Doc. Math. 28 (2023), 1441–1471
DOI 10.4171/DM/935

On the rank of Leopoldt's and Gross's regulator maps

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Abstract. We generalize Waldschmidt's bound for Leopoldt's defect and prove a similar bound for Gross's defect for an arbitrary extension of number fields. As an application, we prove new cases of Gross's finiteness conjecture (also known as the Gross–Kuz'min conjecture) beyond the classical abelian case, and we show that Gross's p-adic regulator has at least half of the conjectured rank. We also describe and compute non-cyclotomic analogues of Gross's defect.

1. Introduction

Let p be a prime number. Given a number field K, we denote by $S_p(K)$ and $S_\infty(K)$ the sets of p-adic places and archimedean places of K respectively. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $A^\wedge = \overline{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} \varprojlim_n A/p^n A$ for any abelian group A. The *Leopoldt regulator map* is the $\overline{\mathbb{Q}}_p$-linear map

$$\iota_K \colon \mathcal{O}_K^{\times,\wedge} \to \prod_{\mathfrak{P} \in S_p(K)} \mathcal{O}_{K_\mathfrak{P}}^{\times,\wedge}$$

induced by the diagonal embedding of the unit group of K into all its p-adic completions.

Conjecture 1.1 (Leopoldt's conjecture for (K, p) [28]). *The Leopoldt regulator map ι_K is injective.*

Ax's method, combined with Brumer's p-adic analogue of Baker's theorem, implies Leopoldt's conjecture for abelian extensions of $\mathbb{Q}$ or of an imaginary quadratic field [1,3]. The same method also proves Leopoldt's conjecture when K is an imaginary A_4-extension of $\mathbb{Q}$ [8].

Another classical result concerning Leopoldt's conjecture is obtained by Waldschmidt as an application of his study of transcendence properties of the exponential function in several variables. Let $\delta_K^{\mathbf{L}}$ be the dimension of $\ker \iota_K$. Leopoldt's conjecture then predicts that $\delta_K^{\mathbf{L}} = 0$ and $\delta_K^{\mathbf{L}}$ is called Leopoldt's defect (we will omit its dependence on p).

Theorem 1.2 (Waldschmidt [37]). *For every number field K, we have*

$$\delta_K^{\mathbf{L}} \leq \big(|S_\infty(K)| - 1\big)/2.$$

2020 Mathematics Subject Classification. Primary 11R27; Secondary 11R23.
Keywords. Leopoldt conjecture, Gross–Kuz'min conjecture, p-adic transcendence theory, Iwasawa theory.

This bound is the best upper bound for Leopoldt's defect so far. The main goal in this work is to derive from Waldschmidt's and Roy's contributions in p-adic transcendence theory similar results concerning the Gross regulator map and to prove new cases of the Gross–Kuz'min conjecture, whose statement is now recalled.

Consider the $\mathbb{Z}_p$-hyperplane $\mathscr{H}$ of $\prod_{\mathfrak{P}|p} \mathbb{Z}_p$ given by the equation $\sum_{\mathfrak{P}|p} s_{\mathfrak{P}} = 0$, and the map

$$\mathscr{L}_K \colon \begin{cases} \mathcal{O}_K\left[\frac{1}{p}\right]^{\times,\wedge} \to \mathscr{H}^\wedge \\ \qquad\qquad x \mapsto \left(-\log_p\left(\mathcal{N}_{\mathfrak{P}}(x)\right)\right)_{\mathfrak{P}}, \end{cases}$$

where $\mathcal{O}_K[\frac{1}{p}]^\times$ is the group of p-units of K, $\mathcal{N}_{\mathfrak{P}}$ is the local norm map for the extension $K_{\mathfrak{P}}/\mathbb{Q}_p$, and $\log_p : \mathbb{Q}_p^\times \to \mathbb{Q}_p$ is the usual Iwasawa p-adic logarithm. By the usual product formula, $\mathscr{L}_K$ is well defined. We shall refer to $\mathscr{L}_K$ as the (cyclotomic) Gross regulator map.

Conjecture 1.3 (Gross–Kuz'min's conjecture for (K, p) [12, 24]). *The Gross regulator map $\mathscr{L}_K$ is surjective.*

Like Leopoldt's conjecture, the Gross–Kuz'min conjecture plays a central role in the formulation of p-adic analogues of Dirichlet's class number formula. Leopoldt's regulator appears in Colmez's formula on the residue at $s = 1$ of the p-adic Dedekind zeta function of a totally real number field [7], whereas Gross's regulator plays the role of an $\mathcal{L}$-invariant in the celebrated Gross–Stark conjecture over a CM number field [12]. When K is neither totally real nor CM, a conjectural interpretation of these regulators in terms of p-adic Artin L-functions is still available (see [29]). Besides their potential applications to the equivariant Tamagawa number conjecture (eTNC) as in [5], the Gross–Kuz'min conjecture and its non-cyclotomic analogue discussed below also yield information on the fine structure of class groups attached to $\mathbb{Z}_p$-extensions of K (see e.g. [9, 19, 23, 27]).

The Gross–Kuz'min conjecture is true when $K/\mathbb{Q}$ is abelian as shown by Greenberg [11]. More recently, Kleine [22] proved the conjecture for any K which has at most two p-adic primes. (We note that Kleine's approach does not use p-adic transcendence theory). A proof of the conjecture for certain Galois extensions $K/\mathbb{Q}$ was obtained by Jaulent [17, 18] and Kuz'min [25] under assumptions on $\mathrm{Gal}(K/\mathbb{Q})$ and on the splitting behavior of p in K (See Remark 4.8).

Our first result gives an upper bound for the Gross defect $\delta_K^{\mathbf{G}} = \dim \operatorname{coker} \mathscr{L}_K$ as well as a slight generalization of Theorem 1.2. See also Hofer–Kleine [13] for other bounds for $\delta_K^{\mathbf{G}}$ when K is a CM field.

Theorem 1.4. *Let K/k be an extension of number fields. The following inequalities hold:*

$$\delta_K^{\mathbf{L}} \leq \delta_k^{\mathbf{L}} + \left(\left|S_\infty(K)\right| - \left|S_\infty(k)\right|\right)/2, \quad \delta_K^{\mathbf{G}} \leq \delta_k^{\mathbf{G}} + \left(\left|S_p(K)\right| - \left|S_p(k)\right|\right)/2.$$

Moreover, if K has at least one real place and $|S_p(K)| \neq |S_p(k)|$, then the second bound is strict.

A key observation that we use in the computation of $\delta_K^{\mathbf{L}}$ and $\delta_K^{\mathbf{G}}$ is that they are compatible with Artin formalism. Fix once and for all an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For any number field $k \subset \overline{\mathbb{Q}}$ of absolute Galois group $G_k = \mathrm{Gal}(\overline{\mathbb{Q}}/k)$, let $\mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ be the set of finite dimensional $\overline{\mathbb{Q}}_p$-valued representations of G_k of finite image. We will define defects $\delta_k^{\mathbf{L}}(\rho)$ and $\delta_k^{\mathbf{G}}(\rho)$ associated with $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ which satisfy the usual Artin formalism. In particular, when $\rho = \mathrm{Ind}_K^k \mathbb{1}_K$ is the induction from G_K to G_k of the trivial representation, they coincide with $\delta_K^{\mathbf{L}}$ and $\delta_K^{\mathbf{G}}$ respectively. We will also define quantities $d(\rho)$, $d^+(\rho)$ and $f(\rho)$ which compute $[K : \mathbb{Q}]$, $|S_\infty(K)|$ and $|S_p(K)|$, respectively, when $\rho = \mathrm{Ind}_K^k \mathbb{1}_K$ (see (3.7)). Our main theorem is the following.

Theorem 1.5. *Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ be an irreducible representation and let $d = d(\rho)$, $d^+ = d^+(\rho)$ and $f = f(\rho)$. If $d^+ = f = 0$, then we have $\delta_{\mathbb{Q}}^{\mathbf{L}}(\rho) = \delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$. Otherwise, we have the following inequalities.*

$$\delta_{\mathbb{Q}}^{\mathbf{L}}(\rho) \le \frac{(d^+)^2}{d + d^+}, \quad \delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) \le \frac{f^2}{d^+ + 2f}.$$

By Artin formalism, this yields the upper bound (e.g. for Leopoldt's defect) $\delta_k^{\mathbf{L}}(\rho) \le d^+(\rho)/2$ for an arbitrary representation $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$. We immediately obtain Theorem 1.4 by choosing ρ such that $\mathrm{Ind}_K^k \mathbb{1}_K = \rho \oplus \mathbb{1}_k$.

The first bound in Theorem 1.5 is Laurent's main theorem in [26], but we will provide a much shorter proof of this result via a lemma on local Galois representations (Lemma 3.13). The second bound, however, does not seem to follow from the classical methods employed by Laurent [26] and Roy [34] to study the p-adic closure of S-units of K, for a given finite set of places S.

Theorem 1.5 together with Artin's formalism easily implies Leopoldt's conjecture for abelian extensions of an imaginary quadratic field. In the same vein, we indicate the two main applications of Theorem 1.5.

Corollary 1.6. *Let k be a totally real field and let V be a totally odd Artin representation of G_k. Then Gross's p-adic regulator matrix $R_p(V)$ defined in* [12, (2.10)] *has rank at least half of its size.*

This corollary strengthens Gross's classical result stating that the matrix $R_p(V)$ has positive rank [12, Proposition 2.13].

Corollary 1.7. *Let K be a number field. The Gross–Kuz'min conjecture holds for K in the following situations:*

(i) *K is an abelian extension of an imaginary quadratic field.*

(ii) *K is an abelian extension of a real quadratic field and has at least one real place.*

(iii) *K is a cubic extension of $\mathbb{Q}$.*

Theorem 4.7 provides a more extensive list of number fields for which the Gross–Kuz'min conjecture holds unconditionally. A proof of the Gross–Kuz'min conjecture for abelian extensions of imaginary quadratic fields when $p \neq 2$ is already given in [29] and the proof of Corollary 1.7 is very close to that of *loc. cit.*.

We highlight in the last part of this article some interesting connections between non-cyclotomic analogues of the Gross–Kuz'min conjecture and *algebraic* independence of p-adic logarithms of units of number fields.

Given an arbitrary $\mathbb{Z}_p$-extension K_∞ of K, we will define a map $\mathscr{L}_{K_\infty/K}$ specializing to $\mathscr{L}_K$ if K_∞ is the cyclotomic extension of K. As noted in [20, 21], there do exist examples of $\mathbb{Z}_p$-extensions K_∞/K for which $\delta^{\mathbf{G}}_{K_\infty/K} > 0$, but a conjectural description of all such K_∞/K is still missing.

In the next theorem, we fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and we let Λ be the $\overline{\mathbb{Q}}$-linear subspace of $\overline{\mathbb{Q}}_p$ generated by 1 and by p-adic logarithms of non-zero algebraic numbers.

Theorem 1.8. *Let k be an imaginary quadratic field, and K an abelian extension of k in which p splits completely. Then there exist at most finitely many distinct $\mathbb{Z}_p$-extensions k_∞ of k for which $\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$. Moreover, no such $\mathbb{Z}_p$-extensions exist if the polynomial*

$$XYZ^2 - (AX - BY)(CX - DY) \in \mathbb{Z}[A, B, C, D, X, Y, Z]$$

does not vanish on any 7-tuple $(a, b, c, d, x, y, z) \in \Lambda^7$ which form a $\overline{\mathbb{Q}}$-linearly independent set.

This last condition should be true according to the weak p-adic Schanuel conjecture. In Proposition 3.1 we illustrate Theorem 1.8 with a classical application to the semi-simplicity of Iwasawa modules attached to Kk_∞/K. Similar results (especially in the case where p is non-split in k) have also been obtained by Bullach and Hofer [4], with applications to the p-part of the eTNC for abelian extensions of imaginary quadratic fields.

Theorem 1.8 can be generalized to arbitrary base fields k having at most r linearly disjoint $\mathbb{Z}_p$-extensions with $r \leq 2$ (Theorem 5.1). The main idea is that, under our assumption on p, one can parameterize $\mathbb{Z}_p$-extensions of k by points on a $(r-1)$-dimensional linear subspace L of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$, where $n = [k : \mathbb{Q}]$. The condition $\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$ then cuts out a closed subvariety $\mathscr{C}$ of L given by polynomial equations with coefficients in

$$\Lambda_0 := \log_p \left(\mathcal{O}_K\left[\tfrac{1}{p}\right]^\times\right) \subset \overline{\mathbb{Q}}_p.$$

We then exploit the fact that any linear (resp. algebraic) independence between elements of Λ_0 implies strong conditions on the $\overline{\mathbb{Q}}$-points (resp. the $\overline{\mathbb{Q}}_p$-points) of $\mathscr{C}$.

Theorem 1.8 was inspired by Betina–Dimitrov's work [2] where the authors show the non-vanishing of a certain $\mathcal{L}$-invariant for Katz's p-adic L-function restricted to the anticyclotomic $\mathbb{Z}_p$-extension. In fact, their result generalizes to any $\mathbb{Z}_p$-extension with non-transcendental slope. We expect that our techniques can give further results on the non-vanishing of $\mathcal{L}$-invariants in more general contexts.

The paper is structured as follows. In Section 2, we recall all the classical results in p-adic transcendence theory which we make use of. In Section 3, we describe Leopoldt's and Gross's defects via class field theory and we show that they are compatible with Artin formalism. Our main results and corollaries are proven in Section 4, except for Theorem 1.8 whose proof is postponed to Section 5.

2. p-adic transcendence theory

Throughout this section we fix an embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, allowing us to view algebraic numbers as p-adic numbers. The following very strong conjecture describes the algebraic dependence between logarithms of algebraic numbers (see [6, Conjecture 3.10]).

Conjecture (Weak p-adic Schanuel conjecture). *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers. If $\log_p(\alpha_1), \dots, \log_p(\alpha_n)$ are linearly independent over $\mathbb{Q}$, then they are algebraically independent over $\mathbb{Q}$.*

We recall some classical results of Brumer, Waldschmidt and Roy and deduce some consequences that turn out to be useful in the study of the Gross–Kuz'min conjecture.

2.1. The Baker–Brumer theorem

Brumer [3] extended Baker's method to the p-adic setting and proved the following theorem on linear independence of logarithms.

Theorem 2.1 (Baker–Brumer theorem). *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers. If $\log_p(\alpha_1), \dots, \log_p(\alpha_n)$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\overline{\mathbb{Q}}$.*

Recall that $\log_p$ is normalized so that we have $\log_p(p) = 0$.

Proposition 2.2. *Let $H \subset \overline{\mathbb{Q}}$ be a number field. The $\overline{\mathbb{Q}}$-linear extension*

$$\log_p\colon \overline{\mathbb{Q}} \otimes_{\mathbb{Z}} H^\times \to \overline{\mathbb{Q}}_p, \quad c \otimes x \mapsto c \log_p \bigl(\iota_p(x)\bigr)$$

of the p-adic logarithm has kernel the line $p^{\overline{\mathbb{Q}}}$ spanned by $1 \otimes p$.

Proof. Let H_p be the completion of $\iota_p(H)$ inside $\overline{\mathbb{Q}}_p$, let $\mathcal{O}_{H_p}^\times$ be its unit group and consider the abelian group

$$\mathcal{T} = \bigl\{x \in H^\times : \iota_p(x) \in \mathcal{O}_{H_p}^\times\bigr\}.$$

Writing any $x \in H^\times$ as $(x/p^v) \cdot p^v$ where v is the valuation of x (seen as an element of H_p), we obtain a direct sum decomposition

$$\overline{\mathbb{Q}} \otimes H^\times = \bigl(\overline{\mathbb{Q}} \otimes \mathcal{T}\bigr) \oplus p^{\overline{\mathbb{Q}}}.$$

Moreover, any $\alpha \in H^\times$ whose p-adic logarithm is 0 must be a root of unity, so multiplicatively independent numbers $\alpha_1, \dots, \alpha_n \in \mathcal{T}$ have $\overline{\mathbb{Q}}$-linearly independent p-adic logarithms by the Baker–Brumer theorem. This shows that the restriction of $\log_p$ to $\overline{\mathbb{Q}} \otimes \mathcal{T}$ is injective, hence $\ker(\log_p) = p^{\overline{\mathbb{Q}}}$. ∎

2.2. Waldschmidt's and Roy's theorem

Recall that Λ is the $\overline{\mathbb{Q}}$-linear subspace of $\overline{\mathbb{Q}}_p$ generated by 1 and by p-adic logarithms of non-zero algebraic numbers.

