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On the Spaces $O(4n)/Sp$ and $Sp(n)/O$, and the Bott Maps

By

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Introduction

Let $O(n)$ and $Sp(n)$ be the orthogonal and symplectic groups respectively, and consider the homogeneous space $O(4n)/Sp=O(4n)/Sp(n)$, where $Sp(n)$ is embedded in $SO(4n)\subset O(4n)$ by the standard representation. The limit space $O(\infty)/Sp=\varinjlim O(4n)/Sp$ is then homotopically equivalent to the 8th iterated loop space $\Omega^8(O(\infty)/Sp)$, as observed in [11] by N. Ray. This equivalence is derived from the Bott periodicity, and indeed, there are homotopy equivalences

$$O(\infty)/Sp\sim\Omega^4(Sp(\infty)/O) \quad \text{and} \quad Sp(\infty)/O\sim\Omega^4(O(\infty)/Sp),$$

where $Sp(\infty)/O=\varinjlim Sp(n)/O$ with $Sp(n)/O=Sp(n)/O(n)$. Using this periodicity one can define a periodic Ω-spectrum of period 8, and hence a periodic cohomology, whose coefficient group is given by the table in [11], (2.1) (see also [8], Appendix II). In the notation of [11] (and [12]), this cohomology was denoted by $O/Sp^*(\ )$.

In fact, this $O/Sp^*(\ )$ is essentially the mod 2 KO-cohomology, as is now known to many (including Ray). More precisely, there is an isomorphism of cohomologies

$$O/Sp^{r+1}(\ )\simeq KO^{r-2}(\ ;\boldsymbol{Z}/2) \quad (r\in\boldsymbol{Z}),$$

that is, there are homotopy equivalences

$$O(\infty)/Sp\sim\tilde{C}(\boldsymbol{P}_2\boldsymbol{R};O(\infty)) \quad \text{and} \quad Sp(\infty)/O\sim\tilde{C}(\boldsymbol{P}_2\boldsymbol{R};Sp(\infty)),$$

where $\boldsymbol{P}_2\boldsymbol{R}$ is the real projective plane and $\tilde{C}(X;Y)$ denotes the space of basepoint-preserving continuous maps from X to Y. These equivalences can be obtained from much more general results of M. Karoubi (see [6], § 3.2 or [7], Chap. IV, § 6 for instance, and see also [1], § 5) (*).

Our main purpose here is to define certain maps

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(*) Originally, this paper was intended to prove the existence of such equivalences. The author thanks Professors I. M. James and M. Karoubi who kindly (and independently) informed him that the equivalences in question can be derived from results of Professor Karoubi.

$$\varphi_n^O : O(4n)/Sp \longrightarrow \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, O(4n)) \quad \text{and} \quad \varphi_n^{Sp} : Sp(n)/O \longrightarrow \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, Sp(n)),$$

and to show (Theorem (3.6)) that these give rise to homotopy equivalences

$$\varphi_\infty^O : O(\infty)/Sp \longrightarrow \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, O(\infty)) \quad \text{and} \quad \varphi_\infty^{Sp} : Sp(\infty)/O \longrightarrow \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, Sp(\infty))$$

upon passage to direct limits. The definition of these maps is in a sense very similar to that of the Bott maps given, for instance, in [4] or [5].

Also, we shall show that the homomorphism

$$(\varphi_n^O)_* : \pi_r(O(4n)/Sp) \longrightarrow \pi_r(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, O(4n)))$$

induced by φ_n^O is isomorphic for $r \leqq 4n-4$, and that

$$(\varphi_n^{Sp})_* : \pi_r(Sp(n)/O) \longrightarrow \pi_r(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, Sp(n)))$$

induced by φ_n^{Sp} is isomorphic for $r \leqq n-1$.

