Ann. Inst. H. Poincaré
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/106

Published by EMS Press

Logarithmic Sobolev and interpolation inequalities on the sphere: Constructive stability results

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Abstract. We consider Gagliardo–Nirenberg inequalities on the sphere which interpolate between the Poincaré inequality and the Sobolev inequality, and include the logarithmic Sobolev inequality as a special case. We establish explicit stability results in the subcritical regime using spectral decomposition techniques, and entropy and *carré du champ* methods applied to nonlinear diffusion flows.

1. Introduction and main results

Functional inequalities are essential in many areas of mathematics. The knowledge of optimal constants, or at least good estimates of them, is crucial for various applications. Whether optimality cases are achieved is a standard issue in analysis. The next natural question is to understand how the deficit, say the difference of the two sides of the functional inequality, measures the distance to the set of optimal functions. Such a question has been actively studied in critical Sobolev inequalities, but much less in subcritical interpolation inequalities. In the case of the sphere, a global stability result based on Bianchi–Egnell-type methods was recently obtained for a family of Gagliardo–Nirenberg inequalities by Frank [32], with the striking observation that only the power 4 of a natural distance is controlled by the deficit. Here we give a more detailed picture, which includes the logarithmic Sobolev inequality, and provide explicit estimates.

On the sphere $\mathbb{S}^d$ with $d \geq 1$, the *logarithmic Sobolev inequality* can be written as

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu), \tag{LS}$$

where $d\mu$ denotes the uniform probability measure. The equality case is achieved by constant functions and $d/2$ is the optimal constant, as shown by taking the test functions $F_\varepsilon(x) = 1 + \varepsilon x \cdot \nu$, for some arbitrary $\nu \in \mathbb{S}^d$, in the limit as $\varepsilon \to 0$. Our first result is an improved inequality under an orthogonality constraint, which improves upon [23, Proposition 5.4].

2020 Mathematics Subject Classification. Primary 26D10; Secondary 58J99, 39B62, 43A90, 49J40, 46E35.
Keywords. Logarithmic Sobolev inequality, Gagliardo–Nirenberg inequalities, stability, sphere, spectral decomposition.

Theorem 1. *Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that*

$$\int_{\mathbb{S}^d} x F \, d\mu = 0, \tag{1.1}$$

we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq \mathcal{C}_d \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu, \tag{1.2}$$

with $\mathcal{C}_d = \frac{2}{d+2}$.

Since equality in (LS) is achieved if and only if F is a constant function, the right-hand side in (1.2) is an estimate of the distance to the set of optimal functions under the constraint $\int_{\mathbb{S}^d} x F \, d\mu = 0$. Alternatively, Theorem 1 amounts to the *improved logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \frac{d+2}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$
$$\forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \text{ such that } \int_{\mathbb{S}^d} x F \, d\mu = 0.$$

Without condition (1.1), there is no such inequality as (1.2). With $F_\varepsilon(x) = 1 + \varepsilon x \cdot \nu$ as above, as $\varepsilon \to 0$ one can indeed check that

$$\|\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{2} \int_{\mathbb{S}^d} F_\varepsilon^2 \log\left(\frac{F_\varepsilon^2}{\|F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu = O(\varepsilon^4) = O(\|\nabla F_\varepsilon\|^4_{\mathrm{L}^2(\mathbb{S}^d)}).$$

In the absence of an additional constraint, like (1.1), such behaviour is in fact optimal. The following estimate arises from the *carré du champ* method.

Proposition 2. *Let $d \geq 1$, $\gamma = 1/3$ if $d = 1$ and $\gamma = (4d-1)(d-1)^2/(d+2)^2$ if $d \geq 2$. Then, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq \frac{1}{2} \frac{\gamma \|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\gamma \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}.$$

With $\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$, notice that the deficit can be estimated from below by

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log(F^2) \, d\mu \geq \frac{\gamma}{2d} \|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)} + o(\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)})$$

if $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ is small enough.

Let $\Pi_1 F$ denote the orthogonal projection of a function $F \in \mathrm{L}^2(\mathbb{S}^d)$ on the spherical harmonics corresponding to the first positive eigenvalue of $-\Delta$ on $\mathbb{S}^d$, i.e.,

$$\Pi_1 F(x) = (d+1) x \cdot \int_{\mathbb{S}^d} y F(y) \, d\mu(y) \quad \forall x \in \mathbb{S}^d.$$

Our main stability result for the logarithmic Sobolev inequality combines the results of Theorem 1 and Proposition 2 as follows.

Theorem 3. *Let $d \ge 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$
$$\ge \mathcal{S}_d \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{2}\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id} - \Pi_1) F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some stability constant $\mathcal{S}_d > 0$.

An explicit estimate of $\mathcal{S}_d$ is given in Section 4.

We also consider the subcritical *Gagliardo–Nirenberg inequalities*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \ge \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu), \tag{GN}$$

for any $p \in [1,2) \cup (2, 2^*)$. Here, $d\mu$ again denotes the uniform probability measure on $\mathbb{S}^d$, the critical Sobolev exponent is $2^* := 2d/(d-2)$ if $d \ge 3$ and we adopt the convention that $2^* = +\infty$ if $d = 1$ or $d = 2$. Inequality (GN) with $p = 1$ is equivalent to the Poincaré inequality. If $d \ge 3$, inequality (GN) also holds for the critical exponent $p = 2^*$ and it is in fact Sobolev's inequality with optimal constant on $\mathbb{S}^d$, but this is out of the scope of our paper which focuses on the subcritical regime $p < 2^*$. The logarithmic Sobolev inequality (LS) is obtained from (GN) by taking the limit as $p \to 2$, and the counterpart of the above results for $p \ne 2$, in the *subcritical range* $p < 2^*$, goes as follows.

Theorem 4. *Assume that $d \ge 1$ and $p \in (1,2) \cup (2, 2^*)$. For any function $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that the orthogonality condition (1.1) holds, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \ge \mathcal{C}_{d,p} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \tag{1.3}$$

with $\mathcal{C}_{d,p} = \frac{2d - p(d-2)}{2(d+p)}$.

Taking $F_\varepsilon(x) = 1 + \varepsilon x \cdot \nu$ as above shows that (1.1) is needed in Theorem 4. We also have a higher-order estimate of the deficit as a consequence of the *carré du champ* method.

Proposition 5. *Let $d \ge 1$ and $p \in (1,2) \cup (2, 2^*)$. There is a convex function ψ on $\mathbb{R}^+$ with $\psi(0) = \psi'(0) = 0$ such that, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \ge \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \, \psi\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right).$$

An explicit expression for ψ will be given in Section 3. The two results of Theorem 4 and Proposition 5 can be combined to prove the analogue of Theorem 3 for $p \ne 2$, with an explicit constant: see Section 4.

Theorem 6. *Let $d \geq 1$ and $p \in (1,2) \cup (2,2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \\ \geq \mathscr{S}_{d,p}\left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id} - \Pi_1)F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$.

Let us give a brief account of the literature. In this paper, we address the distinction between *improved inequalities* (inequalities with improved constants under orthogonality constraints) and *quantitative stability* (as a measure of a distance to the set of optimal functions). There are many adjacent directions of research like, for instance, stability in weaker norms (see for instance [25, 40] for Sobolev's inequality) or notions of stability with no explicit notion of distance. To our knowledge, not so much has been done in subcritical interpolation inequalities (see [11, 32] and some references therein), except for the logarithmic Sobolev inequality, for which we refer to [28, 30] and [31, 38, 39, 41].

The *Gagliardo–Nirenberg inequalities* (GN) on the sphere have been established with optimal constant for any $p \in (2,2^*)$ in [10, Corollary 6.1] and in [7]. In dimension $d = 2$, Onofri's inequality is obtained from (GN) in the limit as $p \to 2^* = +\infty$: see [7, 13]. With $p \in [1,2)$ or $p > 2$ but not too large (if $d \geq 2$), inequality (GN) was known earlier from [3]. A Markovian point of view is presented in [5], with many more references therein on related questions. On Euclidean space, similar inequalities go back to [34, 49, 51]. The *logarithmic Sobolev inequality* (LS) is a well-known limit case as $p \to 2$ and can be considered in a common framework with (GN). Whenever possible, we shall adopt this point of view. For an overview of early results on the sphere, we refer to [36, Section 6, (iv)]. The literature on (LS) on the circle and on the sphere can be traced back at least to [56], [46, Theorem 1, p. 268] with computations based on the ultraspherical operator, and [50] for a more variational approach. The inequality with optimal constant is stated in [3, inequality (13), p. 195] as a consequence of the *carré du champ* method. Also see [4] and [14, p. 342] for related results and [18, 22, 23] for a PDE approach based on entropy estimates and the *carré du champ* method. After Schwarz foliated symmetrization, the problem is reduced to a simpler family of interpolation inequalities involving only the ultraspherical operator.

The interest for stability issues was raised by [12] and the stability result of Bianchi and Egnell in [9], on Euclidean space. Over the years, various approaches have been developed, based on compactness methods and contradiction arguments as in [9, 16], spectral analysis and orthogonality conditions as in [23, Proposition 5.4] and [37], or entropy methods and improved inequalities as in [2, 20, 21, 23, 27]. For spectral methods, a fruitful strategy relies on the Funk–Hecke formula, which is behind (2.2), and the approach of [7, 44], which applies to the stability result for fractional interpolation inequalities of [16] and [29, Corollary 2.3]. This is the method we use in Section 2. Stability issues for (GN) have recently been discussed in [32] with methods of Bianchi–Egnell type, with the drawback that no estimate of the stability constant is known. This drawback can be cured by a *carré du champ* method as we shall see in Section 3. Without entering into details, let us mention some recent progress on stability in [11, 15, 19, 42] for related critical inequalities.

This paper is organized as follows. Section 2 is devoted to the proof by spectral methods of Theorem 7 (see below), which is an extension of Theorems 1 and 4: under orthogonality constraints, these results are reduced to estimates of improved constants in inequalities (LS) and (GN), with various refinements based on a decomposition in spherical harmonics. An explicit stability result without constraints corresponding to Propositions 2 and 5 is proved in Section 3. The proofs of Theorems 3 and 6, in Section 4, is based on the spectral decomposition method developed by Frank [32]. We collect the previous estimates (with and without orthogonality constraints) in global results, with explicit constants. Various additional results are stated in two appendices: the extension of the method to interpolation inequalities for the Gaussian measure on Euclidean space and a discussion of its limitations in Appendix A, the details of the computations of the *carré du champ* method on the sphere and its application in order to establish improved functional inequalities in Appendix B.