Extensions of Baker's method due to Waldschmidt and Roy give a lower bound for the rank of matrices with coefficients in Λ. To each matrix M with coefficients in $\overline{\mathbb{Q}}_p$, of size $m \times \ell$, they assign a number $\theta(M)$ defined as the minimum of all ratios $\frac{\ell'}{m'}$ where (m', ℓ') runs among the pairs of integers satisfying $0 < m' \leq m$ and $0 \leq \ell' \leq \ell$, for which there exist matrices $P \in \mathrm{GL}_m(\overline{\mathbb{Q}})$ and $Q \in \mathrm{GL}_\ell(\overline{\mathbb{Q}})$ such that the product PMQ can be written as

$$\begin{pmatrix} M' & 0 \\ N & M'' \end{pmatrix}$$

with M' of size $m' \times \ell'$. Note that $\theta(M) \leq \frac{\ell}{m}$ with equality if all the entries of M are $\overline{\mathbb{Q}}$-linearly independent. The following theorem is Roy's sharpening of Waldschmidt's theorem ([37, Théorème 2.1.p], [34, Corollary 1]).

Theorem 2.3. *Let M be a matrix with coefficients in Λ, of size $m \times \ell$ with $m, \ell > 0$, and let n be its rank. We have*

$$n \geq \frac{\theta(M)}{1 + \theta(M)} \cdot m.$$

Roy also deduced a useful corollary for 3×2 matrices from Theorem 2.3 in [34, Corollary 2].

Corollary 2.4 (Strong six exponentials theorem). *Let M be a (3×2)-matrix with coefficients in Λ. If the rows of M are $\overline{\mathbb{Q}}$-linearly independent, and if the columns of M are also $\overline{\mathbb{Q}}$-linearly independent, then M has rank 2.*

3. Regulator maps and class groups

3.1. Galois cohomology

For all fields $L \subset \overline{\mathbb{Q}}$ and all finite sets S of places of L containing $S_p(L)$, we let $X(L)$ (resp. $X'_S(L)$) be the Galois group of the maximal abelian pro-p extension of L which is unramified everywhere (resp. unramified everywhere and totally split at all $v \in S$). If $S = S_p(L)$, we simply put $X'(L) = X'_S(L)$.

Given a $\mathbb{Z}_p$-extension K_∞ of K with Galois group Γ, we have $X(K_\infty) = \varprojlim_L X(L)$ and $X'_S(K_\infty) = \varprojlim_L X'_S(L)$, the limit being taken over all intermediate extensions $K \subset L \subset K_\infty$ with $[L : K] < \infty$ and the transition maps being the restriction maps. Therefore, $X(K_\infty)$ and $X'_S(K_\infty)$ are modules over the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$. They are finitely generated torsion as shown by Iwasawa [14, Theorem 5]. In particular, the module $X'_S(K_\infty)_\Gamma$ of Γ-coinvariants, defined as the largest quotient of $X'_S(K_\infty)$ on which Γ acts trivially, is finitely generated over $\mathbb{Z}_p$. We let

$$\delta^{\mathbf{G}}_{K_\infty/K} := \mathrm{rk}_{\mathbb{Z}_p} X'(K_\infty)_\Gamma.$$

If K_∞ is the cyclotomic $\mathbb{Z}_p$-extension K_{cyc} of K we simply write $\delta^{\mathbf{G}}_K$ for $\delta^{\mathbf{G}}_{K_\infty/K}$. We will later see that this definition is compatible with that of the introduction. One motivation in classical Iwasawa theory to compute $\delta^{\mathbf{G}}_{K_\infty/K}$ originates in the following simple result by Jaulent and Sands [20, Proposition 6].

Proposition 3.1. *Let γ be a topological generator of Γ. If no p-adic prime of K splits completely in K_∞ and if $\delta^{\mathrm{G}}_{K_\infty/K} = 0$, then $\gamma - 1$ acts semi-simply on $X(K_\infty)$. That is, $(\gamma-1)^2$ does not divide the elements $P_i \in \mathbb{Z}_p[[\Gamma]]$ appearing in any elementary module $\bigoplus_i \mathbb{Z}_p[[\Gamma]]/(P_i)$ pseudo-isomorphic to $X(K_\infty)$.*

Let $S_0 \supseteq S_p(K) \bigcup S_\infty(K)$ be a finite set of places of K. For any extension L of K and any discrete (resp. compact) G_L-module M which is unramified outside the places of L above S_0, we consider for all $i \geq 0$ the S_0-ramified i-th cohomology group (resp. continuous cohomology group) $\mathrm{H}^i_{S_0}(L, M) = \mathrm{H}^i(\mathrm{Gal}(L_{S_0}/L), M)$, where L_{S_0}/L is the largest extension of L which is unramified outside the places of L above S_0. Given any subset $S \subset S_0$, let

$$Ш^i_S(L, M) = \ker\left[\mathrm{H}^i_{S_0}(L, M) \to \prod_{v \in S} \mathrm{H}^i(L_v, M) \times \prod_{v \in S_0 - S} \mathrm{H}^i(L_v^{\mathrm{ur}}, M)\right],$$

where L_v^{ur}/L_v denotes the maximal unramified extension of L_v (so $\mathbb{R}^{\mathrm{ur}} = \mathbb{R}$ in particular) and the maps above are the usual localization maps. Note that the definition of $Ш^i_S(L, M)$ does not depend on the choice of S_0. We simply write $Ш^i(L, M)$ instead of $Ш^i_S(L, M)$ if $S = S_p(L) \bigcup S_\infty(L)$. We also write M^* for the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of a $\mathbb{Z}_p$-module M.

Lemma 3.2. *Let $L \subset \overline{\mathbb{Q}}$ be a number field and let $S \supset S_p(L) \bigcup S_\infty(L)$ be a finite set of places of L. There are canonical isomorphisms $Ш^2_S(L, \mathbb{Z}_p(1)) \simeq Ш^1_S(L, \mathbb{Q}_p/\mathbb{Z}_p)^* \simeq X'_S(L)$.*

Proof. The first isomorphism is given by the Poitou—Tate duality theorem [31, Theorem 4.10 (a)]. Since $\mathrm{H}^1_S(L, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(\mathrm{Gal}(L_S/L), \mathbb{Q}_p/\mathbb{Z}_p)$, class field theory easily implies $Ш^1(L, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(X'_S(L), \mathbb{Q}_p/\mathbb{Z}_p)$. ■

The isomorphisms provided by Lemma 3.2 are functorial in L in the sense that, given a finite extension L'/L of number fields, the norm map $X(L') \to X(L)$ corresponds to the corestriction map (resp. to the Pontryagin dual of the restriction map) $Ш^2(L', \mathbb{Z}_p(1)) \to Ш^2(L, \mathbb{Z}_p(1))$ (resp. $Ш^1(L', \mathbb{Q}_p/\mathbb{Z}_p)^* \to Ш^1(L, \mathbb{Q}_p/\mathbb{Z}_p)^*$).

Given any $\mathbb{Z}_p$-extension $K_\infty = \bigcup_n K_n$ of K and any finite set $S \supset S_p(K) \bigcup S_\infty(K)$, Lemma 3.2 provides isomorphisms of $\mathbb{Z}_p[[\Gamma]]$-modules

$$\varprojlim_n Ш^2_S\big(K_n, \mathbb{Z}_p(1)\big) \simeq Ш^1_S(K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)^* \simeq X'_S(K_\infty).$$

We now make use of the inflation-restriction exact sequence to study the problem of Galois descent. Noting that $\mathrm{H}^2(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) = \{0\}$ as Γ is pro-cyclic, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & \mathrm{H}^1_{\{p,\infty\}}(K, \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & \mathrm{H}^1_{\{p,\infty\}}(K_\infty, \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigoplus_v \mathrm{Hom}(\Gamma_v, \mathbb{Q}_p/\mathbb{Z}_p) & \to & \bigoplus_v \mathrm{H}^1(K_v, \mathbb{Q}_p/\mathbb{Z}_p) & \to & \bigoplus_w \mathrm{H}^1(K_{\infty,w}, \mathbb{Q}_p/\mathbb{Z}_p), & & \end{array} \tag{3.1}$$

where v (resp. w) in the second row runs over all the p-adic and archimedean places of K (resp. of K_∞).

Proposition 3.3. *We have* $\delta^{\mathbf{G}}_{K_\infty/K} = \dim\ker(\mathrm{Loc}_{K_\infty/K})$, *where* $\mathrm{Loc}_{K_\infty/K}$ *is the localization map*

$$\mathrm{Loc}_{K_\infty/K}\colon \mathrm{H}^1_{\{p,\infty\}}(K,\overline{\mathbb{Q}}_p)/\mathrm{H}^1(\Gamma,\overline{\mathbb{Q}}_p) \to \bigoplus_{\mathfrak{P}\in S_p(K)} \mathrm{H}^1(K_{\mathfrak{P}},\overline{\mathbb{Q}}_p)/\mathrm{H}^1(\Gamma_{\mathfrak{P}},\overline{\mathbb{Q}}_p).$$

Proof. By Lemma 3.2, the kernel of the right vertical map of (3.1) is equal to the Pontryagin dual of $X'(K_\infty)_\Gamma$. Using that $\mathrm{H}^1(K_v,\mathbb{Q}_p/\mathbb{Z}_p)$ is finite when $v\in S_\infty(K)$, $\delta^{\mathbf{G}}_{K_\infty/K}$ is thus equal to the $\mathbb{Z}_p$-rank of the Pontryagin dual of the kernel of the natural map

$$\mathrm{H}^1_{\{p,\infty\}}(K,\mathbb{Q}_p/\mathbb{Z}_p)/\mathrm{H}^1(\Gamma,\mathbb{Q}_p/\mathbb{Z}_p) \to \bigoplus_{\mathfrak{P}\in S_p(K)} \mathrm{H}^1(K_{\mathfrak{P}},\mathbb{Q}_p/\mathbb{Z}_p)/\mathrm{H}^1(\Gamma_{\mathfrak{P}},\mathbb{Q}_p/\mathbb{Z}_p). \quad (3.2)$$

Given $\mathcal{G}\in\{\mathrm{Gal}(K_S/K),\mathrm{Gal}(\overline{K}_{\mathfrak{P}}/K_{\mathfrak{P}}),\Gamma,\Gamma_{\mathfrak{P}}\}$ and $A\in\{\mathbb{Z}_p,\mathbb{Q}_p/\mathbb{Z}_p,\overline{\mathbb{Q}}_p\}$ endowed with the trivial $\mathcal{G}$-action, we have

$$\mathrm{H}^1(\mathcal{G},A) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{ab},(p)},A),$$

where $\mathcal{G}^{\mathrm{ab},(p)}$ is the maximal abelian pro-p-quotient of $\mathcal{G}$. As $\mathrm{rk}_{\mathbb{Z}_p}\mathcal{G}^{\mathrm{ab},(p)}$ is finite, the natural map $\mathrm{H}^1(\mathcal{G},\mathbb{Z}_p)\otimes\mathbb{Q}_p/\mathbb{Z}_p\to\mathrm{H}^1(\mathcal{G},\mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $\mathrm{H}^1(\mathcal{G},\mathbb{Z}_p)\otimes\overline{\mathbb{Q}}_p\to\mathrm{H}^1(\mathcal{G},\overline{\mathbb{Q}}_p)$) has finite kernel and cokernel (resp. is an isomorphism). In particular, the corank of the kernel of (3.2) is equal to $\dim\ker(\mathrm{Loc}_{K_\infty/K})$, as wanted. ∎

Remark 3.4. Since $\mathrm{H}^1_{\{p,\infty\}}(K,\overline{\mathbb{Q}}_p)=\mathrm{Hom}(G_K,\overline{\mathbb{Q}}_p)$ parameterizes the $\mathbb{Z}_p$-extensions of K, the domain of $\mathrm{Loc}_{K_\infty/K}$ has dimension $r_2+\delta^{\mathbf{L}}_K$, where r_2 is the number of complex places of K [38, Theorem 13.4]. In particular, Proposition 3.3 yields an upper bound $\delta^{\mathbf{G}}_{K_\infty/K}\le r_2+\delta^{\mathbf{L}}_K$. Therefore, Leopoldt's conjecture for a totally real field K implies the Gross–Kuz'min conjecture for K, as already noticed by Kolster in [23, Corollary 1.3].

Given a prime $\mathfrak{P}\in S_p(K)$, let $\Gamma_{\mathfrak{P}}$ be the decomposition subgroup of Γ at $\mathfrak{P}$ and denote by $\mathrm{rec}_{\Gamma_{\mathfrak{P}}}\colon K_{\mathfrak{P}}^\times\to\Gamma_{\mathfrak{P}}$ the corresponding local reciprocity map. Define also the $\mathbb{Z}_p$-module

$$\mathcal{H}_{K_\infty/K} := \ker\Big(\bigoplus_{\mathfrak{P}\in S_p(K)}\Gamma_{\mathfrak{P}}\to\Gamma\Big).$$

By the usual product formula in class field theory the regulator map

$$\mathscr{L}_{K_\infty/K}\colon\begin{cases}\mathcal{O}_K\big[\frac{1}{p}\big]^\times \to \mathcal{H}_{K_\infty/K}\\ \qquad x\mapsto \big(\mathrm{rec}_{\Gamma_{\mathfrak{P}}}(x)\big)_{\mathfrak{P}},\end{cases}$$

is well defined, and it extends to a $\overline{\mathbb{Q}}_p$-linear map $\mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}\to\mathcal{H}^\wedge_{K_\infty/K}$ which we still denote by $\mathscr{L}_{K_\infty/K}$. If $K_\infty=K_{\mathrm{cyc}}$, then the character $\log_p\circ\chi_{\mathrm{cyc}}\circ\mathrm{rec}_{\Gamma_{\mathfrak{P}}}\colon K_{\mathfrak{P}}^\times\to\mathbb{Q}_p$ coincides with $-\log_p\circ\mathcal{N}_{\mathfrak{P}}$, where χ_{cyc} is the cyclotomic character. Therefore, $\mathscr{L}_{K_{\mathrm{cyc}}/K}$ is essentially the same as the map $\mathscr{L}_K$ of the introduction, and $\dim\mathrm{coker}(\mathscr{L}_{K_{\mathrm{cyc}}/K})=\delta^{\mathbf{G}}_K$.

Proposition 3.5. *We have* $\delta^{\mathbf{G}}_{K_\infty/K} = \dim \operatorname{coker}(\mathscr{L}_{K_\infty/K})$. *In particular,* $\delta^{\mathbf{G}}_{K_{\mathrm{cyc}}/K} = \delta^{\mathbf{G}}_{K}$.

Proof. For $\mathfrak{P} \in S_p(K)$, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1\big(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p(1)\big) \times & \mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p) \arrow[r, "\cup"] & \mathrm{H}^2\big(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p(1)\big) \arrow[d, "\wr"', "\mathrm{inv}"] \\
K_{\mathfrak{P}}^{\times,\wedge} \arrow[u, "\wr"] \times & \mathrm{Hom}(K_{\mathfrak{P}}^{\times,\wedge}, \overline{\mathbb{Q}}_p) \arrow[u, "\wr"] \arrow[r, "\mathrm{ev}"] & \overline{\mathbb{Q}}_p,
\end{tikzcd} \tag{3.3}$$

where the left vertical map is Kummer's isomorphism, the middle one is induced by the reciprocity map $K_{\mathfrak{P}}^\times \to G^{\mathrm{ab}}_{K_{\mathfrak{P}}}$, ev is the evaluation map and inv is Hasse's invariant in local class field theory. To see that the diagram is commutative, one first shows using e.g. [36, Chapter 14, Section 1, Proposition 3] the commutativity of a similar diagram with $\overline{\mathbb{Q}}_p(1)$, $\overline{\mathbb{Q}}_p$ and $K_{\mathfrak{P}}^{\times,\wedge}$ replaced with μ_{p^n}, $\mathbb{Z}/p^n\mathbb{Z}$ and $K_{\mathfrak{P}}^\times/(K_{\mathfrak{P}}^\times)^{p^n}$ respectively and then take an inverse limit over $n \in \mathbb{N}$.