Our proofs rely heavily on classical results on the Bott maps. The key step is to compare the fibrations

$$U(2n)/Sp \longrightarrow O(4n)/Sp \longrightarrow O(4n)/U$$

and

$$U(n)/O \longrightarrow Sp(n)/O \longrightarrow Sp(n)/U$$

respectively with the fibrations

$$\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R}\,;\, O(4n)) \longrightarrow \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, O(4n)) \longrightarrow \tilde{C}(\boldsymbol{P}_1\boldsymbol{R}\,;\, O(4n))$$

and

$$\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R}\,;\, Sp(n)) \longrightarrow \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\, Sp(n)) \longrightarrow \tilde{C}(\boldsymbol{P}_1\boldsymbol{R}\,;\, Sp(n))$$

associated to the cofibration $\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R} \leftarrow \boldsymbol{P}_2\boldsymbol{R} \leftarrow \boldsymbol{P}_1\boldsymbol{R}$. This will be done in Section 3.

In the following, $\boldsymbol{H}$ stands for the field of quaternions. As usual i and j are the standard generators of the $\boldsymbol{R}$-algebra $\boldsymbol{H}$, and the subfield $\boldsymbol{R}(i)$ of $\boldsymbol{H}$ is identified with the field $\boldsymbol{C}$ of complex numbers. For every field K, the ring of $n \times n$ matrices with elements in K is denoted by $M(n, K)$, and for invertible matrices $A \in GL(n, K)$ and $B \in GL(n, K)$, we denote by $\operatorname{comm}(A, B)$ the commutator $ABA^{-1}B^{-1}$.

§ 1. Preliminaries

We first fix some notations for later use. We put

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \in SO(2n), \qquad K_n = \begin{bmatrix} J_n & 0 \\ 0 & -J_n \end{bmatrix} \in SO(4n),$$

I_n being the identity matrix, and put

$$P_n = \sum_{r=1}^{n} (E_{2r-1,r} + E_{2r,n+r}) \in O(2n), \qquad Q_n = P_{2n} \begin{bmatrix} P_n & 0 \\ 0 & P_n \end{bmatrix} \in O(4n),$$

where $E_{r,s}$ denotes the matrix with a 1 in the (r, s)-position and zeroes elsewhere.

Also we put

$$SpO(2n)=Sp(2n, \boldsymbol{R})\cap O(2n)=\{A\in SO(2n)\,|\,AJ_n=J_nA\},$$
$$SpU(2n)=Sp(2n, \boldsymbol{C})\cap U(2n)=\{A\in SU(2n)\,|\,AJ_n=J_n\bar{A}\},$$

where as usual $Sp(2n, \boldsymbol{R})$ (resp. $Sp(2n, \boldsymbol{C})$) consists of all $2n\times 2n$ matrices A over $\boldsymbol{R}$ (resp. over $\boldsymbol{C}$) such that

$$\det(A)=1 \quad\text{and}\quad AJ_n{}^tA=J_n \quad ({}^tA \text{ being the transpose of } A).$$

Let us define dec: $M(n, \boldsymbol{C})\to M(2n, \boldsymbol{R})$ ("decomplexification") by putting

$$\operatorname{dec}(X+iY)=\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \quad \text{for} \quad X\in M(n, \boldsymbol{R}) \text{ and } Y\in M(n, \boldsymbol{R}),$$

and define deq: $M(n, \boldsymbol{H})\to M(2n, \boldsymbol{C})$ ("dequaternionification") by putting

$$\operatorname{deq}(Z+jW)=\begin{bmatrix} Z & -\bar{W} \\ W & \bar{Z} \end{bmatrix} \quad \text{for} \quad Z\in M(n, \boldsymbol{C}) \text{ and } W\in M(n, \boldsymbol{C}).$$

Then by restriction we get well-known isomorphisms:

$$A\longmapsto \operatorname{dec}(A): U(n)\longrightarrow SpO(2n), \qquad A\longmapsto \operatorname{deq}(A): Sp(n)\longrightarrow SpU(2n).$$

Let $DpO(4n)$ denote the image of $SpU(2n)$ by the isomorphism $A\mapsto\operatorname{dec}(A)$ from $U(2n)$ to $SpO(4n)$, so that

$$DpO(4n)=\{A\in SpO(4n)\,|\,AK_n=K_nA\}.$$

Then $Sp(n)$ is isomorphic to $DpO(4n)$ by $A\mapsto\operatorname{dec}(\operatorname{deq}(A))$.

