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On the existence of uniformly bounded self-adjoint bases in GNS spaces

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Abstract. The Gelfand–Naimark–Segal (GNS) space of a diffuse finite von Neumann algebra with respect to a faithful normal tracial state admits an orthonormal basis consisting of the image inside the GNS space of a uniformly bounded sequence of self-adjoint operators.

1. Introduction

In this paper, all Hilbert spaces are separable, all von Neumann algebras have separable preduals, inclusions of subalgebras are unital and inner products are linear in the left variable. The orthounitary basis problem for II_1 factors is an important and open problem in the subject [4]. This problem has an affirmative solution in the context of group von Neumann algebras of countable discrete groups by construction and finite von Neumann algebras that arise from group measure space construction. In general, nothing is known.

In the classical case, it is well known that the Walsh functions, which take the values ± 1, form an orthonormal basis for $L^2([0,1],\lambda)$, where λ denotes the Lebesgue measure. Note that the multiplication operators associated with the Walsh functions are self-adjoint unitaries. In general, if μ is a non-atomic probability measure (on a standard Borel space), then $L^2(\mu)$ contains an orthonormal basis consisting of images of (self-adjoint) unitaries from $L^\infty(\mu)$.

In the context of von Neumann algebras, it is therefore natural to ask how much the aforesaid classical result still holds for GNS spaces of states given the interplay between the Hilbert norm and operator norm. Given that one cannot leverage much on the algebraic structure of the ambient algebra in general, a compromise with the operator norms of the left multiplication operators associated with an orthonormal basis is forced. Thus, we ask: does the GNS space of a von Neumann algebra with respect to a faithful normal state contains an orthonormal basis consisting of images of a uniformly bounded sequence of self-adjoint operators?

In this paper, we answer this question completely. We first observe that such an orthonormal basis is only possible when the underlying state is tracial. Following [6], we demonstrate that the GNS space of a diffuse finite von Neumann algebra with respect to

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any faithful normal tracial state admits such a basis with the associated self-adjoint operators being bounded (in operator norm) by $(1+\sqrt{2})+\varepsilon$ for every $\varepsilon>0$ (Theorem 3.2). The conspicuous presence of $(1+\sqrt{2})$ in the bound above is due to the effort to control the operator norms by using the nature of Haar transformation matrices. For a more detailed study of uniformly bounded orthonormal bases we refer the interested reader to [2]. This paper has overlap of ideas with [2].

The layout of the paper is as follows: In Section 2, we collect all technical results that are required to address the problem. Section 3 is the main section of this paper. In Section 3, following [6], we provide a necessary and sufficient condition for the existence of a uniformly bounded self-adjoint orthonormal basis in a closed subspace of the GNS space (see Theorem 3.1). Theorem 3.2 is the main result of this paper. In Theorem 3.2, we show that for every $\varepsilon>0$, the GNS space of a diffuse finite von Neumann algebra with respect to a faithful normal tracial state admits an orthonormal basis consisting of a uniformly bounded sequence of self-adjoint operators whose operator norm is bounded by $(1+\sqrt{2})+\varepsilon$.

Finally in Theorem 3.4, we analyze how one can extend a uniformly bounded self-adjoint basis of the GNS space of a subalgebra to a uniformly bounded self-adjoint basis of the GNS space of the original von Neumann algebra.

Now, we record some basic facts which are essential for our purpose. This material is standard and can be found in [8–10].

Let M be a von Neumann algebra equipped with a faithful normal state φ. Let M be represented on the GNS space $\mathcal{H}_\varphi := L^2(M,\varphi)$ in standard form [3]. The inner product and norm on $\mathcal{H}_\varphi$ are denoted by $\langle\cdot,\cdot\rangle_\varphi$ and $\|\cdot\|_{2,\varphi}$ respectively. The operator norm on $\mathbf{B}(\mathcal{H}_\varphi)$ is denoted by $\|\cdot\|$. The self-adjoint part of M is denoted by $M_{\text{s.a.}}$. Let M_1 and $(M_{\text{s.a.}})_1$ denote the unit ball of M and $M_{\text{s.a.}}$ respectively.

Let Ω_φ denote the standard vacuum vector associated with φ. Then,

$$\mathfrak{A}_\varphi = M\Omega_\varphi$$

becomes a (full) left Hilbert algebra endowed with the scalar product induced by $\mathcal{H}_\varphi$, and endowed with the product and involution $\sharp$ respectively given by $(x\Omega_\varphi)(y\Omega_\varphi)=xy\Omega_\varphi$ and $(x\Omega_\varphi)^\sharp = x^*\Omega_\varphi$, $x,y\in M$.

Let S_φ denote the closure of the (densely defined) antilinear operator

$$\mathfrak{A}_\varphi \ni x\Omega_\varphi \mapsto x^*\Omega_\varphi \in \mathcal{H}_\varphi.$$

Let $S_\varphi = J_\varphi\Delta_\varphi^{\frac{1}{2}}$ be the polar decomposition of S_φ. Then the Tomita's modular operator Δ_φ is nonsingular, positive and self-adjoint, and the conjugation operator J_φ is an anti-unitary. It is well-known that φ is a trace if and only if $\Delta_\varphi = 1$ and then $S_\varphi = J_\varphi$.

