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On the Bott Cannibalistic Classes

By

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§1. Introduction

Let G be a compact Lie group, V a (complex) representation of G and $\lambda_V \in K_G(V)$ the Thom class. Since $K_G(V)$ is a free $R(G)$-module generated by λ_V (cf. [1]), there is an element $\theta_k(V) \in R(G)$ such that $\psi^k(\lambda_V) = \theta_k(V) \cdot \lambda_V$. The element $\theta_k(V)$ is called the Bott cannibalistic class of V.

By $K_G^*(\ )$ we denote the $RO(G)$-graded equivariant K-theory. Since $\theta_k(V)$ is not a unit of $R(G)$, the Adams operation ψ^k is not always a (stable) cohomology operation (cf. section 3). So the purpose of this paper is to show the following:

For a finite group G, ψ^k is a cohomology operation of $K_G^*(\ ) \otimes Z[\frac{1}{k}]$ if and only if $(|G|, k) = 1$ (for details see Theorem 3.1).

This paper is organized as follows: In section 2 we show that the element $\theta_k(V)$ is a unit of $R(G) \otimes Z[\frac{1}{k}]$ if $(|G|, k) = 1$. In the next section the main theorem is proved.

If G is a p-group, then Atiyah and Tall showed that $\theta_k(V)$ is a unit of $R(G) \otimes Z_p^{\wedge}$ (cf. [2]).

§2. The Bott Cannibalistic Classes

First recall the following properties of $\theta_k(V)$ (see [2]):

Lemma 2.1. *Let V and W be representations of G and $\varepsilon: R(G) \to Z$ the augmentation. Then*

(i) $\theta_k(V+W) = \theta_k(V)\theta_k(W)$,

(ii) $\theta_k(V) = 1 + V + \cdots + V^{k-1}$ *if* $\dim V = 1$,

(iii) $\varepsilon(\theta_k(V)) = k^n$ *if* $\dim V = n$.

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From now on we assume that G is a finite group, N, k integers such that $(N, k)=1$ and $|G|$ devides N.

Proposition 2.2. *Let V be a one dimensional representation of G. Then there exists a polynomial $f_{N,k}(x) \in Z[\frac{1}{k}][x]$ such that*

$$\theta_k(V) f_{N,k}(V)=1$$

in $R(G) \otimes Z[\frac{1}{k}]$.

To prove Proposition 2.2 we need the following lemma:

Lemma 2.3. *Let R be a (commutative) ring (with unity) such that k is invertible and $r \in R$. If $r^N=1$, then there exists $f_{N,k}(x) \in Z[\frac{1}{k}][x]$ such that*

$$(1+r+\cdots+r^{k-1}) f_{N,k}(r)=1 .$$

Proof of Proposition 2.2. Since $V^N=1$ and $\theta_k(V)=1+V+\cdots+V^{k-1}$ by Lemma 2.1, $\theta_k(V) f_{N,k}(V)=1$ by Lemma 2.3.

Proof of Lemma 2.3. Let ζ be a primitive N-th root of 1 then we have

(i) $1+\zeta^i+\cdots+\zeta^{(N-1)i}=0$ for $1 \leqq i \leqq N-1$

and

(ii) $\prod_{i=1}^{N-1}(1-\zeta^i)=\prod_{i=1}^{N-1}(1-\zeta^{ti}) \neq 0$ if $(t, N)=1$.

If we denote the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \\ x_{N-1} & x_0 & \cdots & x_{N-2} \\ & & \cdots\cdots & \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix}$$

by $A(x_0, x_1, \cdots, x_{N-1})$ then

$$\det A(x_0, x_1, \cdots, x_{N-1})=\prod_{i=0}^{N-1}(x_0+\zeta^i x_1+\cdots+\zeta^{i(N-1)} x_{N-1}) .$$

Since $(k, N)=1$, we can write $k=sN+t$ $(s, t \in Z)$ with $1 \leqq t \leqq N-1$ and $(t, N)=1$. Put $u_i=s+1$ if $0 \leqq i<t$ and $u_i=s$ if $t \leqq i \leqq N-1$. Then

$$1+r+\cdots+r^{k-1}=u_0+u_1 r+\cdots+u_{N-1} r^{N-1}$$

and

$$\det A(u_0, u_1, \cdots, u_{N-1}) = \prod_{i=0}^{N-1} (u_0 + \zeta^i u_1 + \cdots + \zeta^{i(N-1)} u_{N-1})$$

$$= k \prod_{i=1}^{N-1} (1 + \zeta^i + \cdots + \zeta^{i(t-1)} + s(1 + \zeta^i + \cdots + \zeta^{i(N-1)}))$$

$$= k \prod_{i=1}^{N-1} (1 + \zeta^i + \cdots + \zeta^{i(t-1)}) = k((\prod_{1=1}^{N-1} (1 - \zeta^{ti}))/(\prod_{i=1}^{N-1} (1 - \zeta^i))) = k.$$

Since the determinant is a unit of $Z[\frac{1}{k}]$, there exists $a_i \in Z[\frac{1}{k}]$ such that

$$(a_0, a_1, \cdots, a_{N-1}) A(u_0, u_1, \cdots, u_{N-1}) = (1, 0, \cdots, 0).$$

Putting $f_{N,k}(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}$, we have

$$(1 + r + \cdots + r^{k-1}) f_{N,k}(r) = 1. \qquad \text{Q.E.D.}$$

Define $h_{N,k}^{(n)}(X_1, \cdots, X_n) \in Z[\frac{1}{k}][X_1, \cdots, X_n]$ by

$$h_{N,k}^{(n)}(\sigma_1(x_1, \cdots, x_n), \cdots, \sigma_n(x_1, \cdots, x_n)) = \prod_{i=1}^{n} f_{N,k}(x_i),$$

where σ_i is the i-th elementary symmetric function. Then we have

Theorem 2.4. *Let V be an n-dimensional representation of G and k an integer such that $(|G|, k) = 1$. Then*

$$\theta_k(V) h_{N,k}^{(n)}(\Lambda^1(V), \cdots, \Lambda^n(V)) = 1$$

in $R(G) \otimes Z[\frac{1}{k}]$.

