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Braided quantum SU(2) groups

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Abstract. We construct a family of q-deformations of SU(2) for complex parameters $q \neq 0$. For real q, the deformation coincides with Woronowicz' compact quantum $\mathrm{SU}_q(2)$ group. For $q \notin \mathbb{R}$, $\mathrm{SU}_q(2)$ is only a braided compact quantum group with respect to a certain tensor product functor for C*-algebras with an action of the circle group.

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1. Introduction

The q-deformations of SU(2) for real deformation parameters $0 < q < 1$ discovered in [10] are among the first and most important examples of compact quantum groups. Here we construct a family of q-deformations of SU(2) for *complex* parameters $q \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $q \notin \mathbb{R}$, $\mathrm{SU}_q(2)$ is not a compact quantum group, but a braided compact quantum group in a suitable tensor category.

A compact quantum group $\mathbb{G}$ as defined in [11] is a pair $\mathbb{G} = (A, \Delta)$ where $\Delta\colon A \to A \otimes A$ is a coassociative morphism satisfying the cancellation law (1.4) below. The C*-algebra A is viewed as the algebra of continuous functions on $\mathbb{G}$.

The theory of compact quantum groups is formulated within the category $\mathcal{C}^*$ of C*-algebras. This category with the minimal tensor functor $\otimes$ is a monoidal category (see [2]). A more general theory may be formulated within a monoidal category $(\mathcal{D}^*, \boxtimes)$, where $\mathcal{D}^*$ is a suitable category of C*-algebras with additional structure and $\boxtimes\colon \mathcal{D}^* \times \mathcal{D}^* \to \mathcal{D}^*$ is a monoidal bifunctor on $\mathcal{D}^*$. Braided Hopf algebras may be defined in braided monoidal categories (see [4, Definition 9.4.5]). The braiding becomes unnecessary when we work in categories of C*-algebras.

Let A and B be C*-algebras. The multiplier algebra of B is denoted by $\mathrm{M}(B)$. A *morphism* $\pi \in \mathrm{Mor}(A, B)$ is a *-homomorphism $\pi\colon A \to \mathrm{M}(B)$ with $\pi(A)B = B$. If A and B are unital, a morphism is simply a unital *-homomorphism.

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Let $\mathbb{T}$ be the group of complex numbers of modulus 1 and let $\mathcal{C}^*_{\mathbb{T}}$ be the category of $\mathbb{T}$-C*-algebras; its objects are C*-algebras with an action of $\mathbb{T}$, arrows are $\mathbb{T}$-equivariant morphisms. We shall use a family of monoidal structures $\boxtimes_\zeta$ on $\mathcal{C}^*_{\mathbb{T}}$ parametrised by $\zeta \in \mathbb{T}$, which is defined as in [5].

The C*-algebra A of $\mathrm{SU}_q(2)$ is defined as the universal unital C*-algebra generated by two elements α, γ subject to the relations

$$\begin{cases} \alpha^*\alpha + \gamma^*\gamma = \mathrm{I}, \\ \alpha\alpha^* + |q|^2\gamma^*\gamma = \mathrm{I}, \\ \gamma\gamma^* = \gamma^*\gamma, \\ \alpha\gamma = \overline{q}\gamma\alpha, \\ \alpha\gamma^* = q\gamma^*\alpha. \end{cases} \tag{1.1}$$

For real q, the algebra A coincides with the algebra of continuous functions on the quantum $\mathrm{SU}_q(2)$ group described in [10]: $A = \mathrm{C}(\mathrm{SU}_q(2))$. Then there is a unique morphism $\Delta\colon A \to A \otimes A$ with

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma. \end{aligned} \tag{1.2}$$

It is coassociative, that is,

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta, \tag{1.3}$$

and has the following cancellation property:

$$\begin{aligned} A \otimes A &= \Delta(A)(A \otimes \mathrm{I}), \\ A \otimes A &= \Delta(A)(\mathrm{I} \otimes A); \end{aligned} \tag{1.4}$$

here and below, EF for two closed subspaces E and F of a C*-algebra denotes the norm-closed linear span of the set of products ef for $e \in E$, $f \in F$.

If q is not real, then the operators on the right hand sides of (1.2) do not satisfy the relations (1.1), so there is no morphism Δ satisfying (1.2). Instead, (1.2) defines a $\mathbb{T}$-equivariant morphism $A \to A \boxtimes_\zeta A$ for the monoidal functor $\boxtimes_\zeta$ with $\zeta = q/\overline{q}$. This morphism in $\mathcal{C}^*_{\mathbb{T}}$ satisfies appropriate analogues of the coassociativity and cancellation laws (1.3) and (1.4), so we get a braided compact quantum group. Here the action of $\mathbb{T}$ on A is defined by $\rho_z(\alpha) = \alpha$ and $\rho_z(\gamma) = z\gamma$ for all $z \in \mathbb{T}$.

For $X, Y \in \mathrm{Obj}(\mathcal{C}^*)$, $X \otimes Y$ contains commuting copies $X \otimes \mathrm{I}_Y$ of X and $\mathrm{I}_X \otimes Y$ of Y with $X \otimes Y = (X \otimes \mathrm{I}_Y)(\mathrm{I}_X \otimes Y)$. Similarly, $X \boxtimes_\zeta Y$ for $X, Y \in \mathcal{C}^*_{\mathbb{T}}$ is a C*-algebra with injective morphisms $j_1 \in \mathrm{Mor}(X, X \boxtimes_\zeta Y)$ and $j_2 \in \mathrm{Mor}(Y, X \boxtimes_\zeta Y)$ such that $X \boxtimes_\zeta Y = j_1(X)j_2(Y)$. For $\mathbb{T}$-homogeneous elements $x \in X_k$ and $y \in Y_l$ (as defined in (3.4)), we have the commutation relation

$$j_1(x)j_2(y) = \zeta^{kl} j_2(y)j_1(x) \tag{1.5}$$

The following theorem contains the main result of this paper:

Theorem 1.1. *Let* $q \in \mathbb{C} \setminus \{0\}$ *and* $\zeta = q/\overline{q}$. *Then*

(1) *there is a unique* $\mathbb{T}$*-equivariant morphism* $\Delta \in \mathrm{Mor}(A, A \boxtimes_\zeta A)$ *with*

$$\begin{aligned} \Delta(\alpha) &= j_1(\alpha) j_2(\alpha) - q j_1(\gamma)^* j_2(\gamma), \\ \Delta(\gamma) &= j_1(\gamma) j_2(\alpha) + j_1(\alpha)^* j_2(\gamma); \end{aligned} \tag{1.6}$$

(2) Δ *is coassociative, that is,*

$$(\Delta \boxtimes_\zeta \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \boxtimes_\zeta \Delta) \circ \Delta;$$

(3) Δ *obeys the cancellation law*

$$j_1(A)\Delta(A) = \Delta(A) j_2(A) = A \boxtimes_\zeta A.$$

We also describe some important features of the representation theory of $\mathrm{SU}_q(2)$ to explain the definition of $\mathrm{SU}_q(2)$, and we relate $\mathrm{SU}_q(2)$ to the quantum U(2) groups defined by Zhang and Zhao in [12].