The subspace $\mathrm{H}^1(\Gamma_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)$ of $\mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)$ coincides via the middle equality of (3.3) with

$$\mathrm{Hom}(\Gamma_{\mathfrak{P}}, \overline{\mathbb{Q}}_p) = \ker\big[\mathrm{Hom}(K_{\mathfrak{P}}^{\times,\wedge}, \overline{\mathbb{Q}}_p) \to \mathrm{Hom}\,\big(\ker(\mathrm{rec}_{\Gamma_{\mathfrak{P}}}), \overline{\mathbb{Q}}_p\big)\big],$$

so its orthogonal complement under the pairing in (3.3) is equal to $\ker(\mathrm{rec}_{\Gamma_{\mathfrak{P}}})$. Using the fact that $\text{Ш}^1(K, \mathbb{Q}_p) = 0$ by the finiteness of the class group of K, Poitou–Tate's duality then yields a perfect linear pairing

$$\ker\left(\mathrm{H}^1_{\{p,\infty\}}(K, \overline{\mathbb{Q}}_p) \to \bigoplus_{\mathfrak{P} \in S_p(K)} \frac{\mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)}{\mathrm{H}^1(\Gamma_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)}\right)$$
$$\times \operatorname{coker}\Big(\bigoplus_{\mathfrak{P}} \mathrm{rec}_{\Gamma_{\mathfrak{P}}} : \mathcal{O}_K\big[\tfrac{1}{p}\big]^{\times,\wedge} \to \bigoplus_{\mathfrak{P} \in S_p(K)} \Gamma_{\mathfrak{P}}^{\wedge}\Big) \to \overline{\mathbb{Q}}_p.$$

A comparison of dimensions gives

$$\dim \ker(\mathrm{Loc}_{K_\infty/K}) + 1 = \dim \operatorname{coker}(\mathscr{L}_{K_\infty/K}) + 1,$$

so Proposition 3.3 yields the desired equality. ∎

3.2. Isotypic components

We consider in this paragraph the situation where the $\mathbb{Z}_p$-extension K_∞/K comes from the $\mathbb{Z}_p$-extension k_∞/k of a subfield k of K, which means that $K_\infty = Kk_\infty$. Assume that K/k is Galois with Galois group G.

Given an algebraically closed field $\mathbf{Q}$ of characteristic zero (typically, $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}}_p$), we denote by $\mathrm{Art}_{\mathbf{Q}}(G)$ the set of (isomorphism classes of) $\mathbf{Q}$-valued finite dimensional representations of G. With a slight abuse of notation, we sometimes write elements of $\mathrm{Art}_{\mathbf{Q}}(G)$ as (W, ρ), where W is the underlying space of $\rho \in \mathrm{Art}_{\mathbf{Q}}(G)$, and we put $\dim \rho := \dim W$. We denote by $(\mathbf{Q}, \mathbb{1})$ or $(\mathbf{Q}, \mathbb{1}_k)$ the trivial representation of G, and given $(W, \rho) \in \mathrm{Art}_{\mathbf{Q}}(G)$, we write for short $\mathbb{1} \not\subset \rho$ if $\mathbf{Q} \not\subseteq W$ as G-modules.

The $\mathbf{Q}$-valued representations of G are semi-simple and the regular representation splits as

$$\mathbf{Q}[G] = \bigoplus_{\rho} e(\rho) \cdot \mathbf{Q}[G] = \bigoplus_{\rho} W^{\oplus \dim \rho}, \tag{3.4}$$

where (W, ρ) runs through the set of all the $\mathbf{Q}$-valued irreducible representations of G and

$$e(\rho) = \frac{\dim \rho}{|G|} \cdot \sum_{g \in G} \mathrm{Tr}\left(\rho(g^{-1})\right) g \in \mathbf{Q}[G] \tag{3.5}$$

is the usual idempotent attached to ρ.

For any finite set of places S of k containing $S_\infty(k)$, let $\mathcal{O}_K[1/S]^\times$ be the group of S-units of K. Dirichlet's unit theorem implies that we have a decomposition of $\mathbf{Q}[G]$-modules

$$\mathbf{Q} \otimes_{\mathbb{Z}} \mathcal{O}_K[1/S]^\times = \left(\mathbf{Q} \otimes_{\mathbb{Z}} \mathcal{O}_k[1/S]^\times\right) \bigoplus \Big(\bigoplus_{\mathbb{1} \neq \rho} W^{\oplus d_S^+(\rho)}\Big), \tag{3.6}$$

where (W, ρ) runs through the set of all non-trivial irreducible representations of G and $d_S^+(\rho) = \sum_{v \in S} \dim \mathrm{H}^0(k_v, W)$. It will be convenient to introduce the following invariants:

$$\begin{aligned} d(\rho) &:= [k : \mathbb{Q}] \cdot \dim \rho, \\ d^+(\rho) &:= \sum_{v \mid \infty} \dim \mathrm{H}^0(k_v, W), \\ f(\rho) &:= \sum_{\mathfrak{p} \mid p} \dim \mathrm{H}^0(k_{\mathfrak{p}}, W), \end{aligned} \tag{3.7}$$

so that $d^+(\rho) = d^+_{S_\infty(k)}(\rho)$ and $f(\rho) = d^+_{S_\infty(k) \bigcup S_p(k)}(\rho) - d^+_{S_\infty(k)}(\rho)$.

We record in the next lemma a list of useful properties satisfied by the invariants introduced in (3.7) and which we make use of in Sections 4 and 5. Recall that a rule $\rho \mapsto a(\rho) \in \mathbb{Z}$, where ρ runs among all the representations of Galois groups of finite extensions of number fields, is said to be compatible with Artin formalism if, for all finite Galois extensions $N/M/L$:

(a) $a(\tilde{\rho}) = a(\rho)$ if $\tilde{\rho} \in \mathrm{Art}_{\bar{\mathbb{Q}}_p}(\mathrm{Gal}(N/L))$ is the inflation of $\rho \in \mathrm{Art}_{\bar{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$,

(b) $a(\rho_1 \oplus \rho_2) = a(\rho_1) + a(\rho_2)$ for all $\rho_1, \rho_2 \in \mathrm{Art}_{\bar{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$, and

(c) $a(\rho') = a(\rho)$ if ρ' is the induction of ρ from $\mathrm{Gal}(N/M)$ to $\mathrm{Gal}(N/L)$.

The inflation maps identify $\mathrm{Art}_{\bar{\mathbb{Q}}_p}(G_L)$ of the introduction with $\bigcup_L \mathrm{Art}_{\bar{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$, where M runs over all finite Galois extensions of L inside $\bar{\mathbb{Q}}$. Therefore, any rule a satisfying the condition (a) above gives rise to a well-defined map $a \colon \mathrm{Art}_{\bar{\mathbb{Q}}_p}(G_L) \to \mathbb{Z}$.

Lemma 3.6. *The following claims hold.*

(1) *The assignments $\rho \mapsto d(\rho)$, $\rho \mapsto d^+(\rho)$ and $\rho \mapsto f(\rho)$ given by* (3.7) *are compatible with Artin formalism, and therefore they define maps $\mathrm{Art}_{\bar{\mathbb{Q}}_p}(G_L) \to \mathbb{Z}$ for every number field L.*

(2) *For all number fields L, we have $d(\mathbb{1}_L) = [L : \mathbb{Q}]$, $d^+(\mathbb{1}_L) = |S_\infty(L)|$ and $f(\mathbb{1}_L) = |S_p(L)|$.*

(3) *Let M/L be a Galois extension of number fields. For all $(W,\rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$ with $\mathbb{1}_L \not\subset \rho$, we have*

$$\dim \mathrm{Hom}_{\mathrm{Gal}(M/L)}\left(W, \mathcal{O}_M^{\times,\wedge}\right) = d^+(\rho),$$
$$\dim \mathrm{Hom}_{\mathrm{Gal}(M/L)}\left(W, \mathcal{O}_M\left[\tfrac{1}{p}\right]^{\times,\wedge}\right) = d^+(\rho) + f(\rho).$$

Proof. The proof of (1) and (2) follows from the definitions and Shapiro's lemma. The last claim is a consequence of (3.6) for $S = S_\infty(k)$ and $S = S_\infty(k) \cup S_p(k)$. ■

Lemma 3.7. *Let $M/L/E$ be Galois extensions of number fields, let $(W,\rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_L)$ and let $a\colon \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_L) \to \mathbb{Z}$, $a \in \{d, d^+, f\}$ be one of the maps defined by Lemma 3.6 (1).*

(1) *We have $a(\rho) \leq (\dim \rho) \cdot a(\mathbb{1}_L)$.*

(2) *Suppose that M is the field extension $\overline{\mathbb{Q}}^{\ker \rho}$ cut out by ρ. Then $d^+(\rho) = d(\rho)$ if and only if M is totally real. If $d^+(\rho) = 0$, then L is totally real and M is a CM field.*

(3) *Suppose again that $M = \overline{\mathbb{Q}}^{\ker \rho}$. If M has at least one real place, then $d^+(\rho) \geq \dim \rho$ and any subrepresentation θ of $\mathrm{Ind}_L^E \rho$ satisfies $d^+(\theta) \geq 1$. If L has r complex places, then $d^+(\rho) \geq r \cdot \dim \rho$.*

(4) *Let $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(M/L))$ be irreducible and let χ be a multiplicative character of G_L. Then we have*

$$\begin{array}{ll} (\dim\theta)\cdot a(\theta\otimes\chi) \leq a(\chi_{|G_M}) \leq a(\mathbb{1}_M), & \\ (\dim\theta)\cdot a(\theta\otimes\chi) \leq a(\chi_{|G_M}) - a(\chi) \leq a(\mathbb{1}_M) - a(\chi) & \text{if } \theta \neq \mathbb{1}_L. \end{array}$$

Proof. For any place v of L, note that $\dim \mathrm{H}^0(L_v, W) \leq \dim W$, so (1) follows from (3.7) and Lemma 3.6 (2).

We now assume $M = \overline{\mathbb{Q}}^{\ker \rho}$ and we prove (2) and (3). Denote by $\sigma_v \in \mathrm{Gal}(M/L)$ the complex conjugation attached to an infinite place $v \in S_\infty(L)$. Then $\rho(\sigma_v) \in \mathrm{GL}(W)$ is diagonalizable of order 1 or 2 and $\dim \mathrm{H}^0(k_v, W)$ is the number of eigenvalues of $\rho(\sigma_v)$ that are equal to 1. Hence, in view of (3.7), it is clear that $0 \leq d^+(\rho) \leq d(\rho)$, and we have $d^+(\rho) = d(\rho)$ (resp. $d^+(\rho) = 0$) if and only if all places $v \in S_\infty(L)$ are real and $\rho(\sigma_v)$ is a scalar matrix with eigenvalue 1 (resp. -1). Since ρ induces a faithful representation $\mathrm{Gal}(M/L) \hookrightarrow \mathrm{GL}(W)$, $d^+(\rho) = d(\rho)$ is thus equivalent to $\sigma_v = 1$ for all $v \in S_\infty(L)$, and $d^+(\rho) = 0$ implies that all complex conjugations are equal and nontrivial. This proves (2).

Suppose that M has at least one real place w and let v (resp. v_0) be the place of L (resp. of E) lying below w. For notational simplicity, write $\mathrm{H}^0(-, \rho)$ for $\mathrm{H}^0(-, W)$. Then we clearly have $d^+(\rho) \geq \dim \mathrm{H}^0(L_v, \rho) = \dim \rho$. Moreover, if $\theta \subset \mathrm{Ind}_L^E \rho$, then Frobenius reciprocity implies that there exists a subrepresentation $\rho' \subset \rho$ such that $\rho' \subset \theta_{|G_L}$, yielding

$$d^+(\theta) \geq \dim \mathrm{H}^0(E_{v_0}, \theta) = \dim \mathrm{H}^0(L_v, \theta) \geq \dim \mathrm{H}^0(L_v, \rho') = \dim \rho' \geq 1,$$

as claimed. Suppose finally that L has r complex places $v_1, \ldots, v_r$. Then

$$d^+(\rho) \geq \sum_{i=1}^{r} \dim \mathrm{H}^0(L_{v_i}, \rho) = r \cdot \dim \rho,$$

so this ends the proof of (3).

We now prove (4). The upper bounds on $a(\chi_{|G_M})$ follow from (1), so we only prove the lower bounds. Since θ is irreducible, the representation $(\theta \otimes \chi)^{\oplus \dim \theta}$ (and even $(\theta \otimes \chi)^{\oplus \dim \theta} \oplus \chi$ if $\theta \neq \mathbb{1}_L$) occurs as a subrepresentation of $(\mathrm{Ind}_M^L \mathbb{1}_M) \otimes \chi = \mathrm{Ind}_M^L \chi_{|G_M}$. Artin formalism then yields the lower bounds of claim (5), as $a(\mathrm{Ind}_M^L \chi_{|G_M}) = a(\chi_{|G_M})$ and a takes non-negative values. ∎

We now describe the isotypic components of the map $\mathscr{L}_{K_\infty/K}$. For $g \in G$, $\mathfrak{P} \in S_p(K)$ and η a place of K_∞ above $\mathfrak{P}$, the map $K_{\mathfrak{P}} \to K_{g(\mathfrak{P})}$ (resp. $K_{\infty,\eta} \to K_{\infty,\tilde{g}(\eta)}$) induced by g (resp. by a lift $\tilde{g} \in \mathrm{Gal}(K_\infty/k)$ of g) is a field isomorphism which yields a left G-action $x \mapsto g(x)$ (resp. $\gamma \mapsto \tilde{g}\gamma\tilde{g}^{-1}$) on $\bigoplus_{\mathfrak{P}|p} K_{\mathfrak{P}}^\times$ and on $\bigoplus_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}$, respectively. This action also restricts to $\mathscr{H}_{K_\infty/K}$, and G acts trivially on the quotient $(\bigoplus_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}})/\mathscr{H}_{K_\infty/K}$. Moreover, the map $\mathscr{L}_{K_\infty/K}$ is G-equivariant for the natural G-action on $\mathcal{O}_K[\frac{1}{p}]^\times$ and the action on $\mathscr{H}_{K_\infty/K}$ described above.

Fix any $(W, \rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G)$ and let $\mathrm{Hom}_G(X, Y)$ be the $\overline{\mathbb{Q}}_p$-vector space of all G-equivariant linear maps between two $\overline{\mathbb{Q}}_p[G]$-modules X and Y. By definition, the ρ-isotypic component of a G-equivariant $\overline{\mathbb{Q}}_p$-linear map $f : X \to Y$ is the linear map

$$\mathrm{Hom}_G(W, X) \to \mathrm{Hom}_G(W, Y)$$

obtained defined by post-composition with f. Write $\mathscr{L}_{k_\infty/k}(\rho)$ for the ρ-isotypic component of $\mathscr{L}_{K_{\mathrm{cyc}}/K}$ and define

$$\delta^{\mathbf{G}}_{k_\infty/k}(\rho) := \dim \operatorname{coker}\big(\mathscr{L}_{k_\infty/k}(\rho)\big).$$

If $k_\infty = k_{\mathrm{cyc}}$, we abbreviate $\mathscr{L}_{k_\infty/k}(\rho)$ and $\delta^{\mathbf{G}}_{k_\infty/k}(\rho)$ as $\mathscr{L}_k(\rho)$ and $\delta^{\mathbf{G}}_k(\rho)$, respectively.

For all $\mathfrak{p} \in S_p(k)$, fix a place $\mathfrak{P}_0$ of K above $\mathfrak{p}$, let $G_{\mathfrak{p}}$ be the decomposition subgroup of G at $\mathfrak{P}_0$ and let $W_{\mathfrak{p}}^0 = W^{G_{\mathfrak{p}}}$. Note that $\mathrm{Hom}_{G_{\mathfrak{p}}}(W_{\mathfrak{p}}^0, K_{\mathfrak{P}_0}^{\times,\wedge}) = \mathrm{Hom}(W_{\mathfrak{p}}^0, k_{\mathfrak{p}}^{\times,\wedge})$ as $\mathrm{H}^0(G_{\mathfrak{p}}, K_{\mathfrak{P}_0}^{\times,\wedge}) = k_{\mathfrak{p}}^{\times,\wedge}$.

