, and therefore*

$$\mathrm{bSTR}(e^{-tD_\alpha^2}) \xrightarrow{t\to\infty} 0.$$

Proof. Directly from the b-Mishchenko–Fomenko decomposition theorem we prove the invertibility of D_α, α large, exactly as in the closed case, see [40], Section VI. □

A.32 Lemma.

$$\mathrm{bSTR}(e^{-tD_\alpha^2}) - \mathrm{bSTR}(e^{-tD_0^2}) = F(\alpha, t) \quad (= 0),$$

where for each fixed α and t, $F(\alpha,t) = 0$ in A_{ab}.

Proof. Use Duhamel's formula to compute the derivative of $\mathrm{bSTR}(e^{-tD_\alpha^2})$ with respect to α. The usual calculations show that this is the b-supertrace of a supercommutator. Using A.29 we see that this is an explicit term $F(\alpha,t)$, localized on the boundary. As explained in [33], formula (14.16), one can show that $F(\alpha,t) = 0$ in A_{ab}; this is a consequence of the particular structure of the two projections onto $\mathcal{I}_\pm$, namely, that they are residual. □

A.33 Corollary.

$$\lim_{t\to\infty} \mathrm{bSTR}_W(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}) = [I_+]_{[0]} - [I_-]_{[0]} \equiv \mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C}),$$

where part of the assertion is that the limit exists.

A.2.2. The integral operator *b*-index. As in the closed case, we wish to connect the index $\mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C}) \in A_{\mathrm{ab}}$ defined using the index class and the algebraic trace $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$ to the integral-kernel-trace, TR, of the projection operators given by the b-Mishchenko–Fomenko decomposition.

A.34 Definition. We set $P_+ := \Pi_{\mathcal{I}_+}$, $P_- := \Pi_{\mathcal{I}_-}$. We define a second smoothing perturbation of $(D_{\mathcal{L}} + \mathcal{C}_W)^+$ by

$$B_\alpha^+ := (D_{\mathcal{L}}^+ + \mathcal{C}_W^+) - \alpha P_-(D_{\mathcal{L}}^+ + \mathcal{C}^+)P_+,$$

for $\alpha \in \mathbb{R}$; define B_α^- as the (formal) adjoint of B_α^+.

A.35 Remark. Observe that

$$B_\alpha = \begin{pmatrix} 0 & B_\alpha^- \\ B_\alpha^+ & 0 \end{pmatrix}$$

is a residual perturbation of $D_{\mathcal{L}} + \mathcal{C}_W$ in the b-calculus. We shall also use pr := orthogonal projection onto im P_-^*.

A.36 Lemma. $\mathrm{STR}(e^{-tB_\alpha^2})$ *is independent of* α.

Proof. The result follows from Duhamel's formula, the formula for the b-supertrace of a commutator and the fact that the perturbation involved in the definition of B_α is residual. □

Proceeding exactly as in the closed case one proves the analog of lemma A.11 and, in particular, that pr is a residual operator; the analog of lemma A.12 is established using the semi-ideal property of the residual operators (see [49], formula (5.23) and Proposition 5.38). Then, as in the closed case,

$$\mathrm{bSTR}(e^{-tB_1^2}) \xrightarrow{t\to\infty} \mathrm{bTR}(P_+) - \mathrm{bTR}(\mathrm{pr}) = \mathrm{TR}(P_+) - \mathrm{TR}(\mathrm{pr}) \in A_{\mathrm{ab}},$$

with the last equality following once again from the fact that P_+ and pr are residual. Finally, from the trace property we get $\mathrm{TR}(\mathrm{pr}) = \mathrm{TR}(P_-)$, exactly as in the closed case. Using Lemma A.36 we finally get

A.37 Corollary. *The A_{ab}-valued b-index $\mathrm{Ind}_{b,[0]}(D_{\mathcal{L}}, \mathcal{C})$, i.e., the image under the induced map $\mathrm{tr}^{\mathrm{alg}}\colon K_0(A) \to A_{\mathrm{ab}}$ of the index class $\mathrm{Ind}_b(D_{\mathcal{L}}, \mathcal{C})$ can be calculated as*

$$\int_M \mathrm{tr}^{\mathrm{alg}}\, P_+(x,x) - \int_M \mathrm{tr}^{\mathrm{alg}}\, P_-(x,x) \in A_{\mathrm{ab}},$$

where P_+ and P_- are the projections onto $\mathcal{I}_+$ and $\mathcal{I}_-$ as given by the b-Mishchenko–Fomenko decomposition.

Proof. Both expressions are limits for $t \to \infty$ of $\mathrm{bSTR}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2})$ by Lemma A.36 and Corollary A.33. □

A.2.3. Local expansion

A.38 Lemma. *The local supertrace $\mathrm{str}^{\mathrm{alg}}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}(x,x))\,\mathrm{vol}_b(x)$ has a limit for $t \to 0$ which is exactly the differential form*

$$(x \mapsto \mathrm{AS}(D)(x) \wedge \mathrm{ch}(E)(x) \wedge \mathrm{ch}\,\mathcal{L}(x)_{[\dim w]}) \in \Omega^{\dim W}(W, A_{\mathrm{ab}}),$$

where the Chern forms are defined as usual using Chern–Weyl theory and the curvature of the connections.

Proof. The formula holds for $\mathrm{str}(e^{-tD^2_{\mathcal{L}}}(x,x))\,\mathrm{vol}_b(x)$, with proof employing the rescaled b-heat calculus. See [49], Chapter VIII. For the perturbed operator we simply observe that

$$e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2} = e^{-tD^2_{\mathcal{L}}} + tC^{\infty}([0,\infty), \Psi^{-\infty}_{b,A}),$$

see [50], Proposition 8. □

A.2.4. The index formula. The index formula, as stated in Subsection 2.2, follows from the large and short time limits for $\mathrm{bSTR}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2})$ together with the commutator formula A.28, with details as in [50], [34]:

A.39 Proposition.

$$\begin{aligned}\lim_{t\to\infty} \mathrm{bSTR}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}) &- \lim_{t\to 0} \mathrm{bSTR}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}) \\ &= \int_0^{\infty} \frac{d}{\mathrm{bSTR}}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2}) = -\frac{1}{2}\eta_{[0]}(D_{M,\mathcal{L}} + \mathcal{C}) \in A_{\mathrm{ab}}.\end{aligned}$$

Notice that the derivative with respect to t of $\mathrm{bSTR}(e^{-t(D_{\mathcal{L}}+\mathcal{C}_W)^2})$, as given by the commutator formula, is not precisely the eta integrand

$$-\frac{1}{2}\frac{1}{\sqrt{\pi}} t^{-1/2}\, \mathrm{TR}((D_{M,\mathcal{L}} + \mathcal{C})e^{-t(D_{M,\mathcal{L}}+\mathcal{C})^2}).$$

However, as explained in [50], its integral from 0 to $+\infty$ is indeed equal to

$$-\int_0^\infty \frac{1}{2}\frac{1}{\sqrt{\pi}} t^{-1/2}\,\mathrm{TR}((D_{M,\mathcal{L}}+\mathcal{C})e^{-t(D_{M,\mathcal{L}}+\mathcal{C})^2})\,dt \equiv -\frac{1}{2}\eta_{[0]}(D_{M,\mathcal{L}}+\mathcal{C}) \in A_{\mathrm{ab}}.$$

The A_{ab}-valued APS index theorem, as stated in 2.24, now follows immediately from the above results.

