J. Noncommut. Geom. 12 (2018), 439–456
DOI 10.4171/JNCG/282
Journal of Noncommutative Geometry

Large scale index of multi-partitioned manifolds

Thomas Schick and Mostafa Esfahani Zadeh

Abstract. Let M be a complete n-dimensional Riemannian spin manifold, partitioned by q two-sided hypersurfaces which have a compact transverse intersection N and which in addition satisfy a certain coarse transversality condition. Let E be a Hermitean bundle on M with connection. We define a coarse multi-partitioned index of the spin Dirac operator on M twisted by E.

Our main result is the computation of this multi-partitioned index as the Fredholm index of the Dirac operator on the compact manifold N, twisted by the restriction of E to N.

We establish the following main application: if the scalar curvature of M is bounded below by a positive constant everywhere (or even if this happens only on one of the quadrants defined by the partitioning hypersurfaces) then the multi-partitioned index vanishes. Consequently, the multi-partitioned index is an obstruction to uniformly positive scalar curvature on M.

The proof of the multi-partitioned index theorem proceeds in two steps: first we establish a strong new localization property of the multi-partitioned index which is the main novelty of this note. This we establish even when we twist with an arbitrary Hilbert A-module bundle E (for an auxiliary C^*-algebra A). This allows to reduce to the case where M is the product of N with Euclidean space of dimension q. For this special case, standard methods for the explicit calculation of the index in this product situation can be adapted to obtain the result.

Mathematics Subject Classification (2010). 19K56, 58J20, 46L80, 58J22.

Keywords. Large scale index theory, partitioned manifold index theorem, coarse index theory, large scale geometry, coarse geometry, positive scalar curvature, obstruction to positive scalar curvature, Hilbert A-module bundles, K-theory of C^*-algebras, index theory.

1. Introduction

Consider a complete Riemannian spin manifold (M, g). If M is non-compact, which is the situation we are interested in, the classical index theory of elliptic operators (like the Dirac operator) usually can not be applied because of lack of the Fredholm property.

In this situation, using non-commutative geometry and operator algebras, John Roe has initiated an adapted index theory which we call "*large scale index theory*" or sometimes "*coarse index theory*" (compare e.g. [16]). The main player is the

Roe algebra $C^*(X)$, associated to a complete proper metric space X. This is an algebra of operators (acting on function spaces on X), mildly depending on choices. Its K-theory $K_*(C^*(X))$ is canonically and functorially associated to X, with functoriality for proper and uniformly expansive maps.

If M is a complete Riemannian spin manifold of dimension n and $E \to M$ is a Hermitian bundle with connection, we have the twisted Dirac operator D_E. More generally, if A is an auxiliary C^*-algebra and E is a Hilbert A-module bundle with A-linear connection one can form D_E as regular unbounded operator in the Hilbert A-module sense. One of the main virtues of large scale index theory is the construction of its *coarse index* $\operatorname{ind}_c(D_E) \in K_n(C^*(M;A))$. This index contains information about the geometry: on the one hand it depends on the metric only up to bilipschitz equivalence. On the other hand, it vanishes if the metric has uniformly positive scalar curvature and E is flat (actually, it suffices to have this property on sufficiently large subsets of M). It is therefore important to get information about this coarse index. In the present paper, we will do this for multi-partitioned manifolds.

Definition 1.1. A complete n-dimensional Riemannian manifold M is *multi-partitioned* by codimension 1 hypersurfaces $M_1, \dots, M_q$ if

- each of the M_k is two-sided, separating $M = M_k^+ \cup M_k^-$ with $M_k^+ \cap M_K^- = M_k$;
- $N := \bigcap_{k=1}^q M_k$ is compact, and in a neighborhood of N the M_i intersect mutually transversally. In particular, N is itself a submanifold of dimension $n-q$ with trivial normal bundle;
- the collection of hypersurfaces is *coarsely transversal* in the sense that for each $r > 0$ there is $s > 0$ such that $\bigcap_{k=1}^q U_r(M_k) \subset U_s(N)$. Here, for a subset $X \subset M$ we set
$$U_r(X) := \{p \in M \mid d(p,X) \le r\},$$
the r-neighborhood of X.

In Section 3 we will prove

Lemma 1.2. *If M is a multi-partitioned manifold, partitioned by $M_1, \dots, M_q$, the signed distance to the M_k defines a proper and uniformly expansive map $f\colon M \to \mathbb{R}^q$ which is smooth near $N := f^{-1}(0)$ and such that 0 is a regular value.*

The notion of uniform expansiveness is recalled in Definition 2.3. Also the other notions of coarse index theory used in this introduction are recalled in Section 2.

Definition 1.3. Let M be a multi-partitioned spin manifold and $E \to M$ a Hilbert A-module bundle with connection. We define the multi-partitioned index of the Dirac operator twisted by E as

$$\operatorname{ind}_p(D_E) := \kappa\big(f_*\big(\operatorname{ind}_c(D_E)\big)\big) \in K_{n-p}\big(C^*(\mathbb{R}^n;A)\big).$$

Here,
$$f_*\colon K_*\big(C^*(M;A)\big) \to K_*\big(C^*(\mathbb{R}^q;A)\big)$$

is obtained by functoriality of the K-theory of the twisted Roe algebra and

$$\kappa\colon K_*\big(C^*(\mathbb{R}^q;A)\big) \to K_{*-q}(A)$$

is a canonical isomorphism we recall in Corollary 2.11.

Our main result is the calculation of this multi-partitioned manifold index:

Theorem 1.4. *Let M be a complete spin manifold of dimension n with proper continuous uniformly expansive map $f\colon M \to \mathbb{R}^q$. Assume that f is smooth near $N := f^{-1}(0)$ and that 0 is a regular value. This implies that N is a compact submanifold with trivial normal bundle of dimension $n-q$. In particular, N inherits a spin structure from M. For example, M could be multi-partitioned by $M_1,\dots,M_q$. Let $E \to M$ be a Hermitean bundle with connection. Then*

$$\kappa\Big(f_*\big(\mathrm{ind}_c(D_{M,E})\big)\Big) = \mathrm{ind}(D_{N,E|_N}) \in K_{n-q}(\mathbb{C}).$$

Note that $K_{n-q}(\mathbb{C}) = \mathbb{Z}$ if $n-q$ is even, and $K_{n-q}(\mathbb{C}) = \{0\}$ if $n-q$ is odd and $\mathrm{ind}(D_{N,E|_N}) \in K_{n-q}(\mathbb{C})$ *is the index of the Dirac operator on the compact spin manifold N, twisted by the Hermitean bundle $E|_N$.*

Remark 1.5. We strongly expect that Theorem 1.4 generalizes to arbitrary C^*-algebras A and Hilbert A-modules E over M, equating the multi-partitioned index with the Mishchenko–Fomenko index $\mathrm{ind}(D_{N,E|_N}) \in K_{n-q}(A)$ of the Dirac operator on the compact manifold N, twisted with the Hilbert A-module bundle $E|_N$. We comment more on this in Section 4.

