# On the braided Connes–Moscovici construction

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**Abstract.** In 1998, Connes and Moscovici defined the cyclic cohomology of Hopf algebras. In 2010, Khalkhali and Pourkia proposed a braided generalization: to any Hopf algebra $H$ in a braided category $\mathcal{B}$, they associate a paracocyclic object in $\mathcal{B}$. In this paper, we explicitly compute the powers of the paracocyclic operator of this paracocyclic object. Also, we introduce twisted modular pairs in involution for $H$ and derive (co)cyclic modules from them. Finally, we relate the paracocyclic object associated with $H$ to that associated with an $H$-module coalgebra via a categorical version of the Connes–Moscovici trace.

## Contents

## 1. Introduction

Cyclic (co)homology of algebras was introduced in the 1980s by Connes [3, 4] and Tsygan [11] independently. To any algebra is associated a cocyclic vector space (that is, a cocyclic object in the category of vector spaces) whose cohomology is called the cyclic cohomology of the algebra. The notion of a (co)cyclic object in a category, introduced by Connes [3], is a generalization of the notion of a (co)simplicial object in that category.

Cyclic cohomology has been considered in various versions and generalizations. In particular, in [6], Connes and Moscovici defined the Hopf cyclic cohomology by associating a cocyclic vector space to a Hopf algebra $H$ over $\mathbb{C}$ endowed with a modular pair in involution (that is, a pair $(\delta, \sigma)$ where $\delta\colon H \to \mathbb{C}$ is a character and $\sigma \in H$ is a grouplike element satisfying the modular pair condition $\delta(\sigma) = 1$ and a certain involutivity condition). Also, in [5], they relate the Hopf cyclic cohomology of $H$ to the cyclic cohomology of an $H$-module algebra by means of a trace map.

Braided monoidal categories were defined by Street and Joyal in the 1980s and appeared in many areas of mathematics such as low-dimensional topology and representation theory. Several generalizations of cyclic (co)homology were introduced in the braided setting. In this paper, we focus on the braided generalization of the Connes–Moscovici construction due to Khalkhali and Pourkia [8]. Let $H$ be a Hopf algebra in a braided monoidal category $\mathcal{B} = (\mathcal{B}, \otimes, \mathbb{1})$. A modular pair for $H$ is a pair $(\delta, \sigma)$, where $\delta\colon H \to \mathbb{1}$ is an algebra morphism and $\sigma\colon \mathbb{1} \to H$ is a coalgebra morphism such that $\delta\sigma = \mathrm{id}_{\mathbb{1}}$. In [8], Khalkhali and Pourkia associate to any modular pair $(\delta, \sigma)$ for $H$ a paracocyclic object $\mathbf{CM}_\bullet(H, \delta, \sigma) = \{\mathbf{CM}_n(H, \delta, \sigma)\}_{n\in\mathbb{N}}$ in $\mathcal{B}$. This object is cocyclic if it satisfies the cocyclicity condition: for all $n \in \mathbb{N}$,

$$\left(\tau_n(\delta, \sigma)\right)^{n+1} = \mathrm{id}_{H^{\otimes n}},$$

where $\tau_\bullet(\delta, \sigma) = \{\tau_n(\delta, \sigma)\}_{n\in\mathbb{N}}$ is the paracocyclic operator of $\mathbf{CM}_\bullet(H, \delta, \sigma)$. As already noticed in [6] (corresponding to the case $\mathcal{B} = \mathrm{Vect}_{\mathbb{C}}$), verifying the cocyclicity condition (if true) is a rather technical task. Khalkhali and Pourkia proved [8, Theorem 7.3] that if $(\delta, \sigma)$ is a so-called braided modular pair in involution, then $(\tau_2(\delta, \sigma))^3$ is equal to the square of the braiding of $H$. In particular, if $\mathcal{B}$ is symmetric, then $(\tau_2(\delta, \sigma))^3 = \mathrm{id}_{H^{\otimes 2}}$. They also state a similar claim about $(\tau_n(\delta, \sigma))^{n+1}$, which implies the cocyclicity condition when $\mathcal{B}$ is symmetric; see [8, Remark 7.4].

Our first main result is a complete computation (by means of the Penrose graphical calculus) of the powers (up to $n + 1$) of the paracocyclic operator $\tau_n(\delta, \sigma)$ associated with a modular pair $(\delta, \sigma)$ for $H$; see Theorem 2. Next, assume that $\mathcal{B}$ has a twist $\theta$. We introduce the notion of a $\theta$-*twisted modular pair in involution for $H$* (see Section 4.1) and prove (see Corollary 4) that if $(\delta, \sigma)$ is such a pair, then the associated paracocyclic operator satisfies the following twisted cocyclicity condition: for all $n \in \mathbb{N}$,

$$\left(\tau_n(\delta, \sigma)\right)^{n+1} = \theta_{H^{\otimes n}}.$$

When $\mathcal{B}$ is further $\mathbb{k}$-linear, we derive (co)cyclic $\mathbb{k}$-modules from a $\theta$-twisted modular pair $(\delta, \sigma)$ by composing $\mathbf{CM}_\bullet(H, \delta, \sigma)$ with the functors $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, -)$ and $\mathrm{Hom}_{\mathcal{B}}(-, \mathbb{1})$; see Section 4.3. Note that if $\mathcal{B}$ is symmetric, then a braided modular pair in involution in the sense of [8] is an $\mathrm{id}_{\mathcal{B}}$-twisted modular pair in involution, where $\mathrm{id}_{\mathcal{B}}$ is the trivial twist of $\mathcal{B}$. Hence the associated paracocyclic operator satisfies the cocyclicity condition.

Let $H$ be a Hopf algebra in braided category $\mathcal{B}$ with a twist $\theta$. Our second main result is the construction of traces à la Connes–Moscovici. More precisely, let $C$ be an $H$-module coalgebra, that is, a coalgebra in the category of right $H$-modules in $\mathcal{B}$. Inspired by a construction of Akrami and Majid [1], we associate to $C$ a paracocyclic object $\mathbf{C}_\bullet(C)$ in $\mathcal{B}$. We introduce the notion of a $\delta$-*invariant $\sigma$-trace* for $C$ and derive from each such trace a natural transformation from $\mathbf{CM}_\bullet(H, \delta, \sigma)$ to $\mathbf{C}_\bullet(C)$; see Theorem 7. This generalizes the standard Connes–Moscovici trace. We provide examples of traces in the case where $\mathcal{B}$ is a ribbon category and $H$ is its coend (see Section 5.3).

The paper is organized as follows. In Section 2, we review braided monoidal categories, Hopf algebras, and graphical calculus. Section 3 is devoted to preliminaries on

simplicial, paracyclic, and cyclic objects in a category. In Sections 4 and 5, we state our main results and their corollaries. Sections 6 and 7 are devoted to the proofs of Theorems 2 and 7. In the appendix, we provide an alternative proof (by using the Penrose graphical calculus) of the fact that the object $\mathbf{CM}_\bullet(H, \delta, \sigma)$ defined in [8] is paracocyclic.

Throughout the paper, $\Bbbk$ denotes any commutative ring. The class of objects of a category $\mathcal{B}$ is denoted by $\mathrm{Ob}(\mathcal{B})$.

## 2. Preliminaries on monoidal categories and braided Hopf algebras

In this section, we recall some algebraic preliminaries used in the paper. We first recall some facts about braided monoidal categories and the Penrose graphical calculus. Next, we recall definitions of categorical Hopf algebras and related concepts. We finish with a recall on pivotal categories and coends. For a more comprehensive introduction, see [12].

### 2.1. Conventions

In what follows, we suppress in our formulas the associativity and unitality constraints of the monoidal category. This does not lead to any ambiguity since Mac Lane's coherence theorem (see [9]) implies that all possible ways of inserting these constraints give the same results. We will denote by $\otimes$ and $\mathbb{1}$ the monoidal product and unit object of a monoidal category. For any objects $X_1, \ldots, X_n$ of a monoidal category with $n \geq 2$, we set

$$X_1 \otimes X_2 \otimes \cdots \otimes X_n = \big(\cdots\big((X_1 \otimes X_2) \otimes X_3\big) \otimes \cdots \otimes X_{n-1}\big) \otimes X_n$$

and similarly for morphisms. A monoidal category is $\Bbbk$-linear if its Hom sets have a structure of a $\Bbbk$-module such that the composition and monoidal product of morphisms are $\Bbbk$-bilinear.

### 2.2. Braided categories

A *braiding* of a monoidal category $(\mathcal{B}, \otimes, \mathbb{1})$ is a family

$$\tau = \{\tau_{X,Y} \colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathrm{Ob}(\mathcal{B})}$$

of natural isomorphisms such that

$$\tau_{X,Y \otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z), \tag{2.1}$$

$$\tau_{X \otimes Y,Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z}) \tag{2.2}$$

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{B})$. A *braided category* is a monoidal category endowed with a braiding.

A braiding $\tau$ of $\mathcal{B}$ is *symmetric* if for all $X, Y \in \mathrm{Ob}(\mathcal{B})$,

$$\tau_{Y,X} \tau_{X,Y} = \mathrm{id}_{X \otimes Y}.$$

A *symmetric category* is a category endowed with a symmetric braiding.

## 2.3. Twists for braided categories

A *twist* for a braided monoidal category $\mathscr{B}$ is a natural isomorphism

$$\theta = \{\theta_X \colon X \to X\}_{X \in \mathrm{Ob}(\mathscr{B})}$$

such that

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y)\tau_{Y,X}\tau_{X,Y} \tag{2.3}$$

holds for all $X, Y \in \mathrm{Ob}(\mathscr{B})$. Note that this implies that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. For example, any ribbon category (see Section 2.16) has a canonical twist.

Note that when $\mathscr{B}$ is symmetric, a twist for $\mathscr{B}$ is nothing but a monoidal natural isomorphism of the identity functor $\mathscr{B} \to \mathscr{B}$. In particular, $\mathrm{id}_{\mathscr{B}} = \{\mathrm{id}_X \colon X \to X\}_{X \in \mathrm{Ob}(\mathscr{B})}$ is a twist for $\mathscr{B}$.

By a *braided category with a twist*, we mean a braided category endowed with a twist.

## 2.4. Graphical calculus

Throughout this paper, we will use the *Penrose graphical calculus*. For a systematic treatment, one may consult [12]. The diagrams are to be read from bottom to top. In a monoidal category $\mathscr{B}$, the diagrams are made of arcs colored by objects of $\mathscr{B}$ and of boxes, colored by morphisms of $\mathscr{B}$. Arcs colored by $\mathbb{1}$ may be omitted in the pictures. The identity morphism of an object $X$, a morphism $f \colon X \to Y$ in $\mathscr{B}$, and its composition with a morphism $g \colon Y \to Z$ in $\mathscr{B}$ are represented respectively as
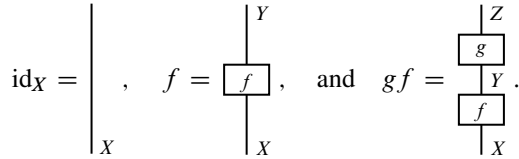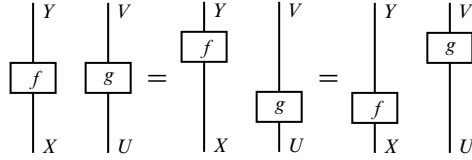


The tensor product of two morphisms $f \colon X \to Y$ and $g \colon U \to V$ is represented by placing a picture of $f$ to the left of the picture of $g$:
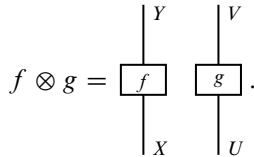


Any diagram represents a morphism. For example, the diagram
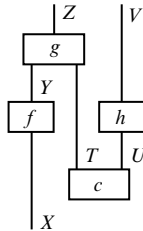
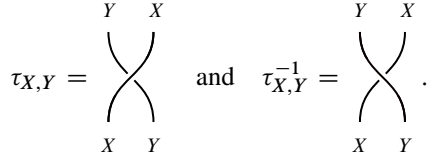represents $(g \otimes \mathrm{id}_V)(f \otimes \mathrm{id}_T \otimes h)(\mathrm{id}_X \otimes c): X \to Z \otimes V$. The morphism associated to a diagram depends only on the isotopy class of the diagram representing it. For example, the *level-exchange property*
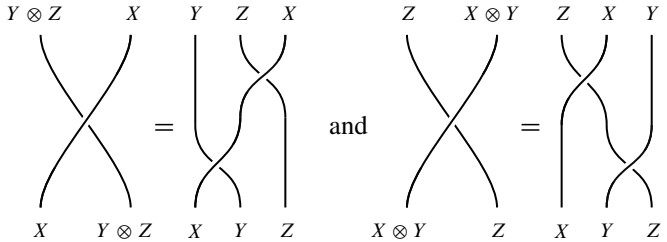


reflects the formula

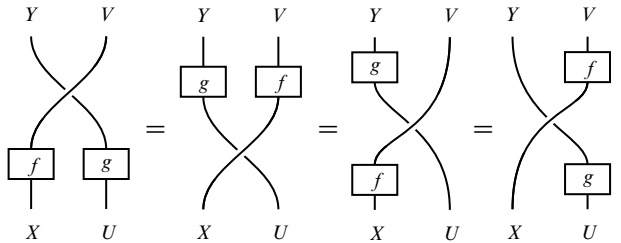$$f \otimes g = (f \otimes \mathrm{id}_V)(\mathrm{id}_X \otimes g) = (\mathrm{id}_Y \otimes g)(f \otimes \mathrm{id}_U).$$

When $\mathcal{B}$ is braided with braiding $\tau$, we depict



$$\tau_{X,Y} = \qquad \text{and} \quad \tau_{X,Y}^{-1} = \qquad .$$

Axioms (2.1) and (2.2) for $\tau$ say that for all $X, Y, Z \in \mathrm{Ob}(\mathcal{B})$,



Naturality of the braiding and the level-exchange property imply that for any two morphisms $f: X \to Y$ and $g: U \to V$ in $\mathcal{B}$,



When $\mathcal{B}$ is braided with a twist $\theta = \{\theta_X: X \to X\}_{X \in \mathrm{Ob}(\mathcal{B})}$, we denote the twist by



$$\theta_X = \qquad .$$

Axiom (2.3) for $\theta$ gives that for any $X, Y \in \mathrm{Ob}(\mathcal{B})$,
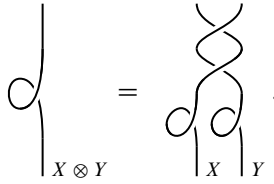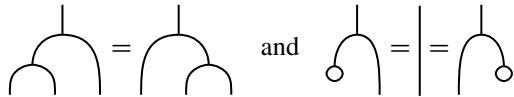


## 2.5. Categorical algebras

An *algebra* in a monoidal category $\mathcal{B}$ is a triple $(A, m, u)$, where $A$ is an object of $\mathcal{B}$, $m \colon A \otimes A \to A$ and $u \colon \mathbb{1} \to A$ are morphisms in $\mathcal{B}$, called *multiplication* and *unit* respectively, which satisfy the associativity and unitality axioms:

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m) \quad \text{and} \quad m(u \otimes \mathrm{id}_A) = \mathrm{id}_A = m(\mathrm{id}_A \otimes u).$$

The multiplication and unit are depicted by
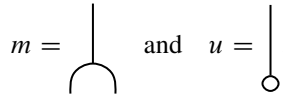


so that the associativity and unitality axioms rewrite graphically as



Here, it is understood that the arcs are colored by the underlying object of the algebra.

An *algebra morphism* between algebras $(A, m, u)$ and $(A', m', u')$ in a monoidal category $\mathcal{B}$ is a morphism $f \colon A \to A'$ in $\mathcal{B}$ such that $fm = m'(f \otimes f)$ and $fu = u'$. The latter conditions are depicted by
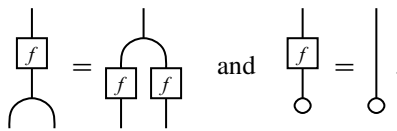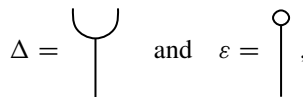


## 2.6. Categorical coalgebras

A *coalgebra* in a monoidal category $\mathcal{B}$ is a triple $(C, \Delta, \varepsilon)$, where $C$ is an object of $\mathcal{B}$, $\Delta \colon C \to C \otimes C$ and $\varepsilon \colon C \to \mathbb{1}$ are morphisms in $\mathcal{B}$, called *comultiplication* and *counit* respectively, which satisfy the coassociativity and counitality axioms:

$$(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta \quad \text{and} \quad (\mathrm{id}_C \otimes \varepsilon)\Delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C)\Delta.$$

The comultiplication and the counit are depicted by

so that the coassociativity and counitality axioms rewrite graphically as

$$\bigcup\hspace{-0.2cm}\bigvee = \bigvee\hspace{-0.2cm}\bigcup \quad \text{and} \quad \bigvee\hspace{-0.1cm}\circ = \Big| = \circ\hspace{-0.1cm}\bigvee.$$

A *coalgebra morphism* between coalgebras $(C, \Delta, \varepsilon)$ and $(C', \Delta', \varepsilon')$ in a monoidal category $\mathcal{B}$ is a morphism $f \colon C \to C'$ in $\mathcal{B}$ such that $\Delta' f = (f \otimes f)\Delta$ and $\varepsilon f = \varepsilon'$.