B. Graded Hermitian complexes and the signature operator

In this appendix we shall make precise our conventions for the signature operator. We follow [22, Section 3.1] and also [32], giving the general definition of graded regular n-dimensional Hermitian complex and its associated signature operator. The only difference between our conventions and those of [22] is that we deal with left modules, whereas [22] deals with right modules. The following material is taken directly from [32].

Let A be a C*-algebra with unit.

B.1 Definition. A graded regular n-dimensional Hermitian complex consists of the following data:

- a $\mathbb{Z}$-graded cochain complex $(\mathcal{E}^*, d)$ of finitely-generated projective left A-modules,
- a nondegenerate quadratic form $Q\colon \mathcal{E}^* \times \mathcal{E}^{n-*} \to A$, and
- an operator $\tau \in \mathrm{Hom}_A(\mathcal{E}^*, \mathcal{E}^{n-*})$ such that $Q(bx, y) = bQ(x, y)$, $Q(x, y)^* = Q(y, x)$, $Q(dx, y) + Q(x, dy) = 0$, $\tau^2 = I$, and $\langle x, y\rangle := Q(x, \tau y)$ defines a Hermitian metric on $\mathcal{E}$.

Let M be a closed oriented n-dimensional Riemannian manifold. Let $\mathcal{V}$ be a flat A-vector bundle on M. We assume that the fibers of $\mathcal{V}$ have A-valued Hermitian inner products which are compatible with the flat structure.

Let $\Omega^*(M; \mathcal{V})$ denote the vector space of smooth differential forms with coefficients in $\mathcal{V}$. If $n = \dim(M) > 0$ then $\Omega^*(M; \mathcal{V})$ is not finitely-generated over A, but we wish to show that it still has all of the formal properties of a graded regular n-dimensional Hermitian complex. If $\alpha \in \Omega^*(M; \mathcal{V})$ is homogeneous, denote its degree by $|\alpha|$. In what follows, α and β will sometimes implicitly denote homogeneous elements of $\Omega^*(M; \mathcal{V})$. Given $m \in M$ and $(\lambda_1 \otimes e_1), (\lambda_2 \otimes e_2) \in \Lambda^*(T_m^*M) \otimes \mathcal{V}_m$, we define $(\lambda_1 \otimes e_1) \wedge (\lambda_2 \otimes e_2)^* \in \Lambda^*(T_m^*M) \otimes A$ by

$$(\lambda_1 \otimes e_1) \wedge (\lambda_2 \otimes e_2)^* = (\lambda_1 \wedge \overline{\lambda_2}) \otimes \langle e_1, e_2\rangle.$$

Extending by linearity (and antilinearity), given $\omega_1, \omega_2 \in \Lambda^*(T_m^*M) \otimes \mathcal{V}_m$, we can define $\omega_1 \wedge \omega_2^* \in \Lambda^*(T_m^*M) \otimes A$.

Define an A-valued quadratic form Q on $\Omega^*(M;\mathcal{V})$ by

$$Q(\alpha,\beta) = i^{-|\alpha|(n-|\alpha|)} \int_M \alpha(m) \wedge \beta(m)^*.$$

It satisfies $Q(\beta,\alpha) = Q(\alpha,\beta)^*$. Using the Hodge duality operator $*$, define $\tau\colon \Omega^p(M;\mathcal{V}) \to \Omega^{n-p}(M;\mathcal{V})$ by

$$\tau(\alpha) = i^{-|\alpha|(n-|\alpha|)} * \alpha.$$

Then $\tau^2 = 1$, and the inner product $\langle\cdot,\cdot\rangle$ on $\Omega^*(M;\mathcal{V})$ is given by $\langle\alpha,\beta\rangle = Q(\alpha,\tau\beta)$. Let $d_{\mathcal{V}}$ be the de Rham differential with values in the flat bundle $\mathcal{V}$; define $d\colon \Omega^*(M;\mathcal{V}) \to \Omega^{*+1}(M;\mathcal{V})$ by

$$d\alpha = i^{|\alpha|} d_{\mathcal{V}}\alpha. \tag{B.2}$$

It satisfies $d^2 = 0$. Its dual $d^\dagger$ with respect to Q, i.e., the operator $d^\dagger$ such that $Q(\alpha, d\beta) = Q(d^\dagger\alpha,\beta)$, is given by $d^\dagger = -d$. The formal adjoint of d with respect to $\langle\cdot,\cdot\rangle$ is $d^* = \tau d^\dagger \tau = -\tau d \tau$.

B.3 Definition. If n is even, the signature operator is

$$\mathcal{D}^{\mathrm{sign}} := d + d^* = d - \tau d \tau. \tag{B.4}$$

It is formally self-adjoint and anticommutes with the $\mathbb{Z}_2$-grading operator τ. If we denote by $\Omega^{\pm}(M,\mathcal{V})$ the ±1-eigenspaces of τ then

$$\mathcal{D}^{\mathrm{sign}} = \begin{pmatrix} 0 & \mathcal{D}^{\mathrm{sign}}_- \\ \mathcal{D}^{\mathrm{sign}}_+ & 0 \end{pmatrix}.$$

The index class of the signature operator in $K_0(A)$ is, by definition, the index class of the elliptic operator $\mathcal{D}^{\mathrm{sign}}_+$. If n is odd, the signature operator is

$$\mathcal{D}^{\mathrm{sign}} = -i(d\tau + \tau d). \tag{B.5}$$

It is formally self-adjoint and defines an index class in $K_1(A)$.

Now suppose that M is a compact oriented manifold of dimension $n = 2m$ with boundary ∂M. We fix a non-negative boundary defining function $x \in C^\infty(M)$ for ∂M and a Riemannian metric on M which is isometrically a product in an (open) collar neighborhood $\mathcal{U} \equiv (0,2)_x \times \partial M$ of ∂M. The signature operator $\mathcal{D}^{\mathrm{sign}}$ is a well-defined differential operator; it is associated to the graded regular n-dimensional Hermitian complex defined by $\Omega^*_c(\mathrm{int}(M);\mathcal{V})$ and the Riemannian metric on M. Let $\mathcal{V}_0$ denote the pullback of $\mathcal{V}$ from M to ∂M; there is a natural isomorphism

$$\mathcal{V}|_{\mathcal{U}} \cong (0,2) \times \mathcal{V}_0.$$

Let $Q_{\partial M}$, $\tau_{\partial M}$, $d_{\partial M}$ and $\mathcal{D}^{\text{sign}}(\partial M)$ denote the expressions defined above on $\Omega^*(\partial M; \mathcal{V}_0^\infty)$. One can decompose Q, τ, d and $\mathcal{D}^{\text{sign}}$, when restricted to compactly-supported forms on $(0,2) \times \partial M$, in terms of $Q_{\partial M}$, $\tau_{\partial M}$, $D_{\partial M}$ and $\mathcal{D}^{\text{sign}}(\partial M)$. This computation is given in great detail in [32]; one proves in particular that with the definitions given above *the operator $\mathcal{D}^{\text{sign}}(\partial M)$ is the boundary operator of $\mathcal{D}_+^{\text{sign}}$ in the sense of Atiyah–Patodi–Singer* [APS, (3.1)].