Braided Hopf algebras that deform $\mathrm{SL}(2, \mathbb{C})$ are already described in [3]. We could, however, find no precise relationship between Majid's braided Hopf algebra BSL(2) and our braided compact quantum group $\mathrm{SU}_q(2)$.

2. The algebra of $\mathrm{SU}_q(2)$

The following elementary lemma explains what the defining relations (1.1) mean:

Lemma 2.1. *Two elements* α *and* γ *of a* C^**-algebra satisfy the relations* (1.1) *if and only if the following matrix is unitary:*

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

There are at least two ways to introduce a C^*-algebra with given generators and relations. One may consider the algebra $\mathcal{A}$ of all non-commutative polynomials in the generators and their adjoints and take the largest C^*-seminorm on $\mathcal{A}$ vanishing on the given relations. The set $\mathfrak{N}$ of elements with vanishing seminorm is an ideal in $\mathcal{A}$. The seminorm becomes a norm on $\mathcal{A}/\mathfrak{N}$. Completing $\mathcal{A}/\mathfrak{N}$ with respect to this norm gives the desired C^*-algebra A. Another way is to consider the operator domain consisting of all families of operators satisfying the relations. Then A is the algebra of all continuous operator functions on that domain (see [1]). Applying one of these procedures to the relations (1.1) gives a C^*-algebra A with two distinguished elements $\alpha, \gamma \in A$ that is universal in the following sense:

Theorem 2.2. *Let $\widetilde{A}$ be a* C^**-algebra with two elements $\widetilde{\alpha}, \widetilde{\gamma} \in \widetilde{A}$ satisfying*

$$\begin{cases} \tilde{\alpha}^*\tilde{\alpha} + \tilde{\gamma}^*\tilde{\gamma} = \mathrm{I}, \\ \tilde{\alpha}\tilde{\alpha}^* + |q|^2\, \tilde{\gamma}^*\tilde{\gamma} = \mathrm{I}, \\ \tilde{\gamma}\tilde{\gamma}^* = \tilde{\gamma}^*\tilde{\gamma}, \\ \tilde{\alpha}\tilde{\gamma} = \overline{q}\tilde{\gamma}\tilde{\alpha}, \\ \tilde{\alpha}\tilde{\gamma}^* = q\tilde{\gamma}^*\tilde{\alpha}. \end{cases} \tag{2.1}$$

Then there is a unique morphism $\rho \in \mathrm{Mor}(A, \widetilde{A})$ with $\rho(\alpha) = \widetilde{\alpha}$ and $\rho(\gamma) = \widetilde{\gamma}$. □

The elements $\widetilde{\alpha} = \mathrm{I}_{\mathrm{C}(\mathbb{T})} \otimes \alpha$ and $\widetilde{\gamma} = z \otimes \gamma$ of $\mathrm{C}(\mathbb{T}) \otimes A$ satisfy (2.1). Here $z \in \mathrm{C}(\mathbb{T})$ denotes the coordinate function on $\mathbb{T}$. (Later, we also denote elements of $\mathbb{T}$ by z.) Theorem 2.2 gives a unique morphism $\rho^A \in \mathrm{Mor}(A, \mathrm{C}(\mathbb{T}) \otimes A)$ with

$$\begin{aligned} \rho(\alpha) &= \mathrm{I}_{\mathrm{C}(\mathbb{T})} \otimes \alpha, \\ \rho(\gamma) &= z \otimes \gamma. \end{aligned} \tag{2.2}$$

This is a continuous $\mathbb{T}$-action, so we may view (A, ρ^A) as an object in the category $\mathcal{C}^*_{\mathbb{T}}$ described in detail in the next section.

Theorem 2.3. *The* C^**-algebras A for different q with $|q| \neq 0, 1$ are isomorphic.*

Proof. During this proof, we write A_q for our C^*-algebra with parameter q.

First, $A_q \cong A_{q'}$ for $q' = q^{-1}$ by mapping $A_q \ni \alpha \mapsto \alpha' = \alpha^* \in A_{q'}$ and $A_q \ni \gamma \mapsto \gamma' = q^{-1}\gamma \in A_{q'}$. Routine computations show that α' and γ' satisfy the relations (1.1), so that Theorem 2.2 gives a unique morphism $A_q \to A_{q'}$ mapping $\alpha \mapsto \alpha'$ and $\gamma \mapsto \gamma'$. Doing this twice gives $q'' = q$, $\alpha'' = \alpha$ and $\gamma'' = \gamma$, so we get an inverse for the morphism $A_q \to A_{q'}$. This completes the first step. It reduces to the case $0 < |q| < 1$, which we assume from now on.

Secondly, we claim that $A_q \cong A_{|q|}$ if $0 < |q| < 1$. Equation (1.1) implies that γ is normal with $\|\gamma\| \leq 1$. So we may use the functional calculus for continuous functions on the closed unit disc $\mathbb{D}^1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

We claim that

$$\alpha f(\gamma) = f(\overline{q}\gamma)\alpha \tag{2.3}$$

for all $f \in \mathrm{C}(\mathbb{D}^1)$. Indeed, the set $B \subseteq \mathrm{C}(\mathbb{D}^1)$ of functions satisfying (2.3) is a norm-closed, unital subalgebra of $\mathrm{C}(\mathbb{D}^1)$. The last two equations in (1.1) say that B contains the functions $f(\lambda) = \lambda$ and $f^*(\lambda) = \overline{\lambda}$. Since these separate the points of $\mathbb{D}^1$, the Stone–Weierstrass Theorem gives $B = \mathrm{C}(\mathbb{D}^1)$.