### 2.7. Graphical calculus and iterated (co)multiplications

Let $(A, m, u)$ and $(C, \Delta, \varepsilon)$ be an algebra and a coalgebra in a monoidal category $\mathcal{B}$. For any $n \in \mathbb{N}$, we define the *$n$-th multiplication $m_n \colon A^{\otimes n} \to A$* and the *$n$-th comultiplication $\Delta_n \colon C \to C^{\otimes n}$* inductively by

$$m_0 = u, \quad m_{n+1} = m(\mathrm{id}_A \otimes m_n), \quad \Delta_0 = \varepsilon, \quad \text{and} \quad \Delta_{n+1} = (\mathrm{id}_C \otimes \Delta_n)\Delta.$$

For $n \geq 1$, we depict them as

$$m_n = \underbrace{\bigtriangleup\hspace{-0.2cm}}_{n \text{ times}} \quad \text{and} \quad \Delta_n = \overbrace{\bigtriangledown}^{n \text{ times}}.$$

The (co)associativity and the (co)unitality of $m$ and $\Delta$ imply that

$$m_{k+1}(m_{n_0} \otimes \cdots \otimes m_{n_k}) = m_{n_0 + \cdots + n_k} \quad \text{and} \quad (\Delta_{n_0} \otimes \cdots \otimes \Delta_{n_k})\Delta_{k+1} = \Delta_{n_0 + \cdots + n_k}$$

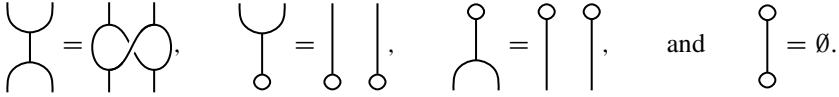for all $k \in \mathbb{N}$ and $n_0, \ldots, n_k \in \mathbb{N}$. For example,

$$m_4 = \bigcap\hspace{-0.1cm} = \bigcap\hspace{-0.1cm} = \bigcap\hspace{-0.1cm}, $$

$$\Delta_3 = \bigcup\hspace{-0.1cm} = \bigcup\hspace{-0.1cm}.$$

### 2.8. Categorical bialgebras

Let $\mathcal{B}$ be a braided monoidal category. A *bialgebra* in $\mathcal{B}$ is a quintuple $(A, m, u, \Delta, \varepsilon)$ such that $(A, m, u)$ is an algebra in $\mathcal{B}$, $(A, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{B}$, and the following compatibility relations hold:

$$\Delta m = (m \otimes m)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta), \quad \Delta u = u \otimes u, \quad \varepsilon m = \varepsilon \otimes \varepsilon, \quad \varepsilon u = \mathrm{id}_{\mathbb{1}}.$$
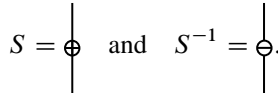
Graphically, these rewrite as



A *bialgebra morphism* between two bialgebras $A$ and $A'$ is a morphism $A \to A'$ in $\mathcal{B}$ which is both an algebra and a coalgebra morphism.
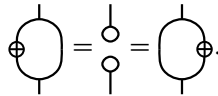
## 2.9.  Categorical Hopf algebras

Let $\mathcal{B}$ be a braided monoidal category. A *Hopf algebra* in $\mathcal{B}$ is a sextuple $(H, m, u, \Delta, \varepsilon, S)$, where $(H, m, u, \Delta, \varepsilon)$ is a bialgebra in $\mathcal{B}$ and $S\colon H \to H$ is an isomorphism in $\mathcal{B}$, called the *antipode*, which satisfies

$$m(S \otimes \mathrm{id}_H)\Delta = u\varepsilon = m(\mathrm{id}_H \otimes S)\Delta.$$

The antipode and its inverse are depicted by

$$S = \quad \text{and} \quad S^{-1} = .$$

Graphically, the antipode axiom is rewritten as



A useful feature of antipodes is that it is *anti-multiplicative*:
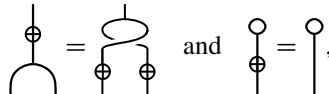


and *anti-comultiplicative*:
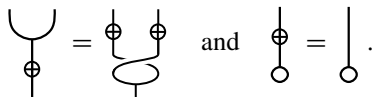


A *Hopf algebra morphism* between two Hopf algebras is a bialgebra morphism between them.

## 2.10.  Categorical modules

Let $(A, m, u)$ be an algebra in a monoidal category $\mathcal{B}$. A *left $A$-module* in $\mathcal{B}$ is a pair $(M, r)$, where $r\colon A \otimes M \to M$ is a morphism in $\mathcal{B}$, called *the action of $A$ on $M$*, which satisfies

$$r(m \otimes \mathrm{id}_M) = r(\mathrm{id}_A \otimes r) \quad \text{and} \quad r(u \otimes \mathrm{id}_M) = \mathrm{id}_M.$$

Graphically, the action $r: A \otimes M \to M$ is denoted by

$$r = \quad \overset{\displaystyle M}{\underset{\displaystyle A \quad M}{\Large\curvearrowright}} \quad ,$$

so that the axioms of a left $A$-module rewrite as



and



A morphism $f: (M, r) \to (M', r')$ between two left $A$-modules $(M, r)$ and $(M', r')$ is a morphism $f: M \to M'$ in $\mathcal{B}$ such that $fr = r'(\mathrm{id}_A \otimes f)$. With composition inherited from $\mathcal{B}$, left $A$-modules and morphisms between them form a category $_A\mathbf{Mod}$.

When $\mathcal{B}$ is braided and $A$ is a bialgebra in $\mathcal{B}$, the category $_A\mathbf{Mod}$ is monoidal: the unit object of $_A\mathbf{Mod}$ is the pair $(\mathbb{1}, \varepsilon)$, the monoidal product of two left $A$-modules $(M, r)$ and $(M', r')$ is given by the pair $(M \otimes M', s)$, where

$$s = (r \otimes r')(\mathrm{id}_A \otimes \tau_{A,M} \otimes \mathrm{id}_{M'})(\Delta \otimes \mathrm{id}_{M \otimes M'}) = \quad \overset{\displaystyle M \quad M'}{\underset{\displaystyle A \quad M \quad M'}{\Large\diagup}} \quad ,$$

and the monoidal product of morphisms is inherited from $\mathcal{B}$.

## 2.11. Categorical comodules

Let $(C, \Delta, \varepsilon)$ be a coalgebra in a monoidal category $\mathcal{B}$. A *left $C$-comodule* in $\mathcal{B}$ is a pair $(N, \gamma)$, where $\gamma: N \to C \otimes N$ is a morphism in $\mathcal{B}$, called *the coaction of $C$ on $N$*, which satisfies

$$(\Delta \otimes \mathrm{id}_N)\gamma = (\mathrm{id}_C \otimes \gamma)\gamma \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}_N)\gamma = \mathrm{id}_N.$$

Graphically, the coaction $\gamma: N \to C \otimes N$ is denoted by

$$\gamma = \quad \overset{\displaystyle C \quad N}{\underset{\displaystyle N}{\Large\diagdown}} \quad ,$$

so that the axioms of a left $A$-comodule rewrite as



and



A morphism $f: (N, \gamma) \to (N', \gamma')$ between two left $A$-comodules $(N, \gamma)$ and $(N', \gamma')$ is a morphism $f: N \to N'$ in $\mathcal{B}$ such that $\gamma' f = (\mathrm{id}_C \otimes f)\gamma$. With composition inherited from $\mathcal{B}$, left $C$-comodules and morphisms between them form a category $_C\mathbf{Comod}$.

When $\mathcal{B}$ is braided and $C$ is a bialgebra in $\mathcal{B}$, the category $_C\mathbf{Comod}$ is monoidal: the unit object of $_C\mathbf{Comod}$ is the pair $(\mathbb{1}, u)$, the monoidal product of two left $C$-comodules $(N, \gamma)$ and $(N', \gamma')$ is given by the pair $(N \otimes N', \delta)$, where

$$\delta = (m \otimes \mathrm{id}_{N \otimes N'})(\mathrm{id}_C \otimes \tau_{N,C} \otimes \mathrm{id}_{N'})(\gamma \otimes \gamma') = $$

and the monoidal product of morphisms is inherited from $\mathcal{B}$.

## 2.12. Diagonal actions

Let $H$ be a bialgebra in a braided category $\mathcal{B}$. The *left diagonal action* of $H$ on $H^{\otimes n}$ is defined inductively by

$$\qquad = \qquad \text{and} \qquad = \qquad \text{for } n \geq 1.$$

Note that

$$\qquad = \qquad \text{and} \qquad = \qquad \text{for } n \geq 2.$$

Similarly, the *right diagonal action* of $H$ on $H^{\otimes n}$ is defined inductively by

$$\qquad = \qquad \text{and} \qquad = \qquad \text{for } n \geq 1.$$

Note that

$$\qquad = \qquad \text{and} \qquad = \qquad \text{for } n \geq 2.$$

It follows from the definitions that, if $\sigma \colon \mathbb{1} \to H$ is a coalgebra morphism, then

$$\qquad = \qquad \cdots \qquad \text{and} \qquad = \qquad \cdots \qquad .$$

## 2.13. Adjoint actions

Let $H$ be a Hopf algebra in a braided category $\mathcal{B}$. The *left adjoint action* of $H$ on $H^{\otimes n}$ is defined inductively by



for $n \geq 1$.

Similarly, the *right adjoint action* of $H$ on $H^{\otimes n}$ is defined inductively by



for $n \geq 1$.

Note that



It follows from the definition that, if $\sigma \colon \mathbb{1} \to H$ is a coalgebra morphism, then



## 2.14. Coadjoint coactions

Let $H$ be a Hopf algebra in a braided category $\mathcal{B}$. The *left coadjoint coaction* of $H$ on $H^{\otimes n}$ is defined inductively by



for $n \geq 1$.

Similarly, the *right coadjoint coaction* of $H$ on $H^{\otimes n}$ is defined inductively by



for $n \geq 1$.

Note that
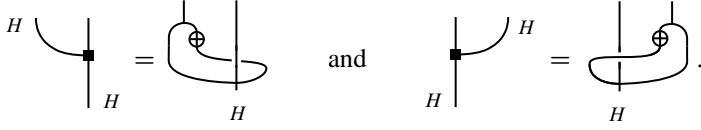


It follows from the definition that, if $\delta\colon H \to \mathbb{1}$ is an algebra morphism, then
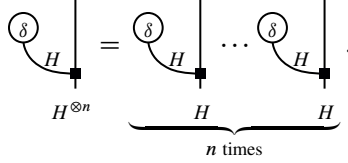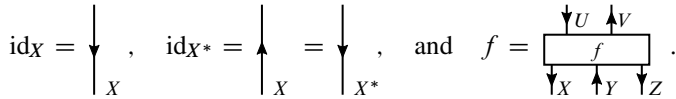


## 2.15. Pivotal categories

A *pivotal* category is a monoidal category $\mathcal{B}$ such that each object $X$ of $\mathcal{B}$ has a *dual* object $X^*$ and four morphisms

$$\mathrm{ev}_X\colon X^* \otimes X \to \mathbb{1}, \quad \mathrm{coev}_X\colon \mathbb{1} \to X \otimes X^*,$$
$$\widetilde{\mathrm{ev}}_X\colon X \otimes X^* \to \mathbb{1}, \quad \widetilde{\mathrm{coev}}_X\colon \mathbb{1} \to X^* \otimes X,$$

satisfying some conditions. Briefly, these say that the associated left/right dual functors coincide as monoidal functors (see [12, Chapter 1] for more details). The latter implies that the dual morphism $f^*\colon Y^* \to X^*$ of a morphism $f\colon X \to Y$ in $\mathcal{B}$ is computed by

$$f^* = (\mathrm{id}_{X^*} \otimes \widetilde{\mathrm{ev}}_Y)(\mathrm{id}_{X^*} \otimes f \otimes \mathrm{id}_{Y^*})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{Y^*})$$
$$= (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*})(\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X).$$

We extend the graphical calculus for monoidal categories (see Section 2.4) to pivotal categories by orienting arcs. If an arc colored by $X$ is oriented upwards, the represented object in source/target of corresponding morphism is $X^*$. For example, $\mathrm{id}_X$, $\mathrm{id}_{X^*}$, and a morphism $f\colon X \otimes Y^* \otimes Z \to U \otimes V^*$ are depicted by

$$\mathrm{id}_X = \quad , \quad \mathrm{id}_{X^*} = \quad = \quad , \quad \text{and} \quad f = \quad .$$

The morphisms $\mathrm{ev}_X$, $\widetilde{\mathrm{ev}}_X$, $\mathrm{coev}_X$, and $\widetilde{\mathrm{coev}}_X$ are respectively depicted by

$$\overset{\frown}{\phantom{X}}X\,, \quad \overset{\frown}{\phantom{X}}{}_X\,, \quad \underset{\smile}{\phantom{X}}{}^X, \quad \text{and} \quad \underset{\smile}{\phantom{X}}{}^X\,.$$

## 2.16. Left and right twists

Let $\mathcal{B}$ be a braided pivotal category. The *left twist* of an object $X$ of $\mathcal{B}$ is defined by

$$\theta_X^l = \quad = (\mathrm{id}_X \otimes \widetilde{\mathrm{ev}}_X)(\tau_{X,X} \otimes \mathrm{id}_{X^*})(\mathrm{id}_X \otimes \mathrm{coev}_X)\colon X \to X,$$

while the *right twist* of $X$ is defined by

$$\theta_X^r = \quad = (\mathrm{ev}_X \otimes \mathrm{id}_X)(\mathrm{id}_{X^*} \otimes \tau_{X,X})(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_X) \colon X \to X.$$

The left and the right twists are natural isomorphisms with inverses

$$(\theta_X^l)^{-1} = \quad \text{and} \quad (\theta_X^r)^{-1} = \quad .$$

The left twist $\theta^l = \{\theta_X^l \colon X \to X\}_{X \in \mathrm{Ob}(\mathcal{B})}$ and the right twist $\theta^r = \{\theta_X^r \colon X \to X\}_{X \in \mathrm{Ob}(\mathcal{B})}$ are twists for $\mathcal{B}$ in the sense of Section 2.3.

A *ribbon* category is a braided pivotal category $\mathcal{B}$ whose left and right twists coincide. Then $\theta = \theta_l = \theta_r$ is called the *twist* of $\mathcal{B}$.

## 2.17. Coends

Let $\mathcal{C}$ and $\mathcal{D}$ be any categories and $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ a functor. A *dinatural transformation* between $F$ and an object $D$ in $\mathcal{D}$ is a function $d$ that assigns to any object $X$ in $\mathcal{C}$ a morphism $d_X \colon F(X, X) \to D$ such that for all morphisms $f \colon X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
F(Y, X) & \xrightarrow{\;F(f, \mathrm{id}_X)\;} & F(X, X) \\[2pt]
{\scriptstyle F(\mathrm{id}_Y, f)}\Big\downarrow & & \Big\downarrow{\scriptstyle d_X} \\[2pt]
F(Y, Y) & \xrightarrow[\;d_Y\;]{} & D.
\end{array}
$$

A *coend* of a functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a pair $(C, i)$, where $C$ is an object of $\mathcal{D}$ and $i$ is a dinatural transformation from $F$ to $C$, which is universal among all dinatural transformations. More precisely, for any dinatural transformation $d$ from $F$ to $D$, there exists a unique morphism $\varphi \colon C \to D$ in $\mathcal{D}$ such that $d_X = \varphi i_X$ for all $X \in \mathrm{Ob}(\mathcal{C})$. A coend $(C, i)$ of a functor $F$, if it exists, is unique up to a unique isomorphism commuting with the dinatural transformation.
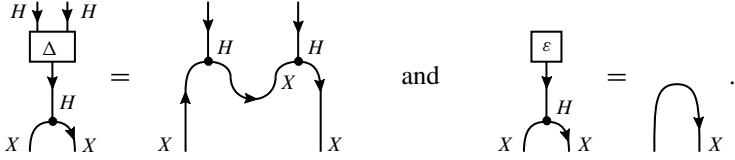
## 2.18. Coend of a pivotal category

Let $\mathcal{B}$ be a pivotal category. The *coend* of $\mathcal{B}$, if it exists, is the coend $(H, i)$ of the functor $F \colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{B}$ defined by

$$F(X, Y) = X^* \otimes Y \quad \text{and} \quad F(f, g) = f^* \otimes g.$$

We depict the universal dinatural transformation $i = \{i_X \colon X^* \otimes X \to H\}_{X \in \mathrm{Ob}(\mathcal{B})}$ as
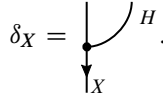
$$i_X = \quad .$$

Note that $H$ is a coalgebra in $\mathcal{B}$ with comultiplication $\Delta\colon H \to H \otimes H$ and counit $\varepsilon\colon H \to \mathbb{1}$, which are unique morphisms such that, for all $X \in \mathrm{Ob}(\mathcal{B})$,
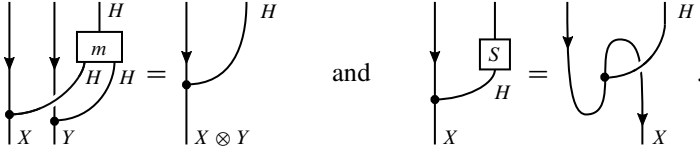


The coalgebra $(H, \Delta, \varepsilon)$ coacts on the objects in $\mathcal{B}$ via the *universal coaction* defined for any $X \in \mathrm{Ob}(\mathcal{B})$ by

$$\delta_X = (\mathrm{id}_X \otimes i_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X).$$

We will denote it graphically as

$$\delta_X = \quad \text{\raisebox{0pt}{}} H \, .$$

Note that $\delta_H$ is the right coadjoint coaction of $H$ on $H$ (see Section 2.14). If $\mathcal{B}$ is braided, then $H$ is a Hopf algebra in $\mathcal{B}$. Its unit is $u = \delta_{\mathbb{1}}\colon \mathbb{1} \to H$ and its multiplication $m\colon H \otimes H \to H$ and antipode $S\colon H \to H$ are characterized as follows: for all $X, Y \in \mathrm{Ob}(\mathcal{B})$,



We refer to [12, Chapter 6] for details.

# 3. Simplicial, paracyclic, and cyclic objects

In this section, we recall the notions of (co)simplicial, para(co)cyclic, and (co)cyclic objects in a category.

## 3.1. The simplicial category

The *simplicial category* $\Delta$ is defined as follows. The objects of $\Delta$ are the nonnegative integers $n \in \mathbb{N}$. A morphism $n \to m$ in $\Delta$ is an increasing map between sets $[n] = \{0, \ldots, n\}$ and $[m] = \{0, \ldots, m\}$. For $n \in \mathbb{N}^*$ and $0 \le i \le n$, the $i$-th *coface* $\delta_i^n\colon n-1 \to n$ is the unique increasing injection from $[n-1]$ into $[n]$ which misses $i \in [n]$. For $n \in \mathbb{N}$ and $0 \le j \le n$, the $j$-th *codegeneracy*

$$\sigma_j^n\colon n+1 \to n$$

is the unique increasing surjection from $[n+1]$ onto $[n]$ which sends both $j$ and $j+1$ to $j$.

