J. Noncommut. Geom. 10 (2016), 1131–1142
DOI 10.4171/JNCG/256

Journal of Noncommutative Geometry

Comparison of KE-theory and KK-theory

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Abstract. We show that the character from the bivariant K-theory KE^G introduced by Dumitraşcu to E^G factors through Kasparov's KK^G for any locally compact group G. Hence KE^G contains KK^G as a direct summand.

Mathematics Subject Classification (2010). 19K35.

Keywords. KK-theory, E-theory.

1. Introduction

K-theory may be generalised in several ways to a bivariant theory. One such bivariant K-theory is Kasparov's KK (see [5]), another is the E-theory of Connes and Higson [1]. Both theories have equivariant versions with respect to second-countable locally compact groups (see [4,6]). These theories are related by a natural transformation $\mathrm{KK}^G(A, B) \to \mathrm{E}^G(A, B)$ because of the universal property of KK^G.

Dumitraşcu defines another equivariant bivariant K-theory $\mathrm{KE}^G(A, B)$ in his thesis [2], which has the same formal properties as KK^G and E^G; in particular, it has an analogue of the Kasparov product and the exterior product. He also constructs explicit natural transformations

$$\mathrm{KK}^G(A, B) \to \mathrm{KE}^G(A, B) \to \mathrm{E}^G(A, B).$$

Hence this also makes the transformation $\mathrm{KK}^G \to \mathrm{E}^G$ explicit.

The construction of a new bivariant K-theory is always laden with technical difficulties, especially the construction of a product. KK-theory and E-theory involve different technicalities, and KE-theory needs a share of both kinds of technicalities. When I was asked to referee the article [3] by Dumitraşcu, I therefore wanted to clarify whether KE^G is really a new theory or equivalent to KK^G or E^G. I expected KE^G to be equivalent to either KK^G or E^G. I came up quickly with a sketch of an argument why KE^G should be equivalent to KK^G, which I communicated to Dumitraşcu, asking him whether he could complete this sketch to a full proof. After a while it became clear that I had to complete this argument myself, which resulted in this note. Its purpose is the following theorem:

Theorem 1.1. *Let G be a second countable locally compact group and let A and B be separable G-C^*-algebras. The natural map* $\mathrm{KE}^G(A, B) \to \mathrm{E}^G(A, B)$ *factors through a map* $\mathrm{KE}^G(A, B) \to \mathrm{KK}^G(A, B)$.

I still expect $\mathrm{KE}^G(A, B) \cong \mathrm{KK}^G(A, B)$, but I do not know how to prove this. A transformation $\mathrm{KE}^G \to \mathrm{KK}^G$ seems enough for applications. It shows that a computation in KE^G gives results in KK^G. For instance, if G has the analogue of a γ-element in KE^G or if $\gamma = 1$ in KE^G, then the same follows in KK^G.

In Section 2, we recall Dumitraşcu's definition of cycles for $\mathrm{KE}^G(A, B)$ and show that we may strengthen his conditions slightly without changing the set of homotopy classes. In Section 3, we show how to get completely positive equivariant asymptotic morphisms from the KE^G-cycles satisfying our stronger conditions.

2. The definition of KE-theory

Throughout this article, G is a second countable, locally compact topological group; A and B are separable C^*-algebras with continuous actions of $G \times \mathbb{Z}/2$. An action of $G \times \mathbb{Z}/2$ is the same as a $\mathbb{Z}/2$-grading together with an action of G by grading-preserving automorphisms; we will frequently combine a $\mathbb{Z}/2$-grading and a G-action in this way.

We first recall the definition of $\mathrm{KE}^G(A, B)$.

Let $L := [1, \infty)$ and $BL := \mathrm{C}_0(L, B)$. A *continuous field of* $G \times \mathbb{Z}/2$-(A, B)-*modules* is a countably generated, $G \times \mathbb{Z}/2$-equivariant Hilbert BL-module $\mathcal{E}$ with a $G \times \mathbb{Z}/2$-equivariant *-homomorphism $\varphi\colon A \to \mathcal{L}(\mathcal{E})$, where $\mathcal{L}(\mathcal{E})$ denotes the C^*-algebra of adjointable operators on $\mathcal{E}$ with its canonical action of $G \times \mathbb{Z}/2$.

Using the evaluation homomorphisms $BL \to B$, $f \mapsto f(t)$, we may view $\mathcal{E}$ as a family of $G \times \mathbb{Z}/2$-equivariant Hilbert B-modules $\mathcal{E}_t$; an operator $x \in \mathcal{L}(\mathcal{E})$ is completely determined by a family of operators $x_t \in \mathcal{L}(\mathcal{E}_t)$. Besides the ideal $\mathcal{K}(\mathcal{E})$ of compact operators on $\mathcal{E}$, we need the two ideals

$$\mathcal{C}(\mathcal{E}) := \{x \in \mathcal{L}(\mathcal{E}) \mid xf \in \mathcal{K}(\mathcal{E}) \text{ for all } f \in \mathrm{C}_0(L)\},$$
$$\mathcal{I}(\mathcal{E}) := \{x \in \mathcal{L}(\mathcal{E}) \mid \lim_{t\to\infty} \|x_t\| = 0\}.$$

We have $\mathcal{C}(\mathcal{E}) \cap \mathcal{I}(\mathcal{E}) = \mathcal{K}(\mathcal{E})$.