Let $q = e^{i\theta}\,|q|$ be the polar decomposition of q. For $\lambda \in \mathbb{D}^1$, let

$$g(\lambda) = \begin{cases} \lambda \mathrm{e}^{\mathrm{i}\theta \log_{|q|}|\lambda|} & \text{for } \lambda \neq 0, \\ 0 & \text{for } \lambda = 0. \end{cases}$$

This is a homeomorphism of $\mathbb{D}^1$ because we get the map g^{-1} if we replace θ by $-\theta$. Thus γ and $\gamma' = g(\gamma)$ generate the same C*-algebra. We also get $g(\overline{q}\lambda) = |q|\, g(\lambda)$, so inserting $f = g$ and $f = \overline{g}$ in (2.3) gives

$$a\gamma' = |q|\,\gamma'\alpha, \qquad a(\gamma')^* = |q|\,(\gamma')^*\alpha.$$

Moreover, $|g(\lambda)| = |\lambda|$ and hence $|\gamma'| = |\gamma|$. Thus we may replace γ by γ' in the first three equations of (1.1). As a result, α and γ' satisfy the relations (1.1) with $|q|$ instead of q. Since g is a homeomorphism, an argument as in the first step now shows that $A_q \cong A_{|q|}$. Finally, [10, Theorem A2.2, page 180] shows that the C*-algebras A_q for $0 < q < 1$ are isomorphic. □

3. Monoidal structure on $\mathbb{T}$-C*-algebras

We are going to describe the monoidal category $(\mathcal{C}^*_{\mathbb{T}}, \boxtimes_\zeta)$ for $\zeta \in \mathbb{T}$ that is the framework for our braided quantum groups. Monoidal categories are defined in [2].

The C*-algebra $\mathrm{C}(\mathbb{T})$ is a compact quantum group with comultiplication

$$\delta\colon \mathrm{C}(\mathbb{T}) \to \mathrm{C}(\mathbb{T}) \otimes \mathrm{C}(\mathbb{T}), \qquad z \mapsto z \otimes z.$$

An object of $\mathcal{C}^*_{\mathbb{T}}$ is, by definition, a pair (X, ρ^X) where X is a C*-algebra and $\rho^X \in \mathrm{Mor}(X, \mathrm{C}(\mathbb{T}) \otimes X)$ makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho^X} & \mathrm{C}(\mathbb{T}) \otimes X \\ \downarrow{\scriptstyle \rho^X} & & \downarrow{\scriptstyle \delta\otimes\mathrm{id}} \\ \mathrm{C}(\mathbb{T}) \otimes X & \xrightarrow[\mathrm{id}_{\mathrm{C}(\mathbb{T})}\otimes\rho^X]{} & \mathrm{C}(\mathbb{T}) \otimes \mathrm{C}(\mathbb{T}) \otimes X \end{array} \tag{3.1}$$

commute and satisfies the *Podleś condition*

$$\rho^X(X)(\mathrm{C}(\mathbb{T}) \otimes 1_X) = \mathrm{C}(\mathbb{T}) \otimes X. \tag{3.2}$$

This is equivalent to a continuous $\mathbb{T}$-action on X by [9, Proposition 2.3].

Let X, Y be $\mathbb{T}$-C*-algebras. The set of morphisms from X to Y in $\mathcal{C}^*_{\mathbb{T}}$ is the set $\mathrm{Mor}_{\mathbb{T}}(X, Y)$ of $\mathbb{T}$-equivariant morphisms $X \to Y$. By definition, $\varphi \in \mathrm{Mor}(X, Y)$ is $\mathbb{T}$-equivariant if and only if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\rho^X} & \mathrm{C}(\mathbb{T}) \otimes X \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{C}(\mathbb{T})}\otimes\varphi} \\ Y & \xrightarrow[\rho^Y]{} & \mathrm{C}(\mathbb{T}) \otimes Y \end{array} \tag{3.3}$$

Let $X \in \mathcal{C}^*_{\mathbb{T}}$. An element $x \in X$ is *homogeneous of degree* $n \in \mathbb{Z}$ if

$$\rho^X(x) = z^n \otimes x. \tag{3.4}$$

The degree of a homogeneous element x is denoted by $\deg(x)$. Let X_n be the set of homogeneous elements of X of degree n. This is a closed linear subspace of X, and $X_n X_m \subseteq X_{n+m}$ and $X_n^* = X_{-n}$ for $n, m \in \mathbb{Z}$. Moreover, finite sums of homogeneous elements are dense in X.

Let $\zeta \in \mathbb{T}$. The monoidal functor $\boxtimes_\zeta : \mathcal{C}^*_{\mathbb{T}} \times \mathcal{C}^*_{\mathbb{T}} \to \mathcal{C}^*_{\mathbb{T}}$ is introduced as in [5]. We describe $X \boxtimes_\zeta Y$ using quantum tori. By definition, the C*-algebra $\mathrm{C}(\mathbb{T}^2_\zeta)$ of the quantum torus is the C*-algebra generated by two unitary elements U, V subject to the relation $UV = \zeta VU$.

There are unique injective morphisms $\iota_1, \iota_2 \in \mathrm{Mor}(\mathrm{C}(\mathbb{T}), \mathrm{C}(\mathbb{T}^2_\zeta))$ with $\iota_1(z) = U$ and $\iota_2(z) = V$. Define $j_1 \in \mathrm{Mor}(X, \mathrm{C}(\mathbb{T}^2_\zeta) \otimes X \otimes Y)$ and $j_2 \in \mathrm{Mor}(Y, \mathrm{C}(\mathbb{T}^2_\zeta) \otimes X \otimes Y)$ by

$$j_1(x) = (\iota_1 \otimes \mathrm{id}_X) \circ \rho^X(x) \qquad \text{for all } x \in X,$$
$$j_2(y) = (\iota_2 \otimes \mathrm{id}_Y) \circ \rho^Y(y) \qquad \text{for all } y \in Y.$$

Let $x \in X_k$ and $y \in Y_l$. Then $j_1(x) = U^k \otimes x \otimes 1$ and $j_2(y) = V^l \otimes 1 \otimes y$, so that we get the commutation relation (1.5). This implies $j_1(X)j_2(Y) = j_2(Y)j_1(X)$, so that $j_1(X)j_2(Y)$ is a C*-algebra. We define

$$X \boxtimes_\zeta Y = j_1(X)j_2(Y).$$

This construction agrees with the one in [5] because $\mathrm{C}(\mathbb{T}^2_\zeta) \cong \mathrm{C}(\mathbb{T}) \boxtimes_\zeta \mathrm{C}(\mathbb{T})$, see also the end of [5, Section 5.2].