We write

$$O(2n)/U=O(2n)/P_nSpO(2n)P_n^{-1}, \qquad Sp(n)/U=Sp(n)/U(n),$$
$$O(4n)/Sp=O(4n)/Q_nDpO(4n)Q_n^{-1}, \qquad Sp(n)/O=Sp(n)/O(n),$$
$$U(2n)/Sp=U(2n)/P_nSpU(2n)P_n^{-1}, \qquad U(n)/O=U(n)/O(n),$$

where

$$P_nSpO(2n)P_n^{-1}=\{P_nAP_n^{-1}\,|\,A\in SpO(2n)\}\subset SO(2n),$$
$$Q_nDpO(4n)Q_n^{-1}=\{Q_nAQ_n^{-1}\,|\,A\in DpO(4n)\}\subset P_{2n}SpO(4n)P_{2n}^{-1}\subset SO(4n),$$
$$P_nSpU(2n)P_n^{-1}=\{P_nAP_n^{-1}\,|\,A\in SpU(2n)\}\subset SU(2n),$$

and we define the limit spaces

$$O(\infty)/U=\varinjlim O(2n)/U, \qquad Sp(\infty)/U=\varinjlim Sp(n)/U,$$
$$O(\infty)/Sp=\varinjlim O(4n)/Sp, \qquad Sp(\infty)/O=\varinjlim Sp(n)/O,$$
$$U(\infty)/Sp=\varinjlim U(2n)/Sp, \qquad U(\infty)/O=\varinjlim U(n)/O,$$

by using the canonical injections $O(2n)/U\to O(2n+2)/U\to\cdots$, etc., induced by

$$A\longmapsto\begin{bmatrix} A & 0 \\ 0 & I_1 \end{bmatrix}\longmapsto\begin{bmatrix} A & 0 \\ 0 & I_2 \end{bmatrix}\longmapsto\cdots.$$

Also, we denote by

$$\xi_n^{O/U}: O(2n)\longrightarrow O(2n)/U, \qquad \xi_n^{Sp/U}: Sp(n)\longrightarrow Sp(n)/U,$$
$$\xi_n^{O/Sp}: O(4n)\longrightarrow O(4n)/Sp, \qquad \xi_n^{Sp/O}: Sp(n)\longrightarrow Sp(n)/O,$$
$$\xi_n^{U/Sp}: U(2n)\longrightarrow U(2n)/Sp, \qquad \xi_n^{U/O}: U(n)\longrightarrow U(n)/O,$$

the canonical surjections, and we define the canonical injections

$$\kappa_n: U(2n)/Sp \longrightarrow O(4n)/Sp \quad \text{and} \quad \iota_n: U(n)/O \longrightarrow Sp(n)/O$$

by putting

$$\kappa_n(\xi_n^{U/Sp}(P_n A P_n^{-1})) = \xi_n^{O/Sp}(Q_n \operatorname{dec}(A) Q_n^{-1}) \qquad \text{for} \quad A \in U(2n)\,,$$

and

$$\iota_n(\xi_n^{U/O}(A)) = \xi_n^{Sp/O}(A) \qquad \text{for} \quad A \in U(n)\,.$$

And letting $\kappa_\infty = \varinjlim \kappa_n$, $\iota_\infty = \varinjlim \iota_n$, we define the injections

$$\kappa_\infty: U(\infty)/Sp \longrightarrow O(\infty)/Sp \quad \text{and} \quad \iota_\infty: U(\infty)/O \longrightarrow Sp(\infty)/O\,.$$

§ 2. Bott Maps

Retaining the notation of Section 1, now let $\Omega(X)$ denote the loop space of X, and consider the maps

$$\omega_n^O: O(2n)/U \longrightarrow \Omega(O(2n))\,, \qquad \omega_n^{Sp}: Sp(n)/U \longrightarrow \Omega(Sp(n))\,,$$
$$\omega_n^{O/U}: U(2n)/Sp \longrightarrow \Omega(O(4n)/U)\,, \qquad \omega_n^{Sp/U}: U(n)/O \longrightarrow \Omega(Sp(n)/U)$$

defined as follows:

$$\omega_n^O(\xi_n^{O/U}(P_n A P_n^{-1}))(t) = P_n \operatorname{comm}(\exp(\pi t J_n),\, A) P_n^{-1}$$

where $A \in O(2n)$, $t \in [0, 1]$;