It is well known (see [10, Lemma 5.1]) that morphisms in $\Delta$ are generated by cofaces $\{\delta_i^n\}_{n \in \mathbb{N}^*, 0 \leq i \leq n}$ and codegeneracies $\{\sigma_j^n\}_{n \in \mathbb{N}, 0 \leq j \leq n}$ subject to the simplicial relations (SR):

$$
\begin{aligned}
\delta_j \delta_i &= \delta_i \delta_{j-1} \quad \text{for } i < j, \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \quad \text{for } i \leq j, \\
\sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j, \\ \mathrm{id}_n & \text{for } i = j, \ i = j + 1, \\ \delta_{i-1} \sigma_j & \text{for } i > j + 1. \end{cases}
\end{aligned}
\tag{SR}
$$

### 3.2. The paracyclic category

The *paracyclic category* $\Delta C_\infty$ is defined as follows. The objects of $\Delta C_\infty$ are the non-negative integers $n \in \mathbb{N}$. The morphisms are generated by morphisms $\{\delta_i^n\}_{n \in \mathbb{N}^*, 0 \leq i \leq n}$, called *cofaces*, morphisms $\{\sigma_j^n\}_{n \in \mathbb{N}, 0 \leq j \leq n}$, called *codegeneracies*, and isomorphisms $\{\tau_n \colon n \to n\}_{n \in \mathbb{N}}$, called *paracocyclic operators*, satisfying the simplicial relations (SR) and the following *paracyclic compatibility relations* (PCR):

$$
\begin{aligned}
\tau_n \delta_i &= \delta_{i-1} \tau_{n-1} \quad \text{for } 1 \leq i \leq n, \\
\tau_n \delta_0 &= \delta_n, \\
\tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} \quad \text{for } 1 \leq i \leq n, \\
\tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2.
\end{aligned}
\tag{PCR}
$$

Note that $\Delta$ is a subcategory of $\Delta C_\infty$.

### 3.3. The cyclic category

The *cyclic category* $\Delta C$ is defined as follows. The objects of $\Delta C$ are the nonnegative integers $n \in \mathbb{N}$. The morphisms are generated by morphisms $\{\delta_i^n\}_{n \in \mathbb{N}^*, 0 \leq i \leq n}$, called *cofaces*, morphisms $\{\sigma_j^n\}_{n \in \mathbb{N}, 0 \leq j \leq n}$, called *codegeneracies*, and isomorphisms $\{\tau_n \colon n \to n\}_{n \in \mathbb{N}}$, called *cocyclic operators*, which satisfy the relations (SR), (PCR), and the *cyclicity condition* (CC):

$$
\tau_n^{n+1} = \mathrm{id}_n.
\tag{CC}
$$

Note that $\Delta C$ is a quotient of $\Delta C_\infty$.

### 3.4. (Co)simplicial, para(co)cyclic, and (co)cyclic objects in a category

Let $\mathcal{C}$ be any category. A *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$, a *paracyclic* object in $\mathcal{C}$ is a functor $\Delta C_\infty^{\mathrm{op}} \to \mathcal{C}$, and a *cyclic object* in $\mathcal{C}$ is a functor $\Delta C^{\mathrm{op}} \to \mathcal{C}$. Dually, a *cosimplicial object* in $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$, a *paracocyclic* object in $\mathcal{C}$ is a functor $\Delta C_\infty \to \mathcal{C}$, and a *cocyclic object* in $\mathcal{C}$ is a functor $\Delta C \to \mathcal{C}$. A (co)simplicial/para(co)cyclic/(co)cyclic object in the category of sets (resp., of $\mathbb{k}$-modules) is called a *(co)simplicial/para(co)cyclic/(co)cyclic set* (resp., $\mathbb{k}$-*module*).

A *morphism* between two (co)simplicial/para(co)cyclic/(co)cyclic objects is a natural transformation between them. One often denotes the image of a morphism $f$ under a (co)simplicial/para(co)cyclic/(co)cyclic object in a category by the same letter $f$.

Since the categories $\Delta, \Delta C_\infty, \Delta C$ are defined by generators and relations, a (co)simplicial/para(co)cyclic/(co)cyclic object in a category is entirely determined by the images of the generators satisfying the corresponding relations. For example, a paracocyclic object $Y$ in $\mathcal{C}$ may be seen as a family $Y_\bullet = \{Y_n\}_{n \in \mathbb{N}}$ of objects of $\mathcal{C}$ equipped with morphisms $\{\delta_i^n \colon Y_{n-1} \to Y_n\}_{n \in \mathbb{N}^*, 0 \le i \le n}$, called *cofaces*, morphisms $\{\sigma_j^n \colon Y_{n+1} \to Y_n\}_{n \in \mathbb{N}, 0 \le j \le n}$, called *codegeneracies*, and isomorphisms $\{\tau_n \colon Y_n \to Y_n\}_{n \in \mathbb{N}}$, called *paracocyclic operators*, subject to the relations (SR) and (PCR). Similarly, a morphism $\beta \colon Y \to Y'$ between two paracocyclic objects $Y_\bullet$ and $Y'_\bullet$ in $\mathcal{C}$ is explicited as a family $\{\beta_n \colon Y_n \to Y'_n\}_{n \in \mathbb{N}}$ of morphisms in $\mathcal{C}$ satisfying

$$\beta_n \delta_i^n = \delta_i^n \beta_{n-1} \quad \text{for any } n \in \mathbb{N}^* \text{ and } 0 \le i \le n,$$
$$\beta_n \sigma_j^n = \sigma_j^n \beta_{n+1} \quad \text{for any } n \in \mathbb{N} \text{ and } 0 \le j \le n,$$
$$\beta_n \tau_n = \tau_n \beta_n \quad \text{for any } n \in \mathbb{N}.$$

Clearly, the composition of a (co)simplicial/para(co)cyclic/(co)cyclic object $X$ in $\mathcal{C}$ with a functor $F \colon \mathcal{C} \to \mathcal{D}$ is a (co)simplicial/para(co)cyclic/(co)cyclic object $FX$ in $\mathcal{D}$. In particular, useful examples are provided by the covariant and the contravariant Hom-functors $\mathrm{Hom}_{\mathcal{C}}(I, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, I)$, where $I$ is an object of $\mathcal{C}$. In this case, we denote

$$\mathrm{Hom}_{\mathcal{C}}(I, X) = \mathrm{Hom}_{\mathcal{C}}(I, -) \circ X$$

and

$$\mathrm{Hom}_{\mathcal{C}}(X, I) = \mathrm{Hom}_{\mathcal{C}}(-, I) \circ X.$$

**Lemma 1.** *Let $\mathcal{B}$ be a $\Bbbk$-linear braided category with a twist $\theta$. If $Y_\bullet = \{Y_n\}_{n \in \mathbb{N}}$ is a paracocyclic object in $\mathcal{B}$ such that its paracocyclic operator satisfies*

$$\tau_n^{n+1} = \theta_{Y_n} \quad \text{for each } n \in \mathbb{N},$$

*then*

(a)  $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_\bullet) = \{\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_n)\}_{n \in \mathbb{N}}$ *is a cocyclic $\Bbbk$-module,*

(b)  $\mathrm{Hom}_{\mathcal{B}}(Y_\bullet, \mathbb{1}) = \{\mathrm{Hom}_{\mathcal{B}}(Y_n, \mathbb{1})\}_{n \in \mathbb{N}}$ *is a cyclic $\Bbbk$-module.*

*Proof.* Let us prove (a). By composition with $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, -) \colon \mathcal{B} \to \mathbf{Mod}_{\Bbbk}$, we obtain that $\{\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_n)\}_{n \in \mathbb{N}}$ is a paracocyclic $\Bbbk$-module. Let us now verify that the condition (CC) holds. Let $n \in \mathbb{N}$. The morphism

$$\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, -)(\theta_{Y_n}) \colon \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_n) \to \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_n)$$

is the $\Bbbk$-linear morphism given by $f \mapsto \theta_{Y_n} f$. The naturality of $\theta$ and the fact that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ imply that

$$\theta_{Y_n} f = f \theta_{\mathbb{1}} = f \quad \text{for all } f \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_n).$$

Thus the morphism $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, -)(\theta_{Y_n})$ is the identity morphism. Now, by definition and the hypothesis that $Y_\bullet(\tau_n^{n+1}) = \theta_{Y_n}$, we have

$$
\begin{aligned}
\big(\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_\bullet)(\tau_n)\big)^{n+1} &= \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, -)\big(Y_\bullet(\tau_n)^{n+1}\big) \\
&= \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, -)(\theta_{Y_n}) = \mathrm{id}_{\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, Y_n)}.
\end{aligned}
$$

Part (b) is proved similarly. ∎

### 3.5. Cyclic (co)homology

To any cyclic $\Bbbk$-module $X \colon \Delta C^{\mathrm{op}} \to \mathbf{Mod}_{\Bbbk}$, one can associate a bicomplex $\mathbf{CC}(X)$ (see [13] for details). The $n$-th *cyclic homology* $HC_n(X)$ of $X$ is defined as the $n$-th homology of the total chain complex associated to the chain bicomplex $\mathbf{CC}(X)$. A morphism between cyclic $\Bbbk$-modules induces a levelwise morphism in cyclic homology.

Similarly, to any cocyclic $\Bbbk$-module $Y \colon \Delta C \to \mathbf{Mod}_{\Bbbk}$, one can associate a cochain bicomplex $\mathbf{CC}(Y)$, obtained by a construction dual to the one of a chain bicomplex. The $n$-th *cyclic cohomology* $HC^n(Y)$ of $Y$ is defined as the $n$-th cohomology of the total cochain complex associated to the cochain bicomplex $\mathbf{CC}(Y)$. A morphism between cocyclic $\Bbbk$-modules induces a levelwise morphism in cyclic cohomology.

## 4. Modular pairs and braided Connes–Moscovici construction

In this section, $\mathcal{B}$ is a braided monoidal category and $H$ is a Hopf algebra in $\mathcal{B}$. We provide a braided generalization of the notion of a modular pair in involution for $H$ and then compute the powers of the paracocyclic operator associated to such a pair (see Theorem 2 and its corollaries).

### 4.1. Modular pairs

A *modular pair* for $H$ is a pair $(\delta, \sigma)$, where $\delta \colon H \to \mathbb{1}$ is an algebra morphism and $\sigma \colon \mathbb{1} \to H$ is a coalgebra morphism such that $\delta\sigma = \mathrm{id}_{\mathbb{1}}$. For instance, $(\varepsilon, u)$ is a modular pair for $H$, where $\varepsilon \colon H \to \mathbb{1}$ and $u \colon \mathbb{1} \to H$ are the counit and the unit of $H$, respectively.

Given a twist $\theta$ for $\mathcal{B}$, a *$\theta$-twisted modular pair in involution* for $H$ is a modular pair $(\delta, \sigma)$ for $H$ such that
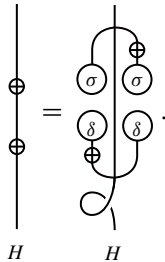


Here we use the graphical conventions from Section 2.4. If $H$ is *involutive* Hopf algebra in $\mathcal{B}$ in the sense that $S^2 = \theta_H$, then $(\varepsilon, u)$ is a $\theta$-twisted modular pair in involution for $H$.

Note that if $\mathcal{B}$ is symmetric with a trivial twist $\mathrm{id}_{\mathcal{B}}$ (see Section 2.3), then an $\mathrm{id}_{\mathcal{B}}$-twisted modular pair in involution corresponds to a braided modular pair in involution in the sense of [8].

## 4.2. Powers of the paracocyclic operators

Let $(\delta, \sigma)$ be a modular pair for $H$. For any $n \geq 0$, define the *paracocyclic operator* $\tau_n(\delta, \sigma) \colon H^{\otimes n} \to H^{\otimes n}$ by

$$\tau_0(\delta, \sigma) = \mathrm{id}_{\mathbb{1}}, \quad \tau_1(\delta, \sigma) = \begin{array}{c}\text{[diagram]}\end{array}, \quad \text{and} \quad \tau_n(\delta, \sigma) = \begin{array}{c}\text{[diagram]}\end{array} \quad \text{for } n \geq 2.$$

Here we use the diagonal actions defined in Section 2.12. Note that the operators $\tau_n(\delta, \sigma)$ are the paracocyclic operators of a paracocyclic object in $\mathcal{B}$ associated with $H$ and $(\delta, \sigma)$ (see Section 4.3). In the following theorem, we compute the powers (up to $n + 1$) of $\tau_n(\delta, \sigma)$.

**Theorem 2.** *For $n \geq 2$ and $2 \leq k \leq n$, we have*

$$\left(\tau_n(\delta, \sigma)\right)^k = \begin{array}{c}\text{[diagram]}\end{array}. \tag{4.1}$$

*In addition,*

$$\left(\tau_n(\delta, \sigma)\right)^{n+1} = \begin{cases} \begin{array}{c}\text{[diagram]}\end{array} & \text{if } n = 1, \\[2em] \begin{array}{c}\text{[diagram]}\end{array} & \text{if } n \geq 2. \end{cases} \tag{4.2}$$

In the statements of the theorem, we use the diagonal actions together with the (co)adjoint (co)actions defined in Sections 2.13 and 2.14. Also, an integer $k$ below an arc denotes the $k$-th tensorand of $H^{\otimes n}$. We prove Theorem 2 in Section 6 by induction and by using properties of modular pairs and twisted antipodes.

In the next corollary, we compute the $(n+1)$-th power of the paracocyclic operator $\tau_n(\delta, \sigma)$ in terms of the $(n+1)$-th power of the paracocyclic operator $\tau_n(\varepsilon, u)$, where $\varepsilon$ and $u$ are the counit and the unit of $H$.

**Corollary 3.** *For any modular pair $(\delta, \sigma)$ and any $n \in \mathbb{N}$,*



*Proof.* We show the result by induction. For $n = 0$, this follows since the adjoint action on $\mathbb{1}$ is given by counit, since the coadjoint coaction on $\mathbb{1}$ is given by unit, and the fact that $\tau_0(\varepsilon, u) = \tau_0(\delta, \sigma) = \mathrm{id}_{\mathbb{1}}$. Let us check the case $n = 1$. By using formula (4.2) of Theorem 2 and the fact that $\tau_1(\varepsilon, u) = S$, we obtain



Suppose that the result is true for an $n \geq 1$ and let us show it for $n + 1$. Indeed, we have

Here (i) follows by applying formula (4.2) for the modular pair $(\delta, \sigma)$, (ii) follows by applying the result for $n = 1$, and (iii) follows by applying formula (4.2) for the modular pair $(\varepsilon, u)$. ∎

The next corollary states that the paracocyclic operator associated with a twisted modular pair in involution satisfies the "twisted cocyclicity condition".

**Corollary 4.** *If $\mathcal{B}$ has a twist $\theta$ and $(\delta, \sigma)$ is a $\theta$-twisted modular pair in involution for $H$, then $(\tau_n(\delta, \sigma))^{n+1} = \theta_{H^{\otimes n}}$ for all $n \in \mathbb{N}$.*

*Proof.* The equality $(\tau_n(\delta, \sigma))^{n+1} = \theta_{H^{\otimes n}}$ is shown by induction. For $n = 0$, this follows by definition and the fact that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. Indeed, $\tau_0(\delta, \sigma) = \mathrm{id}_{\mathbb{1}} = \theta_{\mathbb{1}} = \theta_{H^{\otimes 0}}$. For $n = 1$, we have
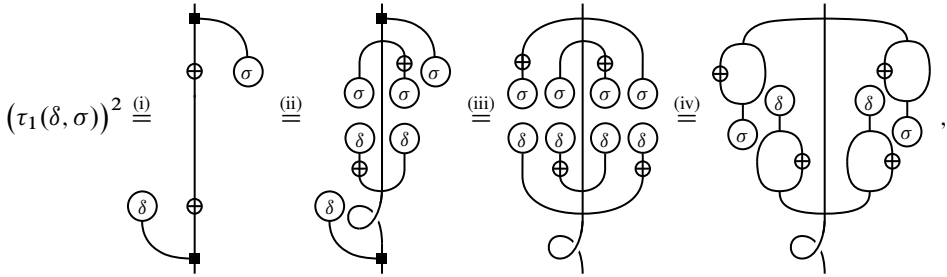


which is $\theta_H$ by the antipode axiom and the (co)unitality. Here (i) follows by formula (4.2) of Theorem 2, (ii) follows by the fact that $(\delta, \sigma)$ is a $\theta$-twisted modular pair in involution for $H$, (iii) follows by the naturality of the twist and the definitions of left coadjoint coaction and right adjoint action, (iv) follows by the (co)associativity and the fact that $\delta$ is an algebra morphism and $\sigma$ is a coalgebra morphism.

Suppose that the statement is true for $n \geq 1$ and let us show it for $n + 1$. We have
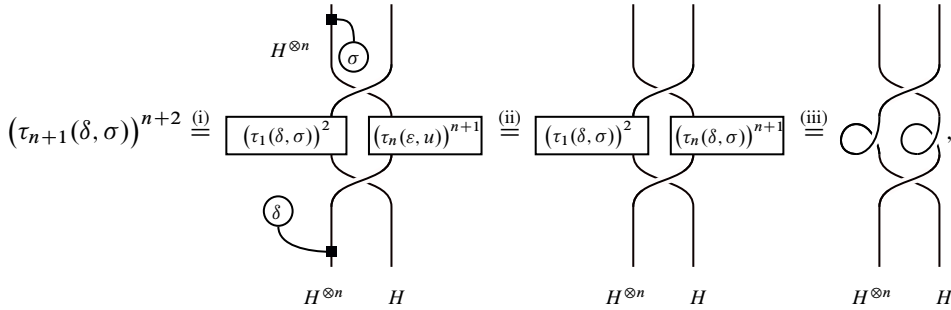


which is equal to $\theta_{H^{\otimes n+1}}$ by the naturality of the braiding and from the axiom of the twist. Here (i) follows from formula (4.2) of Theorem 2, (ii) follows from Corollary 3, (iii) follows from the statement for $n = 1$ and the induction hypothesis. ∎

**Remark 5.** If $\mathcal{B}$ is a symmetric monoidal category endowed with the trivial twist $\mathrm{id}_{\mathcal{B}}$ and $(\delta, \sigma)$ is an $\mathrm{id}_{\mathcal{B}}$-twisted modular pair for $H$, then Corollary 4 gives that $(\tau_n(\delta, \sigma))^{n+1} = \mathrm{id}_{H^{\otimes n}}$ for all $n \in \mathbb{N}$. This was first proved by Khalkhali and Pourkia in [8].

