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Quotients of singular foliations and Lie 2-group actions

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Abstract. Androulidakis–Skandalis (2009) showed that every singular foliation has an associated topological groupoid, called holonomy groupoid. In this note, we exhibit some functorial properties of this assignment: if a foliated manifold $(M, \mathcal{F}_M)$ is the quotient of a foliated manifold $(P, \mathcal{F}_P)$ along a surjective submersion with connected fibers, then the same is true for the corresponding holonomy groupoids. For quotients by a Lie group action, an analogue statement holds under suitable assumptions, yielding a Lie 2-group action on the holonomy groupoid.

1. Introduction

The space of leaves of a foliation is typically not smooth and might fail to be Hausdorff. As a replacement for the leaf space, one often takes a smooth group-like object canonically associated to the foliation, namely the holonomy Lie groupoid, and declares two Lie groupoids to model the same space if they are Morita equivalent.

Singular foliations extend the classical notion of foliation by allowing singularities. In this note, we will adopt the definition of singular foliation that appears in [1] and is inspired by the work of Stefan and Sussmann (among others) in the 1970s. Not only does it entail a smooth partition of the underlying manifold in immersed submanifolds of varying dimension, but also it contains information about the dynamics along the leaves, i.e., the ways that one can flow along them. This notion of singular foliation allows for Lie-theoretic constructions. The most prominent of them is the canonical assignment of a topological groupoid by Androulidakis–Skandalis [1], which is called *holonomy groupoid* since, in the case of (regular) foliations, it recovers the holonomy Lie groupoid mentioned above.

The assignment of the holonomy groupoid to a singular foliation satisfies functoriality properties: under certain conditions, a map π between foliated manifolds induces canonically a morphism of holonomy groupoids, which can be regarded as a replacement for the induced map of leaf spaces. In this paper, we prove properties of this assignment when π satisfies a surjectivity property, and thus it can be regarded as a quotient map. We do so motivated by the desire to establish to which extent the construction of the holonomy groupoid satisfies functorial properties. Since every holonomy groupoid has an associated C^*-algebra [1], this is relevant also in non-commutative geometry, and the natural

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appearance of higher Lie groupoids in our work seems an interesting phenomenon in that context too.

We now outline the main results.

Statement of results. Let $\mathcal{F}$ be a singular foliation on a manifold P, and let $\pi\colon P \to M$ be a surjective submersion with connected fibers. Under mild invariance conditions, $\mathcal{F}$ can be pushed forward along π to a singular foliation $\mathcal{F}_M$ on M. Since the foliation $\mathcal{F}_M$ is obtained from $\mathcal{F}$ by a quotient procedure, it is natural to wonder whether the holonomy groupoid $\mathcal{H}(\mathcal{F}_M)$ is also quotient of the holonomy groupoid $\mathcal{H}(\mathcal{F})$. We show that this is always the case (see Theorem 2.25).

Theorem. *The map π induces a canonical open surjective morphism*

$$\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M).$$

We emphasize that this is a statement about (typically not source simply connected) topological groupoids. It is an analogue of the following fact in Lie groupoid theory: let A, B be integrable Lie algebroids and $\mathcal{G}, \mathcal{H}$ the source simply connected Lie groupoids integrating them. Given a morphism of integrable Lie algebroids $\phi\colon A \to B$ which is fiber-wise surjective and covers a surjective submersion between the manifolds of objects[1], there is a unique Lie groupoid morphism $\Phi\colon \mathcal{G} \to \mathcal{H}$ integrating ϕ, and further Φ is a[2] surjective submersion.

We then refine the above result in the case of Lie group actions. That is, we suppose that $\pi\colon P \to M = P/G$ is the quotient map of the action of a Lie group G on P, which we assume to be free, proper, and preserving the singular foliation $\mathcal{F}$. The action lifts naturally to a G-action on $\mathcal{H}(\mathcal{F})$, but a simple dimension count shows that the quotient cannot be isomorphic to $\mathcal{H}(\mathcal{F}_M)$ in general. In Section 3, we show that when $\mathcal{F}$ contains the infinitesimal generators of the G-action, there is a natural action of a semidirect product Lie group $G \rtimes G$ on $\mathcal{H}(\mathcal{F})$ – not by Lie groupoid automorphisms – with quotient $\mathcal{H}(\mathcal{F}_M)$. Remarkably, this is a Lie 2-group action (see Theorem 3.7). In other words:

Theorem. *When $\mathcal{F}$ contains the infinitesimal generators of the G-action, the induced morphism Ξ is the quotient map of a Lie group action in the category of topological groupoids.*

We expect the above conclusion to hold in greater generality, namely, under a regularity condition on the intersection of $\mathcal{F}$ with the foliation generated by the G-action on P. We hope to address this in a future paper.

In the general case that $\mathcal{F}$ does not necessarily contain the infinitesimal generators of the G-action, the fibers of Ξ coincide with the orbits of a *groupoid* action on $\mathcal{H}(\mathcal{F})$, which we describe in Proposition 5.14.

We also obtain a canonical Lie 2-group action, whose orbits however may be smaller than the fibers of Ξ (see Proposition 4.9 and Corollary 4.11).

[1] Such a morphism is called a Lie algebroid fibration in [10].
[2] Φ may fail to be a Lie groupoid fibration; see, for instance, Example 5.7.

Proposition. *There is a canonical Lie ideal* $\mathfrak{h}$ *of* $\mathfrak{g}$ *which gives rise to a Lie 2-group* $H \rtimes G$ *and a Lie 2-group action on* $\mathcal{H}(\mathcal{F})$*, whose orbits are contained in the* Ξ*-fibers.*

The concrete form of this Lie 2-group action is inspired by the special case in which $\mathcal{F}$ contains the infinitesimal generators of the G-action (hence $H = G$). Indeed, in that case, we recover the Lie 2-group action given in Section 3.

We now return to the general setup of the first theorem (in particular, the surjective submersion $\pi\colon P \to M$ does not necessarily arise from a Lie group action). In Section 5, we address the question of when the open surjective morphism $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$ obtained in the first theorem is a fibration. This is useful because when Ξ is a fibration of Lie groupoids [10], it allows to describe the holonomy groupoid $\mathcal{H}(\mathcal{F}_M)$ without knowing $\mathcal{F}_M$. Namely, $\mathcal{H}(\mathcal{F}_M)$ is the quotient of $\mathcal{H}(\mathcal{F})$ by a normal subgroupoid system; the latter – as we explain just before Example 5.17 – is a set of data defined directly and explicitly in terms of $\mathcal{H}(\mathcal{F})$ and the projection $\pi\colon P \to M$. There is also a notion of fibration of topological groupoids [7], which however appears to be less useful for the purpose of describing the quotient groupoid.

We summarize as follows Example 5.7 and Propositions 5.9, 5.11, and 5.15.

Proposition.

(i) *The open surjective morphism* $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$ *generally fails to be a fibration of topological groupoids. In the smooth case,* Ξ *generally fails to be a fibration of Lie groupoids.*

(ii) *Suppose that* $\mathcal{F}$ *is the pullback foliation of* $\mathcal{F}_M$*. Then the morphism* Ξ *is a fibration of topological groupoids. In the smooth case,* Ξ *is a fibration of Lie groupoids.*

(iii) *Suppose that* π *is the quotient map of a free, proper* G*-action on* P *which preserves* $\mathcal{F}$*. Then the morphism* Ξ *is a fibration of topological groupoids, provided a technical condition is satisfied. In the smooth case,* Ξ *is always a fibration of Lie groupoids.*

Conventions. Given a groupoid $\mathcal{G}$ with space of objects P, its source map is denoted by $\mathbf{s}\colon \mathcal{G} \to P$, its target map is denoted by $\mathbf{t}\colon \mathcal{G} \to P$, and its product (multiplication) $\mathcal{G}{}_{\mathbf{s}}\times_{\mathbf{t}}\mathcal{G} \to \mathcal{G}$ is denoted by $\circ$.

2. Quotients of foliations by surjective submersions

We start reviewing singular foliations in the sense of [1] and the topological groupoids associated to them. As recalled in Section 2.2, given a surjective submersion with connected fibers $P \to M$, an "invariant" singular foliation $\mathcal{F}$ on P induces a singular foliation $\mathcal{F}_M$ on M, which can be regarded as a quotient of the former. The main statement of the paper is that the holonomy groupoid of $\mathcal{F}_M$ is a quotient of the holonomy groupoid of $\mathcal{F}$; see Theorem 2.25. In Section 2.3, we give an explicit characterization of the quotient map when $\mathcal{F}$ is a pullback foliation.

2.1. Background on singular foliations and holonomy groupoids

We review first the notions of singular foliation and holonomy groupoid from the work [1] by Androulidakis–Skandalis.

Singular foliations.

Definition 2.1. A *singular foliation* on a manifold P is a $C^\infty(P)$-submodule $\mathcal{F}$ of the compactly supported vector fields $\mathfrak{X}_c(P)$, closed under the Lie bracket and locally finitely generated. A *foliated manifold* is a manifold with a singular foliation.

Remark 2.2. Let P be a manifold and $\mathcal{F}$ a submodule of $\mathfrak{X}_c(P)$. Take an open set $U \subset P$ and consider

$$\iota_U^{-1}\mathcal{F} := \{X|_U : X \in \mathcal{F} \text{ and } \mathrm{supp}(X) \subset U\}.$$

The module $\mathcal{F}$ is *locally finitely generated* if, for every point of P, there is an open neighborhood U and finitely many vector fields $X_1, \dots, X_n \in \mathfrak{X}(U)$ such that $\iota_U^{-1}\mathcal{F}$ is $\mathrm{Span}_{C_c^\infty(U)}\{X_1, \dots, X_n\}$.

Any singular foliation gives rise to a singular distribution that satisfies the assumptions of the Stefan–Sussmann theorem; therefore, it induces a partition of the manifold into immersed submanifolds called leaves.

Example 2.3. (i) Given an involutive regular distribution $D \subset TP$, which corresponds to a regular foliation by the Frobenius theorem, we obtain a singular foliation $\mathcal{F} := \Gamma_c(D)$.

(ii) If A is a Lie algebroid over P with anchor $\sharp\colon A \to TP$, then $\sharp(\Gamma_c(A))$ is a singular foliation.

The following vector spaces measure the regularity of a singular foliation at a given point.

Definition 2.4. Let $(P, \mathcal{F})$ be a foliated manifold and $p \in P$. Denote $I_p := \{f \in C^\infty(P) : f(p) = 0\}$.

The *tangent of* $\mathcal{F}$ at p is $F_p := \{X(p) : X \in \mathcal{F}\} \subset T_pP$.

The *fiber of* $\mathcal{F}$ at p is $\mathcal{F}_p := \mathcal{F}/I_p\mathcal{F}$.

If $\dim(F_p)$ is constant, then $\mathcal{F}$ is a regular foliation. If $\dim(\mathcal{F}_p)$ is constant, then $\mathcal{F}$ is a projective module and it is isomorphic to the sections of a vector bundle.

Definition 2.5. Let $(M, \mathcal{F}_M)$ be a foliated manifold and $\pi\colon P \to M$ a submersion. Denote

$$\pi^*\mathcal{F}_M := \mathrm{Span}_{C_c^\infty(P)}\{X \circ \pi : X \in \mathcal{F}_M\},$$

a submodule of sections of the pullback vector bundle π^*TM over P. The *pullback foliation* of $\mathcal{F}_M$ under π is

$$\pi^{-1}(\mathcal{F}_M) := (d\pi)^{-1}(\pi^*\mathcal{F}_M),$$

where the preimage is taken with respect to $d\pi\colon \mathfrak{X}_c(P) \to \Gamma(\pi^*TM), Y \mapsto d\pi(Y)$. The pullback foliation is a singular foliation on P [1, Proposition 1.10].

Remark 2.6. The original definition in [1] is given in more generality, for any smooth map π transverse to $\mathcal{F}_M$. When π is a submersion, we also have the description

$$\pi^{-1}(\mathcal{F}_M) = \mathrm{Span}_{C_c^\infty(P)} \{(d\pi)^{-1}(\{X \circ \pi : X \in \mathcal{F}_M\})\},$$

i.e., the pullback foliation is generated by projectable vector fields whose projection lies in $\mathcal{F}_M$.

Definition 2.7. Given a submodule $\mathcal{F}$ of $\mathfrak{X}_c(P)$, its *global hull* is given by

$$\widehat{\mathcal{F}} := \{X \in \mathfrak{X}(P) : fX \in \mathcal{F} \ \forall f \in C_c^\infty(P)\}.$$

Given a surjective submersion $\pi\colon P \to M$ (not necessarily with compact fibers) and a singular foliation $\mathcal{F}_M$ on M, it may happen that the set of projectable vector fields in $\pi^{-1}(\mathcal{F}_M)$ consists just of the zero vector field, but, by definition of a pullback foliation, $\pi^{-1}(\mathcal{F}_M)$ is the $C_c^\infty(P)$-span of projectable vector fields in $\widehat{\pi^{-1}(\mathcal{F}_M)}$.

Holonomy groupoids. Singular foliations as in Definition 2.1 contain more information than just the underlying partition of P into leaves since they carry "dynamics" on P. This extra information was used in [1] to define the holonomy groupoid via the following "building blocks."

Definition 2.8. Given a foliated manifold $(P, \mathcal{F})$, a *bisubmersion* for $\mathcal{F}$ is a triple $(U, \mathbf{t}, \mathbf{s})$, where U is a manifold and $\mathbf{t}\colon U \to P$, $\mathbf{s}\colon U \to P$ are submersions such that

$$\mathbf{s}^{-1}(\mathcal{F}) = \mathbf{t}^{-1}(\mathcal{F}) = \Gamma_c(\ker(d\mathbf{s})) + \Gamma_c(\ker(d\mathbf{t})).$$

Example 2.9. Let $\mathcal{G} \rightrightarrows P$ be a Lie groupoid and $\mathcal{F}_{\mathcal{G}}$ the singular foliation on P given by the Lie algebroid of $\mathcal{G}$. Any Hausdorff open set $U \subset \mathcal{G}$, together with $\mathbf{t}|_U$ and $\mathbf{s}|_U$, is a bisubmersion for $\mathcal{F}_{\mathcal{G}}$. In particular, if $\mathcal{G}$ is a Hausdorff Lie groupoid, then it is a bisubmersion.

The following proposition, proven in [1, §2.3], assures the existence of bisubmersions at any given point $p_0 \in P$.

Proposition 2.10. *Given $p_0 \in P$, let $X_1, \ldots, X_k \in \mathcal{F}$ be vector fields whose classes in the fiber $\mathcal{F}_{p_0}$ form a basis. For $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$, put $\varphi_v := \exp(\sum_i v_i X_i)$, where* exp *denotes the time one flow. Put $W = \mathbb{R}^k \times P$, $\mathbf{t}(v, p) = \varphi_v(p)$, and $\mathbf{s}(v, p) = p$. There is a neighborhood $U \subset W$ of $(0, p_0)$ such that $(U, \mathbf{t}, \mathbf{s})$ is a bisubmersion.*

Definition 2.11. A bisubmersion as in Proposition 2.10, when it has $\mathbf{s}$-connected fibers, is called a *path holonomy bisubmersion*.