$$\omega_n^{O/U}(\xi_n^{U/Sp}(P_n A P_n^{-1}))(t) = \xi_{2n}^{O/U}\Big(Q_n \operatorname{comm}\Big(\exp\Big(\frac{\pi}{2} t K_n\Big),\, \operatorname{dec}(A)\Big) Q_n^{-1}\Big)$$
$$= \xi_{2n}^{O/U}\Big(Q_n \exp\Big(\frac{\pi}{2} t K_n\Big) \operatorname{dec}(A) \exp\Big(-\frac{\pi}{2} t K_n\Big) Q_n^{-1}\Big)$$

where $A \in U(2n)$, $t \in [0, 1]$;

$$\omega_n^{Sp}(\xi_n^{Sp/U}(A))(t) = \operatorname{comm}(\exp(\pi t i I_n),\, A)$$

where $A \in Sp(n)$, $t \in [0, 1]$;

$$\omega_n^{Sp/U}(\xi_n^{U/O}(A))(t) = \xi_n^{Sp/U}\Big(\operatorname{comm}\Big(\exp\Big(\frac{\pi}{2} t j I_n\Big),\, A\Big)\Big)$$
$$= \xi_n^{Sp/U}\Big(\exp\Big(\frac{\pi}{2} t j I_n\Big) A \exp\Big(-\frac{\pi}{2} t j I_n\Big)\Big)$$

where $A \in U(n)$, $t \in [0, 1]$. Here $\operatorname{comm}(A, B)$ denotes $ABA^{-1}B^{-1}$, the commutator, and exp is the exponential map, so that

$$\exp(tJ_n) = I_{2n}\cos(t) + J_n \sin(t)\,, \qquad \exp(tiI_n) = I_n \cos(t) + iI_n \sin(t)\,,$$
$$\exp(tK_n) = I_{4n}\cos(t) + K_n \sin(t)\,, \qquad \exp(tjI_n) = I_n \cos(t) + jI_n \sin(t)\,,$$

for every $t \in \boldsymbol{R}$. By passing to direct limits (and letting $\omega_\infty^O = \varinjlim \omega_n^O$, etc.), we then get maps

$$\omega_\infty^O: O(\infty)/U \longrightarrow \Omega(O(\infty)), \qquad \omega_\infty^{Sp}: Sp(\infty)/U \longrightarrow \Omega(Sp(\infty)),$$
$$\omega_\infty^{O/U}: U(\infty)/Sp \longrightarrow \Omega(O(\infty)/U), \qquad \omega_\infty^{Sp/U}: U(\infty)/O \longrightarrow \Omega(Sp(\infty)/U),$$

and the following theorem is due to Bott and others:

Theorem 2.1 (see [2], [3], [4], [5], and also [10], § 24). *The maps ω_∞^O, $\omega_\infty^{O/U}$, ω_∞^{Sp} and $\omega_\infty^{Sp/U}$ are homotopy equivalences, and*:

(i) *the homomorphism $(\omega_n^O)_*: \pi_r(O(2n)/U) \to \pi_{r+1}(O(2n))$ induced by ω_n^O is isomorphic for $r \leqq 2n-3$;*

(ii) *the homomorphism $(\omega_n^{O/U})_*: \pi_r(U(2n)/Sp) \to \pi_{r+1}(O(4n)/U)$ induced by $\omega_n^{O/U}$ is isomorphic for $r \leqq 4n-3$;*

(iii) *the homomorphism $(\omega_n^{Sp})_*: \pi_r(Sp(n)/U) \to \pi_{r+1}(Sp(n))$ induced by ω_n^{Sp} is isomorphic for $r \leqq 2n$;*

(iv) *the homomorphism $(\omega_n^{Sp/U})_*: \pi_r(U(n)/O) \to \pi_{r+1}(Sp(n)/U)$ induced by $\omega_n^{Sp/U}$ is isomorphic for $r \leqq n-1$.*