### 4.3. Paracocyclic objects associated with modular pairs

Let $(\delta, \sigma)$ be a modular pair for $H$. Let us recall the paracocyclic object $\mathbf{CM}_\bullet(H, \delta, \sigma)$ in $\mathcal{B}$ from [8] associated to this data. For any $n \geq 0$, define

$$\mathbf{CM}_n(H, \delta, \sigma) = H^{\otimes n}.$$

For any $n \geq 1$, define the cofaces $\{\delta_i^n(\sigma): H^{\otimes n-1} \to H^{\otimes n}\}_{0 \leq i \leq n}$ by setting $\delta_0^1 = u$, $\delta_1^1 = \sigma$, and for any $n \geq 2$,

$$
\delta_i^n(\sigma) = \begin{cases}
\quad\quad\quad\quad & \text{if } i = 0, \\
\\
\quad\quad\quad\quad & \text{if } 1 \leq i \leq n - 1, \\
\\
\quad\quad\quad\quad & \text{if } i = n.
\end{cases}
$$



For any $n \geq 0$, define the codegeneracies $\{\sigma_j^n: H^{\otimes n+1} \to H^{\otimes n}\}_{0 \leq j \leq n}$ by



For any $n \geq 0$, the paracocyclic operators $\tau_n(\delta, \sigma): H^{\otimes n} \to H^{\otimes n}$ of $\mathbf{CM}_\bullet(H, \delta, \sigma)$ are those defined in Section 4.2.

Theorem 2 is useful to prove that $\mathbf{CM}_\bullet(H, \delta, \sigma)$ is a paracocyclic object in $\mathcal{B}$. We prove this in the appendix. In particular, we prove that for all $n \in \mathbb{N}$,

$$\tau_n(\delta, \sigma)\sigma_0^n = \sigma_n^n\big(\tau_{n+1}(\delta, \sigma)\big)^2.$$

The next corollary derives (co)cyclic $\Bbbk$-modules from $\mathbf{CM}_\bullet(H, \delta, \sigma)$. It follows directly from Lemma 1 and Corollary 4.

**Corollary 6.** *If $\mathcal{B}$ has a twist $\theta$ and if $(\delta, \sigma)$ is a $\theta$-twisted modular pair in involution for $H$, then*

(a)  $\{\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \mathbf{CM}_n(H, \delta, \sigma))\}_{n \in \mathbb{N}}$ *is a cocyclic $\Bbbk$-module,*

(b)  $\{\mathrm{Hom}_{\mathcal{B}}(\mathbf{CM}_n(H, \delta, \sigma), \mathbb{1})\}_{n \in \mathbb{N}}$ *is a cyclic $\Bbbk$-module.*

## 5. Categorical Connes–Moscovici trace

In this section, $\mathcal{B}$ is a braided category with a twist $\theta$ and $H$ is a Hopf algebra in $\mathcal{B}$. We introduce traces (à la Connes–Moscovici) between paracocyclic objects associated with $H$ (as in Section 4.3) and paracocyclic objects associated with an $H$-module coalgebra. We provide an explicit example of such traces using coends.

### 5.1. Paracocyclic objects associated with coalgebras

Let $C$ be a coalgebra in $\mathcal{B}$. We associate with $C$ a paracocyclic object $\mathbf{C}_\bullet(C)$ in $\mathcal{B}$. It is inspired by the construction of Akrami and Majid from [1]. When $\mathcal{B} = \mathbf{Mod}_\Bbbk$ is the category of $\Bbbk$-modules, one recovers the cocyclic $\Bbbk$-module implicitly defined in the work of Farinati and Solotar [7]. When $\mathcal{B}$ is a symmetric monoidal category endowed with the trivial twist, then the underlying cosimplicial object of $\mathbf{C}_\bullet(C)$ is equal to the one considered in [2, Definition 2.2].

For $n \geq 0$, set $\mathbf{C}_n(C) = C^{\otimes n+1}$. For $n \geq 1$, define the cofaces $\{\delta_i^n : C^{\otimes n} \to C^{\otimes n+1}\}_{0 \leq i \leq n}$ by

$$
\delta_i^n = \begin{cases} \quad \text{if } 0 \leq i \leq n-1, \\[2em] \quad \text{if } i = n. \end{cases}
$$

For $n \geq 0$, define the codegeneracies $\{\sigma_j^n : C^{\otimes n+2} \to C^{\otimes n+1}\}_{0 \leq j \leq n}$ by

$$
\sigma_j^n = \left| \quad \cdots \quad \right| .
$$

For $n \geq 0$, define the paracocyclic operators $\tau_n : C^{\otimes n+1} \to C^{\otimes n+1}$ by

$$
\tau_0 = \quad \text{and} \quad \tau_n = \quad \text{if } n \geq 1.
$$

It follows directly from the definition of a twist (see Section 2.3) and the naturality of the braiding that the paracocyclic operator for $\mathbf{C}_\bullet(C)$ satisfies the relation $\tau_n^{n+1} = \theta_{C^{\otimes n+1}}$ for all $n \in \mathbb{N}$.

### 5.2. Traces

Let $C$ be an *H-module coalgebra* in $\mathcal{B}$, that is, a coalgebra in the category of right $H$-modules in $\mathcal{B}$. In other words, $C$ is a coalgebra in $\mathcal{B}$ endowed with a right action $r : C \otimes H \to C$ of $H$ on $C$ such that the comultiplication $\Delta_C$ and the counit $\varepsilon_C$ of $C$ are both $H$-linear,

that is, morphisms of right $H$-modules. By depicting the right action by

$$r = \qquad,$$

the $H$-linearity of $\Delta_C$ and $\varepsilon_C$ depicts as

$$\qquad = \qquad \text{and} \qquad = \qquad.$$

In these pictures, the red strands are colored by $C$ and the black ones by $H$.

Let $\delta\colon H \to \mathbb{1}$ be an algebra morphism and let $\sigma\colon \mathbb{1} \to H$ be a coalgebra morphism. A $\delta$-invariant $\sigma$-trace for $C$ is a morphism $\alpha\colon \mathbb{1} \to C$ in $\mathcal{B}$ satisfying

$$\qquad = \qquad \text{and} \qquad = \qquad.$$

Given such a morphism, define for any $n \in \mathbb{N}$ the morphism $\alpha_n\colon H^{\otimes n} \to C^{\otimes n+1}$ in $\mathcal{B}$ by setting

$$\alpha_0 = \alpha \quad \text{and} \quad \alpha_n = \qquad \text{for } n \geq 1.$$

Consider the paracocyclic object $\mathbf{CM}_\bullet(H, \delta, \sigma)$ in $\mathcal{B}$ (see Section 4.3) and the paracocyclic object $\mathbf{C}_\bullet(C)$ in $\mathcal{B}$ associated to the coalgebra $C$ in $\mathcal{B}$ (see Section 5.1).

**Theorem 7.** *Let $(\delta, \sigma)$ be a modular pair for $H$ and let $\alpha$ be a $\delta$-invariant $\sigma$-trace for $C$. Then the family $\{\alpha_n\colon H^{\otimes n} \to C^{\otimes n+1}\}_{n \in \mathbb{N}}$ is a morphism of paracocyclic objects from $\mathbf{CM}_\bullet(H, \delta, \sigma)$ to $\mathbf{C}_\bullet(C)$.*

We prove Theorem 7 in Section 7. The next corollary relates the cyclic (co)homologies associated with $\mathbf{CM}_\bullet(H, \delta, \sigma)$ and $\mathbf{C}_\bullet(C)$.

**Corollary 8.** *Assume that $\mathcal{B}$ is $\Bbbk$-linear and has a twist $\theta$. Let $(\delta, \sigma)$ be a $\theta$-twisted modular pair in involution for $H$ and let $\alpha$ be a $\delta$-invariant $\sigma$-invariant trace for $C$. Then*

(a)  *the family $\{\mathrm{Hom}_\mathcal{B}(\mathbb{1}, \alpha_n)\}_{n \in \mathbb{N}}$ induces a morphism in cyclic cohomology*

$$\alpha^*\colon HC^*\big(\mathrm{Hom}_\mathcal{B}\big(\mathbb{1}, \mathbf{CM}_\bullet(H, \delta, \sigma)\big)\big) \to HC^*\big(\mathrm{Hom}_\mathcal{B}\big(\mathbb{1}, \mathbf{C}_\bullet(C)\big)\big);$$

(b)  *the family $\{\mathrm{Hom}_\mathcal{B}(\alpha_n, \mathbb{1})\}_{n \in \mathbb{N}}$ induces a morphism in cyclic homology*

$$\alpha_*\colon HC_*\big(\mathrm{Hom}_\mathcal{B}\big(\mathbf{C}_\bullet(C), \mathbb{1}\big)\big) \to HC_*\big(\mathrm{Hom}_\mathcal{B}\big(\mathbf{CM}_\bullet(H, \delta, \sigma), \mathbb{1}\big)\big).$$

*Proof.* Since $(\mathbf{C}_\bullet(C)(\tau_n))^{n+1} = \theta_{C^{\otimes n+1}}$, Lemma 1 implies that $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \mathbf{C}_\bullet(C))$ is a cocyclic $\Bbbk$-module and that $\mathrm{Hom}_{\mathcal{B}}(\mathbf{C}_\bullet(C), \mathbb{1})$ is a cyclic $\Bbbk$-module. Next, by an application of Theorem 7, the family $\{\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \alpha_n)\}_{n \in \mathbb{N}}$ is a natural transformation between cocyclic $\Bbbk$-modules $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \mathbf{CM}_\bullet(H, \delta, \sigma))$ and $\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \mathbf{C}_\bullet(C))$. This means that for any $n \in \mathbb{N}$, there is a morphism

$$\alpha^n \colon HC^n\big(\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \mathbf{CM}_\bullet(H, \delta, \sigma))\big) \to HC^n\big(\mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, \mathbf{C}_\bullet(C))\big)$$
$$[f] \mapsto [\alpha_n f],$$

where $[f]$ denotes a representative class of an $n$-th cyclic cocycle. This finishes the proof of part (a). The proof of (b) is similar. ∎

### 5.3. Traces from coends

Let $\mathcal{B}$ be a ribbon category with a coend $(H, i)$; see Section 2.18. By [12, Chapter 6], the object $H$ is a Hopf algebra in $\mathcal{B}$ which is involutive (that is $S^2 = \theta_H$, where $\theta$ is the twist of $\mathcal{B}$; see Section 2.16). By Section 4.1, since $H$ is involutive, the pair $(\varepsilon, u)$ is a $\theta$-twisted modular pair in involution for $H$. Since $\mathbf{Mod}_H$ is braided isomorphic to the center of $\mathcal{B}$ (see [12, Section 6.5.3] for details), we obtain that $\mathbf{Mod}_H$ is ribbon. The braiding of $\mathbf{Mod}_H$ is given by



Here, the coaction denoted by a black dot is the universal coaction of $H$ (see Section 2.18). The dual of $(M, r) \in \mathrm{Ob}(\mathbf{Mod}_H)$ is given by $(M^*, r^\dagger)$, where



together with the (co)evaluation morphisms inherited from $\mathcal{B}$:

$$\mathrm{ev}_{(M,r)} = \mathrm{ev}_M, \quad \widetilde{\mathrm{ev}}_{(M,r)} = \widetilde{\mathrm{ev}}_M, \quad \mathrm{coev}_{(M,r)} = \mathrm{coev}_M, \quad \widetilde{\mathrm{coev}}_{(M,r)} = \widetilde{\mathrm{coev}}_M.$$

Note that the last equality in the definition of $r^\dagger$ follows from the involutivity of $H$.

The category $\mathbf{Mod}_H$ has a coend $((C, a), j)$, where $C = H^* \otimes H$ and the action $a \colon C \otimes H \to C$ of $H$ on $C$ is computed by

The universal dinatural transformation $j = \{j_{(M,r)} : (M,r)^* \otimes (M,r) \to (C,a)\}_{(M,r) \in \mathbf{Mod}_H}$ is given by
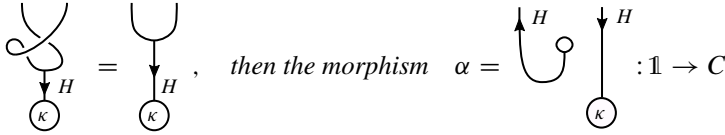
$$j_{(M,r)} = $$
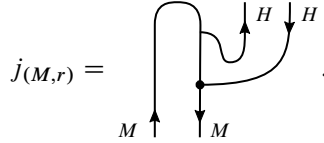
By Section 2.18, the coend $C$ is a Hopf algebra in $\mathbf{Mod}_H$. In particular, it is a coalgebra in $\mathbf{Mod}_H$. The comultiplication $\Delta_C : C \to C \otimes C$ and the counit $\varepsilon_C : C \to \mathbb{1}$ of $C$ are computed by
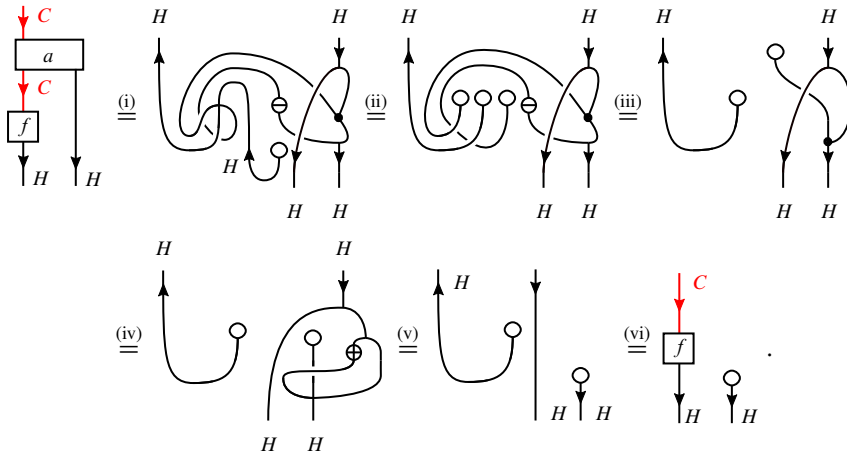
$$\Delta_C = \qquad \text{and} \quad \varepsilon_C = \,.$$
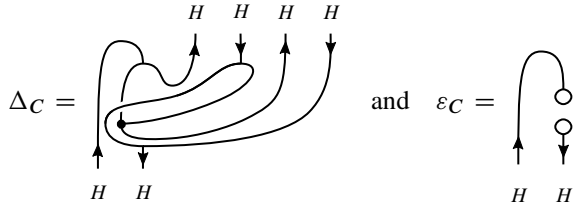
The following lemma gives a way to produce a $\varepsilon$-invariant $u$-trace $\mathbb{1} \to C$, where $\varepsilon$ and $u$ denote the counit and unit of $H$.

**Lemma 9.** *If a morphism $\kappa : \mathbb{1} \to H$ in $\mathcal{B}$ satisfies*

$$\qquad = \qquad , \quad \text{then the morphism} \quad \alpha = \qquad : \mathbb{1} \to C$$

*is a $\varepsilon$-invariant $u$-trace.*

*Proof.* Denote $f = \varepsilon^* \otimes \mathrm{id}_H : H \to C$. Let us first check that $f$ is a morphism between right $H$-modules $(H, \varepsilon)$ and $(C, a)$. Indeed,

$$\overset{(i)}{=} \quad \overset{(ii)}{=} \quad \overset{(iii)}{=} \quad \overset{(iv)}{=} \quad \overset{(v)}{=} \quad \overset{(vi)}{=} \quad .$$

Here (i) follows by definition of $f$ and $a$; (ii) follows from the isotopy of the graphical calculus for pivotal categories and the multiplicativity of the counit; (iii) follows by the naturality of the braiding, the fact that $\varepsilon S = \varepsilon$, and the counitality; (iv) holds since universal coaction of $H$ on itself is the right coadjoint coaction; (v) follows by the naturality of the braiding, the (co)unitality, and the antipode axiom; (vi) follows by definition of $f$.

Next, $f \colon H \to C$ is a coalgebra morphism. Indeed, we have
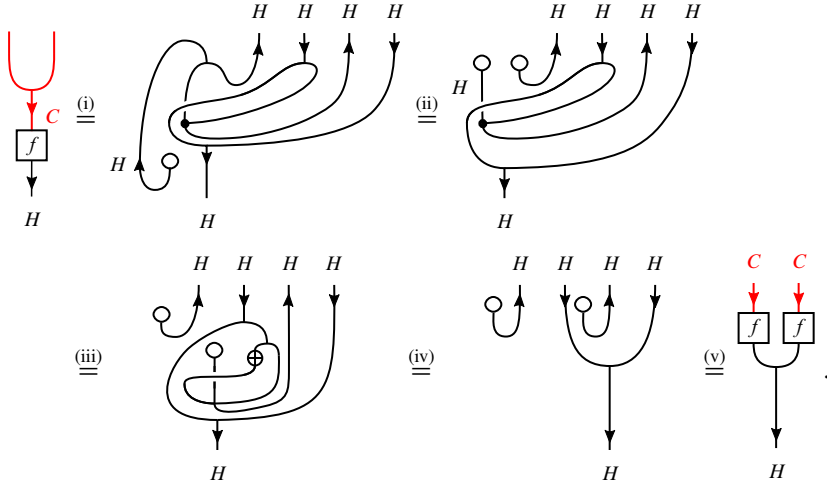


Here (i) and (v) follow by definition of $f$, (ii) follows from the isotopy of the graphical calculus for pivotal categories and the multiplicativity of the counit, (iii) follows by the naturality of the braiding and the fact that universal coaction of $H$ on itself is the right coadjoint coaction, and (iv) follows by the naturality of the braiding, the (co)unitality, and the antipode axiom. Also,
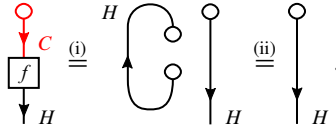


Here (i) follows from definitions of $f$ and $\varepsilon_C$ and (ii) follows from the fact that $\varepsilon u = \mathrm{id}_{\mathbb{1}}$.