A *cycle for* $\mathrm{KE}^G(A, B)$ is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a continuous field of $G \times \mathbb{Z}/2$-(A, B)-modules and F is an odd, adjointable operator on $\mathcal{E}$ that satisfies the following conditions (for all $a \in A$, $g \in G$):

akm1: $(F - F^*)\varphi(a) \in \mathcal{I}(\mathcal{E})$ for all $a \in A$;

akm2: $[F, \varphi(a)] \in \mathcal{I}(\mathcal{E})$ for all $a \in A$;

akm3: $\varphi(a)(F^2 - 1)\varphi(a)^* \geq 0$ modulo $\mathcal{C}(\mathcal{E}) + \mathcal{I}(\mathcal{E})$ for all $a \in A$;

akm4: $(gF - F)\varphi(a) \in \mathcal{I}(\mathcal{E})$ for all $a \in A$, $g \in G$.

Later, we shall meet the following strengthenings of these conditions:

aKm1s: $F = F^*$;

aKm3s: $\|F\| \leq 1$ and $(1 - F^2)\varphi(a) \in \mathcal{C}(\mathcal{E})$ for all $a \in A$;

aKm4s: $gF = F$ for all $g \in G$.

Cycles for $\mathrm{KE}^G(A, \mathrm{C}([0,1], B))$ are called *homotopies* of cycles. We define $\mathrm{KE}^G(A, B)$ as the set of homotopy classes of cycles for $\mathrm{KE}^G(A, B)$.

Lemma 2.1. *Any cycle for* $\mathrm{KE}^G(A, B)$ *is homotopic to one that satisfies* (aKm1s) *and* (aKm3s). *If two cycles satisfying* (aKm1s) *and* (aKm3s) *are homotopic, they are homotopic via a homotopy that satisfies* (aKm1s) *and* (aKm3s).

We will treat condition (aKm4s) below in Lemma 2.2.

Proof. Let $(\mathcal{E}, F)$ be a cycle for $\mathrm{KE}^G(A, B)$. Then $(F + F^*)/2$ is a small perturbation of F and hence gives a homotopic cycle (see [3, Corollary 3.25]) satisfying $F = F^*$.

Now assume $F = F^*$ and (aKm2); then

$$\varphi(a)(1 - F^2)_+\varphi(a)^* \cdot \varphi(a)(1 - F^2)_-\varphi(a)^* \\ \equiv \varphi(a)\varphi(a)^*\varphi(a)(1 - F^2)_+(1 - F^2)_-\varphi(a)^* \equiv 0 \bmod \mathcal{I}(\mathcal{E}).$$

Hence $\varphi(a)(F^2-1)_\pm\varphi(a)^*$ are the positive and negative parts of $\varphi(a)(F^2-1)\varphi(a)^*$ in $\mathcal{L}(\mathcal{E})/\mathcal{I}(\mathcal{E})$. As a result, (aKm3) is equivalent to

$$\varphi(a) \cdot (F^2 - 1)_- \cdot \varphi(a)^* \in \mathcal{C}(\mathcal{E}) + \mathcal{I}(\mathcal{E})$$

for all $a \in A$.

Define $\chi\colon \mathbb{R} \to [-1, 1]$ by $\chi(t) := -1$ for $t \leq -1$, $\chi(t) := t$ for $-1 \leq t \leq 1$, and $\chi(t) := 1$ for $t \geq 1$. Then $\|\chi(F)\| \leq 1$ and $\chi(F)^2 - 1 = (F^2 - 1)_-$. The reformulation of (aKm3) in the previous paragraph shows that $(\mathcal{E}, \chi(F))$ is again a cycle for $\mathrm{KE}^G(A, B)$ and that the linear path $(\mathcal{E}, sF + (1 - s)\chi(F))$ is a homotopy of cycles. Thus any cycle is homotopic to one with $F = F^*$ and $\|F\| \leq 1$.

Next we adapt the standard trick to achieve $F^2 = 1$ for KK-cycles. Let $\mathcal{E}_2 := \mathcal{E} \oplus \mathcal{E}^{\mathrm{op}}$, where op denotes the opposite $\mathbb{Z}/2$-grading. Let A act on $\mathcal{E}_2$ by $\varphi_2 := \varphi \oplus 0$. For $s \in [0, 1]$, let

$$F_{2s} := \begin{pmatrix} F & s\sqrt{1 - u^2}\sqrt{1 - F^2} \\ s\sqrt{1 - F^2}\sqrt{1 - u^2} & -F \end{pmatrix},$$

where $u \in \mathcal{L}(\mathcal{E})^{(0)}$ is an even operator as in [3, Lemma 3.35]; that is, $u \in \mathcal{C}(\mathcal{E})$, $[u, F] \in \mathcal{I}(\mathcal{E})$, $[u, \varphi(a)] \in \mathcal{I}(\mathcal{E})$ for all $a \in A$, $(1 - u^2)(\varphi(a)(F^2 - 1)\varphi(a)^*)_- \in \mathcal{I}(\mathcal{E})$

for all $a \in A$, and $gu - u \in \mathcal{I}(\mathcal{E})$ for all $g \in G$. Since $u \in \mathcal{C}(\mathcal{E})$ and $\mathcal{C}(\mathcal{E}) \cap \mathcal{I}(\mathcal{E}) = \mathcal{K}(\mathcal{E})$, we even have $[u, F] \in \mathcal{K}(\mathcal{E})$, $[u, \varphi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$, and $gu - u \in \mathcal{K}(\mathcal{E})$ for all $g \in G$. Since we already achieved $\|F\| \leq 1$, we also have $(1-u^2)\varphi(a)(1-F^2)\varphi(a)^* \in \mathcal{I}(\mathcal{E})$, hence $(1-u^2)\varphi(aa^*)(1-F^2) \in \mathcal{I}(\mathcal{E})$. This is equivalent to $(1-u^2)\varphi(a)(1-F^2) \in \mathcal{I}(\mathcal{E})$ for all $a \in A$ because elements of the form aa^* span A.