Proposition 3.8. *The map $\mathscr{L}_{k_\infty/k}(\mathbb{1})$ can be naturally identified with $\mathscr{L}_{k_\infty/k}$. If $\mathbb{1} \not\subset \rho$, then the map $\mathscr{L}_{k_\infty/k}(\rho)$ can be naturally identified with the composite map*

$$\begin{aligned}\mathrm{Hom}_G(W, U) &\xrightarrow{\oplus \mathrm{loc}_{\mathfrak{p}}(\rho)} \bigoplus_{\mathfrak{p}} \mathrm{Hom}_{G_{\mathfrak{p}}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}) \\ &\xrightarrow{\oplus \mathrm{res}_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} \mathrm{Hom}(W_{\mathfrak{p}}^0, k_{\mathfrak{p}}^{\times,\wedge}) \\ &\xrightarrow{\oplus\, \mathrm{rec}^0_{\Gamma_{\mathfrak{p}}}} \bigoplus_{\mathfrak{p}} \mathrm{Hom}(W_{\mathfrak{p}}^0, \Gamma_{\mathfrak{p}}^\wedge). \end{aligned} \tag{3.8}$$

Here, $U = \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}$, $\mathfrak{p}$ runs through $S_p(k)$ in each direct sum, $\Gamma_\mathfrak{p}$ is the decomposition subgroup of Γ at $\mathfrak{p}$, $\mathrm{res}_\mathfrak{p}\colon \mathrm{Hom}_{G_\mathfrak{p}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}) \to \mathrm{Hom}(W_\mathfrak{p}^0, k_\mathfrak{p}^{\times,\wedge})$ is the restriction of morphisms from W to $W_\mathfrak{p}^0$ and $\mathrm{rec}_{\Gamma_\mathfrak{p}}^0\colon \mathrm{Hom}(W_\mathfrak{p}^0, k_\mathfrak{p}^{\times,\wedge}) \to \mathrm{Hom}(W_\mathfrak{p}^0, \Gamma_\mathfrak{p}^\wedge)$ is the post-composition with $\mathrm{rec}_\mathfrak{p}$.

Proof. Let $j\colon \mathcal{H}_{K_\infty/K} \hookrightarrow \bigoplus_{\mathfrak{P}|p} \Gamma_\mathfrak{P}$ be the inclusion map and let $j(\rho)$ be its ρ-isotypic component.

Given a prime $\mathfrak{p} \in S_p(k)$ and a fixed prime $\mathfrak{P}_0|\mathfrak{p}$ of K as before, we have

$$\bigoplus_{\mathfrak{P}|\mathfrak{p}} K_\mathfrak{P}^\times = \mathrm{Ind}_{G_\mathfrak{p}}^G K_{\mathfrak{P}_0}^\times, \quad \bigoplus_{\mathfrak{P}|\mathfrak{p}} \Gamma_\mathfrak{P} = \mathrm{Ind}_{G_\mathfrak{p}}^G \Gamma_{\mathfrak{P}_0}$$

as G-modules. Frobenius reciprocity then shows that

$$\mathrm{Hom}_G\Big(W, \bigoplus_{\mathfrak{P}|\mathfrak{p}} K_\mathfrak{P}^{\times,\wedge}\Big) \simeq \mathrm{Hom}_{G_\mathfrak{p}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}),$$
$$\mathrm{Hom}_G\Big(W, \bigoplus_{\mathfrak{P}|\mathfrak{p}} \Gamma_\mathfrak{P}^{\times,\wedge}\Big) \simeq \mathrm{Hom}_{G_\mathfrak{p}}(W, \Gamma_{\mathfrak{P}_0}^{\times,\wedge}),$$

the isomorphisms being the natural projection maps. Therefore, $j(\rho) \circ \mathcal{L}_{k_\infty/k}(\rho)$ can be identified with the map $\bigoplus_{\mathfrak{p}|p} \mathrm{loc}_\mathfrak{p}(\rho)$ post-composed with the composite map

$$\bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_\mathfrak{p}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}) \xrightarrow{\oplus\, \mathrm{rec}_{\mathfrak{P}_0}(\rho)} \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_\mathfrak{p}}(W, \Gamma_{\mathfrak{P}_0}^\wedge) \xrightarrow{\simeq} \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}(W_\mathfrak{p}^0, \Gamma_{\mathfrak{P}_0}^\wedge), \tag{3.9}$$

where the last identification is given by restriction to $W_\mathfrak{p}^0$. In fact, the map in (3.9) is part of a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_\mathfrak{p}}(W, K_{\mathfrak{P}_0}^{\times,\wedge}) \arrow[r] \arrow[d, "\oplus\, \mathrm{rec}_{\mathfrak{P}_0}(\rho)"] & \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_\mathfrak{p}}(W_\mathfrak{p}^0, K_{\mathfrak{P}_0}^{\times,\wedge}) \arrow[d] \\
\bigoplus_{\mathfrak{p}|p} \mathrm{Hom}_{G_\mathfrak{p}}(W, \Gamma_{\mathfrak{P}_0}^\wedge) \arrow[r, "\simeq"] & \bigoplus_{\mathfrak{p}|p} \mathrm{Hom}(W_\mathfrak{p}^0, \Gamma_{\mathfrak{P}_0}^\wedge),
\end{tikzcd}$$

where the horizontal maps are given by restriction to $W_\mathfrak{p}^0$ and the vertical maps are induced by $\mathrm{rec}_{\Gamma_{\mathfrak{P}_0}}$. Furthermore, the functoriality of Artin's reciprocity law shows that the diagram

$$\begin{tikzcd}
K_{\mathfrak{P}_0}^{\times,\wedge} \arrow[r, "\mathrm{rec}_{\mathfrak{P}_0}"] & \Gamma_{\mathfrak{P}_0}^\wedge \\
k_\mathfrak{p}^{\times,\wedge} \arrow[u, hook] \arrow[r, "\mathrm{rec}_\mathfrak{p}"] & \Gamma_\mathfrak{p}^\wedge \arrow[u, "{[n_\mathfrak{p}]}"']
\end{tikzcd}$$

is commutative, where the left vertical map is induced by the inclusion and $[n_\mathfrak{p}]$ is the multiplication by $n_\mathfrak{p} = [K_{\mathfrak{P}_0} : k_\mathfrak{p}]$. Hence, if $\mathbb{1} \not\subseteq \rho$, then the map $j(\rho) \circ \mathcal{L}_{k_\infty/k}(\rho)$ coincides with the map of (3.8) under the identification $[n_\mathfrak{p}^{-1}] : \Gamma_{\mathfrak{P}_0}^\wedge \simeq \Gamma_\mathfrak{p}^\wedge$. Since $j(\rho)$ is an isomorphism in this case, we obtain the desired description of $\mathcal{L}_{k_\infty/k}(\rho)$.

In the case where $\rho = \mathbb{1}$, the map $\mathscr{L}_{K_\infty/K}(\mathbb{1})$ is the map $\mathcal{O}_k[\frac{1}{p}]^{\times,\wedge} \to (\mathcal{H}^{\wedge}_{K_\infty/K})^G$ obtained by restricting $\mathscr{L}_{K_\infty/K}$ to the G-invariants. Under the identifications

$$\Big(\bigoplus_{\mathfrak{P}|p} \Gamma^{\wedge}_{\mathfrak{P}}\Big)^G = \Big(\bigoplus_{\mathfrak{p}|p} \mathrm{Ind}^G_{G_\mathfrak{p}} \Gamma^{\wedge}_{\mathfrak{P}_0}\Big)^G = \bigoplus_{\mathfrak{p}|p} \Gamma^{\wedge}_{\mathfrak{P}_0} \simeq \bigoplus_{\mathfrak{p}|p} \Gamma^{\wedge}_{\mathfrak{p}}$$

induced by Frobenius reciprocity and by $\bigoplus_\mathfrak{p}[n_\mathfrak{p}^{-1}]$, it is clear that $(\mathcal{H}^{\wedge}_{K_\infty/K})^G$ is mapped onto $\mathcal{H}^{\wedge}_{k_\infty/k}$, and that $\mathscr{L}_{K_{\mathrm{cyc}}/K}(\mathbb{1})$ can be naturally identified with $\mathscr{L}_{k_\infty/k}$. ■

Remark 3.9. The map $\mathscr{L}_{k_\infty/k}(\rho)$ admits a more intrinsic description in terms of canonical Selmer groups attached to the arithmetic dual $\widetilde{W} := \mathrm{Hom}(W, \overline{\mathbb{Q}}_p(1))$ of W. Namely, let $\mathrm{H}^1_{\mathrm{f},p}(k, \widetilde{W}) \subset \mathrm{H}^1(k, \widetilde{W})$ be the Selmer group of $\widetilde{W}$ defined by imposing the Bloch–Kato condition at all places not dividing p and no condition at the p-adic places (see [30, Definition 3.2.1]). Using Kummer maps as e.g. in [29, Lemma 3.6.2], one has a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1_{\mathrm{f},p}(k, \widetilde{W}) \arrow[rr] \arrow[d, equal] & & \mathrm{H}^1(k_\mathfrak{p}, \widetilde{W}) \arrow[d, equal] \\
\mathrm{Hom}_G\big(W, \mathcal{O}_K\big[\tfrac{1}{p}\big]^{\times,\wedge}\big) \arrow[rr, "\mathrm{loc}_\mathfrak{p}(\rho)"] & & \mathrm{Hom}_{G_\mathfrak{p}}\big(W, K^{\times,\wedge}_{\mathfrak{P}_0}\big),
\end{tikzcd}$$

where the upper horizontal map is the localization map on cocycles. Moreover, it can be checked that the kernel of $\mathrm{rec}^0_\mathfrak{p} \circ \mathrm{res}_\mathfrak{p} \colon \mathrm{H}^1(k_\mathfrak{p}, \widetilde{W}) \twoheadrightarrow \mathrm{Hom}(W^0_\mathfrak{p}, \Gamma^{\wedge}_\mathfrak{p})$ coincides with the orthogonal complement, say, $Z_\mathfrak{p}$, of $\mathrm{H}^1(\Gamma_\mathfrak{p}, W^0_\mathfrak{p}) \subset \mathrm{H}^1(k_\mathfrak{p}, W)$ under Tate's local pairing. Proposition 3.8 thus identifies $\mathscr{L}_{k_\infty/k}(\rho)$ with the localization map

$$\mathrm{H}^1_{f,p}(k, \widetilde{W}) \to \bigoplus_{\mathfrak{p}|p} \mathrm{H}^1(k_\mathfrak{p}, \widetilde{W})/Z_\mathfrak{p}$$

under these identifications.

Corollary 3.10. *The following claims hold.*

(1) *If $\rho = \mathbb{1}$ then we have $\delta^{\mathbf{G}}_{k_\infty/k}(\mathbb{1}) = \delta^{\mathbf{G}}_{k_\infty/k}$.*

(2) *Assume $k = \mathbb{Q}$, $k_\infty = \mathbb{Q}_{\mathrm{cyc}}$ and $\mathbb{1} \not\subset \rho$. Fix an embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and let $W^0_p = W^{G_{\mathbb{Q}_p}}$. Then,*

$$\delta^{\mathbf{G}}_{\mathbb{Q}}(\rho) = \dim\mathrm{coker}\Big[\mathrm{Hom}_G\big(W, \mathcal{O}_K\big[\tfrac{1}{p}\big]^{\times,\wedge}\big) \to \mathrm{Hom}(W^0_p, \overline{\mathbb{Q}}_p)\Big],$$

the map being the restriction-to-W^0_p map followed by the post-composition by $\log_p \circ \iota_p$.

Proof. The first claim directly follows from Propositions 3.5 and 3.8. The second claim follows from Proposition 3.8 and from the fact that, if $\Gamma_p = \mathrm{Gal}(\mathbb{Q}_{p,\mathrm{cyc}}/\mathbb{Q}_p)$, then the composite map $\log_p \circ \chi_{\mathrm{cyc}} \circ \mathrm{rec}_{\Gamma_p} \colon \mathbb{Q}_p^{\times,\wedge} \to \Gamma_p^{\wedge} \simeq \overline{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} (1 + p\mathbb{Z}_p) \simeq \overline{\mathbb{Q}}_p$ coincides with $-\log_p$ (see e.g. [32, Chapter V, Theorem 2.4]). ■

Corollary 3.11. *The assignment $\rho \mapsto \delta^{\mathbf{G}}_{k_\infty/k}(\rho)$ is compatible with Artin formalism. More precisely,*

(a) $\delta^{\mathbf{G}}_{k_\infty/k}(\rho)$ *does not depend on the choice of the field K such that $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ factors through* $\mathrm{Gal}(K/k)$.

(b) $\delta^{\mathbf{G}}_{k_\infty/k}(\rho_1 \oplus \rho_2) = \delta^{\mathbf{G}}_{k_\infty/k}(\rho_1) + \delta^{\mathbf{G}}_{k_\infty/k}(\rho_2)$ *for any $\rho_1, \rho_2 \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$.*

(c) *If $k_\infty = kk'_\infty$ for some subfield $k' \subset k$ and $\mathbb{Z}_p$-extension k'_∞/k' and if $\rho' = \mathrm{Ind}_k^{k'} \rho$ is induced from $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$, then $\delta^{\mathbf{G}}_{k'_\infty/k'}(\rho') = \delta^{\mathbf{G}}_{k_\infty/k}(\rho)$.*

Proof. Part (b) is obvious from the definition of $\mathscr{L}_{k_\infty/k}(\rho)$. Part (a) is true if ρ is trivial by Corollary 3.10 (1). Let $K'/K/k$ be Galois extensions, take $\mathbb{1} \not\subseteq \rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/k))$ and denote by $\widetilde{\rho}$ its inflation to $\mathrm{Gal}(K'/k)$. Then the maps of (3.8) attached to ρ and $\widetilde{\rho}$ coincide on $\mathrm{Hom}_{\mathrm{Gal}(K/k)}(W, \mathscr{O}_K[\frac{1}{p}]^{\times,\wedge}) = \mathrm{Hom}_{\mathrm{Gal}(K'/k)}(W, \mathscr{O}_{K'}[\frac{1}{p}]^{\times,\wedge})$, so $\delta^{\mathbf{G}}_{k_\infty/k}(\rho) = \delta^{\mathbf{G}}_{k_\infty/k}(\widetilde{\rho})$. Hence, (a) holds for every $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$.

As for (c), take any extension $K/\overline{\mathbb{Q}}^{\ker\rho}$ which is Galois over k' and put $G = \mathrm{Gal}(K/k)$, $G' = \mathrm{Gal}(K/k')$. Then Frobenius reciprocity identifies the ρ'-isotypic component of $\mathscr{L}_{K_\infty/K}$ (seen as a G'-equivariant map) with the ρ-isotypic component of $\mathscr{L}_{K_\infty/K}$ (seen as a G-equivariant map). Hence, the last property follows from Proposition 3.8. ∎

We next define an analogous invariant for Leopoldt's conjecture. For any Galois extension K/k with Galois group G, the localization map

$$\iota_K \colon \mathscr{O}_K^{\times,\wedge} \to \bigoplus_{\mathfrak{P} \in S_p(K)} \mathscr{O}_{K_\mathfrak{P}}^{\times,\wedge}$$

is clearly G-equivariant. For any $(W, \rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G)$ we let $\delta^{\mathbf{L}}_k(\rho)$ be the dimension of the kernel of the ρ-isotypic component

$$\begin{aligned}\iota_k(\rho) \colon \mathrm{Hom}_G(W, \mathscr{O}_K^{\times,\wedge}) &\to \bigoplus_{\mathfrak{p} \in S_p(k)} \mathrm{Hom}_G\Big(W, \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathscr{O}_{K_\mathfrak{P}}^{\times,\wedge}\Big) \\ &\simeq \bigoplus_{\mathfrak{p} \in S_p(k)} \mathrm{Hom}_{G_\mathfrak{p}}(W, \mathscr{O}_{K_{\mathfrak{P}_0}}^{\times,\wedge}) \end{aligned} \tag{3.10}$$

of ι_K. Here (and as in the definition of $\mathscr{L}_{k_\infty/k}(\rho)$), $\mathfrak{P}_0$ is a fixed place of K above $\mathfrak{p}$ for every place $\mathfrak{p}$ of k. The last isomorphism is given by Frobenius reciprocity and is induced by the natural projection map. As in Corollary 3.11, it is easy to see that the rule $\rho \mapsto \delta^{\mathbf{L}}_k(\rho)$ is compatible with Artin formalism.