Historically the first version of Theorem 1.4 is the *partitioned manifold index theorem* of Roe and Higson, for the case $q=1$ with several proofs, e.g. in [16] or [5]. Here only the case n odd is interesting (and treated), as otherwise the target group $K_1(\mathbb{C}) = 0$. In [21,22], the approach of [5] is generalized, still for $q=1$, from $\mathbb{C}$ to arbitrary coefficient algebras A, as long as n is odd. Finally, Siegel treats the case of multi-partitioned manifolds with additional geometric restrictions in [20].

All these proofs consist of two steps. The first is a reduction to the product case $M = N \times \mathbb{R}^q$, and the second is a more or less explicit calculation in this product case.

In this note we develop a new and particularly strong method for the reduction step. Indeed, we prove in particular the following:

Proposition 1.6. *Assume that $f\colon M \to \mathbb{R}^q$ with Hilbert A-module bundle $E \to M$ with connection and $f'\colon M' \to \mathbb{R}^q$ with Hilbert A-module bundle $E' \to M'$ with connection are two complete Riemannian spin manifolds as in Theorem 1.4. Assume there are open neighborhoods U of $f^{-1}(0)$ in M and U' of $f^{-1}(0)$ in M' and a spin-structure preserving isometry $\psi\colon U \to U'$ which is covered by an A-isometry $\Psi\colon E|_U \to E'|_{U'}$ preserving the connections. Then*

$$f_*\big(\mathrm{ind}_c(D_E)\big) = f'_*\big(\mathrm{ind}_c(D_{E'})\big) \in K_n\big(C^*(\mathbb{R}^q;A)\big).$$

Indeed, Proposition 1.6 is a corollary of a localization theorem for classes in $K_*(C^*(\mathbb{R}^q; A))$: if two such are obtained as indices of operators which coincide on an arbitrary non-empty open subset of $\mathbb{R}^q$, then they are already equal.

A consequence of Theorem 1.4 is the following obstruction to positive scalar curvature.

Theorem 1.7. *Let (M, g) be a complete Riemannian spin manifold, partitioned transversally and coarsely by q hypersurfaces whose intersection is a compact manifold N. If $\widehat{A}(N) \neq 0$ then the scalar curvature of g (or of any other complete Riemannian metric which is bilipschitz equivalent to g) can not be uniformly positive outside a compact subset of any quadrant formed by the partitioning hypersurfaces.*

2. Basics of large scale index theory

We start with a very brief review of the Roe algebra $C^*(X; A)$ and a companion, the structure algebra $D^*(X; A)$, inside which $C^*(X; A)$ is an ideal. For simplicity, we assume that X is a positive dimensional Riemannian manifold throughout (possibly with boundary). We follow [14, 16]. For the basics of Hilbert A-modules and their operators (adjointability, A-compactness,...), compare [11].

Definition 2.1. Given a Hilbert A-module bundle $S \to X$, consider the Hilbert A-module $L^2(S)$ of square integrable sections of S with A-valued inner product given by integration of the pointwise A-valued inner product.

One defines the *structure algebra* $D^*(X; A)$ as C^*-closure of the algebra of bounded adjointable A-linear operators T on $L^2(S)$ which satisfy

(1) T has finite propagation, i.e. there is $R > 0$ such that $\operatorname{supp}(Ts) \subset U_R(\operatorname{supp}(s))$ for each $s \in L^2(S)$.

(2) T is pseudolocal: for any compactly supported continuous functions ϕ, ψ with $\phi\psi = 0$ the operator $\phi T \psi$ is an A-compact operator, where we let ϕ act on $L^2(S)$ as multiplication operator.

The *Roe algebra* $C^*(X; A)$ is the norm closure of operators T as above which satisfy

(1) T has finite propagation

(2) T is *locally compact*, i.e. for every compactly supported continuous function ϕ, $T\phi$ and ϕT are A-compact operators.

One checks immediately that $C^*(X; A)$ is an ideal in $D^*(X; A)$.

Remark 2.2. Strictly speaking, one has to enlarge E by tensoring with $l^2(\mathbb{N})$; we gloss over these details, as all we have to do happens in a fixed summand canonically isomorphic to E, as in [2].

Definition 2.3. Let $f: X \to Y$ be a continuous map between complete Riemannian manifolds.

- f is called proper if the inverse image of every compact subset of Y is compact;
- f is uniformly expansive if for every $r > 0$ there is $s > 0$ such that $d(f(x), f(y)) \leq s$ whenever $d(x, y) \leq r$.

Given, in addition, Hilbert A-module bundles $E \to X$ and $F \to Y$, an isometric A-embedding $W: L^2(E) \to L^2(F)$ is said to *cover* f *in the* D^**-sense* if W is a norm-limit of operators V such that

- V has *finite propagation*, i.e. there is $R > 0$ such that $\operatorname{supp}(Vs) \subset U_R(f(\operatorname{supp}(s)))$;
- whenever ϕ is a compactly supported continuous function on X and ψ is a compactly supported continuous function on Y such that $\phi \cdot (\psi \circ f) = 0$ then $\psi V \phi$ is A-compact.

By [7, Lemma 7.7] one can always find an isometry V covering f in the D^*-sense, even itself with prescribed finite propagation $R > 0$.

Given such an isometry V, $Ad_V(T) := VTV^*$ then defines a map

$$Ad_V: D^*(X; A) \to D^*(Y; A)$$

which restricts to a map $C^*(X; A) \to C^*(Y; A)$.

As in [9, Lemma 3], the induced map on K-theory does not depend on the choice of V, but only on f. This implies that

- $C^*(X; A)$ and $D^*(X; A)$, and therefore also the quotient algebra
$$D^*(X; A)/C^*(X; A)$$
are well defined up to non-canonical isomorphism;
- $K_*(C^*(X; A))$, $K_*(D^*(X; A))$ and $K_*(D^*(X; A)/C^*(X; A))$ are well defined up to canonical isomorphism and are functorial for proper continuous uniformly expansive maps.

