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The Mod-5 Splitting of the Compact Exceptional Lie Group E_8

By

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Introduction

J.H. Harper [1] proved that E_8, localized at the prime 5, has the same homotopy type of $K(5)\times B(15, 23, 39, 47)\times B_{13}(5)$ where $H^*(B(15,23,39,47);\ \boldsymbol{Z}_5)=\Lambda(x_{15}, x_{23}, x_{39}, x_{47})$. See [2] resp. [4] for more details about $K(5)$ resp. $B_{13}(5)$. One would like to know whether $B(15, 23, 39, 47)$ is irreducible or of the same homotopy type of $B_7(5)\times B_{19}(5)$. The purpose of this paper is to show that $\mathcal{R}(x_{23})=x_{39}$ where $\mathcal{R}$ is the secondary operation defined by the relation $((1/2)\beta P^1-P^1\beta)(P^1)+P^2(\beta)=0$ and $x_{23}, x_{39}\in H^*(E_8, \boldsymbol{Z}_5)$. This certainly implies that $B(15,23,39,47)$ is indecomposable. Using this fact one can compute the 5-component of $\pi_{38}(E_8)$ which turns out to be zero.

In 1970, H. Toda [4] announced that the following cases of mod-p decompositions of exceptional Lie groups were unknown:

F_4, E_7, E_8	mod 3
F_8	mod 5

Since then, the case F_4 mod 3 was solved in [3], and the cases E_7, E_8 mod 3 were done in [6]. The result of this paper will answer the last question about the mod-p decomposition of a simply connected simple compact exceptional Lie group. This paper is organized as follows: In Part I we prove a result about the cohomology of the classifying space of certain loop spaces; in Part II we compute the Hopf Algebra structure of a certain cover of E_8; in Part III we prove the main result.

Part I

The following result, I believe, is known, but I don't have a reference in the form I need. Let X be a loop space and p an odd prime. Suppose $H^*(X, \boldsymbol{Z}_p)\simeq\boldsymbol{Z}_p[x_1, x_2, \cdots]\otimes\Lambda(y_1, y_2, \cdots)$ where dim x_i are even, dim y_i odd and x_i's, and y_i's are universally transgressive. Call BX the classifying space of X.

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Theorem 1.1. *$H^*(BX, \boldsymbol{Z}_p)$ is isomorphic as an algebra to $\Lambda(z_{1,1}, z_{1,2}, \cdots, z_{i,j}, z_{i,j+1}, \cdots)\otimes\boldsymbol{Z}_p[v_1, \cdots, v_i, \cdots, \mu_{1,2}, \mu_{1,3}, \cdots, \mu_{i,k}, \mu_{i,k+1}, \cdots]$ where $i=1, 2, \cdots$, $j=1, 2, \cdots$, $k=2, 3, \cdots$, $\tau(x_i^{p^j})=z_{i,j}$, $\tau(y_i)=v_i$ and $d_{p^j(p-1)\dim x_i+1}(z_{i,j}\otimes x_i^{p^j(p-1)})=\mu_{i,j+1}$. ($\tau$ is the transgression and d_r is the r-th differential in the Serre spectral sequence associated with the path-loop fibration over BX).*

Sketch of the proof. Let A be the algebra which is a candidate for $H^*(BX, \boldsymbol{Z}_p)$. By [10] chapter 9 we have that $H^*(X, \boldsymbol{Z}_p)\otimes A$ is an acyclic construction where the differentials are defined as stated in the theorem. Now we define a map $f: A\to H^*(BX, \boldsymbol{Z}_p)$. Since A is a free algebra, it suffices to define f on each generator. Let $f(z_{i,j})=\tau(x_i^{p^j})$, $f(v_i)=\tau(y_i)$ and

$$f(\tau v_{i,j+1})=d_{p^j(p-1)\dim x_i+1}(z_{i,j}\otimes x_i^{p^j(p-1)})\,.$$

By the hypothesis and Kudo's transgression theorem (see [7]) the above equalities make sense. So we have a map between the algebraic spectral sequence $H^*(X, \boldsymbol{Z}_p)\otimes A$ and the Serre's spectral sequence associated with the path-loop fibration. The fact that the differentials commute with the map is clear from the definition. So by Zeeman's comparison theorem (see [12]) it follows that $A=H^*(BX, \boldsymbol{Z}_p)$.