There is a unique continuous $\mathbb{T}$-action $\rho^{X \boxtimes_\zeta Y}$ on $X \boxtimes_\zeta Y$ for which j_1 and j_2 are $\mathbb{T}$-equivariant, that is, $j_1 \in \mathrm{Mor}_{\mathbb{T}}(X, X \boxtimes_\zeta Y)$ and $j_2 \in \mathrm{Mor}_{\mathbb{T}}(Y, X \boxtimes_\zeta Y)$. This action is constructed in a more general context in [6]. We always equip $X \boxtimes_\zeta Y$ with this $\mathbb{T}$-action and thus view it as an object of $\mathcal{C}^*_{\mathbb{T}}$.

The construction $\boxtimes_\zeta$ is a bifunctor; that is, $\mathbb{T}$-equivariant morphisms $\pi_1 \in \mathrm{Mor}_{\mathbb{T}}(X_1, Y_1)$ and $\pi_2 \in \mathrm{Mor}_{\mathbb{T}}(X_2, Y_2)$ induce a unique $\mathbb{T}$-equivariant morphism $\pi_1 \boxtimes_\zeta \pi_2 \in \mathrm{Mor}_{\mathbb{T}}(X_1 \boxtimes_\zeta X_2, Y_1 \boxtimes_\zeta Y_2)$ with

$$(\pi_1 \boxtimes_\zeta \pi_2)(j_{X_1}(x_1) j_{X_2}(x_2)) = j_{Y_1}(\pi_1(x_1)) j_{Y_2}(\pi_2(x_2)) \tag{3.5}$$

for all $x_1 \in X_1$ and $x_2 \in X_2$.

Proposition 3.1. *Let $x \in X$ and $y \in Y$ be homogeneous elements. Then*

$$j_1(x)j_2(Y) = j_2(Y)j_1(x),$$
$$j_1(X)j_2(y) = j_2(y)j_1(X).$$

Proof. Equation (1.5) shows that

$$j_1(x)j_2(y) = j_2(y)j_1(\rho^X_{\zeta^{\deg(y)}}(x))$$

for any $x \in X$ and any homogeneous $y \in Y$. Since $\rho^X_{\zeta^{\deg(y)}}$ is an automorphism of X, this implies $j_1(X)j_2(y) = j_2(y)j_1(X)$. Similarly, $j_1(x)j_2(y) = j_2(\rho^Y_{\zeta^{\deg(x)}}(y))j_1(x)$ for homogeneous $x \in X$ and any $y \in Y$ implies $j_1(x)j_2(Y) = j_2(Y)j_1(x)$. □

4. Proof of the main theorem

Let α and γ be the distinguished elements of A. Let $\widetilde{\alpha}$ and $\widetilde{\gamma}$ be the elements of $A \boxtimes_\zeta A$ appearing on the right hand side of (1.6):

$$\begin{aligned} \widetilde{\alpha} &= j_1(\alpha)j_2(\alpha) - qj_1(\gamma)^* j_2(\gamma), \\ \widetilde{\gamma} &= j_1(\gamma)j_2(\alpha) + j_1(\alpha)^* j_2(\gamma). \end{aligned} \tag{4.1}$$

We have $\deg(\alpha) = \deg(\alpha^*) = 0$, $\deg(\gamma) = 1$ and $\deg(\gamma^*) = -1$ by (2.2). Assume $\overline{q}\zeta = q$. Using (1.5) we may rewrite (4.1) in the following form:

$$\begin{aligned} \widetilde{\alpha} &= j_2(\alpha)j_1(\alpha) - \overline{q}j_2(\gamma)j_1(\gamma)^*, \\ \widetilde{\gamma} &= j_2(\alpha)j_1(\gamma) + j_2(\gamma)j_1(\alpha)^*. \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{\alpha}^* &= j_1(\alpha)^* j_2(\alpha)^* - qj_1(\gamma)j_2(\gamma)^*, \\ \widetilde{\gamma}^* &= j_1(\gamma)^* j_2(\alpha)^* + j_1(\alpha)j_2(\gamma)^*. \end{aligned} \tag{4.2}$$

The four equations (4.1) and (4.2) together are equivalent to

$$\begin{pmatrix} \widetilde{\alpha} & -q\widetilde{\gamma}^* \\ \widetilde{\gamma} & \widetilde{\alpha}^* \end{pmatrix} = \begin{pmatrix} j_1(\alpha) & -qj_1(\gamma)^* \\ j_1(\gamma) & j_1(\alpha)^* \end{pmatrix} \begin{pmatrix} j_2(\alpha) & -qj_2(\gamma)^* \\ j_2(\gamma) & j_2(\alpha)^* \end{pmatrix}. \tag{4.3}$$

Lemma 2.1 shows that the matrix

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in \mathrm{M}_2(A) \tag{4.4}$$

is unitary. Hence so is the matrix $j_1(u)j_2(u)$ on the right hand side of (4.3). Now Lemma 2.1 shows that $\widetilde{\alpha}, \widetilde{\gamma} \in A \boxtimes_\zeta A$ satisfy (2.1). So the universal property of A in Theorem 2.2 gives a unique morphism Δ with $\Delta(\alpha) = \widetilde{\alpha}$ and $\Delta(\gamma) = \widetilde{\gamma}$.

The elements α and γ are homogeneous of degrees 0 and 1, respectively, by (2.2). Hence $\widetilde{\alpha}$ and $\widetilde{\gamma}$ are homogeneous of degree 0 and 1 as well. Since α and γ generate A, it follows that Δ is $\mathbb{T}$-equivariant. This proves statement (1) in Theorem 1.1. Here we use the action $\rho^{A\boxtimes_\zeta A}$ of $\mathbb{T}$ with $\rho_z^{A\boxtimes_\zeta A}(j_1(a_1)j_2(a_2)) = j_1(\rho_z^A(a_1))j_2(\rho_z^A(a_2))$. We may rewrite (4.3) as

$$\begin{pmatrix}\Delta(\alpha) & -q\Delta(\gamma)^*\\ \Delta(\gamma) & \Delta(\alpha)^*\end{pmatrix} = \begin{pmatrix}j_1(\alpha) & -qj_1(\gamma)^*\\ j_1(\gamma) & j_1(\alpha)^*\end{pmatrix}\begin{pmatrix}j_2(\alpha) & -qj_2(\gamma)^*\\ j_2(\gamma) & j_2(\alpha)^*\end{pmatrix}.$$

Identifying $\mathrm{M}_2(A)$ with $\mathrm{M}_2(\mathbb{C})\otimes A$, we may further rewrite this as

$$(\mathrm{id}\otimes\Delta)(u) = (\mathrm{id}\otimes j_1)(u)\,(\mathrm{id}\otimes j_2)(u), \tag{4.5}$$

where id is the identity map on $\mathrm{M}_2(\mathbb{C})$.