Suppose τ is a faithful normal tracial state on a finite von Neumann algebra M. Then, for $x,y\in M_{\text{s.a.}}$, one has

$$\langle x\Omega_\tau, y\Omega_\tau\rangle_\tau = \tau(yx) = \tau(xy) = \langle y\Omega_\tau, x\Omega_\tau\rangle_\tau = \overline{\langle x\Omega_\tau, y\Omega_\tau\rangle_\tau} \in \mathbb{R}. \quad (1.1)$$

Thus, $M_{\text{s.a.}}\Omega_\tau \subseteq \mathcal{H}_\tau$ is a real subspace of $\mathcal{H}_\tau$. Moreover, $\operatorname{span}_{\mathbb{C}} M_{\text{s.a.}}\Omega_\tau$ is dense in $\mathcal{H}_\tau$.

2. Technical results

In this section, we discuss technical results that are required for the existence of a uniformly bounded self-adjoint basis in the GNS Hilbert space of a von Neumann algebra with respect to a faithful normal state. We begin with the following result, which shows that orthonormal bases consisting of images of self-adjoint operators are only possible for finite von Neumann algebras.

Theorem 2.1. *Let M be a von Neumann algebra equipped with a faithful normal state φ. Suppose the GNS space $\mathcal{H}_\varphi$ admits an orthonormal basis $\mathcal{O} \subseteq M_{s.a.}\Omega_\varphi$. Then, M is finite and φ is a trace.*

Proof. Let $\{x_n\Omega_\varphi\} \subseteq M_{\text{s.a.}}\Omega_\varphi$ be an orthonormal basis of $\mathcal{H}_\varphi$. Fix $y \in M$ and $k \in \mathbb{N}$. Set

$$\xi = \sum_{n=1}^{k} \langle y\Omega_\varphi, x_n\Omega_\varphi\rangle_\varphi x_n\Omega_\varphi. \tag{2.1}$$

Then, $\xi \in \mathfrak{D}(S_\varphi)$ and

$$\begin{aligned} S_\varphi\xi &= \sum_{n=1}^{k} \overline{\langle y\Omega_\varphi, x_n\Omega_\varphi\rangle_\varphi} S_\varphi x_n\Omega_\varphi \\ &= \sum_{n=1}^{k} \overline{\langle y\Omega_\varphi, x_n\Omega_\varphi\rangle_\varphi} x_n^*\Omega_\varphi \\ &= \sum_{n=1}^{k} \overline{\langle y\Omega_\varphi, x_n\Omega_\varphi\rangle_\varphi} x_n\Omega_\varphi. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), one has

$$\|S_\varphi\xi\|_{2,\varphi}^2 = \sum_{n=1}^{k} \left|\langle y\Omega_\varphi, x_n\Omega_\varphi\rangle_\varphi\right|^2 = \|\xi\|_{2,\varphi}^2.$$

Since $\{\xi = \sum_{n=1}^{k} \langle y\Omega_\varphi, x_n\Omega_\varphi\rangle_\varphi x_n\Omega_\varphi : k \in \mathbb{N},\ y \in M\}$ is dense in $\mathcal{H}_\varphi$, it follows that S_φ extends to a anti-unitary operator on $\mathcal{H}_\varphi$. Consequently, $S_\varphi = J_\varphi$ and $\Delta_\varphi = 1$. This forces that M is finite and φ is tracial. ∎

In the view of Theorem 2.1, from here onwards unless otherwise stated, we assume that M is finite von Neumann algebra equipped with a faithful normal tracial state τ. Further, $\mathcal{H}$ will denote a closed and infinite-dimensional subspace of $\mathcal{H}_\tau$ unless otherwise stated.

Now we prove auxiliary technical lemmas that enable one to renorm M and control both $\|\cdot\|$ and $\|\cdot\|_{2,\tau}$ of operators in M. The following lemma appears in [6], but we describe it for the sake of convenience.

Lemma 2.2 ([6]). *Let $\mathcal{H}_0$ be an inner product space. For $n \geq 0$, if $\xi_0, \xi_1, \ldots, \xi_{2^n-1}$, are pairwise orthogonal in $\mathcal{H}_0$, then there exists a real unitary matrix $(a^n_{k,j}) \in M_{2^n}(\mathbb{R})$ such that if $\zeta_k = \sum_{j=0}^{2^n-1} a^n_{k,j}\xi_j$, $0 \leq k \leq 2^n - 1$, then,*

(1) *ζ_k, $0 \leq k \leq 2^n - 1$, are pairwise orthogonal in $\mathcal{H}_0$;*

(2) $\operatorname{span}_{\mathbb{R}}\{\zeta_k : 0 \leq k \leq 2^n - 1\} = \operatorname{span}_{\mathbb{R}}\{\xi_k : 0 \leq k \leq 2^n - 1\}$*;*

(3) *if $|||\cdot|||$ is any norm on $\mathcal{H}_0$, then*

$$\max_{0\leq k\leq 2^n-1} |||\zeta_k||| < (1+\sqrt{2}) \max_{1\leq k\leq 2^n-1} |||\xi_k||| + 2^{-n/2}|||\xi_0|||.$$

Proof. Conditions (1) and (2) in the statement are true for all $U \in \mathcal{U}(M_{2^n}(\mathbb{R}))$. So we need to exhibit only condition (3) in the statement.