Definition 2.12. Let $(P, \mathcal{F})$ be a foliated manifold, $(U, \mathbf{t}, \mathbf{s})$ a bisubmersion, and $u \in U$. Denote $x = \mathbf{s}(u)$.

- A *bisection* at u consists of an $\mathbf{s}$-section $\sigma\colon V \to U$, defined on an open subset $V \subset P$, whose image is transverse to the fibers of $\mathbf{t}$ and passes through u.

- Given a diffeomorphism $f: P \supset V \to V' \subset P$, a bisubmersion $(U, \mathbf{t}, \mathbf{s})$ is said to *carry* f at $u \in U$ if there exists a bisection σ at u such that $f = \mathbf{t} \circ \sigma$.

Definition 2.13. Let $(P, \mathcal{F})$ be a foliated manifold, and let $(U_1, \mathbf{t}_1, \mathbf{s}_1)$ and $(U_2, \mathbf{t}_2, \mathbf{s}_2)$ be bisubmersions for $\mathcal{F}$.

- The *inverse* bisubmersion of U_1 is $U_1^{-1} := (U_1, \mathbf{s}_1, \mathbf{t}_1)$, obtained by interchanging the source and target maps.
- The *composition* bisubmersion is $U_1 \circ U_2 := (U_{1\,\mathbf{s}_1}\!\times_{\mathbf{t}_2} U_2, \mathbf{t}_1, \mathbf{s}_2)$.
- A *morphism* of bisubmersions is a map $\mu: U_1 \to U_2$ that commutes with the respective source and target maps. We will say that it is a local morphism if it is defined on an open set of U_1.

If there is a morphism of bisubmersions $\mu: U_1 \to U_2$, then any local diffeomorphism carried by U_1 at $u \in U_1$ will be carried by U_2 at $\mu(u)$.

Definition 2.14. Let $(P, \mathcal{F})$ be a foliated manifold, and let $(U_1, \mathbf{t}_1, \mathbf{s}_1)$ and $(U_2, \mathbf{t}_2, \mathbf{s}_2)$ be bisubmersions for $\mathcal{F}$, $u_1 \in U_1$, and $u_2 \in U_2$. We say that u_1 is *equivalent* to u_2 if there is a local morphism of bisubmersions $\mu: U_1 \supset U_1' \to U_2$ with $\mu(u_1) = u_2$.

The previous definition gives an equivalence relation on any family of bisubmersions, as becomes clear from the following useful proposition.

Proposition 2.15. *Let $(P, \mathcal{F})$ be a foliated manifold, and let $(U_1, \mathbf{t}_1, \mathbf{s}_1)$ and $(U_2, \mathbf{t}_2, \mathbf{s}_2)$ be bisubmersions for $\mathcal{F}$, $u_1 \in U_1$, and $u_2 \in U_2$. Then u_1 is equivalent to u_2 if and only if U_1 and U_2 carry the same local diffeomorphism at u_1 and u_2, respectively.*

Proof. This is a direct consequence of [1, Corollary 2.11]. ∎

Definition 2.16. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of bisubmersions for $\mathcal{F}$.

- A bisubmersion U' is *adapted* to $\mathcal{U}$ if, for any $u' \in U'$, there is $u \in U \in \mathcal{U}$ which is equivalent to u'. A family of bisubmersions $\mathcal{U}'$ is adapted to $\mathcal{U}$ if any bisubmersion $U' \in \mathcal{U}'$ is adapted to $\mathcal{U}$.
- We say that $\mathcal{U}$ is an *atlas* if
  (1) for all $p \in P$, there is a $U \in \mathcal{U}$ that carries the identity diffeomorphism nearby p;
  (2) the inverse and finite compositions of elements of $\mathcal{U}$ are adapted to $\mathcal{U}$.

Example 2.17. Let $\mathcal{G} \rightrightarrows P$ be a Lie groupoid and $\mathcal{F}_{\mathcal{G}}$ its associated foliation. Any cover $\mathcal{U}_{\mathcal{G}} := \{U_i\}_{i \in I}$ of $\mathcal{G}$ by open Hausdorff subsets is an atlas of bisubmersions for $\mathcal{F}_{\mathcal{G}}$. In particular, if $\mathcal{G}$ is Hausdorff, it is also an atlas. Any two atlases given by Hausdorff covers of $\mathcal{G}$ are adapted to each other by the identity morphism on $\mathcal{G}$.

Proposition 2.18 (Groupoid of an atlas). *Let $\mathcal{U}$ be an atlas of bisubmersions for $\mathcal{F}$. Denote*

$$\mathcal{G}(\mathcal{U}) := \bigsqcup_{U \in \mathcal{U}} U/\sim,$$

where $\sim$ is the equivalence relation given in Definition 2.14, and endow it with the quotient topology. There is a natural structure of an open topological groupoid on $\mathcal{G}(\mathcal{U})$, where the source and target maps are given by the source and target maps of the elements of $\mathcal{U}$.

The proof of the following statement can be found in [9, §3.1] and in [8].

Proposition 2.19. *Two atlases $\mathcal{U}$ and $\mathcal{U}'$ are adapted to each other if and only if their corresponding topological groupoids are isomorphic: $\mathcal{G}(\mathcal{U}) \cong \mathcal{G}(\mathcal{U}')$.*

Definition 2.20. Let $\mathcal{S}$ be a family of source connected bisubmersions such that $\bigcup_{U \in \mathcal{S}} \mathbf{s}(U) = M$, satisfying this condition: for every $U \in \mathcal{S}$ and $p \in \mathbf{s}(U)$, there is an element $e_p \in U$ carrying the identity diffeomorphism nearby p. The atlas $\mathcal{U}$ generated by $\mathcal{S}$ is called a *source connected atlas*.

For instance, consider a family $\mathcal{S}$ of path holonomy bisubmersions for $\mathcal{F}$ such that $\bigcup_{U \in \mathcal{S}} \mathbf{s}(U) = M$ generates a source connected atlas. It is called a *path holonomy atlas*. Also, any source connected Hausdorff Lie groupoid giving rise to $\mathcal{F}$ constitutes a source connected atlas.

The groupoid of a source connected atlas is source connected, and one can show [8] that it is the same[3] for all source connected atlases.

Definition 2.21. The *holonomy groupoid* $\mathcal{H}(\mathcal{F})$ of $\mathcal{F}$ is the groupoid of any source connected atlas.

In particular, the holonomy groupoid is an open source connected topological groupoid, which does not depend (up to isomorphism) on the choice of source connected atlas.

Remark 2.22. Let $\mathcal{G} \rightrightarrows P$ be a source connected Lie groupoid and $\mathcal{F}_\mathcal{G}$ its associated foliation. Then, as we now explain, the holonomy groupoid $\mathcal{H}(\mathcal{F}_\mathcal{G})$ is a quotient of $\mathcal{G}$.

Let $\mathcal{S}_\mathcal{G}$ be a Hausdorff source connected cover of a neighborhood of the identity bisection in $\mathcal{G}$ and $\mathcal{U}_\mathcal{G}$ the atlas generated by $\mathcal{S}_\mathcal{G}$. Then $\mathcal{U}_\mathcal{G}$ is a source connected atlas; hence $\mathcal{G}(\mathcal{U}_\mathcal{G}) \cong \mathcal{H}(\mathcal{F}_\mathcal{G})$. One can show that $\mathcal{U}_\mathcal{G}$ is adapted to the atlas given in Example 2.17 and, therefore, that $\mathcal{G}(\mathcal{U}_\mathcal{G}) \cong \mathcal{G}/\sim$, where the latter equivalence relation identifies two points when they carry the same local diffeomorphism. Consequently, $\mathcal{H}(\mathcal{F}_\mathcal{G}) \cong \mathcal{G}/\sim$.

When $\mathcal{F}$ is a regular foliation, $\mathcal{H}(\mathcal{F})$ agrees with the classical notion of holonomy groupoid of a regular foliation. Indeed, by the Frobenius theorem, there is a Lie algebroid $D \subset TP$ such that $\mathcal{F} = \Gamma_c(D)$. The monodromy groupoid $\Pi(\mathcal{F})$, consisting of homotopy classes of paths in the leaves of $\mathcal{F}$, is a source connected Lie groupoid integrating D.

[3]Proposition 2.19 then implies that all source connected atlases are adapted to each other.

Therefore, $\mathcal{H}(\mathcal{F}) \cong (\Pi(\mathcal{F})/\sim)$, and the latter is the well-known holonomy groupoid of a regular foliation.

Remark 2.23. A singular foliation $\mathcal{F}$ on M is called *projective* if there is a vector bundle E such that $\mathcal{F} \cong \Gamma_c(E)$ as $C^\infty(M)$-modules. These are exactly the singular foliations for which the holonomy groupoid $H(\mathcal{F})$ is a Lie groupoid. The class of projective foliations contains the regular foliations as a proper subclass.

2.2. The main theorem

We reproduce [2, Lemma 3.2] about quotients of foliated manifolds.

Proposition 2.24. *Let $\pi: P \to M$ be a surjective submersion with connected fibers. Let $\mathcal{F}$ be a singular foliation on P such that $\Gamma_c(\ker d\pi) \subset \mathcal{F}$. Then there is a unique singular foliation $\mathcal{F}_M$ on M with $\pi^{-1}(\mathcal{F}_M) = \mathcal{F}$.*

The following theorem is our main result and will be proven in Appendix A.1.

Theorem 2.25. *Let $\pi: P \to M$ be a surjective submersion with connected fibers. Let $\mathcal{F}$ be a singular foliation on P such that*

$$\left[\Gamma_c(\ker d\pi), \mathcal{F}\right] \subset \Gamma_c(\ker d\pi) + \mathcal{F}. \tag{2.1}$$

Denote by $\mathcal{F}_M$ the singular foliation on M obtained from $\mathcal{F}^{\text{big}} := \Gamma_c(\ker d\pi) + \mathcal{F}$ as in Proposition 2.24. Then there is a canonical, open, surjective morphism of topological groupoids

$$\Xi: \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$$

covering π.

This should be interpreted as follows. The singular foliation $\mathcal{F}_M$ is obtained from $\mathcal{F}$ by a quotient procedure (more precisely $\mathcal{F}_M = \pi_*\mathcal{F}$; see Lemma A.1 (ii)). The theorem states that the same is true for the respective holonomy groupoids.

Remark 2.26. We now give a characterization of the morphism Ξ. By Lemma A.1 (i) and Corollary A.6, any source connected atlas $\mathcal{U}$ for $\mathcal{F}$ satisfies that $\pi\mathcal{U} := \{(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s}) : U \in \mathcal{U}\}$ is an atlas equivalent to a path holonomy atlas for $\mathcal{F}_M$. The map Ξ is characterized by

$$\Xi\big([u]\big) = [u]_M \tag{2.2}$$

for all $u \in \mathcal{U}$, where $[u]_M$ is the class of $u \in (U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$, a bisubmersion for $\mathcal{F}_M$.

Remark 2.27. When $\mathcal{H}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F}_M)$ are Lie groupoids, Ξ is a surjective submersion (cf. Remark A.3).

For regular foliations, the morphism Ξ admits a familiar description.

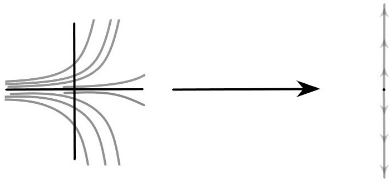

Figure 1. The foliated manifolds in Example 2.29.

Proposition 2.28 (Regular foliations). *When both $\mathcal{F}$ and $\mathcal{F}_M$ are regular foliations, the morphism $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$ can be easily described by*

$$\Xi\big([\gamma]_{\mathrm{hol}}\big) = [\pi \circ \gamma]_{\mathrm{hol}},$$

for each curve $\gamma\colon [0,1] \to P$ inside a leaf of $\mathcal{F}$. Here $[-]_{\mathrm{hol}}$ denotes holonomy classes.

Proof. We may assume that $\mathcal{H}(\mathcal{F})$ is Hausdorff (when not, one needs to argue using a Hausdorff cover of it as in Remark 2.22). The Lie groupoids $(\mathcal{H}(\mathcal{F}), \mathbf{t}, \mathbf{s})$ and $(\mathcal{H}(\mathcal{F}_M), \mathbf{t}_M, \mathbf{s}_M)$ are source connected atlases for $\mathcal{F}$ and $\mathcal{F}_M$, respectively. The map $\hat{\pi}\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$; $[\gamma]_{\mathrm{hol}} \mapsto [\pi \circ \gamma]_{\mathrm{hol}}$ is a submersion since it is a Lie groupoid morphism integrating the fiber-wise surjective Lie algebroid morphism π_*. This implies that $(\mathcal{H}(\mathcal{F}), \mathbf{t}_M \circ \hat{\pi}, \mathbf{s}_M \circ \hat{\pi})$ is a bisubmersion for $\mathcal{F}_M$, by [1, Lemma 2.3]. Notice that the latter triple equals $(\mathcal{H}(\mathcal{F}), \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$, which hence is a bisubmersion.

Using that π has connected fibers, we have that $(\mathcal{H}(\mathcal{F}), \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a source connected atlas for $\mathcal{F}_M$, and, by Remark 2.26, we can thus compute the map Ξ as follows:

$$\Xi\big([\gamma]_{\mathrm{hol}}\big) = \hat{\pi}\big([\gamma]_{\mathrm{hol}}\big) = [\pi \circ \gamma]_{\mathrm{hol}},$$

where the first equality holds by (2.2) and the fact that $\hat{\pi}\colon (\mathcal{H}(\mathcal{F}), \pi \circ \mathbf{t}, \pi \circ \mathbf{s}) \to (\mathcal{H}(\mathcal{F}_M), \mathbf{t}_M, \mathbf{s}_M)$ is a morphism of bisubmersions. ■

We present an example for Theorem 2.25 where $\mathcal{F}$ is a regular foliation and $\mathcal{F}_M$ is a genuinely singular foliation. Notice that the holonomy groupoid of the former foliation has discrete isotropy groups, whereas for the latter the isotropy groups are not all discrete.

Example 2.29. Consider the cylinder $P := S^1 \times \mathbb{R}$ with coordinates (θ, y) and the regular foliation $\mathcal{F}$ given by the integral curves of the nowhere vanishing vector field $X := \frac{\partial}{\partial\theta} + y\frac{\partial}{\partial y}$. The foliation $\mathcal{F}$ is the quotient by the natural $\mathbb{Z}$-action of the foliation on the x-y-plane whose leaves are given by graph(e^{x+c}) on the open upper plane, graph($-e^{x+c}$) on the open lower plane (with c varying through all real numbers), and the line $\{y = 0\}$. The circle $U(1)$ acts on the cylinder P by rotations of the first factor, preserving the foliation $\mathcal{F}$. The singular foliation $\mathcal{F}^{\mathrm{big}}$ on P has three leaves (two open leaves, separated by the middle circle). The quotient map

$$\pi\colon P = S^1 \times \mathbb{R} \to M := P/U(1) \cong \mathbb{R}$$

is the second projection. On the quotient, the induced foliation is $\mathcal{F}_M = \langle y\frac{\partial}{\partial y}\rangle$, a genuinely singular foliation. See Figure 1.