The set of $f \in \mathrm{C}[0,1]$ with $f(1-u^2)\varphi(a)(1-F^2) \in \mathcal{I}(\mathcal{E})$ for all $a \in A$ is a closed ideal because $\mathcal{I}(\mathcal{E})$ is a closed ideal. Since $1-u^2$ and $\sqrt{1-u^2}$ generate the same closed ideal in $\mathrm{C}[0,1]$, namely, the ideal of functions vanishing at 1, our condition is equivalent to $\sqrt{1-u^2}\varphi(a)(1-F^2) \in \mathcal{I}(\mathcal{E})$ for all $a \in A$. We may do the same to F, so our condition is also equivalent to $\sqrt{1-u^2}\varphi(a)\sqrt{1-F^2} \in \mathcal{I}(\mathcal{E})$ for all $a \in A$. Moreover, we may change the order of the three factors here arbitrarily. Therefore, $[F_{2s}, \varphi_2(a)] \in \mathcal{I}(\mathcal{E}_2)$ for all $a \in A$. Furthermore, $[u, F] \in \mathcal{K}(\mathcal{E})$ implies

$$\begin{aligned}&(1-F_{2s}^2)\varphi(a)\\ &\equiv \begin{pmatrix}(1-F^2)(1-s^2+s^2u^2) & 0\\ 0 & (1-F^2)(1-s^2+s^2u^2)\end{pmatrix}\varphi(a) \bmod \mathcal{K}(\mathcal{E}_2).\end{aligned}$$

Hence $(\mathcal{E}_2, F_{2s})$ is a homotopy of cycles for $\mathrm{KE}^G(A, B)$. For $s = 0$, $(\mathcal{E}_2, F_{20})$ is a direct sum of $(\mathcal{E}, F)$ with a degenerate cycle and hence homotopic to $(\mathcal{E}, F)$. Thus $(\mathcal{E}, F)$ is homotopic to $(\mathcal{E}_2, F_{21})$. The diagonal entries of $1 - F_{21}^2$ are $(1-F^2)u^2$, which belongs to $\mathcal{C}(\mathcal{E})$ because $u \in \mathcal{C}(\mathcal{E})$. Hence $1 - F_{21}^2 \in \mathcal{C}(\mathcal{E}_2)$. Thus any cycle for $\mathrm{KE}^G(A, B)$ is homotopic to one satisfying (aKm1s) and (aKm3s).

If we already have $F = F^*$ and $\|F\| \leq 1$, then the canonical homotopy from F to $\chi((F + F^*)/2)$ is constant. And if also $1 - F^2 \in \mathcal{C}(\mathcal{E})$, then the homotopy F_{2s} constructed above satisfies $1 - F_{2s}^2 \in \mathcal{C}(\mathcal{E}_2)$ for any choice of u. If two cycles $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ satisfying (aKm1s) and (aKm3s) are homotopic, then we may apply the modifications above to a homotopy between them; this provides a homotopy between their modifications that satisfies (aKm1s) and (aKm3s); since the canonical homotopies from $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ to their modifications also satisfy (aKm1s) and (aKm3s), we get a homotopy from $(\mathcal{E}_1, F_1)$ to $(\mathcal{E}_2, F_2)$ satisfying (aKm1s) and (aKm3s). Hence restricting to cycles satisfying (aKm1s) and (aKm3s) does not change $\mathrm{KE}^G(A, B)$. □

The *standard $G \times \mathbb{Z}/2$-equivariant Hilbert B-module* is

$$\mathcal{H} = \mathcal{H}_B := L^2(G \times \mathbb{Z}/2) \otimes \ell^2(\mathbb{N}) \otimes B.$$

Lemma 2.2. *We get the same group* $\mathrm{KE}^G(A \otimes \mathcal{K}(L^2G), B \otimes \mathcal{K}(L^2G))$ *if we restrict attention to cycles for* $\mathrm{KE}^G(A \otimes \mathcal{K}(L^2G), B \otimes \mathcal{K}(L^2G))$ *that satisfy* (aKm1s), (aKm3s) *and* (aKm4s) *and with underlying Hilbert module $\mathcal{E} = \mathcal{H}_{B \otimes \mathcal{K}(L^2G)}L$, and homotopies between such cycles with the same properties.*

Proof. The main ideas below already appeared in [7] and as Fell's trick in [2, Lemma 3.3.3]. Let F_0 be the canonical isomorphism $\mathcal{H}_+ \leftrightarrow \mathcal{H}_-$ and let $\varphi_0 = 0$; this gives a degenerate cycle with underlying Hilbert module $\mathcal{H}L$. Hence any $\mathrm{KE}^G(A, B)$-cycle $(\mathcal{E}, F)$ is equivalent to $(\mathcal{E} \oplus \mathcal{H}L, F \oplus F_0)$. Since $\mathcal{E}$ must be countably generated, Kasparov's Stabilisation Theorem gives a G-continuous, $\mathbb{Z}/2$-equivariant unitary operator $V: \mathcal{E} \oplus \mathcal{H}L \to \mathcal{H}L$. (Unless G is compact, we cannot expect V to be G-equivariant.)