One defines

$$K_*(X; A) := K_{*+1}\big(D^*(X; A)/C^*(X; A)\big),$$

the *locally finite K-homology of* X *with coefficients in* A.

In the original papers of Roe the compactness condition on V was forgotten to be mentioned. It is introduced (with this terminology) in [14, Definition 1.7] or (under the name “covers topologically”) in [20, Definition 2.4].

We will use vanishing of these K-groups in a number of situations. The most powerful and useful concept in this context is *flasqueness*.

Definition 2.4. A complete Riemannian manifold M (possibly with boundary) is called *flasque* if there is a continuous, proper and uniformly expansive map $f: M \to M$ with the following properties:

(1) there is a continuous uniformly expansive proper homotopy between f and the identity;

(2) for every compact subset $K \subset M$ there is an $N \in \mathbb{N}$ such that $f^N(M) \cap K = \emptyset$.

Example 2.5. For an arbitrary manifold X, the product $X \times [0, \infty)$ is flasque. Indeed, the map

$$f: X \times [0, \infty) \to X \times [0, \infty); (x, t) \mapsto (x, t + 1)$$

satisfies the conditions required in the definition of flasqueness.

Proposition 2.6. *If M is flasque, then*

$$K_*\big(C^*(M; A)\big) = 0, \quad K_*\big(D^*(M); A\big) = 0, \quad K_*(M; A) = 0.$$

Proof. For $A = \mathbb{C}$, this is proved in [16, Proposition 9.4]. The proof carries over to general A almost literally. □

The final ingredient we will need from the basics of large scale index theory is a Mayer–Vietoris principle.

Definition 2.7. Let M be a complete Riemannian manifold and $Y \subset M$ a closed subset. We define $C^*(Y \subset M; A) \subset C^*(M; A)$ as the closure of the set of all operators $T \in C^*(M; A)$ which have *support near* Y, i.e. such that there is $R > 0$ such that $T\phi = 0$ and $\phi T = 0$ whenever $\mathrm{supp}(\phi) \cap U_R(Y) = \emptyset$.

We define $D^*(Y \subset M; A)$ as the closure of the set of all operators $T \in D^*(M; a)$ such that T has support near Y and in addition $T\phi$ and ϕT are A-compact operators whenever $\mathrm{supp}(\phi) \cap Y = \emptyset$. Then $C^*(Y \subset M; A)$ and $D^*(Y \subset M; A)$ are both ideals in $D^*(M; A)$.

Proposition 2.8. *In the situation of Definition 2.7, the canonical maps*

$$C^*(Y; A) \hookrightarrow C^*(Y \subset M; A), \qquad D^*(Y; A) \hookrightarrow D^*(Y \subset M; A),$$
$$D^*(Y; A)/C^*(Y; A) \to D^*(Y \subset M; A)/C^*(Y \subset M; A)$$

induce isomorphism in K-theory.

Proof. This is proved for $A = \mathbb{C}$ in [20], the proof carries over almost literally to general A. □

Definition 2.9. Assume that $M = M_1 \cup M_2$ with intersection $M_0 := M_1 \cap M_2$ for closed subsets M_1, M_2. This decomposition is called *coarsely excisive* if for each $r > 0$ there is $s > 0$ such that $U_r(M_1) \cap U_r(M_2) \subset U_s(M_0)$.

Theorem 2.10. *Assume that M is a complete Riemannian manifold with a coarsely excisive decomposition $M = M_1 \cup M_2$ into closed subset. Then we have long exact Mayer–Vietoris sequences*

$$\cdots \to K_j\big(C^*(M_1;A)\big) \oplus K_j\big(C^*(M_2;A)\big) \to K_j\big(C^*(M;A)\big) \xrightarrow{\delta_{MV}} K_{j-1}\big(C^*(M_0;A)\big) \to \cdots$$

$$\cdots \to K_j\big(D^*(M_1;A)\big) \oplus K_j\big(D^*(M_2;A)\big) \to K_j\big(D^*(M;A)\big) \xrightarrow{\delta_{MV}} K_{j-1}\big(D^*(M_0;A)\big) \to \cdots$$

$$\cdots \to K_j(M_1;A) \oplus K_j(M_2;A) \to K_j(M;A) \xrightarrow{\delta_{MV}} K_{j-1}(M_0;A) \to \cdots$$

These long exact sequences are compatible with the long exact sequences in K-theory of the extensions $0 \to C^* \to D^* \to D^*/C^* \to 0$.

Proof. For $A = \mathbb{C}$, this is the main result of [20, Section 3]. The proof carries over almost literally to general A. □

Corollary 2.11. *For an arbitrary complete Riemannian manifold M there is a commuting diagram with horizontal isomorphisms*

$$\begin{array}{ccc}
K_*(D^*(M\times\mathbb{R};A)) & \xrightarrow[\cong]{\delta_{MV}} & K_{*-1}(D^*(M;A)) \\
\downarrow & & \downarrow \\
K_*(D^*(M\times\mathbb{R};A)/C^*(M\times\mathbb{R};A)) & \xrightarrow[\cong]{\delta_{MV}} & K_{*-1}(D^*(M;A)/C^*(M;A)) \\
\downarrow & & \downarrow \\
K_{*-1}(C^*(M\times\mathbb{R};A)) & \xrightarrow[\cong]{\delta_{MV}} & K_{*-2}(C^*(M;A)).
\end{array}$$

In particular, for arbitrary j, q we obtain a canonical isomorphism

$$\kappa\colon K_j\big(C^*(\mathbb{R}^q;A)\big) \to K_{j-q}(A).$$

Proof. The decomposition $M \times \mathbb{R} = M \times (-\infty,0] \cup M \times [0,\infty)$ is coarsely excisive, and the half spaces $M \times (-\infty,0]$, $M \times [0,\infty)$ are flasque. Combining 2.6 and Theorem 2.10, the Mayer–Vietoris boundary map gives an isomorphism

$$\delta_{MV}\colon K_j\big(D^*(M\times\mathbb{R};A)\big) \xrightarrow{\cong} K_{j-1}\big(D^*(M;A)\big),$$

and similarly for the other algebras.

