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# Quantum group-twisted tensor products of C*-algebras. II

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**Abstract.** For a quasitriangular $C^*$-quantum group, we enrich the twisted tensor product constructed in the first part of this series to a monoidal structure on the category of its continuous coactions on $C^*$-algebras. We define braided $C^*$-quantum groups, where the comultiplication takes values in a twisted tensor product. We show that compact braided $C^*$-quantum groups yield compact quantum groups by a semidirect product construction.

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## 1. Introduction

Let $C$ and $D$ be $C^*$-algebras with a coaction of a $C^*$-quantum group $\mathbb{G} = (A, \Delta_A)$. As in [12], $C^*$-quantum groups are generated by manageable multiplicative unitaries, and Haar weights are not assumed. If $\mathbb{G}$ is a group, then the $C^*$-tensor product $C \otimes D$ inherits a diagonal coaction. This fails for quantum groups because the diagonal coaction is not compatible with the multiplication in the tensor product. We use the noncommutative tensor products described in [12] to construct a monoidal structure on the category of $\mathbb{G}$-$C^*$-algebras if $\mathbb{G}$ is quasitriangular in a suitable sense.

Such a structure is to be expected from the analogous situation for (co)module algebras over a Hopf algebra. In that context, an R-matrix for the dual Hopf algebra allows to deform the multiplication on the tensor product of two $H$-comodule algebras so as to get an $H$-comodule algebra again. For $C^*$-quantum groups, Hopf module structures are replaced by comodule structures. Hence we call $\mathbb{G}$ quasitriangular if there is a unitary R-matrix $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$ for the dual $C^*$-quantum group.

Since R is a bicharacter, the braided tensor product $C \boxtimes D := (C, \gamma) \boxtimes_{\mathrm{R}} (D, \delta)$ in [12] is defined if $(C, \gamma)$ and $(D, \delta)$ are $C^*$-algebras with continuous coactions of $\mathbb{G}$. We show that $C \boxtimes D$ carries a unique continuous coaction $\gamma \bowtie \delta$ of $\mathbb{G}$ for which the canonical embeddings of $C$ and $D$ are $\mathbb{G}$-equivariant. (We do not denote

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this coaction by $\gamma \boxtimes \delta$ because $\boxtimes$ is a bifunctor, and the *-homomorphism $\gamma \boxtimes \delta$ given by this bifunctoriality is *not* $\gamma \bowtie \delta$.)

It is crucial for the theory here and in [12] that R is unitary. This rules out some important examples of quasitriangular Hopf algebras. For instance, R-matrices for quantum deformations of compact simple Lie groups are non-unitary.

If $(E, \epsilon)$ is another C*-algebra with a continuous coaction of $\mathbb{G}$, then there is a canonical isomorphism $(C \boxtimes D) \boxtimes E \cong C \boxtimes (D \boxtimes E)$. If $C$ or $D$ carries a trivial $\mathbb{G}$-coaction, then $C \boxtimes D = C \otimes D$, and $\gamma \bowtie \delta$ is the obvious induced action, $\gamma \otimes \mathrm{id}_D$ or $\mathrm{id}_C \otimes \delta$. Thus our tensor product on $\mathbb{G}$-coactions is monoidal: the tensor unit is $\mathbb{C}$ with trivial coaction. The tensor product of coactions is braided monoidal if and only if it is symmetric monoidal, if and only if the R-matrix is antisymmetric. This rarely happens, and it should not be expected because this also usually fails on the Hopf algebra level. What should be braided is the category of Hilbert space corepresentations. This is indeed the case, and we use it to prove that the tensor product for coactions is associative and monoidal.

An R-matrix $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$ lifts uniquely to a *universal* R-matrix $\mathrm{R} \in \mathcal{U}(\hat{A}^{\mathrm{u}} \otimes \hat{A}^{\mathrm{u}})$ for the universal quantum group $\hat{A}^{\mathrm{u}}$, so it makes no difference whether we consider R-matrices for $\hat{A}$ or $\hat{A}^{\mathrm{u}}$. Since Hilbert space corepresentations of $A$ are equivalent to Hilbert space representations of $\hat{A}^{\mathrm{u}}$, an R-matrix for $\hat{A}^{\mathrm{u}}$ induces a braiding on the monoidal category of Hilbert space corepresentations of $\mathbb{G}$.

If $\mathbb{G}$ is the quantum group of functions on an Abelian locally compact group $\Gamma$, then its R-matrices are simply bicharacters $\hat{\Gamma} \times \hat{\Gamma} \to \mathrm{U}(1)$. For instance, if $\Gamma$ is $\mathbb{Z}/2$, there are two such bicharacters. One gives the ordinary commutative tensor product with the diagonal coaction, the other gives the skew-commutative tensor product with diagonal $\mathbb{Z}/2$-coaction.

A well-known class of quasitriangular Hopf algebras are Drinfeld doubles; their module algebras are the same as Yetter–Drinfeld algebras. The dual of the Drinfeld double $\mathfrak{D}(\mathbb{G})$ of $\mathbb{G}$ is the Drinfeld codouble $\mathfrak{D}(\mathbb{G})\widehat{\ }$, which we just call quantum codouble. (What we call quantum codouble is called Drinfeld double in [13].)

For our class of C*-quantum groups, quantum codoubles and doubles and corresponding multiplicative unitaries are described in [17]. It is already shown in [17] that quantum codoubles are quasitriangular and that $\mathfrak{D}(\mathbb{G})\widehat{\ }$-C*-algebras are the same as $\mathbb{G}$-Yetter–Drinfeld C*-algebras. In this article, we identify the twisted tensor product for the canonical R-matrix of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ with the twisted tensor product in the category of $\mathbb{G}$-Yetter–Drinfeld C*-algebras constructed in [13].

A *braided* C*-*bialgebra* is a $\mathbb{G}$-C*-algebra $(B, \beta)$ with a comultiplication

$$\Delta_B \colon B \to B \boxtimes B$$

that is coassociative. We call $(B, \beta, \Delta_B)$ a *braided compact quantum group* if $B$ is unital and $\Delta_B$ satisfies an appropriate Podleś condition.

We are particularly interested in braided quantum groups over a codouble $\mathfrak{D}(\mathbb{G})\widehat{\ }$ because they appear in a quantum group version of semidirect products. In the group case, the construction of a semidirect product $G \ltimes H$ requires a conjugation action of $G$ on $H$ such that the multiplication map $H \times H \to H$ is $G$-equivariant. For quantum group semidirect products, the equivariance of the multiplication map on the underlying C*-algebra $B$ of $\mathbb{H}$ only makes sense if we deform the tensor product because there is no canonical $\mathbb{G}$-coaction on $B \otimes B$. A theorem of Radford [15, Theorem 3] for the analogous situation in the world of Hopf algebras suggests that $\mathbb{H}$ should be a braided quantum group over the codouble $\mathfrak{D}(\mathbb{G})\widehat{}$ of $\mathbb{G}$. In this case, we describe an induced C*-bialgebra structure on $A \boxtimes B$. This is a C*-algebraic analogue of what Majid calls "bosonisation" in [10]. We prefer to call the construction of $A \boxtimes B$ a "semidirect product". If $A$ is a compact quantum group and $B$ is a braided compact quantum group, then their semidirect product $A \boxtimes B$ is a compact quantum group.

As a first example, we construct a C*-algebraic analogue of the partial duals studied in [3]. We have constructed braided quantum SU(2) groups with complex deformation parameter $q$ together with Paweł Kasprzak in [6]; their semidirect products are the deformation quantisations of the unitary group U(2) defined in [22].

The construction of the C*-bialgebra $A \boxtimes B$ works in great generality. For this to be a C*-quantum group, we would need a multiplicative unitary for it. Then it is best to work on the level of multiplicative unitaries throughout. That is done in [16]. On that level, one can also go back and decompose a semidirect product into the two factors. Here we limit our attention to the compact case, where bisimplifiability of the comultiplication map is enough to get a quantum group.

We briefly summarise the following sections. In Section 2, we define R-matrices and show that they lift to the universal quantum group. In Section 3, we describe the braided monoidal structure on the category of Hilbert space corepresentations for a quasitriangular C*-quantum group. As an example, we consider the case of Abelian groups. In Section 4 we construct the "diagonal" action of a quasitriangular quantum group on tensor products twisted by the R-matrix and show that it gives a monoidal structure on $\mathbb{G}$-C*-algebras. Section 5 studies the case of quantum codoubles, where coactions on C*-algebras are equivalent to Yetter–Drinfeld algebra structures. Section 6 contains the semidirect product construction for braided C*-bialgebras. The appendix recalls basic results about C*-quantum groups and some results of our previous articles for the convenience of the reader. There are also some new observations about Heisenberg pairs in Appendix A.7, which would fit better into [12] but were left out there.

## 2. R-matrices

Let $\mathbb{G} = (A, \Delta_A)$ be a C*-quantum group and let $\mathrm{W} \in \mathcal{U}(\hat{A} \otimes A)$ be its reduced bicharacter; see Appendix A.1 and Definition A.6.

**Definition 2.1.** A bicharacter $\mathrm{R} \in \mathcal{U}(A \otimes A)$ is called an *R-matrix* if

$$\mathrm{R}(\sigma \circ \Delta_A(a))\mathrm{R}^* = \Delta_A(a) \qquad \text{for all } a \in A. \tag{2.1}$$

**Lemma 2.2.** *The* dual $\hat{\mathrm{R}} := \sigma(\mathrm{R}^*) \in \mathcal{U}(A \otimes A)$ *of a bicharacter* $\mathrm{R} \in \mathcal{U}(A \otimes A)$ *is an R-matrix if and only if* R *is an R-matrix.* □

**Remark 2.3.** The standard convention for Hopf algebras (see [9, Definition 2.1.1]) assumes $\mathrm{R}(\Delta_A(a))\mathrm{R}^* = \sigma \circ \Delta_A(a)$, which is opposite to (2.1). Our convention in (2.1) becomes the standard one if we replace $\Delta_A$ by $\Delta_A^{\mathrm{cop}} := \sigma \circ \Delta_A$ or R by $\mathrm{R}^*$.

In order to simplify proofs later, we lift an R-matrix $\mathrm{R} \in \mathcal{U}(A \otimes A)$ to $\mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}})$:

**Proposition 2.4.** *There is a unique* $\mathrm{R}^{\mathrm{u}} \in \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}})$ *with*

$$(\Lambda \otimes \Lambda)\mathrm{R}^{\mathrm{u}} = \mathrm{R} \qquad \text{in } \mathcal{U}(A \otimes A), \tag{2.2}$$

$$(\Delta_{A^{\mathrm{u}}} \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}} = \mathrm{R}^{\mathrm{u}}_{23}\mathrm{R}^{\mathrm{u}}_{13} \qquad \text{in } \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}} \otimes A^{\mathrm{u}}), \tag{2.3}$$

$$(\mathrm{id}_{A^{\mathrm{u}}} \otimes \Delta_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}} = \mathrm{R}^{\mathrm{u}}_{12}\mathrm{R}^{\mathrm{u}}_{13} \qquad \text{in } \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}} \otimes A^{\mathrm{u}}). \tag{2.4}$$

*This unitary also satisfies*

$$\mathrm{R}^{\mathrm{u}}(\sigma \circ \Delta^{\mathrm{u}}_A(a))(\mathrm{R}^{\mathrm{u}})^* = \Delta^{\mathrm{u}}_A(a) \qquad \text{for all } a \in A^{\mathrm{u}}. \tag{2.5}$$

*Proof.* [11, Proposition 4.7] gives a unique $\mathrm{R}^{\mathrm{u}} \in \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}})$ satisfying (2.2)–(2.4). The nontrivial part is to show that $\mathrm{R}^{\mathrm{u}}$ satisfies (2.5). Let $\mathcal{V} \in \mathcal{U}(\hat{A} \otimes A^{\mathrm{u}})$ be the universal bicharacter as in Appendix A.4. Theorem 25 and Proposition 31 in [19] show that

$$A^{\mathrm{u}} = \{(\omega \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathcal{V} \mid \omega \in \hat{A}'\}^{\mathrm{CLS}} \qquad \text{and} \qquad (\mathrm{id}_{\hat{A}} \otimes \Delta_{A^{\mathrm{u}}})\mathcal{V} = \mathcal{V}_{12}\mathcal{V}_{13}. \tag{2.6}$$

Therefore, (2.5) is equivalent to:

$$\mathrm{R}^{\mathrm{u}}_{23}\mathcal{V}_{13}\mathcal{V}_{12}(\mathrm{R}^{\mathrm{u}}_{23})^* = \mathcal{V}_{12}\mathcal{V}_{13} \qquad \text{in } \mathcal{U}(\hat{A} \otimes A^{\mathrm{u}} \otimes A^{\mathrm{u}}). \tag{2.7}$$

The unitary $\overline{\mathrm{R}} := (\Lambda \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}} \in \mathcal{M}(A \otimes A^{\mathrm{u}})$ is also a bicharacter. Let $X := \mathrm{W}^*_{12}\overline{\mathrm{R}}_{23}\mathcal{V}_{13}\mathrm{W}_{12} \in \mathcal{U}(\hat{A} \otimes A \otimes A^{\mathrm{u}})$. The following computation shows that $X$ is a corepresentation of $(A^{\mathrm{u}}, \Delta_{A^{\mathrm{u}}})$ on $\hat{A} \otimes A$:

$$\begin{aligned}(\mathrm{id}_{\hat{A}} \otimes \mathrm{id}_A \otimes \Delta_{A^{\mathrm{u}}})(\mathrm{W}^*_{12}\overline{\mathrm{R}}_{23}\mathcal{V}_{13}\mathrm{W}_{12}) &= \mathrm{W}^*_{12}\overline{\mathrm{R}}_{23}\overline{\mathrm{R}}_{24}\mathcal{V}_{13}\mathcal{V}_{14}\mathrm{W}_{12}\\ &= (\mathrm{W}^*_{12}\overline{\mathrm{R}}_{23}\mathcal{V}_{13}\mathrm{W}_{12})(\mathrm{W}^*_{12}\overline{\mathrm{R}}_{24}\mathcal{V}_{14}\mathrm{W}_{12})\\ &= X_{123}X_{124}.\end{aligned}$$

The first step uses (2.4) and (2.6), the second step uses that $\overline{\mathrm{R}}_{24}$ and $\mathcal{V}_{13}$ commute, and the last step is trivial. A similar routine computation shows that $Y := \mathcal{V}_{13}\overline{\mathrm{R}}_{23} \in \mathcal{U}(\hat{A} \otimes A \otimes A^{\mathrm{u}})$ satisfies $(\mathrm{id}_{\hat{A}} \otimes \mathrm{id}_A \otimes \Delta_{A^{\mathrm{u}}})Y = Y_{123}Y_{124}$.