Remark. The assertions (i), (ii), (iii) and (iv) can easily be verified if we recall that the homomorphisms

$$\begin{array}{ll} \pi_r(O(n)) \longrightarrow \pi_r(O(\infty)) & \text{for } r \leqq n-2, \\ \pi_r(O(2n)/U) \longrightarrow \pi_r(O(\infty)/U) & \text{for } r \leqq 2n-2, \\ \pi_r(U(2n)/Sp) \longrightarrow \pi_r(U(\infty)/Sp) & \text{for } r \leqq 4n-1, \\ \pi_r(Sp(n)) \longrightarrow \pi_r(Sp(\infty)) & \text{for } r \leqq 4n+1, \\ \pi_r(Sp(n)/U) \longrightarrow \pi_r(Sp(\infty)/U) & \text{for } r \leqq 2n, \\ \pi_r(U(n)/O) \longrightarrow \pi_r(U(\infty)/O) & \text{for } r \leqq n-1, \end{array}$$

induced by the canonical injections, are isomorphic.

§ 3. The Maps φ_n^O and φ_n^{Sp}

Henceforth we use the following notations and conventions:

If X is a compact space with a basepoint, and Y is a topological space with a basepoint, then $\tilde{C}(X; Y)$ denotes the space of basepoint-preserving continuous maps from X to Y, equipped with the compact-open topology.

If $(x_0, x_1, x_2) \in \boldsymbol{R}^3$ and $(x_0, x_1, x_2) \neq (0, 0, 0)$, we write $[x_0: x_1: x_2]$ for the point of $\boldsymbol{P}_2\boldsymbol{R}$ whose homogeneous coordinates are x_0, x_1, x_2. The subspace

$$\{[x_0: x_1: 0] \in \boldsymbol{P}_2\boldsymbol{R} \mid (x_0, x_1) \in \boldsymbol{R}^2, (x_0, x_1) \neq (0, 0)\}$$

of $\boldsymbol{P}_2\boldsymbol{R}$ is identified with the real projective line $\boldsymbol{P}_1\boldsymbol{R}$ in the obvious way, and we write $[x_0: x_1]$ instead of $[x_0: x_1: 0]$.

For any space Y with a basepoint, we identify $\tilde{C}(\boldsymbol{P}_1\boldsymbol{R}; Y)$ with the loop space $\Omega(Y)$ of Y, by the homeomorphism $\tilde{C}(\boldsymbol{P}_1\boldsymbol{R}; Y) \to \Omega(Y)$ induced by the map

$$t \longmapsto [\cos(\pi t): \sin(\pi t)]$$

from $[0, 1]$ to $\boldsymbol{P}_1\boldsymbol{R}$. Also, we identify $\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R}; Y)$ with the double loop space $\Omega^2(Y) = \Omega(\Omega(Y))$ in the following way: Let p be the canonical map from

P_2R onto P_2R/P_1R. Then each element f of $\tilde{C}(P_2R/P_1R; Y)$ is regarded as an element of $\Omega(\Omega(Y))$ by putting

$$f(s)(t)=f(p([\cos(\pi t):\sin(\pi t)\cos(\pi s):\sin(\pi t)\sin(\pi s)]))$$

for $s\in[0, 1]$ and $t\in[0, 1]$.

With these in mind, consider now the diagrams

$$(3.1)\qquad \begin{array}{ccccc}
U(2n)/Sp & \xrightarrow{\kappa_n} & O(4n)/Sp & \longrightarrow & O(4n)/U \\
\downarrow{\scriptstyle \omega_n^{O/U}} & & & & \\
\Omega(O(4n)/U) & (3.1a) & \downarrow{\scriptstyle \varphi_n^O} & (3.1b) & \downarrow{\scriptstyle \omega_{2n}^O} \\
\downarrow{\scriptstyle \Omega(\omega_{2n}^O)} & & & & \\
\Omega^2(O(4n)) & & & & \Omega(O(4n)) \\
\| & & & & \| \\
\tilde{C}(P_2R/P_1R; O(4n)) & \longrightarrow & \tilde{C}(P_2R; O(4n)) & \longrightarrow & \tilde{C}(P_1R; O(4n))
\end{array}$$