Remark 3.12. In terms of Bloch–Kato Selmer groups for $\breve{W} = \mathrm{Hom}(W, \overline{\mathbb{Q}}_p(1))$, the injectivity of $\iota_k(\rho)$ is equivalent to that of the localization map

$$\mathrm{H}^1_\mathrm{f}(k, \breve{W}) \to \prod_{\mathfrak{p}|p} \mathrm{H}^1_\mathrm{f}(k_\mathfrak{p}, \breve{W}).$$

This last statement is Jannsen's conjecture for $\breve{W}$ [15, Question 2].

The following lemma on local Galois representations with finite image will help us describe $\delta_k^{\mathbf{L}}(\rho)$ in another way if $k = \mathbb{Q}$.

Lemma 3.13. *Let $(W, \rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}_p})$ be a local representation factoring through the Galois group of a finite extension $E \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Then the internal multiplication map $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E \to \overline{\mathbb{Q}}_p$ induces an isomorphism*

$$m\colon \mathrm{Hom}_{G_{\mathbb{Q}_p}}(W, \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E) \simeq \mathrm{Hom}_{\overline{\mathbb{Q}}_p}(W, \overline{\mathbb{Q}}_p).$$

Here, we let $G_{\mathbb{Q}_p}$ act on $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E$ via $g(a \otimes x) = a \otimes g(x)$.

Proof. Notice first that $E \simeq \mathbb{Q}_p[\mathrm{Gal}(E/\mathbb{Q}_p)]$ as Galois modules by the normal basis theorem, allowing us to write

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Gal}(E/\mathbb{Q}_p)}(W, \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E) &\simeq \mathrm{Hom}_{\mathrm{Gal}(E/\mathbb{Q}_p)}\big(W, \overline{\mathbb{Q}}_p\big[\,\mathrm{Gal}(E/\mathbb{Q}_p)\big]\big)\\ &\simeq \mathrm{Hom}_{\overline{\mathbb{Q}}_p}(W, \overline{\mathbb{Q}}_p),\end{aligned}$$

the last isomorphism coming from Frobenius reciprocity. (Note that the resulting composite map differs from m.) Therefore, it is enough to show that m is injective.

As ρ has finite image, one may choose a finite Galois extension $L/\mathbb{Q}_p$ which contains E and over which ρ is realizable, i.e., there exist a $L[G_{\mathbb{Q}_p}]$-module W_L and an isomorphism $W_L \otimes_L \overline{\mathbb{Q}}_p \simeq W$ (see, for instance, Brauer's theorem [35, Section 12.3, Théorème 24]). By the base change properties of Hom's, we then have to show that the map $\mathrm{Hom}_{L[G_{\mathbb{Q}_p}]}(W_L, L \otimes_{\mathbb{Q}_p} E) \to \mathrm{Hom}_L(W_L, L)$ is injective. Since any L-linear homomorphism is $\mathbb{Q}_p$-linear, it suffices to show that

$$\mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(W_L, L \otimes_{\mathbb{Q}_p} E) \to \mathrm{Hom}_{\mathbb{Q}_p}(W_L, L)$$

is injective, i.e., the map

$$\begin{aligned}&\big(\mathrm{Hom}_{\mathbb{Q}_p}(W_L, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} E\big)^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Q}_p} L\\ &\quad = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(W_L, L \otimes_{\mathbb{Q}_p} E) \to \mathrm{Hom}_{\mathbb{Q}_p}(W_L, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} L\end{aligned}$$

(sending $h \otimes e \otimes \ell$ to $h \otimes (e\ell)$) is injective, where we let $G_{\mathbb{Q}_p}$ act on both terms of the first tensor product. To this end, put $V = \mathrm{Hom}_{\mathbb{Q}_p}(W_L, \mathbb{Q}_p)$ and apply the exact functor $- \otimes_L \overline{\mathbb{Q}}_p$ to the preceding map. Observing that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$ acts trivially on V, we obtain the map

$$(V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \to V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$$

given by $(v \otimes a) \otimes b \mapsto v \otimes (ab)$. This map turns out to be injective by Fontaine's theory of admissible B-representations, the ring B being, in our case, $\overline{\mathbb{Q}}_p$ equipped with its standard $G_{\mathbb{Q}_p}$-action (see e.g. in [10, Theorem 2.13 (1)]). Hence, m is an isomorphism. ∎

Proposition 3.14. *Let $(W, \rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ factoring through $\mathrm{Gal}(K/\mathbb{Q})$ for some $K \subset \overline{\mathbb{Q}}$ and let $\mathfrak{P}_0$ be the p-adic place of K defined by a fixed embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We have*

$$\delta_{\mathbb{Q}}^{\mathbf{L}}(\rho) = \dim\ker\big[\mathcal{L} :\ \mathrm{Hom}_{G_{\mathbb{Q}}}(W, \mathcal{O}_K^{\times,\wedge}) \to \mathrm{Hom}(W, \overline{\mathbb{Q}}_p)\big],$$

the map $\mathcal{L}$ *being the post-composition by* $\log_p \circ \iota_p : \mathcal{O}_K^{\times,\wedge} \to \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_{\mathfrak{P}_0}$ *followed by the internal multiplication* $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_{\mathfrak{P}_0} \to \overline{\mathbb{Q}}_p$.

Proof. First note that the p-adic logarithm $\log_p : \mathcal{O}_E^{\times} \to E$ over a finite extension E of $\mathbb{Q}_p$ induces an isomorphism $\mathcal{O}_E^{\times,\wedge} \simeq \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E$. Applying Lemma 3.13 to $E = K_{\mathfrak{P}_0}$ we obtain isomorphisms

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(W, \mathcal{O}_{K_{\mathfrak{P}_0}}^{\times,\wedge}) \simeq \mathrm{Hom}_{G_{\mathbb{Q}_p}}(W, \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_{\mathfrak{P}_0}) \simeq \mathrm{Hom}(W, \overline{\mathbb{Q}}_p).$$

The map $\mathcal{L}$ is simply the map $\iota_{\mathbb{Q}}(\rho)$ composed with these isomorphisms, so the claim follows. ∎

4. Bounds on Leopoldt's and Gross's defects

Throughout this section we fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

4.1. Bounds on Leopoldt's defect

Theorem 4.1. *Let* $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ *be irreducible and let* $d = d(\rho)$, $d^+ = d^+(\rho)$. *We have* $\delta_{\mathbb{Q}}^{\mathbf{L}}(\rho) \le \frac{(d^+)^2}{d+d^+}$.

Proof. Since $\delta_{\mathbb{Q}}^{\mathbf{L}}(\mathbb{1}) = \delta_{\mathbb{Q}}^{\mathbf{L}} = 0$, we may assume that $\rho \neq \mathbb{1}$. Let $K/\mathbb{Q}$ be any Galois extension containing $\overline{\mathbb{Q}}^{\ker \rho}$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$. Recall from Lemma 3.6 (3) the equality $\dim \mathrm{Hom}_G(W, \mathcal{O}_K^{\times,\wedge}) = d^+$. By Proposition 3.14 it is enough to show that the rank of the map

$$\mathrm{Hom}_G(W, \mathcal{O}_K^{\times,\wedge}) \to \mathrm{Hom}(W, \overline{\mathbb{Q}}_p)$$

induced by $a \otimes x \mapsto a \log_p(\iota_p(x))$ on $\mathcal{O}_K^{\times,\wedge}$ is at least $\frac{d \cdot d^+}{d+d^+}$.

Consider a $\overline{\mathbb{Q}}$-structure on W, that is, a $\overline{\mathbb{Q}}$-linear representation $W_{\overline{\mathbb{Q}}}$ of G such that $W_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p \simeq W$. Let also $w_1, \dots, w_d$ be a $\overline{\mathbb{Q}}$-basis of $W_{\overline{\mathbb{Q}}}$. As ρ is irreducible, we can consider its idempotent $e(\rho)$ defined in (3.5), and by (3.6) there exists an isomorphism

$$e(\rho) \cdot \big(\overline{\mathbb{Q}} \otimes \mathcal{O}_K^{\times}\big) \simeq W_{\overline{\mathbb{Q}}}^{\oplus d^+},$$

which we fix. Using this isomorphism, we may define for all $1 \le j \le d^+$ a $\overline{\mathbb{Q}}[G]$-linear map

$$\Psi_j : W_{\overline{\mathbb{Q}}} \xrightarrow{\mathrm{incl}_j} W_{\overline{\mathbb{Q}}}^{\oplus d^+} \hookrightarrow \overline{\mathbb{Q}} \otimes \mathcal{O}_K^{\times}, \tag{4.1}$$

where incl_j maps $W_{\overline{\mathbb{Q}}}$ maps onto the j-th component of the direct sum. The homomorphisms $\Psi_1, \dots, \Psi_{d^+}$ form a basis of the space $\mathrm{Hom}_G(W_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}} \otimes \mathcal{O}_K^{\times})$ over $\overline{\mathbb{Q}}$ and of $\mathrm{Hom}_G(W, \mathcal{O}_K^{\times,\wedge})$ over $\overline{\mathbb{Q}}_p$. Moreover, the elements $\Psi_j(w_i) \in \overline{\mathbb{Q}} \otimes \mathcal{O}_K^{\times}$ for $1 \le i \le d$, $1 \le j \le d^+$ are $\overline{\mathbb{Q}}$-linearly independent by construction, as well as their p-adic logarithms by Proposition 2.2. Hence, the matrix $M = (\log_p(\iota_p(\Psi_j(w_i))))_{i,j}$ of size $d \times d^+$ satisfies $\theta(M) = \frac{d^+}{d}$ and Theorem 2.3 implies $\mathrm{rk}\, M \ge \frac{d \cdot d^+}{d+d^+}$ as claimed. ∎

Corollary 4.2. *The following claims hold.*

(1) *Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ and let $d^+ = d^+(\rho)$. We have $\delta_k^{\mathbf{L}}(\rho) \le d^+/2$.*

(2) *For all finite extensions K/k of number fields, we have*

$$\delta_K^{\mathbf{L}} \le \delta_k^{\mathbf{L}} + (\mathrm{rk}\, \mathcal{O}_K^\times - \mathrm{rk}\, \mathcal{O}_k^\times)/2.$$

Proof. Since $d^+ \le d$, the first inequality obviously follows from Theorem 4.1 for irreducible $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$, hence it follows for general ρ from Artin formalism for $\rho \mapsto d(\rho)$ and $\rho \mapsto d^+(\rho)$. By Lemma 3.6 (1)–(2), the unique representation ρ_0 of G_k such that $\mathrm{Ind}_K^k \mathbb{1} = \mathbb{1} \oplus \rho_0$ satisfies $d^+(\rho_0) = \mathrm{rk}\, \mathcal{O}_K^\times - \mathrm{rk}\, \mathcal{O}_k^\times$, so the second inequality follows from the first one applied to ρ_0. ∎

4.2. Bounds on Gross's defect

Theorem 4.3. *Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ be irreducible and let $f = f(\rho)$, $d^+ = d^+(\rho)$. If $d^+ = f = 0$, then $\delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$. Otherwise, we have $\delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) \le \frac{f^2}{d^+ + 2f}$.*

Proof. Recall that $\dim \mathrm{Hom}_G(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}) = d^+ + f$ by Lemma 3.6 (3) and that $\dim W_p^0 = f$. If $\rho = \mathbb{1}$ or $f = 0$, then the codomain of $\mathscr{L}_{\mathbb{Q}}(\rho)$ is $\{0\}$, yielding $\delta_{\mathbb{Q}}^{\mathbf{G}}(\rho) = 0$. We assume henceforth that $\rho \ne \mathbb{1}$ and $f > 0$. Let $K/\mathbb{Q}$ be any Galois extension containing $\overline{\mathbb{Q}}^{\ker \rho}$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$. By Corollary 3.10 (2) it is enough to show that the rank of the map $\mathrm{Hom}_G(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}) \to \mathrm{Hom}(W_p^0, \overline{\mathbb{Q}}_p)$ induced by $a \otimes x \mapsto a \log_p(\iota_p(x))$ on $\mathcal{O}_K[\frac{1}{p}]^{\times,\wedge}$ is at least $\frac{(d^+ + f)\cdot f}{d^+ + 2f}$.

As in the proof of Theorem 4.1, fix a $\overline{\mathbb{Q}}$-structure $W_{\overline{\mathbb{Q}}}$ of W. Fix also a basis $w_1, \dots, w_f$ of the subspace $W_{\overline{\mathbb{Q}},p}^0$ of $G_{\mathbb{Q}_p}$-invariants of $W_{\overline{\mathbb{Q}}}$ and an isomorphism

$$e(\rho) \cdot \left(\overline{\mathbb{Q}} \otimes \mathcal{O}_K[\tfrac{1}{p}]^\times\right) \simeq W_{\overline{\mathbb{Q}}}^{\oplus (d^+ + f)}.$$

Then a similar definition as that in (4.1) yields a basis $\Psi_1, \dots, \Psi_{d^+ + f}$ of the space $\mathrm{Hom}_G(W, \mathcal{O}_K[\frac{1}{p}]^{\times,\wedge})$ such that the elements $\Psi_j(w_i) \in \overline{\mathbb{Q}} \otimes \mathcal{O}_K^\times$ for $1 \le i \le f$, $1 \le j \le d^+ + f$ are $\overline{\mathbb{Q}}$-linearly independent. Since $e(\rho)$ kills $p^{\overline{\mathbb{Q}}}$, we deduce from Proposition 2.2 that the entries of the matrix

$$M' = \left(\log_p\left(\iota_p\left(\Psi_j(w_i)\right)\right)\right)_{i,j}$$

of size $f \times (d^+ + f)$ are $\overline{\mathbb{Q}}$-linearly independent as well. Therefore, $\theta(M') = \frac{d^+ + f}{f}$ and Theorem 2.3 implies $\mathrm{rk}\, M' \ge \frac{(d^+ + f)\cdot f}{d^+ + 2f}$. ∎

Corollary 4.4. *The following claims hold.*

(1) *Let $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$, let K be the field cut out by ρ and let $f = f(\rho)$. Denote by $S_{\mathbb{R}}(K)$ the set of real places of K. Then the following inequalities hold true:*

$$\delta_k^{\mathbf{G}}(\rho) \begin{cases} \le f/2 & \\ < f/2 & \text{if } f \ne 0 \text{ and } S_{\mathbb{R}}(K) \ne \emptyset, \\ \le f/3 & \text{if } K \text{ is totally real.} \end{cases}$$

(2) *Let K/k be a finite extension of number fields. Then the following inequalities hold true:*

$$\delta_K^{\mathbf{G}} \begin{cases} \leq \delta_k^{\mathbf{G}} + \big(|S_p(K)| - |S_p(k)|\big)/2 \\ < \delta_k^{\mathbf{G}} + \big(|S_p(K)| - |S_p(k)|\big)/2 & \text{if } |S_p(K)| \neq |S_p(k)| \text{ and } S_{\mathbb{R}}(K) \neq \emptyset, \\ \leq \delta_k^{\mathbf{G}} + \big(|S_p(K)| - |S_p(k)|\big)/3 & \text{if } K \text{ is totally real.} \end{cases}$$

Proof. We know that $\rho \mapsto \delta_k^{\mathbf{G}}(\rho)$ and $\rho \mapsto f(\rho)$ are compatible with Artin formalism by Corollary 3.11 and Lemma 3.6 (1). We now explain how to prove (1). By Artin formalism, it suffices to prove (1) with ρ replaced by any irreducible subrepresentation $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_{\mathbb{Q}})$ of $\mathrm{Ind}_k^{\mathbb{Q}} \rho$. For such a θ, Lemma 3.7 (2)–(3) implies that $d^+(\theta) \geq 1$ if $S_{\mathbb{R}}(K) \neq \emptyset$, and $d^+(\theta) = d(\theta)$ if K is totally real. Therefore, the inequalities in (1) for θ directly follow from Theorem 4.3. Finally, one proves (2) by applying (1) to

$$\rho = \mathrm{Ind}_K^k \mathbb{1}_K - \mathbb{1}_k$$

and using that $f(\mathrm{Ind}_K^k \mathbb{1}_K) = |S_p(K)|$ and $f(\mathbb{1}_k) = |S_p(k)|$ by Lemma 3.6 (2). ∎

Remark 4.5. The matrices M and M' appearing in the course of the proof of Theorems 4.1 and 4.3 have full rank under the p-adic Schanuel conjecture. Artin formalism thus shows that the p-adic Schanuel conjecture implies Leopoldt's conjecture (resp. Gross–Kuz'min's conjecture) in great generality, as already shown in [16] (resp. in [25]).