Define

$$a^n_{k,j} = \begin{cases} 2^{-n/2}, & \text{if } j = 0 \text{ and } 0 \leq k \leq 2^n - 1, \\ 2^{(s-n)/2}, & \text{if } j = 2^s + r \text{ and } 2^{n-s-1}2r \leq k \leq 2^{n-s-1}(2r+1) - 1, \\ -2^{(s-n)/2}, & \text{if } j = 2^s + r \text{ and } 2^{n-s-1}(2r+1) \leq k \leq 2^{n-s-1}(2r+2) - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $s = 0, 1, \ldots, n-1$, and $r = 0, 1, \ldots, 2^s - 1$. The matrix $(a^n_{k,j})$ is called the *Haar transformation matrix* and transforms the standard basis of $\mathbb{R}^{2^n}$ to the Haar basis [5]. Observe that

$$\sum_{j=1}^{2^n-1} |a^n_{k,j}| = \sum_{s=0}^{n-1} 2^{-(n-s)/2} < 1 + \sqrt{2}, \quad \text{for } 0 \leq k \leq 2^n - 1.$$

Consequently, the result follows. ■

Lemma 2.3. *Let M be a finite von Neumann algebra equipped with a faithful normal tracial state τ. Let $\{x_n\Omega_\tau\} \subseteq M_{s.a.}\Omega_\tau$ be orthonormal in $\mathcal{H}_\tau$. Suppose*

$$\liminf_n \|x_n\| = C < \infty.$$

Then, for any $\varepsilon > 0$, there exists a sequence of operators $\{y_n\} \subseteq M_{s.a.}$ such that

(1) $\sup_n \|y_n\| < (1+\sqrt{2})C + \varepsilon$*;*

(2) $\operatorname{span}_{\mathbb{R}}\{x_n : n \in \mathbb{N}\} = \operatorname{span}_{\mathbb{R}}\{y_n : n \in \mathbb{N}\}$*;*

(3) $\langle y_n\Omega_\tau, y_m\Omega_\tau\rangle_\tau = \delta_{n,m}$ *for all $n, m \in \mathbb{N}$.*

Proof. Note that there is nothing to prove if $\sup_n \|x_n\| \leq (1+\sqrt{2})C$. In that case, one can take $y_n = x_n$, for all n.

Let $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon/2(2+\sqrt{2})$. The hypothesis guarantees that there exists a subsequence $\{x_{n_k}\} \subseteq M_{\text{s.a.}}$ such that $\sup_k \|x_{n_k}\| \leq C + \varepsilon' < \infty$. Write $I = \{n_k : k \in \mathbb{N}\}$. Let $I' = \mathbb{N} \setminus I$ and write

$$I' = \{l_k : k \in \mathbb{N}\}$$

with $l_k < l_{k+1}$ for all k. For $k \in \mathbb{N}$, inductively choose $m_k \in \mathbb{N}$ such that $\|x_{l_k}\| < \varepsilon' 2^{\frac{m_k}{2}}$ and $m_k \leq m_{k+1}$. Partition $\mathbb{N} = \sqcup_{k\in\mathbb{N}} I_k$ with $|I_k| = 2^{m_k}$ as follows:

$$I_1 = \{l_1, n_1, \ldots, n_{2^{m_1}-1}\},$$

$$I_k = \{l_k, n_{2^{m_1}+\cdots+2^{m_{k-1}}-k+2}, \ldots, n_{2^{m_1}+\cdots+2^{m_k}-k}\}, \quad k \geq 2.$$

Equip $M_{\text{s.a.}}\Omega_\tau \subseteq \mathcal{H}_\tau$ with operator norm, i.e., define $\|x\Omega_\tau\|' = \|x\|$, $x \in M_{\text{s.a.}}$. For each $k \in \mathbb{N}$, apply Lemma 2.2 to the vectors $\{x_s\Omega_\tau : s \in I_k\}$ to find $\{y_s\Omega_\tau \in M_{\text{s.a.}}\Omega_\tau : s \in I_k\}$ such that $\bigcup_k \{y_s : s \in I_k\}$ has the desired properties. ∎

Lemma 2.4. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{H}_\tau$ intersecting $M_{s.a.}\Omega_\tau$ nontrivially. Let $\{x_n\Omega_\tau\} \subseteq M_{s.a.}\Omega_\tau$ be a normalized sequence in $\mathcal{H}$ such that $x_n\Omega_\tau \xrightarrow{w} 0$ in $\mathcal{H}_\tau$. Suppose that $\sup_n \|x_n\| = C < \infty$. Let $\varepsilon > 0$. Then, for every finite-dimensional subspace $\mathfrak{F}\Omega_\tau$ of $\mathcal{H} \cap \mathfrak{A}_\tau$ with $\mathfrak{F}$ spanned by self-adjoint operators in M and $k > 0$, there exist $n_0 > k$ and $y \in M_{s.a.}$ such that the following hold:*

(i) $y\Omega_\tau \in (\mathcal{H} \cap \mathfrak{A}_\tau) \cap (\mathfrak{F}\Omega_\tau)^\perp$;

(ii) $\operatorname{span}_{\mathbb{R}}\{\mathfrak{F} \cup \{x_{n_0}\}\} = \operatorname{span}_{\mathbb{R}}\{\mathfrak{F} \cup \{y\}\}$;

(iii) $\|y\Omega_\tau\|_{2,\tau} = 1$;

(iv) $\|y\| < C + \varepsilon 2^{-k}$.

