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Corrigendum to "The variety of polar simplices"

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Abstract. We point out an important error in [Doc. Math. 18 (2013), 469–505] and provide the necessary corrections.

In [3], we tried to describe the compactification of the variety of polar simplices to a quadric. Unfortunately, we wrongly asserted in [3, Corollary 2.2] that being apolar to a quadric is a closed condition in the Hilbert scheme.

Joachim Jelisiejew gave an example of a scheme of length 4 on a line that is not apolar to a quadric Q in $\mathbf{P}^3$, but is a limit of polar simplices for Q, see [2, Example 1]. Consequently the locus of apolar schemes of length n to a nonsingular quadric Q in $\mathbf{P}^{n-1}$ is not closed for any $n \geq 4$. This means that several of the statements of the global properties of $\mathrm{VPS}(Q, n)$ in [3] are wrong. An account of alternative compactifications in the multigraded Hilbert scheme and in the Grassmannian of spaces of quadrics of ideals of polar simplices is given in [2]. Here we explain the errors in [3] and the corrected statements whose proofs can be found in [2].

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring which we view as a homogeneous coordinate ring of the $\mathbf{P}^{n-1} = \mathbf{P}(S_1^*)$. Let $q \in S_2^*$ be a quadric of rank n, then $Q = \{q = 0\} \subset \check{\mathbf{P}}^{n-1}$. A finite subscheme $\Gamma \subseteq \mathbf{P}^{n-1}$ of length n is *apolar* to Q if $I_\Gamma \subseteq q^\perp$, the ideal of forms in S that annihilates q by differentiation. When $\Gamma = \{[\ell_1], \dots, [\ell_n]\}$ is smooth, it is a polar simplex, i.e. $q = \lambda_1 \ell_1^2 + \dots + \lambda_n \ell_n^2$ for suitable nonzero scalars λ_i. This condition may be formulated for ideals in general; an ideal $I \subset S$ is *apolar* to Q if $I \subseteq q^\perp$.

While the condition $I_\Gamma \subset q^\perp$ is *not closed* in the usual Hilbert scheme, the apolarity condition is a *closed* condition in the multigraded Hilbert scheme Hilb^H of ideals I with a fixed Hilbert function H for S/I. Hence, it is more natural to work in the multigraded Hilbert scheme. If $I \subset S$ is a limit of ideals I_Γ of apolar schemes $\Gamma \in \mathrm{Hilb}^n$, then $I \subset I_{\Gamma_0}$, for some $\Gamma_0 \in \mathrm{Hilb}^n$. If Γ_0 is not apolar to q, then the limit ideal $I \neq I_{\Gamma_0}$ and is an unsaturated apolar ideal to q.

We consider the Hilbert function $H := (1, n, n, \dots)$ and a quadric Q of rank n. The locus of saturated ideals in Hilb^H is open by [1, Theorem 2.6], so we consider $\mathrm{VPS}^{\mathrm{sbl}}(Q, H) \subset \mathrm{Hilb}^H$, the closure of the locus of saturated ideals apolar to Q.

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Associating to each ideal $I \in \mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ the space I_2 of quadrics in the ideal defines a forgetful map

$$\pi_G : \mathrm{VPS}^{\mathrm{sbl}}(Q, H) \to \mathbb{G}\left(\binom{n}{2}, q_2^{\perp}\right),$$

into the Grassmannian of $\binom{n}{2}$-dimensional subspaces in $q_2^{\perp}$. Let

$$\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G := \pi_G\big(\mathrm{VPS}^{\mathrm{sbl}}(Q, H)\big).$$

Of course, there is also a natural map $\pi_{\mathrm{Hilb}} : \mathrm{Hilb}^H \to \mathrm{Hilb}^n ; I \mapsto V(I)$, since all ideals with Hilbert function $H = (1, n, n, \ldots)$ defines a scheme of length n. For comparison with the schemes $\mathrm{VPS}(Q, n)$ and $\mathrm{VAPS}(Q, n)$ in [3];

$$\mathrm{VAPS}(Q, n) = \pi_{\mathrm{Hilb}}\big(\mathrm{VPS}^{\mathrm{sbl}}(Q, H)\big),$$

while $\mathrm{VPS}(Q, n)$ is the component of $\mathrm{VAPS}(Q, n)$ that contains the polar simplices.

When $n \geq 4$, the map π_G has positive dimensional fibers, while its restriction to the saturated part of $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ is bijective, even an isomorphism when $n = 4, 5$ (see [2, Remark 4.9]). In particular, the Hilbert scheme compactification and the Grassmannian compactification do not coincide, so [3, Corollary 2.2] is wrong.

The first part of Theorem 1.1 claims that for $2 \leq n \leq 5$ the variety $\mathrm{VPS}(Q, n)$ is smooth of Picard rank 1 and is Fano of index 2. For $n = 2, 3$ we have π_G is an isomorphism, in particular $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ and $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$ are both isomorphic to $\mathrm{VPS}(Q, n)$ and the argument of [3] is correct. For $n = 4, 5$, by [2, Theorem 1.3], $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ is smooth and admits a nontrivial contraction onto the smooth $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$, hence the Picard rank of $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ is at least two. For $\mathrm{VPS}(Q, n)$ we do not know whether it is smooth, but if it were its Picard rank would also be at least two. When replacing $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ by the Grassmannian model $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$, however, we salvage the first part of [3, Theorem 1.1].

Salvaged Theorem 1.1 ([2, Corollary 4.10]). *For $2 \leq n \leq 5$ the variety $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$ is a smooth rational $\binom{n}{2}$-dimensional Fano variety of index 2 and Picard number 1.*

The second part of [3, Theorem 1.1] remains correct, it is not effected by the compactification.

Theorem 1.2 in [3] concerns $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$, the Grassmannian model. It is correct for $n = 4$ after correcting the degree, using a more nuanced machinery of excess intersections. The case $n = 5$ remains open.

Salvaged Theorem 1.2 ([2, Proposition 4.15]). *The variety $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$ contains the image TQ^{-1} of the Gauss map. When $n = 4$, the restriction of the Plücker line bundle generates the Picard group of $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$ and the degree is 362.*

Theorem 1.3 in [3] concerns the linear span of the Grassmannian model $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$, and is wrong. The image of the unsaturated ideals in $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)$ does not lie in the span of TQ^{-1}. Whether $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$ is a linear section of the Grassmannian therefore remains an open problem. It is true for $n = 3$, and a computational proof for $n = 4$ is given in [2].

The remaining results of [3, Sections 1, 2, 3, 4, and 5] are correct. Remark 2.5 in [3] is valid for saturated ideals.

The degree computation [3, Theorem 6.3] is effected by mistakes concerning the compactifications. The degree formula of [3, Theorem 6.3] gives a contribution to the degree of $\mathrm{VPS}^{\mathrm{sbl}}(Q, H)_G$. The remaining contribution can be computed in case $n = 4$ using excess intersection, see [2, Proposition 4.15 and Remark 4.16].

References

[1] J. Jelisiejew and T. Mańdziuk, Limits of saturated ideals. 2022, arXiv:2210.13579

[2] J. Jelisiejew, K. Ranestad, and F.-O. Schreyer, The variety of polar simplices II. 2023, arXiv:2304.00533

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