The q-fold iteration of this then gives an isomorphism

$$K_j\big(C^*(\mathbb{R}^q;A)\big) \to K_{j-q}\big(C^*(\mathbb{R}^0;A)\big)$$

with a canonical isomorphism, $C^*(pt;A) = A \otimes \mathbf{K}$, and of course

$$K_*(A\otimes\mathbf{K}) \cong K_*(A).$$ □

Finally, we recall from [2, 16] how to define the coarse index of a twisted Dirac operator and we mention its main properties. Assume therefore that M is a complete spin manifold (without boundary) and $E \to M$ a Hilbert A-bundle with connection. The twisted Dirac operator D_E with its natural domain is then a self-adjoint unbounded operator on the Hilbert module $L^2(S \otimes E)$ where S is the spinor bundle of M. If we assume that M is even dimensional, $S = S^+ \oplus S^-$ is $\mathbb{Z}/2$-graded and D_E is an odd operator with respect to this grading. Let $\chi\colon \mathbb{R} \to [-1,1]$ an odd continuous function such that $\chi(t) \xrightarrow{t\to\pm\infty} \pm 1$. Then $\chi(D_E)$ is still an odd operator. The Fourier inversion formula and unit propagation of the wave operator imply that $\chi(D_E) \in D^*(M;A)$ and that $\chi(D_E)^2 - 1 \in C^*(M;A)$; compare [2].

Definition 2.12. If $\dim(M)$ is even, choose a measurable fiberwise isometry $S^+ \to S^-$ with induced A-linear isometry of propagation zero

$$V\colon L^2\big(S^+ \otimes E\big) \to L^2\big(S^- \otimes E\big).$$

Because $\chi(D_E)^2 - 1 \in C^*(M;A)$, the operator

$$V^*\chi(D_E)_+\colon L^2\big(S_+ \otimes E\big) \to L^2\big(S_+ \otimes E\big)$$

is a unitary operator in $D^*(M;A)/C^*(M;A)$ and therefore represents a class

$$[D_E] \in K_1\big(D^*(M;A)/C^*(M;A)\big) = K_1(M;A),$$

the *fundamental K-homology class*.

One has the long exact sequence in K-theory associated to the extension

$$0 \to C^*(M;A) \to D^*(M;A) \to D^*(M;A)/C^*(M;A) \to 0$$

with boundary map

$$\partial\colon K_1\big(D^*(M;A)/C^*(M;A)\big) \to K_0\big(C^*(M;A)\big).$$

The *large scale index* is defined to be

$$\mathrm{ind}_c(D_E) := \partial\big([D_E]\big) \in K_0\big(C^*(M;A)\big).$$

Because of homotopy invariance of K-theory, $[D_E]$ as well as $\mathrm{ind}_c(D_E)$ does not depend on the choices made.

If the dimension of M is odd,

$$\big[(\chi(D_E) - 1)/2\big] \in D^*(M;A)/C^*(M;A)$$

is a projector and therefore represents a fundamental class $[D_E] \in K_0(M;A)$. In this case the large scale index is defined by

$$\mathrm{ind}_c(D_E) := \partial\big([D_E]\big) \in K_1\big(C^*(M;A)\big).$$

Remark 2.13. Let M be a compact, closed, even dimensional spin manifold with Dirac operator D_E acting on $L^2(S \otimes E)$. Let $\mathbf{B}$ and $\mathbf{K}$ denote respectively the space of all bounded and all compact operators on $L^2(S \otimes E)$. It is clear that $C^*(M) = \mathbf{K}$. Therefore the coarse index $\operatorname{ind}_c(D_E)$ being an element in $K_0(\mathbf{K}) = \mathbb{Z}$ is an integer. This is the usual Fredholm index of D_E, compare [16, Example after Definition 3.7].

We recall the following well known vanishing result (compare [2, 16]).

Proposition 2.14. *Let M be a complete Riemannian spin manifold and E a Hilbert A-module bundle on M as above. If 0 is not in the spectrum of D_E (in particular by the Schrödinger–Lichnerowicz formula if M has scalar curvature $\geq C > 0$ and E is flat), then*

$$\operatorname{ind}_c(D_E) = 0 \in K_{\dim M}\big(C^*(M; A)\big).$$

Proof. In this case, we can use for χ a function such that $\chi^2 = 1$ on the spectrum of D_E. Consequently, the formula defining $[D_E]$ shows that this class canonically lifts to a class $\rho(D_E) \in K_{\dim M+1}(D^*(M; A))$. Because of the exactness of the K-theory sequence

$$K_*\big(D^*(M; A)\big) \to K_{*-1}(M; A) \xrightarrow{\partial} K_{*-1}\big(C^*(M; A)\big)$$

this implies that $\operatorname{ind}_c(D_E) = \partial([D_E]) = 0$. □

Proposition 2.15. *Given a complete Riemannian spin manifold M of dimension n with Hermitean bundle $E \to M$, write E also for the pullback to $M \times \mathbb{R}$. The Mayer–Vietoris isomorphism δ_{MV} of Corollary 2.11 sends $[D_{M\times\mathbb{R},E}]$ to $[D_{M,E}]$ and consequently also $\operatorname{ind}_c(D_{M\times\mathbb{R},E})$ to $\operatorname{ind}_c(D_{M,E})$:*

$$\begin{array}{ccccc} K_{n+1}(M \times \mathbb{R}) & \xrightarrow[\cong]{\delta_{MV}} & K_n(M); & & [D_{M\times\mathbb{R}}] \mapsto [D_{M,E}] \\ \downarrow \partial & & \downarrow \partial & & \downarrow \\ K_{n+1}(C^*(M \times \mathbb{R})) & \xrightarrow[\cong]{\delta_{MV}} & K_n(C^*(M)); & & \operatorname{ind}_c(D_{M\times\mathbb{R}}) \mapsto \operatorname{ind}_c(D) \end{array}$$

Proof. This crucial property of the Dirac operator is based on the principle that "boundary of Dirac is Dirac". The proof is given in [20, Lemma 4.6] and is based on the precise meaning of "boundary of Dirac is Dirac" as treated in [8, Chapter 11]. □

3. Multi-partitioned manifolds and their large scale index

Throughout this section, assume that M is a complete Riemannian manifold which is multi-partitioned by the separating hypersurfaces $M_1, \dots, M_q$. Recall that this means in particular that the latter are coarsely transversal in the sense of Definition 1.1 and near their common intersection $N := \bigcap_{k=1}^q M_k$ the mutual intersections are transversal in the usual sense, and such that finally N is compact. We now prove Lemma 1.2.

Definition 3.1. We write $M = M_k^+ \cup M_k^-$ for the decomposition of M induced by the hypersurface M_k. Define $h_k\colon M \to \mathbb{R}$ as the signed distance to M_k, i.e.

$$h_k(x) = d(x, M_k) \qquad \text{if } x \in M_k^+$$

and

$$h_k(x) = -d(x, M_k) \qquad \text{if } x \in M_k^-.$$

Set $f\colon M \to \mathbb{R}^q; x \mapsto (h_1(x), \dots, h_q(x))$.