Proof. Let $m = \dim \mathfrak{F}$. Let $z_1\Omega_\tau, \ldots, z_m\Omega_\tau$ be an orthonormal basis of $\mathfrak{F}\Omega_\tau$ with $z_j \in M_{\text{s.a.}}$ for all $1 \leq j \leq m$. Such self-adjoint basis exist by applying the Gram–Schmidt process and using (1.1) on a linearly independent spanning set of $\mathfrak{F}$ consisting of self-adjoint elements. Choose $\delta > 0$ such that

$$0 < \frac{C + \delta \sum_{j=1}^m \|z_j\|}{1 - \delta m} < C + \varepsilon 2^{-k}. \tag{2.3}$$

As $x_n\Omega_\tau \xrightarrow{w} 0$, there exists $n_0 > k$ such that $|\langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau| < \delta$ for $1 \leq j \leq m$ and $x_{n_0}\Omega_\tau \notin \mathfrak{F}\Omega_\tau$ (as $dim(\mathfrak{F}) < \infty$). Let

$$y = \frac{x_{n_0} - \sum_{j=1}^m \langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau z_j}{\left\|x_{n_0}\Omega_\tau - \sum_{j=1}^m \langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau z_j\Omega_\tau\right\|_{2,\tau}}. \tag{2.4}$$

Consequently, y is self-adjoint (see (1.1)), and y satisfies (i), (ii), and (iii) in the statement.

For (iv), observe that

$$\left\|x_{n_0} - \sum_{j=1}^m \langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau z_j\right\| \leq \|x_{n_0}\| + \left\|\sum_{j=1}^m \langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau z_j\right\|$$
$$\leq C + \delta \sum_{j=1}^m \|z_j\|. \tag{2.5}$$

And by reverse triangle inequality,

$$\left\| x_{n_0}\Omega_\tau - \sum_{j=1}^{m} \langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau z_j\Omega_\tau \right\|_{2,\tau} \geq \left| \|x_{n_0}\Omega_\tau\|_{2,\tau} - \left\| \sum_{j=1}^{m} \langle x_{n_0}\Omega_\tau, z_j\Omega_\tau\rangle_\tau z_j\Omega_\tau \right\|_{2,\tau} \right| \geq 1 - \delta m. \tag{2.6}$$

Therefore, from (2.3), (2.4), (2.5), and (2.6), it follows that $\|y\| \leq C + \varepsilon 2^{-k}$. ■

Lemma 2.5. *Let $\mathcal{H} \subseteq \mathcal{H}_\tau$ be a subspace such that $\mathcal{H} \cap (M_{s.a.})_1\Omega_\tau$ is not a totally bounded subset of $\mathcal{H}_\tau$. Then, there exists a normalized sequence $\{x_n\Omega_\tau\} \subseteq \mathcal{H} \cap M_{s.a.}\Omega_\tau$ in $\mathcal{H}$ such that $\sup_n \|x_n\| < \infty$ and $x_n \xrightarrow{w^*} 0$.*

Proof. The proof of this result follows verbatim from [2, Lemma 3.6]. ■

3. Uniformly bounded self-adjoint basis

Now we are in a position to generalize a classical result of Ovsepian and Pełczyński [6] on the existence of a uniformly bounded self-adjoint orthonormal basis in GNS spaces.

Theorem 3.1 (Uniformly bounded self-adjoint basis). *Let $\mathcal{H} \subseteq \mathcal{H}_\tau$ be a subspace. Then $\mathcal{H}$ admits an orthonormal basis consisting of the image inside $\mathcal{H}_\tau$ of a uniformly bounded sequence of self-adjoint operators in M if and only if,*

(1) *$\mathrm{span}_{\mathbb{C}}\{\mathcal{H} \cap M_{s.a.}\Omega_\tau\}$ is dense in $\mathcal{H}$ in $\|\cdot\|_{2,\tau}$;*

(2) *$\mathcal{H} \cap (M_{s.a.})_1\Omega_\tau$ is not a totally bounded subset of $\mathcal{H}_\tau$.*

Moreover, if $\mathcal{H}$ satisfies the condition (1) and if there exists a normalized sequence $\{x_n\Omega_\tau\} \subseteq M_{s.a.}\Omega_\tau \cap \mathcal{H}$ such that $\sup_n \|x_n\| = C < \infty$ and $x_n \xrightarrow{w^*} 0$, then given $\varepsilon > 0$, $\mathcal{H}$ admits an orthonormal basis consisting of the image inside $\mathcal{H}_\tau$ of a uniformly bounded sequence of self-adjoint operators in M bounded by $(1 + \sqrt{2})C + \varepsilon$.

Proof. First, we assume that $\mathcal{H}$ satisfies conditions (1) and (2) in the statement. We show that $\mathcal{H}$ admits an orthonormal basis consisting of the image inside $\mathcal{H}_\tau$ of a uniformly bounded sequence of self-adjoint operators in M. Let

$$\mathcal{K} = \overline{\mathcal{H} \cap M_{\text{s.a.}}\Omega_\tau}^{\|\cdot\|_{2,\tau}}.$$

Then, $\mathcal{K}$ becomes a real Hilbert space. We first show that $\mathcal{K}$ admits an orthonormal basis $\mathcal{O} \subseteq M_{\text{s.a.}}\Omega_\tau$ such that $\sup_{x\Omega_\tau \in \mathcal{O}} \|x\| < \infty$.

Note that condition (1) in the hypothesis implies that there exists an increasing sequence $\mathfrak{F}_m\Omega_\tau \subseteq \mathcal{K} \cap M_{\text{s.a.}}\Omega_\tau$ ($\mathfrak{F}_m \subseteq M_{\text{s.a.}}$), $m = 1, 2, \ldots,$ of finite-dimensional subspaces of $\mathcal{K}$ such that $\dim(\mathfrak{F}_m\Omega_\tau) = m$ for all m and $\bigcup_m \mathfrak{F}_m\Omega_\tau$ is dense in $\mathcal{K}$.