The argument that shows that (2.5) is equivalent to (2.7) also shows that the R-matrix condition (2.1) is equivalent to

$$\mathrm{R}_{23}\mathrm{W}_{13}\mathrm{W}_{12} = \mathrm{W}_{12}\mathrm{W}_{13}\mathrm{R}_{23} \qquad \text{in } \mathcal{U}(\hat{A} \otimes A \otimes A). \tag{2.8}$$

Thus $(\mathrm{id}_{\hat{A}} \otimes \mathrm{id}_A \otimes \Lambda)X = (\mathrm{id}_{\hat{A}} \otimes \mathrm{id}_A \otimes \Lambda)Y$. Now we use Lemma [11, Lemma 4.6], which is a variation on [7, Result 6.1]. It gives $X = Y$ or, equivalently,

$$\mathcal{V}_{13}^{*}\mathrm{W}_{12}^{*}\bar{\mathrm{R}}_{23}\mathcal{V}_{13} = \bar{\mathrm{R}}_{23}\mathrm{W}_{12}^{*} \qquad \text{in } \mathcal{U}(\hat{A} \otimes A \otimes A^{\mathrm{u}}). \tag{2.9}$$

Similarly, $\widetilde{X} := \mathcal{V}_{13}^{*}(\mathrm{R}_{23}^{\mathrm{u}})^{*}\mathcal{V}_{12}\mathcal{V}_{13}$ and $\widetilde{Y} := \mathcal{V}_{12}(\mathrm{R}_{23}^{\mathrm{u}})^{*}$ in $\mathcal{U}(\hat{A} \otimes A^{\mathrm{u}} \otimes A^{\mathrm{u}})$ satisfy $(\mathrm{id}_{\hat{A}} \otimes \Lambda \otimes \mathrm{id}_{A^{\mathrm{u}}})\widetilde{X} = (\mathrm{id}_{\hat{A}} \otimes \Lambda \otimes \mathrm{id}_{A^{\mathrm{u}}})\widetilde{Y}$ by (2.9), and

$$(\mathrm{id}_{\hat{A}} \otimes \Delta_{A^{\mathrm{u}}} \otimes \mathrm{id}_{A^{\mathrm{u}}})(\widetilde{X}) = \widetilde{X}_{124}\widetilde{X}_{134}, \qquad (\mathrm{id}_{\hat{A}} \otimes \Delta_{A^{\mathrm{u}}} \otimes \mathrm{id}_{A^{\mathrm{u}}})(\widetilde{Y}) = \widetilde{Y}_{124}\widetilde{Y}_{134}$$

because of (2.3) and (2.6). Another application of [11, Lemma 4.6] gives $\widetilde{X} = \widetilde{Y}$, which is equivalent to (2.7). □

[19, Proposition 31.2] shows that $(A^{\mathrm{u}}, \Delta_{A^{\mathrm{u}}})$ has a bounded counit: there is a unique morphism $e\colon A^{\mathrm{u}} \to \mathbb{C}$ with

$$(e \otimes \mathrm{id}_{A^{\mathrm{u}}})\Delta_{A^{\mathrm{u}}} = (\mathrm{id}_{A^{\mathrm{u}}} \otimes e)\Delta_{A^{\mathrm{u}}} = \mathrm{id}_{A^{\mathrm{u}}}. \tag{2.10}$$

**Lemma 2.5.** *The unitary* $\mathrm{R}^{\mathrm{u}} \in \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}})$ *in Proposition* 2.4 *satisfies*

$$(e \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}} = (\mathrm{id}_{A^{\mathrm{u}}} \otimes e)\mathrm{R}^{\mathrm{u}} = 1_{A^{\mathrm{u}}} \qquad \text{in } \mathcal{U}(A^{\mathrm{u}}), \tag{2.11}$$

$$\mathrm{R}_{12}^{\mathrm{u}}\mathrm{R}_{13}^{\mathrm{u}}\mathrm{R}_{23}^{\mathrm{u}} = \mathrm{R}_{23}^{\mathrm{u}}\mathrm{R}_{13}^{\mathrm{u}}\mathrm{R}_{12}^{\mathrm{u}} \qquad \text{in } \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}} \otimes A^{\mathrm{u}}). \tag{2.12}$$

*Proof.* Apply $\mathrm{id}_{A^{\mathrm{u}}} \otimes e \otimes \mathrm{id}_{A^{\mathrm{u}}}$ on both sides of (2.3) and (2.4) and then use (2.10). This gives

$$\mathrm{R}^{\mathrm{u}} = (1_{A^{\mathrm{u}}} \otimes (e \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}})\mathrm{R}^{\mathrm{u}} = (((\mathrm{id}_{A^{\mathrm{u}}} \otimes e)\mathrm{R}^{\mathrm{u}}) \otimes 1_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}}.$$

Multiplying with $(\mathrm{R}^{\mathrm{u}})^{*}$ on the right gives $(e \otimes \mathrm{id}_{A^{\mathrm{u}}})\mathrm{R}^{\mathrm{u}} = (\mathrm{id}_{A^{\mathrm{u}}} \otimes e)\mathrm{R}^{\mathrm{u}} = 1_{A^{\mathrm{u}}}$.

The following computation yields (2.12):

$$\mathrm{R}_{12}^{\mathrm{u}}\mathrm{R}_{13}^{\mathrm{u}}\mathrm{R}_{23}^{\mathrm{u}} = ((\mathrm{id}_{A^{\mathrm{u}}} \otimes \Delta^{\mathrm{u}})\mathrm{R}^{\mathrm{u}})\mathrm{R}_{23}^{\mathrm{u}} = \mathrm{R}_{23}^{\mathrm{u}}((\mathrm{id}_{A^{\mathrm{u}}} \otimes \sigma \circ \Delta^{\mathrm{u}})\mathrm{R}^{\mathrm{u}}) = \mathrm{R}_{23}^{\mathrm{u}}\mathrm{R}_{13}^{\mathrm{u}}\mathrm{R}_{12}^{\mathrm{u}};$$

here the first and third step use (2.4) and the second step uses (2.5). □

## 3. Corepresentation categories of quasitriangular quantum groups

**Definition 3.1.** A *quasitriangular* C*-quantum group is a C*-quantum group $\mathbb{G} = (A, \Delta_A)$ with an R-matrix $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$.

Let $\Sigma^{(\mathcal{H}_1,\mathcal{H}_2)}\colon \mathcal{H}_1\otimes\mathcal{H}_2\to\mathcal{H}_2\otimes\mathcal{H}_1$ denote the flip operator. As already pointed out in [19], $\Sigma_{12}^{(\mathcal{H}_1,\mathcal{H}_2)}$ is $\mathbb{G}$-equivariant for all corepresentations of $\mathbb{G}$ if and only if $\mathbb{G}$ is commutative. Hence $\Sigma^{(\cdot,\cdot)}$ does not give a braiding on $\mathfrak{Corep}(\mathbb{G})$ in general.

Let $\mathrm{U}^{\mathcal{H}_i}\in\mathcal{U}(\mathbb{K}(\mathcal{H}_i)\otimes A)$ be corepresentations of $\mathbb{G}$ on $\mathcal{H}_i$ for $i=1,2$. These correspond to representations of the universal quantum group $\hat{A}^{\mathrm{u}}$ by the universal property of $\hat{A}^{\mathrm{u}}$. More precisely, there are unique $\hat{\varphi}_i\in\mathrm{Mor}(\hat{A}^{\mathrm{u}},\mathbb{K}(\mathcal{H}_i))$ such that $(\hat{\varphi}_i\otimes\mathrm{id}_A)\tilde{\mathcal{V}}=\mathrm{U}^{\mathcal{H}_i}$ for $i=1,2$, see Appendix A.4; here $\tilde{\mathcal{V}}^A$ is the universal bicharacter in $\mathcal{U}(\hat{A}^{\mathrm{u}}\otimes A)$.

Define ${}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}\colon \mathcal{H}_1\otimes\mathcal{H}_2\to\mathcal{H}_2\otimes\mathcal{H}_1$ by

$$\mathrm{X}^{(\mathcal{H}_2,\mathcal{H}_1)} := (\hat{\varphi}_2\otimes\hat{\varphi}_1)(\mathrm{R}^{\mathrm{u}})^* \qquad \text{in } \mathcal{U}(\mathcal{H}_2\otimes\mathcal{H}_1), \tag{3.1}$$

$${}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2} := \mathrm{X}^{(\mathcal{H}_2,\mathcal{H}_1)}\circ\Sigma^{\mathcal{H}_1,\mathcal{H}_2} \qquad \text{in } \mathcal{U}(\mathcal{H}_1\otimes\mathcal{H}_2,\mathcal{H}_2\otimes\mathcal{H}_1). \tag{3.2}$$

Here $\mathrm{R}^{\mathrm{u}}\in\mathcal{U}(\hat{A}^{\mathrm{u}}\otimes\hat{A}^{\mathrm{u}})$ is as in Proposition 2.4.

**Proposition 3.2.** *The unitaries* ${}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}\colon \mathcal{H}_1\otimes\mathcal{H}_2\to\mathcal{H}_2\otimes\mathcal{H}_1$ *are* $\mathbb{G}$*-equivariant, that is,*

$${}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}{}_{12}(\mathrm{U}^{\mathcal{H}_1}\mathbin{\textcircled{\scriptsize T}}\mathrm{U}^{\mathcal{H}_2}) = (\mathrm{U}^{\mathcal{H}_2}\mathbin{\textcircled{\scriptsize T}}\mathrm{U}^{\mathcal{H}_1})\,{}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}{}_{12} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}_1\otimes\mathcal{H}_2)\otimes A) \tag{3.3}$$

*for all* $\mathrm{U}^{\mathcal{H}_1},\mathrm{U}^{\mathcal{H}_2}\in\mathfrak{Corep}(\mathbb{G})$*. The tensor product* $\mathbin{\textcircled{\scriptsize T}}$ *is defined in* (A.9)*.*

*The unitaries* ${}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}$ *define a braiding on* $\mathfrak{Corep}(\mathbb{G})$*, that is, the following hexagons commute for all* $\mathrm{U}^{\mathcal{H}_i}\in\mathfrak{Corep}(\mathbb{G})$*,* $i=1,2,3$*:*

$$\begin{array}{ccc} & \mathcal{H}_1\otimes(\mathcal{H}_2\otimes\mathcal{H}_3) \xrightarrow{\;{}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2\otimes\mathcal{H}_3}\;} (\mathcal{H}_2\otimes\mathcal{H}_3)\otimes\mathcal{H}_1 & \\ \nearrow & & \searrow \\ (\mathcal{H}_1\otimes\mathcal{H}_2)\otimes\mathcal{H}_3 & & \mathcal{H}_2\otimes(\mathcal{H}_3\otimes\mathcal{H}_1) \\ {}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}\otimes\mathrm{id}_{\mathcal{H}_3}\searrow & & \nearrow\mathrm{id}_{\mathcal{H}_2}\otimes{}^{\mathcal{H}_1}\!\times^{\mathcal{H}_3} \\ & (\mathcal{H}_2\otimes\mathcal{H}_1)\otimes\mathcal{H}_3 \longrightarrow \mathcal{H}_2\otimes(\mathcal{H}_1\otimes\mathcal{H}_3) & \end{array} \tag{3.4}$$

$$\begin{array}{ccc} & (\mathcal{H}_1\otimes\mathcal{H}_2)\otimes\mathcal{H}_3 \xrightarrow{\;{}^{\mathcal{H}_1\otimes\mathcal{H}_2}\!\times^{\mathcal{H}_3}\;} \mathcal{H}_3\otimes(\mathcal{H}_1\otimes\mathcal{H}_2) & \\ \nearrow & & \searrow \\ \mathcal{H}_1\otimes(\mathcal{H}_2\otimes\mathcal{H}_3) & & (\mathcal{H}_3\otimes\mathcal{H}_1)\otimes\mathcal{H}_2 \\ \mathrm{id}_{\mathcal{H}_1}\otimes{}^{\mathcal{H}_2}\!\times^{\mathcal{H}_3}\searrow & & \nearrow{}^{\mathcal{H}_1}\!\times^{\mathcal{H}_3}\otimes\mathrm{id}_{\mathcal{H}_2} \\ & \mathcal{H}_1\otimes(\mathcal{H}_3\otimes\mathcal{H}_2) \longrightarrow (\mathcal{H}_1\otimes\mathcal{H}_3)\otimes\mathcal{H}_2 & \end{array} \tag{3.5}$$

*Here the unlabelled arrows are the standard associators of Hilbert spaces.*

*Proof.* We have $(\hat{\Delta}_{A^{\mathrm{u}}} \otimes \mathrm{id}_A)\tilde{\mathcal{V}}^A = \tilde{\mathcal{V}}^A_{23}\tilde{\mathcal{V}}^A_{13}$ in $\mathcal{U}(\hat{A}^{\mathrm{u}} \otimes \hat{A}^{\mathrm{u}} \otimes A)$ because $\tilde{\mathcal{V}}^A$ is a character in the first leg. Therefore, the corepresentation $\mathrm{U}^{\mathcal{H}_1} \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U}^{\mathcal{H}_2}$ corresponds to $(\hat{\phi}_1 \otimes \hat{\phi}_2) \circ \sigma \circ \hat{\Delta}^{\mathrm{u}}_A \colon \hat{A}^{\mathrm{u}} \to \mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ through the universal property (A.11) of $\tilde{\mathcal{V}}$:

$$((\hat{\phi}_1 \otimes \hat{\phi}_2) \circ \sigma \circ \hat{\Delta}^{\mathrm{u}}_A \otimes \mathrm{id}_A)\tilde{\mathcal{V}} = \mathrm{U}^{\mathcal{H}_1} \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U}^{\mathcal{H}_2}. \tag{3.6}$$

The following computation yields (3.3):

$$\begin{aligned}
{}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_2}{}_{12}(\mathrm{U}^{\mathcal{H}_1} \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U}^{\mathcal{H}_2}) &= (\hat{\phi}_2 \otimes \hat{\phi}_1 \otimes \mathrm{id}_A)\big((\mathrm{R}^{\mathrm{u}}_{12})^*(\hat{\Delta}^{\mathrm{u}}_A \otimes \mathrm{id}_A)\tilde{\mathcal{V}}\big)\Sigma^{(\mathcal{H}_1,\mathcal{H}_2)}_{12}\\
&= \big(((\hat{\phi}_2 \otimes \hat{\phi}_1) \circ \sigma \circ \hat{\Delta}^{\mathrm{u}}_A) \otimes \mathrm{id}_A)\tilde{\mathcal{V}}\big)\mathrm{X}^{(\mathcal{H}_2,\mathcal{H}_1)}_{12}\Sigma^{(\mathcal{H}_1,\mathcal{H}_2)}_{12}\\
&= (\mathrm{U}^{\mathcal{H}_2} \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U}^{\mathcal{H}_1})\,{}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_2}{}_{12}.
\end{aligned}$$

The first equality uses (3.6) and (3.2), the second equality follows from (2.5) and (3.1), and the last equality uses (3.6) and (3.2).

Equations (3.2) and (3.6) imply

$$\begin{aligned}
{}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_2\otimes\mathcal{H}_3} &:= \mathrm{X}^{(\mathcal{H}_2\otimes\mathcal{H}_3,\mathcal{H}_1)}\Sigma^{(\mathcal{H}_1,\mathcal{H}_2\otimes\mathcal{H}_3)}\\
&= \Big((\hat{\phi}_2 \otimes \hat{\phi}_3 \otimes \hat{\phi}_1)(\sigma \circ \hat{\Delta}^{\mathrm{u}}_A \otimes \mathrm{id}_{\hat{A}^{\mathrm{u}}})(\mathrm{R}^{\mathrm{u}})^*\Big) \circ \Sigma^{(\mathcal{H}_1,\mathcal{H}_2\otimes\mathcal{H}_3)}.
\end{aligned} \tag{3.7}$$

Now we check the first braiding diagram (3.4):

$$\begin{aligned}
&\Big((\hat{\phi}_2 \otimes \hat{\phi}_3 \otimes \hat{\phi}_1)(\sigma \circ \hat{\Delta}^{\mathrm{u}}_A \otimes \mathrm{id}_{\hat{A}^{\mathrm{u}}})(\mathrm{R}^{\mathrm{u}})^*\Big) \circ \Sigma^{(\mathcal{H}_1,\mathcal{H}_2\otimes\mathcal{H}_3)}\\
&\qquad= \Big((\hat{\phi}_2 \otimes \hat{\phi}_3 \otimes \hat{\phi}_1)\big((\mathrm{R}^{\mathrm{u}})^*_{23}(\mathrm{R}^{\mathrm{u}})^*_{13}\big)\Big)\Sigma^{(\mathcal{H}_1,\mathcal{H}_3)}_{23}\Sigma^{(\mathcal{H}_1,\mathcal{H}_2)}_{12}\\
&\qquad= \mathrm{X}^{(\mathcal{H}_3,\mathcal{H}_1)}_{23}\mathrm{X}^{(\mathcal{H}_2,\mathcal{H}_1)}_{13}\Sigma^{(\mathcal{H}_1,\mathcal{H}_3)}_{23}\Sigma^{(\mathcal{H}_1,\mathcal{H}_2)}_{12}\\
&\qquad= \mathrm{X}^{(\mathcal{H}_3,\mathcal{H}_1)}_{23}\Sigma^{(\mathcal{H}_1,\mathcal{H}_3)}_{23}\mathrm{X}^{(\mathcal{H}_2,\mathcal{H}_1)}_{12}\Sigma^{(\mathcal{H}_1,\mathcal{H}_2)}_{12} = {}^{\mathcal{H}_3}\!\!\times^{\mathcal{H}_1}{}_{23}\,{}^{\mathcal{H}_2}\!\!\times^{\mathcal{H}_1}{}_{12};
\end{aligned}$$

here the first equality uses (2.3), the second equality uses (3.1), the third equality uses properties of the flip operator $\Sigma$, and the fourth equality follows from (3.2).

A similar computation for ${}^{\mathcal{H}_1\otimes\mathcal{H}_2}\!\!\times^{\mathcal{H}_3}$ yields the second braiding diagram (3.5). □

**Corollary 3.3.** *If $\mathbb{C}$ carries the trivial corepresentation of $\mathbb{G}$, then*

$${}^{\mathbb{C}}\!\!\times^{\mathcal{H}} \colon \mathbb{C} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathbb{C} \qquad \textit{and} \qquad {}^{\mathcal{H}}\!\!\times^{\mathbb{C}} \colon \mathcal{H} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathcal{H}$$

*are the canonical isomorphisms. For any three corepresentations of $\mathbb{G}$,*

$${}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_2}{}_{23}\,{}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_3}{}_{12}\,{}^{\mathcal{H}_2}\!\!\times^{\mathcal{H}_3}{}_{23} = {}^{\mathcal{H}_2}\!\!\times^{\mathcal{H}_3}{}_{12}\,{}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_3}{}_{23}\,{}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_2}{}_{12}. \tag{3.8}$$

*Proof.* These are general properties of braided monoidal categories, see [5, Proposition 2.1]. They also follow from (2.11), (2.12), and (3.1). □

**Remark 3.4.** The dual $\hat{\mathrm{R}} := \sigma(\mathrm{R}^*)$ of an R-matrix $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$ is again an R-matrix by Lemma 2.2. A routine computation shows that the resulting braiding on $\mathfrak{Corep}(\mathbb{G})$ is the dual braiding, given by the braiding unitaries

$$ {}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2} = \left({}^{\mathcal{H}_2}\!\times^{\mathcal{H}_1}\right)^* : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1. $$

### 3.1. Symmetric braidings.

**Definition 3.5.** An R-matrix $\mathrm{R} \in \mathcal{U}(A \otimes A)$ is called *antisymmetric* if $\mathrm{R}^* = \sigma(\mathrm{R})$ for the flip $\sigma: A \otimes A \to A \otimes A$, $a_1 \otimes a_2 \mapsto a_2 \otimes a_1$.