Finally, let us show that $\alpha$ is a $\varepsilon$-invariant $u$-trace. By definition of $\alpha$ and the fact that

$$f \colon H \to C$$

is an $H$-module morphism, we have that
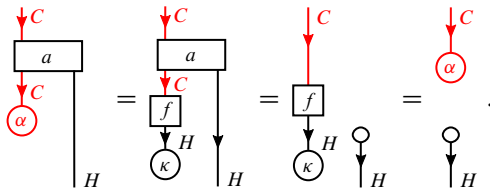
Thus the morphism $\alpha$ is $\varepsilon$-invariant. It remains to show that $\alpha = f\kappa$ is a $u$-trace. Indeed,
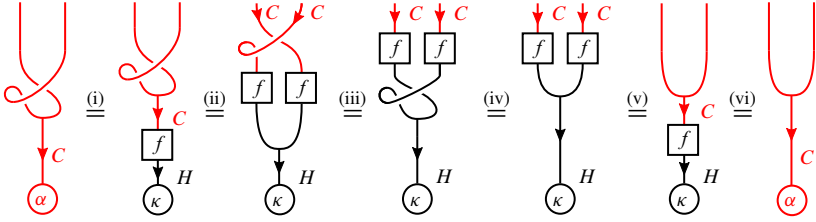


Here (i) and (vi) follow from definition, (ii) and (v) follow by the fact that $f : H \to C$ is a coalgebra morphism, (iii) follows by the naturality of twist and the braiding, and (iv) follows by hypothesis on $\kappa$. $\blacksquare$

Any coalgebra morphism $\mathbb{1} \to H$ satisfies the condition of Lemma 9. Another family of examples satisfying the condition of Lemma 9 is given as follows: for any $X \in \mathrm{Ob}(\mathcal{B})$, set

$$\kappa^X = \quad = i_X \widetilde{\mathrm{coev}}_X : \mathbb{1} \to H.$$
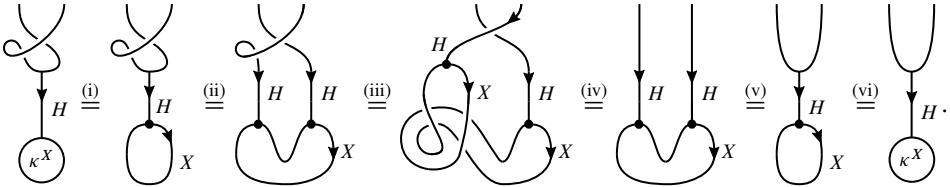
Then,



Here, (i) and (vi) follow by definition of $\kappa^X$, (ii) and (v) follow by definition of comultiplication of $H$, (iii) follows by the naturality of twists, and (iv) follows by the naturality of the braiding and isotopy invariance of graphical calculus.

Thus $\kappa^X$ satisfies the condition of Lemma 9 and so

$$\alpha^X = \quad : \mathbb{1} \to C \tag{5.1}$$

is a $\varepsilon$-invariant $u$-trace for $C$.

**Remark 10.** Note that if $\mathcal{B}$ is $\mathbb{k}$-linear, then any linear combination of $\delta$-invariant $\sigma$-traces is a $\delta$-invariant $\sigma$-trace. In particular, an interesting example of a $\varepsilon$-invariant $u$-trace comes

from topological field theory: if $\mathcal{B}$ is a ribbon fusion $\Bbbk$-linear category and $I$ is a representative set of simple objects of $\mathcal{B}$, then $\alpha = \sum_{k \in I} \dim(k) \alpha^k$ is a $\varepsilon$-invariant $u$-trace. Here $\alpha^k$ is defined in (5.1) and $\dim(k) = \widetilde{\mathrm{ev}}_k \mathrm{coev}_k = \mathrm{ev}_k \widetilde{\mathrm{coev}}_k$ is the dimension of $k$.

## 6. Proof of Theorem 2

Our strategy to compute the $(n + 1)$-th power of the paracocyclic operator $\tau_n(\delta, \sigma)$ is similar to the proof of cocyclicity condition from Connes and Moscovici in [6], where Hopf algebras over $\mathbb{C}$ are considered. We indeed proceed by induction. The difficulty here is that the paracocyclic operators involve the braiding. In our approach, based on graphical calculus, we manage to keep track the powers of paracocyclic operators. In Section 6.1, we list algebraic properties used in our proof of the formulas from Theorem 2. In Section 6.2, we show formula (4.1). In Section 6.3, we show formula (4.2).

Recall that $H$ denotes a Hopf algebra in the braided monoidal category $\mathcal{B}$, $\delta \colon H \to \mathbb{1}$ is an algebra morphism, and $\sigma \colon \mathbb{1} \to H$ is a coalgebra morphism such that $\delta\sigma = \mathrm{id}_{\mathbb{1}}$. Given such a pair, we define the *twisted antipode* $\widetilde{S} \colon H \to H$ by

$$\widetilde{S} = \quad \vcenter{\hbox{}} .$$

For brevity, we denote the twisted antipode $\widetilde{S}$ graphically by

$$\vcenter{\hbox{}} .$$

With this notation, we will rewrite

$$\tau_n(\delta, \sigma) = \vcenter{\hbox{}} \quad \text{if} \quad n \geq 1.$$

Similarly, formula (4.1), which is to be proven, rewrites as

$$\left( \tau_n(\delta, \sigma) \right)^k = \vcenter{\hbox{}} .$$

### 6.1. Preliminary facts

In this section, we state several lemmas, which are used in the proof of Theorem 2. We mention that equalities (a) and (b) from Lemma 11 and the equality from Remark 17 are already stated in [8, Proposition 4.3]. In the lemma that follows, some properties of the twisted antipode are established.

**Lemma 11.** *The following equalities hold:*



(a)    (b)    (c)



(d)    (e)    (f)
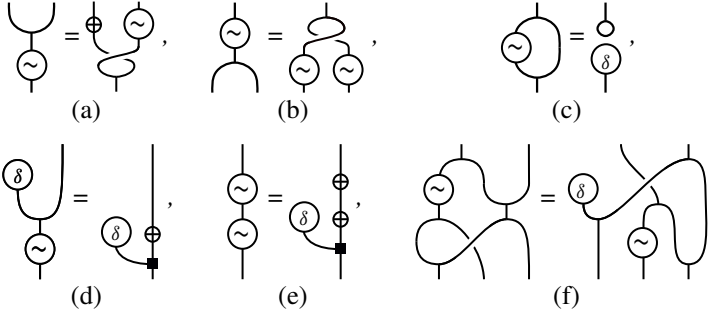
*Proof.* Let us first show equality (a). Indeed, by definition of $\widetilde{S}$, the anti-comultiplicativity of the antipode, the coassociativity, and the naturality of the braiding, we have
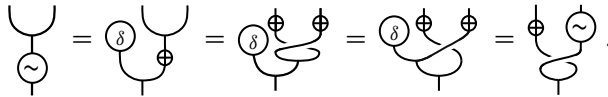


Let us show equality (b). Indeed, by definition of $\widetilde{S}$, the fact that the comultiplication is an algebra morphism, the fact that $\delta$ is an algebra morphism, and the naturality of the braiding we have
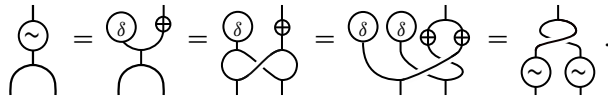


Let us show equality (c). Indeed, this relation follows by the definition of $\widetilde{S}$, the coassociativity, the antipode axiom, and the counitality:
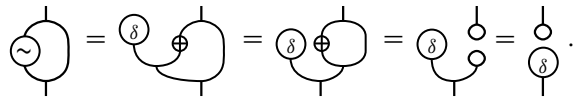


Now we show equality (d). It follows by part (a), the naturality of the braiding, the definition of $\widetilde{S}$, the fact that $\delta$ is an algebra morphism, and the definition of left coadjoint coaction:
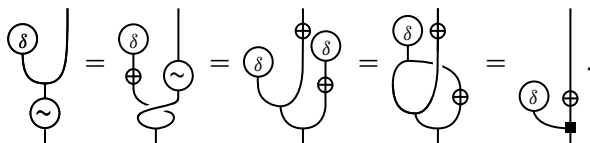
Equality (e) is a consequence of equality (d). To see this, compose the left-hand side of (d) with the antipode $S$ of $H$ and use the definition of $\widetilde{S}$.

Finally, let us show equality (f). Indeed, this equation follows by part (b), the fact that the comultiplication is an algebra morphism, the (co)associativity, the naturality of the braiding, part (c), and the unitality:
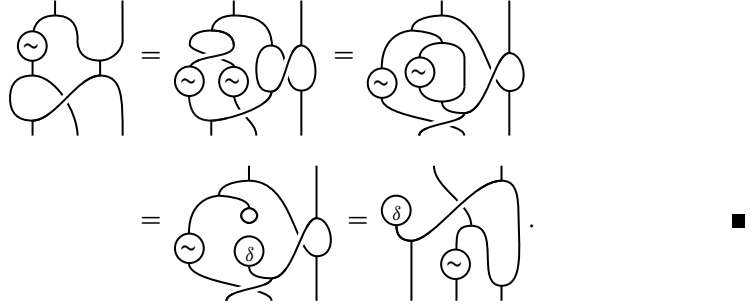


**Remark 12.** Another useful property of the twisted antipode $\widetilde{S}$ is that $\varepsilon\widetilde{S} = \delta$. It follows by the definition of $\widetilde{S}$, the fact that $\varepsilon S = \varepsilon$, and the counitality.

The following lemma gives the expression of the paracocyclic operator $\tau_n(\delta, \sigma)$ in terms of $\tau_{n-1}(\varepsilon, u)$.

**Lemma 13.** If $n \geq 2$, then

(a)
$$\tau_n(\delta, \sigma) = $$


(b)
$$\tau_n(\delta, \sigma) = $$
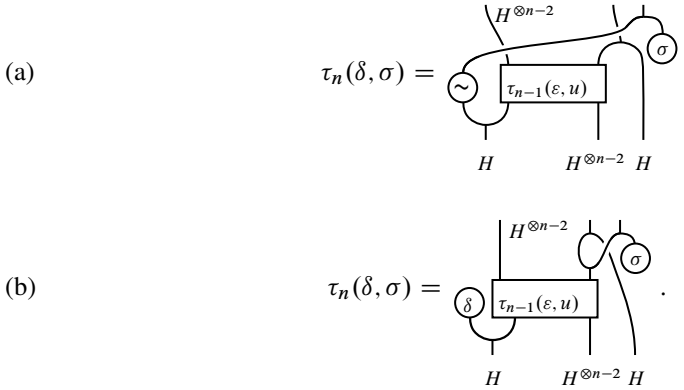

*Proof.* Let us first show equality (a). Indeed, by definition of $\tau_n(\delta, \sigma)$, Lemma 11 (a), the naturality of the braiding, inductive definition of the left diagonal action, and the definition of $\tau_{n-1}(\varepsilon, u)$, we have
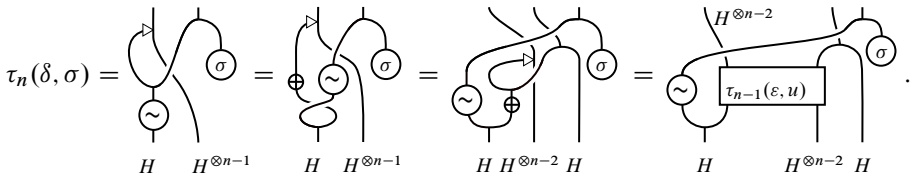
Further, we show equality (b). For $n = 2$, the statement follows by definition. From now on, suppose that $n \geq 3$. By the definition of $\tau_n(\delta, \sigma)$, the definition of the left diagonal action, the coassociativity, and the definition of $\tau_{n-1}(\varepsilon, u)$, we have



The equalities stated in the following lemma are used in computation of squares of the paracocyclic operator $\tau_n(\delta, \sigma)$ in the case $n \geq 3$.

**Lemma 14.** *For any $n \geq 2$, we have*



*Proof.* We begin by showing equality (a). Let us first inspect the case $n = 2$. To see that the equality is true in this case, we use the definition of $\tau_2(\varepsilon, u)$, the fact that the comultiplication is an algebra morphism, the coassociativity, the anti-comultiplicativity of the antipode, and the naturality of the braiding:

From now on, suppose that $n \geq 3$. By definition of $\tau_n(\varepsilon, u)$, the fact that the comultiplication is an algebra morphism, the naturality of the braiding, the coassociativity, and the anti-comultiplicativity of the antipode we have



Let us show equality (b). Indeed, it follows by definition of $\tau_n(\varepsilon, u)$, the naturality of the braiding and the associativity:



Let us show equality (c). It follows from Lemma 13 (b) applied on $\delta = \varepsilon$ and $\sigma = u$ and by the fact that the comultiplication is an algebra morphism:



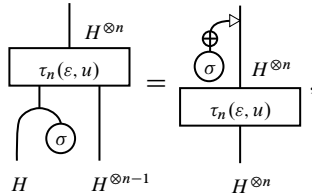Finally, let us show equality (d). Indeed, we have
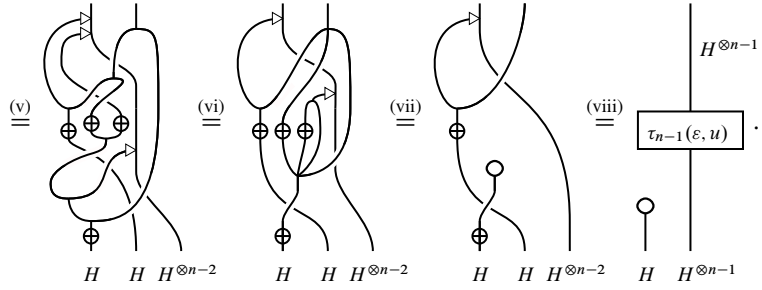
Here (i) follows by definition of $\tau_n(\varepsilon, u)$ and $\tau_{n-1}(\varepsilon, u)$, (ii) follows by inductive definition of left diagonal action, (iii) follows by the anti-multiplicativity of the antipode, (iv) follows by the fact that the comultiplication is an algebra morphism, (v) follows by the axiom of a module and the anti-comultiplicativity of the antipode, (vi) follows by the naturality of the braiding, the (co)associativity, and by the axiom of a module, (vii) follows by applying the antipode axiom twice and by the axiom of a module, (viii) follows by the fact that $\varepsilon S = \varepsilon$, the naturality of the braiding, and definition of $\tau_{n-1}(\varepsilon, u)$. ∎
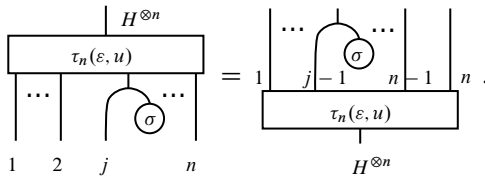
The equalities from the following lemma show how the endomorphism $m(\mathrm{id}_H \otimes \sigma)$ interacts with the paracocyclic operator $\tau_n(\varepsilon, u)$. These equalities are intensively used while proving formula (4.2) by using formula (4.1) from Theorem 2.
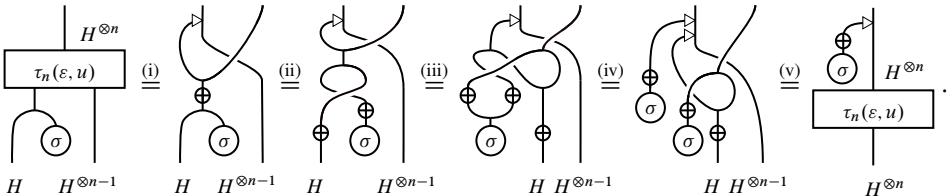
**Lemma 15.** *We have*

(a) *for any $n \geq 1$,*



(b) *for $2 \leq j \leq n$,*



*Proof.* Let us prove equality (a). Indeed, we have

Here (i) follows by definition of $\tau_n(\varepsilon, u)$, (ii) follows by the anti-multiplicativity of the antipode, (iii) follows by the naturality of the braiding, the fact that the comultiplication is an algebra morphism, and the anti-comultiplicativity of the antipode, (iv) follows by the fact that $\sigma$ is a coalgebra morphism and by the axiom of a module, and (v) follows by definition of $\tau_n(\varepsilon, u)$ and the naturality of the braiding.

Let us now show equality (b). Indeed, by definition of $\tau_n(\varepsilon, u)$ and the left diagonal action, by the naturality of the braiding, and the associativity, we have
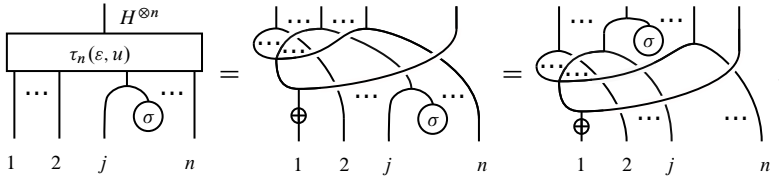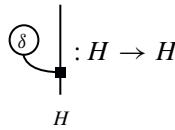


which is the right-hand side of (b) by definition. ∎

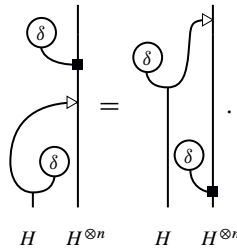Before proving Theorem 2, let us state another auxiliary lemma.