Proof of Lemma 1.2. By the triangle inequality, we have

$$|d(x, X) - d(y, X)| \le d(x, y)$$

for an arbitrary subset $X \subset M$. Moreover, as M has a length metric and M_k is separating, $x \in M_k^+$ and $y \in M_k^-$ satisfy

$$d(x, M_k) + d(y, M_k) \le d(x, y).$$

It follows that $h_k\colon M \to \mathbb{R}$ is a 1-Lipschitz map, therefore f is $\sqrt{q}$-Lipschitz. The condition that N is compact and that the M_k are coarsely transversal implies that the inverse image of every bounded subset of $\mathbb{R}^k$ under f is bounded. This finishes the proof of Lemma 1.2. □

We have now explained all the ingredients for the statement of Theorem 1.4. Indeed, in Section 2 we essentially already proved the model case of this theorem, which reads as follows:

Lemma 3.2. *Let N be a compact n-dimensional spin manifold and $E \to N$ a Hermitean bundle. Write E also for the pullback to $N \times \mathbb{R}^q$. Let $f\colon N \times \mathbb{R}^q \to \mathbb{R}^q$ be the projection. For this special case, the assertion of Theorem 1.4 holds.*

Proof. By naturality of the Mayer–Vietoris sequence, the following diagram is commutative:

$$\begin{array}{ccc}
K_{n+q}(C^*(N \times \mathbb{R}^q)) & \xrightarrow{f_*} & K_{n+q}(C^*(\mathbb{R}^q)) \\
\cong \downarrow \delta^q_{MV} & & \cong \downarrow \delta^q_{MV} \\
K_n(C^*(N)) & \xrightarrow[\cong]{pr_*} & K_n(C^*(\mathbb{R}^0)) \\
\downarrow \cong & & \downarrow \cong \\
K_n(\mathbb{C}) & \xrightarrow{=} & K_n(\mathbb{C})
\end{array}$$

By definition, the right vertical composition is κ so that $\operatorname{ind}_p(D_{N\times\mathbb{R}^q,E}) \in \mathbb{Z}$ is the image of $\operatorname{ind}_c(D_{N\times\mathbb{R}^q,E})$ under the map to the right lower corner. However, by Proposition 2.15,

$$\delta^q_{MV}\big(\operatorname{ind}_c(D_{N\times\mathbb{R}^q,E})\big) = \operatorname{ind}_c(D_{N,E}) \in K_n\big(C^*(N)\big),$$

and the latter is mapped to $\operatorname{ind}(D_{N,E}) \in K_n(\mathbb{C})$ under the isomorphism

$$K_n\big(C^*(N)\big) \to K_n(\mathbb{C}) \cong \mathbb{Z},$$

by Remark 2.13. □

The main novelty of this note is the localization result for the partitioned manifold index. It follows from the following localization result for the K-theory of $C^*(\mathbb{R}^n)$.

Definition 3.3. Two operators $T_1, T_2 \in D^*(\mathbb{R}^q; A)$ are said to coincide on an open set $U \subset \mathbb{R}^q$ if and only if $T_1 s = T_2 s$ for all s with $\operatorname{supp}(s) \subset U$.

Proposition 3.4. *Let $T_1, T_2 \in D^*(\mathbb{R}^q)$ be two operators which coincide on a non-empty open set $U \subset \mathbb{R}^q$. Assume that $[T_1]$ and $[T_2]$ represent elements in $K_j(D^*(\mathbb{R}^q; A)/C^*(\mathbb{R}^q; A))$, i.e. are either idempotents (for j even) or invertible (for j odd) modulo $C^*(\mathbb{R}^q; A)$.*

Just from the fact that T_1 and T_2 coincide on U, it then follows that

$$[T_1] = [T_2] \in K_j\big(D^*(\mathbb{R}^q; A)/C^*(\mathbb{R}^q; A)\big).$$

Proof. By translation invariance of $K_*(D^*(\mathbb{R}^q; A))$ and making U smaller, if necessary, we can assume that $U = B_r(0)$ for some $r > 0$.

We use the auxiliary space $\mathbb{R}^q \setminus U$. We apply the Mayer–Vietoris principle to the decomposition

$$\mathbb{R}^q \setminus B_r(0) = \big(\mathbb{R}^q_- \setminus B_r(0)\big) \cup \big(\mathbb{R}^q_+ \setminus B_r(0)\big).$$

The intersection is $\mathbb{R}^{q-1} \setminus B_r(0)$ and the half spaces are flasque. For $q > 1$ the decomposition is coarsely excisive and, using Proposition 2.6 and Theorem 2.10 we get a Mayer–Vietoris isomorphism

$$K_*\big(\mathbb{R}^q \setminus B_r(0); A\big) \to K_{*-1}\big(\mathbb{R}^{q-1} \setminus B_r(0); A\big).$$

Finally, in the case $q = 1$, write $\mathbb{R}' := (-\infty, -r] \cup [r, \infty) \subset \mathbb{R}$. Using the notation of Definition 2.7, then $D^*(\mathbb{R}'; A)/C^*(\mathbb{R}'; A)$ decomposes as a direct sum

$$\big(D^*((-\infty, -r] \subset \mathbb{R}'; A) + C^*(\mathbb{R}'; A)\big)/C^*(\mathbb{R}'; A) \\ \oplus \big(D^*([r, \infty); A) + C^*(\mathbb{R}'; A)\big)/C^*(\mathbb{R}'; A).$$

By the Noether isomorphism theorems for the two summands we have

$$\begin{aligned}\big(D^*((-\infty,-r]\subset\mathbb{R}';A)+C^*(\mathbb{R}';A)\big)/C^*(\mathbb{R}';A)\\ \cong D^*((-\infty,-r];A)/C^*((-\infty,-r];A),\\ \big(D^*([r,\infty)\subset\mathbb{R}';A)+C^*(\mathbb{R}';A)\big)/C^*(\mathbb{R}';A)\\ \cong D^*([r,\infty);A)/C^*([r,\infty);A).\end{aligned}$$

Therefore

$$K_*(\mathbb{R}';A)=K_*((-\infty,-r];A)\oplus K_*([r,\infty);A)=0$$

again using that the half line is flasque.