B.6 Remark. Let (X, g) be closed oriented and of dimension $2m-1$. Assume that $\mathcal{V} = X \times \mathbb{C}$ is the trivial complex line bundle. We wish to compare the eta-invariant of the signature operator defined with our sign-conventions, denoted D^{sign}, and the eta-invariant of the operator B appearing in the work of Atiyah–Patodi–Singer, see [3]. In order to simplify the notation we set $D := D^{\text{sign}}$.

Recall that $B\phi = (\sqrt{-1})^m(-1)^{p+1}(\varepsilon * d - d *)\phi$ with $\varepsilon = 1$ if $\phi \in \Omega^{2p}(X)$ and $\varepsilon = -1$ if $\phi \in \Omega^{2p-1}(X)$. The operator D is given instead by $D\phi = (-1)^m((\sqrt{-1} * d - d *))\phi$ if $\phi \in \Omega^{2p}(X)$ and $D\phi = - * d\phi - \sqrt{-1} d * \phi$ if $\phi \in \Omega^{2p-1}(X)$. Each operator preserves the parity of the form-degree. Moreover, both commute with the following self-adjoint odd involution: $\Xi := \eta\Theta$ with $\Theta = (-1)^p *$ on both Ω^{2p} and Ω^{2p-1} and $\eta = 1$ if $m = 2k$; $\eta = \sqrt{-1}$ if $m = 2k-1$. Thus $B = B^{\text{even}} \oplus B^{\text{odd}}$, $D = D^{\text{even}} \oplus D^{\text{odd}}$, with $B^{\text{even}} = (\Xi)^{-1} B^{\text{odd}} \Xi$ and similarly for D.[9] The Hodge theorem implies the following orthogonal decomposition of the space of differential forms on X, where H stands for the space of harmonic forms:

$$\begin{aligned}\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus d\Omega^{m-2} \\ \oplus H^{m-1} \oplus d^*\Omega^m \oplus d\Omega^{m-1} \oplus H^m \\ \oplus d^*\Omega^{m+1} \oplus \cdots \oplus \Omega^{2m-1}.\end{aligned}$$

Consider now the following two subspaces of Ω^*:

$$\begin{aligned} V &= H^{m-1} \oplus d^*\Omega^m \oplus d\Omega^{m-1} \oplus H^m, \\ W &= \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus d\Omega^{m-2} \oplus d^*\Omega^{m+1} \oplus \cdots \oplus \Omega^{2m-1}. \end{aligned}$$

Both operators B and D are block diagonal with respect to the orthogonal decomposition $\Omega^* = V \oplus W$; we denote their restrictions by D_V, B_V and D_W, B_W. Notice that both V and W are invariant under the action of Ξ so that, as above, $B_V = B_V^{\text{even}} \oplus B_V^{\text{odd}}$, $B_W = B_W^{\text{even}} \oplus B_W^{\text{odd}}$ and similarly for D_V and D_W. It is easy to check from the explicit expression given above that $B_V^{\text{even}} = -D_V^{\text{even}}$. We will show that

$$\eta(B_W) = 0 = \eta(D_W) \tag{B.7}$$

which implies immediately that

$$\eta(B) = -\eta(D). \tag{B.8}$$

[9] Notice that there is a sign mistake in [36]: the Θ given there in Section 1 must be replaced with the Ξ given here.

In order to establish (B.7) we argue as follows. Define

$$\Omega^{<} = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^{m-2} \oplus d\Omega^{m-2},$$
$$\Omega^{>} = d^*\Omega^{m+1} \oplus \Omega^{m+1} \oplus \cdots \oplus \Omega^{2m-1}$$

so that $W = \Omega^{<} \oplus \Omega^{>}$. There is a natural involution α on W defined as follows:

$$\alpha = \mathrm{id} \text{ on } \Omega^{<}, \quad \alpha = -\mathrm{id} \text{ on } \Omega^{>}.$$

It is immediate from the structure of D and B that

$$D_W \circ \alpha + \alpha \circ D_W = 0, \quad B_W \circ \alpha + \alpha \circ B_W = 0.$$

In other words α gives a grading to W, $W^+ = \Omega^{<}$, $W^- = \Omega^{>}$ and both D_W and B_W are *odd* with respect to such a grading; thus $\eta(B_W) = 0 = \eta(D_W)$ as required. *Conclusion*: the odd-signature operator considered here and the odd signature operator considered in the work of Atiyah–Patodi–Singer are different. However, their eta-invariants are equal up to a sign.

C. Pseudodifferential operators and morphisms of C*-algebras

Let A and B be unital C*-algebras. We assume that there exists a morphism of unital C*-algebras $\lambda\colon A \to B$. Let $\mathcal{E}_A \to M$ be a bundle of finitely generated projective Hilbert A-modules, with fibers isomorphic to a fixed finitely generated projective Hilbert A-module V. In particular, there exists $N \in \mathbb{N}$ and W, a Hilbert A-module, such that $V \oplus W = A^N$ as Hilbert A-modules; moreover V is the image of a self-adjoint projection $p \in M(N \times N, A)$, $p = (p(ij)), i, j = 1, \ldots, N$. The morphism λ induces in a natural way a morphism of matrix algebras $M(N \times N, A) \to M(N \times N, B)$ that will be still denoted by λ. If we define $p_\lambda := (p_\lambda(ij))$ with $p_\lambda(ij) := \lambda(p(ij))$, briefly $p_\lambda = \lambda \circ p$, then p_λ is still a self-adjoint projection in $M(N \times N, B)$ so that $\lambda(V) := \mathrm{im}\, p_\lambda$ is a finitely generated projective Hilbert B-module. Applying this reasoning to a more global situation, we see that λ induces a bundle of finitely generated projective Hilbert B-modules, $\mathcal{E}^\lambda_B$, with fibers diffeomorphic to $\lambda(V)$: in fact, if $\mathcal{E}_A$ is defined by $p \in C^\infty(M, M_{N\times N}(A))$ with $p = p^* = p^2$, then $\mathcal{E}^\lambda_B$ is simply defined by $p_\lambda := \lambda \circ p \in C^\infty(M, M_{N\times N}(B))$. Alternatively, if $\{U_\alpha\}$ is a trivializing covering for $\mathcal{E}_A \to M$, with transition function $g^A_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Iso}_A(V)$ then $\mathcal{E}^\lambda_B \to M$ is defined by $g^B_{\alpha,\beta} := \lambda \circ g^A_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{Iso}_B(\lambda(V))$.