Remarks. 1) This result has been applied when X is the 3-connective cover of some exceptional Lie group, and of course when X is $K(\boldsymbol{Z}_p, n)$.

2) By Kudo's transgression theorem and the fact that τ commutes with the Steenrod operations, we get:

$$\beta z_{i,j+1}=\mu_{i,j+1} \quad\text{and}\quad P^{n_i p^j}(z_{i,j})=z_{i,j+1}\,.$$

3) If we consider the Eilenberg-Moore spectral sequence (see [9]):

$$E_2\simeq \mathrm{EXT}_{H^*(X,\boldsymbol{Z}_p)}(\boldsymbol{Z}_p, \boldsymbol{Z}_p)\Longrightarrow H^*(BX, \boldsymbol{Z}_p)$$

an easy calculation tells us that $E_2\simeq H^*(BX, \boldsymbol{Z}_p)$. So the EilenbergMoore spectral sequence collapses.

Part II

Let $\tilde{E}_8$ be the 3-connective cover of E_8

$$\begin{array}{ccc} \tilde{E}_8 & & \\ \downarrow{\scriptstyle p_1} & & \\ E_8 & \xrightarrow{\ f_1\ } & K(\boldsymbol{Z}, 3) \end{array}$$

where $f_1^*(i_3)=x_3$ and $H^*(E_8, \boldsymbol{Z}_5)\simeq\Lambda(x_3, x_{11})\otimes(\boldsymbol{Z}_5[x_{12}]/(x_{12})^5)\otimes\Lambda(x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$. The cohomology of $\tilde{E}_8$ as an $\mathcal{A}(5)$-algebra is known (see [5]). Namely

$$H^*(\tilde{E}_8, \boldsymbol{Z}_5)\simeq\boldsymbol{Z}_5[y_{50}]\otimes\Lambda(\bar{x}_{15}, \bar{x}_{23}, \bar{x}_{27}, \bar{x}_{35}, \bar{x}_{39}, \bar{x}_{47}, y_{51}, y_{59})$$

where $\bar{x}_i=p_1^*(x_i)$, $\beta y_{50}=y_{51}$ $P^1 y_{51}=y_{59}$ $P^1\bar{x}_i=\bar{x}_{i+8}$ $i=15, 27, 39$. I don't know the Hopf Algebra structure of $H^*(\tilde{E}_8, \boldsymbol{Z}_5)$.

Now let's consider the following fibration

$$\begin{array}{ccc} K(\boldsymbol{Z}, 14)\times K(\boldsymbol{Z}, 26) & \longrightarrow & X \\ & & \downarrow \\ & & \tilde{E}_8 \xrightarrow{f_2} K(\boldsymbol{Z}, 15)\times K(\boldsymbol{Z}, 27) \end{array}$$

where $f_2^*(i_{15})=\bar{x}_{15} f_2^*(i_{27})=\bar{x}_{27}$.