Proof. Let $\{C_\alpha\}$ be all cyclic subgroups of G, then the restriction

$$R(G) \longrightarrow \bigoplus_\alpha R(C_\alpha)$$

is a monomorphism. Since θ_k, $f_{N,k}$ and $h_{N,k}^{(n)}$ commute with restrictions, we may assume that G is cyclic. Since every irreducible representation of a cyclic group is one dimensional, V is a sum of one dimensional representations: $V = V_1 + V_2 + \cdots + V_n$. Note that

$$\Lambda^i(V) = \sigma_i(V_1, V_2, \cdots, V_n)$$

and $\theta_k(V) = \prod_{i=1}^{n} \theta_k(V_i)$. Then Theorem 2.4 is an easy consequence of Proposition 2.2. Q.E.D.

Corollary 2.5. *The element $\theta_k(V)$ is a unit of $R(G) \otimes Z[\frac{1}{k}]$ if $(|G|, k) = 1$.*

§3. Proof of the Main Theorem

If $(|G|, k)=1$, then we can extend the Adams operation ψ^k over $K_G^*(\ )[\frac{1}{k}]$ by the commutative diagram

$$\begin{array}{ccc}
\tilde{K}_G(\Sigma^\omega \wedge X_+)[\frac{1}{k}] & \xrightarrow{\theta_k(\omega)^{-1}\psi^k} & \tilde{K}_G(\Sigma^\omega \wedge X_+)[\frac{1}{k}] \\
\uparrow \cdot\lambda_\omega & & \uparrow \cdot\lambda_\omega \\
K_G(X)[\frac{1}{k}] & \xrightarrow{\psi^k} & K_G(X)[\frac{1}{k}],
\end{array}$$

where ω is the regular representation of G.

Let H be a subgroup of G and $\mathrm{Ind}_H^G\colon R(H)\to R(G)$ the induction homomorphism. Recall that $K_G(G/H)=R(H)$ and $K_G(G/G)=R(G)$. Following Nishida [5], Ind_H^G is the transfer

$$p_*\colon K_G(G/H) \longrightarrow K_G(G/G)$$

for $p\colon G/H\to G/G$ and every transfer commutes with cohomology operations. So if $\psi^k\colon K_G^*(\ )[\frac{1}{k}]\to K_G^*(\ )[\frac{1}{k}]$ is a cohomology operation, $\mathrm{Ind}_H^G\circ\psi^k=\psi^k\circ\mathrm{Ind}_H^G\colon R(H)[\frac{1}{k}]\to R(G)[\frac{1}{k}]$. Moreover since $R(G)$ is torsion free, $\mathrm{Ind}_H^G\circ\psi^k=\psi^k\circ\mathrm{Ind}_H^G\colon R(H)\to R(G)$ if ψ^k is a cohomology operation of $K_G^*(\ )[\frac{1}{k}]$-theory.

Now we can prove the following:

Theorem 3.1. *Let G be a finite group and k an integer. Then the followings are equivalent*:

(i) $(|G|, k)=1$,

(ii) *for any representation V of G, $\theta_k(V)$ is a unit of $R(G)[\frac{1}{k}]$*,

(iii) $\psi^k\colon K_G^*(\ )[\frac{1}{k}]\to K_G^*(\ )[\frac{1}{k}]$ *is a cohomology operation*

and

(iv) $\mathrm{Ind}_H^G\circ\psi^k=\psi^k\circ\mathrm{Ind}_H^G\colon R(H)\to R(G)$ *for any subgroup H of G.*

Proof. Clearly it remains to prove that (iv) implies (i). Suppose (iv) is true and a prime p devides $(|G|, k)$. Then there exists $g_1\in G$ such that g_1 is of order p (cf. [3]). Put $H=\{1\}$, then $\mathrm{Ind}_H^G(1)=\omega$. Note that

$$\chi_\omega(g)=\begin{cases} |G| & \text{if } g=1 \\ 0 & \text{otherwise,}\end{cases}$$

where $g\in G$ and χ_ω is the character of the regular representation ω of G (cf. [6]). Now we have

$$\chi_{\mathrm{Ind}_H^G \circ \psi^k(1)}(g_1) = \chi_{\mathrm{Ind}_H^G(1)}(g_1) = \chi_\omega(g_1) = 0$$

and

$$\chi_{\psi^k \circ \mathrm{Ind}_H^G(1)}(g_1) = \chi_{\psi^k(\omega)}(g_1) = \chi_\omega(g_1^k) = \chi_\omega(1) = |G|\,.$$

This shows that $\mathrm{Ind}_H^G \circ \psi^k \neq \psi^k \circ \mathrm{Ind}_H^G$, which contradicts to (iv). Q.E.D.

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