2. Improvements under orthogonality constraints

In this section we prove Theorems 1 and 4 in the slightly more general framework of Theorem 7 below. Let us consider the generalized entropy functionals

$$\mathcal{E}_2[F] := \frac{1}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu,$$

$$\mathcal{E}_p[F] := \frac{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \quad \text{if } p \neq 2.$$

With this notation, we can rephrase (LS) and (GN) as

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq d\,\mathcal{E}_p[F] \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu),$$

for any $p \in [1, 2^*)$. The optimality case is achieved by considering the test function $F_\varepsilon = 1 + \varepsilon\varphi_1$ in the limit as $\varepsilon \to 0$, where φ_1 is an eigenfunction of the Laplace–Beltrami operator such that $-\Delta\varphi_1 = d\varphi_1$, for instance $\varphi_1(x) = x \cdot \nu$ for some $\nu \in \mathbb{S}^d$ as in Section 1.

Let us consider the decomposition into spherical harmonics of $\mathrm{L}^2(\mathbb{S}^d, d\mu)$,

$$\mathrm{L}^2(\mathbb{S}^d, d\mu) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell,$$

where $\mathcal{H}_\ell$ is the subspace of spherical harmonics of degree $\ell \ge 0$. See for instance [8, 45, 47, 52]. For any integer $k \ge 1$, let us define Π_k as the orthogonal projection with respect to $\mathrm{L}^2(\mathbb{S}^d, d\mu)$ onto $\bigoplus_{\ell=1}^k \mathcal{H}_\ell$. The following statement extends Theorems 1 and 4.

Theorem 7. *Assume that $d \ge 1$, $p \in (1, 2^*)$ and let $k \ge 1$ be an integer. For any function $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \ge \mathcal{C}_{d,p,k} \int_{\mathbb{S}^d} |\nabla (\mathrm{Id} - \Pi_k) F|^2 \, d\mu \tag{2.1}$$

for some explicit constant $\mathcal{C}_{d,p,k} \in (0, 1)$ such that $\mathcal{C}_{d,p,k} \le \mathcal{C}_{d,p,1} = \frac{2d-p(d-2)}{2(d+p)}$.

The expression for $\mathcal{C}_{d,p,k}$ is given below in the proof. Inequality (2.1) can be seen as an improvement of (LS) and (GN), namely

$$(1 - \mathcal{C}_{d,p,k}) \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \ge d\,\mathcal{E}_p[F]$$

for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\Pi_k F = 0$. With $k = 1$, this establishes (1.2) and (1.3), thus proving Theorem 1 if $p = 2$, and Theorem 4 if $p \ne 2$.

Proof of Theorem 7. Let $(F_j)_{j\in\mathbb{N}}$ be the decomposition of F along $\mathcal{H}_j$ for any $j \in \mathbb{N}$. We learn from [7, inequality (19)] or [29, inequality (1.6)] that the *subcritical interpolation inequalities*

$$\mathcal{E}_p[F] \le \sum_{j=1}^{\infty} \zeta_j(p) \int_{\mathbb{S}^d} |F_j|^2 \, d\mu \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \tag{2.2}$$

hold for any $p \in (1, 2) \cup (2, 2^*)$ with

$$\zeta_j(p) := \frac{\gamma_j\big(\frac{d}{p}\big) - 1}{p - 2} \quad \text{and} \quad \gamma_j(x) := \frac{\Gamma(x)\Gamma(j + d - x)}{\Gamma(d - x)\Gamma(x + j)}.$$

This result is based on the Funk–Hecke theorem (see for instance [33, Section 4]) and Lieb's ideas in [44]. Notice that $\zeta_j(p) \ge 0$ for any $p \in (1, 2) \cup (2, 2^*)$. According to [29, Lemma 2.2], the function ζ_j is strictly monotone increasing on $(1, \infty)$ for any $j \ge 2$ and the limits

$$\lambda_j = d \lim_{p\to 2^*} \zeta_j(p)$$

are the eigenvalues of the Laplace–Beltrami on the sphere, with $\lambda_j = j(j + d - 1)$. Hence

$$d\,\mathcal{E}_p[F] \le \sum_{j=1}^{\infty} \lambda_j \int_{\mathbb{S}^d} |F_j|^2 \, d\mu = \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu,$$

which is the essence of the proof of (GN) in [7] and also the main idea for the proof of the stability result for fractional interpolation inequalities of [29, Corollary 2.3]. Here we draw some consequences in standard norms for nonfractional operators and identify estimates of the stability constant in the corresponding stability result.

▷ *The case* $p \neq 2$. Let $x = d/p \in ((d-2)/2, d]$ if $d \geq 2$ and $x \in (0, d]$ if $d = 1$. We consider

$$\xi_j(x) := \frac{|\gamma_j(x) - 1|}{j(j+d-1)} \quad \text{and} \quad h_j(x) = \frac{j(j+d-1)(j+d-x)}{(j+1)(j+d)(j+x)};$$

notice that $\gamma_j(x) > 1$ for $x < d/2$, while $\gamma_j(x) < 1$ for $x > d/2$. An elementary computation shows that $0 < h_j(x) < 1$. Since $\gamma_{j+1}(x)\lambda_\kappa = h_j(x)\lambda_{j+1}\gamma_j(x)$, we obtain

$$\xi_{j+1}(x) = h_j(x)\xi_j(x) + (1 - h_j(x))\xi_j^\star(x), \tag{2.3}$$

where

$$\xi_j^\star(x) := \frac{1}{1 - h_j(x)}\left|\frac{h_j(x)}{\lambda_j} - \frac{1}{\lambda_{j+1}}\right| = \frac{|d - 2x|}{j(j+d)(2x - d + 2) + dx}.$$

Notice that $(\xi_j^\star(x))_{j\geq 2}$ is a monotone decreasing sequence for any fixed, admissible value of x. We start at $j = 2$ with the observation that $\xi_2^\star(x) < \xi_2(x)$ if x is admissible. This gives, by using (2.3), the estimate

$$\xi_3(x) = h_2(x)\xi_2(x) + (1 - h_2(x))\xi_2^\star(x) < \xi_2(x).$$

Using $\xi_3^\star(x) < \xi_2^\star(x)$, we can iterate and conclude by induction that $\xi_j(x) < \xi_2(x)$ for all $j \geq 3$. As a consequence, we obtain

$$\sup_{j\geq 3} \frac{\zeta_j(p)}{j(j+d-1)} < \frac{\zeta_2(p)}{2(d+1)} = \frac{p}{2(d+p)} < \frac{1}{d} \quad \forall p \in (1,2) \cup (2, 2^*).$$

We deduce from (2.2) that

$$\begin{aligned}
\mathcal{E}_p[F] &\leq \int_{\mathbb{S}^d} |F_1|^2 \, d\mu + \frac{p}{2(d+p)} \sum_{j=2}^{\infty} j(j+d-1) \int_{\mathbb{S}^d} |F_j|^2 \, d\mu \\
&= \frac{1}{d} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu + \frac{2d - p(d-2)}{2d(d+p)} \int_{\mathbb{S}^d} |\nabla(\mathrm{Id} - \Pi_1)F|^2 \, d\mu,
\end{aligned}$$

which proves the result with $k = 1$ and gives the expression for $\mathcal{C}_{d,p,1}$.

Let us consider the case $k > 1$. We already know that $\xi_2(x) > \xi_2^\star(x)$. For any $j \geq 2$, we deduce from (2.3) that

$$\xi_{j+1}(x) - \xi_{j+1}^\star(x) = h_j(x)(\xi_j - \xi_j^\star(x)) + \xi_j^\star(x) - \xi_{j+1}^\star(x) \geq h_j(x)(\xi_j - \xi_j^\star(x))$$

because $j \mapsto \xi_j^\star(x)$ is monotone decreasing. By induction, this proves that $\xi_j(x) > \xi_j^\star(x)$ for any $j \geq 2$. As a consequence of (2.3), $j \mapsto \xi_j(x)$ is also monotone decreasing and

$$\sup_{j\geq k+2} \frac{\zeta_j(p)}{j(j+d-1)} < \frac{\zeta_{k+1}(p)}{(k+1)(k+d)} < \frac{1}{d} \quad \forall p \in (1,2) \cup (2, 2^*).$$

Altogether, for any $k \geq 1$, we have

$$d\,\mathcal{E}_p[F] \leq \int_{\mathbb{S}^d} |\nabla \Pi_k F|^2\,d\mu + \frac{d\,\zeta_{k+1}(p)}{(k+1)(k+d)} \int_{\mathbb{S}^d} |\nabla F|^2\,d\mu,$$

and the stability constant in (2.1) is estimated by

$$\mathcal{C}_{d,p,k} = 1 - \frac{d\,\zeta_{k+1}(p)}{(k+1)(k+d)}.$$

In our method, this constant cannot be improved, as shown by a test function such that $F_j = 0$ for any $j \in \mathbb{N}$ such that $j \neq 0$ and $j \neq k+1$, but this does not prove the optimality of $\mathcal{C}_{d,p,k}$.

$\triangleright$ *The case* $p = 2$. By taking the limit as $p \to 2_+$ in (2.2), we obtain

$$\eta_j := \frac{2}{d} \lim_{p \to 2_+} \zeta_j(p) = \psi(j + d/2) - \psi(d/2),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the *digamma function*, and

$$\frac{1}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \leq \frac{d}{2} \sum_{j=1}^{\infty} \eta_j \int_{\mathbb{S}^d} |F_j|^2\,d\mu \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu).$$

From $\psi(z+1) = \psi(z) + 1/z$ obtained by differentiating the identity $\Gamma(z+1) = z\,\Gamma(z)$ with respect to z, we learn that

$$\eta_{j+1} = \eta_j + \frac{2}{d+2j}.$$

We claim that

$$\eta_2 \leq \eta_j \leq \frac{2\lambda_j}{d(d+2)} \quad \forall j \geq 2$$

because there is equality for $j = 2$ as $\eta_2 = \frac{4(d+1)}{d(d+2)}$ and $\lambda_2 = 2(d+1)$ on the one hand, and

$$\eta_{j+1} - \eta_j = \frac{2}{d+2j} \leq \frac{2(d+2j)}{d(d+2)} = \frac{2(\lambda_{j+1} - \lambda_j)}{d(d+2)}$$

on the other hand, so that the result follows by induction.