Since E_8 is a loop space and x_3 is a loop class, it follows that $\tilde{E}_8$ is a loop space. By dimensional reasons $\bar{x}_{15}$, $\bar{x}_{27}$ are also loop classes. So X is also a loop space. We conclude that the tower (1) of fibration below is in fact the loop of the tower (2) below.

$$\begin{array}{ll} K(\boldsymbol{Z}, 14)\times K(\boldsymbol{Z}, 26) \xrightarrow{j} X & K(\boldsymbol{Z}, 15)\times K(\boldsymbol{Z}, 27) \xrightarrow{k} BX \\ \qquad\qquad\qquad\downarrow p_2 & \qquad\qquad\qquad\downarrow h_2 \\ \qquad\qquad\qquad\tilde{E}_8 \longrightarrow K(\boldsymbol{Z}, 15)\times K(\boldsymbol{Z}, 27) & \qquad\qquad B\tilde{E}_8 \xrightarrow{g_2} K(\boldsymbol{Z}, 16)\times K(\boldsymbol{Z}, 28) \\ \qquad\qquad\qquad\downarrow p_1 & \qquad\qquad\qquad\downarrow h_1 \\ \qquad\qquad\qquad E_8 \xrightarrow{f_2} K(\boldsymbol{Z}, 3) & \qquad\qquad BE_8 \xrightarrow{g_1} K(\boldsymbol{Z}, 4)\,. \\ \qquad\qquad\qquad\quad(1) & \qquad\qquad\qquad(2) \end{array}$$

Our purpose now is to describe $H^*(X, \boldsymbol{Z}_5)$ as a Hopf Algebra.

Let $I=(\varepsilon_1, i_1, \cdots, \varepsilon_n, i_n)$ be an admissible sequence. See [11] for more details. Call ϕ_I, $\theta_I \in H^*(X, \boldsymbol{Z}_5)$ the classes such that $j^*(\phi_I)=P^I i_{14}$, $j^*(\theta_I)=P^I i_{26}$, where P^I means $\beta^{\varepsilon_1}P^{i_1}\beta^{\varepsilon_2}P^{i_2}\cdots\beta^{\varepsilon_n}P^{i_n}$.

Theorem 2.1. a) *$H^*(X, \boldsymbol{Z}_5)$ is the free algebra in the following generators: ϕ_I where I runs over the admissible sequences of excess less than 14 with $i_n>1$ or $i_n=\varepsilon_n=1$; θ_J where J runs over the admissible sequences of excess less than 26 with $j_n>0$ or $j_n=\varepsilon_n=1$; $\phi_{(0,7)}$, $\phi_{(0,11,0,1)}$, $\theta_{(0,13)}$, $\theta_{(0,17,0,1)}$, $\bar{x}_{15}\otimes(i_{14})^4$, $\bar{x}_{23}\otimes(P^1 i_{14})^4$, $\bar{x}_{27}\otimes(i_{26})^4$, $\bar{x}_{35}\otimes(P^1 i_{26})^4$, $\bar{y}_{50}$, $\bar{y}_{51}$, $\bar{y}_{59}$, $\bar{\bar{x}}_{39}$, $\bar{\bar{x}}_{47}$ where $p_2^*(y_i)=\bar{y}_i$ and $p_2^*(\bar{x}_i)=\bar{\bar{x}}_i$.*

b) *We have $\beta(\phi_{(0,7)})=x_{15}\otimes(i_{26})^4$, $\beta(\theta_{(0,17,0,1)})=x_{35}\otimes(P^1 i_{26})$ and the set $\{\bar{y}_{50}$, $\bar{\bar{x}}_{39}$, $\phi_{(1,1)}$, $\phi_{(0,2)}$, $\phi_{(0,5)}$, $\phi_{(0,5,0,1)}$, $\theta_{(1,1)}$, $\theta_{(0,2)}$, $\theta_{(0,5)}$, $\theta_{(0,5,0,1)}$ is a set of $\mathcal{A}(5)$-generators for $H^*(X, \boldsymbol{Z}_5)$.*

c) *$H^*(X, \boldsymbol{Z}_5)$ is primitively generated.*

Proof. Part a) is a routine application of the Serre's spectral sequence and Kudo's transgression theorem.

b) Let's consider the fibrations

$$\begin{array}{c} X \longrightarrow \tilde{E}_8 \\ \qquad\downarrow \\ K(\boldsymbol{Z}, 15)\times K(\boldsymbol{Z}, 27)\,. \end{array}$$