**Lemma 3.6.** *If* R *is antisymmetric, then* $(\mathrm{R}^{\mathrm{u}})^* = \sigma(\mathrm{R}^{\mathrm{u}})$ *for the universal lift* $\mathrm{R}^{\mathrm{u}} \in \mathcal{U}(A^{\mathrm{u}} \otimes A^{\mathrm{u}})$ *constructed in Proposition* 2.4.

*Proof.* Both $\sigma(\mathrm{R}^{\mathrm{u}})^*$ and $\mathrm{R}^{\mathrm{u}}$ are bicharacters that lift R. They must be equal because bicharacters lift uniquely by [11, Proposition 4.7]. □

**Proposition 3.7.** *The braiding on* $\mathfrak{Corep}(\mathbb{G})$ *constructed from* $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$ *is symmetric if and only if* R *is antisymmetric.*

*Proof.* Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with corepresentations of $\mathbb{G}$. Let $\hat{\phi}_i: \hat{A}^{\mathrm{u}} \to \mathbb{B}(\mathcal{H}_i)$ be the corresponding $*$-representations. Then

$$ \mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{{}^{\mathcal{H}_1}\times^{\mathcal{H}_2}} \mathcal{H}_2 \otimes \mathcal{H}_1 \xrightarrow{{}^{\mathcal{H}_2}\times^{\mathcal{H}_1}} \mathcal{H}_1 \otimes \mathcal{H}_2 $$

is equal to

$$ (\hat{\phi}_1 \otimes \hat{\phi}_2)(\mathrm{R}^{\mathrm{u}})^* \circ \Sigma^{(\mathcal{H}_2,\mathcal{H}_1)} \circ (\hat{\phi}_2 \otimes \hat{\phi}_1)(\mathrm{R}^{\mathrm{u}})^* \circ \Sigma^{(\mathcal{H}_1,\mathcal{H}_2)} = (\hat{\phi}_1 \otimes \hat{\phi}_2)(\sigma(\mathrm{R}^{\mathrm{u}})\mathrm{R}^{\mathrm{u}})^*. $$

This is the identity operator for all representations $\hat{\phi}_i$ if and only if $\sigma(\mathrm{R}^{\mathrm{u}})\mathrm{R}^{\mathrm{u}} = 1$. □

**3.2. The Abelian case.** Let $B$ be a locally compact group. What is an R-matrix for the commutative quantum group $(\mathrm{C}_0(G), \Delta)$? Since $\mathrm{C}_0(G) \otimes \mathrm{C}_0(G)$ is commutative as well, (2.1) simplifies to the condition $\sigma \circ \Delta = \Delta$, which is equivalent to $G$ being commutative. Hence there is no R-matrix unless $G$ is Abelian, which we assume from now on. Then (2.1) holds for any unitary $\mathrm{R} \in \mathcal{U}(\mathrm{C}_0(G) \otimes \mathrm{C}_0(G))$, so an R-matrix for $\mathbb{G}$ is simply a bicharacter. Equivalently, R is a function $\rho: G \times G \to \mathrm{U}(1)$ satisfying $\rho(xy, z) = \rho(x, z)\rho(y, z)$ and $\rho(x, yz) = \rho(x, y)\rho(x, z)$. Being antisymmetric means $\rho(x, y)\rho(y, x) = 1$ for all $x, y \in G$.

Any bicharacter $\rho$ as above is of the form $\rho(x, y) = \langle \hat{\rho}(x), y \rangle$ for a group homomorphism $\hat{\rho}: G \to \hat{G}$ to the Pontrjagin dual $\hat{G}$, with $\rho(x, \sqcup) = \hat{\rho}$. This is a special case of the interpretation of bicharacters as quantum group homomorphisms in [11].

The category of Hilbert space representations of $G$ is equivalent to the category of corepresentations of $(\mathrm{C}_0(G), \Delta)$ and to the category of representations of

$C^*(G) \cong C_0(\hat{G})$. The tensor category of $G$-representations is already symmetric for the obvious braiding $\Sigma$, which corresponds to the R-matrix 1. What are the braiding operators for a nontrivial R-matrix?

Let $\int_{\hat{G}}^{\oplus} \mathcal{H}_x \,\mathrm{d}\mu(x)$ denote the Hilbert space of $L^2$-sections of a measurable field of Hilbert spaces $(\mathcal{H}_x)_{x\in\hat{G}}$ over $\hat{G}$ with respect to a measure $\mu$, equipped with the action of $C_0(\hat{G})$ by pointwise multiplication. Any representation of $C_0(\hat{G})$ is of this form, where $\mu$ is unique up to measure equivalence and the field $(\mathcal{H}_x)$ is unique up to isomorphism $\mu$-almost everywhere. Let $\mathcal{H}_1 = \int_{\hat{G}}^{\oplus} \mathcal{H}_{1_x} \,\mathrm{d}\mu_1(x)$ and $\mathcal{H}_2 = \int_{\hat{G}}^{\oplus} \mathcal{H}_{2_x} \,\mathrm{d}\mu_2(x)$ be two Hilbert space representations of $G$. Then

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \int_{\hat{G}\times\hat{G}}^{\oplus} \mathcal{H}_{1_x} \otimes \mathcal{H}_{2_y} \,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y)$$

with $C_0(\hat{G}) \otimes C_0(\hat{G}) \cong C_0(\hat{G}\times\hat{G})$ acting by pointwise multiplication. The braiding ${}^{\mathcal{H}_1}\!\!\times^{\mathcal{H}_2}$ maps an $L^2$-section $(\xi_{x,y})_{x,y}$ of the field $(\mathcal{H}_{1_x} \otimes \mathcal{H}_{2_y})_{x,y}$ to the section $(y,x) \mapsto \rho(y,x)^{-1}\xi_{x,y}$ of $(\mathcal{H}_{2_y} \otimes \mathcal{H}_{1_x})_{y,x}$.

**Example 3.8.** Consider $G = \mathbb{Z}/2 = \{\pm 1\}$ and let $\rho(x,y) = xy \in \mathbb{Z}/2 \subseteq \mathrm{U}(1)$; this bicharacter corresponds to the isomorphism $G \cong \hat{G}$. It is both symmetric and antisymmetric. The spectral analysis above writes a $\mathbb{Z}/2$-Hilbert space as a $\mathbb{Z}/2$-graded Hilbert space, splitting it into even and odd elements with respect to the action of the generator of $\mathbb{Z}/2$. The braiding unitary on $\xi \otimes \eta$ is $\Sigma$ if $\xi$ or $\eta$ is even, and $-\Sigma$ if both $\xi$ and $\eta$ are odd. This is the usual Koszul sign rule.

## 4. Coaction categories of quasitriangular quantum groups

Let $\mathbb{G} = (A, \Delta_A, \mathrm{R})$ be a quasitriangular quantum group. Let $(C,\gamma)$ and $(D,\delta)$ be $\mathbb{G}$-C*-algebras. The twisted tensor product $C \boxtimes_{\mathrm{R}} D = C \boxtimes D$ is constructed in [12]. It is a crossed product of $C$ and $D$, that is, there are canonical morphisms $\iota_C : C \to C \boxtimes D$ and $\iota_D : D \to C \boxtimes D$ with

$$\iota_C(C) \cdot \iota_D(D) = \iota_D(D) \cdot \iota_C(C) = C \boxtimes D;$$

here a *morphism* is a nondegenerate *-homomorphism to the multiplier algebra, and $X \cdot Y$ for two subspaces $X$ and $Y$ of a C*-algebra means the *closed linear span* of $x \cdot y$ for $x \in X$, $y \in Y$ as in [12].

Theorem A.9 recalls one of the two equivalent definitions of the twisted tensor product in [12]. Let $(\varphi, \mathrm{U}^{\mathcal{H}})$ and $(\psi, \mathrm{U}^{\mathcal{K}})$ be faithful covariant representations of $(C,\gamma)$ and $(D,\delta)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then $C \boxtimes_{\mathrm{R}} D$ is canonically isomorphic to $\varphi_1(C) \cdot \tilde{\psi}_2(D) \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$, where $\varphi_1(c) = \varphi(c) \otimes 1_{\mathcal{K}}$

and $\tilde{\psi}_2(d) = X(1_{\mathcal{H}} \otimes \psi(d))X^*$ for the unitary $X$ that is characterised by (A.19). The same unitary appears in our construction of the braiding, so

$$\tilde{\psi}_2(d) = {}^{\mathcal{H}}\!\times^{\mathcal{K}}(\psi(d) \otimes 1_{\mathcal{H}})({}^{\mathcal{H}}\!\times^{\mathcal{K}})^*. \tag{4.1}$$

We are going to equip $C \boxtimes D$ with a natural $\mathbb{G}$-coaction and show that this tensor product gives a monoidal structure on the category of $\mathbb{G}$-C*-algebras $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ (see Definition A.4).

**Proposition 4.1.** *There is a unique $\mathbb{G}$-coaction $\gamma \bowtie_{\mathrm{R}} \delta$ on $C \boxtimes_{\mathrm{R}} D$ such that the canonical representation on $\mathcal{H} \otimes \mathcal{K}$ and the corepresentation $\mathrm{U}^{\mathcal{H}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{K}}$ form a covariant representation of $(C \boxtimes_{\mathrm{R}} D, \gamma \bowtie_{\mathrm{R}} \delta)$. This coaction is also the unique one for which the morphisms $\iota_C : C \to C \boxtimes_{\mathrm{R}} D$ and $\iota_D : D \to C \boxtimes_{\mathrm{R}} D$ are $\mathbb{G}$-equivariant.*

*Proof.* We identify $C \boxtimes D$ with its image in $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$. The covariance of this representation of $C \boxtimes D$ with $\mathrm{U}^{\mathcal{H}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{K}}$ means that

$$(\gamma \bowtie_{\mathrm{R}} \delta)(x) = (\mathrm{U}^{\mathcal{H}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{K}})(x \otimes 1_A)(\mathrm{U}^{\mathcal{H}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{K}})^*$$

for all $x \in C \boxtimes D$. Hence there is at most one such coaction $\gamma \bowtie_{\mathrm{R}} \delta$.

The representation $c \mapsto \iota_C(c) = \varphi(c) \otimes 1_{\mathcal{K}}$ is covariant with respect to $\mathrm{U}^{\mathcal{H}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{K}}$ because it is covariant with respect to $\mathrm{U}^{\mathcal{H}} \otimes 1$ by construction and $\iota_C(c)$ acts only on the first leg. Hence $(\gamma \bowtie_{\mathrm{R}} \delta)(\iota_C(c)) = (\iota_C \otimes \mathrm{id}_A)\gamma(c)$ for all $c \in C$. Similarly, the representation $d \mapsto \psi(d) \otimes 1$ on $\mathcal{K} \otimes \mathcal{H}$ is covariant with respect to $\mathrm{U}^{\mathcal{K}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{H}}$. Since the unitary ${}^{\mathcal{H}}\!\times^{\mathcal{K}}$ is $\mathbb{G}$-equivariant by Proposition 3.2, the representation $\tilde{\psi}_2$ on $\mathcal{H} \otimes \mathcal{K}$ is covariant with respect to $\mathrm{U}^{\mathcal{H}} \bigcirc\!\!\!\!\top \mathrm{U}^{\mathcal{K}}$ as well (unlike the representation $d \mapsto 1 \otimes \psi(d)$). Hence $(\gamma \bowtie_{\mathrm{R}} \delta)(\iota_D(d)) = (\iota_D \otimes \mathrm{id}_A)\delta(d)$ for all $d \in D$. As a result, $\gamma \bowtie_{\mathrm{R}} \delta$ maps $C \boxtimes D = \iota_C(C) \cdot \iota_D(D)$ nondegenerately into the multiplier algebra of $(C \boxtimes D) \otimes A$, and the morphisms $\iota_C$ and $\iota_D$ are $\mathbb{G}$-equivariant.

The morphism $\gamma \bowtie_{\mathrm{R}} \delta : C \boxtimes D \to (C \boxtimes D) \otimes A$ is faithful by construction. The Podleś condition for $C \boxtimes D$ follows from those for $C$ and $D$:

$$\begin{aligned}
(\gamma \bowtie_{\mathrm{R}} \delta)(C \boxtimes D) \cdot (1 \otimes A) &= (\iota_C \otimes \mathrm{id}_A)(\gamma(C)) \cdot (\iota_D \otimes \mathrm{id}_A)(\delta(D)) \cdot (1 \otimes A) \\
&= (\iota_C \otimes \mathrm{id}_A)(\gamma(C)) \cdot (\iota_D(D) \otimes A) \\
&= (\iota_C \otimes \mathrm{id}_A)(\gamma(C)) \cdot (1 \otimes A) \cdot (\iota_D(D) \otimes A) \\
&= (\iota_C(C) \otimes A) \cdot (\iota_D(D) \otimes A) = C \boxtimes D \otimes A.
\end{aligned}$$

Thus $\gamma \bowtie_{\mathrm{R}} \delta$ is a continuous $\mathbb{G}$-coaction on $C \boxtimes D$ for which $\iota_C$ and $\iota_D$ are equivariant. Conversely, if $\iota_C$ and $\iota_D$ are $\mathbb{G}$-equivariant, then

$$(\gamma \bowtie_{\mathrm{R}} \delta)(\iota_C(c) \cdot \iota_D(d)) = (\iota_C \otimes \mathrm{id}_A)(\gamma(c)) \cdot (\iota_D \otimes \mathrm{id}_A)(\delta(d))$$

for $c \in C$, $d \in D$; this determines $\gamma \bowtie_{\mathrm{R}} \delta$ because $\iota_C(C) \cdot \iota_D(D) = C \boxtimes D$. □

**Proposition 4.2.** *The coaction $\gamma \bowtie_{\mathrm{R}} \delta$ on $C \boxtimes_{\mathrm{R}} D$ is natural with respect to equivariant morphisms, that is, it gives a bifunctor $\boxtimes_{\mathrm{R}}\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \times \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \to \mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$. It is the only natural coaction for which $\mathbb{C}$ with the obvious isomorphisms $C \boxtimes \mathbb{C} \cong C$ and $\mathbb{C} \boxtimes D \cong D$ is a tensor unit.*

*Proof.* Two $\mathbb{G}$-equivariant morphisms $f\colon C_1 \to C_2$ and $g\colon D_1 \to D_2$ induce a morphism $f \boxtimes g\colon C_1 \boxtimes D_1 \to C_2 \boxtimes D_2$, which is determined uniquely by the conditions $(f \boxtimes g) \circ \iota_{C_1} = \iota_{C_2} \circ f$ and $(f \boxtimes g) \circ \iota_{D_1} = \iota_{D_2} \circ g$ (see [12, Lemma 5.5]). The coactions $\gamma_1 \bowtie_{\mathrm{R}} \delta_1$ and $\gamma_2 \bowtie_{\mathrm{R}} \delta_2$ satisfy $(\gamma_k \bowtie_{\mathrm{R}} \delta_k) \circ \iota_{C_k} = (\iota_{C_k} \otimes \mathrm{id}_A) \circ \gamma_k$ and $(\gamma_k \bowtie_{\mathrm{R}} \delta_k) \circ \iota_{D_k} = (\iota_{D_k} \otimes \mathrm{id}_A) \circ \delta_k$ for $k = 1, 2$. Thus

$$(\gamma_2 \bowtie_{\mathrm{R}} \delta_2) \circ \iota_{C_2} \circ f = (f \boxtimes g \otimes \mathrm{id}_A) \circ (\gamma_1 \bowtie_{\mathrm{R}} \delta_1) \circ \iota_{C_1}$$

and similarly on $D_1$. So $f \boxtimes g$ is equivariant and $\boxtimes_{\mathrm{R}}$ is a bifunctor as asserted. The obvious isomorphisms $C \boxtimes \mathbb{C} \cong C$ and $\mathbb{C} \boxtimes D \cong D$ are $\mathbb{G}$-equivariant and natural and satisfy the triangle axiom for a tensor unit in a monoidal category; so $\mathbb{C}$ with these isomorphisms is a unit for the tensor product $\boxtimes_{\mathrm{R}}$ on $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$.