For the holonomy groupoids, we have $\mathcal{H}(\mathcal{F}) = \mathbb{R} \times P$, the transformation groupoid of the action of the Lie group $\mathbb{R}$ on P by the flow of X, which reads $\phi_t(\theta \bmod 2\pi, y) = (\theta + t \bmod 2\pi, e^t y)$. Further, $\mathcal{H}(\mathcal{F}_M) = \mathbb{R} \times M$, the transformation groupoid of the action of the Lie group $\mathbb{R}$ on M by the flow of $y\frac{\partial}{\partial y}$, which reads $\phi_t(y) = e^t y$. This follows from [2, Example 3.7 (ii)]. The canonical surjective morphism of Theorem 2.25 is

$$\Xi\colon \mathbb{R} \times P \to \mathbb{R} \times M, \quad (t, p) \mapsto \big(t, \pi(p)\big).$$

This can be seen from (2.2) since the vector field X π-projects to $y\frac{\partial}{\partial y}$. Notice that, at points $S^1 \times \{0\}$, the isotropy groups of $\mathcal{H}(\mathcal{F})$ are discrete, as for all regular foliations, while the isotropy group of $\mathcal{H}(\mathcal{F}_M)$ at the point $0 \in M$ is isomorphic to $\mathbb{R}$.

2.3. A characterization of the quotient map for pullback foliations

We make the map Ξ in Theorem 2.25 more explicit in the special case that $\Gamma_c(\ker d\pi) \subset \mathcal{F}$. In this case, $\mathcal{F} = \mathcal{F}^{\mathrm{big}} := \pi^{-1}\mathcal{F}_M$ is the pullback of $\mathcal{F}_M$ by π.

We will need [9, Theorem 3.21], stated as follows.

Theorem 2.30. *Given a foliated manifold $(M, \mathcal{F}_M)$ and a surjective submersion with connected fibers $\pi\colon P \to M$, there is a canonical isomorphism*

$$\varphi\colon \mathcal{H}\big(\pi^{-1}(\mathcal{F}_M)\big) \xrightarrow{\sim} \pi^{-1}\big(\mathcal{H}(\mathcal{F}_M)\big),$$

where the left-hand side denotes the pullback groupoid $P\,{}_{\pi}\!\times_{\mathbf{t}} \mathcal{H}(\mathcal{F}_M)_{\mathbf{s}}\!\times_{\pi} P$.

Remark 2.31. Let $\mathcal{U}$ be a path holonomy atlas for $\mathcal{F}_M$. Then $\pi^{-1}\mathcal{U} := \{\pi^{-1}U : U \in \mathcal{U}\}$, where

$$\pi^{-1}U := P\,{}_{\pi}\!\times_{\mathbf{t}} U_{\mathbf{s}}\!\times_{\pi} P$$

is a source connected atlas for $\pi^{-1}(\mathcal{F}_M)$; see [9]. We describe the isomorphism φ by $[(p, u, q)] \mapsto (p, [u], q)$.

Our alternative description of the map Ξ is as follows.

Proposition 2.32. *Let $\pi\colon P \to M$ be a surjective submersion with connected fibers. Let $\mathcal{F}$ be a singular foliation on P such that $\Gamma_c(\ker d\pi) \subset \mathcal{F}$. Denote by $\mathcal{F}_M$ the unique singular foliation on M such that $\pi^{-1}(\mathcal{F}_M) = \mathcal{F}$. Under the canonical isomorphism $\varphi\colon \mathcal{H}(\mathcal{F}) \xrightarrow{\sim} \pi^{-1}(\mathcal{H}(\mathcal{F}_M))$ given in Theorem 2.30, the following two morphisms coincide:*

- *the morphism $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$ given by Theorem 2.25,*
- *the second projection $\mathrm{pr}_2\colon \pi^{-1}(\mathcal{H}(\mathcal{F}_M)) = P \times_M \mathcal{H}(\mathcal{F}_M) \times_M P \to \mathcal{H}(\mathcal{F}_M)$.*

Proof. Fix a path holonomy atlas $\mathcal{U}_M$ for $\mathcal{F}_M$. By Remark 2.31, the family $\mathcal{U} := \pi^{-1}\mathcal{U}_M$ is a source connected atlas for $\mathcal{F} = \pi^{-1}(\mathcal{F}_M)$. Moreover, one can show that, for all $U \in \mathcal{U}_M$, the triple $(\pi^{-1}(U), \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a bisubmersion for $\mathcal{F}_M$.

We want to show that $\Xi \circ \varphi^{-1} = \mathrm{pr}_2$. To this aim, take any $(p, \zeta, q) \in \pi^{-1}(\mathcal{H}(\mathcal{F}_M))$ (so, in particular, $\zeta \in \mathcal{H}(\mathcal{F}_M)$). Fix a representative $u \in U \in \mathcal{U}$ such that $[u] = \zeta$; then

$\varphi^{-1}(p,\zeta,q) = [(p,u,q)]$, where $(p,u,q) \in \pi^{-1}U$. It is sufficient to show that

$$\Xi\big(\big[(p,u,q)\big]\big) = [u]. \tag{2.3}$$

Note that $\hat{\pi}\colon (\pi^{-1}U, \pi\circ \mathbf{t}, \pi\circ\mathbf{s}) \to U; (p,u,q)\mapsto u$ is a morphism of bisubmersions for $\mathcal{F}_M$, therefore $(p,u,q)\in(\pi^{-1}U,\pi\circ\mathbf{t},\pi\circ\mathbf{s})$ is equivalent to $u\in(U,\mathbf{t},\mathbf{s})$. Then (2.3) holds by the characterization of Ξ given in Remark 2.26. ■

3. Lie 2-group actions on holonomy groupoids

We start reviewing Lie 2-groups and Lie 2-group actions. In Section 3.2, we present an important special case of Theorem 2.25 in which the map Ξ is the quotient map of a Lie 2-group action on $\mathcal{H}(\mathcal{F})$ (see Theorem 3.7 and Proposition 3.8). We will revisit this special case later on, in Section 4.3.

3.1. Background on Lie 2-groups

In the sequel, we will need the notion of Lie 2-group, which we recall here.

Definition 3.1. A *Lie* 2*-group* is a group in the category of Lie groupoids.

In other words, a Lie 2-group is a Lie groupoid $\mathcal{G} \rightrightarrows G$ such that $\mathcal{G}$ and G are Lie groups, so that the group multiplication and group inverse are Lie groupoid morphisms, and the inclusion of the neutral elements is a Lie groupoid morphism.

Remark 3.2. Equivalently, a Lie 2-group is a groupoid in the category of Lie groups.

Example 3.3. Let G be a Lie group and $H\subset G$ a normal Lie subgroup. Then H acts on G by left multiplication, leading to the action Lie groupoid $H\times G\rightrightarrows G$. In particular, the groupoid composition is

$$(h_2, h_1 g)\circ(h_1,g) = (h_2h_1, g).$$

Note that its space of arrows has a group structure, namely the semidirect product by the conjugation action $C_g(h) = ghg^{-1}$ of G on H. Explicitly, the group multiplication is given by

$$(h_1,g_1)\cdot(h_2,g_2) = \big(h_1C_{g_1}(h_2), g_1g_2\big).$$

We write $H\rtimes G$ for $H\times G$ endowed with this group structure.

One can check that $H\rtimes G\rightrightarrows G$ is a Lie 2-group.

Remark 3.4. For the sake of completeness, we provide the description of a Lie 2-group in full generality. A crossed module of Lie groups consists of Lie groups H and G, Lie group morphisms $C\colon G\to \mathrm{Aut}(H); g\mapsto C_g$ and $\mathbf{t}\colon H\to G$ such that $\mathbf{t}(C_g(h)) = g\mathbf{t}(h)g^{-1}$ and $C_{\mathbf{t}(h)}(j) = hjh^{-1}$ for all $g\in G$ and $h,j\in H$. There is a bijection between Lie 2-groups and crossed modules of Lie groups [6]. Given a Lie 2-group $\mathcal{G}\rightrightarrows G$, the associated crossed module is given by G, by $H := \ker(\mathbf{s})$ (a normal subgroup of $\mathcal{G}$), by the restriction $\mathbf{t}\colon H\to G$ of the target map, and the Lie group morphisms $C\colon G\to\mathrm{Aut}(H)$; $g\mapsto C_g(h) := ghg^{-1}$. Then $\mathcal{G}$, as a Lie group, is isomorphic to the semidirect product of

G and H by the action C, and, as a Lie groupoid, it is isomorphic to the transformation groupoid of the H-action on G by left multiplication with $\mathbf{t}(\cdot)$.

Definition 3.5. A *Lie 2-group action* is a group action in the category of Lie groupoids.

Hence an action of a Lie 2-group $\mathcal{G} \rightrightarrows G$ on a Lie groupoid $\mathcal{H} \rightrightarrows P$ consists of group actions of $\mathcal{G}$ on $\mathcal{H}$ and of G on P such that the action map

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{H} & \longrightarrow & \mathcal{H} \\ {\scriptstyle \mathbf{t}\times\mathbf{t}}\downdownarrows{\scriptstyle \mathbf{s}\times\mathbf{s}} & & {\scriptstyle \mathbf{t}}\downdownarrows{\scriptstyle \mathbf{s}} \\ G \times P & \longrightarrow & P \end{array}$$

is a Lie groupoid map. Notice that such an action is not by Lie groupoid automorphisms of $\mathcal{H}$. Nevertheless, the following result holds.

Proposition 3.6. *Consider a free and proper action $\star$ of the Lie 2-group $\mathcal{G} \rightrightarrows G$ on a Lie groupoid $\mathcal{H} \rightrightarrows P$. Then $\mathcal{H}' := \mathcal{H}/\mathcal{G}$ and $M := P/G$ are manifolds, and $\mathcal{H}' \rightrightarrows M$ acquires a canonical Lie groupoid structure. Further, the projection $\mathcal{H} \to \mathcal{H}'$ is a surjective submersion and Lie groupoid morphism.*

Although we do not need the above result, we mention it here because it puts in perspective Theorem 3.7 below. (One can prove Proposition 3.6 and also the stronger statement that the projection is a fibration; see Definition 5.3. To do so, one can follow [10, §2.4]: define $R = \{(gp, p) \in P \times P : p \in P \text{ and } g \in G\}$ and $\mathcal{R} = \{((h,g) \star \xi, \xi) \in \mathcal{H} \times \mathcal{H} : \xi \in \mathcal{H} \text{ and } (h,g) \in \mathcal{G}\}$, and show that $(\mathcal{R}, R)$ is a smooth congruence for $\mathcal{H}$. Then Theorem 5.6 gives the desired conclusion. See [8, §5.3] for more details.)

3.2. Lie 2-group action on the holonomy groupoid of a pullback foliation

Fix a foliated manifold $(P, \mathcal{F})$ and a free and proper action of a connected Lie group G on P preserving $\mathcal{F}$. We denote the quotient map by $\pi\colon M \to M/G$. We assume that the infinitesimal generators of the G-action lie in the global hull $\widehat{\mathcal{F}}$, i.e., $\Gamma_c(\ker d\pi) \subset \mathcal{F}$. This occurs exactly when $\mathcal{F}$ is the pullback of $\mathcal{F}_M$ by π, as in Section 2.3.

Theorem 3.7. *Let G be a connected Lie group acting freely and properly on a foliated manifold $(P, \mathcal{F})$. Assume that $\Gamma_c(\ker d\pi) \subset \mathcal{F}$. Then there is a canonical Lie 2-group action[4] of $G \rtimes G \rightrightarrows G$ on the holonomy groupoid $\mathcal{H}(\mathcal{F})$.*

Here $G \rtimes G \rightrightarrows G$ is endowed with the Lie 2-group structure of Example 3.3.

Proof. We make use of the canonical isomorphism $\mathcal{H}(\mathcal{F}) \cong \pi^{-1}(\mathcal{H}(\mathcal{F}_M))$ given in Theorem 2.30.

There is a canonical Lie 2-group action of $G \rtimes G$ on $\pi^{-1}(\mathcal{H}(\mathcal{F}_M))$, extending the given action of G on the base P, given by

$$(h, g) * \big(p, [v], q\big) = \big(hgp, [v], gq\big). \tag{3.1}$$

[4] Here we use the term "Lie 2-group action" in a loose way since $\mathcal{H}(\mathcal{F})$ is generally not a Lie groupoid.

It can be checked by computations that this defines a group action and groupoid morphism. Alternatively, we can use the isomorphism of Lie 2-groups to $G \times G \rightrightarrows G$ (the pair groupoid, with product group structure) given by $G \rtimes G \cong G \times G$, $(h, g) \mapsto (hg, g)$. Under this isomorphism, (3.1) becomes

$$(G \times G) \times \pi^{-1}\big(\mathcal{H}(\mathcal{F}_M)\big) \to \pi^{-1}\big(\mathcal{H}(\mathcal{F}_M)\big), \quad \big((h, g), (p, [v], q)\big) \mapsto \big(hp, [v], gq\big),$$

which is easily checked to be a Lie 2-group action. ∎

Proposition 3.8. *Assume the setup of Theorem 3.7.*

The orbits of the Lie 2-group action of $G \rtimes G \rightrightarrows G$ on $\mathcal{H}(\mathcal{F})$ are exactly the fibers of the canonical map $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$. In particular, the quotient of $\mathcal{H}(\mathcal{F})$ by the action is canonically isomorphic to $\mathcal{H}(\mathcal{F}_M)$.

Proof. The formula (3.1) makes clear what the orbits are, and Proposition 2.32 shows that they agree with the Ξ-fibers. ∎

4. Quotients of foliations by group actions: the general case

In this section, we consider the following setup:

a foliated manifold $(P, \mathcal{F})$,

a free and proper action of a connected Lie group G on P preserving $\mathcal{F}$.

Condition (2.1) in Theorem 2.25 is satisfied. Hence we obtain an open surjective groupoid morphism

$$\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M) \tag{4.1}$$

covering the projection $\pi\colon P \to M := P/G$, where the latter is endowed with the foliation $\mathcal{F}_M$ specified there.

Unlike the special case considered in Section 3.2, $\Gamma_c(\ker d\pi)$ may not be contained in $\mathcal{F}$; hence the Ξ-fibers are not the orbits of a Lie 2-group action in general. In this section, we make two general statements about the Ξ-fibers.

In Section 4.1, we lift the G-action on P to an action on $\mathcal{H}(\mathcal{F})$ by groupoid automorphisms. Using this action, later in Proposition 5.14 we can characterize the fibers of Ξ as the orbits of a *groupoid* action.

In Section 4.2, we establish the existence of a canonical *Lie 2-group action* on $\mathcal{H}(\mathcal{F})$ whose orbits lie inside the Ξ-fibers but which might fail to be the whole fiber (see Proposition 4.9 and Corollary 4.11).