Observe that $\phi_{(0,7)}$ is transgressive and $\tau(\phi_{(0,7)})=P^7 i_{15}$. So $\beta\phi_{(0,7)}$ is also transgressive and $\tau(\beta\phi_{(0,7)})=\beta P^7 i_{15}$. This implies $\beta\phi_{(0,7)}\neq 0$. On the other hand $j^*(\beta\phi_{(0,7)})=\beta P^7 i_{14}=0$. So $\beta\phi_{(0,7)}\in\ker j^*$. But $\phi_{(0,7)}=P^2\phi_{(0,5)}$. Since $\phi_{(0,5)}$ is primitive by dimensional reasons, it follows that $\beta\phi_{(0,7)}$ is also primitive. But the only non trivial-element of $H^{71}(X, \boldsymbol{Z}_5)$, which may be primitive and belongs to Ker j^*, is $\bar{x}_{15}\otimes(i_{14})^4$. The other cases are similar. Finally the fact that the set $\{\bar{y}_{50}, \bar{\bar{x}}_{39}, \phi_{(1,1)}, \phi_{(0,2)}, \phi_{(0,5)}, \phi_{(0,5,0,1)}, \theta_{(1,1)}, \theta_{(0,2)}, \theta_{(0,5)}, \theta_{(0,5,0,1)}\}$ is a set of $\mathcal{A}(5)$-generators, follows from the description of $H^*(X, \boldsymbol{Z}_5)$ given in part a), the fact that $j^*(\phi_I)=P^I i_{14}$, $j^*(\theta_J)=P^J i_{26}$ and the first part of b).

c) In order to show that $H^*(X, \boldsymbol{Z}_5)$ is primitively generated, it suffices to show that the set of $\mathcal{A}(5)$-generators described in b) is primitive. We certainly have $\bar{y}_{50}$ and $\bar{\bar{x}}_{39}$ primitive. The map $j: K(\boldsymbol{Z}, 14)\times K(\boldsymbol{Z}, 26)\to X$ is certainly an H-map. It is in fact a loop map. Let μ be the multiplication of X and z one of the remaining $\mathcal{A}(5)$-generators for $H^*(X, \boldsymbol{Z}_5)$. Since $j^*(z)$ is primitive, we have that $\bar{\mu}^*(z)\in\text{Ker } j^*\otimes\bar{H}^*(X, \boldsymbol{Z}_5)\oplus H^*(X, \boldsymbol{Z}_5)\otimes\text{Ker } j^*$ where $\bar{\mu}^*$ is the reduced coproduct. The remaining $\mathcal{A}(5)$-generators $\phi_{(1,1)}$, $\phi_{(0,2)}$, $\phi_{(0,5)}$, $\phi_{(0,5,1,0)}$, $\theta_{(1,1)}$, $\theta_{(0,2)}$, $\theta_{(0,5)}$, $\theta_{(0,5,0,1)}$ are in dimension 23, 30, 54, 55, 35, 42, 66, 67 respectively. Up to dimension 45 Ker j^* has $\bar{x}_{39}$ as a $\boldsymbol{Z}_5$-base. Up to dimension 28, $H^*(X, \boldsymbol{Z}_5)$ has $\phi_{(1,1)}$ as a $\boldsymbol{Z}_5$-base. So up to dimension 67, Ker $j^*\otimes\bar{H}^*(X, \boldsymbol{Z}_5)\oplus\bar{H}(X, \boldsymbol{Z}_5)\otimes$Ker j^* has $\bar{\bar{x}}_{39}\otimes\phi_{(1,1)}$ and $\phi_{(1,1)}\otimes\bar{\bar{x}}_{39}$ as a $\boldsymbol{Z}_5$-base. They are in dimension 62. So the above generators are primitive. Q.E.D.