**Lemma 16.** *We have the following assertions.*
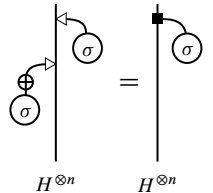
(a) *The morphism*



*is a bialgebra morphism.*

(b) *For all $n \geq 1$,*



(c) *For all $n \geq 1$,*



Equality (b) from Lemma 16 is intensively used while proving both of the equalities from Theorem 2. Equality (c) from Lemma 16 is particularly used in final steps of the computation of $\tau_n(\delta, \sigma)^{n+1}$.

*Proof.* Let us first show part (a). We first show that the morphism from part (a) is an algebra morphism. By using definition of the left coadjoint coaction, the fact that the comultiplication is an algebra morphism, the fact that $\delta$ is an algebra morphism, the anti-multiplicativity of the antipode of $H$, and the naturality of the braiding, we have
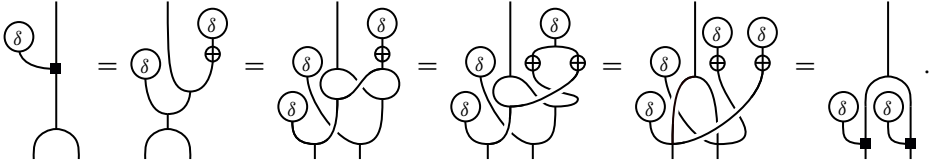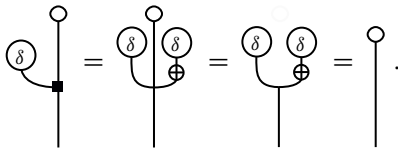
Similarly, by using definition of the left coadjoint coaction, the fact that unit is a coalgebra morphism, the fact that $\delta$ is an algebra morphism, and by the fact that $Su = u$, we have
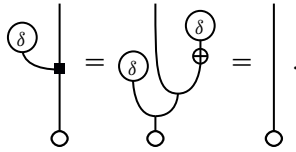
Let us now show that the morphism from (a) is a coalgebra morphism. Indeed, by definition of the left coadjoint coaction, the fact that $\delta$ is an algebra morphism, the naturality of the braiding, the coassociativity, the antipode axiom, and the (co)unitality, we have
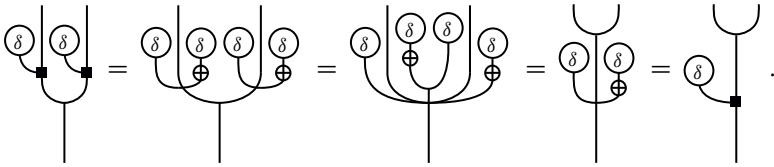
Furthermore, by definition of the left coadjoint coaction, the fact that $\delta$ is an algebra morphism, the naturality of the braiding, the (co)unitality, and the antipode axiom, we have

This completes the proof of part (a).

Let us show equality (b) by induction. For $n = 1$, we prove the statement as follows. By using part (a), the definition of the coadjoint coaction, the fact that $\delta$ is an algebra morphism, the naturality of the braiding, the coassociativity, the antipode axiom, and the (co)unitality, we have
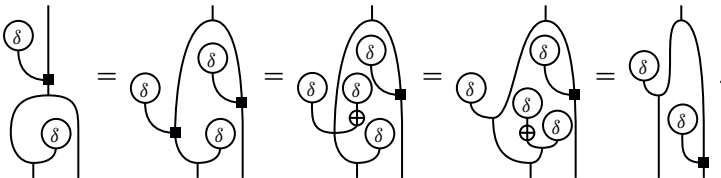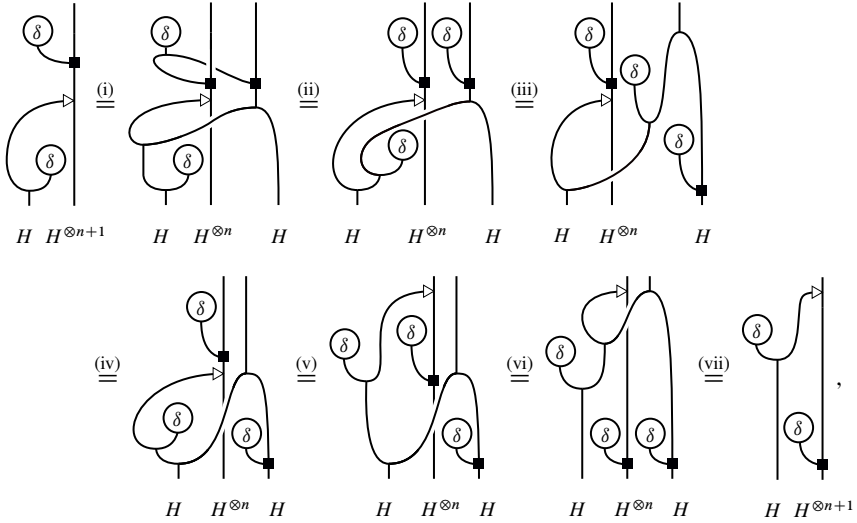
Suppose that the statement is true for an $n \geq 1$ and let us show it for $n + 1$. We have



which shows the desired statement. Here (i) and (vii) both follow by inductive definition of left diagonal action and left coadjoint coaction, (ii) follows by the fact that $\delta$ is an algebra morphism and by the coassociativity, (iii) follows by the naturality of the braiding and the case $n = 1$, (iv) and (vi) both follow by the naturality of the braiding and by the coassociativity and (v) follows by the induction hypothesis.

Finally, we show equality (c) by induction. For $n = 1$, the statement follows by definition of right adjoint action and the fact that $\sigma$ is a coalgebra morphism. Suppose that the statement is true for $n \geq 1$ and let us show it for $n + 1$. Indeed, by using inductive definition of the left and the right diagonal actions, the anti-multiplicativity of the antipode, the fact that $\sigma$ is a coalgebra morphism, the naturality of the braiding, and the induction hypothesis, we have
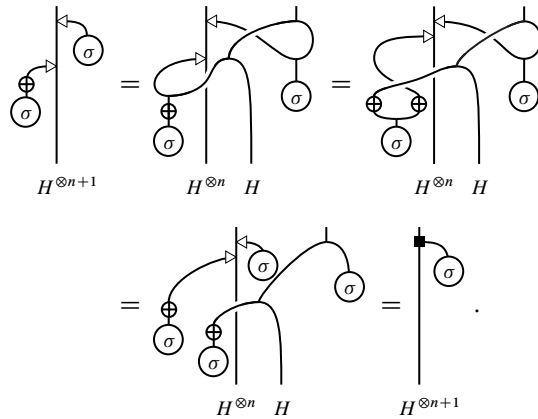
**Remark 17.** If $(\delta, \sigma)$ is a modular pair, then

$$\raisebox{-1em}{\includegraphics{}} = \raisebox{-1em}{\includegraphics{}}.$$

Indeed, this follows by the definition of $\tilde{S}$, the fact that $\sigma$ is a coalgebra morphism, and since the $(\delta, \sigma)$ is a modular pair:

$$\raisebox{-1em}{\includegraphics{}} = \raisebox{-1em}{\includegraphics{}} = \raisebox{-1em}{\includegraphics{}} = \raisebox{-1em}{\includegraphics{}}.$$

## 6.2. Proof of formula (4.1)

The proof of formula (4.1) of Theorem 2 is divided into several steps. For $n = k = 2$, it suffices to calculate the square of $\tau_2(\delta, \sigma)$. For $n \geq 3$, we first calculate the square and then derive formulas for the remaining powers.

**6.2.1. Squares of $\tau_n(\delta, \sigma)$ for $n \geq 2$.** Let us first show that formula (4.1) is true in the case $n = k = 2$. Indeed, we have

$$\left(\tau_2(\delta, \sigma)\right)^2 \overset{\text{(i)}}{=} \cdots \overset{\text{(ii)}}{=} \cdots$$
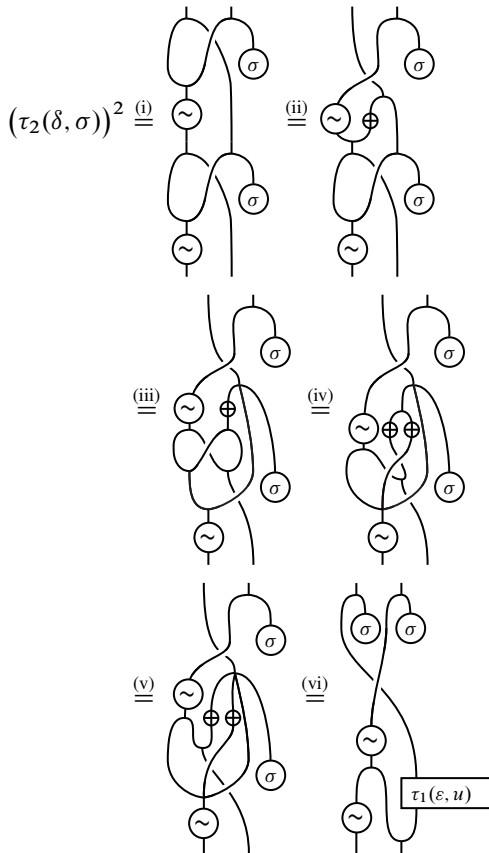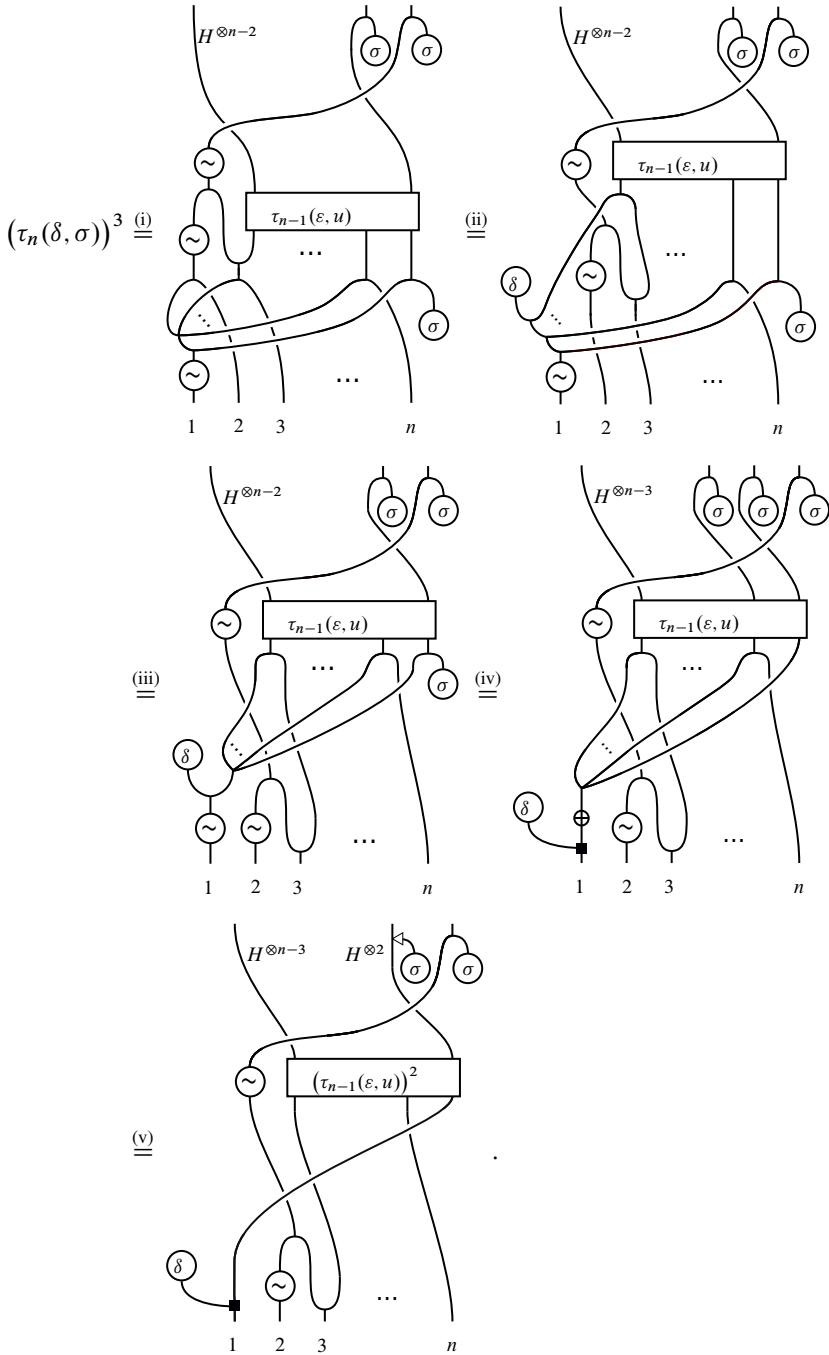
Here (i) follows by definition of $\tau_2(\delta, \sigma)$, (ii) follows from Lemma 13 (a) for $n = 2$ and since $\tau_1(\varepsilon, u) = S$, (iii) follows by the fact that the comultiplication is an algebra morphism and associativity, (iv) follows by the anti-multiplicativity of the antipode, (v) follows by the associativity and the naturality of the braiding, and (vi) follows by the antipode axiom, the naturality of the braiding, the (co)unitality, and the fact that $\tau_1(\varepsilon, u) = S$.

From now on, let us assume that $n \geq 3$. Let us calculate the square of $\tau_n(\delta, \sigma)$. We have
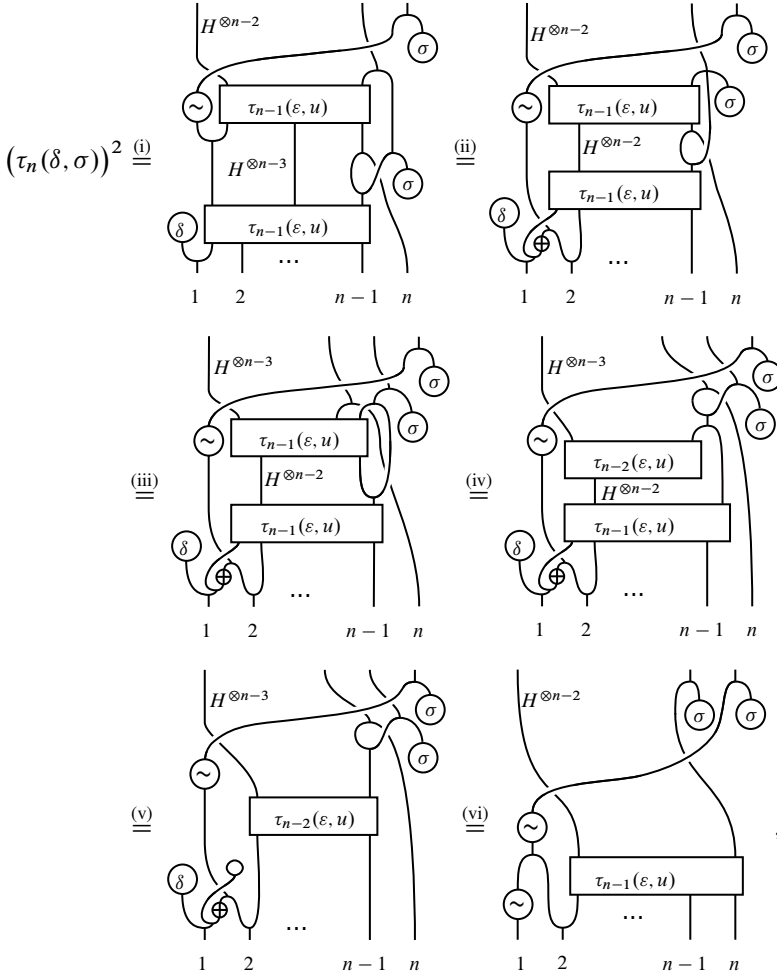


which indeed shows formula (4.1) in the case $n \geq 3$ and $k = 2$. Here (i) follows by parts (a) and (b) of Lemma 13, (ii) follows by the associativity and from Lemma 14 (a) for $n - 1$, (iii) follows by the associativity and from Lemma 14 (b) for $n - 1$, (iv) follows from Lemma 14 (c) for $n - 1$, (v) follows from Lemma 14 (d) for $n - 1$, and (vi) follows by the counitality, definition of the twisted antipode $\widetilde{S}$, and from Lemma 13 (b) applied on $\delta = \varepsilon$ and $\sigma = u$ for $n - 1$.

**6.2.2. Passing from $(\tau_n(\delta, \sigma))^2$ to $(\tau_n(\delta, \sigma))^3$, $n \geq 3$.** From the calculation that has been done in Section 6.2.1, we can easily deduce formula (4.1) for $n \geq 3$ and $k = 3$.

Here (i) follows by developing $\tau_n(\delta, \sigma)^3$ as $\tau_n(\delta, \sigma)^2 \tau_n(\delta, \sigma)$, by definition of $\tau_n(\delta, \sigma)$, and by the computation of $\tau_n(\delta, \sigma)^2$ that has been done in Section 6.2.1, (ii) follows by the coassociativity and Lemma 11 (f), (iii) follows by the naturality of the braiding and the coassociativity, (iv) follows from Lemma 15 (b) for $j = n - 1$ and from Lemma 11 (d), and (v) follows by the naturality of the braiding and the definition of $\tau_{n-1}(\varepsilon, u)$.