Therefore, we get the same set of homotopy classes $\mathrm{KE}^G(A, B)$ if we restrict attention to cycles $(\mathcal{E}, F)$ for which there is a G-continuous, $\mathbb{Z}/2$-grading preserving unitary $V: \mathcal{E} \to \mathcal{H}_B L$. This may be combined with Lemma 2.1, that is, we get the same set of homotopy classes if we assume $(\mathcal{E}, F)$ to satisfy (aKm1s) and (aKm3s) and to have such a unitary V. The unitary V defines a $G \times \mathbb{Z}/2$-equivariant unitary

$$V': L^2(G, \mathcal{E}) \to L^2(G, \mathcal{H}_B L), \quad (V' f)(g) := g(V(f(g))).$$

By a similar formula, any $F \in \mathcal{L}(\mathcal{E})$ defines a G-equivariant adjointable operator F' on $L^2(G, \mathcal{E})$. By [3, Theorem 4.21], the exterior product map

$$\mathrm{KE}^G(A, B) \to \mathrm{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G)), \; (\mathcal{E}, F) \mapsto (\mathcal{E} \otimes \mathcal{K}(L^2 G), F \otimes 1),$$

is an isomorphism. So any cycle for $\mathrm{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G))$ is homotopic to $(\mathcal{E} \otimes \mathcal{K}(L^2 G), F \otimes 1)$ for some cycle $(\mathcal{E}, F)$ for $\mathrm{KE}^G(A, B)$ with (aKm1s) and (aKm3s) and a unitary $V: \mathcal{E} \to \mathcal{H}_B L$ as above; and if two such cycles are homotopic, there is a homotopy of the same form.

As a Hilbert module over itself, $\mathcal{K}(L^2 G) \cong L^2 G \otimes (L^2 G)^*$, where $(L^2 G)^*$ is viewed as a Hilbert $\mathcal{K}(L^2 G)$-module. Hence V' induces a $G \times \mathbb{Z}/2$-equivariant unitary $\mathcal{E} \otimes \mathcal{K}(L^2 G) \to \mathcal{H}_B L \otimes \mathcal{K}(L^2 G) = \mathcal{H}_{B \otimes \mathcal{K}(L^2 G)} L$, and F' induces a G-equivariant odd operator on $\mathcal{E} \otimes \mathcal{K}(L^2 G)$. Since $gF - F \in \mathcal{I}(\mathcal{E})$, F' is a small perturbation of $F \otimes 1$. Thus we get the same group $\mathrm{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G))$ if we use only those cycles and homotopies that satisfy (aKm1s), (aKm3s) and (aKm4s) and have underlying Hilbert module $\mathcal{E} = \mathcal{H}_{B \otimes \mathcal{K}(L^2 G)} L$. □

For the passage from KE^G to E^G, it is harmless to stabilise the C*-algebras A and B. Hence Lemma 2.2 says that it is essentially no loss of generality to restrict attention to those cycles for KE^G that satisfy the stronger assumptions (aKm1s), (aKm3s) and (aKm4s). Furthermore, we may assume that $\mathcal{E} = \mathcal{H}_B L$ is the constant family with fibre the standard G-equivariant Hilbert B-module $\mathcal{H}_B$.

Remark 2.3. If a cycle for $\mathrm{KE}^G(A, B)$ is in the image of $\mathrm{KK}^G(A, B)$, then it satisfies more than (aKm2), namely, $[F, \varphi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$. If KK^G and KE^G were equivalent, then any cycle for KE^G would be homotopic to one with this extra property. I do not know, however, how to prove this.

3. Constructing asymptotic morphisms from KE-cycles

Let $S := C_0((-1,1))$ with the $\mathbb{Z}/2$-grading automorphism $\gamma f(x) = f(-x)$. Dumitraşcu maps a cycle $(\mathcal{H}_B L, \varphi, F)$ for $\mathrm{KE}^G(A, B)$ to an asymptotic morphism from $S \hat{\otimes} A$ to $\mathcal{K}(\mathcal{H}_B)$ in [3, Section 5.1], as follows. Since $\mathcal{I}(\mathcal{H}_B L) \cap \mathcal{C}(\mathcal{H}_B L) = \mathcal{K}(\mathcal{H}_B L)$ and $[F, \varphi(A)] \subseteq \mathcal{I}(\mathcal{H}_B L)$ by (aKm2), the images of A and F in $\mathcal{C}(\mathcal{H}_B L)/\mathcal{K}(\mathcal{H}_B L)$ commute. Hence there is a unique *-homomorphism

$$\Xi\colon S \hat{\otimes} A \to \mathcal{C}(\mathcal{H}_B L)/\mathcal{K}(\mathcal{H}_B L)$$

with $\Xi(h \otimes a) = h(F)\varphi(a)$ for all $h \in S$, $a \in A$ (this works for the maximal C*-norm, which is the only C*-norm here because S is nuclear). We may lift Ξ to a map (of sets) $\bar{\Xi}\colon S \hat{\otimes} A \to \mathcal{C}(\mathcal{H}_B L)$, which we may view as a family of maps $\bar{\Xi}_t\colon S \hat{\otimes} A \to \mathcal{K}(\mathcal{H}_B)$, $t \in L$. These maps $\bar{\Xi}_t$ form an asymptotic morphism. This is used in [3, Section 5.1] to construct a functor $\mathrm{KE}^G \to \mathrm{E}^G$.