and

$$(3.2)\qquad \begin{array}{ccccc}
U(n)/O & \xrightarrow{\iota_n} & Sp(n)/O & \longrightarrow & Sp(n)/U \\
\downarrow{\scriptstyle \omega_n^{Sp/U}} & & & & \\
\Omega(Sp(n)/U) & (3.2a) & \downarrow{\scriptstyle \varphi_n^{Sp}} & (3.2b) & \downarrow{\scriptstyle \omega_n^{Sp}} \\
\downarrow{\scriptstyle \Omega(\omega_n^{Sp})} & & & & \\
\Omega^2(Sp(n)) & & & & \Omega(Sp(n)) \\
\| & & & & \| \\
\tilde{C}(P_2R/P_1R; Sp(n)) & \longrightarrow & \tilde{C}(P_2R; Sp(n)) & \longrightarrow & \tilde{C}(P_1R; Sp(n))
\end{array}$$

where the top rows are the obvious fibration sequences and the bottom rows are induced by the cofibration sequence

$$P_2R/P_1R \longleftarrow P_2R \longleftarrow P_1R,$$

and where the maps φ_n^O and φ_n^{Sp} are defined as follows:

$$\varphi_n^O(\xi_n^{O/Sp}(Q_nAQ_n^{-1}))([u_0:u_1:u_2])=Q_n\mathrm{comm}(u_0I_{4n}+u_1J_{2n}+u_2K_n, A)Q_n^{-1}$$

where $A\in O(4n)$, $(u_0, u_1, u_2)\in S^2$;

$$\varphi_n^{Sp}(\xi_n^{Sp/O}(A))([u_0:u_1:u_2])=\mathrm{comm}(u_0I_n+u_1iI_n+u_2jI_n, A)$$

where $A\in Sp(n)$, $(u_0, u_1, u_2)\in S^2$. Here $S^2=\{(u_0, u_1, u_2)\in R^3\,|\,u_0^2+u_1^2+u_2^2=1\}$ is the unit sphere. By passing to direct limits, we then get diagrams

$$
(3.3)\qquad
\begin{array}{ccccc}
U(\infty)/Sp & \xrightarrow{\kappa_\infty} & O(\infty)/Sp & \longrightarrow & O(\infty)/U \\
\downarrow{\scriptstyle \omega_\infty^{O/U}} & & & & \\
\Omega(O(\infty)/U) & (3.3\mathrm{a}) & \downarrow{\scriptstyle \varphi_\infty^{O}} & (3.3\mathrm{b}) & \downarrow{\scriptstyle \omega_\infty^{O}} \\
\downarrow{\scriptstyle \Omega(\omega_\infty^{O})} & & & & \\
\Omega^2(O(\infty)) & & & & \Omega(O(\infty)) \\
\| & & & & \| \\
\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R};\, O(\infty)) & \longrightarrow & \tilde{C}(\boldsymbol{P}_2\boldsymbol{R};\, O(\infty)) & \longrightarrow & \tilde{C}(\boldsymbol{P}_1\boldsymbol{R};\, O(\infty))
\end{array}
$$

and

$$
(3.4)\qquad
\begin{array}{ccccc}
U(\infty)/O & \xrightarrow{\iota_\infty} & Sp(\infty)/O & \longrightarrow & Sp(\infty)/U \\
\downarrow{\scriptstyle \omega_\infty^{Sp/U}} & & & & \\
\Omega(Sp(\infty)/U) & (3.4\mathrm{a}) & \downarrow{\scriptstyle \varphi_\infty^{Sp}} & (3.4\mathrm{b}) & \downarrow{\scriptstyle \omega_\infty^{Sp}} \\
\downarrow{\scriptstyle \Omega(\omega_\infty^{Sp})} & & & & \\
\Omega^2(Sp(\infty)) & & & & \Omega(Sp(\infty)) \\
\| & & & & \| \\
\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R};\, Sp(\infty)) & \longrightarrow & \tilde{C}(\boldsymbol{P}_2\boldsymbol{R};\, Sp(\infty)) & \longrightarrow & \tilde{C}(\boldsymbol{P}_1\boldsymbol{R};\, Sp(\infty))
\end{array}
$$

where $\varphi_\infty^O = \varinjlim \varphi_n^O$ and $\varphi_\infty^{Sp} = \varinjlim \varphi_n^{Sp}$.

Proposition 3.5. *The diagrams* (3.1), (3.2), (3.3) *and* (3.4) *are all homotopy-commutative. In particular,* (3.1b), (3.2b), (3.3b) *and* (3.4b) *are strictly commutative.*

For the proof, see Section 4, the next section.