Using $\lambda_{j+1} = \lambda_j + (d+2j) = \lambda_j + 2z_j$ where $z_j := j + d/2$, we also have

$$\frac{\eta_{j+1}}{\lambda_{j+1}} = \frac{\eta_j + \frac{1}{z_j}}{\lambda_j + 2z_j} < \frac{\eta_j}{\lambda_j},$$

where the inequality follows from

$$z_j^2 > \frac{\lambda_j}{2\eta_j} \quad \forall j \geq 1.$$

This inequality is indeed true for $j = 1$ because $\eta_1 = 2/d$ and we obtain the result for any $j \geq 1$ by induction using

$$\eta_{j+1} - \eta_j = \frac{2}{d+2j} \geq \frac{\lambda_{j+1}}{2z_{j+1}^2} - \frac{\lambda_j}{2z_j^2} = 2\frac{4j^2 + 2(d^2+2)j + d^3}{(d+2j)^2(d+2+2j)^2}.$$

Altogether, for any $k \geq 1$, we have

$$d\,\mathcal{E}_2[F] \leq \int_{\mathbb{S}^d} |\nabla \Pi_k F|^2 \, d\mu + \frac{d\,\eta_{k+1}}{(k+1)(k+d)} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu$$

and the constant in (2.1) is given by

$$\mathcal{C}_{d,2,k} = 1 - \frac{d\,\eta_{k+1}}{(k+1)(k+d)}.$$

In the framework of our method, this estimate of the constant cannot be improved, as shown by a test function such that $F_j = 0$ for any $j \in \mathbb{N}$ such that $j \neq 0$ and $j \neq k+1$, but again this does not prove the optimality of $\mathcal{C}_{d,p,k}$. ■

3. Improvements by the *carré du champ* method

We improve upon Frank's stability result in [32] by giving a constructive estimate based on the *carré du champ* method, without assuming any additional constraint. Various computations that are needed for a complete proof, most of them already known in the literature, are collected in Appendix B.

3.1. A simple estimate based on the heat flow, below the Bakry–Emery exponent

Let us consider the constant γ given by

$$\gamma := \left(\frac{d-1}{d+2}\right)^2 (p-1)(2^\# - p) \ \text{ if } d \geq 2, \quad \gamma := \frac{p-1}{3} \ \text{ if } d = 1, \tag{3.1}$$

where $2^\# := \frac{2d^2+1}{(d-1)^2}$ is the *Bakry–Emery exponent*. Notice that $\gamma = 2 - p$ with $1 \leq p \leq 2^\#$ means that

$$d = 1 \quad \text{and} \quad p = 7/4 = p_*(1),$$

$$d > 1 \quad \text{and} \quad p = p_*(d) := \frac{3 + d + 2d^2 - 2\sqrt{4d + 4d^2 + d^3}}{(d-1)^2}.$$

Let us define

$$s_\star := \frac{1}{p-2} \ \text{ if } p > 2 \quad \text{and} \quad s_\star := +\infty \ \text{ if } p \leq 2. \tag{3.2}$$

For any $s \in [0, s_\star)$, let

$$\varphi(s) = \begin{cases} \dfrac{1-(p-2)s-(1-(p-2)s)^{-\frac{\gamma}{p-2}}}{2-p-\gamma} & \text{if } \gamma \neq 2-p \text{ and } p \neq 2, \\ \dfrac{1}{2-p}(1+(2-p)s)\log(1+(2-p)s) & \text{if } \gamma = 2-p \neq 0, \\ \dfrac{1}{\gamma}(e^{\gamma s}-1) & \text{if } p = 2. \end{cases} \tag{3.3}$$

In [20, Theorem 2.1] (also see [21] and earlier related references therein) the *improved Gagliardo–Nirenberg inequalities*

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq d\,\varphi\left(\frac{\mathcal{E}_p[F]}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d) \tag{3.4}$$

are stated with γ given by (3.1) under the conditions

$$d \geq 1 \text{ and } 1 \leq p \leq 2^\# \ \text{ if } d \geq 2, \quad p \geq 1 \ \text{ if } d = 1.$$

Why this estimate is based on the heat flow is explained in Appendix B. Additional justifications and discussion of the case $p = 2$ are also given in Appendix B.

Since $\varphi(0) = 0$, $\varphi'(0) = 1$, and φ is convex increasing, with an asymptote at $s = s_\star$ if $p \in (2, 2^\#)$, we know that $\varphi\colon [0, s_\star) \to \mathbb{R}^+$ is invertible and $\psi\colon \mathbb{R}^+ \to [0, s_\star)$, $s \mapsto \psi(s) := s - \varphi^{-1}(s)$, is convex increasing with $\psi(0) = \psi'(0) = 0$, $\lim_{t\to+\infty}(t - \psi(t)) = s_\star$, and

$$\psi''(0) = \varphi''(0) = \frac{(d-1)^2}{(d+2)^2}(2^\# - p)(p-1) > 0 \quad \forall p \in (1, 2^\#).$$

Proposition 8. *With the above notation, $d \geq 1$ and $p \in (1, 2^\#)$, we have*

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d).$$

If $p = 2$, notice that ψ is explicit and given by

$$\psi(t) := t - \frac{1}{\gamma}\log(1+\gamma t) \quad \forall t \geq 0.$$

The proof of Proposition 2 follows from the observation that $\psi(t) \geq \frac{\gamma}{2}\frac{t^2}{1+\gamma t}$ for any $t \geq 0$.

3.2. An estimate based on the fast diffusion flow, valid up to the critical exponent

The subcritical range $p \in [2^\#, 2^*)$ corresponding to exponents between the *Bakry–Emery exponent* and the critical Sobolev exponent is not covered in Section 3.1. In that case, we rely on entropy methods based on a fast diffusion or porous medium equation of exponent m, which are detailed in Appendix B (with corresponding references), to establish that an improved inequality (3.4) holds for any $\varphi = \varphi_{m,p}$, where

$$\varphi_{m,p}(s) := \int_0^s \exp\big[-\zeta\big((1-(p-2)z)^{1-\delta} - (1-(p-2)s)^{1-\delta}\big)\big]\,dz, \tag{3.5}$$

provided $m \in \mathcal{A}_p := \mathcal{A}_p := \{m \in [m_-(d,p), m_+(d,p)] \colon \frac{2}{p} \le m < 1 \text{ if } p < 4\}$, where

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\Big(d\,p + 2 \pm \sqrt{d\,(p-1)(2\,d-(d-2)\,p)}\Big), \tag{3.6}$$

while the parameters δ and ζ are defined by

$$\delta := 1 + \frac{(m-1)\,p^2}{4\,(p-2)},$$
$$\zeta := \frac{(d+2)^2\,p^2 m^2 - 2\,p\,(d+2)(d\,p+2)\,m + d^2(5\,p^2-12\,p+8) + 4\,d\,(3-2\,p)\,p + 4}{(1-m)(d+2)^2\,p^2}.$$

Let $s_\star := 1/(p-2)$ as in (3.2) and consider the inverse function $\varphi_{m,p}^{-1}\colon \mathbb{R}^+ \to [0, s_\star)$ and $\psi_{m,p}(s) := s - \varphi_{m,p}^{-1}(s)$. Exactly as in the case $m = 1$, we have the improved entropy – entropy production inequality

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\varphi_{m,p}\left(\frac{\mathcal{E}_p[F]}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \quad \forall\,F \in \mathrm{H}^1(\mathbb{S}^d),$$

which provides us with the following stability estimate.

Proposition 9. *With the above notation, $d \ge 1$, $p \in (2, 2^*)$ and $m \in \mathcal{A}_p$, we have*

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi_{m,p}\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \quad \forall\,F \in \mathrm{H}^1(\mathbb{S}^d).$$

The function $\varphi_{m,p}$ can be expressed in terms of the *incomplete* Γ *function*, while $\psi_{m,p}$ is known only implicitly.

3.3. Comparison with other estimates

Let us assume that $p \in (2, 2^*)$. In [32], Frank proves the existence of a positive constant $\mathsf{c}_\star(d,p)$ such that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge \mathsf{c}_\star(d,p)\,\frac{\Big(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F - \bar{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\Big)^2}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \quad \forall\,F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where $\bar{F} := \int_{\mathbb{S}^d} F\,d\mu$, which in particular implies the existence of a positive constant $\mathsf{c}(d,p)$ such that

$$\begin{aligned}&\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F]\\ &\quad \ge \mathsf{c}(d,p)\,\frac{\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \quad \forall F \in \mathrm{H}^1(\mathbb{S}^d, d\mu),\end{aligned} \tag{3.7}$$

for all $p \in (2, 2^*)$. The value of the constant $\mathsf{c}_\star(d,p)$ found in [32] is unknown as it follows from a compactness argument, in the spirit of [9], but the exponent 4 in the right-hand side of (3.7) is optimal. With the test functions $F_\varepsilon(x) = 1 + \varepsilon x \cdot \nu$ for some arbitrary $\nu \in \mathbb{S}^d$, we can indeed check that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^4}\left(\|\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F_\varepsilon]\right) = \frac{(d+p)(p-1)}{2d(d+3)},$$

which gives the upper bounds

$$\mathsf{c}(d,p) \le \frac{(p-1)(d+p)}{2(p-2)(d+3)} \quad \text{and} \quad \mathsf{c}_\star(d,p) \le \frac{d^2}{(d+1)^2}\,\frac{(p-1)(d+p)}{2(p-2)(d+3)}.$$

Let us notice that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\|F - \bar{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ by the Poincaré inequality, so that we have

$$\begin{aligned}\left(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F - \bar{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2 &\ge \|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\\ &\ge \frac{d^2}{(d+1)^2}\left(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F - \bar{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2\end{aligned}$$

and, at least if $\mathsf{c}_\star(d,p)$ and $\mathsf{c}(d,p)$ are the optimal constants,

$$\frac{d^2}{(d+1)^2}\mathsf{c}(d,p) \le \mathsf{c}_\star(d,p) \le \mathsf{c}(d,p).$$

We claim that the *carré du champ* method provides us with a constructive estimate of $\mathsf{c}(d,p)$. Let

$$\phi_c(s) := \frac{d}{2(1-c)}\left(2cs - s_\star + \sqrt{s_\star^2 + 4cs(s - s_\star)}\right).$$

Corollary 10. *Let $p \in (2, 2^*)$. With the notation of Proposition 9, inequality (3.7) holds with*

$$\mathsf{c} = \sup\left\{c > 0 \colon \exists\, m \in \mathcal{A}_p \text{ such that } \phi_c(s) \le \varphi_{m,p}(s)\ \forall s \in [0, s_\star)\right\}.$$

Proof. With no loss of generality, let us assume that $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$ and define

$$\mathsf{i} = \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{and} \quad \mathsf{e} := \frac{1 - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2},$$

so that $\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1 - (p-2)\mathsf{e}$. With $c = \mathsf{c}(d,p)$, we can rewrite (3.7) as

$$\mathsf{i} - d\,\mathsf{e} \geq \frac{c\,\mathsf{i}^2}{\mathsf{i} + \frac{d}{p-2} - d\,\mathsf{e}},$$

which amounts to

$$\mathsf{i} - d\,\mathsf{e} \geq \phi_c(\mathsf{e}).$$

Since we know that $\mathsf{i} - d\,\mathsf{e} \geq \varphi_{m,p}(\mathsf{e})$, the conclusion follows for the largest possible $c > 0$ such that $\varphi_{m,p} \geq \phi_c$. ■

4. Global stability results

We collect the statements of Theorems 3 and 6 into a single result. The whole section is devoted to its proof.

Theorem 11. *Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - d\,\mathcal{E}_p[F] \geq \mathcal{S}_{d,p}\left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id} - \Pi_1)F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \tag{4.1}$$

for some explicit stability constant $\mathcal{S}_{d,p} > 0$.