4.3. Applications

Theorem 4.6. *Let k be a totally real number field and let $(V, \rho) \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$ be such that $d^+(\rho) = 0$. Then Gross's p-adic regulator matrix $R_p(V)$ [12, (2.10)] is of size $f(\rho)$ and of rank at least $f(\rho)/2$.*

Proof. Let K be the CM field cut out by ρ and let $(\mathrm{Hom}_{\overline{\mathbb{Q}}_p}(V, \overline{\mathbb{Q}}_p), \rho^*)$ be the contragredient representation of ρ. Gross's regulator map λ_p defined in [12, (1.18)] can be identified with the "minus part" of $\mathscr{L}_K$, which is, by definition, the restriction of $\mathscr{L}_K$ to the subspace where the complex conjugation acts by -1. This means that λ_p and $\mathscr{L}_K$ share the same θ-isotypic component for every representation $\theta \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/k))$ such that $d^+(\theta) = 0$. Since taking $(V \otimes -)^{G_k}$ amounts to taking ρ^*-isotypic components, we conclude that $\mathrm{rk}\, R_p(V) = \mathrm{rk}\, \mathscr{L}_k(\rho^*) = f(\rho^*) - \delta_k^{\mathbf{G}}(\rho^*)$, so $\mathrm{rk}\, R_p(V) \geq f(\rho^*)/2 = f(\rho)/2$ by Theorem 4.3. ∎

In the next theorem, we write k^+ for the maximal totally real subfield of a number field k, and $\mathbb{Q}^{\mathrm{ab}}$ for the maximal abelian extension of $\mathbb{Q}$.

Theorem 4.7. *Let K/k be an abelian extension of number fields. The Gross–Kuz'min conjecture holds for K in each of the following cases.*

(a) *Either $|S_p(K)| \leq 2$, or $|S_p(K)| \leq 3$ and K has at least one real place, or $|S_p(K)| \leq 4$ and $K/\mathbb{Q}$ is Galois, or $|S_p(K)| \leq 6$ and $K/\mathbb{Q}$ is a real Galois extension.*

(b) *$|S_p(k)| = 1$, or $|S_p(k)| \leq 2$ and K has at least one real place.*

(c) *$K \subset k \cdot \mathbb{Q}^{\mathrm{ab}}$, $k/\mathbb{Q}$ is Galois, and either $|S_p(k)| \leq 3$, or $|S_p(k)| \leq 5$ and K is real.*

(d) *k is an imaginary quadratic field, or k is a real quadratic field and K has at least one real place.*

(e) *$k/\mathbb{Q}$ is Galois, $|S_p(k)| \leq 2$, $|S_p(k^+)| = 1$ and $[K : k]$ and $[k : \mathbb{Q}]$ are coprime.*

Proof. Recall that $\delta^{\mathbf{G}}(-)$ is compatible with Artin formalism by Corollary 3.11. We shall often appeal to Lemmas 3.6 and 3.7 and to the following consequence of Theorem 4.3 without further notice. For any irreducible representation

$$\theta \in \mathrm{Art}_{\bar{\mathbb{Q}}_p}(G_{\mathbb{Q}}),$$

we have $\delta^{\mathbf{G}}_{\mathbb{Q}}(\theta) = 0$ if $f(\theta) \leq 1$, or if $f(\theta) = 2$ and $d^+(\theta) \geq 1$. In particular, $\delta^{\mathbf{G}}_{\mathbb{Q}}(\theta) = 0$ if θ is a multiplicative character of $G_{\mathbb{Q}}$, so $\delta^{\mathbf{G}}_{M} = 0$ for any abelian extension $M/\mathbb{Q}$ by Artin formalism.

Since $\delta^{\mathbf{G}}_{\mathbb{Q}} = 0$, it follows from Corollary 4.4 that $\delta^{\mathbf{G}}_{K} = 0$ for K satisfying one of the two first assumptions in case (a). Consider the two last assumptions in (a) and assume that $K/\mathbb{Q}$ is Galois. We claim that $\delta^{\mathbf{G}}_{\mathbb{Q}}(\theta) = 0$ for all irreducible $\theta \in \mathrm{Art}_{\bar{\mathbb{Q}}_p}(\mathrm{Gal}(K/\mathbb{Q}))$. We may assume that $\dim\theta \geq 2$, so $f(\theta) \leq (f(\mathbb{1}_K) - f(\mathbb{1}_{\mathbb{Q}}))/(\dim\theta) \leq (|S_p(K)| - 1)/2$. The two last assumptions in (a) ensure that we either have $f(\theta) \leq 1$, or $f(\theta) \leq 2$ and $d^+(\theta) = d(\theta) \geq 1$, so we indeed have $\delta^{\mathbf{G}}_{\mathbb{Q}}(\theta) = 0$. Therefore, $\delta^{\mathbf{G}}_{K} = 0$ in case (a).

Let $G = \mathrm{Gal}(K/k)$ and let $\widehat{G} = \mathrm{Hom}(G, \bar{\mathbb{Q}}_p^{\times})$ be the group of linear characters of G. We place ourselves in cases (b), (c) and (d), we fix $\chi \in \widehat{G}$ and we show that $\delta^{\mathbf{G}}_{k}(\chi) = 0$. Since $f(\chi) \leq |S_p(k)|$, Corollary 4.4 (1) implies $\delta^{\mathbf{G}}_{k}(\chi) = 0$ in case (b). Suppose now we are in case (c). Then χ descends to a character $\chi_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$. Moreover, as $k/\mathbb{Q}$ is Galois, any irreducible subrepresentation ρ of $\mathrm{Ind}_k^{\mathbb{Q}}\chi \simeq (\mathrm{Ind}_k^{\mathbb{Q}}\mathbb{1}_k) \otimes \chi_{\mathbb{Q}}$ occurs $(\dim\rho)$ times, so it satisfies $f(\rho) \leq |S_p(k)|/(\dim\rho)$. Moreover, if K is totally real, then any such ρ satisfies $d^+(\rho) = d(\rho) \geq 1$, so we can conclude $\delta^{\mathbf{G}}_{\mathbb{Q}}(\rho) = 0$. Therefore, $\delta^{\mathbf{G}}_{k}(\chi) = 0$ in case (c). We now assume to be in case (d). Then $\mathrm{Ind}_k^{\mathbb{Q}}\chi$ has dimension 2, so it is either irreducible or it is the sum of two characters of $G_{\mathbb{Q}}$, say η_1 and η_2. In the latter case, we already know that $\delta^{\mathbf{G}}_{\mathbb{Q}}(\eta_i) = 0$ for $i = 1, 2$, so $\delta^{\mathbf{G}}_{k}(\chi) = \delta^{\mathbf{G}}_{\mathbb{Q}}(\mathrm{Ind}_k^{\mathbb{Q}}\chi) = 0$. If $\mathrm{Ind}_k^{\mathbb{Q}}\chi$ is irreducible, then the assumptions on K and k imply that $f(\chi) \leq 2$ and $d^+(\chi) \geq 1$ by Lemma 3.7 (3), yielding $\delta^{\mathbf{G}}_{k}(\chi) = 0$. Therefore, $\delta^{\mathbf{G}}_{K} = 0$ in the cases (b), (c), and (d).

We now make the assumptions in (e) and we assume without loss of generality that K is Galois over $\mathbb{Q}$ with Galois group $\mathcal{G}$. By the Schur–Zassenhaus theorem, $\mathcal{G}$ is the semi-direct product of $H := \mathrm{Gal}(k/\mathbb{Q})$ acting on G. Let $\rho \in \mathrm{Art}_{\bar{\mathbb{Q}}_p}(\mathcal{G})$ be irreducible and let us prove that $\delta^{\mathbf{G}}_{\mathbb{Q}}(\rho) = 0$. By [35, Chapitre II, Section 8.2], ρ can be written as $\mathrm{Ind}_{k'}^{\mathbb{Q}}(\theta \otimes \chi)$, where $k'/\mathbb{Q}$ is a subextension of $k/\mathbb{Q}$, θ an irreducible representation of $\mathrm{Gal}(k/k')$ and χ a character of $\mathrm{Gal}(K/k')$. Note that $f(\rho) \leq |S_p(k)|/(\dim\theta) \leq 2/(\dim\theta)$, so we may assume without loss of generality that $\dim\theta = 1$. If k' is totally real, then

$$f(\rho) \leq \left|S_p(k')\right| \leq \left|S_p(k^+)\right| = 1,$$

and otherwise we have $d^+(\rho) \geq 1$. In any case, $\delta^{\mathbf{G}}_{\mathbb{Q}}(\rho) = 0$ so we may infer $\delta^{\mathbf{G}}_{K} = 0$ in case (e) as well. ∎

Remark 4.8. We compare in this remark our results with earlier works mentioned in the introduction.

(i) Theorem 4.7 (c) and (a) imply the Gross–Kuz'min conjecture in the cases of abelian extensions of $\mathbb{Q}$ (as already shown in [11]) and for number fields with at most 2 p-adic places (which is proven in [22]).

(ii) Jaulent [18] proves that if K/k is such that k satisfies the Gross–Kuz'min conjecture and $|S_p(K)| = |S_p(k)|$, then K also satisfies the Gross–Kuz'min conjecture. In fact, given a $\mathbb{Z}_p$-extension k_∞/k and $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(G_k)$, one has

$$\delta^{\mathbf{G}}_{k_\infty/k}(\rho) \le \sum_{\mathfrak{p}|p} \dim \mathrm{Hom}(W^0_{\mathfrak{p}}, \Gamma^\wedge_{\mathfrak{p}}) \le \sum_{\mathfrak{p}|p} \dim W^0_{\mathfrak{p}} = f(\rho)$$

by Proposition 3.8. Applying this to $\rho = \mathrm{Ind}^k_K \mathbb{1}_K - \mathbb{1}_k$, we obtain

$$\delta^{\mathbf{G}}_{Kk_\infty/K} - \delta^{\mathbf{G}}_{k_\infty/k} \le |S_p(K)| - |S_p(k)|$$

by Artin formalism (Corollary 3.11). Jaulent's claim then follows from this inequality with $k_\infty = k_{\mathrm{cyc}}$ and Proposition 3.5.

(iii) Jaulent [17] proves the Gross–Kuz'min conjecture for real Galois extensions $K/\mathbb{Q}$ in which p splits completely, under the additional assumption that the group algebra $\mathbb{Q}_p[\mathrm{Gal}(K/\mathbb{Q})]$ does not contain a matrix algebra $M_n(D)$ over a skew-field D with $n > 2$. If we slightly strengthen this last assumption by supposing that $\mathrm{Gal}(K/\mathbb{Q})$ only has one- or two-dimensional irreducible representations over $\overline{\mathbb{Q}}_p$ but we do not make any splitting assumption on p in K, then K still satisfies the Gross–Kuz'min conjecture. Indeed, any irreducible $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/\mathbb{Q}))$ satisfies $f(\rho) \le d(\rho) \le 2$, while $d^+(\rho) = d(\rho) \ge 1$, so $\delta^{\mathbf{G}}_{\mathbb{Q}}(\rho) = 0$ by Theorem 4.3. Therefore, $\delta^{\mathbf{G}}_K = 0$ for such a K.

(iv) Kuz'min [25] proved that any S_4-extension K of $\mathbb{Q}$ satisfies the Gross–Kuz'min conjecture under the assumptions that the unique quadratic subfield E is imaginary and that the decomposition subgroup D_p of p in K is the 3-Sylow V_3 of S_4. In fact, the first assumption can be relaxed and it is enough that D_p contains V_3. The same conclusion holds if we instead assume that K is real, and that p splits in E but not completely in K. The proof of these facts is similar to that in Example 4.9.

Example 4.9. Any totally real S_5-extension $K/\mathbb{Q}$ satisfies the Gross–Kuz'min conjecture, provided that the decomposition subgroup of p (say D_p) in K contains the 5-Sylow of S_5. Indeed, assume first that $|D_p| = 5$, so that $|S_p(K)| = 24$. The irreducible representations of S_5 are $\mathbb{1}$, sgn (the sign character), Std (the standard representation, 4-dimensional), Std $\otimes$ sgn, and π_1, π_2, π_3 (of respective dimensions $5, 5, 6$). As sgn is of prime-to-5 order, $\mathrm{sgn}_{|G_{\mathbb{Q}_p}}$ is trivial, so $f(\mathrm{sgn}) = 1$ and $f(\mathrm{Std}) = f(\mathrm{Std} \otimes \mathrm{sgn})$. In light of (3.4), we obtain

$$24 = f(\mathbb{1}_K) = f(\mathrm{Ind}^{\mathbb{Q}}_K \mathbb{1}_K) = 1 + 1 + 8 \cdot f(\mathrm{Std}) + 5 \cdot f(\pi_1) + 5 \cdot f(\pi_2) + 6 \cdot f(\pi_3).$$

It is then clear that all the f's involved are at most 2. Moreover, $K/\mathbb{Q}$ being totally real, Lemma 3.7 (2) shows that $d^+(\rho) \geq 1$ for all $\rho \in \mathrm{Art}_{\overline{\mathbb{Q}}_p}(\mathrm{Gal}(K/\mathbb{Q}))$, so $\delta^{\mathbf{G}}_{\mathbb{Q}}(\rho) = 0$ by Theorem 4.3. Hence, $\delta^{\mathbf{G}}_K = 0$. The case where $|D_p| > 5$ is treated similarly, noticing that $|S_p(K)| \leq 12$.

5. Vanishing locus of Gross's defect

5.1. Preliminaries

This section is devoted to the proof of the following theorem which, in turn, implies Theorem 1.8 stated in the introduction.

Theorem 5.1. *Let k be a number field and let $\varphi\colon G_k \to \overline{\mathbb{Q}}_p^\times$ be a finite-order character. Assume also that p completely splits in the number field cut out by φ.*

1. *Assume that $r_2 + \delta^{\mathbf{L}}_k \leq 1$, where r_2 is the number of complex places of k. If there exists at least one $\mathbb{Z}_p$-extension k_∞ of k such that $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) = 0$, then there are at most $[k:\mathbb{Q}]$ (and at most $[k:\mathbb{Q}] - 1$ if $\varphi = \mathbb{1}$) $\mathbb{Z}_p$-extensions k_∞/k such that $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) \neq 0$.*
2. *Assume that k is an imaginary quadratic field. There is at most one $\mathbb{Z}_p$-extension k_∞/k for which $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) \neq 0$, and the slope of k_∞/k defined in Example 5.5 is finite and transcendental. Moreover, if φ cuts out an abelian extension of $\mathbb{Q}$ or if the polynomial $XYZ^2 - (AX - BY)(CX - DY)$ does not vanish on any 7-tuple $(a, b, c, d, x, y, z) \in \Lambda^7$ which form a $\overline{\mathbb{Q}}$-linearly independent set, then $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) = 0$ for any $\mathbb{Z}_p$-extension k_∞ of k.*

Proof of Theorem 1.8, assuming Theorem 5.1. Let K be an abelian extension of an imaginary quadratic field k and let $K^{\mathrm{ab}} \subset K$ be its maximal absolutely abelian subfield. By Artin formalism (Corollary 3.11) and by Theorem 5.1, we have

$$\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$$

for a given $\mathbb{Z}_p$-extension k_∞/k if and only if there exists a character φ of $\mathrm{Gal}(K/k)$ such that $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) > 0$. Such a character cannot be a character of $\mathrm{Gal}(K^{\mathrm{ab}}/k)$, and moreover, k_∞ is uniquely determined by φ. Therefore, we have

$$\delta^{\mathbf{G}}_{Kk_\infty/K} > 0$$

for at most $[K:k] - [K^{\mathrm{ab}}:k]$ distinct $\mathbb{Z}_p$-extensions of k. ∎

In the rest of Section 5, we fix once and for all an abelian extension K/k with Galois group G such that p totally splits in K. We let n be the degree of k and (r_1, r_2) its signature, and we put $r = r_1 + r_2 - 1 - \delta^{\mathbf{L}}_k$ so that the maximal multiple $\mathbb{Z}_p$-extension of k has rank $n - r$. Write $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ for the p-adic primes of k. Finally, denote by $\mathscr{Z}(k)$ the set of all $\mathbb{Z}_p$-extensions of k.