Again, by virtue of Lemma 2.5, condition (2) in the hypothesis implies that there exists a sequence

$$\{x_n\Omega_\tau\} \subseteq \mathcal{K} \cap M_{\text{s.a.}}\Omega_\tau$$

such that $C = \sup_n \|x_n\| < \infty$, $\|x_n\Omega_\tau\|_{2,\tau} = 1$ for all n, and $x_n \xrightarrow{w^*} 0$.

Fix $\varepsilon > 0$. Choose $\varepsilon' > 0$ such that $(2+\sqrt{2})\varepsilon' \leq \frac{\varepsilon}{2}$ holds. Now for $\varepsilon' > 0$, we will inductively define an orthonormal sequence $y_n\Omega_\tau \in \mathcal{K} \cap M_{\text{s.a.}}\Omega_\tau$, $n \in \mathbb{N}$, such that for any $m = 1, 2, \ldots,$

$$\mathfrak{F}_m\Omega_\tau \subseteq \mathcal{H}_{2m-1}, \quad \|y_{2m}\| < C + \varepsilon' 2^{-m}, \tag{3.1}$$

where $\mathcal{H}_m = \operatorname{span}_{\mathbb{R}}\{y_j\Omega_\tau : 1 \leq j \leq m\}$.

To start with, pick $y_1 \in \mathfrak{F}_1$ such that $\|y_1\Omega_\tau\|_{2,\tau} = 1$. Suppose that for each k with $1 \leq k \leq n-1$, we have chosen $y_k \in M_{\text{s.a.}}$ that satisfies (3.1) and that

$$\langle y_i\Omega_\tau, y_j\Omega_\tau\rangle_\tau = \delta_{i,j}$$

for all $1 \leq i, j \leq n-1$. In the n-th step, we analyze two cases separately depending on the dimension of $\mathfrak{F}_n$.

Case 1: $n = 2m$ for some $m \in \mathbb{N}$. In this case, apply Lemma 2.4 with $\mathfrak{F}\Omega_\tau = \operatorname{span}_{\mathbb{C}} \mathcal{H}_{n-1}$ and $k = m$ and $\{x_l\Omega_\tau\}$ to extract $y \in M_{\text{s.a.}}$ satisfying the statement of Lemma 2.4. Put $y_n = y$. Thus, $\|y_n\| < C + \varepsilon' 2^{-m}$.

Case 2: $n = 2m-1$ for some $m \geq 2$. If $\mathfrak{F}_m\Omega_\tau \subseteq \mathcal{H}_{n-1}$, then define $y_n = y$ (as before), where y is obtained by applying Lemma 2.4 to $\mathfrak{F}\Omega_\tau = \operatorname{span}_{\mathbb{C}} \mathcal{H}_{n-1}$, $k = 1$ and $\{x_l\Omega_\tau\}$.

Now we work inside the real Hilbert space $\mathcal{K}$ and repeatedly use (1.1). If $\mathfrak{F}_m\Omega_\tau \nsubseteq \mathcal{H}_{n-1}$, then there exists $z \in M_{\text{s.a.}}$ such that $z\Omega_\tau \in \mathfrak{F}_m\Omega_\tau \setminus \mathcal{H}_{n-1}$ linearly independent of $\mathfrak{F}_{m-1}\Omega_\tau$. Let $\tilde{z} \in M_{\text{s.a.}}$ be such that $\tilde{z}\Omega_\tau$ is the orthogonal projection of $z\Omega_\tau$ from $\mathcal{K}$ onto $\mathcal{H}_{n-1}$. Put

$$y_n = \frac{z - \tilde{z}}{\|(z-\tilde{z})\Omega_\tau\|_{2,\tau}}.$$

Clearly, $\|y_n\Omega_\tau\|_{2,\tau} = 1$ and $y_n\Omega_\tau \in \mathcal{H}_{n-1}^{\perp}$. By induction hypothesis,

$$\mathfrak{F}_{m-1}\Omega_\tau \subseteq \mathcal{H}_{2m-3} = \mathcal{H}_{n-2} \subseteq \mathcal{H}_{n-1}$$

holds.

Note that $y_n\Omega_\tau \in (\mathfrak{F}_{m-1}\Omega_\tau)^{\perp}$. Also, by construction $y_n\Omega_\tau \in \mathfrak{F}_m\Omega_\tau \ominus \mathfrak{F}_{m-1}\Omega_\tau$. As $\mathfrak{F}_{m-1} \subseteq \mathfrak{F}_m$, so $\mathfrak{F}_m\Omega_\tau = \mathfrak{F}_{m-1}\Omega_\tau \oplus \mathbb{R}y_n\Omega_\tau$. Consequently, we have

$$\mathfrak{F}_m\Omega_\tau \subseteq \mathcal{H}_n = \mathcal{H}_{2m-1}.$$

Therefore, $\{y_n\}$ can be constructed by induction such that (3.1) is satisfied for all m.

Clearly, $\{y_n\Omega_\tau\}$ is an orthonormal basis of $\mathcal{K}$. Note that $\{y_{2n}\}$ satisfies the hypothesis of Lemma 2.3. Now apply Lemma 2.3 to get the desired orthonormal basis $\mathcal{O} \subseteq M_{\text{s.a.}}\Omega_\tau$ of $\mathcal{K}$ bounded by

$$(1+\sqrt{2})(C+\varepsilon') + \varepsilon' < (1+\sqrt{2})C + \varepsilon.$$

Clearly, $\mathcal{O}$ is an orthonormal basis of $\mathcal{H}$.