Now we prove statement (2) in Theorem 1.1. Let j_1, j_2, j_3 be the natural embeddings of A into $A\boxtimes_\zeta A\boxtimes_\zeta A$. Since Δ is $\mathbb{T}$-equivariant, we may form $\Delta\boxtimes_\zeta \mathrm{id}$ and $\mathrm{id}\boxtimes_\zeta\Delta$. The values of $\mathrm{id}\otimes\big(\Delta\boxtimes_\zeta\mathrm{id}_A\big)$ and $\mathrm{id}\otimes\big(\mathrm{id}_A\boxtimes_\zeta\Delta\big)$ on the right hand side of (4.5) are equal:

$$\begin{aligned}\big(\mathrm{id}\otimes(\Delta\boxtimes_\zeta\mathrm{id}_A)\circ\Delta\big)(u) &= (\mathrm{id}\otimes j_1)(u)\,(\mathrm{id}\otimes j_2)(u)\,(\mathrm{id}\otimes j_3)(u),\\ \big(\mathrm{id}\otimes(\mathrm{id}_A\boxtimes_\zeta\Delta)\circ\Delta\big)(u) &= (\mathrm{id}\otimes j_1)(u)\,(\mathrm{id}\otimes j_2)(u)\,(\mathrm{id}\otimes j_3)(u).\end{aligned}$$

Thus $(\Delta\boxtimes_\zeta\mathrm{id}_A)\circ\Delta$ and $(\mathrm{id}_A\boxtimes_\zeta\Delta)\circ\Delta$ coincide on $\alpha,\gamma,\alpha^*,\gamma^*$. Since the latter generate A, this proves statement (2) of Theorem 1.1.

Now we prove statement (3). Let

$$S = \{x\in A : j_1(x)\in\Delta(A)j_2(A)\}.$$

This is a closed subspace of A. We may also rewrite (4.5) as

$$\begin{pmatrix}j_1(\alpha) & -qj_1(\gamma)^*\\ j_1(\gamma) & j_1(\alpha)^*\end{pmatrix} = \begin{pmatrix}\Delta(\alpha) & -q\Delta(\gamma)^*\\ \Delta(\gamma) & \Delta(\alpha)^*\end{pmatrix}\begin{pmatrix}j_2(\alpha) & -qj_2(\gamma)^*\\ j_2(\gamma) & j_2(\alpha)^*\end{pmatrix}^*. \tag{4.6}$$

Thus $\alpha,\gamma,\alpha^*,\gamma^*\in S$. Let $x, y\in S$ with homogeneous y. Proposition 3.1 gives

$$\begin{aligned}j_1(xy) = j_1(x)j_1(y) \in \Delta(A)j_2(A)j_1(y) &= \Delta(A)j_1(y)j_2(A)\\ &\subseteq \Delta(A)\Delta(A)j_2(A)j_2(A) = \Delta(A)j_2(A).\end{aligned}$$

That is, $xy\in S$. Therefore, all monomials in $\alpha,\gamma,\alpha^*,\gamma^*$ belong to S, so that $S = A$. Hence $j_1(A)\subseteq\Delta(A)j_2(A)$. Now $A\boxtimes_\zeta A = j_1(A)j_2(A)\subseteq\Delta(A)j_2(A)j_2(A) = \Delta(A)j_2(A)$, which is one of the Podleś conditions. Similarly, let

$$R = \{x\in A : j_2(x)\in j_1(A)\Delta(A)\}.$$

Then R is a closed subspace of A. We may also rewrite (4.5) as

$$\begin{pmatrix}j_2(\alpha) & -qj_2(\gamma)^*\\ j_2(\gamma) & j_2(\alpha)^*\end{pmatrix} = \begin{pmatrix}j_1(\alpha) & -qj_1(\gamma)^*\\ j_1(\gamma) & j_1(\alpha)^*\end{pmatrix}^*\begin{pmatrix}\Delta(\alpha) & -q\Delta(\gamma)^*\\ \Delta(\gamma) & \Delta(\alpha)^*\end{pmatrix}. \tag{4.7}$$

Thus $\alpha, \gamma, \alpha^*, \gamma^* \in R$. Let $x, y \in R$ with homogeneous x. Proposition 3.1 gives

$$\begin{aligned} j_2(xy) = j_2(x) j_2(y) \in j_2(x) j_1(A)\Delta(A) = j_1(A) j_2(x)\Delta(A) \\ \subseteq j_1(A) j_1(A)\Delta(A)\Delta(A) = j_1(A)\Delta(A). \end{aligned}$$

Thus $xy \in R$. Therefore, all monomials in $\alpha, \gamma, \alpha^*, \gamma^*$ belong to R, so that $R = A$, that is, $j_2(A) \subseteq j_1(A)\Delta(A)$. This implies $A \boxtimes_\zeta A = j_1(A) j_2(A) \subseteq j_1(A) j_1(A)\Delta(A) = j_1(A)\Delta(A)$ and finishes the proof of Theorem 1.1.

5. The representation theory of SU_q

For real q, the relations defining the compact quantum group $\mathrm{SU}_q(2)$ are dictated if we stipulate that the unitary matrix in Lemma 2.1 is a representation and that a certain vector in the tensor square of this representation is invariant. Here we generalise this to the complex case. This is how we found $\mathrm{SU}_q(2)$.

Let $\mathcal{H}$ be a $\mathbb{T}$-Hilbert space, that is, a Hilbert space with a unitary representation $U\colon \mathbb{T} \to \mathcal{U}(\mathcal{H})$. For $z \in \mathbb{T}$ and $x \in \mathcal{K}(\mathcal{H})$ we define

$$\rho_z^{\mathcal{K}(\mathcal{H})}(x) = U_z x U_z^*.$$

Thus $(\mathcal{K}(\mathcal{H}), \rho^{\mathcal{K}(\mathcal{H})})$ is a $\mathbb{T}$-C*-algebra. Let $(X, \rho^X) \in \mathrm{Obj}(\mathcal{C}^*_{\mathbb{T}})$. Since $\rho^{\mathcal{K}(\mathcal{H})}$ is inner, the braided tensor product $\mathcal{K}(\mathcal{H}) \boxtimes_\zeta X$ may (and will) be identified with $\mathcal{K}(\mathcal{H}) \otimes X$; see [5, Corollary 5.18] and [5, Example 5.19].