**6.2.3. Computation of $(\tau_n(\delta, \sigma))^{j+1}, 2 \leq j \leq n - 1$.** Note that by now, we have completely shown formula (4.1) in the cases $n = 2$ and $n = 3$. Also, the square and the cube of $\tau_n(\delta, \sigma)$ are calculated for each $n \geq 3$. In this section, we finish the proof of (4.1), by focusing on the case $n \geq 4$. As it has been already noted, for $j = 2$, $(\tau_n(\delta, \sigma))^{j+1}$ is already computed in Section 6.2.2. From now on, we assume that $3 \leq j \leq n - 1$. If (4.1) is established for $k = j$, then
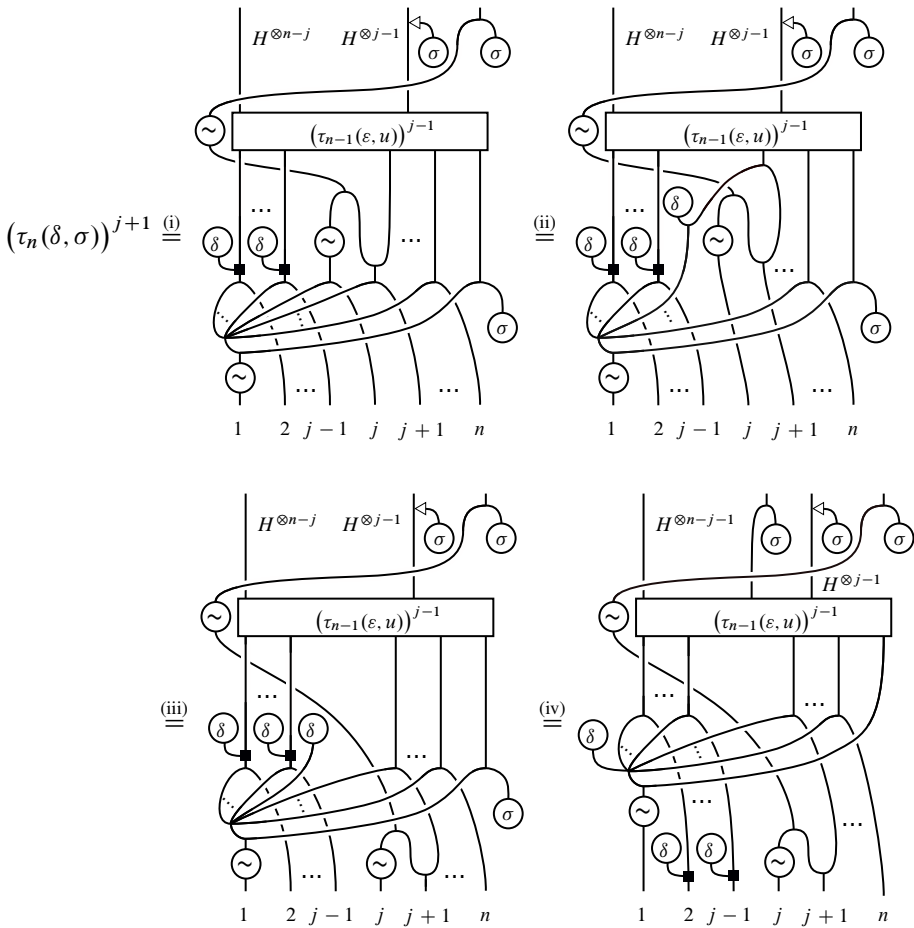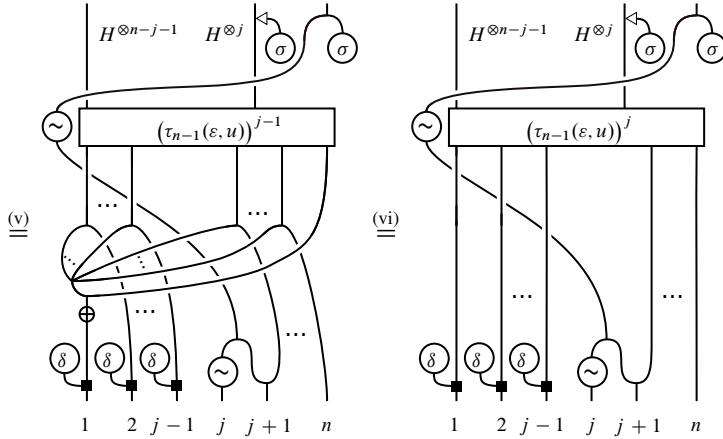
which finishes the proof of (4.1). Here (i) follows by decomposing $(\tau_n(\delta,\sigma))^{j+1}$ in the composition $(\tau_n(\delta,\sigma))^j \tau_n(\delta,\sigma)$, by definition of $\tau_n(\delta,\sigma)$, and by the hypothesis that (4.1) is established for $k = j$, (ii) follows by the coassociativity and Lemma 11 (f), (iii) follows by the naturality of the braiding and the coassociativity, (iv) follows from Lemma 16 (b) and by applying $j - 1$ times Lemma 15 (b), (v) follows from Lemma 11 (d) and the coassociativity, and finally (vi) follows by the naturality of the braiding and the definition of $\tau_{n-1}(\varepsilon, u)$.

### 6.3. Proof of formula (4.2)

In order to show formula (4.2) of Theorem 2, we will separately consider three cases: $n = 1$, $n = 2$, and $n \geq 3$.

   If $n = 1$, then we have
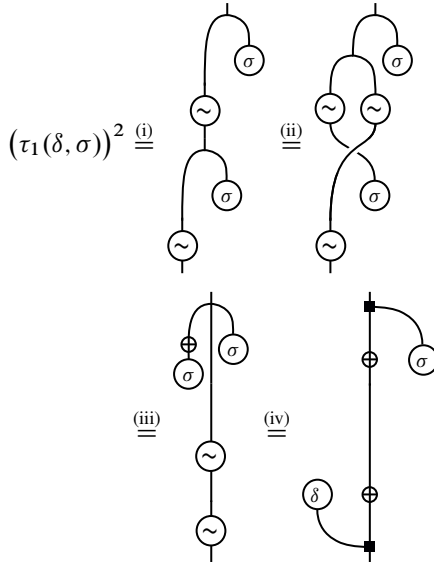
Here (i) follows by definition $\tau_1(\delta, \sigma)$, (ii) follows from Lemma 11 (b), (iii) follows by the naturality of the braiding, the associativity, and Remark 17, and (iv) follows by definition of right adjoint action, the fact that $\sigma$ is a coalgebra morphism, and Lemma 11 (e).

If $n = 2$, then we have
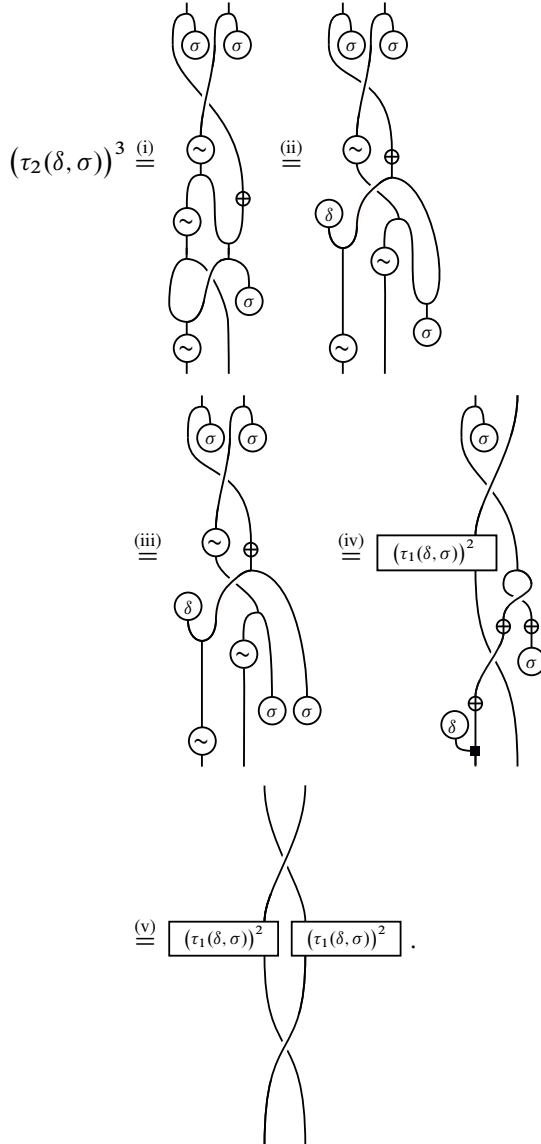


Here (i) follows by expanding $(\tau_2(\delta, \sigma))^3$ as $(\tau_2(\delta, \sigma))^2 \tau_2(\delta, \sigma)$, using the definition of $\tau_2(\delta, \sigma)$, and formula (4.1) in the case $n = k = 2$, (ii) follows from Lemma 11 (f), (iii) follows by using the fact that $\sigma$ is a coalgebra morphism, (iv) follows by the naturality of the braiding, definition of $\tau_1(\delta, \sigma)$, and the anti-multiplicativity of the antipode, and (v) fol-

lows by the naturality of the braiding, by definition of right adjoint action, and the case $n = 1$ of formula (4.2), which is shown above.

From now on, let us assume that $n \geq 3$. We have
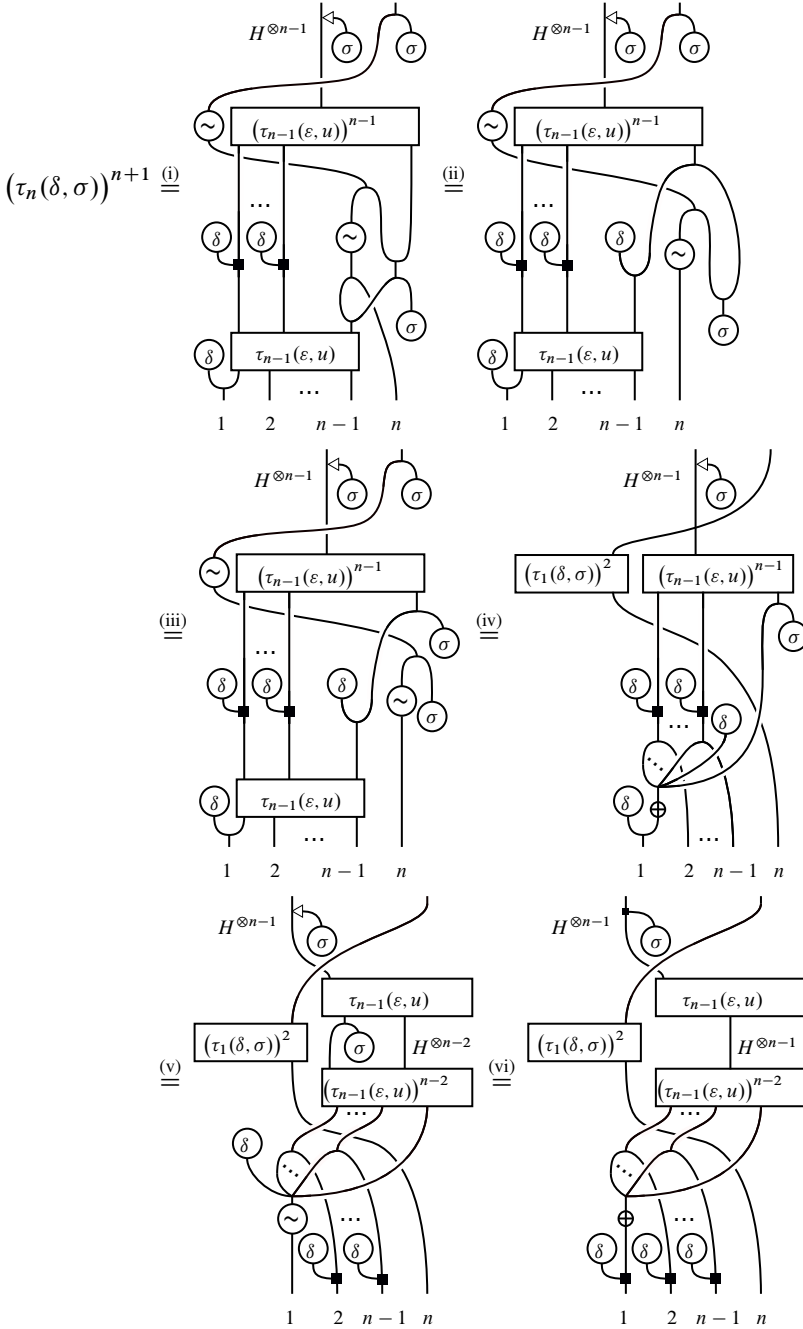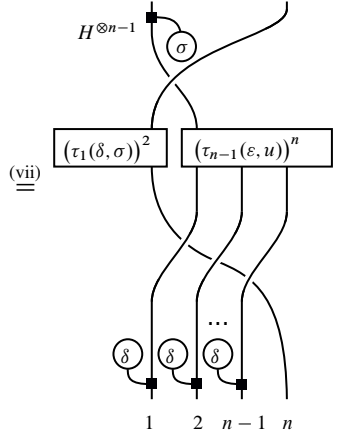
which shows formula (4.2). Here (i) follows by writing $(\tau_n(\delta,\sigma))^{n+1}$ as the composition $(\tau_n(\delta,\sigma))^n \tau_n(\delta,\sigma)$, by using Lemma 13 (b), and using formula (4.1) for $k=n$, (ii) follows from Lemma 11 (f), (iii) follows by the fact that $\sigma$ is a coalgebra morphism, (iv) follows by definition of $\tau_{n-1}(\varepsilon,u)$, the naturality of the braiding, definition of $\tau_1(\delta,\sigma)$, and the coassociativity, (v) follows by definition of the twisted antipode, Lemma 16 (b), and by applying $n-2$ times Lemma 15 (b), (vi) follows by the naturality of the braiding, by combining Lemma 15 (a) with Lemma 16 (c) for $n-1$, by the coassociativity, and Lemma 11 (d), and (vii) follows by the naturality of the braiding and definition of $\tau_{n-1}(\varepsilon,u)$.

## 7. Proof of Theorem 7

In order to show that the family $\{\alpha_n: H^{\otimes n} \to C^{\otimes n+1}\}_{n\in\mathbb{N}}$ is a morphism between the paracocyclic objects $\mathbf{CM_\bullet}(H,\delta,\sigma)$ and $\mathbf{C_\bullet}(C)$ in $\mathcal{B}$, we will directly check that

$$\alpha_n \delta_i^n(\sigma) = \delta_i^n \alpha_{n-1} \quad \text{for } 0 \le i \le n, n \ge 1, \tag{7.1}$$

$$\alpha_n \sigma_j^n = \sigma_j^n \alpha_{n+1} \quad \text{for } 0 \le j \le n, n \ge 0, \tag{7.2}$$

$$\alpha_n \tau_n(\delta,\sigma) = \tau_n \alpha_n \quad \text{for } n \ge 0. \tag{7.3}$$

Note that we abusively use the same notation for cofaces, codegeneracies, and paracocyclic operators of two different constructions. These should be understood from context. Roughly described, equalities (7.1) and (7.2) follow by the fact that $C$ is a coalgebra in the category of right $H$-modules. In order to show equality (7.3) in the case $n \ge 2$, we will need the following computation.
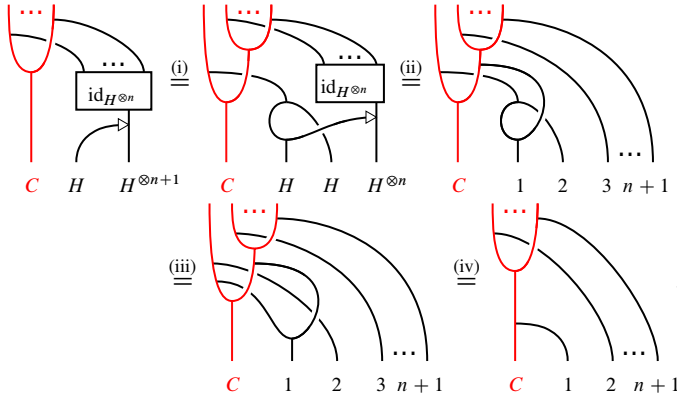
**Lemma 18.** *If $n \ge 2$, then*
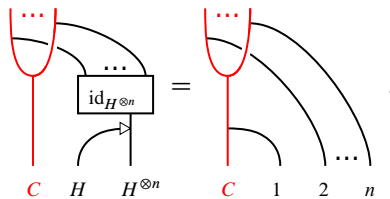
*Proof.* We prove the claim by induction. Let us first show it for $n = 2$. Indeed, by the right module axiom, the naturality of braiding, and the fact that the comultiplication $\Delta_C : C \to C \otimes C$ is an $H$-linear morphism, we have
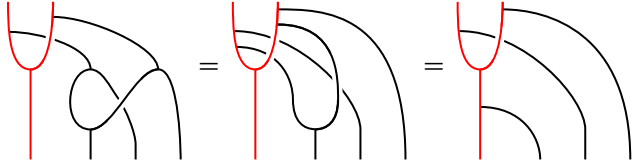


Suppose that the claim is true for $n \geq 2$ and let us show it for $n + 1$. We have



which indeed proves the claim for $n + 1$. Here (i) follows by the inductive definition of the left diagonal action, (ii) follows by the induction hypothesis, (iii) follows by the right module axiom, and (iv) follows by the naturality of the braiding and the fact that the comultiplication $\Delta_C : C \to C \otimes C$ is an $H$-linear morphism. ∎

## 7.1. Proof of equality (7.1)

Let us show equality (7.1). If $n = 1$ and $i = 0$, then this equality writes as $\alpha_1 \delta_0^1 = \delta_0^1 \alpha_0$, which follows by definitions and the right module axiom. Let $n = i = 1$. In this case, equality (7.1) writes as $\alpha_1 \delta_1^1 = \delta_1^1 \alpha_0$, which is exactly the condition that $\alpha : \mathbb{1} \to C$ is a $\sigma$-trace. This shows equality (7.1) for $n = 1$ and $0 \leq i \leq 1$.