The converse is obvious, as $\mathcal{H}$ is infinite-dimensional. The proof of the last statement is contained in the above argument. ■

The following theorem is the main result of this paper. Here, we discuss the existence of a uniformly bounded self-adjoint orthonormal basis in the GNS Hilbert space of a diffuse finite von Neumann algebra with respect to a faithful normal tracial state.

Theorem 3.2. *Let M be a diffuse finite von Neumann algebra equipped with a faithful normal tracial state τ. Then for any $\varepsilon > 0$, $\mathcal{H}_\tau$ admits an orthonormal basis consisting of the image inside the GNS space of a uniformly bounded sequence of self-adjoint operators in M bounded by $(1+\sqrt{2})+\varepsilon$.*

Proof. Fix $\varepsilon > 0$. First note that $\text{span}_{\mathbb{C}}\, M_{\text{s.a.}}\Omega_\tau$ is dense in $\mathcal{H}_\tau$ in $\|\cdot\|_{2,\tau}$. Let $\mathcal{A}$ be a masa in M. Then, $\mathcal{A} \simeq L^\infty([0,1],\mu)$ and $\tau_{1_{\mathcal{A}}} = \mu$, where μ is the Lebesgue measure on $[0,1]$ (see [7, Theorem 3.5.2 and Corollary 3.5.3]). Recall that the Rademacher system $\{r_n\}_{n\in\mathbb{N}}$ on $[0,1]$ is defined by

$$r_n(t) = \text{sign}\sin(2^n\pi t), \quad t \in [0,1].$$

It is well known that the Rademacher system $\{r_n\}_{n\in\mathbb{N}}$ forms an orthonormal set in $L^2([0,1],\mu)$. Since r_n is ± 1 valued, by functional calculus, r_n can be realized as a self-adjoint unitary in $\mathcal{A}$. Thus, the associated operator u_n corresponding to r_n is self-adjoint unitary in M and $u_n \xrightarrow{w^*} 0$. In other words, $\|u_n\Omega_\tau\|_{2,\tau} = 1$ for all n, and $\sup_n\|u_n\| = \sup_n\|r_n\|_\infty = 1$. Consequently, by Theorem 3.1, there exists an orthonormal basis $\mathcal{O} \subseteq M_{\text{s.a.}}\Omega_\tau$ of $\mathcal{H}_\tau$ such that

$$\sup_{x\Omega_\tau\in\mathcal{O}} \|x\| < (1+\sqrt{2})+\varepsilon.$$

This completes the proof. ■

In general, we have the following result.

Theorem 3.3. *Let M be a finite von Neumann algebra with a diffuse central summand and equipped with a faithful normal tracial state τ. Then, $\mathcal{H}_\tau$ admits an orthonormal basis consisting of uniformly bounded sequence of self-adjoint operators.*

Proof. By Theorem 3.1, it is enough to show that $(M_{\text{s.a.}})_1\Omega_\tau$ is not a totally bounded subset of $\mathcal{H}_\tau$. Let B be a diffuse central summand of M. Let $p \in \mathcal{Z}(M)$ be the central projection in M such that $Mp = B$. Set $\tau_{1_B} = \tau(\cdot p)$ on B. By scaling if necessary, one can assume that τ_{1_B} is a faithful normal tracial state on B. By Theorems 3.2 and 3.1, it follows that $(B_{\text{s.a.}})_1\Omega_{\tau_{1_B}}$ is not a totally bounded subset of $L^2(B,\tau_{1_B})$. Since $(B_{\text{s.a.}})_1\Omega_{\tau_{1_B}} \subseteq (M_{\text{s.a.}})_1\Omega_\tau$, and $L^2(B,\tau_{1_B}) \subseteq \mathcal{H}_\tau$, we have $(M_{\text{s.a.}})_1\Omega_\tau$ is not a totally bounded subset of $\mathcal{H}_\tau$. This completes the proof. ■

Finally, one can extend a uniformly bounded self-adjoint orthonormal basis of a closed subspace to that of the full GNS space when the closed subspace arises as the GNS space of a diffuse subalgebra. This is the content of the following theorem.

Theorem 3.4. *Let M be a finite von Neumann algebra equipped with a faithful normal tracial state τ. Let $B \subseteq M$ be a subalgebra of M. Then the following hold:*

(i) *Suppose $L^2(B, \tau_{\restriction B})$ admits an orthonormal basis consisting of a sequence of uniformly bounded self-adjoint operators in B bounded by k. Then given $\varepsilon > 0$, $\mathcal{H}_\tau$ admits an orthonormal basis consisting of a uniformly bounded sequence of self-adjoint operators in M bounded by $(1 + \sqrt{2})k + \varepsilon$.*

(ii) *Suppose B is diffuse. Then, $L^2(B, \tau_{\restriction B})$ admits a uniformly bounded self-adjoint orthonormal basis. Moreover, the aforesaid orthonormal basis has an extension to an orthonormal basis $\mathcal{O} \subseteq M_{s.a.}\Omega_\tau$ of $\mathcal{H}_\tau$ such that*

$$\sup_{x\Omega_\tau \in \mathcal{O}} \|x\| < \infty.$$

Proof. Note that $\tau_{\restriction B}$ is a faithful normal tracial state on B and $L^2(B, \tau_{\restriction B})$ is identified with $\overline{B\Omega_\tau}^{\|\cdot\|_{2,\tau}} \subseteq \mathcal{H}_\tau$.