Part III

Now let us consider the following diagram:

$$\begin{array}{ccccc}
X & \longrightarrow & PBX & \longrightarrow & BX \\
\downarrow{\scriptstyle p_2} & & \downarrow & & \downarrow{\scriptstyle h_2} \\
\tilde{E}_8 & \longrightarrow & P B\tilde{E}_8 & \longrightarrow & B\tilde{E}_8
\end{array}$$

where p_2 and h_2 were defined in part II and the horizontal lines are the path-loop fibration.

Theorem 3.1. a) *$H^*(X, \boldsymbol{Z}_5)$ has a set of generators, which are universally transgressive, as an algebra.*

b) $P^1(\tau(\bar{\bar{x}}_{47}))=\beta(\tau(\bar{\bar{x}}_{47}))=0$ *and* $P^5(\tau(\bar{\bar{x}}_{47}))=\tau(\bar{\bar{x}}_{39})\tau(\bar{\bar{x}}_{47})$.

Proof. To show that $H^*(X, \boldsymbol{Z}_5)$ has generators as an algebra which are universally transgressive, it suffices to show that it has a set of $\mathcal{A}(5)$-generators which are universally transgressive. So let's consider the Eilenberg-Moore spectral sequence, see [9], namely

$$E_2\simeq \text{EXT}_{H_*(X,\boldsymbol{Z}_5)}(\boldsymbol{Z}_5, \boldsymbol{Z}_5)\Longrightarrow H^*(BX, \boldsymbol{Z}_5).$$

Since the $\mathcal{A}(5)$-generators described in Theorem 2.1 part b) are primitive we have $d_1(z)=0$ where d_1 is the first differential of the Eilenberg-Moore spectral sequence and z is one of the $\mathcal{A}(5)$-generators.

Since $d_2(z)$ is an element of a quotient of

$$\Sigma\bar{H}^*(X, \boldsymbol{Z}_5)\otimes\Sigma\bar{H}^*(X, \boldsymbol{Z}_5)\otimes\Sigma\bar{H}^*(X, \boldsymbol{Z}_5),$$

and the first non zero element appears in the dimension, 72, we must have $d_2(z)=0$. By a similar argument $d_i(z)=0$ for all i. So z is a permanent cycle.

b) By Theorem 1.1 we know that $H^*(BX, \boldsymbol{Z}_5)$ is a free algebra. Call $w_{40}=\tau(\bar{\bar{x}}_{39})$ and $w_{48}=\tau(\bar{\bar{x}}_{47})$. So we must have $P^{24}w_{48}=w_{48}^5$. Now let's consider the commutative diagram above. We have that $\bar{x}_{47}$ is transgressive by dimensional reasons. So $w_{48}=h_2^*(\tau(\bar{x}_{47}))$. This implies that βw_{48}, P^1w_{48} and P^5w_{48} belong to $\operatorname{Im} h_2^*$. By the description of $H^*(BX, \boldsymbol{Z}_5)$ we must have $\beta w_{48}=P^1w_{48}=0$ and $P^5w_{48}=\lambda w_{40}w_{48}$, $\lambda\in\boldsymbol{Z}_5$. Suppose $\lambda=0$. Then by the Adem's relations it follows that $P^{24}w_{48}=0$ which is a contradiction. So $\lambda\neq 0$. Without loss of generality let us assume $\lambda=1$. Q.E.D.

Call $z_{24}=\tau(\phi_{(1,1)})$, $z_{31}=\tau(\phi_{(0,2)})$, $z_{55}=\tau(\phi_{(0,5)})$, $z_{63}=\tau(\phi_{(0,5,0,1)})$, $t_{36}=\tau(\theta_{(1,1)})$, $t_{43}=\tau(\theta_{(1,1)})$, $t_{67}=\tau(\theta_{(0,5)})$, $t_{75}=\tau(\theta_{(0,5,0,1)})$.