4.1. The lifted group action

We show that the G-action on P admits a canonical lift to $\mathcal{H}(\mathcal{F})$. We start with the following lemma.

Lemma 4.1. *Let $\hat{g}\colon P \to P$ be the diffeomorphism given by the action of $g \in G$. Take a path holonomy atlas $\mathcal{U}$ and a bisubmersion $W \in \mathcal{U}$. The triple*

$$gW := (W, \mathbf{t}_g := \hat{g} \circ \mathbf{t}, \mathbf{s}_g := \hat{g} \circ \mathbf{s})$$

is a bisubmersion. Moreover, gW is adapted to $\mathcal{U}$.

Proof. Because the G-action preserves $\mathcal{F}$, the pullback foliation $\hat{g}^{-1}\mathcal{F}$ equals $\mathcal{F}$, implying that gW is a bisubmersion.

We prove that gW is adapted to $\mathcal{U}$. Notice that $g(W_1 \circ W_2) = gW_1 \circ gW_2$ for any $W_1, W_2 \in \mathcal{U}$. Hence it is sufficient to assume that W is a path holonomy bisubmersion as any element in the path holonomy atlas is a composition of such elements.

Denote by $v_1, \dots, v_n \in \mathcal{F}$ the vector fields that give rise to the path holonomy bisubmersion W (hence $W \subset \mathbb{R}^n \times P$). Consider the push-forward vector fields $\hat{g}_* v_1, \dots, \hat{g}_* v_n \in \mathcal{F}$. The associated path holonomy bisubmersion is defined on

$$W' := \{(v, gp) : (v, p) \in W\} \subset \mathbb{R}^n \times P.$$

Since $gW \to W'$, $(v, p) \mapsto (v, gp)$ is an isomorphism of bisubmersions and since W' is adapted to $\mathcal{U}$ (being a path holonomy bisubmersion), we conclude that gW is adapted to $\mathcal{U}$. ∎

Now we introduce the *lifted action*

$$\vec{\star}\colon G \times \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}), \quad g\vec{\star}[v] := [v]_{gW} \tag{4.2}$$

where, for any v in a path holonomy bisubmersion W, we denote by $[v]_{gW}$ the class of v regarded as an element of gW. This is clearly well defined and indeed a Lie group action. Further, this action is by groupoid automorphisms.

Lemma 4.2. *For all $g \in G$, the map $g\vec{\star}(-)\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$ is a groupoid morphism covering the diffeomorphism $\hat{g}\colon P \to P$.*

Proof. Using the construction of $g\vec{\star}(-)$, it is clear that the source and target map commute with the map $\hat{g}$. Further, $g\vec{\star}(-)$ preserves the groupoid composition since $g(W_1 \circ W_2) = gW_1 \circ gW_2$ for any path holonomy bisubmersions W_1, W_2. ∎

Lemma 4.3. *The orbits of the lifted action $\vec{\star}$ lie in the fibers of $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$.*

Proof. Let $\mathcal{U}$ be a source connected atlas for $\mathcal{F}$. The morphism Ξ is induced by the identity map $\mathcal{U}$ to $\pi\mathcal{U}$; see the characterization of Ξ given in Remark 2.26.

Fix $g \in G$, and $u \in U \in \mathcal{U}$. By the above and since $\pi \circ \hat{g} = \pi$, the images under Ξ of both $[u]$ and $g\vec{\star}[u] = [u]_{gU}$ are the class of the element $u \in (U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$. In particular, $\Xi([u]) = \Xi(g\vec{\star}[u])$, showing the desired statement. ∎

Example 4.4 (Regular foliations). As discussed in Remark 2.22, if $\mathcal{F}$ is a regular foliation, then $\mathcal{H}(\mathcal{F})$ coincides with the classical notion of holonomy groupoid given by holonomy classes of paths in the leaves. The Lie group G acts canonically on $\mathcal{H}(\mathcal{F})$ by translating paths: $G \times \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$; $(g, [\gamma]_{\mathrm{hol}}) \mapsto [g\gamma]_{\mathrm{hol}}$. It is easy to see that this action agrees with the lifted action (4.2), i.e., that $[g\gamma]_{\mathrm{hol}} = g\vec{\star}[\gamma]_{\mathrm{hol}}$, because $[g\gamma]_{\mathrm{hol}} \in (\mathcal{H}(\mathcal{F}), \mathbf{t}, \mathbf{s})$ is equivalent to $[\gamma]_{\mathrm{hol}} \in (\mathcal{H}(\mathcal{F}), \mathbf{t}_g, \mathbf{s}_g)$. Moreover, using the characterization of Ξ given by Proposition 2.28, it is clear that $\Xi(g\vec{\star}[\gamma]_{\mathrm{hol}}) = \Xi([\gamma]_{\mathrm{hol}})$, in accordance with Lemma 4.3.

4.2. A canonical Lie 2-group action on the holonomy groupoid

In this subsection, we prove that there is always a Lie 2-group action on $\mathcal{H}(\mathcal{F})$ whose orbits lie inside the fibers of the morphism Ξ. In general, however, the orbits do not coincide with the (connected components of) the Ξ-fibers. The formulae for this Lie 2-group action are suggested by the special case we will spell out in Section 4.3.

Denote the Lie algebra of G by $\mathfrak{g}$, and by $v_x \in \mathfrak{X}(P)$ the generator of the action corresponding to $x \in \mathfrak{g}$.

Lemma 4.5. *The subspace* $\mathfrak{h} := \{x \in \mathfrak{g} : v_x \in \widehat{\mathcal{F}}\}$ *is a Lie ideal of* $\mathfrak{g}$.

Proof. Since $\mathcal{F}$ is G-invariant, for all $y \in \mathfrak{g}$ we have $[v_y, \mathcal{F}] \subset \mathcal{F}$, or equivalently $[v_y, \widehat{\mathcal{F}}] \subset \widehat{\mathcal{F}}$. Let $x \in \mathfrak{h}$. Then for all $y \in \mathfrak{g}$ we have $v_{[y,x]} = [v_y, v_x] \in \widehat{\mathcal{F}}$; that is, $[y, x] \in \mathfrak{h}$. ■

Denote by H the unique connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Lemma 4.5 implies that H is a normal subgroup; hence, as in Example 3.3, we obtain a Lie 2-group $H \rtimes G \rightrightarrows G$.

We define a Lie group action of H of $\mathcal{H}(\mathcal{F})$. It is not by groupoid automorphisms, unlike the lifted G-action $\vec{\star}$ introduced in Section 4.1, but rather it preserves every source fiber. In order to do so, we need a lemma.

Lemma 4.6. *There is a canonical groupoid morphism*

$$\phi \colon H \times P \to \mathcal{H}(\mathcal{F}),$$

where $H \times P$ denotes the transformation groupoid of the H-action on P obtained restricting the action of G.

The morphism ϕ can be described as follows: take $(h, p) \in H \times P$ and denote by $\hat{h} \colon P \to P$ the diffeomorphism corresponding to h under the G-action. Then $\phi(h, p)$ is the unique element of $\mathcal{H}(\mathcal{F})$ carrying the diffeomorphism $\hat{h}$ near p.

Proof of Lemma 4.6. Denote by $\mathcal{F}_H$ the regular foliation on P by orbits of the H-action. Its holonomy groupoid is exactly $H \times P$, as follows from [1, Example 3.4 (4)] (use that the Lie groupoid $H \times P$ gives rise to the foliation $\mathcal{F}_H$ and is effective; i.e., the identity diffeomorphism on M is carried only by identity elements of the Lie groupoid, due to the freeness of the action).

Since $\mathcal{F}_H \subset \mathcal{F}$, we are done applying [11, Lemma 4.4] in the special case of the pair groupoid over P.

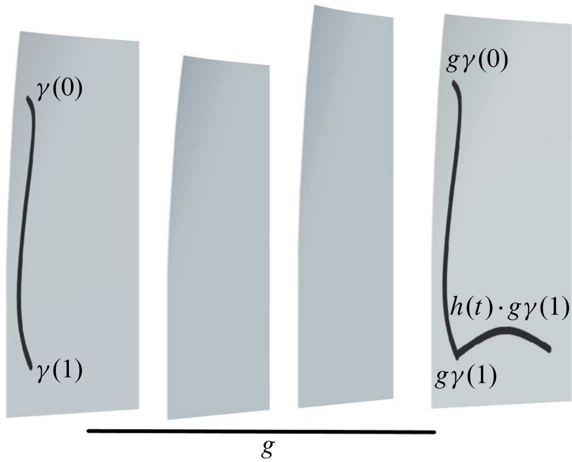

Figure 2. On the left, a curve representing the holonomy class $(h, g) \star [\gamma]$.

The description of ϕ given in the statement holds since ϕ is a groupoid morphism covering Id_P. ∎

Lemma 4.7. *One has* $\phi(H \times P) \subset \mathcal{K} := \ker(\Xi)$.

Proof. We use Lemma 4.6. A point $\phi(h, p)$ of the left-hand side carries near p the diffeomorphism $\hat{h}$ (the diffeomorphism corresponding to h under the G-action). If $h \in H$ is sufficiently close to the unit element, $\phi(h, p)$ admits a representative u in a path holonomy bisubmersion $(U, \mathbf{t}, \mathbf{s})$ for $\mathcal{F}$ satisfying the properties of Remark 2.26. The point u, viewed as a point in $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$, carries Id_M since $\hat{h}$ preserves each π-fiber. This implies that $\Xi([u]) = 1_{\pi(q)}$, by the characterization of Ξ given in Remark 2.26. ∎

Example 4.8 (Regular foliations). As discussed in Remark 2.22, if $\mathcal{F}$ is a regular foliation, there is a groupoid morphism $Q\colon \Pi(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$. The orbits of the H-action lie inside the leaves of $\mathcal{F}$; hence for every path $h(t)$ in H and $p \in P$, the homotopy class $[h(t)p]$ is an element of $\Pi(\mathcal{F})$. Moreover, the freeness of the G-action implies that the elements $[h(t)p], [\tilde{h}(t)p] \in \Pi(\mathcal{F})$ have the same holonomy if and only if $h(1) = \tilde{h}(1)$ and $h(0) = \tilde{h}(0)$. Therefore, there is a well-defined injective groupoid morphism

$$H \times P \to \mathcal{H}(\mathcal{F}); \quad (h, p) \mapsto Q\big[h(t)p\big],$$

where $h(t)$ is any path in H with $h(0) = e$ and $h(1) = h$. This morphism is precisely ϕ. It is clear using Proposition 2.28 that its image lies inside $\mathcal{K} = \ker(\Xi)$.

Consider now the following map, obtained applying the morphism ϕ of Lemma 4.6 and left-multiplying:

$$\overleftarrow{\star}\colon H \times \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}), \quad h \overleftarrow{\star} \xi := \phi\big(h, \mathbf{t}(\xi)\big) \circ \xi. \tag{4.3}$$

Notice that ϕ being a groupoid morphism implies that $\overleftarrow{\star}$ is a group action. We now assemble the group action $\overleftarrow{\star}$ and the lifted action $\overrightarrow{\star}$. See also Figure 2.

Proposition 4.9. *The map*

$$\star\colon (H \rtimes G) \times \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}), \quad (h, g) \star \xi := h \overleftarrow{\star} (g \overrightarrow{\star} \xi)$$

is a Lie 2-group action.

Proof. We first observe that if $\xi \in \mathcal{H}(\mathcal{F})$ carries a diffeomorphism ψ, then $g \overrightarrow{\star} \xi$ carries the diffeomorphism $\hat{g}\psi\hat{g}^{-1}$.

We also observe the following two facts, which hold because both the left and the right sides carry the same diffeomorphism (as can be seen using the above observation) and because of Proposition 2.15,

(i) the map $\phi\colon H \times P \to \mathcal{H}(\mathcal{F})$ satisfies the following equivariance property:

$$g \overrightarrow{\star} \phi(h, p) = \phi(c_g h, gp),$$

where c_g denotes conjugation by g,

(ii) for all $h \in H$ and $\xi \in \mathcal{H}(\mathcal{F})$, we have

$$h \overrightarrow{\star} \xi = \phi\big(h, \mathbf{t}(\xi)\big) \circ \xi \circ \phi\big(h^{-1}, h\mathbf{s}(\xi)\big).$$

To show that $\star$ defines a group action, the main requirement is to show that $((h_1, g_1)(h_2, g_2)) \star \xi$ equals $(h_1, g_1) \star ((h_2, g_2) \star \xi)$ for all $(h_i, g_i) \in H \rtimes G$ and $\xi \in \mathcal{H}(\mathcal{F})$. This holds by a straightforward computation, in which the second term is re-written using the fact that ϕ is a groupoid morphism and is G-equivariant (fact (i) above).

To show that $\star$ defines a groupoid morphism, the main requirement is to show that $(h_1h_2, g_2) \star (\xi_1 \circ \xi_2)$ and $((h_1, g_1) \star \xi_1) \circ ((h_2, g_2) \star \xi_2)$ agree, where $g_1 = h_2 g_2$ and $\mathbf{s}(\xi_1) = \mathbf{t}(\xi_2)$. Upon using that ϕ is a groupoid morphism and the action $\overrightarrow{\star}$ is by groupoid automorphisms, this boils down to applying[5] fact (ii) above. ■

Example 4.10 (Regular foliations). When both $\mathcal{F}$ and $\mathcal{F}_M$ are regular foliations, using Examples 4.4 and 4.8, the Lie 2-group action $\star$ can be described as follows: for any path γ in a leaf of $\mathcal{F}$,

$$(h, g) \star [\gamma] = \big[h(t) \cdot g\gamma(1)\big] \circ [g\gamma],$$

where $t \mapsto h(t)$ is any path in the Lie group H starting at the unit element and ending in h and the dot denotes the group action of H on P. Thus the right-hand side is the (holonomy class of the) concatenation of the following two paths in P: the translate $g\gamma$ of the original curve γ by the group element g and the path obtained acting on its endpoint $g\gamma(1)$ by the path $h(t)$ in H.

Lemmas 4.3 and 4.7 imply the following.

Corollary 4.11. *The orbits of the Lie 2-group action $\star$ of Proposition 4.9 lie inside the Ξ-fibers.*

[5] with $h := h_2$ and $\xi := g_2 \overrightarrow{\star} \xi_1$.

4.3. An alternative description for the Lie 2-group action of Section 3.2

We obtained the formula for the Lie 2-group action of $H \rtimes G$ in Section 4.2 by considering a special case, as we now explain. Assume the setup of Theorem 3.7, in particular, that $\Gamma_c(\ker d\pi) \subset \mathcal{F}$ (i.e., $\widehat{\mathcal{F}}$ contains the infinitesimal generators of the G-action). There we defined an action of $G \rtimes G$ on $\mathcal{H}(\mathcal{F})$ by means of the canonical isomorphism $\varphi: \mathcal{H}(\mathcal{F}) \xrightarrow{\sim} \pi^{-1}\mathcal{H}(\mathcal{F}_M)$.

The goal of this subsection is to prove the following proposition.