Instead of working with the map $\mathscr{L}_{k_\infty/k}(\varphi)$, it will be more convenient to consider the following alternative description of $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi)$.

Lemma 5.2. *Let k_∞/k be a $\mathbb{Z}_p$-extension with Galois group Γ. The quantity $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi)$ is the dimension of the kernel of the φ-isotypic component of the map $\mathrm{Loc}_{K_\infty/K}$ of Proposition 3.3.*

Proof. This follows from Poitou–Tate duality as in the proof of Proposition 3.5 where one replaces the G_K-module $\mathbb{Q}_p$ by the module $\overline{\mathbb{Q}}_p(\varphi)$ on which G_k acts by φ. ∎

Since φ is a multiplicative character, the φ-isotypic component of a $\overline{\mathbb{Q}}_p[G]$-module X is canonically isomorphic to the linear subspace $X[\varphi]$ of X consisting of elements $x \in X$ such that $g \cdot x = \varphi(g)x$ for all $g \in G$. Hence Lemma 5.2 asserts that, if $\varphi \neq \mathbb{1}_k$, then $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi)$ is equal to the dimension of the kernel of the localization map

$$\mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p)[\varphi] \to \bigoplus_{i=1}^{n} \Big(\bigoplus_{\mathfrak{P}|\mathfrak{p}_i} \mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)/\, \mathrm{Hom}(\Gamma_{\mathfrak{P}}, \overline{\mathbb{Q}}_p)\Big)[\varphi]. \tag{5.1}$$

5.2. Matrices in logarithms of algebraic numbers

For any $\mathfrak{P} \in S_p(K)$, the local Artin reciprocity map identifies the maximal pro-p quotient of the abelianization of $G_{K_{\mathfrak{P}}}$ with the p-adic completion of $K^\times_{\mathfrak{P}}$, so under this identification we have

$$\mathrm{H}^1(K_{\mathfrak{P}}, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(K^\times_{\mathfrak{P}}, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(\mathbb{Q}_p^\times, \overline{\mathbb{Q}}_p),$$

where Hom refers to continuous group homomorphisms. We also see $\log_p$ and the p-adic valuation map ord_p as characters $K^\times_{\mathfrak{P}} \simeq \mathbb{Q}_p^\times \to \overline{\mathbb{Q}}_p$. In order to describe elements in the domain of the map (5.1) we make use of the short exact sequence of $\overline{\mathbb{Q}}_p[G]$-modules

$$0 \to \mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p) \xrightarrow{A} \bigoplus_{i=1}^{n} \mathrm{Hom}\Big(\prod_{\mathfrak{P}|\mathfrak{p}_i} K^\times_{\mathfrak{P}}, \overline{\mathbb{Q}}_p\Big) \to \mathrm{Hom}\big(\mathcal{O}_K\big[\tfrac{1}{p}\big]^\times, \overline{\mathbb{Q}}_p\big), \tag{5.2}$$

where A is induced by the Artin map. The exactness of (5.2) follows from that of the second row in [33, Lemma 10.3.13] by taking $\overline{\mathbb{Q}}_p$-linear duals, after noticing that

$$\mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(G_{S_p(K)}, \overline{\mathbb{Q}}_p),$$

where $G_{S_p(K)}$ is the Galois group of the maximal extension of K unramified outside p.

Let $1 \le i \le n$ and fix a prime $\mathfrak{P}_i$ of K above $\mathfrak{p}_i$. We define a basis $\{\eta_{i,\varphi}, \widetilde{\eta}_{i,\varphi}\}$ of the φ-component of $\mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K^\times_{\mathfrak{P}}, \mathbb{Q}_p)$ as follows. First define characters η_i and $\widetilde{\eta}_i$ of $\prod_{\mathfrak{P}|\mathfrak{p}_i} K^\times_{\mathfrak{P}}$ by imposing that they are supported on $K^\times_{\mathfrak{P}_i}$ and that $\eta_{i|K^\times_{\mathfrak{P}_i}} = -\log_p$ and $\widetilde{\eta}_{i|K^\times_{\mathfrak{P}_i}} = \mathrm{ord}_p$.

Noting that $\{\eta_i, \widetilde{\eta}_i\}$ is a basis of $\mathrm{Hom}(K^\times_{\mathfrak{P}_i}, \overline{\mathbb{Q}}_p)$, we see from $\prod_{\mathfrak{P}|\mathfrak{p}_i} K^\times_{\mathfrak{P}} \simeq \mathrm{Ind}^G_{\{1\}} K^\times_{\mathfrak{P}_i}$ that the translates $\{\eta_i \circ \sigma^{-1}, \widetilde{\eta}_i \circ \sigma^{-1}\}_{\sigma\in G}$ form a basis of $\mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K^\times_{\mathfrak{P}}, \mathbb{Q}_p)$. The φ-component of this latter space is therefore generated by the characters $\eta_{i,\varphi}$ and $\widetilde{\eta}_{i,\varphi}$ given by

$$\eta_{i,\varphi} = \sum_{\sigma\in G} \varphi(\sigma) \cdot \eta_i \circ \sigma, \quad \widetilde{\eta}_{i,\varphi} = \sum_{\sigma\in G} \varphi(\sigma) \cdot \widetilde{\eta}_i \circ \sigma.$$

We next define two matrices L_φ and M_φ which, roughly speaking, consist of p-adic logarithms of linearly independent φ-units. As in the proof of Theorems 4.1 and 4.3, it will be essential in order to apply results of Section 2 to consider a "$\bar{\mathbb{Q}}$-structure" for φ, which simply means here that we see $\varphi\colon G_k \to \bar{\mathbb{Q}}_p^\times$ as taking values in $\bar{\mathbb{Q}}^\times$. This is of course possible using $\iota_p\colon \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, as the values of φ are roots of unity.

Let u_i be any $\mathfrak{P}_i$-unit of K which is not a unit (take for example a generator of $\mathfrak{P}_i^h$, where h is the class number of K). The $\mathfrak{P}_i$-adic valuation map $\mathcal{O}_K[1/\mathfrak{P}_i]^\times \to \mathbb{Z}$ has kernel $\mathcal{O}_K^\times$, so the choice of u_i with a given $\mathfrak{P}_i$-valuation is unique, up to multiplication by a unit of K. Consider

$$u_{i,\varphi} = \prod_{\sigma\in G} \varphi(\sigma) \otimes \sigma^{-1}(u_i) \in \bar{\mathbb{Q}} \otimes K^\times.$$

It is clear that $u_{i,\varphi}$ is a unit away from the primes above $\mathfrak{p}_i$, and $u_{1,\varphi}, \ldots, u_{n,\varphi}$ form a basis of $(\bar{\mathbb{Q}} \otimes \mathcal{O}_K[1/\mathfrak{p}_i]^\times)[\varphi]$ modulo $(\bar{\mathbb{Q}} \otimes \mathcal{O}_K^\times)[\varphi]$. We also fix a basis $\{\varepsilon_{1,\varphi}, \ldots, \varepsilon_{r(\varphi),\varphi}\}$ of $(\bar{\mathbb{Q}} \otimes \mathcal{O}_K^\times)[\varphi]$ modulo the kernel of Leopoldt's map $\iota_k(\varphi)$ of (3.10), where

$$r(\varphi) = d^+(\varphi) - \delta_k^{\mathrm{L}}(\varphi).$$

For all $j = 1, \ldots, n$, one can see via $\iota_{\mathfrak{P}_j}\colon K \hookrightarrow K_{\mathfrak{P}_j} = \mathbb{Q}_p$ the elements $u_{i,\varphi}$ and $\varepsilon_{i,\varphi}$ inside $\bar{\mathbb{Q}} \otimes \mathbb{Q}_p^\times$. We then define two matrices $L_\varphi = (L_{i,j,\varphi})$ and $M_\varphi = (M_{i,j,\varphi})$ of respective sizes $n \times n$ and $r(\varphi) \times n$ by letting

$$L_{i,j,\varphi} = \frac{\log_p\left(\iota_{\mathfrak{P}_j}(u_{i,\varphi})\right)}{\mathrm{ord}_p\left(\iota_{\mathfrak{P}_i}(u_i)\right)}, \quad M_{i,j,\varphi} = \log_p\left(\iota_{\mathfrak{P}_j}(\varepsilon_{i,\varphi})\right), \tag{5.3}$$

where we extended $\log_p$ to $\bar{\mathbb{Q}} \otimes \mathbb{Q}_p^\times$ by linearity. Notice that M_φ has full rank by construction.

Let η' be an element in the φ-component of $\bigoplus_{i=1}^n \mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K_{\mathfrak{P}}^\times, \bar{\mathbb{Q}}_p)$, which we write as $\sum t_i \eta_{i,\varphi} + \tilde{t}_i \tilde{\eta}_{i,\varphi}$ in the basis $\{\eta_{i,\varphi}, \tilde{\eta}_{i,\varphi}\colon 1 \le i \le n\}$. Denote by T and $\widetilde{T}$ the column matrices of respective coordinates $(t_1, \ldots, t_n)$ and $(\tilde{t}_1, \ldots, \tilde{t}_n)$.

Lemma 5.3. *η' belongs to the image of the map A of (5.2) if and only if $\widetilde{T} = L_\varphi T$ and $M_\varphi T = 0$.*

Proof. By the exactness of (5.2) such an η' is characterized by its vanishing at all the u_i's and the $\varepsilon_{i,\varphi}$'s. Going back to the definitions yields, for all $1 \le i, j \le n$, the identities $\eta_{j,\varphi}(\iota_{\mathfrak{P}_j}(u_i)) = -\log_p(\iota_{\mathfrak{P}_j}(u_{i,\varphi}))$, $\tilde{\eta}_{j,\varphi}(\iota_{\mathfrak{P}_j}(u_i)) = \mathrm{ord}_p(\iota_{\mathfrak{P}_j}(u_i))$, $\mathrm{ord}_p(\iota_{\mathfrak{P}_j}(u_i)) = 0$ (for $i \neq j$). Hence, we find

$$\begin{aligned}\eta'(u_i) = 0 &\Leftrightarrow -\sum_{1\le j\le n} t_j \cdot \log_p\left(\iota_{\mathfrak{P}_j}(u_i)\right) + \sum_{1\le j\le n} \tilde{t}_j \cdot \mathrm{ord}_p\left(\iota_{\mathfrak{P}_j}(u_i)\right) = 0 \\ &\Leftrightarrow \tilde{t}_i = \sum_{1\le j\le n} t_j \cdot L_{i,j,\varphi} = [L_\varphi T]_i,\end{aligned}$$

and likewise, $\eta'(\varepsilon_i) = 0 \Leftrightarrow [M_\varphi T]_i = 0$. These two conditions hold for all $1 \le i \le n$ if and only if $\widetilde{T} = L_\varphi T$ and $M_\varphi T = 0$, as wanted. ∎

In what follows we repeatedly use the following elementary fact. For all compact topological groups $\mathcal{G}$, any non-trivial continuous group homomorphism $\eta\colon \mathcal{G} \to \mathbb{Q}_p$ factors through a quotient Z_η isomorphic to $\mathbb{Z}_p$, and two such homomorphisms η and η' are proportional (*i.e.*, $\eta = \lambda \cdot \eta'$ for some $\lambda \in \mathbb{Q}_p$) if and only if $Z_\eta = Z_{\eta'}$. Conversely, any topological group Z isomorphic to $\mathbb{Z}_p$ which arises as a quotient of $\mathcal{G}$ defines a continuous homomorphism $\eta\colon \mathcal{G} \to \mathbb{Q}_p$, which is unique up to scaling.

Fix $k_\infty \in \mathscr{Z}(k)$ and put $\Gamma = \mathrm{Gal}(k_\infty/k)$. The above argument attaches to Γ a non-zero element $\eta \in \mathrm{Hom}(G_k, \mathbb{Q}_p)$, unique up to scaling. Since the restriction map induces an isomorphism

$$\mathrm{Hom}(G_K, \mathbb{Q}_p)[\mathbb{1}] \simeq \mathrm{Hom}(G_k, \mathbb{Q}_p),$$

one can write $A(\eta)$ as $\sum_i (s_i \eta_{i,\mathbb{1}} + \tilde{s}_i \tilde{\eta}_{i,\mathbb{1}})$. We shall refer to the column matrices $S = (s_1, \ldots, s_n)^{\mathrm{t}} \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ and $\widetilde{S} = (\tilde{s}_1, \ldots, \tilde{s}_n)^{\mathrm{t}} \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ as the coordinates of k_∞.

Proposition 5.4. *The following claims hold.*

(1) *The map sending a $\mathbb{Z}_p$-extension k_∞/k to its coordinates S defines a bijection between $\mathscr{Z}(k)$ and $\{S \in \mathbb{P}^{n-1}(\mathbb{Q}_p) : M_{\mathbb{1}} S = 0\}$.*

(2) *Let $k_\infty \in \mathscr{Z}(k)$ with coordinates $S \in \ker(M_{\mathbb{1}})$ and consider the matrix of size $(n + r(\varphi)) \times n$ given in block notation by:*

$$N_\varphi(S) = \begin{bmatrix} \mathrm{Diag}(S) L_\varphi - \mathrm{Diag}(L_{\mathbb{1}} S) \\ M_\varphi \end{bmatrix}, \tag{5.4}$$

where $\mathrm{Diag}(U)$ denotes the diagonal matrix associated with a column matrix U. Then

$$\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) \leq \dim \ker N_\varphi(S) - \gamma,$$

where $\gamma = 1$ if $\varphi = \mathbb{1}$ and $\gamma = 0$ otherwise. If, moreover, none of the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ splits completely in k_∞, then $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) = \dim \ker N_\varphi(S) - \gamma$.

Proof. The inverse of the map in (1) takes $S = (s_1, \ldots, s_n)^{\mathrm{t}} \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ to the $\mathbb{Z}_p$-extension k_∞/k constructed as follows. Lemma 5.3 applied to $\varphi = \mathbb{1}$ tells us that there exists $\eta \in \mathrm{Hom}(G_K, \mathbb{Q}_p)[\mathbb{1}] = \mathrm{Hom}(G_k, \mathbb{Q}_p)$ such that $A(\eta)$ has coordinates S and $\widetilde{S} := L_{\mathbb{1}} S$. As A is injective, η is unique up to scaling. Then k_∞ is defined as the kernel field of η.