Now note that all the rows in the diagrams (3.1), (3.2), (3.3) and (3.4) are Hurewicz fibration sequences. If we combine (2.1) and (3.5), we obtain:

Theorem 3.6 (compare with [6], § 3.2 or [7], Chap. IV, § 6). *The maps* φ_∞^O *and* φ_∞^{Sp} *are homotopy equivalences, and:*

(i) *the homomorphism* $(\varphi_n^O)_*: \pi_r(O(4n)/Sp) \to \pi_r(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R};\, O(4n)))$ *induced by* φ_n^O *is isomorphic for* $r \leqq 4n-4$;

(ii) *the homomorphism* $(\varphi_n^{Sp})_*: \pi_r(Sp(n)/O) \to \pi_r(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R};\, Sp(n)))$ *induced by* φ_n^{Sp} *is isomorphic for* $r \leqq n-1$.

Proof. The assertions (i) and (ii) follow immediately from (2.1) and (3.5) by the five-lemma. It also follows that

$$(\varphi_\infty^O)_*: \pi_r(O(\infty)/Sp) \longrightarrow \pi_r(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R};\, O(\infty)))$$

and

$$(\varphi_\infty^{Sp})_*: \pi_r(Sp(\infty)/O) \longrightarrow \pi_r(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}; Sp(\infty)))$$

are isomorphic for all r. By J. H. C. Whitehead's theorem (and by [9], Theorem 3), the map φ_∞^{Sp} is therefore a homotopy equivalence, since $Sp(\infty)/O$ and $\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}; Sp(\infty))$ are both connected. To conclude that φ_∞^{O} is also a homotopy equivalence, we must be more careful, since $O(\infty)/Sp$ and $\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}; O(\infty))$ are not connected. But by the same argument as in [5], § 1 we can easily see that φ_∞^{O} is a homomorphism of Hopf spaces, and hence, noting that

$$(\varphi_\infty^{O})_*: \pi_0(O(\infty)/Sp) \longrightarrow \pi_0(\tilde{C}(\boldsymbol{P}_2\boldsymbol{R}; O(\infty)))$$

is bijective, we see that φ_∞^{O} is a homotopy equivalence. This completes the proof.

§ 4. Proof of Proposition 3.5

This section is devoted to the proof of (3.5). First notice the following:

$$\varphi_n^{O}(\xi_n^{O/Sp}(Q_nAQ_n^{-1}))([\cos(\pi t): \sin(\pi t)\cos(\pi s): \sin(\pi t)\sin(\pi s)])$$
$$= Q_n \mathrm{comm}\Big(\exp\Big(\frac{\pi}{2}sJ_{2n}K_n\Big)\exp(\pi tJ_{2n})\exp\Big(-\frac{\pi}{2}sJ_{2n}K_n\Big),\ A\Big)Q_n^{-1}$$

where $A\in O(4n)$, $s\in[0, 1]$, $t\in[0, 1]$;

$$\varphi_n^{Sp}(\xi_n^{Sp/O}(A))([\cos(\pi t): \sin(\pi t)\cos(\pi s): \sin(\pi t)\sin(\pi s)])$$
$$= \mathrm{comm}\Big(\exp\Big(\frac{\pi}{2}sijI_n\Big)\exp(\pi tiI_n)\exp\Big(-\frac{\pi}{2}sijI_n\Big),\ A\Big)$$

where $A\in Sp(n)$, $s\in[0, 1]$, $t\in[0, 1]$. Hence we easily see that (3.1b), (3.2b), (3.3b) and (3.4b) strictly commute.