Proposition 4.12. *Consider these Lie 2-group actions of $G \rtimes G$ on $\mathcal{H}(\mathcal{F})$:*

- *the action*[6] $\star$ *described in Proposition 4.9,*
- *the action described in Theorem 3.7.*

These two actions coincide. In other words, under the canonical isomorphism $\varphi: \mathcal{H}(\mathcal{F}) \xrightarrow{\sim} \pi^{-1}\mathcal{H}(\mathcal{F}_M)$ given in Theorem 2.30, one has

$$\varphi\big((h,g) \star \xi\big) = (h,g) * \varphi(\xi)$$

for all $(g,h) \in G \rtimes G$ and $\xi \in \mathcal{H}(\mathcal{F})$, where $*$ is the action given in (3.1).

We will prove this statement by analyzing the restrictions of the actions to $\{e\} \times G$ and $G \times \{e\}$. (Notice that these two subgroups generate $G \rtimes G$ as a group since every element (h,g) can be written as $(h,e)(e,g)$.) We denote by $\varphi: \mathcal{H}(\mathcal{F}) \xrightarrow{\sim} \pi^{-1}\mathcal{H}(\mathcal{F}_M)$ the canonical isomorphism given in Theorem 2.30.

Lemma 4.13. *Under the isomorphism φ, the lifted action $\vec{\star}$ of G introduced in Section 4.1 and the restriction of the Lie 2-group action $*$ of (3.1) agree:*

$$\varphi(g \vec{\star} \xi) = (e,g) * \varphi(\xi)$$

for all $g \in G$ and $\xi \in \mathcal{H}(\mathcal{F})$.

Proof. Let $\mathcal{U}$ be a path holonomy atlas for $\mathcal{F}_M$. Recall that, by Remark 2.31, the pullback atlas $\pi^{-1}\mathcal{U}$ is an atlas for $\pi^{-1}(\mathcal{F}_M)$ equivalent to a path holonomy atlas. Fix $\xi \in \mathcal{H}(\mathcal{F})$. Take a representative w in a bisubmersion W in the path holonomy atlas of $\pi^{-1}(\mathcal{F}_M)$. By the above, there is a path holonomy bisubmersion U in $\mathcal{U}$ and a locally defined morphism of bisubmersions

$$\tau: (W, \mathbf{t}, \mathbf{s}) \to (\pi^{-1}U, \mathbf{t}, \mathbf{s})$$

mapping w to some point (p, v, q). By definition, $\varphi([w]) = (p, [v], q)$.

Now fix $g \in G$. Recall that the bisubmersion $gW := (W, \hat{g} \circ \mathbf{s}, \hat{g} \circ \mathbf{t})$ was defined in Lemma 4.1. The same map τ is also a morphism of bisubmersions

$$gW \to (\pi^{-1}U, \hat{g} \circ \mathbf{t}, \hat{g} \circ \mathbf{s}).$$

[6] Note that, under our assumptions on $\mathcal{F}$, in the setting of Proposition 4.9 we have $H = G$, because the Lie subalgebra $\mathfrak{h}$ introduced in Lemma 4.5 equals the whole of $\mathfrak{g}$.

The latter bisubmersion is isomorphic to $(\pi^{-1}U, \mathbf{t}, \mathbf{s})$ via $(p', v', q') \mapsto (gp', v', gq')$. By composition, we obtain a morphism of bisubmersions $gW \to (\pi^{-1}U, \mathbf{t}, \mathbf{s})$ mapping w to (gp, v, gq). Hence $g\vec{\star}[w] := [w]_{gW}$ agrees with $[(gp, v, gq)] \in \mathcal{H}(\mathcal{F})$, and therefore, under φ, it is mapped to $(gp, [v], gq) = (e, g) * \varphi([w])$. ∎

Lemma 4.14. *Under the isomorphism φ, the action $\overleftarrow{\star}$ of H introduced in (4.3) and the restriction of the Lie 2-group action $*$ of (3.1) agree:*

$$\varphi(h\overleftarrow{\star}\xi) = (h, e) * \varphi(\xi)$$

for all $h \in G$ and $\xi \in \mathcal{H}(\mathcal{F})$.

Proof. Take an arbitrary element $\xi \in \mathcal{H}(\mathcal{F})$ and a path holonomy atlas $\mathcal{U}$ for $\mathcal{F}_M$. Let $(p, u, q) \in \pi^{-1}U \in \pi^{-1}\mathcal{U}$ be a representative of ξ, and let f be a local diffeomorphism carried at (p, u, q).

Fix $h \in G$, and denote by $\hat{h}\colon P \to P$ the diffeomorphism corresponding to h under the G-action. Note that the transformation groupoid $G \times P$ carries the diffeomorphism $\hat{h}$ at (h, p). Hence any representative of $\phi(h, p) \in \mathcal{H}(\mathcal{F})$ in $\pi^{-1}\mathcal{U}$ carries this diffeomorphism, where ϕ is the groupoid morphism of Lemma 4.6. In turn, this implies that any representative of $\phi(h, p) \circ [(p, u, q)] \in \mathcal{H}(\mathcal{F})$ will carry $\hat{h} \circ f$. Note that $(hp, u, q) \in \pi^{-1}U$ also carries $\hat{h} \circ f$. By the definition of a holonomy groupoid (using Proposition 2.15) it follows that $h\overleftarrow{\star}[(p, u, q)] = [(hp, u, q)]$.

We conclude that

$$\varphi\big(h\overleftarrow{\star}[(p, u, q)]\big) = \varphi\big([(hp, u, q)]\big) = \big(hp, [u], q\big) = (h, e) * \varphi\big([(p, u, q)]\big),$$

using in the second equality the description of the isomorphism φ given in Remark 2.31. ∎

Proof of Proposition 4.12. The proposition follows from

$$\varphi\big((h, g) \star \xi\big) = \varphi\big(h\overleftarrow{\star}(g\vec{\star}\xi)\big) = (h, e) * \big((e, g) * \varphi(\xi)\big) = (h, g) * \varphi(\xi),$$

where we used Lemmas 4.13 and 4.14 in the second equality. ∎

Remark 4.15. The image of $\phi\colon G \times P \to \mathcal{H}(\mathcal{F})$ is the kernel of $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$. Indeed, for every $(h, q) \in G \times P$, we have $\phi(h, q) = h\overleftarrow{\star}1_q \in \mathcal{H}(\mathcal{F})$. Under the identification $\varphi\colon \mathcal{H}(\mathcal{F}) \xrightarrow{\sim} \pi^{-1}\mathcal{H}(\mathcal{F}_M)$, this element corresponds to $(h, e) * \varphi(1_q) = (hq, 1_{\pi(q)}, q) \in \pi^{-1}\mathcal{H}(\mathcal{F}_M)$ by Lemma 4.14. Under the same identification, $\ker(\Xi)$ corresponds to $\ker(\mathrm{pr}_2) = P \times_\pi M \times_\pi P$, by Proposition 2.32.

5. Groupoid fibrations

In this section, we investigate when the map Ξ of Theorem 2.25 is a fibration. Loosely speaking, fibrations are the notion of a "nice" quotient map in the category of (topological or Lie) groupoids.

5.1. Background on fibrations

We first present the recent notion of fibration for open topological groupoids.

Definition 5.1 ([7, Definition 2.1]). Let $\mathcal{H} \rightrightarrows P$ and $\mathcal{H}' \rightrightarrows M$ be open topological groupoids. A morphism of topological groupoids $\Xi\colon \mathcal{H} \to \mathcal{H}'$ covering a continuous map $\pi\colon P \to M$ is called *fibration* if

$$\Xi_{\mathbf{s}}\colon \mathcal{H} \to \mathcal{H}'{}_{\mathbf{s}}\times_{\pi} P; \quad \xi \mapsto \big(\Xi(\xi), \mathbf{s}(\xi)\big) \tag{5.1}$$

is a surjective open map.

Remark 5.2. In Definition 5.1, the base map $\pi\colon P \to M$ is required to be neither open nor surjective; see [7, Remark 2.7] for a motivation of this choice. In all the instances of fibration appearing in this note, the base map is a surjective submersion between manifolds, thus also open.

We now review fibrations of Lie groupoids and equivalent characterizations, after [10].

Definition 5.3 ([10, Definition 2.4.3]). Let $\mathcal{H} \rightrightarrows P$ and $\mathcal{H}' \rightrightarrows M$ be Lie groupoids. A morphism of Lie groupoids $\Xi\colon \mathcal{H} \to \mathcal{H}'$ covering a smooth map $\pi\colon P \to M$ is called *fibration* if π and $\Xi_{\mathbf{s}}\colon \mathcal{H} \to \mathcal{H}'{}_{\mathbf{s}} \times_{\pi} P$ (as in (5.1)) are both surjective submersions.

The surjectivity conditions in the previous definition assure that, for any two composable elements in $\mathcal{H}'$, there exist composable preimages in $\mathcal{H}$, and thus the composition on $\mathcal{H}'$ is entirely determined by the composition in $\mathcal{H}$.

In [10], two notions are used to describe fibrations of Lie groupoids: smooth congruences and normal subgroupoid systems.

Before introducing them, recall that a *smooth equivalence relation* on a manifold P is an embedded wide Lie subgroupoid R of the pair groupoid $P \times P$. By the Godement criterium, P/R is a smooth manifold such that the projection map from P is a submersion if and only if R is a smooth equivalence relation.

Definition 5.4 ([10, §2.4.5]). Let $\mathcal{H} \rightrightarrows P$ be a Lie groupoid. A *smooth congruence* on $\mathcal{H}$ consists of two smooth equivalence relations $\mathcal{R}$ on $\mathcal{H}$ and R on P such that

- $\mathcal{R} \rightrightarrows R$ is a Lie subgroupoid of the Cartesian product $\mathcal{H} \times \mathcal{H} \rightrightarrows P \times P$,
- the map $\mathcal{R} \to \mathcal{H}\,{}_{\mathbf{s}}\times_{\mathrm{Pr}_1} R; (\xi_2, \xi_1) \mapsto (\xi_2, \mathbf{s}(\xi_2), \mathbf{s}(\xi_1))$ is a surjective submersion[7].

Normal subgroupoid systems allow to describe a Lie groupoid fibration in terms of data on the domain, analogously to how a surjective group morphism can be described by its kernel. Note that if $\mathcal{K}$ is a closed embedded wide Lie subgroupoid of $\mathcal{H}$, then, by the Godement criterion, the set of left cosets

$$\mathcal{K}\backslash\mathcal{H} = \{\mathcal{K} \circ \xi : \xi \in \mathcal{H}\}$$

[7] In [10], Mackenzie describes this condition as a certain square diagram being "versal."

has a unique manifold structure making the quotient map $\mathcal{H} \to \mathcal{K}\backslash\mathcal{H}; \xi \mapsto \mathcal{K}\xi := (\mathcal{K} \circ \xi)$ a surjective submersion. Note also that the source map $\mathbf{s}\colon \mathcal{H} \to P$ quotients to a well-defined surjective submersion $\mathcal{K}\backslash\mathcal{H} \to P$, which we also denote by $\mathbf{s}$.

Definition 5.5 ([10, Definition 2.4.7]). A *normal subgroupoid system* in $\mathcal{H} \rightrightarrows P$ is a triple $(\mathcal{K}, R, \theta)$, where $\mathcal{K}$ is a closed, embedded, wide Lie subgroupoid of $\mathcal{H}$; R is a smooth equivalence relation on P; and θ is an action of R on the map $\mathbf{s}\colon \mathcal{K}\backslash\mathcal{H} \to P$ such that, for all $(p,q) \in R$, the following holds.

(1) Let $\xi \in \mathcal{H}$ with $\mathbf{s}(\xi) = q$ and $\xi_1 \in \mathcal{H}$ with $\theta(p,q)(\mathcal{K}\xi) = \mathcal{K}\xi_1$; then $(\mathbf{t}(\xi_1), \mathbf{t}(\xi)) \in R$.

(2) $\theta(p,q)(\mathcal{K}e_q) = \mathcal{K}e_p$.

(3) Let ξ_1 and ξ_2 be composable elements in $\mathcal{H}$ such that $\mathbf{s}(\xi_2) = q$. Consider ξ_1' and ξ_2' such that $\theta(p,q)(\mathcal{K}\xi_2) = \mathcal{K}\xi_2'$ and $\theta(\mathbf{t}(\xi_2'), \mathbf{t}(\xi_2))(\mathcal{K}\xi_1) = \mathcal{K}\xi_1'$; then
$$\theta(p,q)\big(\mathcal{K}(\xi_1 \circ \xi_2)\big) = \mathcal{K}(\xi_1' \circ \xi_2').$$

The following theorem says that smooth congruences, fibrations, and normal subgroupoids systems are equivalent descriptions for the quotients of Lie groupoids.

Theorem 5.6 ([10, Theorems 2.4.6 and 2.4.8]). *Let $\mathcal{H} \rightrightarrows P$ be a Lie groupoid.*

(1) *If Ξ is a fibration defined on $\mathcal{H}$ covering π, then the pair $(\mathcal{R}, R)$ is a congruence, where $\mathcal{R} := \mathcal{H} \times_\Xi \mathcal{H}$ and $R := P \times_\pi P$. Conversely, given a congruence, the induced quotient map is a fibration.*

(2) *If $(\mathcal{K}, R, \theta)$ is a normal subgroupoid system on $\mathcal{H}$, then R, together with the equivalence relation on $\mathcal{H}$ given by*
$$\mathcal{R} := \big\{(\xi', \xi) \in \mathcal{H} \times \mathcal{H} : \big(\mathbf{s}(\xi'), \mathbf{s}(\xi)\big) \in R \text{ and } \theta\big(\mathbf{s}(\xi'), \mathbf{s}(\xi)\big)\mathcal{K}\xi = \mathcal{K}\xi'\big\},$$
is a smooth congruence on $\mathcal{H}$.
Conversely, let $(\mathcal{R}, R)$ be a smooth congruence on $\mathcal{H}$. Let $\mathcal{K}$ consist of elements of $\mathcal{H}$ which are related to an identity element e_p, where $p \in P$.
Let θ be the following action of R on $\mathcal{K}\backslash\mathcal{H}$: for every $(p,q) \in R$ and $\xi \in \mathcal{H}$ with source q,
$$\theta(p,q)(\mathcal{K}\xi) = \mathcal{K}\xi',$$
where ξ' is any element related to ξ and with source p. Then $(\mathcal{K}, R, \theta)$ is a normal subgroupoid system on $\mathcal{H}$.

5.2. The morphism Ξ is not always a fibration

Let $\pi\colon P \to M$ be a surjective submersion with connected fibers, let $\mathcal{F}$ be a singular foliation on P satisfying condition (2.1), and denote by $\mathcal{F}_M$ the induced singular foliation on M. In Theorem 2.25, we obtained an open surjective morphism of topological groupoids $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$.

This morphism is not always a fibration, as the following example shows. What fails here is the surjectivity of the map $\Xi_\mathbf{s}$ (as in (5.1)).