By assumption, T_1, T_2 coincide on U. We claim that this implies that

$$T_1-T_2\in D^*(Y\subset\mathbb{R}^q;A)$$

with $Y=\mathbb{R}^q\setminus U$. The support condition is automatic, as $U_r(Y)=\mathbb{R}^q$. If $\phi\colon\mathbb{R}^q\to\mathbb{C}$ has support on U then $(T_1-T_2)\phi=0$ by assumption, therefore $(T_1-T_2)\phi$ is compact. Because T_1, T_2 are in $D^*(\mathbb{R}^q;A)$, the commutator $[\phi, T_1-T_2]$ is compact. This gives then the required remaining compactness of $\phi(T_1-T_2)$.

From this, we conclude that the images of T_1 and T_2 in

$$D^*(\mathbb{R}^q;A)/\big(D^*(Y\subset\mathbb{R}^q;A)+C^*(\mathbb{R}^q;A)\big)$$

coincide.

By Proposition 2.8,

$$K_*\big(D^*(Y\subset\mathbb{R}^q;A)/C^*(Y\subset\mathbb{R}^q;A)\big)\cong K_{*-1}(Y;A)=0.$$

Therefore, the long exact K-theory sequence of the extension

$$\begin{aligned}0\to D^*(Y\subset\mathbb{R}^q;A)/C^*(Y\subset\mathbb{R}^q;A)\to D^*(\mathbb{R}^q;A)/C^*(\mathbb{R}^q;A)\\ \to D^*(\mathbb{R}^q;A)/\big(D^*(Y\subset\mathbb{R}^q;A)+C^*(\mathbb{R}^q;A)\big)\to 0\end{aligned}$$

gives the isomorphism, induced by the projection,

$$K_*(\mathbb{R}^q;A)\xrightarrow{\cong}K_*\big(D^*(\mathbb{R}^q;A)/\big(D^*(Y\subset\mathbb{R}^q;A)+C^*(\mathbb{R}^q;A)\big)\big).$$

We observed above that the images of T_1 and T_2 in the right hand algebra coincide. Because of the isomorphism, $[T_1]=[T_2]\in K_*(\mathbb{R}^q;A)$, as we had to prove. □

The localization theorem, Proposition 1.6, now is a rather direct corollary, as we want to prove next. Assume therefore the situation of Proposition 1.6, with two manifolds $f\colon M\to\mathbb{R}^q$, $f'\colon M'\to\mathbb{R}^q$ together with Hilbert A-module bundles with connection which are locally isometric on open neighborhoods U, U' of the inverse images N, N' of 0 via isometries ψ, Ψ. As f is proper and continuous and U is an open neighborhood of N (and the corresponding situation for M'), if we choose $t > 0$ sufficiently small then

$$f^{-1}(B_t(0)) \subset U \quad \text{and} \quad (f')^{-1}(B_t(0)) \subset U'.$$

Choose $r > 0$ such that

$$U_r(f^{-1}(B_{t/2}(0))) \subset f^{-1}(B_t(0)).$$

Because U, U' are isometric, the same is then also true for M'. Next, choose a smooth chopping function χ as for the definition of $[D_E]$ such that its Fourier transform $\widehat{\chi}$ (which is a distribution which is smooth outside 0, as explained in [2]) has support in $(-r/4, r/4)$. By the Fourier inversion formula and unit propagation speed of the wave operator (which implies that $\chi(D_E), \chi(D_{E'})$ have propagation $r/4$), $\chi(D_E)s = \Psi\chi(D_{E'})\Psi^{-1}s$ for each s with support in $f^{-1}(B_{t/2}(0))$. Next, for the construction of

$$f_*: D^*(M; A) \to D^*(\mathbb{R}^q; A) \quad \text{and} \quad f'_*: D^*(M'; A) \to D^*(\mathbb{R}^q; A)$$

choose isometries V, V' as in Definition 2.3 with propagation smaller than $r/4$. These isometries can be constructed locally and patched together. We can therefore in addition arrange that

$$Vs = V'\Psi s \quad \text{if } \operatorname{supp}(s) \subset U_{3r/4}\big(f^{-1}(B_{t/2}(0))\big) \subset U. \tag{3.1}$$

As

$$\langle V^*u, s\rangle_{L^2(S\otimes E)} = \langle u, Vs\rangle_{L^2(\mathbb{R}^q)},$$

the fact that V has propagation $r/4$ implies that if $\operatorname{supp}(u) \subset B_{t/2}(0)$ then the support of V^*u is contained in $U_{r/4}(f^{-1}(B_{t/2}(0)))$. Then Equation (3.1) implies that $\Psi V^*u = (V')^*u$ for these u.

Taken together, we get that

$$V\chi(D_E)V^*u = V'\chi(D_{E'})(V')^*u$$

if u is supported on $B_{t/2}(0)$. This implies that

$$f_*\big((\chi(D_E)+1)/2\big) \quad \text{and} \quad f'_*\big((\chi(D_{E'})+1)/2\big)$$

coincide on $B_{t/2}(0) \subset \mathbb{R}^q$. If M has even dimension, in addition we have to choose the measurable bundle isomorphisms $V_S, V'_{S'}$ between the positive and negative spinor bundles. Again, this construction is local and we can therefore arrange that for sections supported on U the isometry Ψ intertwines these bundle isomorphisms, i.e. $\Psi V_S s = V'_{S'}\Psi s$ if $\operatorname{supp}(s) \subset U$.

It then follows that, if u is supported on $B_{t/2}(0) \subset \mathbb{R}^q$ then

$$\begin{aligned} f_*(V_S^* \chi(D_E)_+)u &\overset{\text{Def}}{=} V(V_S^* \chi(D_E)_+)V^* u \\ &= V'(V'_{S'} \chi(D_{E'})_+)(V')^* u = f'_*(V'_{S'} \chi(D_{E'})_+)(u). \end{aligned}$$

To summarize: in all dimensions the representatives of the classes $f_*[D_E]$ and $f'_*([D'_{E'}])$ coincide on the non-empty open subset $B_{t/2}(0) \subset \mathbb{R}^q$. By Proposition 3.4,

$$f_*\big([D_E]\big) = f'_*\big([D_{E'}]\big) \in K_{\dim M}(\mathbb{R}^q; A).$$

By naturality of the boundary map of the K-theory long exact then also

$$f_*\big(\operatorname{ind}_c(D_E)\big) = f'_*\big(\operatorname{ind}_c(D_{E'})\big) \in K_{\dim M}\big(C^*(\mathbb{R}^q; A)\big),$$

as we have to prove.

Finally, we now can give the proof of the multi-partitioned manifold index Theorem 1.4.