The value of $\mathcal{S}_{d,p}$ is elementary and explicit but its expression is lengthy. We explain in the proof how to compute it with all necessary details to obtain a numerical expression for $\mathcal{S}_{d,p}$ for given p and d, if needed.

Proof of Theorem 11. By homogeneity of (4.1), we can assume that $\|F\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$ without loss of generality. For clarity, we subdivide the proof into various steps. Let us start with the case $p > 2$.

• *An estimate based on the carré du champ method.* If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} / \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \geq \vartheta_0 > 0$, we know by the convexity of $\psi_{m,p}$ that

$$\begin{aligned}\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] &\geq d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi_{m,p}\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\\ &\geq \frac{d}{\vartheta_0}\,\psi_{m,p}\left(\frac{\vartheta_0}{d}\right)\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.\end{aligned} \tag{4.2}$$

In that case, we conclude, from $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \|\nabla \Pi_1 F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|\nabla(\mathrm{Id} - \Pi_1)F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ and

$$\|\nabla \Pi_1 F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}.$$

Let us assume now that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0 \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$. By taking into account (GN), we obtain

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0 \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \le \vartheta_0 \Big(\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{p-2}{d} \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \Big).$$

Using $\|F\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$, under the assumption that $\vartheta_0 < d/(p-2)$, we know that

$$\vartheta := \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \frac{d\,\vartheta_0}{d-(p-2)\vartheta_0}. \tag{4.3}$$

Notice that the parameter ϑ_0 still has to be chosen.

• *An estimate of the average.* Let us estimate $\Pi_0 F := \int_{\mathbb{S}^d} F\,d\mu$. By the Poincaré inequality we have

$$1 = \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \Big(\int_{\mathbb{S}^d} F\,d\mu \Big)^2 + \|(\mathrm{Id}-\Pi_0)F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \Big(\int_{\mathbb{S}^d} F\,d\mu \Big)^2 + \frac{\vartheta}{d},$$

and on the other hand we know that $(\int_{\mathbb{S}^d} F\,d\mu)^2 \le \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$ by the Cauchy–Schwarz inequality, so that

$$\frac{d-\vartheta}{d} < \Big(\int_{\mathbb{S}^d} F\,d\mu \Big)^2 \le 1. \tag{4.4}$$

We assume in the sequel that

$$\vartheta < d. \tag{4.5}$$

• *Partial decomposition on spherical harmonics.* With no loss of generality, let us write

$$F = \mathcal{M}(1 + \varepsilon\mathcal{Y} + \eta G) \tag{4.6}$$

such that $\mathcal{M} = \Pi_0 F$ and $\Pi_1 F = \varepsilon\mathcal{M}\mathcal{Y}$, where $\mathcal{Y}(x) = \sqrt{\frac{d+1}{d}}\,x\cdot\nu$ for some given $\nu \in \mathbb{S}^d$. Here the functions $\mathcal{Y}$ and G are normalized so that $\|\nabla\mathcal{Y}\|_{\mathrm{L}^2(\mathbb{S}^d)} = \|\nabla G\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$ and

$$\mathcal{M}^{-2}\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \varepsilon^2 + \eta^2 = \vartheta \quad \text{and} \quad \mathcal{M}^{-2}\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1 + \frac{1}{d}\varepsilon^2 + \eta^2\|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.$$

We observe that $\Pi_0(F-\mathcal{M}) = 0$. Using (GN) and the Poincaré inequality, we have

$$\|F-\mathcal{M}\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \le \|F-\mathcal{M}\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{p-2}{d}\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \frac{p-1}{d}\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.$$

Similarly, by (2.1), i.e.,

$$\frac{dp}{2(d+p)}\|\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \Big(1 - \frac{2d-p(d-2)}{2(d+p)}\Big)\|\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[G],$$

and the improved Poincaré inequality (2.1), written with $p=1$ and $k=1$,

$$\|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \frac{1}{2(d+1)}\|\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \frac{1}{2(d+1)},$$

we have

$$\|G\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \le \|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{p(p-2)}{2(d+p)}\|\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le C_{p,d}$$

using $\|\nabla G\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$, with $C_{p,d} := \frac{1}{2(d+1)} + \frac{p(p-2)}{2(d+p)}$. By the Cauchy–Schwarz inequality, we also have

$$\|G\|_{\mathrm{L}^1(\mathbb{R}^d)} \le \frac{1}{\sqrt{2(d+1)}},$$

We recall that the eigenvalues of $-\Delta$ on $\mathbb{S}^d$ are $\lambda_k = k(k+d-1)$ with $k \in \mathbb{N}$. In preparation for a detailed Taylor expansion as in [32], let us consider the function

$$\mathcal{Y}(x) := \sqrt{\frac{d+1}{d}}\, x\cdot\nu,$$

which is such that $-\Delta\mathcal{Y} = \lambda_1\mathcal{Y}$ with $\lambda_1 = d$ and

$$\|\nabla\mathcal{Y}\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1, \qquad \|\mathcal{Y}\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \frac{1}{d},$$
$$\|\mathcal{Y}\|^4_{\mathrm{L}^4(\mathbb{S}^d)} = \frac{3(d+1)}{(d+3)d^2}, \quad \|\mathcal{Y}\|^6_{\mathrm{L}^6(\mathbb{S}^d)} = \frac{15(d+1)^2}{(d+3)(d+5)d^2}.$$

The function $\mathcal{Y}_2 := \mathcal{Y}^2 - \frac{1}{d}$ is such that $-\Delta\mathcal{Y}_2 = \lambda_2\mathcal{Y}_2$ with $\lambda_2 = 2(d+1)$ and

$$\|\mathcal{Y}_2\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \frac{2}{d(d+3)}, \quad \|\nabla\mathcal{Y}_2\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \frac{4(d+1)}{d(d+3)}.$$

The function $\mathcal{Y}_3 := \mathcal{Y}^3 - \frac{3(d+1)}{d(d+3)}\mathcal{Y}$ is such that $-\Delta\mathcal{Y}_3 = \lambda_3\mathcal{Y}_3$ with $\lambda_3 = 3(d+2)$ and

$$\|\mathcal{Y}_3\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \frac{6(d+1)^2}{(d+5)(d+3)^2d^2}, \quad \|\nabla\mathcal{Y}_3\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \frac{18(d+2)(d+1)^2}{(d+5)(d+3)^2d^2}.$$

As a consequence of (4.6), we know that $\Pi_0 G = \Pi_1 G = 0$ and $\|\nabla G\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$. Let

$$g_2 := \frac{\int_{\mathbb{S}^d}\nabla\mathcal{Y}_2\cdot\nabla G\,d\mu}{\|\nabla\mathcal{Y}_2\|_{\mathrm{L}^2(\mathbb{S}^d)}} \quad\text{and}\quad g_3 := \frac{\int_{\mathbb{S}^d}\nabla\mathcal{Y}_3\cdot\nabla G\,d\mu}{\|\nabla\mathcal{Y}_3\|_{\mathrm{L}^2(\mathbb{S}^d)}}.$$

With $k = 1, 2$, using $-\Delta\mathcal{Y}_k = \lambda_k\mathcal{Y}_k$ with $\lambda_k = \|\nabla\mathcal{Y}_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}/\|\mathcal{Y}_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$, we compute

$$\int_{\mathbb{S}^d}\mathcal{Y}^k G\,d\mu = \int_{\mathbb{S}^d}\mathcal{Y}_k G\,d\mu = \frac{\|\mathcal{Y}_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla\mathcal{Y}_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\int_{\mathbb{S}^d}\nabla\mathcal{Y}_k\cdot\nabla G\,d\mu = g_k\frac{\|\mathcal{Y}_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla\mathcal{Y}_k\|_{\mathrm{L}^2(\mathbb{S}^d)}}$$

and obtain

$$\int_{\mathbb{S}^d}\mathcal{Y}^2 G\,d\mu = \int_{\mathbb{S}^d}\mathcal{Y}_2 G\,d\mu = \frac{g_2}{\sqrt{d(d+1)(d+3)}},$$
$$\int_{\mathbb{S}^d}\mathcal{Y}^3 G\,d\mu = \int_{\mathbb{S}^d}\mathcal{Y}_3 G\,d\mu = c_3 g_3 \quad\text{with } c_3 := \frac{d+1}{d(d+3)}\sqrt{\frac{2}{(d+2)(d+5)}}.$$

• *Taylor expansions* (1). Let us start with elementary estimates of $\|1+\varepsilon\mathcal{Y}\|_{\mathrm{L}^p(\mathbb{S}^d)}$. If it holds that $2 \le p < 3$ and $|s|<1$, we have

$$\frac{1}{2}\big((1+s)^p+(1-s)^p\big) \le 1+\frac{p}{2}(p-1)\,s^2\Big(1+\frac{1}{12}(p-2)(p-3)\,s^2\Big)$$

because all other terms in the series expansion of the left-hand side around $s=0$ correspond to even powers of s and appear with nonpositive coefficients. If either $1 \le p < 2$ or $p > 3$ and $|s| < 1/2$, let

$$f_p(s) := \frac{1}{2}\big((1+s)^p+(1-s)^p\big)-\Big(1+\frac{p}{2}(p-1)\,s^2\Big)$$

and notice that $f_p''(s)=\frac{p}{2}(p-1)\big((1+s)^{p-2}+(1-s)^{p-2}-2\big)\ge 0$ by convexity of the function $y\mapsto y^{p-2}$, so that $c_p^{(+)}$ defined as the maximum of $s\mapsto f_p(s)/s^6$ on $[-1/2,1/2]\ni s$ is finite and we have

$$\frac{1}{2}\big((1+s)^p+(1-s)^p\big) \le 1+\frac{p}{2}(p-1)\,s^2\Big(1+\frac{1}{12}(p-2)(p-3)\,s^2\Big)+c_p^{(+)}\,s^6. \tag{4.7}$$

We adapt the convention that $c_p^{(+)}=0$ if $p\in[2,3)$. Using the fact that $\mathcal{Y}(-x)=-\mathcal{Y}(x)$,

$$\|1+\varepsilon\mathcal{Y}\|^p_{\mathrm{L}^p(\mathbb{S}^d)}=\frac{1}{2}\Big(\|1+\varepsilon\mathcal{Y}\|^p_{\mathrm{L}^p(\mathbb{S}^d)}+\|1-\varepsilon\mathcal{Y}\|^p_{\mathrm{L}^p(\mathbb{S}^d)}\Big).$$

For any $\varepsilon\in(0,1/2)$ we use (4.7) to write

$$\|1+\varepsilon\mathcal{Y}\|^p_{\mathrm{L}^p(\mathbb{S}^d)}-\Big(1+\frac{p}{2}(p-1)\Big(\|\mathcal{Y}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{1}{12}(p-2)(p-3)\,\|\mathcal{Y}\|^4_{\mathrm{L}^4(\mathbb{S}^d)}\,\varepsilon^2\Big)\varepsilon^2\Big)$$
$$\le c_p^{(+)}\,\|\mathcal{Y}\|^6_{\mathrm{L}^6(\mathbb{S}^d)}\,\varepsilon^6.$$