Definition 5.1. Let $\mathcal{H}$ be a $\mathbb{T}$-Hilbert space and let $v \in \mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes A)$ be a unitary element which is $\mathbb{T}$-invariant, that is, $(\rho_z^{\mathcal{K}(\mathcal{H})} \otimes \rho_z^X)(v) = v$. We call v a *representation* of $\mathrm{SU}_q(2)$ on $\mathcal{H}$ if

$$(\mathrm{id}_{\mathcal{H}} \otimes \Delta)(v) = (\mathrm{id}_{\mathcal{H}} \otimes j_1)(v)\,(\mathrm{id}_{\mathcal{H}} \otimes j_2)(v).$$

Theorem 6.1 below will show that representations of $\mathrm{SU}_q(2)$ are equivalent to representations of a certain compact quantum group. This allows us to carry over all the usual structural results about representations of compact quantum groups to $\mathrm{SU}_q(2)$. In particular, we may tensor representations. To describe this directly, we need the following result:

Proposition 5.2. *Let X, Y, U, T be $\mathbb{T}$-C*-algebras. Let $v \in X \otimes T$ and $w \in Y \otimes U$ be homogeneous elements of degree 0. Denote the natural embeddings by*

$$\begin{aligned} i_1\colon X \to X \boxtimes_\zeta Y, \qquad i_2\colon Y \to X \boxtimes_\zeta Y, \\ j_1\colon U \to U \boxtimes_\zeta T, \qquad j_2\colon T \to U \boxtimes_\zeta T. \end{aligned}$$

Then $(i_1 \otimes j_2)(v)$ and $(i_2 \otimes j_1)(w)$ commute in $(X \boxtimes_\zeta Y) \otimes (U \boxtimes_\zeta T)$.

Proof. We may assume that $v = x \otimes t$ and $w = y \otimes u$ for homogeneous elements $x \in X$, $t \in T$, $y \in Y$ and $u \in U$. Since $\deg(v) = \deg(w) = 0$, we get $\deg(x) = -\deg(t)$ and $\deg(y) = -\deg(u)$. The following computation completes the proof:

$$\begin{aligned}&(i_1 \otimes j_2)(v)\,(i_2 \otimes j_1)(w) = (i_1(x) \otimes j_2(t))\,(i_2(y) \otimes j_1(u))\\ &= i_1(x)i_2(y) \otimes j_2(t)j_1(u) = \zeta^{\deg(x)\deg(y)-\deg(t)\deg(u)} i_2(y)i_1(x) \otimes j_1(u)j_2(t)\\ &= (i_2(y) \otimes j_1(u))\,(i_1(x) \otimes j_2(t)) = (i_2 \otimes j_1)(w)\ (i_1 \otimes j_2)(v). \qquad \square\end{aligned}$$

Proposition 5.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be $\mathbb{T}$-Hilbert spaces and let $v_i \in \mathrm{M}(\mathcal{K}(\mathcal{H}_i) \otimes A)$ for $i = 1, 2$ be representations of $\mathrm{SU}_q(2)$. Define*

$$v = (\iota_1 \otimes \mathrm{id}_A)(v_1)(\iota_2 \otimes \mathrm{id}_A)(v_2) \in \mathrm{M}(\mathcal{K}(\mathcal{H}_1) \boxtimes_\zeta \mathcal{K}(\mathcal{H}_2) \otimes A)$$

and identify $\mathcal{K}(\mathcal{H}_1) \boxtimes_\zeta \mathcal{K}(\mathcal{H}_2) \cong \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then $v \in \mathrm{M}(\mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes A)$ is a representation of $\mathrm{SU}_q(2)$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. It is denoted $v_1 \oplus\!\!\!\!\top\, v_2$ and called the tensor product *of v_1 and v_2.*

Proof. It is clear that v is $\mathbb{T}$-invariant. We compute

$$\begin{aligned}(\mathrm{id}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes \Delta)(v) &= (\mathrm{id}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes \Delta)((\iota_1 \otimes \mathrm{id}_A)(v_1)(\iota_2 \otimes \mathrm{id}_A)(v_2))\\ &= (\iota_1 \otimes j_1)(v_1)\ (\iota_1 \otimes j_2)(v_1)\ (\iota_2 \otimes j_1)(v_2)\ (\iota_2 \otimes j_2)(v_2)\\ &= (\iota_1 \otimes j_1)(v_1)\ (\iota_2 \otimes j_1)(v_2)\ (\iota_1 \otimes j_2)(v_1)\ (\iota_2 \otimes j_2)(v_2)\\ &= (\mathrm{id}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes j_1)(v)\ (\mathrm{id}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes j_2)(v),\end{aligned}$$

where the third step uses Proposition 5.2. □

Now consider the Hilbert space $\mathbb{C}^2$, let $\{e_0, e_1\}$ be its canonical orthonormal basis. We equip it with the representation $U\colon \mathbb{T} \to \mathcal{U}(\mathbb{C}^2)$ defined by $U_z e_0 = z e_0$ and $U_z e_1 = e_1$. Let $\rho^{\mathrm{M}_2(\mathbb{C})}$ be the action implemented by U:

$$\rho_z^{\mathrm{M}_2(\mathbb{C})}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix} a_{11} & z a_{12} \\ \bar{z} a_{21} & a_{22}\end{pmatrix},$$

where $a_{ij} \in \mathbb{C}$. We claim that

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^*\end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \otimes A$$

is a representation of $\mathrm{SU}_q(2)$ on $\mathbb{C}^2$. By Lemma 2.1, the relations defining A are equivalent to u being unitary. The $\mathbb{T}$-action on A is defined so that u is $\mathbb{T}$-invariant. The comultiplication is defined exactly so that u is a representation, see (4.5).

The particular shape of u contains further assumptions, however. To explain these, we consider an arbitrary compact quantum group $\mathbb{G} = (\mathrm{C}(\mathbb{G}), \Delta_{\mathbb{G}})$ in $\mathcal{C}^*_{\mathbb{T}}$ with a unitary representation

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathrm{C}(\mathbb{G})),$$

such that a, b, c, d generate the C^*-algebra $\mathrm{C}(\mathbb{G})$. We assume that u is $\mathbb{T}$-invariant for the above $\mathbb{T}$-action on $\mathbb{C}^2$. Thus $\deg(a) = \deg(d) = 0$, $\deg(b) = -1$, $\deg(c) = 1$.