Conversely, assume that $\gamma \boxdot \delta$ is a natural $\mathbb{G}$-coaction on $C \boxtimes D$ for which $\mathbb{C}$ with the obvious isomorphisms $C \boxtimes \mathbb{C} \cong C$ and $\mathbb{C} \boxtimes D \cong D$ is a tensor unit. That is, these two isomorphisms are $\mathbb{G}$-equivariant. The unique morphisms $1_C\colon \mathbb{C} \to C$ and $1_D\colon \mathbb{C} \to D$ given by the unit multiplier are equivariant with respect to the trivial $\mathbb{G}$-coaction on $\mathbb{C}$. We have $\iota_C = \mathrm{id}_C \boxtimes 1_D$ and $\iota_D = 1_C \boxtimes \mathrm{id}_D$. Hence $\iota_C$ and $\iota_D$ are equivariant for $\gamma \boxdot \delta$. This forces $\gamma \boxdot \delta = \gamma \bowtie_{\mathrm{R}} \delta$. □

**Theorem 4.3.** *If $C_1, C_2, C_3$ are objects of $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$, then there is a unique isomorphism of triple crossed products $C_1 \boxtimes (C_2 \boxtimes C_3) \cong (C_1 \boxtimes C_2) \boxtimes C_3$, which is also $\mathbb{G}$-equivariant. Thus $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ with the tensor product $\boxtimes_{\mathrm{R}}$ is a monoidal category.*

*Proof.* An isomorphism of triple crossed products is an isomorphism that intertwines the embeddings of $C_1$, $C_2$ and $C_3$. Since the images of these embeddings generate the crossed product, such an isomorphism is unique if it exists.

Let $(C_i, \gamma_i)$ be $\mathbb{G}$-C*-algebras and let $(\varphi_i, \mathrm{U}^{\mathcal{H}_i})$ be faithful covariant representations of $(C_i, \gamma_i)$, respectively, for $i = 1, 2, 3$. The construction of the $\mathbb{G}$-coaction on $C_i \boxtimes C_j$ shows that $(\varphi_i \boxtimes \varphi_j, \mathrm{U}^{\mathcal{H}_i} \mathbin{\text{ⓣ}} \mathrm{U}^{\mathcal{H}_j})$ is a faithful covariant representation of $C_i \boxtimes C_j$ on $\mathcal{H}_i \otimes \mathcal{H}_j$. Therefore, Theorem A.9 gives a faithful representation $\varphi_1 \boxtimes (\varphi_2 \boxtimes \varphi_3)$ of $C_1 \boxtimes (C_2 \boxtimes C_3)$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, which is characterised by:

$$\iota_{C_1}(c_1) \mapsto \varphi_1(c_1) \otimes 1_{\mathcal{H}_2} \otimes 1_{\mathcal{H}_3},$$
$$\iota_{C_2}(c_2) \mapsto ({}^{\mathcal{H}_2 \otimes \mathcal{H}_3}\!\times^{\mathcal{H}_1})\big(\varphi_2(c_2) \otimes 1_{\mathcal{H}_3} \otimes 1_{\mathcal{H}_1}\big)({}^{\mathcal{H}_2 \otimes \mathcal{H}_3}\!\times^{\mathcal{H}_1})^*,$$
$$\iota_{C_3}(c_3) \mapsto ({}^{\mathcal{H}_2 \otimes \mathcal{H}_3}\!\times^{\mathcal{H}_1})({}^{\mathcal{H}_3}\!\times^{\mathcal{H}_2}{}_{12})\big(\varphi_3(c_3) \otimes 1_{\mathcal{H}_2} \otimes 1_{\mathcal{H}_1}\big)({}^{\mathcal{H}_3}\!\times^{\mathcal{H}_2}{}_{12})^*({}^{\mathcal{H}_2 \otimes \mathcal{H}_3}\!\times^{\mathcal{H}_1})^*$$

for $c_i \in C_i$, $i = 1, 2, 3$. The diagrams in Proposition 3.2 and Corollary 3.3 give

$$ {}^{\mathcal{H}_2\otimes\mathcal{H}_3}\!\times^{\mathcal{H}_1} = {}^{\mathcal{H}_2}\!\times^{\mathcal{H}_1}{}_{12} \cdot {}^{\mathcal{H}_3}\!\times^{\mathcal{H}_1}{}_{23}, $$

$$ {}^{\mathcal{H}_2\otimes\mathcal{H}_3}\!\times^{\mathcal{H}_1} \cdot {}^{\mathcal{H}_3}\!\times^{\mathcal{H}_2}{}_{12} = {}^{\mathcal{H}_2}\!\times^{\mathcal{H}_1}{}_{12} \cdot {}^{\mathcal{H}_3}\!\times^{\mathcal{H}_1}{}_{23} \cdot {}^{\mathcal{H}_3}\!\times^{\mathcal{H}_2}{}_{12} = {}^{\mathcal{H}_3}\!\times^{\mathcal{H}_1\otimes\mathcal{H}_2} \cdot {}^{\mathcal{H}_2}\!\times^{\mathcal{H}_1}{}_{23}. $$

Hence the above characterisation of $\varphi_1 \boxtimes (\varphi_2 \boxtimes \varphi_3)$ simplifies to

$$ \begin{aligned} \big(\varphi_1 \boxtimes (\varphi_2 \boxtimes \varphi_3)\big) \circ \iota_{C_1}(c_1) &= \varphi_1(c_1) \otimes 1, \\ \big(\varphi_1 \boxtimes (\varphi_2 \boxtimes \varphi_3)\big) \circ \iota_{C_2}(c_2) &= {}^{\mathcal{H}_2}\!\times^{\mathcal{H}_1}{}_{12}(\varphi_2(c_2) \otimes 1)({}^{\mathcal{H}_2}\!\times^{\mathcal{H}_1}{}_{12})^*, \\ \big(\varphi_1 \boxtimes (\varphi_2 \boxtimes \varphi_3)\big) \circ \iota_{C_3}(c_3) &= ({}^{\mathcal{H}_3}\!\times^{\mathcal{H}_1\otimes\mathcal{H}_2})(\varphi_3(c_3) \otimes 1)({}^{\mathcal{H}_3}\!\times^{\mathcal{H}_1\otimes\mathcal{H}_2})^*. \end{aligned} \tag{4.2} $$

Similarly, we get a faithful representation $(\varphi_1 \boxtimes \varphi_2) \boxtimes \varphi_3$ of $(C_1 \boxtimes C_2) \boxtimes C_3$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. A computation as above shows that the combinations of braiding unitaries in it are equal to those in (4.2). Thus $\varphi_1 \boxtimes (\varphi_2 \boxtimes \varphi_3)$ and $(\varphi_1 \boxtimes \varphi_2) \boxtimes \varphi_3$ are the same representation on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. This gives an isomorphism $C_1 \boxtimes (C_2 \boxtimes C_3) \cong (C_1 \boxtimes C_2) \boxtimes C_3$ that intertwines the canonical embeddings of $C_1$, $C_2$ and $C_3$. It is $\mathbb{G}$-equivariant because our $\mathbb{G}$-coactions are uniquely determined by their actions on the tensor factors $C_1$, $C_2$ and $C_3$.

The natural isomorphisms above provide the associators needed for a monoidal category. The pentagon condition for these associators and the compatibility with the unit transformations $\mathbb{C} \boxtimes D \cong D$, $C \boxtimes \mathbb{C} \cong C$ follow by checking the relevant commuting diagram on each tensor factor separately. □

**Proposition 4.4.** *The monoidal structure $\boxtimes_{\mathrm{R}}$ is braided monoidal if and only if it is symmetric monoidal, if and only if $\mathrm{R}$ is antisymmetric. In that case, the braiding is the unique isomorphism of crossed products $(C \boxtimes_{\mathrm{R}} D, \iota_C, \iota_D) \cong (D \boxtimes_{\mathrm{R}} C, \iota_C, \iota_D)$.*

*Proof.* [12, Proposition 5.1] shows that $D \boxtimes_{\mathrm{R}} C \cong C \boxtimes_{\hat{\mathrm{R}}} D$ as crossed products, where $\hat{\mathrm{R}} := \sigma(\mathrm{R}^*)$. If $\mathrm{R}$ is antisymmetric, this gives an isomorphism of crossed products between $(C \boxtimes_{\mathrm{R}} D, \iota_C, \iota_D)$ and $(D \boxtimes_{\mathrm{R}} C, \iota_C, \iota_D)$. This isomorphism is equivariant because our coactions are determined by what they do on the embedded copies of $C$ and $D$. Thus we get a braided monoidal category in this case.

Conversely, any braiding must give the identity map on $C \boxtimes_{\mathrm{R}} \mathbb{C}$ and $\mathbb{C} \boxtimes_{\mathrm{R}} D$ because $\mathbb{C}$ is the tensor unit. Since $\iota_C = \mathrm{id}_C \boxtimes 1_D$ and $\iota_D = 1_C \boxtimes \mathrm{id}_D$, any braiding must be an isomorphism of crossed products $C \boxtimes_{\mathrm{R}} D \cong D \boxtimes_{\mathrm{R}} C$. By the argument above, this happens if and only if $C \boxtimes_{\mathrm{R}} D = C \boxtimes_{\hat{\mathrm{R}}} D$ as crossed products. Corollary A.13 says that the crossed product $A \boxtimes_{\mathrm{R}} A$ determines $\mathrm{R}$ uniquely. Hence $\mathrm{R} = \hat{\mathrm{R}}$. □

## 5. Quantum codoubles

Quantum codoubles of compact quantum groups were introduced by Podleś and Woronowicz in [14] to construct an example of a quantum Lorentz group. The definition was extended to the non-compact case in [21]. Quantum codoubles were also described by Baaj and Vaes in [2, Proposition 9.5], assuming the underlying quantum group to be generated by a regular multiplicative unitary. We call the dual of the quantum codouble Drinfeld double. Some authors use different notation, exchanging doubles and codoubles.

We shall refer to [17] for the definition of the quantum codouble $\mathfrak{D}(\mathbb{G})\widehat{\ }$ and the Drinfeld double $\mathfrak{D}(\mathbb{G})$ of a $\mathrm{C}^*$-quantum group $\mathbb{G} = (A, \Delta_A)$. It is shown in [17] that $\mathfrak{D}(\mathbb{G})\widehat{\ }$ and $\mathfrak{D}(\mathbb{G})$ are again $\mathrm{C}^*$-quantum groups and that $\mathfrak{D}(\mathbb{G})\widehat{\ }$-coactions on $\mathrm{C}^*$-algebras are equivalent to $\mathbb{G}$-Yetter–Drinfeld $\mathrm{C}^*$-algebras. To simplify notation, we shall only consider the special case of the results in [17] where $B = \hat{A}$ and the bicharacter $\mathrm{V}$ is $\mathrm{W} \in \mathcal{U}(\hat{A} \otimes A)$ because that is all we need below.

The main result in this section is that this equivalence of categories between $\mathfrak{D}(\mathbb{G})\widehat{\ }$-coactions and $\mathbb{G}$-Yetter–Drinfeld $\mathrm{C}^*$-algebras turns the tensor products $\boxtimes_{\mathrm{R}}$ for $\mathfrak{D}(\mathbb{G})\widehat{\ }$-$\mathrm{C}^*$-algebras and a canonical R-matrix for $\mathfrak{D}(\mathbb{G})\widehat{\ }$ into the tensor product $\boxtimes_{\mathrm{W}}$ for $\mathbb{G}$-Yetter–Drinfeld algebras. We also show that the tensor product $\boxtimes_{\mathrm{R}}$ for a general quasitriangular quantum group is a special case of the same operation for its codouble (see Theorem 5.7).

The *quantum codouble* $\mathfrak{D}(\mathbb{G})\widehat{\ } = (\hat{\mathcal{D}}, \Delta_{\hat{\mathcal{D}}})$ of $\mathbb{G}$ is defined by $\hat{\mathcal{D}} := A \otimes \hat{A}$ and

$$\begin{aligned} \sigma^{\mathrm{W}}\colon A \otimes \hat{A} &\to \hat{A} \otimes A, & a \otimes \hat{a} &\mapsto \mathrm{W}(\hat{a} \otimes a)\mathrm{W}^*, \\ \Delta_{\hat{\mathcal{D}}}\colon \hat{\mathcal{D}} &\to \hat{\mathcal{D}} \otimes \hat{\mathcal{D}}, & a \otimes \hat{a} &\mapsto \sigma^{\mathrm{W}}_{23}(\Delta_A(a) \otimes \hat{\Delta}_A(\hat{a})), \end{aligned}$$

for $a \in A$, $\hat{a} \in \hat{A}$. We may generate $\mathfrak{D}(\mathbb{G})\widehat{\ }$ by a manageable multiplicative unitary by [17, Theorem 4.1]. So it is a $\mathrm{C}^*$-quantum group and has a dual $\mathfrak{D}(\mathbb{G}) = (\mathcal{D}, \Delta_{\mathcal{D}})$, which is called the *Drinfeld double* of $\mathbb{G}$. We have

$$\mathcal{D} = \rho(A) \cdot \theta(\hat{A}) \quad \text{and} \quad \Delta_{\mathcal{D}}(\rho(a) \cdot \theta(\hat{a})) = (\rho \otimes \rho)\Delta_A(a) \cdot (\theta \otimes \theta)\hat{\Delta}_A(\hat{a})$$

for a certain pair of representations $\rho$ and $\theta$ of $A$ and $\hat{A}$ on the same Hilbert space. The formulas for $\rho$ and $\theta$ will not be needed in the following.

It is crucial that $\rho$ and $\theta$ give Hopf *-homomorphisms from $\mathbb{G}$ and $\widehat{\mathbb{G}}$ to $\mathfrak{D}(\mathbb{G})$. These induce dual morphisms $\mathfrak{D}(\mathbb{G})\widehat{\ } \to \mathbb{G}$ and $\mathfrak{D}(\mathbb{G})\widehat{\ } \to \widehat{\mathbb{G}}$ (compare Theorem A.8). These quantum group morphisms induce a map on corepresentations (see [11, Proposition 6.5] and the proof of [12, Theorem 5.2] for a correct general proof). Thus a corepresentation of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ induces corepresentations of $\mathbb{G}$ and $\widehat{\mathbb{G}}$ on the same Hilbert space. It is shown in [17] that this gives a bijection between corepresentations of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ and certain pairs of corepresentations of $\mathbb{G}$ and $\widehat{\mathbb{G}}$:

**Proposition 5.1** ([17, Proposition 6.11]). *Let $\mathcal{K}$ be a Hilbert space. The corepresentations $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes A)$ and $\mathrm{V} \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \hat{A})$ of $\mathbb{G}$ and $\widehat{\mathbb{G}}$ associated to a corepresentation $\mathrm{X} \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \widehat{\mathcal{D}})$ of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ satisfy*

$$\sigma^{\mathrm{W}}_{23}(\mathrm{U}_{12}\mathrm{V}_{13}) = \mathrm{V}_{12}\mathrm{U}_{13} \qquad \textit{in } \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes \hat{A} \otimes A);$$

*we call a pair* $(\mathrm{U},\mathrm{V})$ *with this property* $\mathfrak{D}(\mathbb{G})\hat{}$-compatible. *The map* $\mathrm{X} \mapsto (\mathrm{U},\mathrm{V})$ *above is a bijection from corepresentations of* $\mathfrak{D}(\mathbb{G})\hat{}$ *to* $\mathfrak{D}(\mathbb{G})\hat{}$*-compatible pairs of corepresentations of* $\mathbb{G}$ *and* $\widehat{\mathbb{G}}$*, with inverse*

$$\mathrm{X} := \mathrm{U}_{12}\mathrm{V}_{13} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes A \otimes \hat{A}).$$

A quantum group morphism also induces a functor between the coaction categories, see Theorem A.8. Thus a continuous coaction of $\mathfrak{D}(\mathbb{G})\hat{}$ on a $\mathrm{C}^*$-algebra $C$ induces coactions of $\mathbb{G}$ and $\widehat{\mathbb{G}}$. Once again, this gives a bijection from coactions of $\mathfrak{D}(\mathbb{G})\hat{}$ to certain pairs of coactions of $\mathbb{G}$ and $\widehat{\mathbb{G}}$:

**Proposition 5.2.** *Let* $C$ *be a* $\mathrm{C}^*$*-algebra. The continuous coactions* $\gamma$ *and* $\delta$ *of* $\mathbb{G}$ *and* $\widehat{\mathbb{G}}$ *associated to a continuous coaction* $\xi$ *of* $\mathfrak{D}(\mathbb{G})\hat{}$ *satisfy*

$$\sigma^{\mathrm{W}}_{23}\big((\gamma \otimes \mathrm{id}_{\hat{A}})\delta\big) = (\delta \otimes \mathrm{id}_A)\gamma; \tag{5.1}$$

*A* $\mathrm{C}^*$*-algebra with such a pair of coactions is called a* $\mathbb{G}$-Yetter–Drinfeld $\mathrm{C}^*$-algebra*.*

*The map* $\xi \mapsto (\gamma,\delta)$ *above is a bijection from continuous coactions of* $\mathfrak{D}(\mathbb{G})\hat{}$ *to the set of pairs of continuous coactions* $(\gamma,\delta)$ *satisfying* (5.1)*; the inverse maps* $(\gamma,\delta)$ *to* $\xi = (\gamma \otimes \mathrm{id}_{\hat{A}})\delta$*.*

Yetter–Drinfeld $\mathrm{C}^*$-algebras were defined by Nest and Voigt in [13, Definition 3.1] (assuming Haar weights on $\mathbb{G}$), and Proposition 5.2 is essentially [13, Proposition 3.2]. For $\mathrm{C}^*$-quantum groups without Haar weights, Proposition 5.2 is [17, Proposition 6.8], with an explicit description of the bijection taken from the proof of [17, Proposition 6.8].