For cycles with extra properties as in Lemma 2.2, we are going to construct a completely positive, contractive and $G \times \mathbb{Z}/2$-equivariant choice for $\bar{\Xi}$ in a natural way. Using Thomsen's picture for KK^G, this will give a functor $\mathrm{KE}^G \to \mathrm{KE}^G$, by essentially the same arguments as in [3].

First we approximate the identity map on S by $\mathbb{Z}/2$-equivariant, completely positive contractions of finite rank. Let $n \in \mathbb{N}$. Let

$$I_n := \{-2^n + 1, -2^n + 2, \dots, 2^n - 1\}.$$

For $k \in I_n$, define $\psi_{n,k} \in S$ by

$$\psi_{n,k}(x) := \begin{cases} \sqrt{2^n x - (k-1)} & \text{for } k - 1 \le 2^n x \le k, \\ \sqrt{k + 1 - 2^n x} & \text{for } k \le 2^n x \le k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\psi_{n,k}^2$ is the unique continuous, piecewise linear function with singularities in $2^{-n} \cdot \{k-1, k, k+1\}$ and $\psi_{n,k}^2(2^{-n}k) = 1$ and $\psi_{n,k}^2(2^{-n}l) = 0$ for $k \ne l$. We have $\gamma(\psi_{n,k}) = \psi_{n,-k}$ for all $k \in I_n$. Define

$$\Psi_n\colon S \to S, \quad f \mapsto \sum_{k \in I_n} f(2^{-n}k) \cdot \psi_{n,k}^2.$$

Equivalently,

$$\Psi_n f(2^{-n}(k+t)) = (1-t) \cdot f(2^{-n}k) + t \cdot f(2^{-n}(k+1)) \tag{3.1}$$

for $k \in \{-2^n, -2^n + 1, \dots, 2^n - 1\}$, $t \in [0,1]$, because $f(\pm 1) = 0$.

By construction, Ψ_n is a completely positive map of finite rank. It is grading-preserving because $\gamma(\psi_{n,k}) = \psi_{n,-k}$, and contractive because $\sum_{k \in I_n} \psi_{n,k}^2 \le 1$.

We have $\lim\|f-\Psi_n(f)\|_\infty=0$ for each $f\in S$, and this holds uniformly for f in a compact subset of S because all the operators Ψ_n are contractions.

Now let A and B be $\mathbb{Z}/2$-graded C*-algebras. Let $\hat{\otimes}$ be the graded-commutative tensor product. This is functorial for grading-preserving completely positive contractions. Hence we get a grading-preserving completely positive contraction $\Psi_n^A=\Psi_n\mathbin{\hat{\otimes}}\mathrm{id}_A\colon S\mathbin{\hat{\otimes}}A\to S\mathbin{\hat{\otimes}}A$. The sequence $\Psi_n^A(f)$ converges in norm to f for any $f\in S\mathbin{\hat{\otimes}}A$ because Ψ_n converges to id_S uniformly on compact subsets.

To make use of Lemma 2.2, we assume $A=A_0\otimes\mathcal{K}(L^2G)$ and $B=B_0\otimes\mathcal{K}(L^2G)$ for some $\mathbb{Z}/2$-graded C*-algebras A_0 and B_0. Then we get the same group $\mathrm{KE}^G(A,B)$ if we use cycles and homotopies that satisfy (aKm1s), (aKm2), (aKm3s) and (aKm4s), and where the underlying family of Hilbert modules $\mathcal{E}$ is the constant family $\mathcal{H}_BL$ with the standard G-equivariant Hilbert B-module $\mathcal{H}_B$ as its fibre. (Actually, $\mathcal{H}_B$ is G-equivariantly isomorphic to $(B^\infty)\oplus(B^\infty)^{\mathrm{op}}$.)

Let (φ,F) be such a special cycle for $\mathrm{KE}^G(A,B)$. That is, $\varphi\colon A\to\mathcal{L}(\mathcal{H}_BL)$ is a $G\times\mathbb{Z}/2$-equivariant *-homomorphism and $F\in\mathcal{L}(\mathcal{H}_BL)$, such that $\gamma(F)=-F$, $F=F^*$, $\|F\|\le1$, $g(F)=F$ for all $g\in G$, $\lim_{t\to\infty}\|[F_t,\varphi_t(a)]\|=0$ for all $a\in A$, and $(1-F^2)\varphi(a)\in\mathcal{C}(\mathcal{H}_BL)$. Since $[(1-F^2)\varphi(a^*)]^*=\varphi(a)(1-F^2)$, it is equivalent to require $(1-F^2)\varphi(a)\in\mathcal{C}(\mathcal{H}_BL)$ or $\varphi(a)(1-F^2)\in\mathcal{C}(\mathcal{H}_BL)$ for all $a\in A$. Furthermore, this implies $h(F)\varphi(a)\in\mathcal{C}(\mathcal{H}_BL)$ and $\varphi(a)h(F)\in\mathcal{C}(\mathcal{H}_BL)$ for all $h\in S$.