Theorem 5.4. *Let $\mathbb{G}$ be a braided compact quantum group with a unitary representation u as above. Assume $b \neq 0$ and that the vector $e_0 \otimes e_1 - qe_1 \otimes e_0 \in \mathbb{C}^2 \otimes \mathbb{C}^2$ for $q \in \mathbb{C}$ is invariant for the representation $u \oplus u$. Then $q \neq 0$, $\overline{q}\zeta = q$, $d = a^*$, $b = -qc^*$, and there is a unique morphism $\pi\colon \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathbb{G})$ with $\pi(\alpha) = a$ and $\pi(\gamma) = c$. This is $\mathbb{T}$-equivariant and satisfies $(\pi \boxtimes_\zeta \pi) \circ \Delta_{\mathrm{SU}_q(2)} = \Delta_{\mathbb{G}} \circ \pi$.*

Proof. The representation $u \oplus u \in \mathrm{M}_4(\mathrm{C}(\mathbb{G}))$ is given by Proposition 5.3, which uses a canonical isomorphism $\mathrm{M}_2(\mathbb{C}) \boxtimes_\zeta \mathrm{M}_2(\mathbb{C}) \cong \mathrm{M}_4(\mathbb{C})$. This comes from the following standard representation of $\mathrm{M}_2(\mathbb{C}) \boxtimes_\zeta \mathrm{M}_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$. For $T, S \in \mathrm{M}_2(\mathbb{C})$ of degree k, l and $x, y \in \mathbb{C}^2$ of degree m, n, we let $\iota_1(T)\iota_2(S)(x \otimes y) = \overline{\zeta}^{lm} Tx \otimes Sy$. By construction, $u \oplus u$ is $(\iota_1 \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})(u) \cdot (\iota_2 \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})(u)$. So we may rewrite the invariance of $e_0 \otimes e_1 - qe_1 \otimes e_0$ as

$$(\iota_1 \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})(u^*)(e_0 \otimes e_1 - qe_1 \otimes e_0) = (\iota_2 \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})(u)(e_0 \otimes e_1 - qe_1 \otimes e_0) \quad (5.1)$$

in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathrm{C}(\mathbb{G})$. The left and right hand sides of (5.1) are

$$e_0 \otimes e_1 \otimes a^* + e_1 \otimes e_1 \otimes b^* - qe_0 \otimes e_0 \otimes c^* - qe_1 \otimes e_0 \otimes d^*,$$
$$e_0 \otimes e_0 \otimes b + e_0 \otimes e_1 \otimes d - qe_1 \otimes e_0 \otimes a - q\overline{\zeta} e_1 \otimes e_1 \otimes c,$$

respectively. These are equal if and only if $b = -qc^*$, $d = a^*$, and $b^* = -q\overline{\zeta}c$. Since $b \neq 0$, this implies $q \neq 0$ and $\overline{q}\zeta = q$, and u has the form in Lemma 2.1. Since u is a representation, it is unitary. So a, c satisfy the relations defining $\mathrm{SU}_q(2)$ and Theorem 2.2 gives the unique morphism π. The conditions on u in Definition 5.1 imply that π is $\mathbb{T}$-equivariant and compatible with comultiplications. □

The proof also shows that q is uniquely determined by the condition that $e_0 \otimes e_1 - qe_1 \otimes e_0$ should be $\mathrm{SU}_q(2)$-invariant. An invariant vector for $\mathrm{SU}_q(2)$ should also be homogeneous for the $\mathbb{T}$-action. There are three cases of homogeneous vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$: multiples of $e_0 \otimes e_0$, multiples of $e_1 \otimes e_1$, and linear combinations of $e_0 \otimes e_1$ and $e_1 \otimes e_0$. If a non-zero multiple of $e_i \otimes e_j$ for $i, j \in \{0, 1\}$ is invariant, then the representation u is reducible. Ruling out such degenerate cases, we may normalise the invariant vector to have the form $e_0 \otimes e_1 - qe_1 \otimes e_0$ assumed in

Theorem 5.4. Roughly speaking, $\mathrm{SU}_q(2)$ is the universal family of braided quantum groups generated by a 2-dimensional representation with an invariant vector in $u \oplus u$.

Up to scaling, the basis e_0, e_1 is the unique one consisting of joint eigenvectors of the $\mathbb{T}$-action with degrees 1 and 0. Hence the braided quantum group $(\mathrm{C}(\mathrm{SU}_q(2)), \Delta)$ determines q uniquely. There is, however, one extra symmetry that changes the $\mathbb{T}$-action on $\mathrm{C}(\mathrm{SU}_q(2))$ and that corresponds to the permutation of the basis e_0, e_1. Given a $\mathbb{T}$-algebra A, let $S(A)$ be the same C^*-algebra with the $\mathbb{T}$-action by $\rho_z^{S(A)} = (\rho_z^A)^{-1}$. Since the commutation relation (1.5) is symmetric in k, l, there is a unique isomorphism

$$S(A \boxtimes_\zeta B) \cong S(A) \boxtimes_\zeta S(B), \qquad j_1(a) \mapsto j_1(a), \quad j_2(b) \mapsto j_2(b).$$

Hence the comultiplication on $\mathrm{C}(\mathrm{SU}_q(2))$ is one on $S(\mathrm{C}(\mathrm{SU}_q(2)))$ as well.

Proposition 5.5. *The braided quantum groups $S(\mathrm{C}(\mathrm{SU}_q(2)))$ and $\mathrm{C}(\mathrm{SU}_{\tilde{q}}(2))$ for $\tilde{q} = \overline{q}^{-1}$ are isomorphic as braided quantum groups.*

Proof. Let α, γ be the standard generators of $A_q = \mathrm{C}(\mathrm{SU}_q(2))$ and let $\tilde{\alpha}, \tilde{\gamma}$ be the standard generators of $A_{\tilde{q}}$. We claim that there is an isomorphism $\varphi\colon A_q \to A_{\tilde{q}}$ that maps $\alpha \mapsto \tilde{\alpha}^*$ and $\gamma \mapsto \tilde{q}\tilde{\gamma}^*$ and that is an isomorphism of braided quantum groups from $S(A_q)$ to $A_{\tilde{q}}$. Lemma 2.1 implies that the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} & -\tilde{q}\tilde{\gamma}^* \\ \tilde{\gamma} & \tilde{\alpha}^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}^* & -\tilde{\gamma} \\ \tilde{q}\tilde{\gamma}^* & \tilde{\alpha} \end{pmatrix} = \begin{pmatrix} \varphi(\alpha) & \varphi(-q\gamma^*) \\ \varphi(\gamma) & \varphi(\alpha^*) \end{pmatrix}$$

is unitary. Now Lemma 2.1 and Theorem 2.2 give the desired morphism φ. Since the inverse of φ may be constructed in the same way, φ is an isomorphism. On generators, it reverses the grading, so it is $\mathbb{T}$-equivariant as a map $S(A_q) \to A_{\tilde{q}}$.