Let $\mathcal{YD}\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ denote the category with $\mathbb{G}$-Yetter–Drinfeld $\mathrm{C}^*$-algebras as objects and morphisms that are both $\mathbb{G}$- and $\widehat{\mathbb{G}}$-equivariant as arrows.

The following unitary is an R-matrix for $\mathfrak{D}(\mathbb{G})\hat{}$ by [17, Lemma 5.11]:

$$\mathrm{R} = (\theta \otimes \rho)\mathrm{W} \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D}).$$

Thus $\mathfrak{D}(\mathbb{G})\hat{}$ is quasitriangular and the construction in the previous section gives a monoidal structure $\boxtimes_{\mathrm{R}}$ on $\mathfrak{C}^*\mathfrak{alg}(\mathfrak{D}(\mathbb{G})\hat{})$. What happens when we translate this to the equivalent setting of $\mathbb{G}$-Yetter–Drinfeld $\mathrm{C}^*$-algebras?

The reduced bicharacter $\mathrm{W} \in \mathcal{U}(\hat{A} \otimes A)$, being a bicharacter, gives a tensor product $\boxtimes_{\mathrm{W}}$ for two $\mathbb{G}$-Yetter–Drinfeld $\mathrm{C}^*$-algebras. This tensor product is also used by Nest and Voigt in [13] (they require, however, that $\mathbb{G}$ has Haar weights).

**Theorem 5.3.** *Let* $C_1$ *and* $C_2$ *be* $\mathfrak{D}(\mathbb{G})\hat{}$-$\mathrm{C}^*$*-algebras, view them also as* $\mathbb{G}$*-Yetter–Drinfeld* $\mathrm{C}^*$*-algebras. There is an equivariant isomorphism of crossed products* $C_1 \boxtimes_{\mathrm{R}} C_2 \cong C_1 \boxtimes_{\mathrm{W}} C_2$.

*Proof.* First we describe the braiding on $\mathfrak{Corep}(\mathfrak{D}(\mathbb{G})\hat{})$ induced by R in terms of W and compatible pairs of corepresentations of $A$ and $\hat{A}$.

Recall that the maps $\rho\colon A \to \mathcal{D}$ and $\theta\colon \hat{A} \to \mathcal{D}$ are Hopf *-homomorphisms. Thus they lift to the universal quantum groups: $\rho^{\mathrm{u}}\colon A^{\mathrm{u}} \to \mathcal{D}^{\mathrm{u}}$ and $\theta^{\mathrm{u}}\colon \hat{A}^{\mathrm{u}} \to \mathcal{D}^{\mathrm{u}}$. Let $\mathrm{W}^{\mathrm{u}} \in \mathcal{U}(\hat{A}^{\mathrm{u}} \otimes A^{\mathrm{u}})$ be the universal lift of W. The unitary $(\theta^{\mathrm{u}} \otimes \rho^{\mathrm{u}})(\mathrm{W}^{\mathrm{u}}) \in \mathcal{U}(\mathcal{D}^{\mathrm{u}} \otimes \mathcal{D}^{\mathrm{u}})$ is a bicharacter and lifts R. Hence it is *the* universal lift $\mathrm{R}^{\mathrm{u}}$ of R.

A corepresentation of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ is equivalent to a representation of $\mathcal{D}^{\mathrm{u}}$. Composing this with the morphisms $\theta^{\mathrm{u}}$ and $\rho^{\mathrm{u}}$ gives representations $\hat{\pi}$ and $\pi$ of $\hat{A}^{\mathrm{u}}$ and $A^{\mathrm{u}}$. These are, in turn, equivalent to corepresentations U and V of $A$ and $\hat{A}$. The construction of (U, V) is exactly the bijection to $\mathfrak{D}(\mathbb{G})\widehat{\ }$-compatible pairs of corepresentations in Proposition 5.1.

Now take two corepresentations of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ on Hilbert spaces $\mathcal{H}_k$. These correspond to representations $\Pi_k$ of $\mathcal{D}^{\mathrm{u}}$, which determine representations $\hat{\pi}_k = \Pi_k \circ \theta^{\mathrm{u}}$ and $\pi_k = \Pi_k \circ \rho^{\mathrm{u}}$ of $\hat{A}^{\mathrm{u}}$ and $A^{\mathrm{u}}$ on $\mathcal{H}_k$ for $k = 1, 2$. The braiding unitary ${}^{\mathcal{H}_1}\!\times^{\mathcal{H}_2}$ is given by (3.1) and (3.2) and involves the unitary

$$(\Pi_1 \otimes \Pi_2)(\mathrm{R}^{\mathrm{u}})^* = (\Pi_1\theta^{\mathrm{u}} \otimes \Pi_2\rho^{\mathrm{u}})(\mathrm{W}^{\mathrm{u}})^* = (\hat{\pi}_1 \otimes \pi_2)(\mathrm{W}^{\mathrm{u}})^*.$$

Let $(C_i, \lambda_i)$ be $\mathfrak{D}(\mathbb{G})\widehat{\ }$-C*-algebras. Proposition 5.2 gives a unique pair of coactions $\gamma_i\colon C_i \to C_i \otimes A$ and $\delta_i\colon C_i \to C_i \otimes \hat{A}$ such that $(C_i, \gamma_i, \delta_i)$ is a $\mathbb{G}$-Yetter-Drinfeld C*-algebra and $\lambda_i = (\gamma_i \otimes \mathrm{id}_{\hat{A}}) \circ \delta_i$ for $i = 1, 2$. There are faithful covariant representations $(\mathrm{X}^{\mathcal{H}_i}, \varphi_i)$ of $(C_i, \lambda_i, \widehat{\mathcal{D}})$ on Hilbert spaces $\mathcal{H}_i$ for $i = 1, 2$. We use these faithful covariant representations to define $C_1 \boxtimes_{\mathrm{R}} C_2$ as a C*-subalgebra of $\mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, see Theorem A.9.

Proposition 5.1 turns $\mathrm{X}^{\mathcal{H}_i}$ into a $\mathfrak{D}(\mathbb{G})\widehat{\ }$-compatible pair of corepresentations $(\mathrm{U}^{\mathcal{H}_i}, \mathrm{V}^{\mathcal{H}_i})$. The maps on corepresentations and coactions induced by a quantum group morphism preserve covariance of representations. Hence $(\varphi_i, \mathrm{U}^{\mathcal{H}_i})$ is a covariant representation of $(C_i, \gamma_i, A)$ on $\mathcal{H}_i$ and $(\varphi_i, \mathrm{V}^{\mathcal{H}_i})$ is a covariant representation of $(C_i, \delta_i, \hat{A})$ on $\mathcal{H}_i$ for $i = 1, 2$, respectively. We use these faithful covariant representations to define $C_1 \boxtimes_{\mathrm{W}} C_2$ as a C*-subalgebra of $\mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, see Theorem A.9.

The representation of $C_1 \boxtimes_{\chi} C_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ comes from the representations

$$C_1 \ni c_1 \mapsto \varphi_1(c_1) \otimes 1, \qquad C_2 \ni c_2 \mapsto Z_{\chi}(1 \otimes \varphi_2(c_2))Z_{\chi}^*,$$

where $Z_{\mathrm{R}} = (\Pi_1 \otimes \Pi_2)(\mathrm{R}^{\mathrm{u}})^*$ and $Z_{\mathrm{W}} = (\hat{\pi}_1 \otimes \pi_2)(\mathrm{W}^{\mathrm{u}})^*$. The computation above shows that $Z_{\mathrm{W}} = Z_{\mathrm{R}}$, so $C_1 \boxtimes_{\mathrm{R}} C_2 = C_1 \boxtimes_{\mathrm{W}} C_2$ as C*-subalgebras of $\mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. □

Since $\mathbb{G}$-Yetter–Drinfeld C*-algebras are equivalent to $\mathfrak{D}(\mathbb{G})\widehat{\ }$-C*-algebras, the isomorphism in Theorem 5.3 shows that $C_1 \boxtimes_{\mathrm{W}} C_2$ for two $\mathbb{G}$-Yetter–Drinfeld C*-algebras $C_1$ and $C_2$ carries a unique $\mathbb{G}$-Yetter–Drinfeld C*-algebra structure for which the embeddings of $C_1$ and $C_2$ are equivariant. This extra structure is natural, and $(\mathcal{YD}\mathfrak{C}^*\mathfrak{alg}(\mathbb{G}), \boxtimes_{\mathrm{W}})$ is a monoidal category. By construction, the equivalence between $\mathcal{YD}\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ and $\mathfrak{C}^*\mathfrak{alg}(\mathfrak{D}(\mathbb{G})\widehat{\ })$ is an equivalence of monoidal categories between $(\mathcal{YD}\mathfrak{C}^*\mathfrak{alg}(\mathbb{G}), \boxtimes_{\mathrm{W}})$ and $(\mathfrak{C}^*\mathfrak{alg}(\mathfrak{D}(\mathbb{G})\widehat{\ }), \boxtimes_{\mathrm{R}})$.

**Remark 5.4.** Propositions 3.7 and 4.4 show that $\boxtimes_{\mathrm{R}}$ admits a braiding if and only if it is symmetric, if and only if the braiding on $\mathfrak{Corep}(\mathfrak{D}(\mathbb{G})\widehat{\ })$ associated to R is symmetric if and only if R is antisymmetric, that is, $\mathrm{R}^* = \sigma(\mathrm{R})$. For the codouble, this is equivalent to $\mathrm{W} = \widehat{\mathrm{W}}$. We know no non-trivial multiplicative unitary with this property. Since $\mathrm{W} = \widehat{\mathrm{W}}$ implies that $A$ and $\hat{A}$ are the same C*-algebra, such a multiplicative unitary cannot be regular.

**Example 5.5.** The tensor product $A \boxtimes_{\mathrm{W}} \hat{A}$ is the *canonical Heisenberg double* of a C*-quantum group $\mathbb{G}$, in the sense that its representations are the Heisenberg pairs of $\mathbb{G}$ (see Proposition A.12). Theorem 5.3 says that $A \boxtimes_{\mathrm{W}} \hat{A} \cong A \boxtimes_{\mathrm{R}} \hat{A}$; this is a C*-algebraic version of [4, Proposition 5.1].

It is already shown in [13, Section 3] that $\mathcal{YDC}^*\mathfrak{alg}(\mathbb{G})$ is a monoidal category for the tensor product $\boxtimes_{\mathrm{W}}$ if $\mathbb{G}$ has Haar weights.

Now let $\mathbb{G}$ be a quasitriangular quantum group with R-matrix $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$. We view R as a quantum group morphism from $A$ to $\hat{A}$. Theorem A.8 explains how R gives an induced coaction $\delta\colon C \to C \otimes \hat{A}$ on any $\mathbb{G}$-C*-algebra $(C, \gamma)$.

**Lemma 5.6.** *Any pair $(\gamma, \delta)$ as above is $\mathbb{G}$-Yetter–Drinfeld.*

*Proof.* Any object $C \in \mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ is equivariantly isomorphic to a subobject of $D \otimes A$ with coaction $\mathrm{id}_D \otimes \Delta_A$ for some C*-algebra $D$ by [12, Lemma 2.9]. Since the tensor factor $D$ causes no problems, it suffices to prove the lemma for $(C, \gamma) = (A, \Delta_A)$; here $\delta = \Delta_R$ is the right coaction characterised by (A.16) for $\chi = \mathrm{R}$.

The relation (2.8) has to be modified for $\mathrm{R} \in \mathcal{U}(\hat{A} \otimes \hat{A})$ by taking the dual multiplicative unitaries because we use $\hat{A}$ instead of $A$. This gives

$$\mathrm{R}_{12}\mathrm{W}_{13}\mathrm{W}_{23} = \mathrm{W}_{23}\mathrm{W}_{13}\mathrm{R}_{12} \qquad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes A). \tag{5.2}$$

Equations (A.16) for R, (A.4) and (5.2) give

$$\begin{aligned}
\sigma^{\mathrm{W}}_{34}\big((\mathrm{id}_{\hat{A}} \otimes (\Delta_A \otimes \mathrm{id}_{\hat{A}})\Delta_R)\mathrm{W}\big) &= \mathrm{W}_{12}\sigma^{\mathrm{W}}_{34}(\mathrm{W}_{13}\mathrm{R}_{14}) \\
&= \mathrm{W}_{12}\mathrm{W}_{34}\mathrm{W}_{14}\mathrm{R}_{13}\mathrm{W}^*_{34} \\
&= \mathrm{W}_{12}\mathrm{R}_{13}\mathrm{W}_{14} = (\mathrm{id}_{\hat{A}} \otimes (\Delta_R \otimes \mathrm{id}_A)\Delta_A)\mathrm{W}.
\end{aligned}$$

Finally, we slice the first leg of this equation with $\omega \in \hat{A}'$. This gives (5.1) for the pair $(\Delta_A, \Delta_R)$ because slices of W generate $A$. □

Since a morphism between two $\mathbb{G}$-C*-algebras is $\mathbb{G}$-equivariant if and only if it is $\mathbb{G}$- and $\widehat{\mathbb{G}}$-equivariant, Lemma 5.6 gives a fully faithful embedding of $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ into $\mathcal{YDC}^*\mathfrak{alg}(\mathbb{G}) \cong \mathfrak{C}^*\mathfrak{alg}(\mathfrak{D}(\mathbb{G})\widehat{\ })$ that leaves the underlying C*-algebras unchanged. Thus an R-matrix for $\mathbb{G}$ induces a quantum group morphism $\mathbb{G} \to \mathfrak{D}(\mathbb{G})\widehat{\ }$.

**Theorem 5.7.** *The embedding $(\mathfrak{C}^*\mathfrak{alg}(\mathbb{G}), \boxtimes_{\mathrm{R}}) \to (\mathcal{YDC}^*\mathfrak{alg}(\mathbb{G}), \boxtimes_{\mathrm{W}})$ is monoidal.*

*Proof.* Let $C$ and $D$ be two $\mathbb{G}$-C*-algebras, equip them with the $\mathbb{G}$-Yetter–Drinfeld structure described above. As a C*-algebra, we have an isomorphism of crossed products $C \boxtimes_{\mathrm{R}} D \cong C \boxtimes_{\mathrm{W}} D$ by [12, Example 5.4]. The induced $\mathbb{G}$-Yetter–Drinfeld algebra structure on $C \boxtimes_{\mathrm{W}} D$ is the unique one for which the embeddings of $C$ and $D$ are equivariant: combine Proposition 4.1 with the equivalence of $\mathbb{G}$-Yetter–Drinfeld algebra structures and $\mathfrak{D}(\mathbb{G})\widehat{\ }$-coactions. Similarly, the induced $A$-coaction on $C \boxtimes_{\mathrm{R}} D$ is the unique one for which the embeddings of $C$ and $D$ are $A$-equivariant. Since the $\hat{A}$-coactions constructed from R and an $A$-coaction are natural, the embeddings of $C$ and $D$ into $C \boxtimes_{\mathrm{R}} D$ are also $\hat{A}$-equivariant. Hence the $\mathbb{G}$-Yetter–Drinfeld algebra structures on $C \boxtimes_{\mathrm{R}} D$ and $C \boxtimes_{\mathrm{W}} D$ are the same as well. Since the isomorphism between these tensor products is one of crossed products, it automatically satisfies the coherence conditions required for a monoidal functor. □

## 6. Braided $C^*$-bialgebras and braided compact quantum groups

We are going to define braided C*-bialgebras and use them to construct ordinary C*-bialgebras by a semidirect product construction, which is the C*-analogue of what Majid calls "bosonisation" in [10]. We check that the semidirect product C*-bialgebra is bisimplifiable if and only if the braided C*-bialgebra is bisimplifiable. Thus we may construct compact quantum groups from two pieces: an ordinary compact quantum group and a braided quantum group over its codouble.