(i) Fix $\varepsilon > 0$. Let $\{x_n\Omega_\tau\} \subseteq B_{\text{s.a.}}\Omega_\tau$ be an orthonormal basis of $L^2(B, \tau_{\restriction B})$ such that $\sup_n \|x_n\| \leq k$. Note that $\{x_n\Omega_\tau\} \subseteq M_{\text{s.a.}}\Omega_\tau$ is an orthonormal sequence such that $\sup_n \|x_n\| \leq k$ and $x_n \xrightarrow{w^*} 0$. Thus, by Theorem 3.1, it follows that $\mathcal{H}_\tau$ admits an orthonormal basis consisting of a uniformly bounded sequence of self-adjoint operators in M bounded by $(1 + \sqrt{2})k + \varepsilon$. This completes the proof.

(ii) Suppose B is diffuse. By Theorem 3.2, $L^2(B, \tau_{\restriction B})$ admits a uniformly bounded self-adjoint orthonormal basis $\mathcal{O}_1 \subseteq B_{\text{s.a.}}\Omega_\tau$. Note that $\mathcal{O}_1 \subseteq M_{\text{s.a.}}\Omega_\tau$. Now we show that $L^2(B, \tau_{\restriction B})^\perp$ admits an orthonormal basis $\mathcal{O}_2 \subseteq M_{\text{s.a.}}\Omega_\tau$ such that $\sup_{y\Omega_\tau \in \mathcal{O}_2} \|y\| < \infty$. Note that

$$\begin{aligned}\operatorname{span}_{\mathbb{C}}\big(L^2(B, \tau_{\restriction B})^\perp \cap M_{\text{s.a.}}\Omega_\tau\big) &= \operatorname{span}_{\mathbb{C}}\big\{x\Omega_\tau \in M_{\text{s.a.}}\Omega_\tau : \mathbb{E}_B(x) = 0\big\} \\ &= \big\{x\Omega_\tau \in \mathcal{H}_\tau : x \in M \text{ and } \mathbb{E}_B(x) = 0\big\}\end{aligned}$$

is dense in $L^2(B, \tau_{\restriction B})^\perp$, where $\mathbb{E}_B$ denotes the τ-preserving faithful normal conditional expectation from M onto B.

The argument is obvious if $\dim L^2(B, \tau_{\restriction B})^\perp < \infty$. Thus, without loss of generality, one can assume that $\dim L^2(B, \tau_{\restriction B})^\perp = \infty$. We show that $L^2(B, \tau_{\restriction B})^\perp$ satisfies the hypothesis of Theorem 3.1.

Denote $\mathcal{D} = L^2(B, \tau_{\restriction B})^\perp \cap (M_{\text{s.a.}})_1\Omega_\tau$ and $\mathcal{S} = L^2(B, \tau_{\restriction B})^\perp \cap M_1\Omega_\tau$. Then, $\mathcal{S} \subseteq \mathcal{D} + i\mathcal{D}$. Indeed, let $x\Omega_\tau \in \mathcal{S}$. There exist $x_1, x_2 \in (M_{\text{s.a.}})_1$ such that

$$x = x_1 + ix_2.$$

Since $\mathbb{E}_B(x_1) + i\mathbb{E}_B(x_2) = \mathbb{E}_B(x) = 0$, we have

$$\mathbb{E}_B(x_1) = \mathbb{E}_B(x_2) = 0.$$

Thus, $x_1\Omega_\tau, x_2\Omega_\tau \in \mathcal{D}$.

Suppose to the contrary, $\mathcal{D}$ is a totally bounded subset of $\mathcal{H}_\tau$. Since $\mathcal{S} \subseteq \mathcal{D} + i\,\mathcal{D}$, so $\mathcal{S}$ is a totally bounded subset of $\mathcal{H}_\tau$ as well. Fix $0 \neq z \in M_1$ such that $\mathbb{E}_B(z) = 0$. Note that $B_1 z\Omega_\tau \subseteq \mathcal{S}$ is a totally bounded subset of $\mathcal{H}_\tau$, where B_1 denotes the unit ball of B.

Since B is diffuse, there exists a sequence of unitaries $u_n \in B$ such that $u_n \overset{w^*}{\longrightarrow} 0$. Thus, $u_n z\Omega_\tau \overset{w}{\longrightarrow} 0$. Since $B_1 z\Omega_\tau$ is totally bounded, there exists a subsequence $\{n_k\}$ such that $u_{n_k} z\Omega_\tau \to 0$ in $\|\cdot\|_{2,\tau}$. But $\|u_{n_k} z\Omega_\tau\|_{2,\tau} = \|z\Omega_\tau\|_{2,\tau}$ for all k. This is a contradiction. Thus, $\mathcal{S}$ cannot be totally bounded subset of $\mathcal{H}_\tau$. Hence, $\mathcal{D}$ is not a totally bounded subset of $\mathcal{H}_\tau$.