Example 5.7. Let $P = \mathbb{R}^2 - (\{0\} \times \mathbb{R}_+)$ (the plane with a vertical half-line removed), $M = \mathbb{R}$, and $\pi\colon P \to M$ the first projection. Take $\mathcal{F}$ to be the foliation on P given by the horizontal lines and half-lines; then $\mathcal{F}_M$ is the full foliation on M. In particular, the foliation $\mathcal{F}$ has no holonomy, and $\mathcal{H}(\mathcal{F}_M) = M \times M$.

The map $\Xi_{\mathbf{s}}\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)_{\mathbf{s}}\times_\pi P$ (as in (5.1)) is not surjective. Indeed, take $\zeta \in \mathcal{H}(\mathcal{F}_M)$ such that $\mathbf{s}(\zeta) = 1$ and $\mathbf{t}(\zeta) = -1$. For all $y > 0$, the element $(\zeta, (1, y))$ lies in $\mathcal{H}(\mathcal{F}_M)_{\mathbf{s}}\times_\pi P$, but there is no ξ in $\mathcal{H}(\mathcal{F})$ satisfying $\mathbf{s}(\xi) = (1, y)$ and $\Xi(\xi) = \zeta$ (otherwise $\mathbf{t}(\xi)$ would be of the form $(-1, *)$ and thus not lie in the same leaf as $\mathbf{s}(\xi)$, leading to a contradiction).

Remark 5.8. The above example also shows that, in general, Lie algebroid fibrations do not integrate to Lie groupoid fibrations. To explain this, recall that a Lie algebroid fibration [10] is a morphism of Lie algebroids $\phi\colon A \to B$ which is fiber-wise surjective and covers a surjective submersion between the manifolds of objects. Suppose A, B integrate to source simply connected Lie groupoids $\mathcal{G}, \mathcal{H}$. Then the unique Lie groupoid morphism $\Phi\colon \mathcal{G} \to \mathcal{H}$ integrating ϕ is a surjective submersion, but, in general, it fails to be a Lie groupoid fibration. An instance is the above example, in which A and B are the involutive distributions tangent to the foliations, $\phi = \pi_*$, and consequently $\Phi = \Xi$.

It was pointed out to us that a sufficient condition for a Lie algebroid fibration to integrate to Lie groupoid fibration is the existence of a complete Ehresmann connection, as realized by Brahic in [3]. See [3, Example 4.12] for an instance involving foliations.

5.3. Fibrations from pullback foliations

In this and the next subsection, we present two cases in which Ξ is a fibration. A simple instance is when $\mathcal{F}$ is a pullback foliation.

Proposition 5.9. *As in Proposition 2.24, let $\pi\colon P \to M$ be a surjective submersion with connected fibers, and let $\mathcal{F}$ be a singular foliation on P satisfying $\Gamma_c(\ker d\pi) \subset \mathcal{F}$. Then the map $\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M)$ of Theorem 2.25 is a fibration of topological groupoids.*

Proof. There is an isomorphism of topological groupoids $\varphi\colon \mathcal{H}(\mathcal{F}) \to \pi^{-1}(\mathcal{H}(\mathcal{F}_M))$ by Theorem 2.30. By Proposition 2.32, the map Ξ corresponds to the projection

$$\mathrm{pr}_2\colon \pi^{-1}\big(\mathcal{H}(\mathcal{F}_M)\big) \to \mathcal{H}(\mathcal{F}_M),$$

which is clearly a fibration. ∎

Remark 5.10. As the proof of Proposition 5.9 shows, when $\mathcal{H}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F}_M)$ are smooth, then Ξ is a fibration of Lie groupoids. The normal subgroupoid system $(\mathcal{K}, R, \theta)$ corresponding to it (as in Theorem 5.6) is given by $\mathcal{K} = P \times_M 1_M \times_M P$, by $R = P \times_M P$, and the following Lie groupoid action θ of R on $\mathcal{K}\backslash\mathcal{G} \cong \mathcal{H}(\mathcal{F}_M) \times_M P$:

$$\theta(p, q)\big[(\xi, q)\big] = \big[(\xi, p)\big].$$

Here the square bracket denotes the equivalence class in $\mathcal{K}\backslash\mathcal{G}$.

5.4. Fibrations from group actions

We exhibit one more case when Ξ is a fibration. In this whole subsection, we assume the setup at the beginning of Section 4, that is, a foliated manifold $(P, \mathcal{F})$, and a free and proper action of a connected Lie group G on P preserving $\mathcal{F}$.

Denote $M := P/G$, denote by $\mathcal{F}_M$ the induced singular foliation there, and use the short-hand notation $\mathcal{H} := \mathcal{H}(\mathcal{F})$ and $\mathcal{H}' := \mathcal{H}(\mathcal{F}_M)$. The groupoid morphism Ξ defined in (4.1) and the lifted Lie group action $\vec{\star}$ in (4.2), thanks to Lemmas 4.2 and 4.3, provide us with the following data:

(1) a free and proper action of a Lie group G on a manifold P, with quotient map $\pi\colon P \to M := P/G$,

(2) an open surjective morphism of holonomy groupoids $\Xi\colon \mathcal{H} \to \mathcal{H}'$ covering π,

(3) a group action $\vec{\star}$ of G on $\mathcal{H}$ by groupoid automorphisms covering the G-action on P and preserving each fiber of Ξ:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Xi} & \mathcal{H}' \\ \downdownarrows & & \downdownarrows \\ P & \xrightarrow{\pi} & M. \end{array}$$

Fibrations of topological groupoids.

Proposition 5.11. *Assume the setup at the beginning of Section 4. Make the following technical assumption: the diffeomorphism $G \times P \to G \times P$, $(g, p) \mapsto (g, \hat{g}(p))$ is an automorphism of the product foliation $0 \times \mathcal{F}$. (This is a reasonable assumption since, for every $g \in G$, $\hat{g}$ is an automorphism of $(P, \mathcal{F})$.) Then the map Ξ is a fibration of topological groupoids.*

Proof. According to Definition 5.3, we need to show that the map

$$\Xi_{\mathbf{s}}\colon \mathcal{H} \to \mathcal{H}'{}_{\mathbf{s}}\times_{\pi} P; \quad \xi \mapsto \big(\Xi(\xi), \mathbf{s}(\xi)\big)$$

is surjective and open. For the surjectivity, we argue as follows. Let $(\zeta, p) \in \mathcal{H}'{}_{\mathbf{s}}\times_{\pi} P$. There exists a $\xi \in \mathcal{H}$ such that $\Xi(\xi) = \zeta$. Then $\pi(\mathbf{s}(\xi)) = \pi(p)$, which means that there exists a (unique) $g \in G$ such that $g\mathbf{s}(\xi) = p$. This implies $\Xi_{\mathbf{s}}(g\vec{\star}\xi) = (\zeta, p)$, i.e., $\Xi_{\mathbf{s}}$ is surjective.

To prove that $\Xi_{\mathbf{s}}$ is an open map, let $\mathcal{O} \subset \mathcal{H}$ be an open subset. It suffices to prove that, for any point $\xi \in \mathcal{O}$, there is a subset $V \subset \mathcal{O}$ containing ξ such that $\Xi_{\mathbf{s}}(V)$ is open. Take a slice through $\mathbf{s}(\xi)$, i.e., a sufficiently small submanifold $S \subset P$ through $\mathbf{s}(\xi)$ transverse to the π-fibers. We will construct such V as $\widetilde{G}\vec{\star}\sigma$, where $\widetilde{G}$ is a neighborhood of the identity element in G and σ is a neighborhood of ξ in $\mathbf{s}^{-1}(S)$.

Step 1. *There is a relatively compact neighborhood σ of ξ in $\mathbf{s}^{-1}(S)$ such that its closure satisfies $\bar{\sigma} \subset \mathcal{O}$.*

To construct σ, take a bisubmersion U and a point $u \in U$ such that the quotient map $q_U\colon U \to H$ satisfies $q_U(u) = \xi$. As U is a manifold, there exists a relatively compact (open) neighborhood U_0 of u, and furthermore we can assume that its closure $\bar{U}_0$ lies in the open subset $q_U^{-1}(\mathcal{O})$. Since q_U is an open map by [9, Lemma 3.1], $q(U_0)$ is open. Hence

$$\sigma := q_U(U_0) \cap \mathbf{s}^{-1}(S)$$

is an open subset of $\mathbf{s}^{-1}(S)$.

We now show that $\bar{\sigma}$, the closure of σ in $\mathbf{s}^{-1}(S)$, is compact. To do so, we make use of $\breve{\sigma}$, the closure of σ in $\mathbf{s}^{-1}(\bar{S})$. We have $\breve{\sigma} \subset q_U(\bar{U}_0) \cap \mathbf{s}^{-1}(\bar{S})$. This shows that $\breve{\sigma}$ is compact, being a closed subset of a compact set. Further, it shows that, shrinking U_0 if necessary, we can arrange that $\breve{\sigma}$ is contained in $\mathbf{s}^{-1}(S)$. This in turn implies that $\bar{\sigma} \subset \breve{\sigma}$: the right-hand side is a closed subset of $\mathbf{s}^{-1}(S)$ containing σ, and, by definition, the left-hand side is the smallest such subset. Hence $\bar{\sigma}$, being a closed subset of a compact set, is itself compact.

Step 2. *There is a neighborhood $\widetilde{G} \subset G$ of the identity element e such that $\widetilde{G} \vec{\star} \sigma \subset \mathcal{O}$.*

The preimage of $\mathcal{O}$ under the action map $\vec{\star}: G \times \mathcal{H} \to \mathcal{H}$ is open, and, by Step 1, it contains $\{e\} \times \bar{\sigma}$. Hence it contains $\widetilde{G} \times \bar{\sigma}$ for some neighborhood $\widetilde{G}$ of e, using the compactness of $\bar{\sigma}$.

Step 3. *$V := \widetilde{G} \vec{\star} \sigma$ is an open subset of $\mathcal{H}$.*

Notice that, for every $g \in \widetilde{G}$,

$$g \vec{\star} \sigma = \left\{ q_{gU}(v) : v \in \mathbf{s}_U^{-1}(S) \cap U_0 \right\}, \tag{5.2}$$

where q_{gU} denotes the quotient map of the bisubmersion obtained as in Lemma 4.1. We want to describe this set in terms of the bisubmersion U. This is possible because $q_U(U)$ is an open subset of $\mathcal{H}$ containing σ, thus $g \vec{\star} \sigma$ will lie in $q_U(U)$ provided g is close enough to the unit element $e \in G$. In the following claim, we might need to shrink $\widetilde{G}$ to a smaller neighborhood of e.

Claim. Let $g \in \widetilde{G}$. There is a morphism of bisubmersions $\phi^g : gU \to U$ (defined only on an open subset of gU) which is a diffeomorphism onto its image and depends smoothly[8] on g.

Assume first that U is a path holonomy bisubmersion, associated to local generators $X_1, \dots, X_k$ of $\mathcal{F}$ near $\mathbf{s}(\xi)$. Fix $Y_i \in \Gamma(\ker(d\mathbf{s}_U))$ such that $d\mathbf{t}_U(Y_i) = X_i$ (this is possible by the proof of [1, Proposition 2.10 (b)]). We then have $d\mathbf{t}_{gU}(Y_i) = \hat{g}_* X_i$. There exist smooth functions c_{ij}^g, defined on an open subset of $\mathbf{s}_U(U) \cap \hat{g}(\mathbf{s}_U(U))$, such that $\hat{g}_* X_i = \sum_j c_{ij}^g X_j$. Further, these functions can be chosen to vary smoothly with g, by the technical assumption in the statement of the proposition and since $X_1, \dots, X_k$ – viewed as vector fields on $G \times P$ – locally generate the singular foliation $0 \times \mathcal{F}$.

Consider the map

$$\phi^g : gU \to U, \quad \exp_{(0,x)} \left(\sum \lambda_i Y_i \right) \mapsto \exp_{(0,\hat{g}x)} \left(\sum \lambda_i (\mathbf{t}_U^* c_{ij}^g) Y_j \right), \tag{5.3}$$

[8] Recall that the manifold underlying the bisubmersion gU is U, and thus it is independent of g.

which actually is defined only on an open subset of gU (namely, when $x \in \mathbf{s}_U(U) \cap \hat{g}^{-1}(\mathbf{s}_U(U))$). It preserves source fibers since the Y_i are tangent to the source fibers of both bisubmersions. Applying $\mathbf{t}_{gU}$ to the above point of the domain and applying $\mathbf{t}_U$ to its image, we obtain the same point of P (namely, $\exp_{(0,\hat{g}x)}(\sum \lambda_i \hat{g}_* X_i)$). Thus ϕ^g is a morphism of bisubmersions. Further, ϕ^g is a diffeomorphism onto its image (see the proof of [1, Proposition 2.10 (b)]; this was also checked explicitly in [2, Lemma 2.6]). Finally, looking at (5.3), it is clear that ϕ^g depends smoothly on g.

In the general case, $U = U_l \circ \cdots \circ U_1$ is a composition of path holonomy bisubmersions. We can apply the above construction to obtain, for each $k = 1, \ldots, l$, a morphism of bisubmersions $\phi_k : gU_k \to U_k$. Assembling them, we obtain a morphism of bisubmersions $\phi^g : gU \to U$ satisfying the properties of the claim. This proves the claim.

From (5.2) and the claim, we deduce that $g \vec{\star} \sigma = \{q_U(\phi^g v) : v \in \mathbf{s}_U^{-1}(S) \cap U_0\}$. Define

$$\Phi : \widetilde{G} \times \big(\mathbf{s}_U^{-1}(S) \cap U_0\big) \to U, \quad (g, v) \mapsto \phi^g(v).$$

Then $\widetilde{G} \vec{\star} \sigma = q_U(\mathrm{image}(\Phi))$. As q_U is an open map, it suffices to argue that image(Φ) is open in U.

The map Φ fits in the commutative diagram

$$\begin{array}{ccc} \widetilde{G} \times \big(\mathbf{s}_U^{-1}(S) \cap U_0\big) & \xrightarrow{\ \Phi\ } & U \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & \widetilde{G}, & \end{array}$$

where pr_1 is the first projection and α is mapping u to the unique group element g such that $\mathbf{s}_U(u) \in \hat{g}S$. The maps pr_1 and α are submersions. Further, for every $g \in \widetilde{G}$, the map

$$\Phi|_{\mathrm{pr}_1^{-1}(g)} : \mathbf{s}_U^{-1}(S) \cap U_0 \to \alpha^{-1}(g) = \mathbf{s}_U^{-1}(\hat{g}S)$$

equals ϕ^g, which is a diffeomorphism onto its image by the above claim. Hence Φ is a diffeomorphism onto its image, which consequently is open in U by dimension reasons.