Proposition 3.2. *The following relation holds in* $H^*(BX, \boldsymbol{Z}_5)$:

$$P^2z_{24}-\beta P^1z_{31}=\alpha w_{40} \qquad 0\neq\alpha\in\boldsymbol{Z}_5.$$

Proof. Since we have $\beta w_{48}=P^1w_{48}=0$, by Liulevicius's decomposition of P^p, see [8], we have:

$$\alpha P^5w_{48}=P^3\mathcal{R}(w_{48})+\beta\Gamma(w_{48})$$

modulo the total indeterminancy and $\alpha\neq 0$. By Theorem 3.1 part b) the left hand side is $\alpha w_{40}w_{48}$. Let I be the ideal of $H^*(BX, \boldsymbol{Z}_5)$ generated by all generators but w_{40}, w_{48}. By dimensional reasons $\mathcal{R}(w_{48})$ and $\Gamma(w_{48})\in I$. Now one would like to know if $\gamma x\in I$ for $\gamma\in\mathcal{A}(5)$, $x\in I$ and $\dim(\gamma x)\leqq 88$. Among the $\mathcal{A}(5)$-generators, we have the following relations up to dimension 88:

dim:	relation
25	βz_{24}
32	$P^1z_{24}-\beta z_{31}$
40	$P^2z_{24}-\beta P^1z_{31}$
55	P^3z_{31}
56	$\beta z_{55}-P^4z_{24}$
64	$P^5z_{24}-\beta z_{63}$
71	$P^3z_{31}+P^2z_{55}-P^1z_{63}$

dim:	relation
79	$P^3 z_{55}$
37	βt_{36}
44	$P^1 t_{36} - \beta t_{43}$
52	$P^2 t_{36} - \beta P^1 t_{43}$
67	$P^3 t_{43}$
68	$\beta t_{67} - P^4 t_{36}$
76	$P^5 t_{36} - \beta t_{75}$
83	$P^3 t_{43} + P^2 t_{67} - P^1 t_{75}$.

So the only element which might not be in I is $P^2 z_{24} - \beta P^1 z_{31}$. On the other hand, if $P^2 z_{24} - \beta P^1 z_{31} \in I$ we have that I is closed under the action of the Steenrod Algebra at least up to dimension 88, so $w_{40} w_{45} \in I$ which is a contradiction. So $P^2 z_{24} - \beta P^1 z_{31} = \lambda w_{40}$, $0 \neq \lambda \in \boldsymbol{Z}_5$. Q.E.D.

Theorem 3.3. *$\mathcal{R}(x_{23}) = x_{39}$ where $\mathcal{R}$ is the secondary cohomology operation defined by the relation $((1/2)\beta P^1 - P^1\beta)P^1 - P^2\beta = 0$.*

Proof. Let's apply the suspension homomorphism to the equality $P^2 z_{24} - \beta P^1 z_{31} = w_{40}$. We get $P^2 \phi_{(1,1)} - \beta P^1 \phi_{(0,2)} = \bar{\bar{x}}_{39}$. Now by the second Peterson-Stein formula we have:

$$\bar{\bar{x}}_{39} = p_2^*(\mathcal{R}(\bar{\bar{x}}_{23}))$$

modulo the total indeterminancy. Since the total indeterminancy is zero we have:

$$\bar{\bar{x}}_{39} = p_2^*(\mathcal{R}(\bar{x}_{23})) = p_2^*(\bar{x}_{39}).$$

So $\mathcal{R}(\bar{x}_{23}) = \bar{x}_{39}$. But $\bar{x}_{23} = p_1^*(x_{23})$ and $\bar{x}_{39} = p_1^*(x_{39})$. So $\mathcal{R}(p_1^*(x_{23})) = p_1^*(\mathcal{R}(x_{23})) = p_1^*(x_{39})$. So $\mathcal{R}(x_{23}) = x_{39}$.

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