Now let $n \geq 2$ and $i = 0$. By definitions, the naturality of the braiding, the axiom of a right module, and the coassociativity, we have
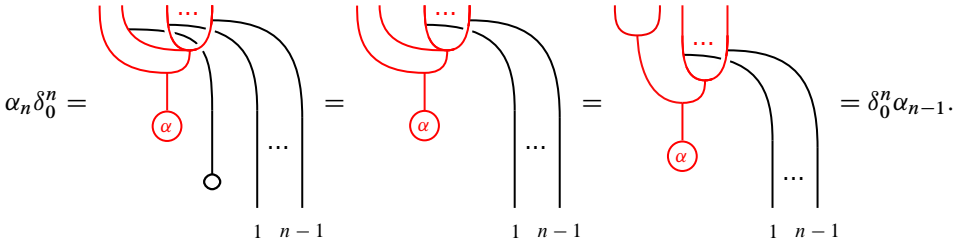
Let $n \geq 2$ and $1 \leq i \leq n-1$. In this case, equation (7.1) follows by definitions, the coassociativity, and the fact that $\Delta_C \colon C \to C \otimes C$ is an $H$-linear morphism:



Finally, let $n \geq 2$ and $i = n$. Equation (7.1) in this case follows by the case $n \geq 2$ and $i = 0$, which is written above, the equation (7.3), which is proven in Section 7.3, and by the paracyclic compatibility relation $\tau_n \delta_0^n = \delta_n^n$ (see Section 3.2). Indeed, we have

$$\alpha_n \delta_n^n = \alpha_n\big(\tau_n(\delta,\sigma)\delta_0^n\big) = \big(\alpha_n \tau_n(\delta,\sigma)\big)\delta_0^n = (\tau_n \alpha_n)\delta_0^n$$
$$= \tau_n(\alpha_n \delta_0^n) = \tau_n(\delta_0^n \alpha_{n-1}) = (\tau_n \delta_0^n)\alpha_{n-1} = \delta_n^n \alpha_{n-1}.$$

## 7.2. Proof of equality (7.2)

Let us show equality (7.2). We consider the three following cases: $j = 0$, $1 \leq j \leq n-1$, and $j = n$. In each case, the desired equality follows by definition, the counitality, the fact that the counit $\varepsilon_C \colon C \to \mathbb{1}$ is an $H$-linear morphism, and the naturality of the braiding.

Indeed, if $j = 0$, then we have

which is $\sigma_0^n \alpha_{n+1}$ by definition. If $1 \leq j \leq n-1$, then



$$= \sigma_j^n \alpha_{n+1}.$$

Finally, if $j = n$, then we have



$$= \sigma_n^n \alpha_{n+1}.$$

## 7.3. Proof of equality (7.3)

Let us verify that equation (7.3) holds. When $n = 0$, this holds since twist morphisms are natural, $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, and since $\tau_0(\delta, \sigma) = \mathrm{id}_{\mathbb{1}}$. Indeed,

$$\tau_0 \alpha_0 = \theta_C \alpha_0 = \alpha_0 = \alpha_0 \tau_0(\delta, \sigma).$$

Let us check it for the case $n = 1$. Indeed, we have



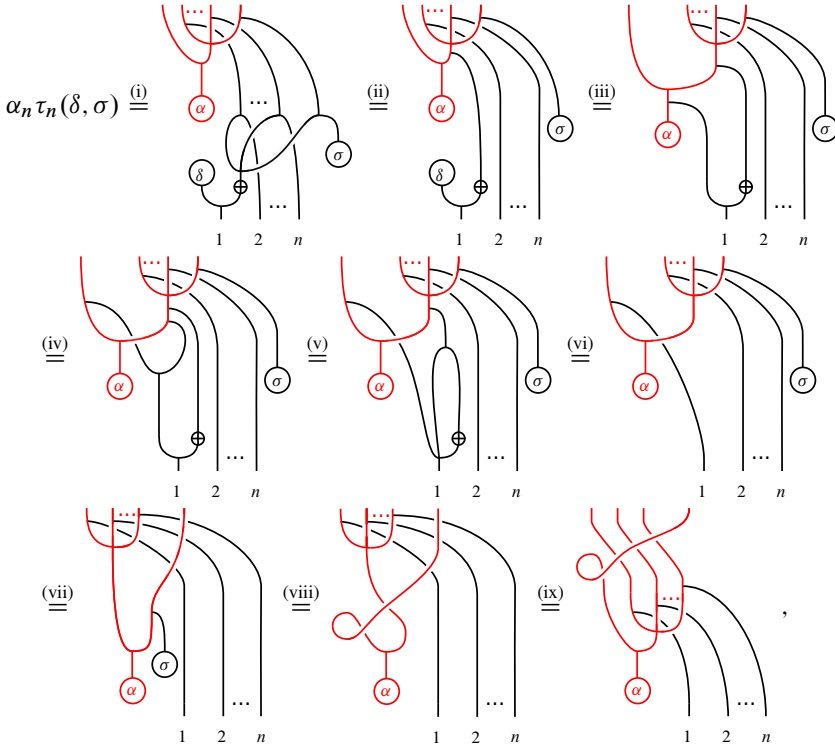$$\alpha_1 \tau_1(\delta, \sigma) \stackrel{(i)}{=} \cdots \stackrel{(ii)}{=} \cdots \stackrel{(iii)}{=} \cdots \stackrel{(iv)}{=} \cdots$$

$$\stackrel{(v)}{=} \cdots \stackrel{(vi)}{=} \cdots \stackrel{(vii)}{=} \cdots \stackrel{(viii)}{=} \tau_1 \alpha_1.$$
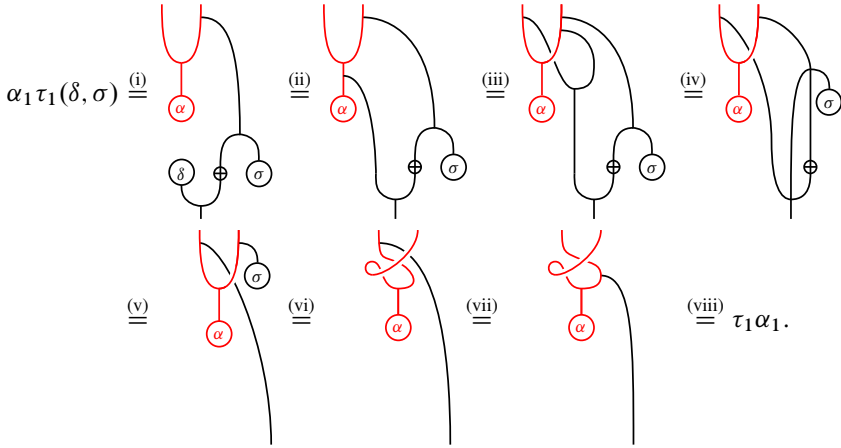
Here (i) and (viii) follow by definition, (ii) follows by the fact that $\alpha$ is $\delta$-invariant, (iii) follows from the fact that the comultiplication $\Delta_C \colon C \to C \otimes C$ is $H$-linear. Equality (iv) follows by the (co)associativity, (v) follows by the antipode axiom and the (co)unitality, (vi) follows from the fact that $\alpha$ is a $\sigma$-trace, and (vii) follows by the naturality of the braiding.

Finally, let us check equality (7.3) when $n \geq 2$. Indeed, we have



$$\alpha_n \tau_n(\delta, \sigma) \stackrel{(i)}{=} \cdots \stackrel{(ii)}{=} \cdots \stackrel{(iii)}{=} \cdots$$

$$\stackrel{(iv)}{=} \cdots \stackrel{(v)}{=} \cdots \stackrel{(vi)}{=} \cdots$$

$$\stackrel{(vii)}{=} \cdots \stackrel{(viii)}{=} \cdots \stackrel{(ix)}{=} \cdots,$$

which equals to $\tau_n \alpha_n$ by definition. Here (i) follows by definition, (ii) follows by applying Lemma 18, (iii) follows from the fact that $\alpha$ is $\delta$-invariant, and (iv) follows from the fact that the comultiplication $\Delta_C \colon C \to C \otimes C$ is $H$-linear. Equality (v) follows from the coassociativity and the right module axiom, (vi) follows by the antipode axiom, the (co)unitality, and the right module axiom, (vii) follows by the coassociativity and the naturality of the braiding, (viii) follows from the fact that $\alpha$ is a $\sigma$-trace, and finally (ix) follows by the naturality of the braiding.

## Appendix

In this appendix, we verify paracyclic compatibility relations (PCR) of the paracocyclic object $\mathbf{CM}_\bullet(H, \delta, \sigma)$ defined in Section 4.3. Note that the verification of simplicial relations (SR) for this object is an easy task. One can show it graphically by using the level-exchange property (see Section 2.4), the coassociativity, and the counitality. Also, the relation $\tau_n \sigma_i^n = \sigma_{i-1}^n \tau_{n+1}$ for $1 \le i \le n$ follows from the bialgebra axiom and by the naturality of the braiding. In this appendix, we show that

(a)  $\tau_n \delta_i^n = \delta_{i-1}^n \tau_{n-1}$ for $n \in \mathbb{N}^*$ and $1 \le i \le n$,

(b)  $\tau_n \delta_0^n = \delta_n^n$ for $n \in \mathbb{N}^*$,

(c)  $\tau_n(\delta, \sigma)\sigma_0^n = \sigma_n^n(\tau_{n+1}(\delta, \sigma))^2$ for $n \in \mathbb{N}$.
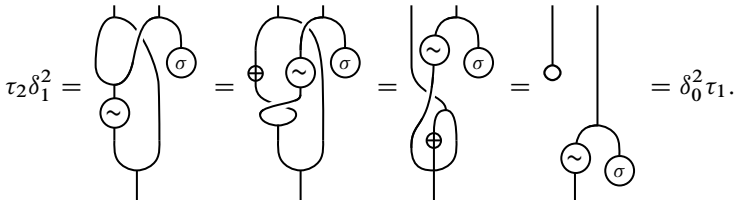
Verifications of (a)–(c) are somewhat involved. Note that (c) is proved for $n = 2$ in [8, Theorem 7.1]. Recall the notion of twisted antipodes and the corresponding notation from Section 6.

### The relation $\tau_n \delta_i^n = \delta_{i-1}^n \tau_{n-1}$

If $n = 1$ and $i = 1$, the relation rewrites as $\tau_1 \delta_1^1 = \delta_0^1 \tau_0$ and it follows by definitions, the fact that $\sigma$ is a coalgebra morphism, the fact that $(\delta, \sigma)$ is a modular pair, and by the antipode axiom:
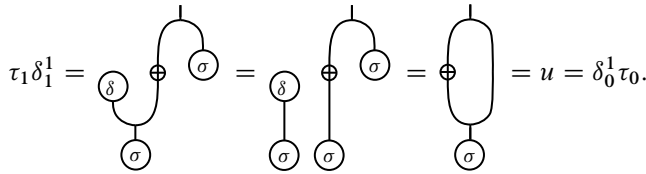


If $n = 2$ and $i = 1$, the relation rewrites as $\tau_2 \delta_1^2 = \delta_0^2 \tau_1$ and it follows by definitions, Lemma 11 (a), the coassociativity, the antipode axiom, and the naturality of the braiding:

If $n = 2$ and $i = 2$, the relation rewrites as $\tau_2 \delta_2^2 = \delta_1^2 \tau_1$ and it follows by definitions, the fact that $\sigma$ is a coalgebra morphism, and the fact that the comultiplication is an algebra morphism:

$$\tau_2 \delta_2^2 = \quad\quad = \quad\quad = \quad\quad = \delta_1^2 \tau_1.$$

Now let $n \geq 3$. For $i = 1$, the relation rewrites as $\tau_n \delta_1^n = \delta_0^n \tau_{n-1}$ and it is true since

$$\tau_n \delta_1^n \overset{(i)}{=} \quad\quad \overset{(ii)}{=} \quad\quad \overset{(iii)}{=}$$

$$\overset{(iv)}{=} \quad\quad \overset{(v)}{=} \quad\quad \overset{(vi)}{=}$$

$$\overset{(vii)}{=} \quad\quad \overset{(viii)}{=} \quad\quad \overset{(ix)}{=} \delta_0^n \tau_{n-1}.$$

Here (i) and (ix) follow by definition, (ii) follows by inductive definition of the left diagonal action, (iii) and (viii) follow from Lemma 11 (a), (iv) and (vi) follow by the naturality of the braiding and the coassociativity, (v) follows by the anti-comultiplicativity of the antipode, and (vii) follows by the antipode axiom, the counitality, and the naturality of the braiding.

For $2 \leq i \leq n - 1$, the relation $\tau_n \delta_i^n = \delta_{i-1}^n \tau_{n-1}$ is a consequence of the fact that the comultiplication is an algebra morphism and the naturality of the braiding. Finally, let us check the case $i = n$. The relation follows by definition, Lemma 13 (b), by the fact that $\sigma$

is a coalgebra morphism, and by the fact that the comultiplication is an algebra morphism:



$$\tau_n \delta_n^n = \cdots = \cdots = \cdots = \delta_{n-1}^n \tau_{n-1}.$$

## The relation $\tau_n \delta_0^n = \delta_n^n$

For $n = 1$, the relation rewrites as $\tau_1 \delta_0^1 = \delta_1^1$ and it follows by definition, the fact that $\delta$ is an algebra morphism, by $\varepsilon S = \varepsilon$, and the fact that the unit is a coalgebra morphism:



$$\tau_1 \delta_0^1 = \cdots = \cdots = \sigma = \delta_1^1.$$

Now let $n \geq 2$. By definition, the fact that $\delta$ is an algebra morphism, by $\varepsilon S = \varepsilon$, the fact that the unit is a coalgebra morphism, the left module axiom, and the naturality of the braiding, we have



$$\tau_n \delta_0^n = \cdots = \cdots = \cdots = \delta_n^n.$$

## The relation $\tau_n(\delta, \sigma)\sigma_0^n = \sigma_n^n(\tau_{n+1}(\delta, \sigma))^2$

In order to show the relation, one can use Theorem 2. We first prove the case $n = 0$. It is true since



$$\sigma_0^0(\tau_1(\delta, \sigma))^2 \overset{\text{(i)}}{=} \cdots \overset{\text{(ii)}}{=} \cdots \overset{\text{(iii)}}{=} \cdots \overset{\text{(iv)}}{=} \cdots \overset{\text{(v)}}{=} \cdots \overset{\text{(vi)}}{=} \tau_0(\delta, \sigma)\sigma_0^0.$$

Here (i) follows by using formula (4.2) of Theorem 2 and the definition of $\sigma_0^0$ and (ii) follows from definition of right adjoint action and left coadjoint action of $H$ on itself, the naturality of the braiding, and the fact that $\delta$ is an algebra morphism and $\sigma$ is a coalgebra

morphism. Equality (iii) follows by applying twice the fact that the counit is an algebra morphism, (iv) follows by the fact that $\varepsilon S = \varepsilon$ and since $\sigma$ is a coalgebra morphism, (v) follows by the fact that $\delta$ is an algebra morphism and by the antipode axiom, and (vi) follows by definition of $\tau_0(\delta, \sigma)$ and $\sigma_0^0$.

Let us now consider the case when $n \geq 1$. Indeed, the relation still holds since
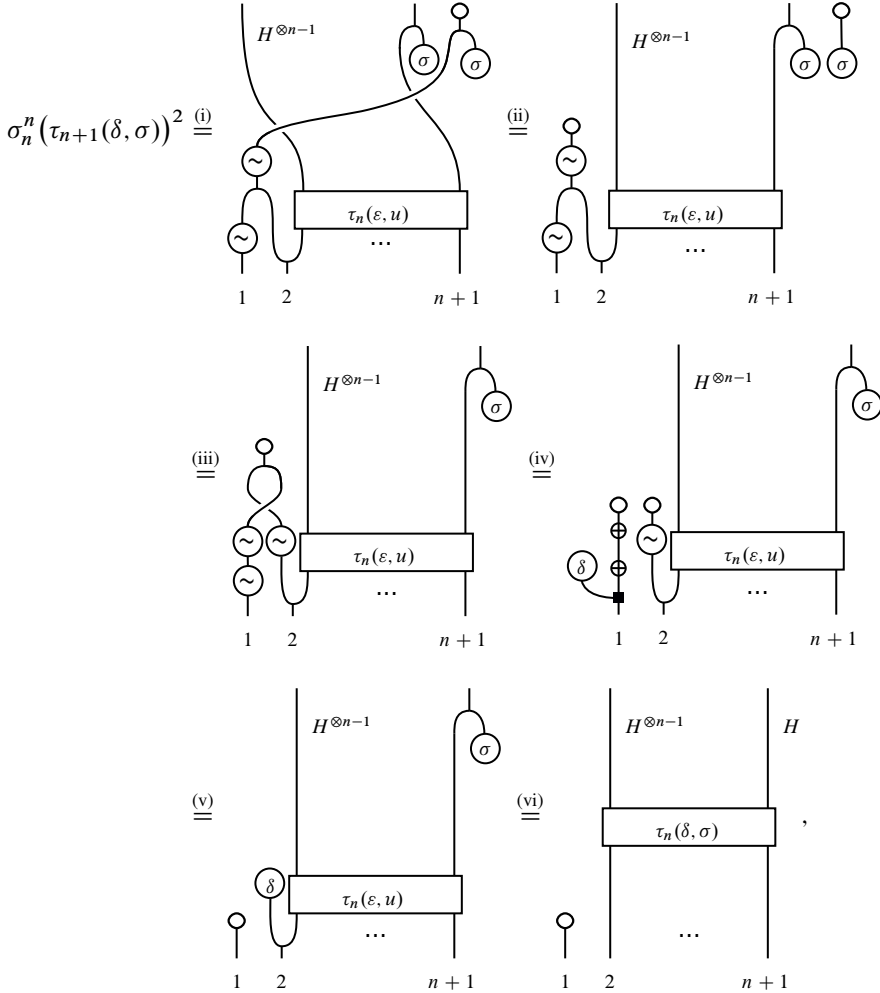


which by definitions equals to $\tau_n(\delta, \sigma)\sigma_0^n$. Here (i) follows by Theorem 2 applied for $n + 1$ and $k = 2$ and by definition of $\sigma_n^n$, (ii) follows by the fact that counit is an algebra morphism and the naturality of the braiding, (iii) follows from Lemma 11 (b) and the fact that $\sigma$ is a coalgebra morphism, (iv) follows from Lemma 11 (e), by the fact that counit is an algebra morphism, and by the naturality of the braiding, (v) follows by the fact that $\varepsilon S = \varepsilon$, by Remark 12, and Lemma 16 (a), and finally (vi) follows by definition of the paracocyclic operator $\tau_n(\delta, \sigma)$.

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