For similar reasons, one can prove that there is another constant $c_p^{(-)}$ which provides us with a lower bound $c_p^{(-)}\,\|\mathcal{Y}\|^6_{\mathrm{L}^6(\mathbb{S}^d)}\,\varepsilon^6$. Altogether, this amounts to

$$c_{p,d}^{(-)}\,\varepsilon^6 \le \|1+\varepsilon\mathcal{Y}\|^p_{\mathrm{L}^p(\mathbb{S}^d)}-\big(1+a_{p,d}\,\varepsilon^2+b_{p,d}\,\varepsilon^4\big)\le c_{p,d}^{(+)}\,\varepsilon^6, \tag{4.8}$$

with

$$a_{p,d}:=\frac{p\,(p-1)}{2\,d},\quad b_{p,d}:=\frac{1}{4}(p-2)(p-3)\,\frac{d+1}{d\,(d+3)}\,a_{p,d},$$
$$c_{p,d}^{(\pm)}:=\frac{15\,(d+1)^2}{(d+3)(d+5)\,d^2}\,c_p^{(\pm)}.$$

Estimate (4.8) is valid under the condition that $\varepsilon<1/2$. We shall therefore request that

$$\vartheta<\frac{1}{4}, \tag{4.9}$$

which is an obvious sufficient condition according to (4.3). Now we draw two consequences of (4.8). First, let us give an upper estimate of $\|1+\varepsilon\mathcal{Y}\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$. Using

$$(1+s)^{\frac{2}{p}} \le 1 + 2\frac{s}{p} - (p-2)\frac{s^2}{p^2} + \frac{2}{3}(p-1)(p-2)\frac{s^3}{p^3},$$

we obtain

$$\|1+\varepsilon\mathcal{Y}\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \le 1 + \frac{2}{p}a_{p,d}\,\varepsilon^2 + \frac{1}{p^2}\left(2p\,b_{p,d} - (p-2)a_{p,d}^2\right)\varepsilon^4 + r^{(+)}\varepsilon^6, \tag{4.10}$$

where the remainder term $r^{(+)}$ is explicitly estimated by

$$\begin{aligned} 96p^3 r^{(+)} = {} & 64a_{p,d}^3(p^2-3p+2) + 48a_{p,d}^2(p^2-3p+2)(2b_{p,d}+c_{p,d}) \\ & + 12a_{p,d}(p-2)(2b_{p,d}+c_{p,d})(2b_{p,d}(p-1)+c_{p,d}(p-1)-8p) \\ & + 8b_{p,d}^3(p^2-3p+2) + 12b_{p,d}^2(p-2)(c_{p,d}(p-1)-4p) \\ & + 6b_{p,d}c_{p,d}(p-2)(c_{p,d}(p-1)-8p) \\ & + c_{p,d}\big(c_{p,d}^2(p^2-3p+2) - 12c_{p,d}(p-2)p + 192p^2\big). \end{aligned}$$

To do this estimate, we simply write that $\varepsilon^\alpha \le 2^{6-\alpha}\varepsilon^6$ for any $\alpha > 6$ using the (nonoptimal) bound $\varepsilon^2 < 1/2$. Similarly, using

$$\begin{aligned} (1+s)^{\frac{2}{p}-1} \le {} & 1 - (p-2)\frac{s}{p} + (p-1)(p-2)\frac{s^2}{p^2} - \frac{1}{3}(p-1)(p-2)(3p-2)\frac{s^3}{p^3} \\ & + \frac{1}{6}(p-1)(p-2)(3p-2)(2p-1)\frac{s^4}{p^4}, \end{aligned}$$

we obtain

$$\|1+\varepsilon\mathcal{Y}\|^{2-p}_{\mathrm{L}^p(\mathbb{S}^d)} \le 1 + \frac{p-2}{p}a_{p,d}\,\varepsilon^2 - \frac{p-2}{p^2}\left(p\,b_{p,d} - (p-1)a_{p,d}^2\right)\varepsilon^4 + r^{(-)}\varepsilon^6, \tag{4.11}$$

where the remainder term $r^{(-)}$ also has an explicit expression in terms of $a_{p,d}$, $b_{p,d}$ and $c^{(-)}_{p,d}$, which is not given here.

• *Taylor expansions* (2). With $u \ge 0$, $u + r \ge 0$ and $p > 2$, we claim that

$$(u+r)^p \le u^p + p\,u^{p-1}r + \frac{p}{2}(p-1)u^{p-2}r^2 + \sum_{2<k<p} C^p_k\,u^{p-k}|r|^k + K_p|r|^p$$

for some constant $K_p > 0$, where the coefficients

$$C^p_k := \frac{\Gamma(p+1)}{\Gamma(k+1)\Gamma(p-k+1)}$$

are the binomial coefficients if p is an integer. It is proved in [19] that $K_p = 1$ if $p \in (2,4] \cup \{6\}$. The proof is similar to the above analysis and is left to the reader. Let us integrate this inequality and raise both sides to the power $2/p$ to get

$$\|u + r\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \le \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} (1+s)^{\frac{2}{p}},$$

with

$$s = \frac{1}{\|u\|^p_{\mathrm{L}^p(\mathbb{S}^d)}} \Bigg(p \int_{\mathbb{S}^d} u^{p-1} r \, d\mu + \frac{p}{2}(p-1) \int_{\mathbb{S}^d} u^{p-2} r^2 \, d\mu \\ + \sum_{2<k<p} C^p_k \int_{\mathbb{S}^d} u^{p-k} |r|^k \, d\mu + K_p \int_{\mathbb{S}^d} |r|^p \, d\mu \Bigg).$$

By assumption $2/p < 1$ so that we may use the identity $(1+s)^{2/p} \le 1 + 2s/p$ for any $s \ge -1$. Notice that we can assume that $u + r \ge 0$ and deduce from (1) that $s \ge -1$. As a consequence, we have

$$\|u + r\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \le \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \\ + \frac{2}{p} \|u\|^{2-p}_{\mathrm{L}^p(\mathbb{S}^d)} \Bigg(p \int_{\mathbb{S}^d} u^{p-1} r \, d\mu + \frac{p}{2}(p-1) \int_{\mathbb{S}^d} u^{p-2} r^2 \, d\mu \\ + \sum_{2<k<p} C^p_k \int_{\mathbb{S}^d} u^{p-k} |r|^k \, d\mu + K_p \int_{\mathbb{S}^d} |r|^p \, d\mu \Bigg).$$

We apply these computations to $u = 1 + \varepsilon \mathcal{Y}$ and $r = \eta G$ to obtain

$$\mathcal{M}^{-2} \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|1 + \varepsilon \mathcal{Y}\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \\ \le \frac{2}{p} \|1 + \varepsilon \mathcal{Y}\|^{2-p}_{\mathrm{L}^p(\mathbb{S}^d)} \eta \Bigg(p \int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-1} G \, d\mu \\ + \frac{p}{2}(p-1)\eta \int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-2} |G|^2 \, d\mu \\ + \sum_{2<k<p} C^p_k \eta^k \int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-k} |G|^k \, d\mu \\ + K_p \eta^p \int_{\mathbb{S}^d} |G|^p \, d\mu \Bigg).$$

Let us detail the expansion of each of the terms involving G in the right-hand side of this estimate. For any $s \in (-1/2, 1/2)$, using the expansion

$$(1+s)^{p-1} \le 1 + (p-1)s + \frac{1}{2}(p-1)(p-2)s^2 \\ + \frac{1}{6}(p-1)(p-2)(p-3)s^3 + R_p s^4$$

for some constant $R_p > 0$ applied with $s = 1 + \varepsilon \mathcal{Y}$, we obtain

$$\eta \int_{\mathbb{S}^d} (1+\varepsilon\mathcal{Y})^{p-1} G\,d\mu \le \frac{1}{2}(p-1)(p-2)\frac{g_2}{\sqrt{d(d+1)(d+3)}}\eta\varepsilon^2 + \frac{1}{6}(p-1)(p-2)(p-3)c_3 g_3 \eta\varepsilon^3 + \frac{R_p\eta\varepsilon^4}{\sqrt{2(d+1)}}.$$

The other terms admit simpler expansions:

$$\eta^2 \int_{\mathbb{S}^d} (1+\varepsilon\mathcal{Y})^{p-2}|G|^2\,d\mu \le \eta^2(1+\varepsilon)^{p-2}\|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le \|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\eta^2 + \frac{1}{2(d+1)}\eta^2\left((1+\varepsilon)^{p-2}-1\right)$$

and

$$\sum_{2<k<p} C_k^p \eta^k \int_{\mathbb{S}^d} (1+\varepsilon\mathcal{Y})^{p-k}|G|^k\,d\mu + K_p\eta^p\int_{\mathbb{S}^d}|G|^p\,d\mu \le \sum_{2<k<p} C_k^p\eta^k(1+\varepsilon)^{p-k}\|G\|^k_{\mathrm{L}^p(\mathbb{S}^d)} + K_p\eta^p\|G\|^p_{\mathrm{L}^p(\mathbb{S}^d)} \le \sum_{2<k<p} C_k^p\eta^k(1+\varepsilon)^{p-k}C_{p,d}^{k/p} + K_p\eta^p C_{p,d}\,.$$

Collecting (4.10) and (4.11) with the above estimates, we arrive at

$$\begin{aligned}
&\mathcal{M}^{-2}\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\\
&\le 1 + \frac{2}{p}a_{p,d}\varepsilon^2 + \frac{1}{p^2}\left(2pb_{p,d}-(p-2)a_{p,d}^2\right)\varepsilon^4 + r^{(+)}\varepsilon^6\\
&\quad + \left(1+\frac{p-2}{p}a_{p,d}\varepsilon^2 - \frac{p-2}{p^2}\left(pb_{p,d}-(p-1)a_{p,d}^2\right)\varepsilon^4 + r^{(-)}\varepsilon^6\right)\\
&\quad\cdot\Bigg[(p-1)(p-2)\frac{g_2}{\sqrt{d(d+1)(d+3)}}\eta\varepsilon^2 + \frac{1}{3}(p-1)(p-2)(p-3)c_3g_3\eta\varepsilon^3\\
&\qquad + \frac{2R_p\eta\varepsilon^4}{\sqrt{2(d+1)}} + (p-1)\left(\|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\eta^2 + \frac{1}{2(d+1)}\eta^2\left((1+\varepsilon)^{p-2}-1\right)\right)\\
&\qquad + \frac{2}{p}\left(\sum_{2<k<p} C_k^p\eta^k(1+\varepsilon)^{p-k}C_{p,d}^{k/p} + K_p\eta^pC_{p,d}\right)\Bigg].
\end{aligned}$$