**Definition 6.1.** A *braided* C*-*bialgebra* over a quasitriangular quantum group $\mathbb{G} = (A, \Delta_A, \mathrm{R})$ is a $\mathbb{G}$-C*-algebra $(B, \beta)$ with a $\mathbb{G}$-equivariant morphism $\Delta_B \colon B \to B \boxtimes_{\mathrm{R}} B$ which is coassociative:

$$(\Delta_B \boxtimes_{\mathrm{R}} \mathrm{id}_B) \circ \Delta_B = (\mathrm{id}_B \boxtimes_{\mathrm{R}} \Delta_B) \circ \Delta_B. \tag{6.1}$$

We call $(B, \Delta_B)$ *bisimplifiable* if it satisfies the braided Podleś conditions

$$\Delta_B(B) \cdot \iota_1(B) = B \boxtimes_{\mathrm{R}} B = \Delta_B(B) \cdot \iota_2(B), \tag{6.2}$$

where $\iota_1$ and $\iota_2$ denote the two canonical maps $B \rightrightarrows B \boxtimes_{\mathrm{R}} B$.

A *braided compact quantum group* over $\mathbb{G}$ is a unital, bisimplifiable braided C*-bialgebra $(B, \Delta_B)$ over $\mathbb{G}$.

In the following, we let $\mathbb{G} = (A, \Delta_A)$ be *any* C*-quantum group, and we let $(B, \Delta_B)$ be a braided C*-bialgebra over the codouble $\mathfrak{D}(\mathbb{G})\widehat{\ }$ with its canonical R-matrix. Equivalently, $B$ is a C*-bialgebra in the category of $\mathbb{G}$-Yetter–Drinfeld C*-algebras (see Theorem 5.3). Thus we do not assume $\mathbb{G}$ to be quasitriangular any more. Since we may embed the coaction category of a quasitriangular C*-quantum group into the one for its codouble by Theorem 5.7, our new setting is more general than the one in Definition 6.1.

The monoidal structure on $\mathbb{G}$-Yetter–Drinfeld algebras is given by the tensor product $C \boxtimes_{\mathrm{W}} D$ for the bicharacter W by Theorem 5.3. So the underlying C*-algebra only uses the coaction of $A$ on $C$ and the coaction of $\hat{A}$ on $D$. Both coactions are used to equip $C \boxtimes_{\mathrm{W}} D$ with a Yetter–Drinfeld algebra structure, which we need to form tensor products of more than two factors. We abbreviate $\boxtimes = \boxtimes_{\mathrm{W}}$.

The C*-algebra $A$ carries the canonical continuous coaction $\Delta_A$ of $A$ and a canonical coaction of $\hat{A}$ by $\mathrm{Ad}(\widehat{\mathrm{W}})\colon a \mapsto \widehat{\mathrm{W}}(a \otimes 1_{\hat{A}})\widehat{\mathrm{W}}^*$. These two coactions satisfy the Yetter–Drinfeld compatibility condition. The Podleś condition for the $\hat{A}$-coaction on $A$ is not automatic, however: it is a weak form of regularity. Since we do not want to impose any regularity condition on $\mathbb{G}$, we make sure that we do not use the coaction of $\hat{A}$ on $A$ in the following constructions. The $A$-coaction $\Delta_A$ on $A$ is enough to define the twisted tensor products $A \boxtimes B$ and $A \boxtimes (B \boxtimes B)$.

**Lemma 6.2.** *There are unique coactions of $A$ and $\hat{A}$ on $A \boxtimes B$ and $A \boxtimes (B \boxtimes B')$ for which the canonical embeddings of $A$, $B$ and $B'$ are equivariant; the $A$-coactions are continuous, the $\hat{A}$-coactions are injective, but do not necessarily satisfy the Podleś condition. The coactions of $A$ and $\hat{A}$ are compatible. There is a canonical isomorphism of triple crossed products $A \boxtimes (B \boxtimes B') \cong (A \boxtimes B) \boxtimes B'$, which is equivariant for the coactions of $A$ and $\hat{A}$.*

*Proof.* If the $\hat{A}$-coaction on $A$ were continuous, our previous theory for coactions of the codouble of $\mathbb{G}$ would give all the statements immediately.

Let $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a manageable multiplicative unitary generating $(A, \Delta_A)$. Let $\pi\colon A \to \mathbb{B}(\mathcal{H})$ and $\hat{\pi}\colon \hat{A} \to \mathbb{B}(\mathcal{H})$ be the resulting representations. The unitaries $(\hat{\pi} \otimes \mathrm{id}_A)(\mathrm{W}) \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$ and $(\pi \otimes \mathrm{id}_{\hat{A}})(\widehat{\mathrm{W}}) \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes \hat{A})$ are corepresentations because $\mathrm{W} \in \mathcal{U}(\hat{A} \otimes A)$ and $\widehat{\mathrm{W}} \in \mathcal{U}(A \otimes \hat{A})$ are bicharacters. These two corepresentations together with $\pi$ form a faithful covariant representation of $(A, \Delta_A, \mathrm{Ad}(\widehat{\mathrm{W}}))$. Moreover, the corepresentations $(\hat{\pi} \otimes \mathrm{id}_A)(\mathrm{W})$ and $(\pi \otimes \mathrm{id}_{\hat{A}})(\widehat{\mathrm{W}})$ satisfy the Yetter–Drinfeld compatibility condition, so they give a corepresentation of the codouble $\mathfrak{D}(\mathbb{G})\widehat{\ }$.

Let $\beta\colon B \to B \otimes A$ and $\hat{\beta}\colon B \to B \otimes \hat{A}$ denote the coactions of $A$ and $\hat{A}$ on $B$ that give the Yetter–Drinfeld algebra structure on $B$. We may choose a faithful covariant representation $(\rho, U, V)$ of $(B, \beta, \hat{\beta})$ on some Hilbert space $\mathcal{K}$. Thus $U$ and $V$ satisfy the Yetter–Drinfeld compatibility condition, so they give a corepresentation of the codouble $\mathfrak{D}(\mathbb{G})\widehat{\ }$.

Now we represent $A \boxtimes B$ faithfully on $\mathcal{H} \otimes \mathcal{K}$. This gives a C*-algebra even if $\mathrm{Ad}(\widehat{\mathrm{W}})$ is not continuous because the construction of $A \boxtimes B = A \boxtimes_{\mathrm{W}} B$ only uses the $A$-coaction $\Delta_A$ on $A$ and the $\hat{A}$-coaction $\hat{\beta}$ on $B$. The codouble of $\mathbb{G}$ acts on $\mathcal{H} \otimes \mathcal{K}$ by the usual tensor product corepresentation. As in the proof of Proposition 4.1, we get a unique coaction of $\mathfrak{D}(\mathbb{G})\widehat{\ }$ on $A \boxtimes B$ for which its representation on $\mathcal{H} \otimes \mathcal{K}$ and the embeddings of $A$ and $B$ are equivariant. Only the proof of the Podleś condition breaks down because we do not know the Podleś condition for the $\mathfrak{D}(\mathbb{G})\widehat{\ }$-coaction on $A$. We may, however, split the $\mathfrak{D}(\mathbb{G})\widehat{\ }$-coaction into compatible coactions of $A$

and $\hat{A}$, and prove the Podleś condition for the coaction of $A$, just as in the proof of Proposition 4.1, using only the $A$-equivariance of the embeddings of $A$ and $B$ into $A \boxtimes B$ and the Podleś conditions for the $A$-coactions on $A$ and $B$.

Similarly, the construction of the associator $(A \boxtimes B) \boxtimes B' \cong A \boxtimes (B \boxtimes B')$ in the proof of Theorem 4.3 still works, using the covariant representation of $A \boxtimes B$ just constructed, and gives the remaining statements. □

Our goal is to construct a coassociative comultiplication on $C := A \boxtimes B$ from a braided comultiplication $\Delta_B\colon B \to B \boxtimes B$. The first ingredient is the morphism

$$\mathrm{id}_A \boxtimes \Delta_B\colon A \boxtimes B \to A \boxtimes (B \boxtimes B),$$

which is the unique one with $(\mathrm{id}_A \boxtimes \Delta_B) \circ \iota_A = \iota_A$ and $(\mathrm{id}_A \boxtimes \Delta_B) \circ \iota_B = \iota_{B\boxtimes B} \circ \Delta_B$. Next we construct a canonical map

$$\Psi\colon A \boxtimes B \boxtimes B \to (A \boxtimes B) \otimes (A \boxtimes B).$$

Under regularity assumptions on $\mathbb{G}$, we could construct this by composing the canonical morphism

$$j_{124}\colon A \boxtimes B \boxtimes B \to A \boxtimes B \boxtimes A \boxtimes B$$

with an isomorphism $A \boxtimes B \boxtimes A \boxtimes B \cong (A \boxtimes B) \otimes (A \boxtimes B)$, which exists because the $\hat{A}$-coaction on $A$ is inner (see [12, Corollary 5.16]). The following proposition constructs $\Psi$ directly without regularity assumptions on $\mathbb{G}$:

**Proposition 6.3.** *Let $B$ and $B'$ be $\mathbb{G}$-Yetter–Drinfeld algebras, let $\beta\colon B \to B \otimes A$ be the $A$-coaction. There is a unique injective morphism*

$$\Psi\colon A \boxtimes B \boxtimes B' \to (A \boxtimes B) \otimes (A \boxtimes B')$$

*that satisfies, for $a \in A$, $b \in B$, $b' \in B'$,*

$$\begin{aligned} \Psi \circ \iota_A(a) &= (\iota_A \otimes \iota_A) \circ \Delta_A(a), \\ \Psi \circ \iota_B(b) &= (\iota_B \otimes \iota_A) \circ \beta(b), \\ \Psi \circ \iota_{B'}(b') &= 1_{A\boxtimes B} \otimes \iota_{B'}(b'). \end{aligned} \tag{6.3}$$

Before we prove this technical result, we state our main result and give a simple example. Another example is the construction of quantum U(2) groups from braided quantum SU(2) groups in [6] (the conventions in [6] are, however, slightly different).

**Theorem 6.4.** *Let $C := A \boxtimes B$ and $\Delta_C := \Psi \circ (\mathrm{id}_A \boxtimes \Delta_B)\colon C \to C \otimes C$. Then $(C, \Delta_C)$ is a bisimplifiable* C*-*bialgebra whenever $(B, \Delta_B)$ is a bisimplifiable braided* C*-*bialgebra over $\mathbb{G}$. If $\Delta_B$ is injective, then so is $\Delta_C$, and vice versa.*

**Corollary 6.5.** *$(C, \Delta_C)$ is a compact quantum group if $\mathbb{G}$ is a compact quantum group and $(B, \Delta_B)$ is a braided compact quantum group over $\mathbb{G}$.*

*Proof.* The $C^*$-algebra $C$ is unital if and only if $A$ and $B$ are both unital. In the unital (compact) case, the Podleś conditions suffice to characterise compact quantum groups. □

**Example 6.6.** The following example is inspired by the construction of partial duals in [3] in the setting of Hopf algebras. Let $K$ be a compact group, let $\Gamma$ be a discrete group, and let $\varphi\colon \Gamma \to \mathrm{Aut}(K)$ be a group homomorphism. Let $\Gamma$ act on $B := \mathrm{C}(K)$ by $\varphi_g^* f(k) := f(\varphi_g^{-1}(k))$ for all $k \in K$, $g \in \Gamma$, $f \in \mathrm{C}(K)$. Equip $\mathrm{C}(K)$ with the trivial coaction of $\Gamma$. Since the $\Gamma$-coaction on $\mathrm{C}(K)$ is trivial,

$$\mathrm{C}_0(\Gamma) \boxtimes \mathrm{C}(K) \cong \mathrm{C}_0(\Gamma) \otimes \mathrm{C}(K) \cong \mathrm{C}_0(\Gamma \times K).$$

The comultiplication $\Delta_C$ is the one that is induced by the multiplication in the semidirect product group $\Gamma \ltimes K$.

We may also view $\mathrm{C}(K)$ as a Yetter–Drinfeld algebra over $\mathrm{C}^*_\mathrm{r}(\Gamma)$ instead of $\mathrm{C}_0(\Gamma)$. This gives a compact quantum group $\mathrm{C}^*_\mathrm{r}(\Gamma) \boxtimes \mathrm{C}(K)$ by Corollary 6.5. Its underlying $C^*$-algebra is canonically isomorphic to the reduced crossed product $\Gamma \ltimes \mathrm{C}(K)$ (see [12, Section 6.3]). Thus the unital $C^*$-algebra $\Gamma \ltimes \mathrm{C}(K)$ becomes a compact quantum group by Corollary 6.5. This is the partial dual of the group $\Gamma \ltimes K$, where we dualise $\mathrm{C}_0(\Gamma)$ to $\mathrm{C}^*_\mathrm{r}(\Gamma)$ and leave $\mathrm{C}(K)$ unchanged. This example is also a special case of a bicrossed product (see [1, Proposition 8.22]).

In the remainder of this section, we will prove Proposition 6.3 and Theorem 6.4. First we name the coactions on our Yetter–Drinfeld algebras: call them

$$\beta\colon B \to B \otimes A, \quad \hat{\beta}\colon B \to B \otimes \hat{A}, \quad \beta'\colon B' \to B' \otimes A, \quad \hat{\beta}'\colon B' \to B' \otimes \hat{A}.$$

We choose faithful covariant, $\mathfrak{D}(\mathbb{G})\widehat{\ }$-compatible representations $(\varphi, \mathrm{U}, \mathrm{V})$ and $(\varphi', \mathrm{U}', \mathrm{V}')$ of $(B, \beta, \hat{\beta})$ and $(B', \beta', \hat{\beta}')$ on some Hilbert spaces $\mathcal{L}$ and $\mathcal{L}'$. We choose a manageable multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ generating $\mathbb{G}$; it induces representations $\pi\colon A \to \mathbb{B}(\mathcal{H})$ and $\hat{\pi}\colon \hat{A} \to \mathbb{B}(\mathcal{H})$ with $\mathbb{W} = (\hat{\pi} \otimes \pi)\mathrm{W}$. Then $\pi$ and the corepresentation $(\hat{\pi} \otimes \mathrm{id}_A)W \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$ form a covariant representation of $(A, \Delta_A)$. We use these covariant representations of $A$, $B$ and $B'$ to realise $A \boxtimes B \boxtimes B'$ as a $C^*$-subalgebra of $\mathbb{B}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{L}')$; this gives

$$A \boxtimes B \boxtimes B' = \iota_A(A) \cdot \iota_B(B) \cdot \iota_{B'}(B')$$

for three representations $\iota_A$, $\iota_B$, $\iota_{B'}$ of $A$, $B$ and $B'$ on $\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{L}'$. We describe these representations as in the proof of Theorem 4.3.

The representation $\iota_A$ is most easy:

$$\iota_A(a) = \pi(a) \otimes 1_{\mathcal{L} \otimes \mathcal{L}'}.$$

To describe $\iota_B$, we must represent the universal $R$-matrix, which is essentially $\mathrm{W}^\mathrm{u}$, on the Hilbert space $\mathcal{H} \otimes \mathcal{L}$. The bijection between corepresentations of $A$ and representations of $\hat{A}^{\mathrm{u}}$ maps the left corepresentation $\mathrm{W}_{1\pi} \in \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}))$ to the representation $\hat{\pi}\colon \hat{A}^{\mathrm{u}} \to \hat{A} \to \mathbb{B}(\mathcal{H})$. The bijection between corepresentations of $\hat{A}$ and representations of $A^{\mathrm{u}}$ maps the right corepresentation $\mathrm{V} \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes \hat{A})$ to the unique representation $\rho\colon A^{\mathrm{u}} \to \mathbb{B}(\mathcal{L})$ with $\mathrm{W}_{1\rho} = \hat{\mathrm{V}} := \sigma(\mathrm{V})^* \in \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{L}))$. Hence the resulting representation $\hat{\pi} \otimes \rho\colon \hat{A}^{\mathrm{u}} \otimes A^{\mathrm{u}} \to \mathbb{B}(\mathcal{H} \otimes \mathcal{L})$ maps $\mathrm{W}^{\mathrm{u}}$ to $\hat{\mathbb{V}} := \hat{\mathrm{V}}_{\hat{\pi}2}$. Thus the braiding unitary ${}^{\mathcal{L}}\!\times^{\mathcal{H}}$ is $\hat{\mathbb{V}}^*\Sigma$ and

$$\iota_B(b) = \hat{\mathbb{V}}_{12}^*(1_{\mathcal{H}} \otimes \varphi(b) \otimes 1_{\mathcal{L}})\hat{\mathbb{V}}_{12}.$$

The representations $\iota_A$ and $\iota_B$ without the trivial third leg in $\mathcal{L}'$ also realise $A \boxtimes B$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{L})$. Similarly, we realise $A \boxtimes B'$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{L}')$ using $\hat{\mathbb{V}}' := (\hat{\pi} \otimes \mathrm{id}_{\mathcal{L}'})\hat{\mathrm{V}}' \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L}')$ instead of $\hat{\mathbb{V}}$.