If $\mathcal{E}_A \to M$ is endowed with a connection ∇^A, then $\mathcal{E}_B \to M$ inherits in a natural way a connection $\nabla^{B,\lambda}$ defined as follows. If ∇^A is equal to $p \circ d \circ p$ then $\nabla^{B,\lambda}$ is simply equal to $p_\lambda \circ d \circ p_\lambda$; if, on the other hand, ∇^A is arbitrary, then $\nabla^A = p \circ d \circ p + \omega$, with $\omega \in C^\infty(M, \mathrm{End}_A(\mathcal{E}_A))$ and we then define

$$\nabla^{B,\lambda} = p_\lambda \circ d \circ p_\lambda + \omega_\lambda$$

with $\omega_\lambda \in C^\infty(M, \mathrm{End}_B(\mathcal{E}_B^\lambda))$ the 1-form induced by ω and λ. Alternatively, if ∇^A is defined by a collection of local 1-forms $\{\omega_\alpha\}$ associated to the trivializing cover $\{U_\alpha\}$, with

$$\omega_\alpha \in \Omega^1(U_\alpha, \mathrm{End}_A(V)), \quad \omega_\beta = g_{\alpha,\beta}^{-1} dg_{\alpha,\beta} + g_{\alpha,\beta}^{-1}\omega_\beta g_{\alpha,\beta}$$

then we define the connection $\nabla^{B,\lambda}$ through the local 1-forms $\{\omega_\alpha^\lambda\}$ where, once again, $\omega_\alpha^\lambda \in \Omega^1(U, \mathrm{End}_B(\lambda(V)))$ is defined in a natural way by ω_α and the extension of λ to a morphism $M(N \times N, A) \to M(N \times N, B)$.

Consider now the graded vector space of pseudodifferential Hilbert A-module bundle operators $\Psi_A^*(M, \mathcal{E}_A, \mathcal{F}_A)$, with $\mathcal{F}_A$ a second bundle of finitely generated projective Hilbert A-modules. This space, defined for the first time in [53], is nothing but

$$\Psi^*(M) \otimes_{C^\infty(M\times M)} C^\infty(M \times M, \mathrm{HOM}_A(\mathcal{F}_A, \mathcal{E}_A)) \tag{C.1}$$

where we are considering the bundle $\mathrm{HOM}_A(\mathcal{F}_A, \mathcal{E}_A)) \longrightarrow M\times M$ with fiber at (x, y) equal to $\mathrm{Hom}_A((\mathcal{E}_A)_y, (\mathcal{F}_A)_x)$. Here we are using the $C^\infty(M\times M)$-module structure of both $\Psi^*(M)$ and $C^\infty(M\times M, \mathrm{HOM}_A(\mathcal{F}_A, \mathcal{E}_A))$. The morphism λ induces in a natural way a morphism of graded vector spaces $\lambda_\natural\colon \Psi_A^*(M, \mathcal{E}_A, \mathcal{F}_A) \to \Psi_B^*(M, \mathcal{E}_B^\lambda, \mathcal{F}_B^\lambda)$ which is simply induced by the natural map

$$C^\infty(M \times M, \mathrm{HOM}_A(\mathcal{F}_A, \mathcal{E}_A)) \longrightarrow C^\infty(M \times M, \mathrm{HOM}_B(\mathcal{F}_B^\lambda, \mathcal{E}_B^\lambda)).$$

If $\mathcal{F}_A = \mathcal{E}_A$ then $\lambda_\natural$ is in fact a morphism of graded algebras; we shall often denote $\Psi_A^*(M, \mathcal{E}_A, \mathcal{E}_A)$ by $\Psi_A^*(M, \mathcal{E}_A)$ or, more simply, by Ψ_A^*. Thus the following simple conclusion is true.

C.2 Corollary. *If $\mathcal{Q} \in \Psi_A^*$ is invertible as an element in Ψ_A^*, then $\lambda_\natural(\mathcal{Q})$ is invertible in Ψ_B^*.*

As an important example we shall consider twisted Dirac operators. Thus $\mathcal{E}_A = E \otimes \mathcal{V}_A$ with E a vector bundle over M and $\mathcal{V}_A$ a line bundle with typical fiber A and endowed with a flat connection ∇^A. We can then choose trivializations of $\mathcal{V}_A$ with locally constant transition functions $\{a_{\alpha,\beta} \in U_1(A) \subset A\}$. Let $D \in \mathrm{Diff}^1(M, E)$ be a Dirac-type operator acting on the sections of E and consider $D_{\mathcal{V}_A} \in \mathrm{Diff}_A^1(M, \mathcal{E}_A)$, the twisted Dirac operator defined by the connection ∇^A. Then $\lambda_\natural(D_{\mathcal{V}_A}) = D_{\mathcal{V}_B^\lambda}$ with $\mathcal{V}_B^\lambda$ the flat bundle induced by λ. Notice that the transition functions of the flat bundle $\mathcal{V}_B^\lambda$ will only involve the image $\lambda(A)$.

Let now $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$. The operator $D_{\mathcal{V}_A}$ defines a self-adjoint unbounded regular operator on the (full) Hilbert A-module $L_A^2(M, \mathcal{E}_A)$. It is well known that there is a continuous functional calculus for regular operators on (full) Hilbert A-modules; in particular $\chi(D_{\mathcal{V}_A})$ is a well-defined bounded A-linear endomorphism on $L_A^2(M, \mathcal{E}_A)$. As observed in [34], following the arguments in [63, p. 300], one can show that $\chi(D_{\mathcal{V}_A})$ is in fact a smoothing operator: $\chi(D_{\mathcal{V}_A}) \in \Psi_A^{-\infty}(M, \mathcal{E}_A)$. Since $\chi(D_{\mathcal{V}_A})$ is defined through a functional integral, we see immediately that

$$\lambda_\natural(\chi(D_{\mathcal{V}_A}) = \chi(\lambda_\natural(D_{\mathcal{V}_A})) = \chi(D_{\mathcal{V}_B^\lambda}). \tag{C.3}$$

The same is true for other functions and other operators for the same reason. E.g. if $\mathcal{C}$ is a smoothing operator, then

$$\lambda_\sharp((D_{\mathcal{V}_A} + \mathcal{C})\exp(t(D_{\mathcal{V}_A} + \mathcal{C})^2)) = (D_{\mathcal{V}_B^\lambda} + \lambda_\sharp\mathcal{C})\exp(t(D_{\mathcal{V}_B^\sharp} + \lambda_\sharp\mathcal{C})^2). \tag{C.4}$$

C.5 Remark. Let $\phi \in C_c^\infty(\mathbb{R}, [0,1])$ be any real function equal to 1 on $[-\varepsilon, \varepsilon]$ and equal to 0 on $(-\infty, -2\varepsilon] \cup [2\varepsilon, \infty)$. Let $\mathcal{C}_A \in \Psi_A^{-\infty}$ and let $\mathcal{C}_B^\lambda := \lambda_\natural(\mathcal{C}_A)$. Then the following implication holds:

$$\text{if } (\mathrm{id} - \phi(D_{\mathcal{V}_A})) \circ \mathcal{C}_A = 0 \text{ then } (\mathrm{id} - \phi(D_{\mathcal{V}_B^\lambda})) \circ \mathcal{C}_B^\lambda = 0. \tag{C.6}$$

Notice that if $(\mathrm{id} - \phi(D_{\mathcal{V}_A})) \circ \mathcal{C}_A = 0$ then, taking adjoints, $\mathcal{C}_A \circ (\mathrm{id} - \phi(D_{\mathcal{V}_A})) = 0$.