The next step is easier to write down for trivially graded A, so we assume this for a moment to explain our idea. Then $S\mathbin{\hat{\otimes}}A\cong\mathrm{C}_0((-1,1),A)$. Since $\psi_{n,k}\in S$, we get $\psi_{n,k}(F)\varphi(a)\in\mathcal{C}(\mathcal{H}_BL)$ for all $n\in\mathbb{N}$, $k\in I_n$, $a\in A$. Hence

$$\xi_n(f):=\sum_{k=-2^n+1}^{2^n-1}\psi_{n,k}(F)\varphi(f(k\cdot2^{-n}))\psi_{n,k}(F) \tag{3.2}$$

for $f\colon(-1,1)\to A$ continuous with $f(\pm1)=0$ defines a map $\xi_n\colon S\mathbin{\hat{\otimes}}A\to\mathcal{C}(\mathcal{H}_BL)$. This map is grading-preserving, completely positive and G-equivariant because $F=F^*$, $\|F\|\le1$ and F is G-equivariant. If $f\ge0$, then

$$\xi_n(f)\le\sum_{k=-2^n+1}^{2^n-1}\psi_{n,k}(F)\cdot\|f(k\cdot2^{-n})\|\cdot\psi_{n,k}(F)\le\|f\|_\infty\sum_{k=-2^n+1}^{2^n-1}\psi_{n,k}(F)^2\le\|f\|_\infty;$$

thus ξ_n is contractive. If $\pi\colon\mathcal{C}(\mathcal{H}_BL)\to\mathcal{C}(\mathcal{H}_BL)/\mathcal{K}(\mathcal{H}_BL)$ denotes the quotient map, then $\pi\circ\xi_n=\Xi\circ\Psi_n^A$ because $\pi(A)$ and $\pi(F)$ commute. Now we remove the assumption that A is trivially graded:

Lemma 3.1. *There is a sequence of $G\times\mathbb{Z}/2$-equivariant completely positive contractive maps $\xi_n\colon S\mathbin{\hat{\otimes}}A\to\mathcal{C}(\mathcal{H}_BL)$ with $\pi\circ\xi_n=\Xi\circ\Psi_n^A$ for $n\in\mathbb{N}$, even if A is $\mathbb{Z}/2$-graded.*

Proof. We fix $n \in \mathbb{N}$. To make the proof of complete positivity easy, we directly construct the Stinespring dilation of our map ξ_n. Let

$$\mathcal{E} := \bigoplus_{k=0}^{2^n-1} \big(\mathcal{H}_B L \oplus (\mathcal{H}_B L)^{\mathrm{op}}\big).$$

Let A act by $\varphi \oplus \varphi \circ \gamma$ on each summand $\mathcal{H}_B L \oplus (\mathcal{H}_B L)^{\mathrm{op}}$. Let $x: \mathcal{E} \to \mathcal{E}$ be the operator that acts by

$$\begin{pmatrix} 0 & 2^{-n}k \\ 2^{-n}k & 0 \end{pmatrix}$$

on the kth summand. This operator is self-adjoint, and it graded-commutes with the representation of A because we take $\varphi\gamma$ for the second summands. Thus the functional calculus for x provides a *-homomorphism $S \to \mathcal{L}(\mathcal{E})$ that graded-commutes with A. Hence we get a $G \times \mathbb{Z}/2$-equivariant *-homomorphism $\alpha: S \hat{\otimes} A \to \mathcal{L}(\mathcal{E})$. We let $\xi_n(f) := V^*\alpha(f)V$ for all $f \in S \hat{\otimes} A$, where $V = (V_k)_{k \in I_n}: \mathcal{H}_B L \to \mathcal{E}$ has the components

$$2^{-1/2}(\psi_{n,k}(F) + \psi_{n,-k}(F)): \mathcal{H}_B L \to \mathcal{H}_B L,$$
$$2^{-1/2}(\psi_{n,k}(F) - \psi_{n,-k}(F)): \mathcal{H}_B L \to (\mathcal{H}_B L)^{\mathrm{op}}$$

for $k > 0$, and

$$\psi_{n,0}(F) = 2^{-1}(\psi_{n,k}(F) + \psi_{n,-k}(F)): \mathcal{H}_B L \to \mathcal{H}_B L,$$
$$0 = 2^{-1}(\psi_{n,k}(F) - \psi_{n,-k}(F)): \mathcal{H}_B L \to (\mathcal{H}_B L)^{\mathrm{op}}$$

for $k = 0$. Notice that V_k is grading-preserving because $\psi_{n,k} + \psi_{n,-k}$ is an even function and $\psi_{n,k} - \psi_{n,-k}$ is an odd function. Since V is G-invariant as well, ξ_n is $G \times \mathbb{Z}/2$-equivariant. The map ξ_n is completely positive. Since

$$\begin{aligned} V^*V &= \Big(\psi_{n,0}^2 + \frac{1}{2}\sum_{k=1}^{2^n-1} (\psi_{n,k} + \psi_{n,-k})^2 + (\psi_{n,k} - \psi_{n,-k})^2\Big)(F) \\ &= \Big(\psi_{n,0}^2 + \sum_{k=1}^{2^n-1} \psi_{n,k}^2 + \psi_{n,-k}^2\Big)(F) = \Big(\sum_{k \in I_n} \psi_{n,k}^2\Big)(F) \le 1, \end{aligned}$$

the map ξ_n is completely contractive.