Step 4. *$\Xi_{\mathbf{s}}(V)$ is open in $\mathcal{H}'{}_{\mathbf{s}}\times_\pi P$.*

As $\Xi \times \mathbf{s} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}' \times P$ is an open map, and by Step 3, the image of $V \times V$ under this map is open. Hence its intersection with $\mathcal{H}'{}_{\mathbf{s}}\times_\pi P$ is open there. This intersection is exactly

$$A := \big\{\big(\Xi(v_1), \mathbf{s}(v_2)\big) : v_1, v_2 \in V \text{ and } \pi\big(\mathbf{s}(v_1)\big) = \pi\big(\mathbf{s}(v_2)\big)\big\}.$$

We now show that $A = \Xi_{\mathbf{s}}(V)$. We just need to prove "$\subset$" since the other inclusion is obvious. To this aim, take an arbitrary element $a := (\Xi(v_1), \mathbf{s}(v_2))$ of A. Since $\mathbf{s}(v_1)$ and $\mathbf{s}(v_2)$ lie in the same π-fiber, there is a unique $g \in G$ such that $\hat{g} \cdot \mathbf{s}(v_1) = \mathbf{s}(v_2)$. It is immediate to check that $\Xi_{\mathbf{s}}(g \vec{\star} v_1) = a$; hence we are done if we show that $g \vec{\star} v_1 \in V$. To this aim, write $v_i = g_i \vec{\star} \xi_i$ for unique $g_i \in \widetilde{G}$ and $\xi_i \in \sigma$ $(i = 1, 2)$. We have $\mathbf{s}(\xi_1) = \mathbf{s}(\xi_2)$ since these elements belong to the same π-fiber and $\mathbf{s}(\sigma)$ lies inside a slice transverse to such fibers. This implies that $g = g_2 g_1^{-1}$. Hence $g \vec{\star} v_1 = g_2 \vec{\star} \xi_1$, which by construction lies in V. ∎

Assuming the setup at the beginning of Section 4, in Lemma 5.12 and Proposition 5.14 we describe the fibers of $\Xi\colon \mathcal{H} \to \mathcal{H}'$. Denote by $\mathcal{K} := \ker(\Xi)$ a topological subgroupoid of $\mathcal{H}$ with space of objects P. Given a fibration of open topological groupoids, the kernel – called "fiber" in the terminology of [7] – is an open topological subgroupoid [7, Lemma 2.3]. Hence $\mathcal{K}$ is an open topological subgroupoid when the technical assumption in Proposition 5.11 is satisfied.

Lemma 5.12. *The fibers of Ξ are given by the orbits of the action of G on $\mathcal{H}$ composed*[9] *with elements in $\mathcal{K}$. More precisely, the fiber through $\xi \in \mathcal{H}$ is*

$$\mathcal{K} \circ (G \vec{\star} \xi) := \{\chi \circ (g \vec{\star} \xi) : \chi \in \mathcal{K} \text{ and } g \in G\}.$$

Proof. Fix $\xi_1 \in \mathcal{H}$. Thanks to Lemma 4.3, the above subset $\mathcal{K} \circ (G \vec{\star} \xi_1)$ is certainly contained in the Ξ-fiber through ξ_1.

To show the converse, let ξ_2 lie in the same Ξ-fiber as ξ_1; then $\mathbf{s}(\xi_2)$ and $\mathbf{s}(\xi_1)$ lie in the same π-fiber. Let $g \in G$ such that $\hat{g}\mathbf{s}(\xi_1) = \mathbf{s}(\xi_2)$. As this equals $\mathbf{s}(g \vec{\star} \xi_1)$, the groupoid composition $\xi_2 \circ (g \vec{\star} \xi_1)^{-1}$ is well defined, and

$$\Xi\big(\xi_2 \circ (g \vec{\star} \xi_1)^{-1}\big) = \Xi(\xi_2) \circ \Xi(g \vec{\star} \xi_1)^{-1} = \Xi(\xi_2) \circ \Xi(\xi_1)^{-1} = 1_{\pi(\mathbf{t}(\xi_2))},$$

where we used that Ξ is a groupoid morphism and the action of G preserves the Ξ-fibers, respectively, in the first and second equalities. As a consequence, $\xi_2 \circ (g \vec{\star} \xi_1)^{-1} \in \ker \Xi = \mathcal{K}$. ∎

Remark 5.13. While the fibers of a group morphism are just translates of the kernel, for morphisms of groupoids over different bases this is no longer true. This explains why the description of the fibers in Lemma 5.12 is slightly involved.

A first consequence of Lemma 5.12 is that the fibers of $\Xi\colon \mathcal{H} \to \mathcal{H}'$ are orbits of a groupoid action.

Proposition 5.14. *There is a topological groupoid structure on $\mathcal{K} \times G$ and a groupoid action of $\mathcal{K} \times G$ on $\mathbf{t}\colon \mathcal{H} \to P$, whose orbits coincide with the fibers of Ξ.*

An instance of this proposition is Example 2.29, where we have $G = U(1)$ and $\ker \Xi = 1_P$.

Proof. Since the Lie group G acts by groupoid automorphisms on the groupoid $\mathcal{K}$, we can form the semidirect product groupoid (see [4, §2] and [5, §11.4]). We obtain a groupoid structure on $\mathcal{K} \times G$ with space of objects P, for which

(a) the source and target maps are, respectively, $(\xi, g) \mapsto g^{-1}\mathbf{s}(\xi)$ and $(\xi, g) \mapsto \mathbf{t}(\xi)$,

(b) the composition is $(\xi_2, g_2) \circ (\xi_1, g_1) \mapsto (\xi_2 \circ (g_2 \vec{\star} \xi_1), g_2 g_1)$.

One checks that the groupoid $\mathcal{K} \times G$ acts on the map $\mathbf{t}\colon \mathcal{H} \to P$ via

$$\star\colon (\mathcal{K} \times G) \times_P \mathcal{H} \to \mathcal{H}; \quad \big((\chi, g), \xi\big) \mapsto (\chi, g) \star \xi := \chi \circ (g \vec{\star} \xi). \tag{5.4}$$

The orbits of this groupoid action are precisely the fibers of Ξ, by Lemma 5.12. ∎

[9] Recall that the composition (multiplication) of the groupoid $\mathcal{H}$ is denoted by $\circ$.

Fibrations of Lie groupoids. To conclude this section, we show that for Lie groupoids Ξ is a fibration in the sense of Definition 5.3. Denote $\mathcal{K} := \ker(\Xi)$. Denote

$$R = P \times_M P \cong G \times P,$$

where the isomorphism is given by $(q, p) \mapsto (g, p)$ for $g \in G$, the unique element satisfying $gp = q$. Note that since the action $\vec{\star}$ of G on $\mathcal{H}$ preserves each fiber of Ξ, we obtain by restriction a group action of G on $\mathcal{K}$, also by groupoid automorphisms. This ensures that the following groupoid action θ of R on $\mathbf{s}\colon \mathcal{K}\backslash\mathcal{H} \to P$ is well defined:

$$\theta(g, p)(\mathcal{K}\xi) = \mathcal{K}(g \vec{\star} \xi).$$

Hence θ essentially amounts to the lifted G-action.

Proposition 5.15. *Assume the setup at the beginning of Section 4, and that $\mathcal{H}$ and $\mathcal{H}'$ are Lie groupoids. Then*

(i) *Ξ is a fibration of Lie groupoids,*

(ii) *$(\mathcal{K}, R, \theta)$ is the normal subgroupoid system corresponding to it, via Theorem 5.6.*

Proof. We first prove that $(\mathcal{K}, R, \theta)$ is a normal subgroupoid system. Because Ξ and π are surjective submersions (see Remark 2.27), we have that $\mathcal{K}$ is a closed, embedded wide Lie subgroupoid of $\mathcal{H}$ and R is a smooth equivalence relation. Because $\vec{\star}$ is a smooth Lie group action, we have that θ is also a smooth action. The three conditions in Definition 5.5 are satisfied because $\vec{\star}$ is a group action on $\mathcal{H}$ by Lie groupoid automorphisms, covering the group action of G on P.

Since $(\mathcal{K}, R, \theta)$ is a normal subgroupoid system, by Theorem 5.6 $(\mathcal{R}, R)$ is a smooth congruence, where

$$\mathcal{R} = \big\{(\xi', \xi) \in \mathcal{H} \times \mathcal{H} : \big(\mathbf{s}(\xi'), \mathbf{s}(\xi)\big) \in R \text{ and } \theta\big(\mathbf{s}(\xi'), \mathbf{s}(\xi)\big)\mathcal{K}\xi = \mathcal{K}\xi'\big\}.$$

Lemma 5.12 shows that $(\xi', \xi) \in \mathcal{R}$ if and only if $\Xi(\xi') = \Xi(\xi)$. This means that the quotient map of the smooth congruence $(\mathcal{R}, R)$ is exactly Ξ. In particular, by Theorem 5.6, Ξ is a fibration. ■

Remark 5.16. Proposition 5.15 holds also replacing the hypothesis that $\mathcal{H}'$ is an embedded Lie groupoid with the hypothesis that $\mathcal{K}$ is a Lie subgroupoid of $\mathcal{H}$, with the same proof.

When $\mathcal{H}(\mathcal{F})$ is a Hausdorff Lie groupoid, Remark 2.26 allows to give a description of $\mathcal{K}$ that does not make reference to the morphism Ξ. Namely, $\mathcal{K}$ consists of the elements $\xi \in \mathcal{H}(\mathcal{F})$ that carry a diffeomorphism φ such that, for some slice S through $\mathbf{s}(\xi)$ transverse to the π-fibers, this diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\quad\varphi\quad} & \varphi(S) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M. & \end{array}$$

In the following simple example, we describe $\mathcal{H}(\mathcal{F}_M)$ using Proposition 5.15.

Example 5.17. Let $P = S^1 \times \mathbb{R}$ be the cylinder with coordinates θ and y, endowed with the (free) action of $G = (\mathbb{R}, +)$ by vertical translations. The action preserves the (regular) foliation

$$\mathcal{F} := \left\langle \frac{\partial}{\partial \theta} + \lambda \frac{\partial}{\partial y} \right\rangle$$

by spirals, where λ is a fixed non-zero real number. The quotient $M := P/G$ is S^1. It is easy to see that the induced foliation $\mathcal{F}_M$ is the full foliation, but we want to describe $\mathcal{H}(\mathcal{F}_M)$ without using this fact.

We have $\mathcal{H}(\mathcal{F}) = \mathbb{R} \times P$, the transformation groupoid of the flow of $\frac{\partial}{\partial \theta} + \lambda \frac{\partial}{\partial y}$. By the above characterization of $\mathcal{K}$, we have $\mathcal{K} = \mathbb{Z} \times P$, which is an embedded Lie subgroupoid of $\mathcal{H}(\mathcal{F})$. Hence $\mathcal{K} \backslash \mathcal{H}(\mathcal{F}) = (\mathbb{R}/\mathbb{Z}) \times P$, and quotienting by the groupoid action θ we obtain $(\mathbb{R}/\mathbb{Z}) \times S^1$. One checks easily that the induced groupoid structure is given by the transformation groupoid of the Lie group $\mathbb{R}/\mathbb{Z}$ acting by rotations on S^1, which is isomorphic to the pair groupoid structure on $S^1 \times S^1$. Proposition 5.15 and Remark 5.16 state that it is isomorphic to $\mathcal{H}(\mathcal{F}_M)$.

A. A lemma about generating sets

We prove the following statement, which will be used in Appendix A.1. Recall that $\widehat{\mathcal{F}}$ denotes the global hull of $\mathcal{F}$; see Definition 2.7.

Lemma A.1. *Let $\pi \colon P \to M$ be a surjective submersion with connected fibers. Let $\mathcal{F}$ be a singular foliation on P such that $[\Gamma_c(\ker d\pi), \mathcal{F}] \subset \Gamma_c(\ker d\pi) + \mathcal{F}$.*

(i) *The set*

$$\widehat{\mathcal{F}}^{\mathrm{proj}} := \{X \in \widehat{\mathcal{F}} : X \textit{ is } \pi\textit{-projectable to a vector field on } M\}$$

generates $\mathcal{F}$ as a $C_c^\infty(P)$-module.

(ii) *The singular foliation $\mathcal{F}_M$ on M introduced in Theorem 2.25 admits the following description:*

$$\mathcal{F}_M = \pi_*(\mathcal{F}) := \mathrm{Span}_{C_c^\infty(M)}\{\pi_* X : X \in \widehat{\mathcal{F}}^{\mathrm{proj}}\}.$$

Proof. We first make a claim.

Claim. Lemma A.1 holds in the special case that $\Gamma_c(\ker d\pi) \subset \mathcal{F}$.

Indeed, in this special case, by Proposition 2.24, there is a unique singular foliation $\mathcal{F}_M$ on M with $\pi^{-1}(\mathcal{F}_M) = \mathcal{F}$. Given this, (i) is a consequence of Definition 2.5. For (ii), note that $\pi^{-1}(\pi_*(\mathcal{F})) = \mathcal{F}$, as can be checked using (i). Since $\mathcal{F} = \pi^{-1}\mathcal{F}_M$, we obtain $\mathcal{F}_M = \pi_*(\mathcal{F})$ by the uniqueness statement in Proposition 2.24. This proves the claim.

Take $\mathcal{F}^{\mathrm{big}} := \Gamma_c(\ker d\pi) + \mathcal{F}$, a singular foliation satisfying the condition of the above claim. We now proceed to prove the two items of the lemma.

(i) By the claim, $\widehat{\mathcal{F}^{\text{bigproj}}}$ generates $\mathcal{F}^{\text{big}}$ as a $C_c^\infty(P)$-module. Take $X \in \mathcal{F} \subset \mathcal{F}^{\text{big}}$. There exist finitely many $Y_j \in \widehat{\mathcal{F}^{\text{bigproj}}}$ and $f_j \in C_c^\infty(P)$ such that $X = \sum_j f_j Y_j$. By definition of $\mathcal{F}^{\text{big}}$, we can write $Y_j = \widehat{Y}_j + Z_j$ with $\widehat{Y}_j \in \widehat{\mathcal{F}}^{\text{proj}}$ and $Z_j \in \Gamma(\text{Ker}(d\pi))$. Then

$$X = \sum_j f_j Y_j = \sum_j f_j \widehat{Y}_j + \sum_j f_j Z_j.$$

The last term $\sum_j f_j Z_j = X - \sum_j f_j \widehat{Y}_j$ lies in $\mathcal{F}$ as the difference of two elements of $\mathcal{F}$ and is π-projectable (to the zero vector field on M). Hence this last term lies in $\widehat{\mathcal{F}}^{\text{proj}}$, and we have proven (i).

(ii) We have $\pi^{-1}(\pi_*(\mathcal{F})) = \pi^{-1}(\pi_*(\mathcal{F}^{\text{big}})) = \mathcal{F}^{\text{big}}$, using the claim in the second equality, and $\mathcal{F}^{\text{big}} = \pi^{-1}(\mathcal{F}_M)$ by definition. The uniqueness in Proposition 2.24 implies that $\mathcal{F}_M = \pi_*(\mathcal{F})$. ∎

A.1. Proof of Theorem 2.25

The following proposition is a special case[10] of [11, Proposition D.4], augmented with the statement that Ξ is an open map.