Let now $K \in \Psi_A^{-\infty}(M, \mathcal{E}_A)$ be a smoothing operator, and let us consider $\mathrm{TR}(K) \in A_{\mathrm{ab}}$. As an immediate consequence of the definitions,

$$\lambda(\mathrm{TR}(K)) = \mathrm{TR}(\lambda_\natural K) \in B_{\mathrm{ab}}. \tag{C.7}$$

If now $\mathcal{C}_B^\lambda = \lambda_\natural(\mathcal{C}_A)$, then

C.8 Lemma.

$$\lambda(\eta_{[0]}(D_{\mathcal{V}_A} + \mathcal{C}_A)) = \eta_{[0]}(D_{\mathcal{V}_B^\lambda} + \mathcal{C}_B^\lambda)) \quad \text{in } B_{\mathrm{ab}} \tag{C.9}$$

Proof. This simply follows from the definition of $\eta_{[0]}$ as a convergent integral. Then, using for the second equality (C.7),

$$\begin{aligned}
\lambda\eta_{[0]}(D_{\mathcal{V}_A} + \mathcal{C}_A) &= \frac{1}{\sqrt{\pi}}\lambda \int_0^\infty \mathrm{TR}((D_{\mathcal{V}_A} + \mathcal{C}_A)\exp(t(D_{\mathcal{V}_A} + \mathcal{C}_A)^2))\frac{dt}{\sqrt{t}} \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{TR}(\lambda_\sharp((D_{\mathcal{V}_A} + \mathcal{C}_A)\exp(t(D_{\mathcal{V}_A} + \mathcal{C}_A)^2)))\frac{dt}{\sqrt{t}} \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{TR}((D_{\mathcal{V}_B^\lambda} + \mathcal{C}_B^\lambda)\exp(t(D_{\mathcal{V}_B^\lambda} + \mathcal{C}_B^\lambda)^2))\frac{dt}{\sqrt{t}} \\
&= \eta_{[0]}(D_{\mathcal{V}_B^\lambda} + \mathcal{C}_B^\lambda).
\end{aligned}$$

□

D. Twists with finite dimensional representations

As a particular case of the principles explained in the previous subsection, we consider a discrete group Γ and a finite dimensional unitary representation $\lambda\colon \Gamma \to \mathrm{U}(d) \subset \mathrm{M}(d \times d, \mathbb{C})$. This induces a morphism of C*-algebras (we also call it λ) $\lambda\colon C^*\Gamma \to \mathrm{M}(d \times d, \mathbb{C})$. Here $C^*\Gamma$ is the maximal C*-algebra of Γ, and the extension of λ follows from the universal property of $C^*\Gamma$.

Given, as in Section 3.1, a classifying map $u\colon M \to B\Gamma$ defined on a closed manifold M, we obtain the corresponding Mishchenko–Fomenko bundle $\mathcal{L}$ and the twisted Dirac operator $D_{\mathcal{L}}$; thus in this case

$$A = C^*\Gamma, \quad \mathcal{V}_A \equiv \mathcal{L} := \widetilde{M} \times_\Gamma C^*\Gamma, \quad B = \mathrm{M}(d \times d, \mathbb{C}).$$

Let V_λ be the flat *vector bundle* $\widetilde{M} \times_\lambda \mathbb{C}^d$ and let D_λ be the Dirac operator twisted by V_λ. Then it is easy to see that the bundle $\mathcal{V}_B^\lambda$ introduced in the previous subsection is equal to the direct sum of d copies of V_λ and that $D_{\mathcal{V}_B^\lambda}$ is the diagonal operator $D_\lambda \otimes I_d$, with I_d the $d \times d$-identity matrix. If $\mathcal{C} \in \Psi^{-\infty}_{C^*\Gamma}$ is a trivializing perturbation, then $\lambda_\natural \mathcal{C} = C_\lambda \otimes I_d$, with $C_\lambda \in \Psi^{-\infty}$ and $D_\lambda + C_\lambda$ invertible. Let $\mathrm{tr}\colon \mathrm{M}(d \times d, \mathbb{C}) \to \mathbb{C}$ be the usual trace and consider $\mathrm{tr}_d := d^{-1}\,\mathrm{tr}$; let $\mathrm{tr}_\lambda := \mathrm{tr}_d \circ \lambda\colon C^*\Gamma_{\mathrm{ab}} \to \mathbb{C}$. Then, by Lemma C.8,

$$\mathrm{tr}_\lambda(\eta_{[0]}(D_{\mathcal{L}} + \mathcal{C}) = \eta(D_\lambda + C_\lambda). \tag{D.1}$$

Let us go back to closed manifolds and let $X = M \sqcup (-M')$ with M and M' homotopy equivalent. Let us fix a unitary representation $\lambda\colon \Gamma \to U(d)$ and let $D^{\mathrm{sign}}_{X,\lambda}$ be signature operator twisted by the flat vector bundle associated to λ. Let $0 < \varepsilon$ such that $\mathrm{spec}(D^{\mathrm{sign}}_{\lambda,X}) \cap (-\varepsilon, \varepsilon) \subseteq \{0\}$. Let $\phi \in C^\infty_c(\mathbb{R}, [0,1])$ be a real function equal to 1 on $[-\varepsilon, \varepsilon]$ and equal to 0 on $(-\infty, -2\varepsilon] \cup [2\varepsilon, \infty)$. Let $\mathcal{D}^{\mathrm{sign}}_X$ be the Mishchenko–Fomenko signature operator on X. We know from Section 10 that in this situation we can construct a trivializing perturbation $\mathcal{C}$ of $\mathcal{D}^{\mathrm{sign}}_X$ such that $(\mathrm{id} - \phi(\mathcal{D}^{\mathrm{sign}}_X)) \circ \mathcal{C} = 0$. Thus, using (C.6), we see that $\mathcal{C}_\lambda$ is a smoothing operator such that $(\mathrm{id} - \phi(D^{\mathrm{sign}}_{\lambda,X})) \circ C_\lambda = 0$; by Remark 10.16 we can now conclude as in Section 10.1, but without making use of the limiting procedure of Section 10.2, that $\eta(D_\lambda) = (\eta(D_\lambda + C_\lambda) + \eta(D_\lambda - C_\lambda))/2$. Using Remark 10.8 we can even conclude that the stable and unstable APS-rho invariants coincide, without making use of the limiting procedure of Section 10.2.

E. Classical L^2-invariants versus $\mathcal{N}\Gamma$-invariants

E.1. Classical L^2-invariants. In Section 1 we introduced (delocalized) L^2-invariants by working directly on a normal covering $\widetilde{M}$ of a closed manifold M. However, all the main arguments of this paper are given in terms of $A/\overline{[A, A]}$-valued invariants for Dirac operators twisted by Hilbert A-module bundles. We briefly called these invariants *degree zero invariants*. In this appendix, we will explain how the classical L^2-invariants can be derived from these degree zero invariants, where the C*-algebra A in question is $\mathcal{N}\Gamma$, and the bundle to twist with is the Mishchenko–Fomenko line bundle $\mathcal{N} = \widetilde{M} \times_\Gamma \mathcal{N}\Gamma$. This procedure is well known to the experts. In this appendix, we will give a detailed description of it, since we are not aware of a place in the literature where this would be covered, in particular when dealing with eta-invariants (the focus of interest here).