Let $f \in S$ and $a \in A$. If $f \in S$ is even, then

$$\begin{aligned} \xi_n(f \hat{\otimes} a) = \psi_{n,0}(F) f(0)\varphi(a)\psi_{n,0}(F) \\ + \sum_{k=1}^{2^n-1} (\psi_{n,k}(F) + \psi_{n,-k}(F)) f(2^{-n}k)\varphi(a)(\psi_{n,k}(F) + \psi_{n,-k}(F)) \\ + (\psi_{n,k}(F) - \psi_{n,-k}(F)) f(2^{-n}k)\varphi\gamma(a)(\psi_{n,k}(F) - \psi_{n,-k}(F)); \end{aligned}$$

if $f \in S$ is odd, then

$$\xi_n(f \hat{\otimes} a) = \sum_{k=1}^{2^n-1} (\psi_{n,k}(F) - \psi_{n,-k}(F)) f(2^{-n}k)\varphi(a)(\psi_{n,k}(F) + \psi_{n,-k}(F)) + (\psi_{n,k}(F) + \psi_{n,-k}(F)) f(2^{-n}k)\varphi\gamma(a)(\psi_{n,k}(F) - \psi_{n,-k}(F))$$

Now we use that $\pi(F)$ graded-commutes with $\pi\varphi(A)$ to simplify $\pi \circ \xi_n(f \hat{\otimes} a)$. For even f, this is equal to the π-image of

$$\frac{1}{2}\sum_{k=1}^{2^n-1} (\psi_{n,k} + \psi_{n,-k})^2(F) f(2^{-n}k)\varphi(a) + (\psi_{n,k} - \psi_{n,-k})^2(F) f(2^{-n}k)\varphi(a) + \psi_{n,0}^2(F) f(0)\varphi(a) = \sum_{k\in I_n} \psi_{n,k}^2(F) f(2^{-n}k)\varphi(a) = \Psi_n^A(f)(F)\cdot\varphi(a),$$

which is $\Xi \circ \Psi_n^A(f \hat{\otimes} a)$. For odd f, $\pi \circ \xi_n(f \hat{\otimes} a)$ is equal to the π-image of

$$\begin{aligned}&\sum_{k=1}^{2^n-1} (\psi_{n,k} + \psi_{n,-k})(F)(\psi_{n,k} - \psi_{n,-k})(F) f(2^{-n}k)\varphi(a)\\ &= \sum_{k=1}^{2^n-1} (\psi_{n,k}^2 - \psi_{n,-k}^2)(F) f(2^{-n}k)\varphi(a)\\ &= \sum_{k\in I_n} \psi_{n,k}^2(F) f(2^{-n}k)\varphi(a) = \Psi_n^A(f)(F)\cdot\varphi(a),\end{aligned}$$

which is $\Xi \circ \Psi_n^A(f \hat{\otimes} a)$ once again. Thus $\Xi \circ \Psi_n^A(f \hat{\otimes} a) = \pi(\Psi_n^A(f)(F)\cdot\varphi(a))$ for all $f \in S$, $a \in A$, as desired. □

Let $\xi_{n+s} = (1-s)\xi_n + s\xi_{n+1}$ for $n \in \mathbb{N}$, $s \in [0,1]$. The maps $(\xi_s)_{s\in L}$ form a continuous family of grading-preserving, G-equivariant, completely positive contractions $\xi_s\colon S \hat{\otimes} A \to \mathcal{C}(\mathcal{H}_B L)$. In the following, we view ξ_s as a family of functions $\xi_{s,t}\colon S \hat{\otimes} A \to \mathcal{K}(\mathcal{H}_B)$, and we lift Ξ to an asymptotic morphism $\bar{\Xi}_t\colon S \hat{\otimes} A \to \mathcal{K}(\mathcal{H}_B)$.

Lemma 3.2. *For separable A, there is a continuous increasing function $t_0\colon L \to L$ with $\lim_{s\to\infty} t_0(s) = \infty$ such that for all $t \geq t_0$, $\xi_{s,t(s)}\colon S \hat{\otimes} A \to \mathcal{K}(\mathcal{H}_B)$ is asymptotically equal to the reparametrisation $\Xi_{t(s)}$ of Ξ and hence an asymptotic morphism in the same class as Ξ.*

Proof. Since $S \hat{\otimes} A$ is separable, there is a sequence (f_i) whose closed linear span is $S \hat{\otimes} A$. For the asymptotic equality we need $\pi\xi_{s,t(s)}(f_i) = \Xi_{t(s)}(f_i)$ for $i \in \mathbb{N}$. We have norm convergence $\lim_{s\to\infty} \Psi_s^A(f_i) = f_i$ for all $i \in \mathbb{N}$. Since $\|f_i\| \to 0$

and Ψ_s^A is uniformly bounded, this convergence is uniform. Hence for each $n \in \mathbb{N}$ there is $s_n \in L$ such that $\|\Psi_s^A(f_i) - f_i\| < 1/n$ for all $s \geq s_n$, $i \in \mathbb{N}$. We may assume that the sequence (s_n) is strictly increasing with $\lim_{n\to\infty} s_n = \infty$.