Thus, by Theorem 3.1, $L^2(B, \tau_{\restriction B})^\perp$ admits an orthonormal basis $\mathcal{O}_2 \subseteq M_{\text{s.a.}}\Omega_\tau$ such that $\sup_{y\Omega_\tau \in \mathcal{O}_2} \|y\| < \infty$. Clearly, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ is the required orthonormal basis of $\mathcal{H}_\tau$. This completes the proof. ∎

Remark 3.5. The condition set out in Theorem 3.4 (ii) is sufficient but not necessary. Let N and $\mathcal{R}$ be finite von Neumann algebras equipped with faithful normal tracial states τ_1 and τ_2, respectively. Set

$$M := N \oplus \mathcal{R}, \quad B := N \oplus \mathbb{C}1_{\mathcal{R}}, \quad \text{and} \quad \tau := \frac{1}{2}(\tau_1 \oplus \tau_2) \text{ on } M.$$

Then, $B \subseteq M$ is a von Neumann subalgebra of M. Note that

$$\begin{aligned} L^2(B, \tau_{\restriction B})^\perp &= L^2(M, \tau) \ominus L^2(B, \tau_{\restriction B}) \\ &= \big(L^2(N, \tau_1) \oplus L^2(\mathcal{R}, \tau_2)\big) \ominus \big(L^2(N, \tau_1) \oplus \mathbb{C}\Omega_{\tau_2}\big) \\ &= 0 \oplus \big(L^2(\mathcal{R}, \tau_2) \ominus \mathbb{C}\Omega_{\tau_2}\big). \end{aligned}$$

Suppose N is diffuse. Then, by Theorem 3.3, $L^2(B, \tau_{\restriction B})$ admits a uniformly bounded self-adjoint orthonormal basis, though B is not diffuse.

(i) Set $\mathcal{R} = M_2(\mathbb{C})$. In this case, it is easy to see that $L^2(B, \tau_{\restriction B})^\perp$ admits a uniformly bounded self-adjoint orthonormal basis as $\dim L^2(B, \tau_{\restriction B})^\perp < \infty$. In other words, any uniformly bounded self-adjoint orthonormal basis of $L^2(B, \tau_{\restriction B})$ has an extension to a uniformly bounded self-adjoint orthonormal basis of the full space $L^2(M, \tau)$.

(ii) Set $\mathcal{R} = \oplus_{n\geq 1} M_n(\mathbb{C})$. Suppose $L^2(B, \tau_{\restriction B})^\perp$ admits a uniformly bounded self-adjoint orthonormal basis, i.e., $L^2(\mathcal{R}, \tau_2) \ominus \mathbb{C}\Omega_{\tau_2}$ admits a uniformly bounded self-adjoint orthonormal basis $\mathcal{O}_{\mathcal{R}}$. Consequently, $\mathcal{O} = \mathcal{O}_{\mathcal{R}} \cup \{\Omega_{\tau_2}\}$ is a uniformly bounded self-adjoint orthonormal basis of $L^2(\mathcal{R}, \tau_2)$. Consequently, the symmetric embedding $\mathcal{R} \ni x \mapsto x\Omega_{\tau_2} \in L^2(\mathcal{R}, \tau_2)$ is not compact, which is a contradiction to [1, Theorem 5.6]. Thus, any uniformly bounded self-adjoint orthonormal basis of $L^2(B, \tau_{\restriction B})$ does not have any extension to a uniformly bounded self-adjoint orthonormal basis of the full space $L^2(M, \tau)$.

Remark 3.6. In view of Theorem 3.2 it is natural to ask if this statement generalizes in the context of faithful normal states. Let

$$\mathfrak{P}_\varphi = \overline{\Delta_\varphi^{1/4} M_+ \Omega_\varphi}^{\|\cdot\|_{2,\varphi}}$$

denote the standard positive cone in $\mathcal{H}_\varphi$ (see [9, Section 10.23] for details). Then

$$\mathcal{D} := \overline{\mathfrak{P}_\varphi - \mathfrak{P}_\varphi}^{\|\cdot\|_{2,\varphi}}$$

becomes a real Hilbert space. Observe that $\mathcal{D}$ is invariant with respect to J_φ and $M_1\Omega_\varphi \cap \mathcal{D} \subseteq M_1\Omega_\varphi \cap M_1'\Omega_\varphi$ from Tomita's fundamental theorem. Note that if $\xi \in M_1\Omega_\varphi \cap M_1'\Omega_\varphi$, then there exist unique $\xi_1, \xi_2 \in \mathcal{D} \cap M_1\Omega_\varphi$ such that $\xi = \xi_1 + i\xi_2$ (see [9, Theorem 10.12] and [9, Section 10.23]). Thus,

$$M_1\Omega_\varphi \cap M_1'\Omega_\varphi \subseteq (M_1\Omega_\varphi \cap \mathcal{D}) + i(M_1\Omega_\varphi \cap \mathcal{D}).$$

Thus, $M_1\Omega_\varphi \cap \mathcal{D}$ is *not* a totally bounded subset of $\mathcal{D}$ if and only if $M_1\Omega_\varphi \cap M_1'\Omega_\varphi$ is not a totally bounded subset of $\mathcal{H}_\varphi$. And the latter happens if and only if M is not purely atomic [2, Theorem 7.12]. Now by replacing the roles of $\mathcal{H}$, $M_{\text{s.a.}}\Omega_\tau$ and $(M_{\text{s.a.}})_1\Omega_\tau$ with $\mathcal{D}$, $M\Omega_\varphi$ and $M_1\Omega_\varphi$ in Theorem 3.1, the GNS space $\mathcal{H}_\varphi$ admits an orthonormal basis $\mathcal{O} \subseteq M\Omega_\varphi$ such that $J_\varphi x\Omega_\varphi = x\Omega_\varphi$, for all $x\Omega_\varphi \in \mathcal{O}$, and $\sup_{x\Omega_\varphi \in \mathcal{O}} \|x\| < \infty$ if and only if M is *not* purely atomic.

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