E.2. Twisting with $l^2\Gamma$. The passage from the covering situation to the twist with the $\mathcal{N}\Gamma$ bundle is achieved by using an intermediate step: instead of twisting with $\mathcal{N} := \widetilde{M} \times_\Gamma \mathcal{N}\Gamma$ we twist with $\mathcal{H} := \widetilde{M} \times_\Gamma l^2\Gamma$.

The typical fiber of this bundle is isomorphic to $l^2\Gamma$, which we consider as an $\mathcal{N}\Gamma$-Hilbert space (in the notation of [60]).

We start with giving the basic definitions, repeated from [60, Section 8].

E.1 Definition. A *finitely generated projective $\mathcal{N}\Gamma$-Hilbert space V* is a Hilbert space together with a right action of $\mathcal{N}\Gamma$ such that V embeds isometrically preserving the $\mathcal{N}\Gamma$-module structure as a direct summand into $l^2(\Gamma)^n$ for some n. A (general) *$\mathcal{N}\Gamma$-Hilbert space V* satisfies the same conditions a finitely generated projective $\mathcal{N}\Gamma$-Hilbert space does, with the exception that $l^2(\Gamma)^n$ is replaced by $H \otimes l^2(\Gamma)$ for some Hilbert space H with trivial $\mathcal{N}\Gamma$-action (the tensor product has to be completed).

E.2 Definition. An $\mathcal{N}\Gamma$-Hilbert space morphism is a bounded $\mathcal{N}\Gamma$-linear map between two $\mathcal{N}\Gamma$-Hilbert spaces. If it is an isometry for the Hilbert space structure, it is called $\mathcal{N}\Gamma$-Hilbert space isometry. An $\mathcal{N}\Gamma$-Hilbert space bundle $\mathcal{H}$ on a space X is a locally trivial bundle of $\mathcal{N}\Gamma$-Hilbert spaces, the transition functions being $\mathcal{N}\Gamma$-Hilbert space isometries. A smooth structure is given by a trivializing atlas where all the transition functions are smooth.

If the fibers are finitely generated projective $\mathcal{N}\Gamma$-Hilbert space, the bundle is called a *finitely generated projective $\mathcal{N}\Gamma$-Hilbert space bundle*.

E.3 Lemma. *The L^2-sections of an $\mathcal{N}\Gamma$-Hilbert space bundle $\mathcal{W}$ on a Riemannian manifold X form themselves an $\mathcal{N}\Gamma$-Hilbert space.*

Proof. Compare [60, Lemma 8.10] □

E.4 Example. Essentially the only example we will be dealing with is the $\mathcal{N}\Gamma$-Hilbert space $l^2(\Gamma)$. This is obviously finitely generated projective. Since the left regular representation acts by $\mathcal{N}\Gamma$-Hilbert space isometries, $\mathcal{H} := \widetilde{M} \times_\Gamma l^2(\Gamma)$ is a finitely generated projective $\mathcal{N}\Gamma$-Hilbert space bundle on M.

The trivial connection on $\widetilde{M} \times l^2(\Gamma)$ descends to a canonical flat connection on $\widetilde{M} \times_\Gamma l^2(\Gamma)$.

E.5 Remark. Note that $\mathcal{N}\Gamma$ is canonically a subset of $l^2(\Gamma)$ (the inclusion given by $b \mapsto 1 \cdot b$), and $l^2(\Gamma)$ is the completion of $\mathcal{N}\Gamma$ with respect to the inner product $\langle b_1, b_2\rangle := \mathrm{tr}_\Gamma(b_1^* b_2)$. In the same way, $\widetilde{M} \times_\Gamma l^2(\Gamma)$ is the fiberwise completion of $\widetilde{M} \times_\Gamma \mathcal{N}\Gamma$. For all of this, compare [60, Section 8.6]

Given a Dirac type operator $D\colon C^\infty(M, E) \to C^\infty(M, E)$ on M, we can now also form the twisted Dirac operator $D_{\mathcal{H}}$ as usual.

If M is closed, this is an elliptic differential operator of order 1 on $\mathcal{N}\Gamma$-Hilbert space bundles in the sense of [8]. In any case, it extends to an unbounded operator on $L^2(E \otimes \mathcal{H})$. The main point, as observed in [60, Section 8], is now the following:

E.6 Proposition. *The operator $D_{\mathcal{H}}$ maps the sections of the subbundle $E \otimes \mathcal{N}$ of $E \otimes \mathcal{H}$ to sections of $E \otimes \mathcal{N}$, and the restriction is exactly the operator $D_{\mathcal{N}}$. This holds for smooth sections and all kinds of completions. Moreover, all the functions of the operator $D_{\mathcal{H}}$ we are considering (like $D_{\mathcal{H}} \exp(-tD_{\mathcal{H}}^2)$) restrict to the corresponding functions of $D_{\mathcal{N}}$.*

Vice versa, we get $D_{\mathcal{H}}$ by applying the procedure of completion of [60, Section 8] *to $D_{\mathcal{N}}$.*

The above properties together imply that there is a canonical bijection of algebras

$$\{\psi(D_{\mathcal{N}}) \mid \psi \text{ is a Schwartz function}\} \quad \text{and} \quad \{\psi(D_{\mathcal{H}}) \mid \psi \text{ is a Schwartz function}\}.$$

From now on, making use of this identification, we will write $D_{\mathcal{H}}$ and $D_{\mathcal{N}}$ interchangeably. Put it differently, we now have defined the invariants $\eta_\tau(D_{\mathcal{H}}) := \eta_\tau(D_{\mathcal{N}})$, $\eta_{\tau_{\langle g\rangle}}(D_{\mathcal{H}}) := \eta_{\tau_{\langle g\rangle}}(D_{\mathcal{N}})$.

We are now going to explain why it is equivalent to consider $\widetilde{D}$ acting on $\widetilde{E} \to \widetilde{M}$ and $D_{\mathcal{H}}$, $\mathcal{H} = \widetilde{M} \times_\Gamma l^2(\Gamma)$, acting on $E \otimes \mathcal{H} \to M$. In fact, there is a dictionary that allows to pass back and forth between objects on the covering and twisted objects.

For the sake of completeness we indicate the constructions. Other accounts (where certain aspects are explained in more detail) can be found e.g. in [40, Section III], [58, Section 3.1] and [59, Example 3.39].