Since $\pi \circ \xi_s = \Xi \circ \Psi_s^A$ and Ξ is a *-homomorphism, we get $\|\pi \circ \xi_s(f_i) - \Xi(f_i)\| < 1/n$ for all $s \geq s_n$, $i \in \mathbb{N}$. By definition of the quotient norm in $\mathcal{C}(\mathcal{H}_B L)/\mathcal{K}(\mathcal{H}_B L)$, we may find $t_i(s,n) \in L$ with $\|\xi_{s,t}(f_i) - \bar{\Xi}_t(f_i)\| < 1/n$ for $s \geq s_n$, $t \geq t_i(s,n)$. Since $\|f_i\| \to 0$ for $i \to \infty$, there are only finitely many i with $\|\xi_{s,t}(f_i)\| \geq 1/2n$ and $\bar{\Xi}_t(f_i) \geq 1/2n$; hence we may find $t(s,n)$ independent of i with $\|\xi_{s,t}(f_i) - \bar{\Xi}_t(f_i)\| < 1/n$ for all $i \in \mathbb{N}$, $s \geq s_n$, $t \geq t(s,n)$.

Now choose $t_0(s)$ increasing and continuous with $\lim_{s\to\infty} t_0(s) = \infty$ and $t_0(s) \geq t(s,n)$ for $s \in [s_n, s_{n+1}]$. If $t(s) \geq t_0(s)$ for all $s \in L$, then $\|\xi_{s,t(s)}(f_i) - \bar{\Xi}_{t(s)}(f_i)\| < 1/n$ for all $s \in [s_n, s_{n+1}]$ and all $i \in \mathbb{N}$. Thus $\xi_{s,t(s)}$ and $\bar{\Xi}_{t(s)}$ are asymptotically equal. This implies that $\xi_{s,t(s)}$ is an asymptotic morphism because $\bar{\Xi}_t$ is one. □

The asymptotic morphism $\xi_{s,t(s)}$ from $S \hat{\otimes} A$ to $\mathcal{K}(\mathcal{H}_B)$ in Lemma 3.2 is also linear, completely positive contractive and $G \times \mathbb{Z}/2$-equivariant. Thomsen [8] describes $\mathrm{KK}^G(A, B)$ using asymptotic homomorphisms with these extra properties. We cannot directly appeal to [8] because we have replaced the ungraded suspension on both A and B by the graded suspension S on A alone. It is well-known, however, that both approaches give the same definition of equivariant E-theory. For the same reason, both approaches with added complete positivity requirements give $\mathrm{KK}^G(A, B)$. Let us make this more explicit.

An asymptotic morphism (ξ_t) from $S \hat{\otimes} A$ to $\mathcal{K}(\mathcal{H}_B)$ gives an extension

$$0 \to \mathrm{C}_0(L, \mathcal{K}(\mathcal{H}_B)) \to E \to S \hat{\otimes} A \to 0,$$

where $E = \mathrm{C}_0(L, \mathcal{K}(\mathcal{H}_B)) + \xi(S \hat{\otimes} A)$; it comes with evaluation homomorphisms $\epsilon_t \colon E \to \mathcal{K}(\mathcal{H}_B)$ for $t \in L$. If the asymptotic morphism is $G\times\mathbb{Z}/2$-equivariant, completely positive and contractive, then the extension above has a $G \times \mathbb{Z}/2$-equivariant, completely positive and contractive cross-section. Hence there is a long exact sequence in $\mathrm{KK}^{G\times\mathbb{Z}/2}$ for this extension. Since the kernel is contractible, we get that the quotient map in the extension is invertible in $\mathrm{KK}^{G\times\mathbb{Z}/2}$. Composing its inverse with the evaluation homomorphism, we get a class in

$$\mathrm{KK}^{G\times\mathbb{Z}/2}(S \hat{\otimes} A, \mathcal{K}(\mathcal{H}_B)) \cong \mathrm{KK}^{G\times\mathbb{Z}/2}(S \hat{\otimes} A, B) \cong \mathrm{KK}^G(A, B).$$

Here we use a description of KK^G for $\mathbb{Z}/2$-graded C*-algebras in terms of $G \times \mathbb{Z}/2$-equivariant Kasparov theory that goes back to Haag in the non-equivariant case and is extended to the equivariant case in [7].

Thus we attach a class in $\mathrm{KK}^G(A, B)$ to a cycle for $\mathrm{KE}^G(A, B)$. Since the same construction applies to homotopies, this construction descends to a well-defined map $\xi\colon \mathrm{KE}^G(A, B) \to \mathrm{KK}^G(A, B)$. By design, the composite map

$$\mathrm{KE}^G(A, B) \to \mathrm{KK}^G(A, B) \to \mathrm{E}^G(A, B)$$

is the functor Ξ of [3].

The Kasparov product in KK^G becomes the composition of completely positive equivariant asymptotic morphisms in the above picture. A composite of two completely positive equivariant asymptotic morphisms is again completely positive and equivariant. So the same argument as in [2] shows that ξ is a functor.

Proposition 3.3. *The composite map*

$$\mathrm{KK}^G(A, B) \to \mathrm{KE}^G(A, B) \to \mathrm{KK}^G(A, B)$$

is the identity on $\mathrm{KK}^G(A, B)$.

Proof. This clearly holds on the class in $\mathrm{KK}^G(A, B)$ of a grading-preserving equivariant *-homomorphism $f\colon S \hat{\otimes} A \to B$. If this f is a KK^G-equivalence, then $[f]^{-1}$ is mapped to $[f]^{-1}$ as well by functoriality. Hence any composite of such classes is mapped to itself by functoriality. Any class in KK^G may be written as such a composition of classes of $[f]$ and $[f]^{-1}$. This follows from the Cuntz picture for $\mathrm{KK}^G(A, B) \cong \mathrm{KK}^{G\times\mathbb{Z}/2}(S \hat{\otimes} A, B)$ in [7]. □

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Received 19 March, 2015; revised 22 May, 2015

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