Given $f : M \to \mathbb{R}^q$ as in Theorem 1.4, by homotopy invariance of the index we can deform the metric on M and connection on E in a neighborhood U of N such that it is isometric to a neighborhood of $N \times \{0\}$ in $N \times \mathbb{R}^q$ (with product structure) without changing $\operatorname{ind}(D_{M,E})$.

By Proposition 1.6, which we just proved,

$$f_*\big(\operatorname{ind}_c(D_{M,E})\big) = f_*\big(\operatorname{ind}_c(D_{N \times \mathbb{R}^q, E|_N})\big).$$

But for the latter we already proved in Lemma 3.2 that

$$\kappa\big(\operatorname{ind}_c(D_{N \times \mathbb{R}^q, E|_N})\big) = \operatorname{ind}(D_{N, E|_N}) \in \mathbb{Z}.$$

Therefore Theorem 1.4 is established.

We next apply the multi-partitioned manifold index theorem to prove non-existence theorems for metrics with positive scalar curvature.

Lemma 3.5. *Let M be a spin manifold with spinor bundle S and let E be a flat Hilbert A-module bundle bundle on M. Let D_E be a Dirac operator twisted by E.*

(1) *If there is a constant $C > 0$ such that the scalar curvature of g is grater than C outside Y, then $\operatorname{ind}_c(D_E)$ is in the image of*

$$K_*\big(C^*(Y \subset M; A)\big) \to K_*\big(C^*(M; A)\big).$$

(2) *Let (M', g') be another complete manifold. If $f : M \to M'$ is a proper and uniformly expanding map with $f(Y) \subset Y' \subset M'$ then $f_*(\operatorname{ind}_c D) \in K_*(C^*(M'; A))$ takes its value in the image of*

$$K_*\big(C^*(Y' \subset M'; A)\big) \to K_*\big(C^*(M'; A)\big).$$

Proof. The first part for the case $A = \mathbb{C}$ is stated in [16, page 22] without proof. In [17] a proof of this special case is provided using the Friedrichs extension of symmetric operators which are bounded below. A sketch of a proof using similar ideas for general A is given in [22]. Simultaneously, a detailed proof using Fourier inversion techniques was given in the Göttingen thesis of Daniel Pape [13] and appeared in [2].

The second part is a direct consequence of the first part and of naturality. □

With this lemma, we are in the position to prove Theorem 1.7. Assume therefore that $f: M \to \mathbb{R}^q$ is a proper and uniformly expansive map defined on a complete spin manifold M and assume that f is smooth near $N := f^{-1}(0)$ such that 0 is a regular value. Let D stand for the Dirac operator of M. If the scalar curvature of M is uniformly positive on the quadrant $\bigcap_{k=1}^{q} M_k^+ = f^{-1}([0,\infty)^q)$, then by Lemma 3.5 $f_*(\mathrm{ind}_c(D))$ lies in the image of

$$K_*\big(C^*(\mathbb{R}^q \setminus [0,\infty)^q \subset \mathbb{R}^q)\big) \to K_*\big(C^*(\mathbb{R}^q)\big).$$

Now, $\mathbb{R}^q \setminus [0,\infty)^q$ is a flasque space, therefore by Proposition 2.6,

$$K_*\big(C^*(\mathbb{R}^p \setminus [0,\infty)^q)\big) = K_*\big(C^*(\mathbb{R}^q \setminus [0,\infty)^q \subset \mathbb{R}^q)\big)$$

vanishes. It follows, under the positivity assumption on the scalar curvature, that $f_*(\mathrm{ind}_c(D)) = 0$. On the other hand, by Theorem 1.4,

$$\kappa f_*(\mathrm{ind}_c(D)) = \mathrm{ind}(D_N) = \hat{A}(N) \neq 0.$$

This contradiction proves the theorem.

4. Generalization to arbitrary coefficient C^*-algebras and open questions

The partitioned manifold index theorem ($q = 1$) for Hilbert A-module coefficients is established in [21] and plays an important role in [2] in the proof of the codimension 2-obstruction to positive scalar curvature. Generalizations of this obstruction to higher codimensions, which would be very interesting, most likely would require to generalize Theorem 1.4 to arbitrary C^*-algebras A. The most natural example is $A = C^*(\pi_1(M))$, the (reduced or maximal) group C^*-algebra, and $E = \widetilde{M} \times_{\pi_1 M} C^*(\pi_1(M))$, the Mishchenko line bundle, in line with the principle that the optimal C^*-index information can be obtained twisting with this bundle [19].

Our proof of Theorem 1.4 generalizes, provided the following two facts of input are established:

The generalization of Remark 2.13, namely the identification of the coarse index of a compact space with the Mishchenko–Fomenko index (or any other version classically used) — this is folk knowledge, but it is hard to find suitable references.

The product formula for the A-module twisted index generalizing Proposition 2.15. Actually, one should establish a more general product formula for coarse indices as follows:

Theorem 4.1. *Given complete Riemannian manifolds M_1 with Hilbert A_1-module bundle $E_1 \to M_1$ and M_2 with Hilbert A_2-module bundle $E_2 \to M_2$, the external tensor product of operators on*

$$L^2(M_1, S_1 \otimes E_1) \otimes L^2(M_2, S_2 \otimes E_2) \xrightarrow{\cong} L^2(M_1 \times M_2, S_1 \otimes S_2 \otimes E_1 \otimes E_2)$$

defines an algebra homomorphism

$$C^*(M_1; A_1) \otimes C^*(M_2; A_2) \to C^*(M_1 \times M_2; A_1 \otimes A_2)$$

(which in general is not an isomorphism) and an induced map in K-theory

$$\alpha\colon K_i\big(C^*(M_1; A_1)\big) \otimes K_j\big(C^*(M_2; A_2)\big) \to K_{i+j}\big(C^*(M_1 \times M_2; A_1 \otimes A_2)\big).$$

In this situation, we should have a product formula for the index of the Dirac operator: If M_1, M_2 (and therefore also $M_1 \times M_2$) are spin manifolds then for the coarse indices we obtain

$$\alpha\big(\operatorname{ind}_c(D_{M_1,E_1}) \otimes \operatorname{ind}_c(D_{M_2,E_2})\big) = \operatorname{ind}_c(D_{M_1 \times M_2, E_1 \otimes E_2}) \\ \in K_{\dim(M_1)+\dim(M_2)}\big(C^*(M_1 \times M_2; A_1 \otimes A_2)\big).$$

Indeed, this is the most non-trivial generalization as this index formula at some point requires explicit calculations, using e.g. basics of Hilbert C^*-index theory as in [18]. However, the best route to prove this is probably to give a new description of the coarse index, e.g. using the picture of [6] or Kasparov theory, which is more directly adapted to such product situations. Of course, then the task remains to identify the new definition of the index with the old one. This will follow from the methods of Rudolf Zeidler's [23].