The translation is summarized in the following table.

no.	$\widetilde{M}$	$\cdot \otimes \mathcal{H}$
1	$L^2(\widetilde{M}, \widetilde{E})$	$L^2(M, E \otimes \mathcal{H})$
2	$\{s \in C^\infty(\widetilde{M}, \widetilde{E}) \mid \sum_{\gamma\in\Gamma} \lvert s(\gamma x)\rvert^2 < \infty \text{ for all } x \in \widetilde{M}\}$	$C^\infty(M, E \otimes \mathcal{H})$
3	$\widetilde{D}$	$D_{\mathcal{H}}\ (= D_{\mathcal{N}})$
4	$\widetilde{D} \exp(-t\widetilde{D}^2)$	$D_{\mathcal{H}} \exp(-tD_{\mathcal{H}}^2)$
5	$\mathrm{Tr}_{\langle g\rangle}$	$\tau_{\langle g\rangle} \circ \mathrm{TR}$
6	$\mathrm{Tr}_{(2)}$	$\tau_\Gamma \circ \mathrm{TR}$
7	$\eta_{\langle g\rangle}(\widetilde{D})$	$\eta_{\tau_{\langle g\rangle}}(D_{\mathcal{H}})$

Recall that for an operator $\widetilde{A}$ given by the smooth integral kernel $\tilde{k}$, by definition

$$\operatorname{Tr}_{\langle g\rangle}(\widetilde{A}) = \sum_{\gamma\in[g]} \int_{\widetilde{M}/\Gamma} \operatorname{tr}_x k(x,\gamma x)\, dx, \tag{E.7}$$

and in particular $\operatorname{Tr}_{(2)}(\widetilde{A}) = \int_{\widetilde{M}/\Gamma} \operatorname{tr}_x \tilde{k}(x,x)\, dx$.

We give some explanation how to translate between the two sides.

(1) A section s of $\widetilde{E}$ corresponds to the section $\hat{s}$ of $E \otimes \mathcal{H}$ with $\hat{s}(x) = \sum_{\gamma\in\Gamma} s(\gamma\tilde{x}) \otimes [\tilde{x},\gamma]$, where $\tilde{x} \in \widetilde{M}$ is an arbitrary lift of $x \in M$ along the covering projection. We identify the fibers E_x and $\tilde{E}_{\gamma\tilde{x}}$. Moreover, by definition $\mathcal{H}_x = \Gamma\tilde{x} \times_\Gamma l^2(\Gamma)$. This construction is well defined by the very definition of the twisted bundle $\mathcal{H}$, with fiber identified with $l^2(\Gamma)$ using the chosen lift $\tilde{x}$.

(2) The above identification defines an isometry of the spaces of L^2-sections. Moreover, it is compatible with the Γ-actions. $L^2(\widetilde{M},\widetilde{E})$ is well known to be an $\mathcal{N}\Gamma$-Hilbert space, and the identification is an identification of $\mathcal{N}\Gamma$-Hilbert spaces. In addition, it preserves smoothness and continuity, where the condition as given in the table is used to really get a section of $E \otimes \mathcal{H}$.

(3) The operators $\widetilde{D}$ and $D_{\mathcal{H}}$ are conjugated to each other under the isomorphism of the section spaces. This follows from their local definition as follows. For a small connected neighborhood U of $x \in M$, we can choose a lift $\widetilde{U}$, a connected neighborhood of a lift $\tilde{x}$, such that there is a unique section $U \to \widetilde{U}$ of the restriction of the covering $\widetilde{M} \to M$ to U, and then $y \mapsto [\tilde{y},\gamma]$ is by definition of the connection a *flat* section of $\mathcal{H}|_U$ for each $\gamma \in \Gamma$. Consequently, using the identification of (2) and this flatness, we see that $\widetilde{D}(\tilde{s})$ corresponds on the set U to

$$\sum_{\gamma\in\Gamma} \widetilde{D}\tilde{s}(\gamma\tilde{x}) \otimes [\tilde{x},\gamma] = D_{\mathcal{H}}\Big(\sum_{\gamma\in\Gamma} \tilde{s}(\gamma\tilde{x}) \otimes [\tilde{x},\gamma]\Big),$$

i.e. to $D_{\mathcal{H}}$ applied to the section corresponding to $\tilde{s}$.

(4) Since the self-adjoint unbounded operators $\widetilde{D}$ and $D_{\mathcal{H}}$ are unitarily equivalent, the same is true for all bounded measurable functions of them, using functional calculus. In particular, this is the case for $\widetilde{D}\exp(-t\widetilde{D}^2)$, but also for any other bounded measurable function $\phi\colon \mathbb{R} \to \mathbb{R}$.

(5) Choose a subset $U \subset M$ such that $M \setminus U$ has measure zero and such that the restriction of the covering $\widetilde{M} \to M$ to U is trivial. If we choose an appropriate lift of U then $\widetilde{M}|U \cong U \times \Gamma$. This induces a trivialization $\mathcal{H}|_U \cong U \times l^2(\Gamma)$. Using this, we identify $L^2(U, E \otimes \mathcal{H}|_U) = L^2(U, E|_U) \otimes l^2(\Gamma)$.

On the other hand, using the corresponding trivialization of the covering $\widetilde{M}|U \cong U \times \Gamma$ we get the identification $L^2(\widetilde{U}, \widetilde{E}|U) \cong L^2(U, E|U) \otimes l^2(\Gamma)$, and our unitary isomorphism defined above becomes the identity under these identifications.

So, in this picture, also the operators $\widetilde{D}$ and $D_{\mathcal{H}}$ are identical. It remains to show that the two ways to define the trace are equal. We take traces of operators like

$S = \widetilde{D}e^{-t\widetilde{D}^2}$, which have integral kernels $\tilde{k}(\tilde{x}, \tilde{y})$ on $\widetilde{M} \times \widetilde{M}$. Note that it is well known that these operators are of Γ-trace class, by the results of [1]. They are also well known to be smoothing operators in the $\mathcal{N}\Gamma$-Mishchenko–Fomenko calculus, and therefore $\mathrm{TR}(S)$ is defined.

Using the identification $\widetilde{U} = U \times \Gamma$, $\tilde{k}(u, g, v, h)$ has the property that for a section s of $E|_U$ and $u \in U$, $g, \gamma \in \Gamma$,

$$S(s(\cdot) \otimes \gamma)(u, g) = \int_U \tilde{k}(u, g, x, \gamma)s(x)\, dx.$$

On the other hand, if we look at the integral kernel $k_{\mathcal{H}}$ on $M \times M$ of S, which we now consider as being obtained from $D_{\mathcal{H}}$, using again the identification $L^2(U, E|_U \otimes \mathcal{H}) = L^2(U, E|_U) \otimes l^2(\Gamma)$, we get for a section s of $E|_U$:

$$\begin{aligned} S(s(\,\cdot\,) \otimes \gamma)(u) &= \int_U k_{\mathcal{H}}(u, x)(s(u) \otimes \gamma)\, dx \\ &= \sum_{g \in \Gamma} \int_U k_{\mathcal{H}}(u, x)(\gamma, g)(s(u) \otimes \gamma) \otimes g\, dx. \end{aligned}$$

Here we write the homomorphism $k_{\mathcal{H}}(u, x)$ from $E_x \otimes l^2(\Gamma)$ to $E_u \otimes l^2(\Gamma)$ as a matrix $(k_{\mathcal{H}}(u, x)(\gamma, g))_{\gamma, g \in \Gamma}$ of homomorphisms from E_x to E_u, using the orthonormal basis Γ of $l^2(\Gamma)$.