Using $|g_2|<1$, $|g_3|<1$, and $2(d+1)\|G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}<1$, this gives rise to an explicit although lengthy expression for a positive constant $\mathcal{R}_{p,d}$ such that

$$\mathcal{M}^{-2}\left(\int_{\mathbb{S}^d}|\nabla F|^2\,d\mu - d\,\mathcal{E}_p[F]\right) \ge A\varepsilon^4 - B\varepsilon^2\eta + C\eta^2 - \mathcal{R}_{p,d}\left(\vartheta^p + \vartheta^{5/2}\right),$$

with $A := \frac{(p-1)(d+p)}{2d(d+3)}$, $B := \frac{d(p-1)}{\sqrt{d(d+1)(d+3)}}$ and $C := \frac{d+2}{2(d+1)}$. The discriminant

$$B^2 - 4AC = -\frac{1}{d(d+3)}(p-1)(2d - p(d-2))$$

is negative if (and only if) $p \in (1, 2^*)$, so that we can write

$$As^2 - Bs + C = (A-\lambda)s^2 - Bs + (C-\lambda) + \lambda(s^2+1) \geq \lambda(s^2+1),$$

where

$$\lambda := \frac{1}{2}\big(A + C + \sqrt{(A-C)^2 + B^2}\big)$$

is given by the condition that $B^2 - 4(A-\lambda)(C-\lambda) = 0$. Altogether, we obtain

$$\mathcal{M}^{-2}\left(\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F]\right) \geq \lambda(\varepsilon^4 + \eta^2) - \mathcal{R}_{p,d}(\vartheta^p + \vartheta^{5/2}).$$

• *Conclusion if $p > 2$.* We choose $\vartheta > 0$ such that (4.5) and (4.9) are fulfilled. With the additional assumption that

$$\vartheta \leq \vartheta_{p,d} := \big\{\theta > 0 \colon \mathcal{R}_{p,d}(\theta^p + \theta^{5/2}) = \tfrac{\lambda}{4}\theta^2\big\},$$

using $\eta^4 \leq \eta^2$ and $2\varepsilon^2\eta^2 \leq \varepsilon^4 + \eta^2$ if $\eta < 1$, we have

$$\mathcal{R}_{p,d}(\vartheta^p + \vartheta^{5/2}) \leq \frac{\lambda}{4}\vartheta^2 = \frac{\lambda}{4}(\varepsilon^2 + \eta^2)^2 \leq \frac{\lambda}{2}(\varepsilon^4 + \eta^2) \leq \frac{\lambda}{2}\Big(\frac{\varepsilon^4}{\varepsilon^2 + \eta^2 + 1} + \eta^2\Big).$$

For any F such that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \vartheta$, we obtain

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \geq \frac{\lambda}{2}\left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id} - \Pi_1)F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right).$$

Using (4.2) and (4.4), this completes the proof of Theorem 11 if $p > 2$ with

$$\vartheta \leq \min\Big\{\frac{d}{2}, \frac{1}{4}, \vartheta_{p,d}\Big\} = \frac{d\,\vartheta_0}{d - (p-2)\vartheta_0} \quad \text{and} \quad \mathcal{S}_{d,p} = \min\Big\{\frac{d}{\vartheta_0}\psi_{m,p}\Big(\frac{\vartheta_0}{d}\Big), \frac{\lambda}{2}\Big\}.$$

• *The case $p \leq 2$.* The strategy is the same, with some simplifications, so we only sketch the proof and emphasize the changes compared to the case $p > 2$. Let us notice that

$$(1+s)^p \leq 1 + ps + \frac{p}{2}(p-1)s^2 \quad \text{if } 1 \leq p < 2$$

and $(1+s)^2 \log((1+s)^2) \leq 2s + 2s^2 + \frac{2}{3}s^3$ in the limit case $p = 2$. The estimates involving $1 + \varepsilon\mathcal{Y}$ are therefore essentially the same if we assume $\varepsilon < 1/2$, while the computation of $\|u + r\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$ is in fact simpler, when applied to $u = 1 + \varepsilon\mathcal{Y}$ and $r = \eta G$. The estimate on the average is simplified because $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} \leq \|F\|_{\mathrm{L}^2(\mathbb{S}^d)}$ by Hölder's inequality, since $d\mu$ is a probability measure on $\mathbb{S}^d$. Spectral estimates are exactly the same and the Taylor expansions present no additional difficulty, as we can use (GN) for some exponent $q \in (2, 2^*)$ to control the remainder terms if $p = 2$, so that $(1+s)^2 \log((1+s)^2) \leq 2s + 2s^2 + \kappa_q s^q$ for some $\kappa_q > 0$. The conclusion is the same as for $p > 2$ except that we have to replace $\psi_{m,p}$ by ψ defined as in Proposition 8. ∎

A. Improved Gaussian inequalities, hypercontractivity and stability

Whether the results of Theorems 1, 4 and 7 can be extended to the Euclidean case with the Gaussian measure is a very natural question. Spherical harmonics can indeed be replaced by Hermite polynomials and there is a clear correspondence for spectral estimates. The answer is yes for a whole family of interpolation inequalities, but it is no for the logarithmic Sobolev inequality, which is an endpoint of the family.

Let us consider the *normalized Gaussian measure* on $\mathbb{R}^d$ defined by

$$d\sigma(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2}\, dx.$$

For any $p \in [1,2)$, Beckner [6] established the family of interpolation inequalities

$$\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\sigma)}}{2-p} \le \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} \quad \forall f \in \mathrm{H}^1(\mathbb{R}^d, d\sigma). \tag{A.1}$$

With $p = 1$, inequality (A.1) is the *Gaussian Poincaré inequality*, while one recovers the Gaussian logarithmic Sobolev inequality of [35] in the limit as $p \to 2$. For any $p \in [1,2)$, the inequality is optimal: using $f_\varepsilon := 1 + \varepsilon\varphi$ as a test function, where φ is such that $\int_{\mathbb{R}^d} \varphi\, d\sigma = 0$, we recover the *Gaussian Poincaré inequality* with optimal constant in the limit as $\varepsilon \to 0$, so that the constant in (A.1) cannot be improved. Based on [1, 43], the improved version of the inequality

$$\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\sigma)}}{2-p} \le \frac{p}{2}\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} \quad \forall f \in \mathrm{H}^1(\mathbb{R}^d, d\sigma) \tag{A.2}$$

holds under the additional condition

$$\int_{\mathbb{R}^d} x f(x)\, d\sigma = 0. \tag{A.3}$$

Let us give a short proof of (A.2). Assume that $f = \sum_{k\in\mathbb{N}} f_k$ is a decomposition on Hermite functions such that $\mathcal{L} f_k = -k f_k$ where $\mathcal{L} = \Delta - x\cdot\nabla$ is the Ornstein–Uhlenbeck operator, and let $a_k := \|f_k\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)}$ for any $k \in \mathbb{N}$, so that

$$\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} = \sum_{k\in\mathbb{N}} a_k \quad \text{and} \quad \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} = \sum_{k\in\mathbb{N}} k\, a_k.$$

Let us consider the solution of

$$\frac{\partial u}{\partial t} = \mathcal{L} u \tag{A.4}$$

with initial datum $u(t=0,\cdot) = f$ and notice that

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} = \sum_{k\in\mathbb{N}} a_k e^{-2kt}.$$

Hence, if (A.3) holds, $a_1 = 0$ and

$$\begin{aligned} \|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} &= \sum_{k\ge 2} a_k (1 - e^{-2kt}) \\ &\le \frac{1}{2}(1 - e^{-4t}) \sum_{k\in\mathbb{N}} k a_k \\ &= \frac{1}{2}(1 - e^{-4t}) \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)}, \end{aligned} \tag{A.5}$$

because $k \mapsto (1 - e^{-2kt})/k$ is monotone nonincreasing for any given $t \ge 0$. Next we use Nelson's hypercontractivity estimate in [48, Theorem 3] to find $t_* > 0$ such that

$$\|u(t_*,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} \le \|f\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\sigma)}.$$

As noted in [35], this estimate can be seen as a consequence of the *Gaussian logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)}}\right) d\sigma \le 2 \int_{\mathbb{R}^d} |\nabla v|^2 \, d\sigma \quad \forall v \in \mathrm{H}^1(\mathbb{R}^d, d\sigma), \tag{A.6}$$

and the argument goes as follows. With $h(t) := \|u(t,\cdot)\|_{\mathrm{L}^{q(t)}(\mathbb{R}^d,d\sigma)}$ for some exponent q depending on t and u solving (A.4), we have

$$\frac{h'}{h} = \frac{q'}{q^2} \int_{\mathbb{R}^d} \frac{|u|^q}{h^q} \log\left(\frac{|u|^q}{h^q}\right) d\sigma - \frac{4}{h^q} \frac{q-1}{q^2} \int_{\mathbb{R}^d} |\nabla(|u|^{q/2})|^2 \, d\sigma \le 0$$

by (A.6) applied to $v = |u|^{q/2}$, if $t \mapsto q(t)$ solves the ordinary differential equation

$$q' = 2(q-1).$$

With $q(0) = p < 2$, we obtain $q(t) = 1 + (p-1)e^{2t}$ and find that Nelson's time t_* is determined by the condition $q(t_*) = 2$ which means $e^{-2t_*} = p - 1$. Replacing $t = t_*$ in (A.5) completes the proof of (A.2), which can be recast in the form of a stability result for (A.1).

Theorem 12. *Let $d \ge 1$ and $p \in [1,2)$. For any $f \in \mathrm{H}^1(\mathbb{R}^d, d\sigma)$ such that (A.3) holds,*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \frac{1}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\sigma)}\right) \ge \frac{2-p}{2} \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)}.$$

As a by-product of the proof, with $t = t_*$ in (A.5), we have the mode-by-mode interpolation inequality

$$\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\sigma)}}{2-p} \le \sum_{k\ge 1} \frac{1-(p-1)^k}{k(2-p)} \|\nabla f_k\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} \quad \forall f \in \mathrm{H}^1(\mathbb{R}^d, d\sigma),$$

without imposing condition (A.3), for any $p \in [1,2)$. For any $k \ge 1$,

$$\lim_{p\to 2_-} \frac{1-(p-1)^k}{k(2-p)} = \lim_{p\to 2_-} \frac{1-(1-(2-p))^k}{k(2-p)} = 1,$$

so that no improvement should be expected by this method. This is very similar to the case of the critical exponent on the sphere of dimension $d \ge 3$. In this sense $p = 2$ is the critical case in the presence of a Gaussian weight, as *all modes* are equally involved in the estimate of the constant. This is a limitation of the method which does not forbid a stability result for (A.6), to be established by other methods.