Proposition A.2. *Let $\pi\colon P \to M$ be a surjective submersion.*

Let $\mathcal{F}$ be a singular foliation on P, and assume that it satisfies the following condition:

$$\widehat{\mathcal{F}}^{\text{proj}} := \{X \in \widehat{\mathcal{F}} : X \text{ is } \pi\text{-projectable to a vector field on } M\}$$
$$\text{generates } \mathcal{F} \text{ as a } C_c^\infty(P)\text{-module.} \tag{A.1}$$

Let $\pi_*(\mathcal{F}) := \text{Span}_{C_c^\infty(M)}\{\pi_* X : X \in \widehat{\mathcal{F}}^{\text{proj}}\}$, which is a singular foliation on M.

Then there is a canonical, open, surjective morphism of topological groupoids

$$\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}\big(\pi_*(\mathcal{F})\big)$$

covering π.

Remark A.3. If $\mathcal{H}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F}_M)$ are Lie groupoids, then Ξ is a submersion. This follows from the proof of Proposition A.2 given below since, in that case, the quotient maps from bisubmersions to holonomy groupoids are submersive.

We can now prove Theorem 2.25.

Proof of Theorem 2.25. Apply Proposition A.2. Notice that condition (A.1) is satisfied by Lemma A.1 (i) and $\pi_*\mathcal{F} = \mathcal{F}_M$ by Lemma A.1 (ii). ∎

In order to keep this paper self-contained, we now provide a proof for Proposition A.2. It differs from the proof found in [11, Proposition D.4], in that it allows to write down the morphism Ξ more explicitly, and makes clear that Ξ is an open map. We start with a lemma.

[10] This special case is obtained from [11, Proposition D.4] taking $\mathcal{B}_1$ to be the singular foliation $\mathcal{F}$, G_i to be the pair groupoid $M_i \times M_i$ and $F := \pi \times \pi$.

Lemma A.4. *Let $\pi: P \to M$ and $\mathcal{F}$ be as in Proposition A.2, and let $\mathcal{F}_M := \pi_*(\mathcal{F})$. Then there exists a family of path holonomy bisubmersions $\mathcal{S}$ for $\mathcal{F}$ so that $\bigcup_{U \in \mathcal{S}} \mathbf{s}(U) = P$, with the following properties.*

(i) *For any $U \in \mathcal{S}$ one has that $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a bisubmersion for $\mathcal{F}_M$. Further, it is adapted to a path holonomy bisubmersion for $\mathcal{F}_M$.*

(ii) *Let $\mathcal{U}$ be the atlas generated by $\mathcal{S}$. Then for any $U \in \mathcal{U}$ one has that $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a (source connected) bisubmersion for $\mathcal{F}_M$. Further, the atlas generated by*

$$\pi\mathcal{U} := \{(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s}) : U \in \mathcal{U}\},$$

is equivalent to a path holonomy atlas for $\mathcal{F}_M$.

Proof. We first make a claim.

Claim. $\mathcal{F}$ is locally generated by *finitely many* π-projectable vector fields in $\widehat{\mathcal{F}}$.

Indeed, for any $p \in P$ there is a neighborhood $V_0 \subset P$ and finitely many generators $Y_1, \dots, Y_k \in \mathfrak{X}(V_0)$ of $\iota_{V_0}^{-1}(\mathcal{F})$, for ι_{V_0} the inclusion. Take any precompact open set $V \subset V_0$ containing p and $\rho_V \in C_c^\infty(V_0)$ such that $\rho_V = 1$ on V. For each i, since $\rho_V Y_i \in \mathcal{F}$, condition (A.1) assures that there is a finite number of π-projectable elements $X_i^j \in \widehat{\mathcal{F}}^{\text{proj}}$ and $f_i^j \in C_c^\infty(P)$ such that

$$\rho_V Y_i = \sum_j f_i^j X_i^j.$$

Therefore, every element of $\iota_V^{-1}(\mathcal{F})$ is a $C_c^\infty(V)$-linear combination of the X_i^j, which are π-projectable and lie in $\widehat{\mathcal{F}}$. This proves the claim.

(i) Now, for every point p_0 of P, take a minimal set of π-projectable elements $\{X_1, \dots, X_n\}$ in $\widehat{\mathcal{F}}$ that are local generators of $\mathcal{F}$ nearby that point. Let $(U, \mathbf{t}, \mathbf{s})$ be the corresponding path holonomy bisubmersion, where $U \subset \mathbb{R}^n \times P$. Then $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a bisubmersion for $\mathcal{F}_M$, with source map $(\lambda, p) \mapsto \pi(p)$ and target map $(\lambda, p) \mapsto \exp_{\pi(p)}(\sum \lambda_i \pi_* X_i)$. A way to see this is to apply [1, Lemma 2.3] to the path holonomy bisubmersion $W \subset \mathbb{R}^n \times M$ for $\mathcal{F}_M$ corresponding to the generators $\{\pi_* X_1, \dots, \pi_* X_n\}$ and to the submersion[11] $(\mathrm{Id}_{\mathbb{R}^n}, \pi): \mathbb{R}^n \times P \to \mathbb{R}^n \times M$. We observe that $(\mathrm{Id}_{\mathbb{R}^n}, \pi)$ is a morphism of bisubmersions from $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ to W. This shows that the former bisubmersion is adapted (see Definition 2.16) to the latter.

(ii) Take $U \in \mathcal{U}$; without loss of generality, assume $U = U_1 \circ \cdots \circ U_k$ for $U_i \in \mathcal{S}$. Denote $\pi U_i := (U_i, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$, which are bisubmersions for $\mathcal{F}_M$ by (i). Note that the inclusion map $\omega: U_1 \circ \cdots \circ U_k \to \pi U_1 \circ \cdots \circ \pi U_k$ makes the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{\ \omega\ } & \pi U_1 \circ \cdots \circ \pi U_k \\ {\scriptstyle \mathbf{t}}\downdownarrows{\scriptstyle \mathbf{s}} & & {\scriptstyle \tilde{\mathbf{t}}}\downdownarrows{\scriptstyle \tilde{\mathbf{s}}} \\ P & \xrightarrow{\ \pi\ } & M. \end{array} \tag{A.2}$$

[11] More precisely, to its restriction to $(\mathrm{Id}_{\mathbb{R}^n}, \pi)^{-1}(W) \cap U$.

Because of this commutative diagram and since $\pi U_1 \circ \cdots \circ \pi U_k$ is a bisubmersion for $\mathcal{F}_M$, one gets that

$$A := (\pi \circ \mathbf{t})^{-1}\mathcal{F}_M = (\pi \circ \mathbf{s})^{-1}\mathcal{F}_M. \tag{A.3}$$

The left-hand side is

$$\begin{aligned}(\pi \circ \mathbf{t})^{-1}\mathcal{F}_M &= \mathbf{t}^{-1}\big(\mathcal{F} + \ker_c(d\pi)\big) = \mathbf{t}^{-1}(\mathcal{F}) + \mathbf{t}^{-1}\big(\ker_c(d\pi)\big)\\ &= \ker_c(d\mathbf{s}) + \ker_c(d\mathbf{t}) + \mathbf{t}^{-1}\big(\ker_c(d\pi)\big)\\ &= \ker_c(d\mathbf{s}) + \mathbf{t}^{-1}\big(\ker_c(d\pi)\big),\end{aligned}$$

using, respectively, that $\pi^{-1}\mathcal{F}_M = \mathcal{F} + \ker_c(d\pi)$, that $\mathbf{t}$ is a submersion, and that U is a bisubmersion for $\mathcal{F}$. Here we use the short-hand notation $\ker_c(d\pi) := \Gamma_c(\ker(d\pi))$. Repeating for the right-hand side, from (A.3) we obtain

$$A = \ker_c(d\mathbf{s}) + \mathbf{t}^{-1}\big(\ker_c(d\pi)\big) = \ker_c(d\mathbf{t}) + \mathbf{s}^{-1}\big(\ker_c(d\pi)\big).$$

This implies that

$$A = \mathbf{t}^{-1}\big(\ker_c(d\pi)\big) + \mathbf{s}^{-1}\big(\ker_c(d\pi)\big) = \ker_c\,\big(d(\pi \circ \mathbf{t})\big) + \ker_c\,\big(d(\pi \circ \mathbf{s})\big),$$

i.e., that $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a bisubmersion for $\mathcal{F}_M$.

By construction, the bisubmersion $\pi U_1 \circ \cdots \circ \pi U_k$ lies in the atlas for $\mathcal{F}_M$ generated by $\pi\mathcal{S}$. Thus the commutative diagram (A.2) shows that $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is adapted to the atlas for $\mathcal{F}_M$ generated by $\pi\mathcal{S}$, which in turn is adapted to a path holonomy atlas by (i). This shows that the atlas generated by $\pi\mathcal{U}$ is adapted to a path holonomy atlas for $\mathcal{F}_M$. It is actually equivalent to such an atlas because a path holonomy atlas is adapted to any other atlas. ■

Remark A.5. Not every bisubmersion $(U, \mathbf{t}, \mathbf{s})$ for $\mathcal{F}$ satisfies that $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$ is a bisubmersion for $\mathcal{F}_M$. For instance, take $P := \mathbb{R}^2$, $\mathcal{F} = 0$, and $M = \mathbb{R}$ with map $\pi\colon P \to M$ given by the first projection. Then $\mathcal{F}_M = 0$. Now take any diffeomorphism $\phi\colon P \to P$ that does not preserve the foliation $\pi^{-1}(\mathcal{F}_M)$ by the fibers of π. Then (P, Id, ϕ) is a bisubmersion for $\mathcal{F}$ but $(P, \pi, \pi \circ \phi)$ is not a bisubmersion for $\mathcal{F}_M$.

Proof of Proposition A.2. Let $\mathcal{U}$ be an atlas for $\mathcal{F}$ as in Lemma A.4. For every $U \in \mathcal{U}$, we use the short-hand notation πU to denote $(U, \pi \circ \mathbf{t}, \pi \circ \mathbf{s})$, a bisubmersion for $\mathcal{F}_M$. Define Ξ as follows:

$$\Xi\colon \mathcal{H}(\mathcal{F}) \to \mathcal{H}(\mathcal{F}_M); \quad [u] \mapsto [u]_M, \tag{A.4}$$

where $u \in U \in \mathcal{U}$ and $[u]_M$ is the class of u seen as an element of $\pi U \in \pi\mathcal{U}$. Here we used that, by Lemma A.4, $\mathcal{H}(\mathcal{F}_M)$ agrees with the groupoid associated to the atlas generated by $\pi\mathcal{U}$.

The map Ξ is well defined. If $u_1 \in U_1 \in \mathcal{U}$ and $u_2 \in U_2 \in \mathcal{U}$ are equivalent, then there exists a morphism of bisubmersions sending u_1 to u_2. Using the same morphism, it is clear that $u_1 \in \pi U_1 \in \pi\mathcal{U}$ is equivalent to $u_2 \in \pi U_2 \in \pi\mathcal{U}$, i.e., that $[u_1]_M = [u_2]_M$. The same argument shows that Ξ is independent of the specific choice of the atlas $\mathcal{U}$, thus canonical.

It is clear that Ξ covers π and sends the identity bisection of $\mathcal{H}(\mathcal{F})$ to the identity bisection of $\mathcal{H}(\mathcal{F}_M)$. To prove that it is a morphism of set theoretic groupoids, we only need to prove that it preserves the composition. It does because, for any $U_1, U_2 \in \mathcal{U}$, the inclusion map $\pi(U_1 \circ U_2) \to \pi U_1 \circ \pi U_2$ is a morphism of bisubmersions for $\mathcal{F}_M$.

We check that Ξ is a continuous open map. This holds because in the following commutative diagram the quotient maps Q and Q_M are continuous and open (see [9, Lemma 3.1]) and because Q is surjective:

$$\begin{array}{ccc} \bigsqcup_{U\in\mathcal{U}} U & \xrightarrow{\mathrm{Id}} & \bigsqcup_{U\in\mathcal{U}} \pi U \\ \big\downarrow Q & & \big\downarrow Q_M \\ \mathcal{H}(\mathcal{F}) & \xrightarrow{\Xi} & \mathcal{H}(\mathcal{F}_M). \end{array}$$

The map Ξ is surjective. By the above, $\Xi(\mathcal{H}(\mathcal{F}))$ is a neighborhood of the identities of $\mathcal{H}(\mathcal{F}_M)$. Because Ξ is a morphism of topological groupoids, $\Xi(\mathcal{H}(\mathcal{F}))$ is a symmetric set closed under compositions. It is well known that any **s**-connected topological groupoid is generated by any symmetric neighborhood of the identities [10]. Because $\mathcal{H}(\mathcal{F}_M)$ is **s**-connected, we obtain $\Xi(\mathcal{H}(\mathcal{F})) = \mathcal{H}(\mathcal{F}_M)$, i.e., Ξ is surjective. ■

The following corollary extends the conclusions of Lemma A.4 (ii) to arbitrary source connected atlases (as defined in Definition 2.20).

Corollary A.6. *Let $\pi\colon P \to M$, $\mathcal{F} \subset \mathfrak{X}_c(P)$, and $\mathcal{F}_M \subset \mathfrak{X}_c(M)$ as in Proposition A.2. Then for any source connected atlas $\mathcal{U}'$ for $\mathcal{F}$ one has that $\pi\mathcal{U}' := \{\pi U' : U' \in \mathcal{U}'\}$ is an atlas equivalent to a path holonomy atlas for $\mathcal{F}_M$.*

Proof. We first observe that, in Lemma A.4, actually $\pi\mathcal{U}$ is already an atlas. This follows from the fact that the map Ξ, given as in (A.4), is surjective.

Now let $\mathcal{U}'$ be any source connected atlas for $\mathcal{F}$. Then $\mathcal{U}'$ is adapted to $\mathcal{U}$ (see [8, §4]). This means that, for any element $u \in U' \in \mathcal{U}'$, there exists a morphism of bisubmersions ω_u from a neighborhood $U'_u \subset U'$ to a bisubmersion $U \in \mathcal{U}$. In particular, the following diagram commutes:

$$\begin{array}{ccc} U'_u & \xrightarrow{\omega_u} & U \\ \mathbf{t}\big\downarrow\big\downarrow\mathbf{s} & & \pi\circ\mathbf{t}\big\downarrow\big\downarrow\pi\circ\mathbf{s} \\ P & \xrightarrow{\pi} & M. \end{array}$$

The triple $(U'_u, \mathbf{t}\circ\pi, \mathbf{s}\circ\pi)$ is a bisubmersion for $\mathcal{F}_M$ since the argument following diagram (A.2) can be applied identically to the diagram above. Moreover, since being a bisubmersion is a local property, we have that $\pi U' := (U', \mathbf{t}\circ\pi, \mathbf{s}\circ\pi)$ is a bisubmersion for $\mathcal{F}_M$.

The families $\pi\mathcal{U}' := \{\pi U' : U' \in \mathcal{U}'\}$ and $\pi\mathcal{U}$ are adapted to each other since the atlases $\mathcal{U}'$ and $\mathcal{U}$ are equivalent. This implies that $\pi\mathcal{U}'$ is already an atlas for $\mathcal{F}_M$, equivalent to $\pi\mathcal{U}$. The latter is equivalent to a path holonomy atlas by Lemma A.4 (ii), so we are done. ■

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