Let us prove the second claim. Fix $k_\infty \in \mathscr{Z}(k)$ and denote by $\eta \in \mathrm{Hom}(G_K, \mathbb{Q}_p)[\mathbb{1}]$ the corresponding continuous homomorphism. By Lemma 5.2 and by (5.1), $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) + \gamma$ is the dimension of the space consisting of all elements $\eta' \in \bigoplus_{i=1}^n (\mathrm{Hom}(\prod_{\mathfrak{P}|\mathfrak{p}_i} K_{\mathfrak{P}}^\times, \overline{\mathbb{Q}}_p))[\varphi]$ satisfying the conditions of Lemma 5.3 and such that

$$\eta'_{|K^\times_{\mathfrak{P}_i}} \in \mathrm{Hom}(\Gamma_{\mathfrak{P}_i}, \overline{\mathbb{Q}}_p) \tag{5.5}$$

for all $1 \leq i \leq n$. For a given homomorphism η' of coordinates T and $\widetilde{T}$ and a fixed index i, we reinterpret (5.5) as follows. First recall that $\mathrm{Hom}(\Gamma_{\mathfrak{P}_i}, \overline{\mathbb{Q}}_p)$ is spanned by $A(\eta)_{|K^\times_{\mathfrak{P}_i}}$, which has coordinates $(s_i, \tilde{s}_i)$ in the basis $\{\eta_{i|K^\times_{\mathfrak{P}_i}}, \tilde{\eta}_{i|K^\times_{\mathfrak{P}_i}}\} = \{-\log_p, \mathrm{ord}_p\}$

of $\mathrm{Hom}(K_{\mathfrak{P}_i}^\times, \overline{\mathbb{Q}}_p)$, whereas $\eta'_{|K_{\mathfrak{P}_i}^\times}$ has coordinates $(t_i, \tilde{t}_i)$ in this basis. Hence, (5.5) is equivalent to

$$t_i = \lambda \cdot s_i \quad \text{and} \quad \tilde{t}_i = \lambda \cdot \tilde{s}_i \quad \text{for some } \lambda_i \in \overline{\mathbb{Q}}_p$$

and therefore implies

$$s_i \cdot \tilde{t}_i = \tilde{s}_i \cdot t_i,$$

an equality which can be rephrased as $[\mathrm{Diag}(S)L_\varphi T]_i = [\mathrm{Diag}(L_{\mathbb{1}} S)T]_i$ since $\widetilde{T} = L_\varphi T$ and $\widetilde{S} = L_{\mathbb{1}} S$. Moreover, the converse implication holds unless $(s_i, \tilde{s}_i) = (0,0)$, that is, $\Gamma_{\mathfrak{p}_i} = \Gamma_{\mathfrak{P}_i} = \{1\}$. Therefore, $\delta^{\mathbf{G}}_{k_\infty/k}(\varphi) + \gamma$ is at most the dimension of the space of all $T \in \overline{\mathbb{Q}}_p^n$ that lie in the kernel of both $\mathrm{Diag}(S)L_\varphi - \mathrm{Diag}(L_{\mathbb{1}} S)$ and M_φ, with equality if none of the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ splits completely in k_∞. ∎

Example 5.5. If k is an imaginary quadratic field, then $M_{\mathbb{1}}$ is of size 0 and Proposition 5.4 (1) provides a bijection between $\mathscr{Z}(k)$ and $\mathbb{P}^1(\mathbb{Q}_p)$. Explicitly, take $k_\infty \in \mathscr{Z}(k)$ and let $\eta\colon G_k \to \mathbb{Q}_p$ be a non-trivial continuous group homomorphism factoring through $\mathrm{Gal}(k_\infty/k)$. Then applying the map A of (5.2) to η and restricting to units yields a nonzero element of

$$\mathrm{Hom}(\mathcal{O}_{k_{\mathfrak{p}_1}}^\times \times \mathcal{O}_{k_{\mathfrak{p}_2}}^\times, \mathbb{Q}_p) = \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Q}_p) \oplus \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Q}_p).$$

We write this element as $s_1\eta_1 + s_2\eta_2$, where η_i is $\log_p$ on the i-th summand of the preceding direct sum, and the coordinates of k_∞ are $S = (s_1, s_2)$.

As η is unique up to a scalar, the ratio $s_1/s_2 \in \mathbb{Q}_p \cup \{\infty\}$ is well defined. We refer to s_1/s_2 as the *slope* of k_∞. For instance, the cyclotomic extension of k has slope 1, whereas its anticyclotomic extension has slope -1. Moreover, slope 0 (resp. slope ∞) corresponds to the unique $\mathbb{Z}_p$-extension of k that is unramified at $\mathfrak{p}_1$ (resp. at $\mathfrak{p}_2$).

If the roles of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are exchanged, then the slope is changed to its reciprocal, and in particular, it makes sense to say that the slope is transcendental.

5.3. Proof of Theorem 5.1

We keep the notations of the preceding sections and, in particular, (5.3) and (5.4). We abbreviate $L_{\mathbb{1}}$, $M_{\mathbb{1}}$ and $N_{\mathbb{1}}(S)$ as L, M and $N(S)$, respectively. By Proposition 5.4 (1), the set $\mathscr{Z}(k)$ can be identified with a closed linear subvariety of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ of dimension $n - r - 1 = r_2 + \delta_k^{\mathbf{L}}$.

Proposition 5.6. *Assume that $n - r \leq 2$ and let*

$$\mathscr{C}(k) = \left\{k_\infty \in \mathscr{Z}(k) : \delta^{\mathbf{G}}_{k_\infty/k} \neq 0\right\},$$
$$\mathscr{C}_\varphi(k) = \left\{k_\infty \in \mathscr{Z}(k) : \delta^{\mathbf{G}}_{k_\infty/k}(\varphi) \neq 0\right\}.$$

(1) *If k is an imaginary quadratic field, then $\mathscr{C}(k) = \emptyset$.*

(2) *If there exists $k_\infty \in \mathscr{Z}(k) \setminus \mathscr{C}(k)$ in which no prime above p totally splits, then $\mathscr{C}(k)$ is finite with $|\mathscr{C}(k)| \leq n - 1$.*

(3) *If there exists $k_\infty \in \mathscr{Z}(k) \setminus \mathscr{C}_\varphi(k)$ in which no prime above p totally splits, then $\mathscr{C}_\varphi(k)$ is finite with $|\mathscr{C}_\varphi(k)| \leq n$.*

Proof. Let us prove (1) and (2). Notice first that $\mathrm{Diag}(S)LS = \mathrm{Diag}(LS)S$, so by (5.4), $MS = 0$ implies $N(S)S = 0$. By Proposition 5.4 (2) with $\varphi = \mathbb{1}$ and $K = k$, $\mathscr{C}(k)$ embeds into the set of all $S \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ such that $MS = 0$ and $\operatorname{rk} N(S) < n-1$.

Assume first that k is quadratic, so $n = 2$ and $r = 0$. Given any $S = (s_1, s_2) \in \mathbb{P}^1(\mathbb{Q}_p)$, the matrix $N(S) = \mathrm{Diag}(S)L - \mathrm{Diag}(LS)$ has the form

$$\begin{pmatrix} -s_2 L_{1,2} & s_1 L_{1,2} \\ s_2 L_{2,1} & -s_1 L_{2,1} \end{pmatrix}.$$

As $\log_p$ is injective on $\mathbb{Z}_p^\times$, $\log_p(\iota_{\mathfrak{p}_1}(u_2))$ and $\log_p(\iota_{\mathfrak{p}_2}(u_1))$ both are non-zero, so at least one of the two non-diagonal entries of $N(S)$ is non-zero. Therefore, this matrix has rank one for any $S \in \mathbb{P}^1(\mathbb{Q}_p)$, and $\mathscr{C}(k) = \emptyset$.

We no longer assume that k is imaginary quadratic, but we still assume that $n - r \le 2$. The case where $n - r = 1$ is trivial, because it forces $\mathscr{Z}(k) = \{k_{\mathrm{cyc}}\}$. We may then assume that $n - r = 2$. Since M has full rank r, there exist invertible matrices P, Q such that $PMQ = (I_r \mid 0)$, where I_r is the identity matrix of size r. The change of variables $S' = Q^{-1}S$ induces a linear bijection between $\ker M \subset \mathbb{P}^{n-1}(\overline{\mathbb{Q}}_p)$ and the projective line $\{0\} \times \mathbb{P}^1(\overline{\mathbb{Q}}_p) \subset \mathbb{P}^{n-1}(\overline{\mathbb{Q}}_p)$. Now consider the list $P_1(S'), \dots, P_t(S')$ of all $(n-1) \times (n-1)$-minors of the matrix

$$N(S) = N(QS') = \begin{bmatrix} \mathrm{Diag}(QS')L - \mathrm{Diag}(LQS') \\ M \end{bmatrix}.$$

All the P_k's are two-variable homogeneous polynomials of degree $\le n-1$. The assumption of (2) implies, by Proposition 5.4 (2), that there exists S with $\operatorname{rk} N(S) = n-1$. Therefore, at least one of the P_k's is not the zero polynomial, and hence, it has at most $n-1$ zeros in $\{0\} \times \mathbb{P}^1(\overline{\mathbb{Q}}_p)$. Again by Proposition 5.4 (2), we thus may conclude that $|\mathscr{C}(k)| \le n-1$.

The proof of point (2) is very similar to the previous one. Indeed, by (2) of Proposition 5.4, $\mathscr{C}_\varphi(k)$ embeds into the set of all $S \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ such that $MS = 0$ and $\operatorname{rk} N_\varphi(S) < n$. The same argument with the $(n-1) \times (n-1)$ minors of $N(S)$ replaced by the $n \times n$ minors of $N_\varphi(S)$ shows that, under the assumption of (3), one has $|\mathscr{C}(k)| \le n$. ∎

We end the proof of Theorem 5.1 with the case where k is imaginary quadratic. We let τ be the complex conjugation of k. Recall that τ acts on φ via $\varphi^\tau(g) = \varphi(\tau g \tau)$ and that $\varphi^\tau = \varphi$ if and only if φ cuts out an extension of k which is abelian over $\mathbb{Q}$.

Proposition 5.7. *Assume that k is an imaginary quadratic field, $\varphi \ne \mathbb{1}$ and that p splits completely in the field cut out by φ.*

(1) *If $\varphi^\tau = \varphi$, then $\mathscr{C}_\varphi(k) = \emptyset$.*

(2) *If $\varphi^\tau \ne \varphi$, then any $k_\infty \in \mathscr{C}_\varphi(k)$ has transcendental slope.*

(3) *If $\varphi^\tau \ne \varphi$, then $|\mathscr{C}_\varphi(k)| \le 1$. Moreover, if $XYZ^2 - (AX - BY)(CX - DY)$ does not vanish on any 7-tuple in Λ^7 which form a $\overline{\mathbb{Q}}$-linearly independent set, then $\mathscr{C}_\varphi(k) = \emptyset$.*

Proof. Let $k_\infty \in \mathscr{Z}(k)$ of coordinates $S = (s_1, s_2)$. Take K to be the Galois closure over $\mathbb{Q}$ of the field cut out by φ. Note that $K \neq k$ and that p totally splits in K. By Proposition 5.4, k_∞ belongs to $\mathscr{C}_\varphi(k)$ if and only if the matrix

$$N_\varphi(S) = \begin{pmatrix} s_1(L_{1,1,\varphi} - L_{1,1}) - s_2 L_{1,2} & s_1 L_{1,2,\varphi} \\ s_2 L_{2,1,\varphi} & s_2(L_{2,2,\varphi} - L_{2,2}) - s_1 L_{2,1} \\ M_{1,1,\varphi} & M_{1,2,\varphi} \end{pmatrix}$$

has rank 1. The definition of L_φ (and also L_{φ^τ}) involves the choice, for $i = 1, 2$, of a prime $\mathfrak{P}_i$ of K above $\mathfrak{p}_i$ and a $\mathfrak{P}_i$-unit u_i of non-zero valuation. Since $\tau(\mathfrak{p}_1) = \mathfrak{p}_2$, one may take $\tau(\mathfrak{P}_1) = \mathfrak{P}_2$ and $\tau(u_1) = u_2$, so that

$$L_{1,1} = L_{2,2}, \quad L_{1,2} = L_{2,1}, \quad L_{1,1,\phi} = L_{2,2,\phi^\tau}, \quad \text{and} \quad L_{1,2,\phi} = L_{2,1,\phi^\tau},$$

for $\phi \in \{\varphi, \varphi^\tau\}$. We may also take $\tau(\varepsilon_{1,\varphi})$ to be $\varepsilon_{1,\varphi^\tau}$, so that $M_{1,2,\varphi} = M_{1,1,\varphi^\tau}$. Moreover, the elements $u_{1,\varphi}, u_{1,\mathbb{1}}, u_{2,\varphi}, u_{2,\mathbb{1}}, \varepsilon_{1,\varphi}$ and, under the additional condition $\varphi \neq \varphi^\tau$, the elements $u_{1,\varphi}, u_{1,\varphi^\tau}, u_{1,\mathbb{1}}, u_{2,\varphi}, u_{2,\varphi^\tau}, u_{2,\mathbb{1}}, \varepsilon_{1,\varphi}, \varepsilon_{1,\varphi^\tau}$ form two sets of $\overline{\mathbb{Q}}$-linearly independent p-units. To see this, consider their valuations at $\mathfrak{P}_1$ and $\mathfrak{P}_2$, and use the fact that units belonging to distinct isotypic components are linearly independent.

Now, consider first the case where $\varphi = \varphi^\tau$. Then $N_\varphi(S)$ has rank 1 if and only if its two columns are equal. This last condition easily implies that $s_1 = \pm s_2$ and that $u_{1,\varphi}/u_{1,\mathbb{1}}$, $u_{2,\mathbb{1}}$ and $u_{2,\varphi}$ have $\overline{\mathbb{Q}}$-linearly dependent $\mathfrak{P}_1$-adic logarithms. But all of these units have trivial $\mathfrak{P}_1$-valuation, so they must be linearly dependent by Proposition 2.2. We have already justified that this is not the case, so $N_\varphi(S)$ has rank 2 and $\mathscr{C}_\varphi(k) = \emptyset$. This proves (1).

Assume now that $\varphi \neq \varphi^\tau$ and that k_∞ has an algebraic slope, i.e., $s_1/s_2 \in \mathbb{P}^1(\overline{\mathbb{Q}})$. We may assume that both s_1 and s_2 are algebraic numbers, hence $N_\varphi(S)$ has coefficients in the $\overline{\mathbb{Q}}$-linear subspace Λ of $\overline{\mathbb{Q}}_p$ introduced in Section 2. Moreover, $N_\varphi(S)$ has $\overline{\mathbb{Q}}$-linearly independent rows and columns. Indeed, the sets $\{\varepsilon_{1,\varphi}, \varepsilon_{1,\varphi^\tau}\}$ and $\{u_{1,\varphi}/u_{1,\mathbb{1}}, u_{2,\mathbb{1}}, u_{2,\varphi}, \varepsilon_{1,\varphi}\}$ are two sets of independent units with trivial $\mathfrak{P}_1$-valuation. Therefore, their images under $\log_p \circ \iota_{\mathfrak{P}_1}$ are again linearly independent by Proposition 2.2. We thus may apply Corollary 2.4 and conclude that $\operatorname{rk} N_\varphi(S) = 2$. This proves (2).

Finally, assume that $\varphi \neq \varphi^\tau$ and that $k_\infty \in \mathscr{C}_\varphi(k)$, i.e., $N_\varphi(S)$ has rank 1. We already know that $s_1/s_2 \in \mathbb{P}^1(\mathbb{Q}_p) - \mathbb{P}^1(\overline{\mathbb{Q}})$ and in particular, both s_1 and s_2 are non-zero. Letting $\mu = s_1/s_2$, the vanishing of the minors of $N_\varphi(S)$ yields the relations

$$\begin{aligned} M_{1,1,\varphi^\tau} \cdot (L_{1,1,\varphi} - L_{1,1} - L_{2,1} \cdot \mu^{-1}) &= L_{2,1,\varphi^\tau} \cdot M_{1,1,\varphi}, \\ M_{1,1,\varphi} \cdot (L_{1,1,\varphi^\tau} - L_{1,1} - L_{2,1} \cdot \mu) &= L_{2,1,\varphi} \cdot M_{1,1,\varphi^\tau}. \end{aligned}$$

Since $L_{2,1} \cdot M_{1,1,\varphi} \neq 0$, the second equality uniquely determines the slope μ, so $|\mathscr{C}_\varphi(k)| \leq 1$. Furthermore, eliminating μ in the above equations then yields the polynomial equation

$$\begin{aligned} M_{1,1,\varphi} \cdot M_{1,1,\varphi^\tau} \cdot L_{2,1}^2 = \big((L_{1,1,\varphi^\tau} - L_{1,1}) \cdot M_{1,1,\varphi} - L_{2,1,\varphi} \cdot M_{1,1,\varphi^\tau}\big) \\ \cdot \big(L_{2,1,\varphi^\tau} \cdot M_{1,1,\varphi} - (L_{1,1,\varphi} - L_{1,1}) \cdot M_{1,1,\varphi^\tau}\big). \end{aligned}$$

The elements of the set $\{u_{1,\varphi}/u_{1,\mathbb{1}}, u_{1,\varphi^\tau}/u_{1,\mathbb{1}}, u_{2,\varphi}, u_{2,\varphi^\tau}, u_{2,\mathbb{1}}, \varepsilon_{1,\varphi}, \varepsilon_{1,\varphi^\tau}\}$ are linearly independent, and they all have a trivial $\mathfrak{P}_1$-adic valuation, so their images under $\log_p \circ \iota_{\mathfrak{P}_1}$ are also $\overline{\mathbb{Q}}$-linearly independent. Therefore, the above polynomial identity contradicts our assumption. This ends the proof of (3). ■

Acknowledgments. The author would like to thank Dominik Bullach and Lassina Dembélé for reading and providing comments on a preliminary version of this paper.

Funding. This research is supported by the Luxembourg National Research Fund, Luxembourg, INTER/ANR/18/12589973 GALF.

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Communicated by Otmar Venjakob

Received 16 February 2023; revised 2 October 2023.

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