Next we shall show that (3.1a), (3.2a), (3.3a) and (3.4a) commute up to homotopy. Put

$$F_n(r, s, t)$$
$$= \exp\Big(\frac{\pi}{4}rJ_{2n}\Big)\exp\Big(\frac{\pi}{2}sJ_{2n}K_n\Big)\exp(\pi tJ_{2n})\exp\Big(-\frac{\pi}{2}sJ_{2n}K_n\Big)\exp\Big(-\frac{\pi}{4}rJ_{2n}\Big),$$

$$G_n(r, s, t)$$
$$= \exp\Big(\frac{\pi}{4}riI_n\Big)\exp\Big(\frac{\pi}{2}sijI_n\Big)\exp(\pi tiI_n)\exp\Big(-\frac{\pi}{2}sijI_n\Big)\exp\Big(-\frac{\pi}{4}riI_n\Big),$$

and for each $r\in[0, 1]$, define the maps

$$\Theta_n^{O}(r): U(2n)/Sp \longrightarrow \Omega^2(O(4n)) \quad\text{and}\quad \Theta_n^{Sp}(r): U(n)/O \longrightarrow \Omega^2(Sp(n))$$

as follows:

$$\Theta_n^{O}(r)(\xi_n^{U/Sp}(P_nAP_n^{-1}))(s)(t)$$
$$= Q_n\exp\Big(\frac{\pi}{2}rsK_n\Big)\mathrm{comm}(F_n(r, s, t), \mathrm{dec}(A))\exp\Big(-\frac{\pi}{2}rsK_n\Big)Q_n^{-1}$$

where $A\in U(2n)$, $s\in[0, 1]$, $t\in[0, 1]$;

$$\begin{aligned}&\Theta_n^{Sp}(r)(\xi_n^{U/O}(A))(s)(t)\\&\quad=\exp\left(\frac{\pi}{2}rsjI_n\right)\mathrm{comm}(G_n(r,\ s,\ t),\ A)\exp\left(-\frac{\pi}{2}rsjI_n\right)\end{aligned}$$

where $A\in U(n)$, $s\in[0,\ 1]$, $t\in[0,\ 1]$. Then as is easily seen, the diagrams

$$\begin{array}{ccc}U(2n)/Sp & \xrightarrow{\kappa_n} & O(4n)/Sp\\ \downarrow{\scriptstyle \Theta_n^O(0)} & & \\ \Omega^2(O(4n)) & & \downarrow{\scriptstyle \varphi_n^O}\\ \| & & \\ \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R}\,;\ O(4n)) & \longrightarrow & \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\ O(4n))\end{array}$$

and

$$\begin{array}{ccc}U(n)/O & \xrightarrow{\iota_n} & Sp(n)/O\\ \downarrow{\scriptstyle \Theta_n^{Sp}(0)} & & \\ \Omega^2(Sp(n)) & & \downarrow{\scriptstyle \varphi_n^{Sp}}\\ \| & & \\ \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R}\,;\ Sp(n)) & \longrightarrow & \tilde{C}(\boldsymbol{P}_2\boldsymbol{R}\,;\ Sp(n))\end{array}$$

strictly commute, where, as in (3.1a) and (3.2a), the bottom maps are induced by the canonical map from $\boldsymbol{P}_2\boldsymbol{R}$ onto $\boldsymbol{P}_2\boldsymbol{R}/\boldsymbol{P}_1\boldsymbol{R}$. On the other hand,

$$F_n(1,\ s,\ t)=\exp\left(-\frac{\pi}{2}sK_n\right)\exp(\pi tJ_{2n})\exp\left(\frac{\pi}{2}sK_n\right),$$

$$G_n(1,\ s,\ t)=\exp\left(-\frac{\pi}{2}sjI_n\right)\exp(\pi tiI_n)\exp\left(\frac{\pi}{2}sjI_n\right),$$

and direct calculations show that

$$\Theta_n^O(1)=\Omega(\omega_{2n}^O)\circ\omega_n^{O/U}\quad\text{and}\quad\Theta_n^{Sp}(1)=\Omega(\omega_n^{Sp})\circ\omega_n^{Sp/U}.$$

Hence the homotopy-commutativity of (3.1a) and (3.2a) is clear, and considering

$$\Theta_\infty^O(r)=\varinjlim\Theta_n^O(r)\quad\text{and}\quad\Theta_\infty^{Sp}(r)=\varinjlim\Theta_n^{Sp}(r),$$

we see that (3.3a) and (3.4a) are also homotopy-commutative.

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Added in proof: After submitting this paper, the author became aware of the following paper, in which one can find a generalisation of our result about O/Sp: T. Bier and U. Schwardmann, Räume normierter Bilinearformen und Cliffordstrukturen, *Math. Z.*, **180** (1982), 203-215.