Let us conclude this appendix with some bibliographic comments on the literature on inequality (A.1), for the Gaussian measure. The analogue of Proposition 5 in the Gaussian case is known from [2]; also see [26, Section 2.5]. Assuming that not only condition (A.3) is satisfied, but also orthogonality conditions with all modes up to order $k_0 \ge 2$, then an improvement of the order of

$$\frac{1-(p-1)^{k_0}}{k_0(2-p)}$$

can be achieved for inequality (A.1), which is the counterpart of Theorem 4 in the Gaussian case. This has been studied in [43] but we can refer to [1] for a more abstract setting and later papers, e.g., to [53, 55] for results on compact manifolds and generalizations involving weights. For an overview of interpolation between Poincaré and logarithmic Sobolev inequalities from the point of view of Markov processes, and for some spectral considerations, we refer to [54, Chapter 6]. Notice that hypercontractivity appears as one of the main motivations of the founding paper [3] of the *carré du champ* method.

B. *Carré du champ* method and improved inequalities

For the sake of completeness, we collect various results of [20–22, 24] and draw some new consequences. Computations similar to those of Section B.1 can be found in [10] for the study of rigidity results in elliptic equations. For nonlinear parabolic flows, also see [17, 18]. Other sections of this appendix collect results which are scattered in the literature, but additional details needed in Section 3 are given, for instance a sketch of the proof of Proposition 16 or the computations in the case $p = 2$.

B.1. Algebraic preliminaries

Let us denote the *Hessian* by $\mathrm{H}v$ and define the *trace-free Hessian* by

$$\mathrm{L}v := \mathrm{H}v - \frac{1}{d}(\Delta v) g_d\,.$$

We also consider the trace-free tensor

$$\mathrm{M}v := \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{d}\frac{|\nabla v|^2}{v} g_d\,,$$

where $(\nabla v \otimes \nabla v)_{ij} := \partial_i v \partial_j v$ and $\|\nabla v \otimes \nabla v\|^2 = |\nabla v|^4 = (g_d^{ij} \partial_i v \partial_j v)^2$ using Einstein's convention. Using

$$\mathrm{L} : g_d = 0, \quad \mathrm{M} : g_d = 0,$$

where $a : b$ denotes $a^{ij} b_{ij}$ and $\|a\|^2 := a : a$, and

$$\|\mathrm{L}v\|^2 = \|\mathrm{H}v\|^2 - \frac{1}{d}(\Delta v)^2,$$
$$\|\mathrm{M}v\|^2 = \left\|\frac{\nabla v \otimes \nabla v}{v}\right\|^2 - \frac{1}{d}\frac{|\nabla v|^4}{v^2} = \frac{d-1}{d}\frac{|\nabla v|^4}{v^2},$$

we deduce from

$$\begin{aligned}\int_{\mathbb{S}^d} \Delta v \frac{|\nabla v|^2}{v}\, d\mu &= \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2}\, d\mu - 2\int_{\mathbb{S}^d} \mathrm{H}v : \frac{\nabla v \otimes \nabla v}{v}\, d\mu \\ &= \frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{M}v\|^2\, d\mu - 2\int_{\mathbb{S}^d} \mathrm{L}v : \frac{\nabla v \otimes \nabla v}{v}\, d\mu \\ &\quad - \frac{2}{d}\int_{\mathbb{S}^d} \Delta v \frac{|\nabla v|^2}{v}\, d\mu\end{aligned}$$

a first identity that reads

$$\int_{\mathbb{S}^d} \Delta v \frac{|\nabla v|^2}{v}\, d\mu = \frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{M}v\|^2\, d\mu - 2\int_{\mathbb{S}^d} \mathrm{L}v : \frac{\nabla v \otimes \nabla v}{v}\, d\mu\right). \tag{B.1}$$

The Bochner–Lichnerowicz–Weitzenböck formula on $\mathbb{S}^d$ takes the simple form

$$\frac{1}{2}\Delta(|\nabla v|^2) = \|\mathrm{H}v\|^2 + \nabla(\Delta v)\cdot\nabla v + (d-1)|\nabla v|^2,$$

where the last term, i.e., $\mathrm{Ric}(\nabla v, \nabla v) = (d-1)|\nabla v|^2$, accounts for the Ricci curvature tensor contracted with $\nabla v \otimes \nabla v$. An integration of this formula on $\mathbb{S}^d$ shows a second identity,

$$\int_{\mathbb{S}^d} (\Delta v)^2\, d\mu = \frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{L}v\|^2\, d\mu + d\int_{\mathbb{S}^d} |\nabla v|^2\, d\mu. \tag{B.2}$$

Hence,

$$\begin{aligned}\mathcal{K}[v] &:= \int_{\mathbb{S}^d} \left(\Delta v + \kappa\frac{|\nabla v|^2}{v}\right)\left(\Delta v + (\beta-1)\frac{|\nabla v|^2}{v}\right) d\mu \\ &= \int_{\mathbb{S}^d} (\Delta v)^2\, d\mu + (\kappa+\beta-1)\int_{\mathbb{S}^d} \Delta v\frac{|\nabla v|^2}{v}\, d\mu + \kappa(\beta-1)\int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2}\, d\mu\end{aligned}$$

can be rewritten using (B.1) and (B.2) as

$$\begin{aligned}\mathcal{K}[v] = \frac{d}{d-1}&\int_{\mathbb{S}^d} \|\mathrm{L}v\|^2\, d\mu + d\int_{\mathbb{S}^d} |\nabla v|^2\, d\mu \\ &+ (\kappa+\beta-1)\frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{M}v\|^2\, d\mu - 2\int_{\mathbb{S}^d} \mathrm{L}v : \mathrm{M}v\, d\mu\right) \\ &+ \kappa(\beta-1)\frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{M}v\|^2\, d\mu\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{d-1} \int_{\mathbb{S}^d} \left(\|\mathrm{L}v\|^2 - 2b\,\mathrm{L}v : \mathrm{M}v + c\|\mathrm{M}v\|^2 \right) d\mu + d \int_{\mathbb{S}^d} |\nabla v|^2 \, d\mu \\
&= \frac{d}{d-1} \int_{\mathbb{S}^d} \left(\|\mathrm{L}v - b\mathrm{M}v\|^2 + (c-b^2)\|\mathrm{M}v\|^2 \right) d\mu + d \int_{\mathbb{S}^d} |\nabla v|^2 \, d\mu \\
&= \frac{d}{d-1} \int_{\mathbb{S}^d} \|\mathrm{L}v - b\mathrm{M}v\|^2 \, d\mu + (c-b^2) \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} \, d\mu + d \int_{\mathbb{S}^d} |\nabla v|^2 \, d\mu,
\end{aligned}
$$

where

$$
b = (\kappa + \beta - 1) \frac{d-1}{d+2} \quad \text{and} \quad c = \frac{d}{d+2}(\kappa + \beta - 1) + \kappa(\beta - 1).
$$

Let $\kappa = \beta(p-2) + 1$. The condition $\gamma := c - b^2 \geq 0$ amounts to

$$
\gamma = \frac{d}{d+2}\beta(p-1) + (1 + \beta(p-2))(\beta - 1) - \left(\frac{d-1}{d+2}\beta(p-1) \right)^2, \qquad \text{(B.3)}
$$

where $\gamma = -(A\beta^2 - 2B\beta + C)$ with

$$
A = \left(\frac{d-1}{d+2}(p-1) \right)^2 + 2 - p, \quad B = \frac{d+3-p}{d+2} \quad \text{and} \quad C = 1.
$$

A necessary and sufficient condition for the existence of a β such that $\gamma \geq 0$ is that the reduced discriminant is nonnegative, which amounts to

$$
B^2 - AC = \frac{4d(d-2)}{(d+2)^2}(p-1)(2^* - p) \geq 0.
$$

Summarizing, we have the following result, which can be found in [22] for a general manifold with positive Ricci curvature.

Lemma 13. *With the above notation, for any smooth function v on $\mathbb{S}^d$, we have*

$$
\mathcal{K}[v] \geq \gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} \, d\mu + d \int_{\mathbb{S}^d} |\nabla v|^2 \, d\mu
$$

for some $\gamma > 0$ given in terms of β by (B.3) if $p \in (1, 2^*)$.

Notice that we recover the expression for γ in (3.1) if we take $\beta = 1$. The case $p = 2$ does not add any difficulty compared to $p \neq 2$.

B.2. Diffusion flow and monotonicity

Assume that u is a positive solution of

$$
\frac{\partial u}{\partial t} = u^{-p(1-m)} \left(\Delta u + (mp - 1) \frac{|\nabla u|^2}{u} \right). \qquad \text{(B.4)}
$$

In the linear case $m=1$, u^p solves the heat equation. Otherwise we deal with the nonlinear case either of a fast diffusion flow with $m<1$ or of a solution of the porous media equation with $m>1$. We claim that

$$\frac{d}{dt}\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}=0 \quad\text{and}\quad \frac{d}{dt}\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=2(p-2)\int_{\mathbb{S}^d}u^{-p(1-m)}|\nabla u|^2\,d\mu.$$

Let us assume that the parameters β and m are related by

$$m=1+\frac{2}{p}\Big(\frac{1}{\beta}-1\Big). \tag{B.5}$$

If v is a function such that $u=v^\beta$, then v solves

$$\frac{\partial v}{\partial t}=v^{2-2\beta}\Big(\Delta v+\kappa\frac{|\nabla v|^2}{v}\Big),$$

with $\kappa=\beta(p-2)+1$, and as a consequence we find that

$$\frac{d}{dt}\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=2(p-2)\beta^2\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu.$$

Similarly, we find that

$$\frac{d}{dt}\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=-2\int_{\mathbb{S}^d}\Big(\beta v^{\beta-1}\frac{\partial v}{\partial t}\Big)(\Delta v^\beta)\,d\mu=-2\beta^2\mathcal{K}[v]. \tag{B.6}$$

By eliminating β in (B.3) using (B.5), we obtain

$$\gamma=\frac{\gamma_0+\gamma_1 d+\gamma_2 d^2}{(d+2)^2(2-p(1-m))^2}, \tag{B.7}$$

with $\gamma_0=4(mp-1)^2$, $\gamma_1=-4p(m-3+p(2-m)(1+m))$, $\gamma_2=(m^2-2m+5)p^2-12p+8$. The condition $\gamma\ge0$ determines the range $m_-(d,p)\le m\le m_+(d,p)$ of *admissible* parameters m, where $m_\pm(d,p)$ is given by (3.6). Summarizing, we have the following result (also see [22]).

Lemma 14. *Assume that $p\in(1,2^*)$ and $m\in[m_-(d,p),m_+(d,p)]$. If u solves (B.4), then we have*

$$\frac{1}{2\beta^2}\frac{d}{dt}\Big(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[u]\Big)\le-\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu, \tag{B.8}$$

where $v=u^{1/\beta}$ with β and γ given in terms of m by (B.3) and (B.7) respectively.

Notice that the case of the linear flow corresponds to the case $m=\beta=1$ and $v=u$.