It would perhaps also be desirable to give a written account of the details of the generalizations of 2.6, 2.8, 2.10, on the other hand this is really straightforward.

Remark 4.2. Note that the proof of Theorem 1.7 works without any change replacing $\mathbb{C}$ by a general C^*-algebra A, as soon as Theorem 1.4 is generalized.

Question 4.3. (1) It would be interesting to work out explicit example situations where our theorem, in particular the obstruction to positive scalar curvature, can be applied.

(2) In particular, can one establish a higher version of the codimension 2-obstruction to positive scalar curvature, using e.g. intersections of several transversal codimension 2-submanifolds, or an iterative procedure (with a second codimension 2-submanifold inside the first,. . .).

Very speculatively, can one perhaps pass to even higher codimension submanifold obstructions?

(3) In [14], the secondary invariants $\rho(g) \in K_{*+1}(D^*M)$ are studied and a secondary partitioned manifold index theorem for them is established. However, Rudolf Zeidler, in his Göttingen doctoral thesis, provides examples which show that the localization methods of this paper do not carry over to the secondary invariants. This is despite the fact that a product formula for the secondary invariants is proved in [23].

References

[1] M. Gromov and H. Blaine Lawson, Jr, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, *Inst. Hautes Études Sci. Publ. Math.*, (1983), no. 58, 83–196 (1984). Zbl 0627.53036 MR 720933

[2] B. Hanke, D. Pape, and T. Schick, Codimension two index obstructions to positive scalar curvature, *Ann. Inst. Fourier (Grenoble)*, **65** (2015), no. 6, 2681–2710. Zbl 1344.58012 MR 3449594

[3] B. Hanke and T. Schick, Enlargeability and index theory, *J. Differential Geom.*, **74** (2006), no. 2, 293–320. Zbl 1122.58011 MR 2259056

[4] B. Hanke and T. Schick, Enlargeability and index theory: infinite covers, *K-Theory*, **38** (2007), no. 1, 23–33. Zbl 1128.58012 MR 2353861

[5] N. Higson, A note on the cobordism invariance of the index, *Topology*, **30** (1991), no. 3, 439–443. Zbl 0731.58065 MR 1113688

[6] N. Higson and E. Guentner, Group C^*-algebras and K-theory, in *Noncommutative geometry*, 137–251, Lecture Notes in Math., 1831, Springer, Berlin, 2004. Zbl 1053.46048 MR 2058474

[7] N. Higson and J. Roe, On the coarse Baum-Connes conjecture, in *Novikov conjectures, index theorems and rigidity (Oberwolfach, 1993). Vol. 2*, 227–254, London Math. Soc. Lecture Note Ser., 227, Cambridge Univ. Press, Cambridge, 1995. Zbl 0957.58019 MR 1388312

[8] N. Higson and J. Roe, *Analytic K-homology*, Oxford University Press, 2000. Zbl 0968.46058 MR 1817560

[9] N. Higson, J. Roe, and G. Yu, A coarse Mayer–Vietoris principle, *Math. Proc. Cambridge Philos. Soc.*, **114** (1993), no. 1, 85–97. Zbl 0792.55001 MR 1219916

[10] D. Kucerovsky, Functional calculus and representations of $C_0(\mathbb{C})$ on a Hilbert module, *Q. J. Math.*, **53** (2002), no. 4, 467–477. Zbl 1036.46043 MR 1949157

[11] C. E. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series, 210, Cambridge University Press, 1995. Zbl 0822.46080 MR 1325694

[12] A. S. Miščenko and A. T. Fomenko, The index of elliptic operators over C^*-algebras, *Izv. Akad. Nauk SSSR Ser. Mat.*, **43** (1979), no. 4, 831–859, 967. Zbl 0416.46052 MR 548506

[13] D. Pape, *Index theory and positive scalar curvature*, Dissertation, Mathematisches Institut, Georg-August-Universität Göttingen, 2011.

[14] P. Piazza and T. Schick, Rho-classes, index theory and Stolz' positive scalar curvature sequence, *J. Topol.*, **7** (2014), no. 4, 965–1004. Zbl 1320.58012 MR 3286895

[15] J. Roe, Partitioning noncompact manifolds and the dual Toeplitz problem, in *Operator algebras and applications, Vol. 1*, 187–228, London Math. Soc. Lecture Note Ser., 135, Cambridge Univ. Press, Cambridge, 1988. Zbl 0677.58042 MR 996446

[16] J. Roe, *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conference Series in Mathematics, 90, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. Zbl 0853.58003 MR 1399087

[17] J. Roe, Positive curvature, partial vanishing theorems, and coarse indices, *Proc. Edinb. Math. Soc.*, **59** (2016), no. 20, 223–233. Zbl 1335.58017 MR 3439130

[18] T. Schick, L^2-index theorems, KK-theory, and connections, *New York J. Math.*, **11** (2005), 387–443. Zbl 1093.58009 MR 2188248

[19] T. Schick, The topology of positive scalar curvature, in *Proceedings of the International Congress of Mathematicians (ICM, Seoul, 2014). Vol. II. Invited lectures*, 227–254, KM Kyung Moon Sa, Seoul, 2014. Zbl 1373.53053

[20] P. Siegel, The Mayer–Vietoris Sequence for the Analytic Structure Group, 2012. arXiv:1212.0241

[21] M. E. Zadeh, Index theory and partitioning by enlargeable hypersurfaces, *J. Noncommut. Geom.*, **4** (2010), no. 3, 459–473. Zbl 1201.58020 MR 2670972

[22] M. E. Zadeh, A note on some classical results of Gromov–Lawson, *Proc. Amer. Math. Soc.*, **140** (2012), no. 10, 3663–3672. Zbl 1306.58007 MR 2929034

[23] R. Zeidler, Positive scalar curvature and product formulas for secondary index invariants, *J. Topol.*, **9** (2016), no. 3, 687–724. Zbl 1354.58019 MR 3551834

Received 14 December, 2014; revised 15 June, 2015

T. Schick, Georg-August-Universität Göttingen, Mathematisches Institut,
Bunsenstrasse 3, 37073 Göttingen, Germany

E-mail: thomas.schick@math.uni-goettingen.de

M. E. Zadeh, Department of Mathematical Sciences, Sharif University of Technology,
P.O. Box 11155-9415, Tehran 14588-89694, Iran

E-mail: esfahani@sharif.ir