Since these are two representations of the same operator S, it follows that

$$k_{\mathcal{H}}(u, x)(\gamma, g) = \tilde{k}(u, \gamma, x, g). \tag{E.8}$$

Now, by definition of $\mathrm{Tr}_{\langle g \rangle}$, we get

$$\mathrm{Tr}_{\langle g \rangle}(S) = \sum_{h \in \langle g \rangle} \int_U \mathrm{tr}_x\, \tilde{k}(x, 1, x, h)\, dx. \tag{E.9}$$

On the other hand, by the definition of $\tau_{\langle g \rangle}$ in terms of a matrix decomposition of elements of $\mathcal{N}\Gamma \subset \mathcal{B}(l^2\Gamma)$, using the basis Γ of $l^2(\Gamma)$,

$$\tau_{\langle g \rangle}\, \mathrm{TR}(S) = \int_U \tau_{\langle g \rangle}\, \mathrm{tr}_x^{\mathrm{alg}}(k_{\mathcal{H}}(x, x))\, dx = \int_U \sum_{h \in \langle g \rangle} \mathrm{tr}_x(k_{\mathcal{H}}(x, x)(1, h))\, dx. \tag{E.10}$$

From equation (E.8), the right-hand sides of equations (E.9) and (E.10) are equal. Consequently,

$$\mathrm{Tr}_{\langle g \rangle}(S) = \tau_{\langle g \rangle} \circ \mathrm{TR}(S),$$

as claimed.

(6) Since $\mathrm{Tr}_{(2)} = \mathrm{Tr}_{\langle 1 \rangle}$, this is a special case of (5).

(7) This is a direct consequence of (4) and of (5), using the definition of the eta-invariants.

For more details, in particular with respect to the delocalized trace, compare also [41, Section 4] and [40, Section 3]

We want to single out the relevant proposition which is used in the body of this paper, using the results of the above table and of Section :

E.11 Proposition. *Let M be a compact manifold with Γ-covering $\widetilde{M}$, and D a Dirac type operator on M. Let $\langle g\rangle$ be a finite conjugacy class in Γ. If $\widetilde{D}$ is the lift of D to $\widetilde{M}$, and $D_{\mathcal{N}}$ is the twist of D with the Mishchenko–Fomenko line bundle $\mathcal{N} = \widetilde{M} \times_\Gamma \mathcal{N}\Gamma$, then*

$$\eta_{\langle g\rangle}(\widetilde{D}) = \eta_{\tau_{\langle g\rangle}}(D_{\mathcal{N}}).$$

In particular, for $g = e$ we compute the classical L^2-eta invariant

$$\eta_{(2)}(\widetilde{D}) = \eta_{\tau_\Gamma}(D_{\mathcal{N}}).$$

E.12 Remark. There is an alternative route to Proposition E.11, due to John Lott, and based on particular properties of the heat kernel. We briefly explain the main ideas, referring to [40] for the details.

There is a sequence of inclusions of algebras

$$\mathcal{B}^\omega_\Gamma \subset \mathcal{B}^\infty_\Gamma \subset C^*_r\Gamma \subset \mathcal{N}\Gamma, \tag{E.13}$$

where $\mathcal{B}^\omega_\Gamma$ is the algebra of functions on Γ which are exponentially rapidly decreasing and where $\mathcal{B}^\infty_\Gamma$ is the Connes–Moscovici algebra. A generalized Dirac operator can be twisted with the corresponding flat bundles

$$\mathcal{V}^\omega := \widetilde{M} \times_\Gamma \mathcal{B}^\omega_\Gamma,\ \mathcal{V}^\infty := \widetilde{M} \times_\Gamma \mathcal{B}^\infty_\Gamma,\ \mathcal{V} := \widetilde{M} \times_\Gamma C^*_r\Gamma,\ \mathcal{N} := \widetilde{M} \times_\Gamma \mathcal{N}\Gamma \tag{E.14}$$

producing $\mathcal{D}^\omega$, $\mathcal{D}^\infty$, $\mathcal{D} := D_{\mathcal{V}}$, $D_{\mathcal{N}}$. There are obvious compatibility conditions for these operators, coming from the inclusions of vector spaces

$$C^\infty(M, E \otimes \mathcal{V}^\omega) \subset C^\infty(M, E \otimes \mathcal{V}^\infty) \subset C^\infty(M, E \otimes \mathcal{V}) \subset C^\infty(M, E \otimes \mathcal{N}).$$

We also have natural inclusions for the corresponding Mishchenko–Fomenko calculi

$$\Psi^*_{\mathcal{B}^\omega_\Gamma} \xrightarrow{j_\omega} \Psi^*_{\mathcal{B}^\infty_\Gamma} \xrightarrow{j_\infty} \Psi^*_{C^*_r\Gamma} \xrightarrow{j_r} \Psi^*_{\mathcal{N}\Gamma}.$$

For the smoothing operators we have natural traces and a commutative diagram

$$\begin{array}{ccccccc}
\Psi^{-\infty}_{\mathcal{B}^\omega_\Gamma} & \xrightarrow{j_\omega} & \Psi^{-\infty}_{\mathcal{B}^\infty_\Gamma} & \xrightarrow{j_\infty} & \Psi^{-\infty}_{C^*_r\Gamma} & \xrightarrow{j_r} & \Psi^{-\infty}_{\mathcal{N}\Gamma} \\
\downarrow{\scriptstyle \mathrm{TR}_\omega} & & \downarrow{\scriptstyle \mathrm{TR}_\infty} & & \downarrow{\scriptstyle \mathrm{TR}} & & \downarrow{\scriptstyle \mathrm{TR}_{\mathcal{N}\Gamma}} \\
(\mathcal{B}^\omega_\Gamma)_{\mathrm{ab}} & \longrightarrow & (\mathcal{B}^\infty_\Gamma)_{\mathrm{ab}} & \longrightarrow & (C^*_r\Gamma)_{\mathrm{ab}} & \longrightarrow & (\mathcal{N}\Gamma)_{\mathrm{ab}}.
\end{array}$$

Using Proposition 6 and Proposition 7 in [40] we have that

$$\tau_{\langle g\rangle}\, \mathrm{TR}_\omega(\mathcal{D}^\omega e^{-t(\mathcal{D}^\omega)^2}) = \mathrm{Tr}_{\langle g\rangle}(\widetilde{D} e^{-t(\widetilde{D})^2}).$$

Since from the commutative diagram and the definition of $\tau_{\langle g\rangle}$ we clearly have

$$\tau_{\langle g\rangle}\, \mathrm{TR}_\omega(\mathcal{D}^\omega e^{-t(\mathcal{D}^\omega)^2}) = \tau_{\langle g\rangle}\, \mathrm{TR}_{\mathcal{N}\Gamma}(D_{\mathcal{N}} e^{-tD^2_{\mathcal{N}}}),$$

we see that Proposition E.11 is proved once again. Notice that we have also established that

$$\tau_{\langle g\rangle}\, \mathrm{TR}_\infty(\mathcal{D}^\infty e^{-t(\mathcal{D}^\infty)^2}) = \mathrm{Tr}_{\langle g\rangle}(\widetilde{D} e^{-t(\widetilde{D})^2}).$$

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