Proof of Lemma 14. For a smooth solution, the result follows from (B.6) and Lemma 13. The result for a general solution is obtained by standard regularization procedures. ∎

B.3. Interpolation

Depending on the value of p, we shall consider various interpolation inequalities. Let us define

$$\delta := \frac{2-(4-p)\beta}{2\beta(p-2)} \text{ if } p>2, \quad \delta := 1 \text{ if } p \in [1,2]. \tag{B.9}$$

Lemma 15. *If one of the conditions*

(i) $p \in (1, 2^\#)$ *and* $\beta = 1$ *(so that* $\delta = 1$*),*

(ii) $p \in (1, 2^*)$*,* $\beta > 1$*, and* $\beta \le 2/(4-p)$ *if* $p > 4$*,*

is satisfied, then $u = v^\beta$ *is such that*

$$\int_{\mathbb{S}^d} \frac{|\nabla v|^4}{|v|^2}\,d\mu \ge \frac{1}{\beta^2}\,\frac{\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu \int_{\mathbb{S}^d}|\nabla v|^2\,d\mu}{\left(\int_{\mathbb{S}^d}|u|^2\,d\mu\right)^\delta \left(\int_{\mathbb{S}^d}|u|^p\,d\mu\right)^{\frac{\beta-1}{\beta(p-2)}}}. \tag{B.10}$$

Case (ii) was originally proved in [17, 18] and we refer to [21] for a proof in the case of the ultraspherical operator.

Proof of Lemma 15. In case (i), $v = u$ and inequality (B.10) is a consequence of the Cauchy–Schwarz inequality

$$\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu = \int_{\mathbb{S}^d}\frac{|\nabla v|^2}{v}\cdot v\,d\mu \le \left(\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu\right)^{\frac12}\left(\int_{\mathbb{S}^d}|u|^2\,d\mu\right)^{\frac12},$$

Cases (i) and (ii) follow from two Hölder inequalities.

(1) With $\frac12 + \frac{\beta-1}{2\beta} + \frac{1}{2\beta} = 1$, we deduce from

$$\begin{aligned}\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu &= \int_{\mathbb{S}^d}\frac{|\nabla v|^2}{v}\cdot 1\cdot v\,d\mu \\ &\le \left(\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu\right)^{\frac12}\left(\int_{\mathbb{S}^d}1\,d\mu\right)^{\frac{\beta-1}{2\beta}}\cdot\left(\int_{\mathbb{S}^d}|u|^2\,d\mu\right)^{\frac{1}{2\beta}},\end{aligned}$$

and the assumption that $d\mu$ is a probability measure, the first estimate

$$\left(\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu\right)^{\frac12} \ge \frac{\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu}{\left(\int_{\mathbb{S}^d}|u|^2\,d\mu\right)^{\frac{1}{2\beta}}}.$$

(2) With $\frac12 + \frac{\beta-1}{\beta(p-2)} + \frac{2-(4-p)\beta}{2\beta(p-2)} = 1$, Hölder's inequality shows that

$$\begin{aligned}\frac{1}{\beta^2}\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu &= \int_{\mathbb{S}^d} v^{2(\beta-1)}|\nabla v|^2\,d\mu = \int_{\mathbb{S}^d}\frac{|\nabla v|^2}{v}\cdot v^{\frac{p(\beta-1)}{p-2}}\cdot v^{2\beta\delta}\,d\mu \\ &\le \left(\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu\right)^{\frac12}\left(\int_{\mathbb{S}^d}|u|^p\,d\mu\right)^{\frac{\beta-1}{\beta(p-2)}}\left(\int_{\mathbb{S}^d}|u|^2\,d\mu\right)^{\delta},\end{aligned}$$

from which we deduce the second estimate

$$\left(\int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2}\,d\mu\right)^{\frac{1}{2}} \geq \frac{1}{\beta^2}\,\frac{\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu}{\left(\int_{\mathbb{S}^d} |u|^2\,d\mu\right)^{\delta}\left(\int_{\mathbb{S}^d} |u|^p\,d\mu\right)^{\frac{\beta-1}{\beta(p-2)}}}.$$

The combination of our two estimates proves (B.10). ■

Using (B.5), condition (ii) in Lemma 15 is changed into the condition that $2/p \leq m < 1$ and we may notice as in [17, 18] that it is always satisfied if we choose $\beta = 4/(6-p)$ corresponding to an *admissible* fast diffusion exponent $m = (p+2)/(2p)$, for any $p \in (2, 2^*)$. By "admissible", one should understand $m_-(d,p) \leq m \leq m_+(d,p)$, so that γ is nonnegative. With the choice of $m = (p+2)/(2p)$, we find $\delta = 1/4$.

B.4. Improved functional inequalities

Let us denote the *entropy* and the *Fisher information* respectively by

$$\mathsf{e} := \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \text{and} \quad \mathsf{i} := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)},$$

and let γ and δ be given respectively by (B.3) and (B.9). Up to the replacement of u by $u/\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}$, with no loss of generality, we shall assume that

$$\|u\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1.$$

We learn from (B.8) and (B.10) that

$$(\mathsf{i} - d\,\mathsf{e})' \leq \frac{\gamma\,\mathsf{i}\,\mathsf{e}'}{\beta^2(1-(p-2)\mathsf{e})^{\delta}}. \tag{B.11}$$

Solving the ordinary differential equation in the equality case of (B.11) is equivalent to solving

$$\frac{d}{dt}(\mathsf{i} - d\,\varphi(\mathsf{e})) = \frac{\gamma}{\beta^2}\,\frac{\mathsf{e}'}{(1-(p-2)\mathsf{e})^{\delta}}(\mathsf{i} - d\,\varphi(\mathsf{e})),$$

where φ solves

$$\varphi'(s) = 1 + \frac{\gamma}{\beta^2}\,\frac{\varphi(s)}{(1-(p-2)s)^{\delta}}. \tag{B.12}$$

The reader is invited to check that the solution of (B.12) with initial datum $\varphi(0) = 0$ is given by (3.3) if $m = 1$ and by (3.5) with $\zeta = 2\gamma/(\beta(1-\beta))$ in the nonlinear case. We learn from (B.11) that

$$(\mathsf{i} - d\,\varphi(\mathsf{e}))' \leq \frac{\gamma}{\beta^2}\,\frac{\mathsf{e}'}{(1-(p-2)\mathsf{e})^{\delta}}(\mathsf{i} - d\,\varphi(\mathsf{e})).$$

This is enough to prove the following result.

Proposition 16. *With the above notation, we claim that*

$$\mathsf{i} \geq d\,\varphi(\mathsf{e}).$$

Proof. Let us give the scheme of a proof. Let $\tilde\gamma := \gamma/\beta^2$ in order to simplify notation. We can argue as follows:

(1) $\mathsf{i}' + 2\,d\,\mathsf{i} = (\mathsf{i} - d\,\mathsf{e})' \le 0$ shows that

$$0 \le \mathsf{i}(t) \le \mathsf{i}(0)e^{-2\,dt}$$

and in particular $\lim_{t\to+\infty} \mathsf{i}(t) = 0$.

(2) As $t \to +\infty$, e converges to a constant, hence $\lim_{t\to+\infty} \mathsf{e}(t) = 0$.

(3) From (B.11), we learn that

$$(\mathsf{i} - d\,\mathsf{e})' \le d\,\tilde\gamma\,\mathsf{e}\mathsf{e}' = \frac{1}{2}d\,\tilde\gamma\,(\mathsf{e}^2)',$$

where the inequality follows from $1 - (p-2)\mathsf{e} \le 1$ and $\mathsf{i} \ge d\,\mathsf{e}$.

(4) It follows from $(\mathsf{i} - d\,\mathsf{e})' \le 0$ that $\mathsf{i} \ge d\,\mathsf{e}$ using an integration from any $t \ge 0$ to $+\infty$.

(5) Unless u is a constant, we read from $(\mathsf{i} - d\,\mathsf{e})' \le \frac{1}{2}\tilde\gamma\,d\,(\mathsf{e}^2)'$ that $\mathsf{i} - d\,\mathsf{e} > \frac{1}{2}\tilde\gamma\,d\,\mathsf{e}^2$, using again an integration from any $t \ge 0$ to $+\infty$.

(6) Take some $\vartheta \in (0,1)$ and consider the solution of

$$\bar\varphi'(s) = 1 + \frac{\vartheta\tilde\gamma\bar\varphi(s)}{(1-(p-2)s)^\delta}, \quad \bar\varphi(0) = 0. \tag{B.13}$$

In the spirit of (B.11), we have a following chain of elementary estimates:

$$(\mathsf{i} - d\vartheta\,\bar\varphi(\mathsf{e}))' \le (\mathsf{i} - d\bar\varphi(\mathsf{e}))' + d(1-\vartheta)(\bar\varphi(\mathsf{e}))' \le (\mathsf{i} - d\bar\varphi(\mathsf{e}))'$$

and obtain

$$(\mathsf{i} - d\vartheta\,\bar\varphi(\mathsf{e}))' \le \frac{\tilde\gamma\mathsf{e}'}{(1-(p-2)\mathsf{e})^\delta}(\mathsf{i} - d\vartheta\,\bar\varphi(\mathsf{e})). \tag{B.14}$$

We know that $\bar\varphi(0) = 0$ and read from (B.13) that $\bar\varphi'(0) = 1$ and $\bar\varphi''(0) = \vartheta\tilde\gamma\bar\varphi'(0) = \vartheta\tilde\gamma$ so that $\bar\varphi(\mathsf{e}) - \mathsf{e} \sim \frac{1}{2}\vartheta\tilde\gamma\mathsf{e}^2$ as $\mathsf{e} \to 0$. Using $\mathsf{i} - d\,\mathsf{e} > \frac{1}{2}\tilde\gamma d\,\mathsf{e}^2$, we learn that

$$\mathsf{i} - d\bar\varphi(\mathsf{e}) \ge \frac{1}{2}\tilde\gamma d(1-\vartheta)\mathsf{e}^2(1 + O(\mathsf{e}))$$

for $\mathsf{e} = \mathsf{e}(t)$ small enough, i.e., for $t > 0$ large enough.

(7) It is simple to check from (B.14) that $\mathsf{i} - d\vartheta\,\bar\varphi(\mathsf{e})$ cannot change sign.

(8) We conclude as above that $\mathsf{i} - d\vartheta\,\bar\varphi(\mathsf{e}) \ge 0$ using an integration from any $t \ge 0$ to $+\infty$.

(9) Finally, we consider the limit as $\vartheta \to 1_-$.

Altogether, we conclude that $\mathsf{i} \ge d\varphi(\mathsf{e})$, where φ solves (B.12). This completes the scheme of the proof of Proposition 16. ∎

Funding. This work has been partially supported by Project EFI (ANR-17-CE40-0030) of the French National Research Agency (ANR). G.B. has been funded by the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement no. 754362.

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Received 23 November 2022; revised 30 September 2023; accepted 6 October 2023.

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