Let Δ and $\tilde{\Delta}$ denote the comultiplications on $S(A_q)$ and $A_{\tilde{q}}$. We compute

$$\begin{aligned}
(\varphi \boxtimes_\zeta \varphi)\Delta(\alpha) &= (\varphi \boxtimes_\zeta \varphi)(j_1(\alpha)j_2(\alpha) - q j_1(\gamma^*)j_2(\gamma)) \\
&= j_1(\varphi(\alpha))j_2(\varphi(\alpha)) - q j_1(\varphi(\gamma^*))j_2(\varphi(\gamma)) \\
&= j_1(\tilde{\alpha}^*)j_2(\tilde{\alpha}^*) - \tilde{q} j_1(\tilde{\gamma})j_2(\tilde{\gamma}^*), \\
\tilde{\Delta}(\varphi(\alpha)) = \tilde{\Delta}(\tilde{\alpha}^*) &= j_2(\tilde{\alpha})^* j_1(\tilde{\alpha})^* - q^{-1} j_2(\tilde{\gamma})^* j_1(\tilde{\gamma}) \\
&= j_1(\tilde{\alpha})^* j_2(\tilde{\alpha})^* - q^{-1}\zeta j_1(\tilde{\gamma}) j_2(\tilde{\gamma})^*.
\end{aligned}$$

These are equal because $\tilde{q} = \overline{q}^{-1} = q^{-1}\zeta$. Similarly, $(\varphi \boxtimes_\zeta \varphi)\Delta(\gamma) = \tilde{\Delta}(\varphi(\gamma))$. Thus φ is an isomorphism of braided quantum groups. □

6. The semidirect product quantum group

A quantum analogue of the semidirect product construction for groups turns the braided quantum group $\mathrm{SU}_q(2)$ into a genuine compact quantum group (B, Δ_B), see [6, Section 6]. Here B is the universal C^*-algebra with three generators α, γ, z with the $SU_q(2)$-relations for α and γ and

$$\begin{aligned} z\alpha z^* &= \alpha, \\ z\gamma z^* &= \zeta^{-1}\gamma, \\ zz^* &= z^*z = \mathrm{I}; \end{aligned}$$

the comultiplication is defined by

$$\begin{aligned} \Delta_B(z) &= z \otimes z, \\ \Delta_B(\alpha) &= \alpha \otimes \alpha - q\gamma^* z \otimes \gamma, \\ \Delta_B(\gamma) &= \gamma \otimes \alpha + \alpha^* z \otimes \gamma. \end{aligned}$$

There are two embeddings $\iota_1, \iota_2 \colon A \rightrightarrows B \otimes B$ defined by

$$\begin{aligned} \iota_1(\alpha) &= \alpha \otimes \mathrm{I} & \iota_2(\alpha) &= \mathrm{I} \otimes \alpha, \\ \iota_1(\gamma) &= \gamma \otimes \mathrm{I} & \iota_2(\gamma) &= z \otimes \gamma. \end{aligned}$$

Homogeneous elements $x, y \in A$ satisfy

$$\iota_1(x)\iota_2(y) = \zeta^{\deg(x)\deg(y)}\iota_2(y)\iota_1(x). \tag{6.1}$$

Thus we may rewrite the comultiplication as

$$\begin{aligned} \Delta_B(z) &= z \otimes z, \\ \Delta_B(\alpha) &= \iota_1(\alpha)\iota_2(\alpha) - q\iota_1(\gamma)^*\iota_2(\gamma), \\ \Delta_B(\gamma) &= \iota_1(\gamma)\iota_2(\alpha) + \iota_1(\alpha)^*\iota_2(\gamma). \end{aligned}$$

In particular, Δ_B respects the commutation relations for (α, γ, z), so it is a well-defined *-homomorphism $B \to B \otimes B$. It is routine to check the cancellation conditions (1.4) for B, so (B, Δ_B) is a compact quantum group.

This is a compact quantum group with a projection as in [7, 8]. Here the projection $\pi \colon B \to B$ is the unique *-homomorphism with $\pi(\alpha) = 1_B$, $\pi(\gamma) = 0$ and $\pi(z) = z$; this is an idempotent bialgebra morphism. Its “image” is the copy of $C(\mathbb{T})$ generated by z, its “kernel” is the copy of A generated by α and γ.

For $q = 1$, $B \cong C(\mathbb{T} \times SU(2))$ as a C^*-algebra, which is commutative. The representation on $\mathbb{C}^2$ combines the standard embedding of SU(2) and the representation of $\mathbb{T}$ mapping z to the diagonal matrix with entries $z, 1$. This gives a homeomorphism $\mathbb{T} \times SU(2) \cong U(2)$. So (B, Δ_B) is the group U(2), written as a semidirect product of SU(2) and $\mathbb{T}$.

For $q \neq 1$, (B, Δ_B) is the coopposite of the quantum $U_q(2)$ group described previously by Zhang and Zhao in [12]: the substitutions $a = \alpha^*$, $b = \gamma^*$ and $D = z^*$ turn our generators and relations into those in [12], and the comultiplications differ only by a coordinate flip.

Theorem 6.1. *Let* $U \in \mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes \mathrm{C}(\mathbb{T}))$ *be a unitary representation of* $\mathbb{T}$ *on a Hilbert space* $\mathcal{H}$*. There is a bijection between representations of* $\mathrm{SU}_q(2)$ *on* $\mathcal{H}$ *and representations of* (B, Δ_B) *on* $\mathcal{H}$ *that restrict to the given representation on* $\mathbb{T}$*.*

Proof. Let $v \in \mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes A)$ be a unitary representation of $\mathrm{SU}_q(2)$ on $\mathcal{H}$. Since B contains copies of A and $\mathrm{C}(\mathbb{T})$, we may view $u = vU^*$ as an element of $\mathrm{M}(\mathcal{K}(\mathcal{H}) \otimes B)$. The $\mathbb{T}$-invariance of v,

$$(\mathrm{id} \otimes \rho^A)(v) = U_{12}^* v_{13} U_{12}$$

and the formula for ι_2 (which is basically given by the action ρ^A) show that

$$U_{12}(\mathrm{id} \otimes \iota_2)(v)U_{12}^* = v_{13}.$$

Using $(\mathrm{id} \otimes \iota_2)(v) = v_{12}$, we conclude that u is a unitary representation of (B, Δ_B):

$$(\mathrm{id} \otimes \Delta_B)(u) = v_{12}(\mathrm{id} \otimes \iota_2)(v)U_{12}^* U_{13}^* = v_{12}U_{12}^* v_{13} U_{13}^* = u_{12}u_{13}.$$

Going back and forth between u and v is the desired bijection. □

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