The braiding for $\mathcal{L}$ and $\mathcal{L}'$ involves a unitary $Z$ given by (A.18). This unitary may also be characterised uniquely by the condition

$$\mathbb{U}_{12}(\hat{\mathbb{V}}')_{23}^* Z_{13} = (\hat{\mathbb{V}}')_{23}^*\mathbb{U}_{12} \qquad \text{in } \mathcal{U}(\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}'), \tag{6.4}$$

where $\mathbb{U} := (\mathrm{id}_{\mathcal{L}} \otimes \pi)\mathrm{U} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H})$; compare (A.19). As in the proof of Theorem 4.3, we see that the representation $\iota_{B'}$ is

$$\iota_{B'}(b') = Z_{23}(\hat{\mathbb{V}}')_{13}^*(1_{\mathcal{H}\otimes\mathcal{L}} \otimes \varphi'(b'))(\hat{\mathbb{V}}')_{13}Z_{23}^*.$$

*Proof of Proposition* 6.3. Define

$$\Psi(x) := \mathbb{W}_{13}\mathbb{U}_{23}(\hat{\mathbb{V}}')_{34}^* x_{124}\hat{\mathbb{V}}'_{34}\mathbb{U}_{23}^*\mathbb{W}_{13}^* \qquad \text{for } x \in \mathbb{B}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{L}').$$

This is an injective *-homomorphism from $\mathbb{B}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{L}')$ to $\mathbb{B}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}')$. We compute $\Psi \circ \iota_A$, $\Psi \circ \iota_B$, $\Psi \circ \iota_{B'}$; this will show that $\Psi$ maps $A \boxtimes B \boxtimes B'$ into

$$(A \boxtimes B) \otimes (A \boxtimes B') \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}')$$

and has the expected values on $\iota_A(A)$, $\iota_B(B)$ and $\iota_{B'}(B')$.

Since $\iota_A(a) = \pi(a)_1$, we get

$$\Psi \circ \iota_A(a) = \mathbb{W}_{13}\pi(a)_1\mathbb{W}_{13}^* = (\iota_A \otimes \iota_A) \circ \Delta_A(a).$$

Next, $\Psi \circ \iota_B(b) = \mathbb{W}_{13}\mathbb{U}_{23}\hat{\mathbb{V}}_{12}^*\varphi(b)_2\hat{\mathbb{V}}_{12}\mathbb{U}_{23}^*\mathbb{W}_{13}^*$. The Yetter–Drinfeld compatibility condition for U and V is equivalent to $\mathbb{W}_{13}\mathbb{U}_{23}\hat{\mathbb{V}}_{12}^* = \hat{\mathbb{V}}_{12}^*\mathbb{U}_{23}\mathbb{W}_{13}$ in $\mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H})$. Using this and the covariance condition for $(\varphi, \mathrm{U})$ with respect to $\beta$, we compute

$$\Psi \circ \iota_B(b) = \hat{\mathbb{V}}_{12}^*\mathbb{U}_{23}\varphi(b)_2\mathbb{U}_{23}^*\hat{\mathbb{V}}_{12} = (\iota_B \otimes \iota_A) \circ \beta(b).$$

Finally, we compute

$$\begin{aligned}\Psi \circ \iota_{B'}(b') &= \mathbb{W}_{13}\mathbb{U}_{23}(\hat{\mathbb{V}}')^*_{34}Z_{24}(\hat{\mathbb{V}}')^*_{14}\varphi'(b')_4\hat{\mathbb{V}}'_{14}Z^*_{24}\hat{\mathbb{V}}'_{34}\mathbb{U}^*_{23}\mathbb{W}^*_{13}\\ &= \mathbb{W}_{13}(\hat{\mathbb{V}}')^*_{34}\mathbb{U}_{23}(\hat{\mathbb{V}}')^*_{14}\varphi'(b')_4\hat{\mathbb{V}}'_{14}\mathbb{U}^*_{23}\hat{\mathbb{V}}'_{34}\mathbb{W}^*_{13}\\ &= \mathbb{W}_{13}(\hat{\mathbb{V}}')^*_{34}(\hat{\mathbb{V}}')^*_{14}\varphi'(b')_4\hat{\mathbb{V}}'_{14}\hat{\mathbb{V}}'_{34}\mathbb{W}^*_{13}\\ &= (\hat{\mathbb{V}}')^*_{34}\mathbb{W}_{13}\varphi'(b')_4\mathbb{W}^*_{13}\hat{\mathbb{V}}'_{34}\\ &= (\hat{\mathbb{V}}')^*_{34}\varphi'(b')_4\hat{\mathbb{V}}'_{34} = 1_{A\boxtimes B} \otimes \iota_{B'}(b').\end{aligned}$$

The first equality is trivial; the second equality uses (6.4); the third equality commutes $\mathbb{U}_{23}$ with $\hat{\mathbb{V}}_{14}$ and $\varphi'(b)_4$; the fourth equality uses that $\mathrm{V}'$ is a corepresentation of $\hat{A}$; the fifth equality commutes $\mathbb{W}_{13}$ and $\varphi'(b')_4$; and the last equality is the definition of the embedding of $A \boxtimes B'$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{L}')$. □

*Proof of Theorem* 6.4. Let $C := A \boxtimes B$ and $\Delta_C := \Psi \circ (\mathrm{id}_A \boxtimes \Delta_B): C \to C \otimes C$, where we use $\Psi$ from Proposition 6.3 in the special case $B = B'$. We first check that this comultiplication is coassociative. It suffices to check $(\Delta_C \otimes \mathrm{id}_C)\Delta_C \circ \iota_A = (\mathrm{id}_C \otimes \Delta_C)\Delta_C \circ \iota_A$ and $(\Delta_C \otimes \mathrm{id}_C)\Delta_C \circ \iota_B = (\mathrm{id}_C \otimes \Delta_C)\Delta_C \circ \iota_B$. The first statement holds because on elements of the form $\iota_A(a)$ with $a \in A$, we get $(\mathrm{id}_A \otimes \Delta_A)\Delta_A(a)$ and $(\Delta_A \otimes \mathrm{id}_A)\Delta_A(a)$, respectively, embedded into $C \otimes C \otimes C$ via $\iota_A \otimes \iota_A \otimes \iota_A$.

To check the formula on $B$, we also need the two maps

$$\begin{aligned}\Psi'&: A \boxtimes B \boxtimes (B \boxtimes B) \to (A \boxtimes B) \otimes (A \boxtimes B \boxtimes B),\\ \Psi''&: A \boxtimes (B \boxtimes B) \boxtimes B \to (A \boxtimes B \boxtimes B) \otimes (A \boxtimes B)\end{aligned}$$

that we get from Proposition 6.3 for $B$, $B \boxtimes B$ and $B \boxtimes B$, $B$, respectively. These satisfy, among others,

$$\begin{aligned}&\Psi'(b_2) = \beta(b)_{23}, \quad &&\Psi'(b_3) = b_4, \quad &&\Psi'(b_4) = b_5,\\ &\Psi''(b_2) = \beta(b)_{24}, \quad &&\Psi''(b_3) = \beta(b)_{34}, \quad &&\Psi''(b_4) = b_5.\end{aligned}$$

Here we use leg numbering notation to distinguish the different copies of $B$ more clearly. For instance, $\beta(b)_{23}$ means $(\iota_B \otimes \iota_A)\beta(b)$. With these maps, we may write

$$\begin{aligned}(\mathrm{id}_C \otimes (\mathrm{id}_A \boxtimes \Delta_B)) \circ \Psi|_{B\boxtimes B} &= \Psi' \circ (\mathrm{id}_B \boxtimes \Delta_B),\\ ((\mathrm{id}_A \boxtimes \Delta_B) \otimes \mathrm{id}_C) \circ \Psi|_{B\boxtimes B} &= \Psi'' \circ (\Delta_B \boxtimes \mathrm{id}_B).\end{aligned}$$

The second formula uses that $\Delta_B$ is $\mathbb{G}$-equivariant with respect to the actions $\beta$ and $\beta \bowtie \beta$ and that $\Psi''$ on $\iota_2(B)\iota_3(B)$ is $\beta \bowtie \beta$. Since $\beta$ is a coaction, we get

$$(\mathrm{id}_C \otimes \Psi) \circ \Psi'|_{B\boxtimes B\boxtimes B} = (\Psi \otimes \mathrm{id}_C) \circ \Psi''|_{B\boxtimes B\boxtimes B}.$$

This and the coassociativity of $\Delta_B$ imply that $\Delta_C$ is coassociative also on $B$.

Now we turn to the Podleś conditions. We have $A \boxtimes B = \iota_A(A)\iota_B(B) = \iota_B(B)\iota_A(A)$, $\beta(B)\cdot(1\otimes A) = B\otimes A$ because $\beta$ satisfies the Podleś condition, and $\Delta_A(A)(1\otimes A) = \Delta_A(A)(A\otimes 1) = A\otimes A$ and $\Delta_B(B)\cdot B_2 = \Delta_B(B)B_1 = B\boxtimes B$ because $A$ and $B$ are bisimplifiable. Thus

$$\begin{aligned}\Delta_C(C)\cdot(1\otimes C) &= \Delta_C(\iota_B(B))\cdot\Delta_C(\iota_A(A))\cdot(1\otimes\iota_A(A))\cdot(1\otimes\iota_B(B))\\ &= \Delta_C(\iota_B(B))\cdot(\iota_A\otimes\iota_A)(\Delta_A(A)\cdot(1\otimes A))\cdot(1\otimes\iota_B(B))\\ &= \Delta_C(\iota_B(B))\cdot(\iota_A(A)\otimes C) = \Psi(\Delta_B(B)_{23})\cdot(\iota_A(A)\otimes C).\end{aligned}$$

Since $\Psi\circ\iota_{B'}(b') = 1\otimes\iota_{B'}(b')$ is a multiplier of $\iota_A(A)\otimes C$, we may rewrite this as

$$\begin{aligned}&\Psi(\Delta_B(B)_{23}\cdot B_3)\cdot(\iota_A(A)\otimes C)\\ &\qquad = \Psi((B\boxtimes B)_{23})\cdot(\iota_A(A)\otimes C) = \Psi(B_2)\cdot(\iota_A(A)\otimes C),\end{aligned}$$

using that $(B,\Delta_B)$ is bisimplifiable. Finally, the formula for $\Psi$ on the second leg and the Podleś condition for $\beta$ show that this is $C\otimes C$, as desired.

The other Podleś condition is proved similarly. Since $\Psi$ is injective, $\Delta_C$ is injective if and only if $\mathrm{id}_A\boxtimes\Delta_B$ is injective. This is equivalent to $\Delta_B$ being injective by [11, Proposition 5.6]. □

## A. Preliminaries

### A.1. Multiplicative unitaries and quantum groups.

**Definition A.1** ([1]). Let $\mathcal{H}$ be a Hilbert space. A unitary $\mathbb{W}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is *multiplicative* if it satisfies the *pentagon equation*

$$\mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}). \tag{A.1}$$

Technical assumptions such as manageability [20] are needed to construct C*-algebras out of a multiplicative unitary.

**Theorem A.2** ([18–20]). *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{W}\in\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ a manageable multiplicative unitary. Let*

$$A := \{(\omega\otimes\mathrm{id}_{\mathcal{H}})\mathbb{W} : \omega\in\mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}, \tag{A.2}$$
$$\hat{A} := \{(\mathrm{id}_{\mathcal{H}}\otimes\omega)\mathbb{W} : \omega\in\mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}. \tag{A.3}$$

(1) *$A$ and $\hat{A}$ are separable, nondegenerate* C*-*subalgebras of* $\mathbb{B}(\mathcal{H})$.

(2) $\mathbb{W}\in\mathcal{U}(\hat{A}\otimes A)\subseteq\mathcal{U}(\mathcal{H}\otimes\mathcal{H})$. *We write* $\mathrm{W}^A$ *for* $\mathbb{W}$ *viewed as a unitary multiplier of* $\hat{A}\otimes A$ *and call it* reduced bicharacter.

(3) *There is a unique morphism* $\Delta_A: A \to A \otimes A$ *such that*

$$(\mathrm{id}_{\hat{A}} \otimes \Delta_A)\mathrm{W}^A = \mathrm{W}^A_{12}\mathrm{W}^A_{13} \qquad \text{in } \mathcal{U}(\hat{A} \otimes A \otimes A); \tag{A.4}$$

*it is* coassociative *and* bisimplifiable*:*

$$(\Delta_A \otimes \mathrm{id}_A) \circ \Delta_A = (\mathrm{id}_A \otimes \Delta_A) \circ \Delta_A, \tag{A.5}$$

$$\Delta_A(A) \cdot (1_A \otimes A) = A \otimes A = (A \otimes 1_A) \cdot \Delta_A(A). \tag{A.6}$$

A $\mathrm{C}^*$*-quantum group* is a $\mathrm{C}^*$-bialgebra $\mathbb{G} = (A, \Delta_A)$ constructed from a manageable multiplicative unitary. This class contains the locally compact quantum groups of Kustermans and Vaes [8], which are defined by the existence of left and right Haar weights.

The *dual* multiplicative unitary is $\widehat{\mathbb{W}} := \Sigma\mathbb{W}^*\Sigma \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$, where $\Sigma(x \otimes y) = y \otimes x$. It is manageable if $\mathbb{W}$ is. The $\mathrm{C}^*$-quantum group $\widehat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$ generated by $\widehat{\mathbb{W}}$ is the *dual* of $\mathbb{G}$. Its comultiplication is characterised by

$$(\hat{\Delta}_A \otimes \mathrm{id}_A)\mathrm{W}^A = \mathrm{W}^A_{23}\mathrm{W}^A_{13} \qquad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes A). \tag{A.7}$$

### A.2. Corepresentations.

**Definition A.3.** A (right) *corepresentation* of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$ is a unitary $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$ with

$$(\mathrm{id}_{\mathbb{K}(\mathcal{H})} \otimes \Delta_A)\mathrm{U} = \mathrm{U}_{12}\mathrm{U}_{13} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A \otimes A). \tag{A.8}$$

Let $\mathrm{U}^1 \in \mathcal{U}(\mathbb{K}(\mathcal{H}_1) \otimes A)$ and $\mathrm{U}^2 \in \mathcal{U}(\mathbb{K}(\mathcal{H}_2) \otimes A)$ be corepresentations of $\mathbb{G}$. An element $t \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called an *intertwiner* if $(t \otimes 1_A)\mathrm{U}^1 = \mathrm{U}^2(t \otimes 1_A)$. The set of all intertwiners between $\mathrm{U}^1$ and $\mathrm{U}^2$ is denoted $\mathrm{Hom}(\mathrm{U}^1, \mathrm{U}^2)$. This gives corepresentations a structure of $\mathrm{W}^*$-category (see [19, Sections 3.1–2]).

The *tensor product* of two corepresentations $\mathrm{U}^{\mathcal{H}_1}$ and $\mathrm{U}^{\mathcal{H}_2}$ is defined by

$$\mathrm{U}^{\mathcal{H}_1} \mathbin{\top\!\!\!\!\bigcirc} \mathrm{U}^{\mathcal{H}_2} := \mathrm{U}^{\mathcal{H}_1}_{13}\mathrm{U}^{\mathcal{H}_2}_{23} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes A). \tag{A.9}$$

Routine computations show the following: $\mathrm{U}^{\mathcal{H}_1} \mathbin{\top\!\!\!\!\bigcirc} \mathrm{U}^{\mathcal{H}_2}$ is a corepresentation; $\top\!\!\!\!\bigcirc$ is associative; and the trivial 1-dimensional representation is a tensor unit. Thus corepresentations form a monoidal $\mathrm{W}^*$-category, which we denote by $\mathfrak{Corep}(\mathbb{G})$; see [19, Section 3.3] for more details.

### A.3. Coactions.

**Definition A.4.** A *continuous* (*right*) *coaction* of $\mathbb{G}$ on a $\mathrm{C}^*$-algebra $C$ is a morphism $\gamma: C \to C \otimes A$ with the following properties:

(1) $\gamma$ is injective;

(2) $\gamma$ is a comodule structure, that is, $(\mathrm{id}_C \otimes \Delta_A)\gamma = (\gamma \otimes \mathrm{id}_A)\gamma$;

(3) $\gamma$ satisfies the *Podleś condition* $\gamma(C)\cdot(1_C\otimes A) = C\otimes A$.

We call $(C,\gamma)$ a $\mathbb{G}$-*C*-algebra*. We often drop $\gamma$ from our notation.

A morphism $f\colon C\to D$ between two $\mathbb{G}$-C*-algebras $(C,\gamma)$ and $(D,\delta)$ is $\mathbb{G}$-*equivariant* if $\delta\circ f = (f\otimes \mathrm{id}_A)\circ\gamma$. Let $\mathrm{Mor}^{\mathbb{G}}(C,D)$ be the set of $\mathbb{G}$-equivariant morphisms from $C$ to $D$. Let $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ be the category with $\mathbb{G}$-C*-algebras as objects and $\mathbb{G}$-equivariant morphisms as arrows.

**Definition A.5.** A *covariant representation* of $(C,\gamma,A)$ on a Hilbert space $\mathcal{H}$ is a pair consisting of a corepresentation $\mathrm{U}\in\mathcal{U}(\mathbb{K}(\mathcal{H})\otimes A)$ and a representation $\varphi\colon C\to\mathbb{B}(\mathcal{H})$ that satisfy the covariance condition

$$(\varphi\otimes\mathrm{id}_A)\circ\gamma(c) = \mathrm{U}(\varphi(c)\otimes 1_A)\mathrm{U}^* \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H})\otimes A) \tag{A.10}$$

for all $c\in C$. A covariant representation is called *faithful* if $\varphi$ is faithful.

Faithful covariant representations always exist by [12, Example 4.5].

**A.4. Universal quantum groups.** The universal quantum group

$$\mathbb{G}^{\mathrm{u}} := (A^{\mathrm{u}}, \Delta_{A^{\mathrm{u}}})$$

associated to $\mathbb{G}=(A,\Delta_A)$ is introduced in [19]. By construction, it comes with a *reducing map* $\Lambda\colon A^{\mathrm{u}}\to A$ and a universal bicharacter $\mathcal{V}\in\mathcal{U}(\hat{A}\otimes A^{\mathrm{u}})$. This may also be characterised as the unique bicharacter in $\mathcal{U}(\hat{A}\otimes A^{\mathrm{u}})$ that lifts $\mathrm{W}^A\in\mathcal{U}(\hat{A}\otimes A)$ in the sense that $(\mathrm{id}_{\hat{A}}\otimes\Lambda)\mathcal{V}=\mathrm{W}^A$.

Similarly, there are unique bicharacters $\tilde{\mathcal{V}}\in\mathcal{U}(\hat{A}^{\mathrm{u}}\otimes A)$ and $\mathrm{W}^{\mathrm{u}}\in\mathcal{U}(\hat{A}^{\mathrm{u}}\otimes A^{\mathrm{u}})$ that lift $\mathrm{W}^A\in\mathcal{U}(\hat{A}\otimes A)$; the latter is constructed in [7] assuming a Haar measure and in [11] in the more general setting of manageable multiplicative unitaries. The universality of $\tilde{\mathcal{V}}\in\mathcal{U}(\hat{A}^{\mathrm{u}}\otimes A)$ says that for any corepresentation $\mathrm{U}\in\mathcal{U}(\mathbb{K}(\mathcal{H})\otimes A)$ of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, there is a unique representation $\rho\colon\hat{A}^{\mathrm{u}}\to\mathbb{B}(\mathcal{H})$ with

$$(\rho\otimes\mathrm{id}_A)\tilde{\mathcal{V}} = \mathrm{U} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H})\otimes A). \tag{A.11}$$

**A.5. Bicharacters as quantum group morphisms.** Let $\mathbb{G}=(A,\Delta_A)$ and $\mathbb{H}=(B,\Delta_B)$ be C*-quantum groups. Let $\widehat{\mathbb{G}}=(\hat{A},\hat{\Delta}_A)$ and $\widehat{\mathbb{H}}=(\hat{B},\hat{\Delta}_B)$ be their duals.

**Definition A.6** ([11, Definition 16]). A *bicharacter from* $\mathbb{G}$ *to* $\widehat{\mathbb{H}}$ is a unitary $\chi\in\mathcal{U}(\hat{A}\otimes\hat{B})$ with

$$(\hat{\Delta}_A\otimes\mathrm{id}_{\hat{B}})\chi = \chi_{23}\chi_{13} \qquad \text{in } \mathcal{U}(\hat{A}\otimes\hat{A}\otimes\hat{B}), \tag{A.12}$$

$$(\mathrm{id}_{\hat{A}}\otimes\hat{\Delta}_B)\chi = \chi_{12}\chi_{13} \qquad \text{in } \mathcal{U}(\hat{A}\otimes\hat{B}\otimes\hat{B}). \tag{A.13}$$

Bicharacters in $\mathcal{U}(\hat{A}\otimes B)$ are interpreted as quantum group morphisms from $\mathbb{G}$ to $\mathbb{H}$ in [11]. We mainly use bicharacters in $\mathcal{U}(\hat{A}\otimes\hat{B})$ and rewrite some definitions in [11] in this setting.

**Definition A.7.** A *right quantum group morphism* from $\mathbb{G}$ to $\widehat{\mathbb{H}}$ is a morphism $\Delta_R\colon A \to A \otimes \hat{B}$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta_R} & A \otimes \hat{B} \\ {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \Delta_A \otimes \mathrm{id}_{\hat{B}}} \\ A \otimes A & \xrightarrow[\mathrm{id}_A \otimes \Delta_R]{} & A \otimes A \otimes \hat{B} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta_R} & A \otimes \hat{B} \\ {\scriptstyle \Delta_R}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \hat{\Delta}_B} \\ A \otimes \hat{B} & \xrightarrow[\Delta_R \otimes \mathrm{id}_{\hat{B}}]{} & A \otimes \hat{B} \otimes \hat{B} \end{array} \tag{A.14}$$

The following theorem summarises some of the main results of [11].

**Theorem A.8.** *There are natural bijections between the following sets:*

(1) *bicharacters* $\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})$ *from* $\mathbb{G}$ *to* $\widehat{\mathbb{H}}$*;*

(2) *bicharacters* $\hat{\chi} \in \mathcal{U}(\hat{B} \otimes \hat{A})$ *from* $\mathbb{H}$ *to* $\widehat{\mathbb{G}}$*;*

(3) *right quantum group homomorphisms* $\Delta_R\colon A \to A \otimes \hat{B}$*;*

(4) *functors* $F\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \to \mathfrak{C}^*\mathfrak{alg}(\widehat{\mathbb{H}})$ *with* $\mathsf{For}_{\widehat{\mathbb{H}}} \circ F = \mathsf{For}_{\mathbb{G}}$ *for the forgetful functor* $\mathsf{For}_{\mathbb{G}}\colon \mathfrak{C}^*\mathfrak{alg}(\mathbb{G}) \to \mathfrak{C}^*\mathfrak{alg}$*;*

(5) *Hopf *-homomorphisms* $f\colon A^{\mathrm{u}} \to \hat{B}^{\mathrm{u}}$ *between universal quantum groups;*

(6) *bicharacters* $\chi^{\mathrm{u}} \in \mathcal{U}(\hat{A}^{\mathrm{u}} \otimes \hat{B}^{\mathrm{u}})$.

*The first bijection maps a bicharacter* $\chi$ *to*

$$\hat{\chi} := \sigma(\chi^*). \tag{A.15}$$

*A bicharacter* $\chi$ *and a right quantum group homomorphism* $\Delta_R$ *determine each other uniquely via*

$$(\mathrm{id}_{\hat{A}} \otimes \Delta_R)(\mathrm{W}^A) = \mathrm{W}^A_{12}\chi_{13}. \tag{A.16}$$

*The functor* $F$ *associated to* $\Delta_R$ *is the unique one that maps* $(A, \Delta_A)$ *to* $(A, \Delta_R)$*. In general,* $F$ *maps a continuous* $\mathbb{G}$*-coaction* $\gamma\colon C \to C \otimes A$ *to the unique* $\widehat{\mathbb{H}}$*-coaction* $\delta\colon C \to C \otimes \hat{B}$ *for which the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C \otimes A \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_R} \\ C \otimes \hat{B} & \xrightarrow[\gamma \otimes \mathrm{id}_{\hat{B}}]{} & C \otimes A \otimes \hat{B} \end{array} \tag{A.17}$$

*The bicharacter in* $\mathcal{U}(\hat{A} \otimes \hat{B})$ *associated to a Hopf *-homomorphism* $f\colon A^{\mathrm{u}} \to \hat{B}^{\mathrm{u}}$ *is* $\chi := (\mathrm{id}_{\hat{A}} \otimes \Lambda_{\hat{B}} f)(\mathcal{V}^A)$*, where* $\mathcal{V}^A \in \mathcal{U}(\hat{A} \otimes A^{\mathrm{u}})$ *is the unique bicharacter lifting* $\mathrm{W}^A \in \mathcal{U}(\hat{A} \otimes A)$ *and* $\Lambda_{\hat{B}}\colon \hat{B}^{\mathrm{u}} \to \hat{B}$ *is the reducing map.*

**A.6. Twisted tensor products.** Let $\gamma\colon C \to C \otimes A$ and $\delta\colon D \to D \otimes B$ be coactions of $\mathbb{G}$ and $\mathbb{H}$ on C*-algebras $C$ and $D$, respectively. We are going to describe the twisted tensor product

$$C \boxtimes D := (C, \gamma) \boxtimes_{\chi} (D, \delta)$$

for a bicharacter $\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})$. Let $(\varphi, \mathrm{U}^{\mathcal{H}})$ and $(\psi, \mathrm{U}^{\mathcal{K}})$ be faithful covariant representations of $(C, \gamma)$ and $(D, \delta)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Thus $\mathrm{U}^{\mathcal{H}} \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$ and $\mathrm{U}^{\mathcal{K}} \in \mathcal{U}(\mathbb{K}(\mathcal{K}) \otimes B)$ are corepresentations of $\mathbb{G}$ and $\mathbb{H}$. Let $\rho^{\mathcal{H}}\colon \hat{A}^{\mathrm{u}} \to \mathbb{B}(\mathcal{H})$ and $\rho^{\mathcal{K}}\colon \hat{B}^{\mathrm{u}} \to \mathbb{B}(\mathcal{K})$ be the corresponding representations of the universal duals. Let $\chi^{\mathrm{u}} \in \mathcal{U}(\hat{A} \otimes \hat{B})$ lift $\chi$, see Theorem A.8. Let

$$Z := (\rho^{\mathcal{H}} \otimes \rho^{\mathcal{K}})(\chi^{\mathrm{u}})^* \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K}). \tag{A.18}$$

The proof of [12, Theorem 4.1] shows that this is the unique $Z \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K})$ with

$$\mathrm{U}^{\mathcal{H}}_{1\alpha} \mathrm{U}^{\mathcal{K}}_{2\beta} Z_{12} = \mathrm{U}^{\mathcal{K}}_{2\beta} \mathrm{U}^{\mathcal{H}}_{1\alpha} \qquad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L}) \tag{A.19}$$

for any $\chi$-Heisenberg pair $(\alpha, \beta)$ on any Hilbert space $\mathcal{L}$.

Define representations $\iota_C = \varphi_1$ and $\iota_D = \tilde{\psi}_2$ of $C$ and $D$ on $\mathcal{H} \otimes \mathcal{K}$ by

$$\begin{aligned} \iota_C(c) = \varphi_1(c) &:= \varphi(c) \otimes 1_{\mathcal{K}}, \\ \iota_D(d) = \tilde{\psi}_2(d) &:= Z(1_{\mathcal{H}} \otimes \psi(d))Z^*. \end{aligned} \tag{A.20}$$

**Theorem A.9** ([12, Lemma 3.20, Theorem 4.3, Theorem 4.9]). *The subspace*

$$C \boxtimes D := \varphi_1(C) \cdot \tilde{\psi}_2(D) \subset \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$$

*is a nondegenerate* C**-subalgebra. The crossed product* $(C \boxtimes D, \iota_C, \iota_D)$*, up to equivalence, does not depend on the faithful covariant representations* $(\mathrm{U}^{\mathcal{H}}, \varphi)$ *and* $(\mathrm{U}^{\mathcal{K}}, \psi)$*.*

We call $C \boxtimes_{\chi} D$ the *twisted tensor product* of $C$ and $D$. It generalises the minimal tensor product of C*-algebras.

**A.7. Heisenberg pairs.** Let $\mathbb{G} = (A, \Delta_A)$ and $\mathbb{H} = (B, \Delta_B)$ be C*-quantum groups and let $\mathrm{W}^A \in \mathcal{U}(\hat{A} \otimes A)$ and $\mathrm{W}^B \in \mathcal{U}(\hat{B} \otimes B)$ be their reduced bicharacters, respectively. Let $\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})$ be a bicharacter from $\mathbb{G}$ to $\widehat{\mathbb{H}}$.

**Definition A.10** ([12, Definition 3.1]). Let $E$ be a C*-algebra and let $\alpha\colon A \to E$ and $\beta\colon B \to E$ be morphisms. The pair $(\alpha, \beta)$ is a *$\chi$-Heisenberg pair* or briefly *Heisenberg pair* on $E$ if

$$\mathrm{W}^A_{1\alpha} \mathrm{W}^B_{2\beta} = \mathrm{W}^B_{2\beta} \mathrm{W}^A_{1\alpha} \chi_{12} \qquad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes E); \tag{A.21}$$

here $\mathrm{W}^A_{1\alpha} := ((\mathrm{id}_{\hat{A}} \otimes \alpha)\mathrm{W}^A)_{13}$ and $\mathrm{W}^B_{2\beta} := ((\mathrm{id}_{\hat{B}} \otimes \beta)\mathrm{W}^B)_{23}$.

The following lemma is routine to check:

**Lemma A.11.** *Let $(\alpha, \beta)$ be a $\mathcal{X}$-Heisenberg pair on a* C*-*algebra $E$. Define $\alpha' : A \to A \otimes B \otimes E$ and $\beta' : B \to A \otimes B \otimes E$ by $\alpha'(a) := \big((\mathrm{id}_A \otimes \alpha)\Delta_A(a)\big)_{13}$ and $\beta'(b) := \big((\mathrm{id}_B \otimes \beta)\Delta_B(b)\big)_{23}$. This is again a $\mathcal{X}$-Heisenberg pair.*

Equip $A$ and $B$ with the standard coactions $\Delta_A$ and $\Delta_B$ of $\mathbb{G}$ and $\mathbb{H}$, respectively, and form the tensor product $A \boxtimes_{\mathcal{X}} B$. This plays a special role, as explained after Proposition 5.6 in [12]: coactions $\gamma : C \to C \otimes A$ and $\delta : D \to D \otimes B$ induce a canonical map

$$\gamma \boxtimes \delta : C \boxtimes_{\mathcal{X}} D \to (C \otimes A) \boxtimes_{\mathcal{X}} (D \otimes B) \cong C \otimes D \otimes (A \boxtimes_{\mathcal{X}} B).$$

**Proposition A.12.** *Let $\mathcal{H}$ be a Hilbert space and let $\varphi : A \boxtimes_{\mathcal{X}} B \to \mathbb{B}(\mathcal{H})$ be a representation. Define $\alpha := \varphi \circ \iota_A : A \to \mathbb{B}(\mathcal{H})$ and $\beta := \varphi \circ \iota_B : B \to \mathbb{B}(\mathcal{H})$. The pair $(\alpha, \beta)$ is a $\mathcal{X}$-Heisenberg pair.*

*Proof.* This is the only place where we use the construction of twisted tensor products through Heisenberg pairs in [12, Section 3]. Let $(\alpha', \beta')$ be a $\mathcal{X}$-Heisenberg pair on a C*-algebra $E$. Define morphisms

$$\begin{aligned} \iota_A &: A \to A \otimes B \otimes E, & a &\mapsto \big((\mathrm{id}_A \otimes \alpha')\Delta_A(a)\big)_{13}, \\ \iota_B &: B \to A \otimes B \otimes E, & b &\mapsto \big((\mathrm{id}_B \otimes \beta')\Delta_B(b)\big)_{23}. \end{aligned}$$

Then $A \boxtimes_{\mathcal{X}} B \cong \iota_A(A) \cdot \iota_B(B)$. Lemma A.11 shows that $(\iota_A, \iota_B)$ is a $\mathcal{X}$-Heisenberg pair on $A \boxtimes_{\mathcal{X}} B$. Hence $(\varphi \circ \iota_A, \varphi \circ \iota_B)$ is a $\mathcal{X}$-Heisenberg pair on $\mathcal{H}$. □

**Corollary A.13.** *If $A \boxtimes_{\mathcal{X}} B \cong A \boxtimes_{\mathcal{X}'} B$ as crossed products for two bicharacters $\mathcal{X}, \mathcal{X}' \in \mathcal{U}(\hat{A} \otimes \hat{B})$, then $\mathcal{X} = \mathcal{X}'$.*

*Proof.* Let $\mathcal{H}, \varphi, \alpha, \beta$ as in Proposition A.12. The pair $(\alpha, \beta)$ is both a $\mathcal{X}$-Heisenberg pair and a $\mathcal{X}'$-Heisenberg pair by Proposition A.12. The commutation relation (A.21) that characterises Heisenberg pairs gives

$$\mathrm{W}^A_{2\beta} \mathrm{W}^A_{1\alpha} \mathcal{X}_{12} = \mathrm{W}^A_{1\alpha} \mathrm{W}^A_{1\beta} = \mathrm{W}^A_{2\beta} \mathrm{W}^A_{1\alpha} \mathcal{X}'_{12} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes \mathbb{K}(\mathcal{H})).$$

Thus $\mathcal{X} = \mathcal{X}'$. □

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